

ENUMERATIONS AND BIJECTIONS FOR STANLEY POLYOMINOES

JEAN-LUC BARIL, AUBREY BLECHER, AND JOSÉ L. RAMÍREZ

ABSTRACT. Stanley polyominoes are a subclass of parallelogram polyominoes in which each row begins strictly to the right of the beginning of the previous row and ends strictly to the right of the end of the previous row. In this paper, we derive generating functions for Stanley polyominoes based on the numbers of columns and rows, area, semiperimeter, and numbers of interior points and edges. We also establish combinatorial connections through bijections with other combinatorial structures such as Dyck paths, skew Ferrer diagrams, and peakless Motzkin paths. As a byproduct, we answer the open question of finding a bijection between parallelogram polyominoes of area n and coin fountains with n coins in the even-numbered rows and $n - k$ coins in the odd-numbered rows.

1. INTRODUCTION

A *polyomino* is a finite connected set of cells of the square lattice, where connectivity is defined by edge adjacency (see [19]). A *parallelogram polyomino* is a polyomino such that the intersection with any line perpendicular to the main diagonal is a connected segment; equivalently, it is delimited by two non-intersecting lattice paths consisting of north and east steps and having the same endpoints. Parallelogram polyominoes form a well-studied subclass of polyominoes, with strong connections to lattice paths and Catalan-type structures. They have been extensively studied from several enumerative points of view including generating functions, bijections, and algebraic methods (see [6, 7, 10, 11, 17, 23]).

In this paper, we study a subclass of parallelogram polyominoes, which we call *Stanley polyominoes*, as introduced in Stanley's book [22]. These are parallelogram polyominoes in which each row, read from bottom to top, begins strictly to the right of the beginning of the previous row and ends strictly to the right of the end of the previous row. See Figure 1 for an example. As noted in [22], the number of Stanley polyominoes with $n + 1$ columns is the n th Catalan number

$$C_n = \frac{1}{n+1} \binom{2n}{n}.$$

This result is naturally explained by a bijection ϕ (defined below) from Stanley polyominoes with $n + 1$ columns to Dyck paths of semilength n , that is, lattice paths of $\mathbb{N}^2$ starting at $(0, 0)$, ending at $(2n, 0)$, and consisting of up-steps $u = (1, 1)$ and down-steps $d = (1, -1)$.

Date: April 12, 2026.

2010 Mathematics Subject Classification. 05A05, 05A15, 05A19, 05A30.

Key words and phrases. Stanley polyomino, Catalan number, Generating function, Kernel method, Bijections.

Let P be a Stanley polyomino with k rows and $n + 1$ columns. Define the sequence $a(P) = (a_1, a_2, \dots, a_k)$ as follows: a_1 is the number of cells in the first row minus one, and for $2 \leq i \leq k$, a_i is the number of cells in the i th row that are not directly above a cell of the $(i - 1)$ th row. Similarly, define the sequence $b(P) = (b_1, b_2, \dots, b_k)$ as follows: b_k is the number of cells in the last row minus one, and for $1 \leq i \leq k - 1$, b_i is the number of cells in the i th row that are not directly below a cell of the $(i + 1)$ th row.

By definition of a Stanley polyomino, we have

$$a_1 + a_2 + \dots + a_k = n = b_1 + b_2 + \dots + b_k.$$

Moreover, for any i with $1 \leq i \leq k$, we have $\sum_{j=1}^i a_j \geq \sum_{j=1}^i b_j$.

We now define the map ϕ by

$$\phi(P) = u^{a_1} d^{b_1} u^{a_2} d^{b_2} \dots u^{a_k} d^{b_k}.$$

We refer to Figure 1 for an illustration of the bijection ϕ on a Stanley polyomino. For the polyomino shown in Figure 1, we have

$$a(P) = (5, 3, 2, 2, 3) \quad \text{and} \quad b(P) = (3, 1, 6, 1, 4).$$

Therefore,

$$\phi(P) = u^5 d^3 u^3 d u^2 d^6 u^2 d u^3 d^4.$$

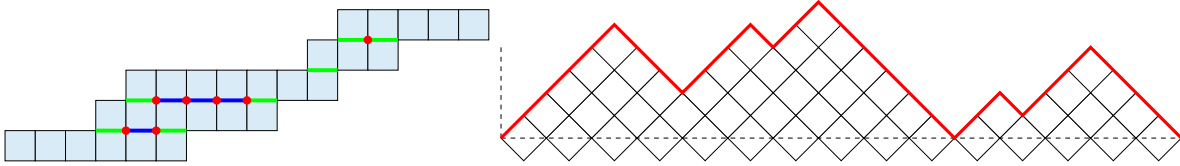


FIGURE 1. A Stanley Polyomino and its associated Dyck path by ϕ .

We now introduce several classical statistics on polyominoes. We refer to [19] for a historical review of polyominoes, and to [18] for definitions of several statistics and enumerative methods related to polyominoes.

Let P be a Stanley polyomino. We denote by $\mathbf{first}(P)$ the number of cells in the first row. We denote by $\mathbf{col}(P)$ and $\mathbf{row}(P)$ the number of columns and rows of P , respectively. The *semiperimeter* of P , denoted by $\mathbf{sper}(P)$, is half of the *perimeter* of P , where the perimeter is the number of cell borders that are not shared by two cells of P . Thus, $\mathbf{sper}(P) = \mathbf{col}(P) + \mathbf{row}(P)$. We denote by $\mathbf{area}(P)$ the area of P , that is, the number of cells of P . An *interior point* of P is a point belonging to exactly four cells of P , and we denote by $\mathbf{point}(P)$ the number of interior points of P . We denote by $\mathbf{edgint}(P)$ the number of strictly internal edges, that is, horizontal edges joining two interior points. We denote by $\mathbf{adja}(P)$ the number of weakly internal edges, that is, horizontal edges joining two adjacent cells. All these parameters have been extensively investigated across various classes of polyominoes in the literature (see, for example, [3–5, 8, 9, 11, 13, 15, 17, 18]).

For the polyomino shown in Figure 1, we have $\mathbf{col}(P) = 16$, $\mathbf{row}(P) = 5$, $\mathbf{sper}(P) = 16 + 5 = 21$, $\mathbf{area}(P) = 27$, $\mathbf{point}(P) = 7$, and $\mathbf{edgint}(P) = 4$.

To describe how these statistics on Stanley polyominoes are transported by ϕ to Dyck paths, we recall a few standard definitions for Dyck paths. A *peak* (resp. *valley*) is an occurrence of the consecutive subword ud (resp. du). The *height* of a peak (resp. valley) is the ordinate of the middle point of the occurrence. A 1-valley is a valley of height at least one. The semilength of a Dyck path is the number of its up-steps.

Let P be a Stanley polyomino, and let $a = (a_1, a_2, \dots, a_k)$, $b = (b_1, b_2, \dots, b_k)$ be the sequences defined above. Let $\phi(P)$ be the image of P under ϕ . By construction, the number of rows of P is k , while the number of columns is $1 + a_1 + a_2 + \dots + a_k$. These correspond, respectively, to the number of peaks and to the semilength plus one in $\phi(P)$. Since the semiperimeter is the sum of the numbers of columns and rows, it corresponds to the number of peaks plus the semilength plus one. Now let c_i denote the number of cells in the i th row of P . Then $c_1 = 1 + a_1$ and, for $i \geq 2$,

$$c_i = 1 + \sum_{j=1}^{i-1} (a_j - b_j) + a_i.$$

Observe that c_i is equal to the height of the i th peak of $\phi(P)$ plus one. Since the area of P is the sum of the row lengths c_i , it follows that $\text{area}(P)$ corresponds to the sum of peak heights plus the number of peaks in $\phi(P)$. The correspondences for $\text{point}(P)$, $\text{adja}(P)$, and $\text{edgint}(P)$ are obtained similarly. Table 1 summarizes the main correspondences between the statistics on Stanley polyominoes and Dyck paths.

Notation	Statistics on polyominoes	Statistics on Dyck paths
$\text{col}(P)$	Number of columns	Semilength + 1
$\text{row}(P)$	Number of rows	Number of peaks
$\text{sper}(P)$	Semiperimeter	Number of peaks + semilength + 1
$\text{first}(P)$	Number of cells in the first row	Height of the first peak + 1
$\text{area}(P)$	Area	Sum of peak heights + number of peaks
$\text{point}(P)$	Number of interior points	Sum of valley heights
$\text{adja}(P)$	Number of weakly internal edges	Sum of valley heights + number of valleys
$\text{edgint}(P)$	Number of strictly internal edges	Sum of 1-valley heights - number of 1-valleys

TABLE 1. Correspondence between statistics on Stanley polyominoes and Dyck paths.

Motivation and outline of the paper. The enumeration of parallelogram polyominoes with respect to geometric and combinatorial parameters has been extensively studied in the literature. In this paper, we continue this line of research for Stanley polyominoes. Our starting point is a bijection ϕ with Dyck paths, which allows us to transport several natural statistics and to derive a multivariate generating function for Stanley polyominoes with respect to the numbers of columns and rows, the area, the semiperimeter, and the numbers of interior points and edges.

In Section 2, we derive a generating function for Stanley polyominoes with respect to the numbers of columns and rows, area, semiperimeter, and the numbers of interior points and edges.

In Section 2.1, we study the enumeration with respect to the number of columns and recover the fact that Stanley polyominoes are counted by the Catalan numbers. Further results are obtained depending on the values of the other parameters.

In Sections 2.2 and 2.3, we carry out a similar analysis with respect to the semiperimeter and the area, respectively. We also establish combinatorial connections via bijections with other structures, such as Dyck paths, skew Ferrers diagrams, and peakless Motzkin paths. As a byproduct, we answer an open question by constructing a bijection between parallelogram polyominoes of area n and coin fountains with n coins in the even-numbered rows and $n - k$ coins in the odd-numbered rows.

Finally, in Section 3, we exploit the bijection between Stanley polyominoes and Dyck paths to derive several generating functions expressed as continued fractions.

2. AREA, SEMIPERIMETER, INTERIOR POINTS, AND EDGES

In this section, we study several statistics on the class of Stanley polyominoes: the area, the numbers of columns and rows, the number of interior points, the number of strictly internal edges, and the number of cells in the first row. These are denoted by $\text{area}(P)$, $\text{col}(P)$, $\text{row}(P)$, $\text{point}(P)$, $\text{edgint}(P)$, and $\text{first}(P)$, respectively. Note that the semiperimeter $\text{sper}(P)$ is the sum of the numbers of columns and rows, and that the number of weakly internal edges $\text{adja}(P)$ is equal to $\text{point}(P) + \text{row}(P) - 1$.

Let $\mathcal{P}$ denote the set of all Stanley polyominoes. For $m, n, r, k \geq 1$ and $s, t \geq 0$, let $\mathcal{P}_{m,n,r,s,t,k}$ be the set of Stanley polyominoes satisfying

$$\begin{array}{lll} \text{col}(P) = m, & \text{row}(P) = n, & \text{area}(P) = r, \\ \text{edgint}(P) = s, & \text{point}(P) = t, & \text{first}(P) = k. \end{array}$$

Then

$$\mathcal{P} = \bigcup_{\substack{m,n,r,k \geq 1 \\ s,t \geq 0}} \mathcal{P}_{m,n,r,s,t,k}.$$

We first consider the generating function

$$F(x, y, z, p, q) = \sum_{P \in \mathcal{P}} x^{\text{col}(P)} y^{\text{row}(P)} z^{\text{area}(P)} p^{\text{edgint}(P)} q^{\text{point}(P)}.$$

For $h \geq 1$, let $F_h(x, y, z, p, q)$ denote the generating function for Stanley polyominoes whose first row contains exactly h cells. Then

$$F(x, y, z, p, q) = \sum_{h \geq 1} F_h(x, y, z, p, q).$$

We introduce a catalytic variable u that marks the length of the first row, and we define the generating function

$$\begin{aligned} F(x, y, z, p, q; u) &= \sum_{h \geq 1} F_h(x, y, z, p, q) u^h \\ &= \sum_{\substack{m, n, r, h \geq 1 \\ s, t \geq 0}} \sum_{P \in \mathcal{P}_{m, n, r, s, t, h}} x^m y^n z^r p^s q^t u^h, \end{aligned}$$

so that $F(x, y, z, p, q; 1) = F(x, y, z, p, q)$. For brevity, we write $F(u)$ for $F(x, y, z, p, q; u)$, $F(1)$ for $F(x, y, z, p, q)$, and F_h for $F_h(x, y, z, p, q)$.

Our next goal is to obtain an expression for the generating function $F(1) = F(x, y, z, p, q)$.

Theorem 2.1. *The generating function for the number of non-empty Stanley polyominoes according to the number of columns, the number of rows, the area, the number of strictly internal edges, and the number of interior points is*

$$\begin{aligned} F(x, y, z, p, q) &= \frac{G(x, y, z, p, q)}{1 + H(x, y, z, p, q)}, \quad \text{where} \\ G(x, y, z, p, q) &= \sum_{\ell \geq 0} \frac{x^{\ell+2} y^{\ell+2} (pq)^{\frac{\ell(\ell+3)}{2}} (p-1) z^{\frac{(\ell+3)(\ell+2)}{2}} - y^{\ell+1} x^{\ell+1} z^{\frac{(\ell+2)(\ell+1)}{2}} (pq)^{\frac{\ell(\ell+1)}{2}} p}{p^{2\ell+1} q^{\ell} (xz (zpq)^{\ell} - 1)} \Delta_{\ell}, \\ H(x, y, z, p, q) &= \sum_{\ell \geq 0} \frac{\left(- (pq)^{\frac{\ell(\ell+3)}{2}} z^{\frac{(\ell+4)(\ell+1)}{2}} + z^{\frac{(\ell+6)(\ell+1)}{2}} q (p-1) (pq)^{\frac{\ell(\ell+5)}{2}} \right) x^{\ell+1} y^{\ell+1}}{p^{2\ell} q^{\ell} (xz (zpq)^{\ell} - 1) ((zpq)^{\ell} zqp - 1)} \Delta_{\ell}, \\ \Delta_{\ell} &= \frac{1}{\prod_{i=0}^{\ell-1} (1 - xz (zpq)^i) \prod_{i=0}^{\ell-1} ((zpq)^i zqp - 1)} \\ &= \frac{(-1)^{\ell}}{(xz, zpq)_{\ell} \cdot (zpq, zpq)_{\ell}}, \end{aligned}$$

and where $(a, b)_k$ is the Pochhammer symbol defined by

$$(a, b)_k = \prod_{j=0}^{k-1} (1 - ab^j).$$

Proof. We decompose Stanley polyominoes according to the number of rows.

If P consists of a single row with $h \geq 1$ cells, then

$$\begin{aligned} \text{col}(P) &= h, & \text{row}(P) &= 1, & \text{area}(P) &= h, \\ \text{edgint}(P) &= 0, & \text{point}(P) &= 0, & \text{first}(P) &= h. \end{aligned}$$

Hence its contribution to the generating function is $yx^h z^h u^h$, and summing over $h \geq 1$ gives

$$A_1 := y \sum_{h \geq 1} (xzu)^h = \frac{yxzu}{1 - xzu}.$$

Now consider a Stanley polyomino P with at least two rows. Let k and h be the numbers of cells of the first and second rows, respectively, and let i be the number of cells in the first row that are adjacent to a cell of the second row. Then, $1 \leq i \leq h - 1$ and $k - i \geq 1$ with $h \geq 2$ and $k \geq 2$.

We construct P from a Stanley polyomino Q counted by $F_h(x, y, z, p, q)$ in two steps: (1) we add below Q a row of i cells in such a way that the first and second rows are left-aligned, and (2) we add $k - i \geq 1$ cells to the left of the first row. See Figure 2 for an illustration.

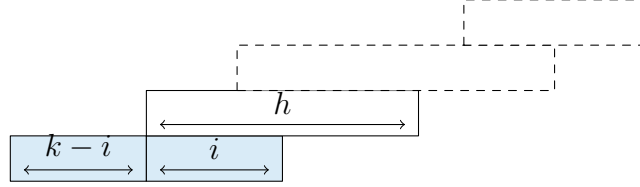


FIGURE 2. Decomposition according to the cell numbers of the first two rows.

After applying operation (1), the new bottom row contributes one new row, i new cells, $i - 1$ new interior points, and $i - 2$ new strictly internal edges when $i \geq 2$. Hence the corresponding weight is $\alpha_1 = yzu$ and $\alpha_i = yz^i p^{i-2} q^{i-1} u^i$ with $i \geq 2$. Therefore, summing over $i = 1, \dots, h - 1$ and then over h , we obtain

$$\begin{aligned} A_2 &:= \sum_{h \geq 2} F_h \alpha_1 + \sum_{h \geq 3} F_h \sum_{i=2}^{h-1} \alpha_i = \sum_{h \geq 2} F_h yzu + \sum_{h \geq 3} F_h \sum_{i=2}^{h-1} yz^i p^{i-2} q^{i-1} u^i \\ &= yzu (F(1) - xyz) + y \sum_{h \geq 3} F_h \frac{(zpq u)^h - z^2 p^2 q^2 u^2}{qp^2 (zpq u - 1)} \\ &= yzu (F(1) - xyz) + y \frac{F(zpq u) - F_2(zpq u) - F_1(zpq u)}{qp^2 (zpq u - 1)} \\ &\quad - y \frac{z^2 p^2 q^2 u^2 (F(1) - F_2(1) - F_1(1))}{qp^2 (zpq u - 1)}, \end{aligned}$$

where $F_1(u)$ (resp. $F_2(u)$) are the contributions for polyominoes having one (resp. 2) cells in the first row. Then, we have $F_1(u) = xyz u$ and $F_2(u) = xyz^2 (F(u) - F_1(u))$.

Applying operation (2), that is, adding $k - i \geq 1$ cells to the left of the first row, contributes the factor

$$A_2 \cdot \frac{xzu}{1 - xzu}.$$

Adding the single-row contribution, we obtain

$$\begin{aligned} F(u) &= A_1 + A_2 \cdot \frac{xzu}{1-xzu} \\ &= \frac{xzu(xy z^2(p-1)u-p)y}{p(uxz-1)} - \frac{u^2 z^2(-1+qz(p-1)u)yx}{(uxz-1)(uzqp-1)} F(1) + \\ &\quad \frac{yxuz}{(-uxz+1)(uzqp-1)qp^2} F(zpqu). \end{aligned}$$

Define

$$\begin{aligned} A(u) &= \frac{xzu(xy z^2(p-1)u-p)y}{p(uxz-1)}, \quad B(u) = \frac{u^2 z^2(1-qz(p-1)u)yx}{(uxz-1)(uzqp-1)}, \quad \text{and} \\ C(u) &= \frac{yxuz}{(-uxz+1)(uzqp-1)qp^2}. \end{aligned}$$

Then $F(u)$ can be written

$$(1) \quad F(u) = A(u) + B(u)F(1) + C(u)F(uzqp).$$

Iterating this relation (with the convention that an empty product equals 1) and assuming that $|u| \leq 1$, $|z| \leq 1$, $|q| \leq 1$, and $|p| \leq 1$, we obtain (after setting $v = zqp$)

$$F(u) = \sum_{\ell \geq 0} A(uv^\ell) \prod_{i=0}^{\ell-1} C(uv^i) + \left(\sum_{\ell \geq 0} B(uv^\ell) \prod_{i=0}^{\ell-1} C(uv^i) \right) F(1).$$

Now set $u = 1$ and solve for $F(1) = F(x, y, z, p, q)$:

$$F(1) = \frac{\sum_{\ell \geq 0} A(v^\ell) \prod_{i=0}^{\ell-1} C(v^i)}{1 - \sum_{\ell \geq 0} B(v^\ell) \prod_{i=0}^{\ell-1} C(v^i)}.$$

Substituting the explicit expressions for $A(v^\ell)$, $B(v^\ell)$, and $C(v^i)$ and simplifying gives the closed form stated in the theorem. $\square$

The first few terms of the series expansion of $F(x, y, z, p, q)$ in powers of x are

$$\begin{aligned} F(x, y, z, p, q) &= xyz + z^2 y x^2 + y z^3 (y z + 1) x^3 + (\mathbf{y z^4} + \mathbf{y^3 z^6} + \mathbf{q z^6 y^2} + \mathbf{2 z^5 y^2}) x^4 + \\ &\quad z^5 (1 + y^3 z^3 + z^2 (q^2 z^2 + 2 q z + 3) y^2 + (p q^2 z^3 + 2 q z^2 + 3 z) y) y x^5 + O(x^6). \end{aligned}$$

The bold terms correspond to the Stanley polyominoes shown in Figure 3.

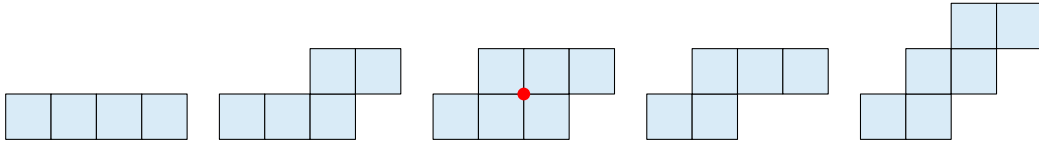


FIGURE 3. The five Stanley polyominoes with four columns. From left to right, the contributions are respectively yz^4x^4 , $y^2z^5x^4$, $qy^2z^6x^4$, $y^2z^5x^4$, and $y^3z^6x^4$.

2.1. Enumeration with respect to the number of columns. Setting $y = z = p = q = 1$ in (1) and writing $G(u) = F(x, 1, 1, 1, 1; u)$, we obtain the following equation

$$(2) \quad G(u) \left(1 + \frac{xu}{(xu-1)(u-1)} \right) = \frac{xu}{1-xu} + \frac{G(1)u^2x}{(xu-1)(u-1)}.$$

We use the kernel method (see [2,21]) to determine $G(1)$. This method consists in cancelling the kernel

$$K(u) = 1 + \frac{xu}{(xu-1)(u-1)}$$

by finding u as an algebraic function r of x . Thus, setting $u = r$, the right-hand side must also vanish, which determines $G(1)$.

We factor the kernel as

$$K(u) = \frac{x(u-r)(u-s)}{(xu-1)(u-1)},$$

where

$$r = \frac{1 - \sqrt{1-4x}}{2x} \quad \text{and} \quad s = \frac{1 + \sqrt{1-4x}}{2x}.$$

Note that r is the generating function of the Catalan numbers, which ensures that we remain in the ring of formal power series. Substituting $u = r$ into (2), we obtain

$$G(1) = \frac{r-1}{r}.$$

Thus $G(1)$ is the generating function of the Catalan numbers, up to a shift. More precisely,

$$[x^n]G(1) = \frac{1}{n} \binom{2n-2}{n-1}.$$

Now, substituting the above expression for $G(1)$ into (2) and cancelling the factor $(u-r)$, we obtain the following result.

Theorem 2.2. *The generating function for Stanley polyominoes with respect to the number of columns (marked by x) and the number of cells in the first row (marked by u) is*

$$G(u) = \frac{u}{r(s-u)} = \frac{(2-u-\sqrt{1-4x})xu}{2u^2x-2u+2}.$$

Moreover we have for $n \geq 2$ and $1 \leq k \leq n$,

$$[u^k]G(u) = x(rx)^{k-1} \quad \text{and} \quad [x^n u^k]G(u) = \frac{k-1}{2n-k-1} \binom{2n-k-1}{n-k}.$$

The first few terms of the series expansion of $G(u)$ as a power series in x

$$xu + x^2u^2 + u^2(1+u)x^3 + \mathbf{u^2(u^2 + 2u + 2)}x^4 + u^2(u^3 + 3u^2 + 5u + 5)x^5 + \\ u^2(u^4 + 4u^3 + 9u^2 + 14u + 14)x^6 + u^2(u^5 + 5u^4 + 14u^3 + 28u^2 + 42u + 42)x^7 + O(x^8).$$

The bold coefficient of x^4 corresponds to the Stanley polyominoes shown in Figure 3.

By differentiating $G(u)$ with respect to u and then setting $u = 1$, we obtain the following result.

Corollary 2.3. *The generating function, with respect to the number of columns, for the total number of cells in the first row over all Stanley polyominoes is*

$$\frac{s}{r(s-1)^2} = \frac{1 - \sqrt{1-4x}}{2x} - 1$$

and the coefficient of x^n is the n th Catalan number $C_n = \frac{1}{n+1} \binom{2n}{n}$. Consequently, the average number of cells in the first row among Stanley polyominoes tends to 4 as $n \rightarrow \infty$.

Note that, by the bijection ϕ described above, the coefficient of $x^n u^k$ in $G(u)$ counts Dyck paths of semilength $n-1$ whose first peak has height $k-1$.

Corollary 2.4. *The generating function, with respect to the number of columns, for Stanley polyominoes P satisfying $\mathbf{edgint}(P) = 0$ is*

$$\frac{x(1-2x)}{x^2-3x+1}.$$

Moreover, for $n \geq 2$, the coefficient of x^n in this series is F_{2n-3} , where F_n is the Fibonacci sequence defined by $F_n = F_{n-1} + F_{n-2}$ with the initial conditions $F_0 = 0$ and $F_1 = 1$.

Proof. Let $R(x)$ be the generating function, with respect to the number of columns, for Stanley polyominoes P satisfying $\mathbf{edgint}(P) = 0$. If P consists of a single row, then its contribution is $\frac{x}{1-x}$. Now assume that P has at least two rows, and let Q be obtained from P by deleting the first row. Since $\mathbf{edgint}(P) = 0$, the second row can overlap the first row in at most two cells.

If the overlap has size 1, then the first row contributes $\frac{x}{1-x}$ and Q must have at least two columns, so the contribution is

$$\frac{x}{1-x} (R(x) - x).$$

If the overlap has size 2, then Q must have at least three columns. Since the contribution of polyominoes counted by $R(x)$ with one or two columns is $x + x^2 + x(R(x) - x)$, the contribution in this case is

$$\frac{x}{1-x} (R(x) - x - x^2 - x(R(x) - x)).$$

Collecting the above contributions, we obtain the claimed generating function. $\square$

With a similar, but simpler, argument, we obtain the following result.

Corollary 2.5. *The generating function, with respect to the number of columns, for Stanley polyominoes P satisfying $\mathbf{point}(P) = 0$ is*

$$\frac{x(1-x)}{1-2x},$$

and, for $n \geq 2$, the coefficient of x^n in this series is 2^{n-2} .

Since Stanley polyominoes with a given number of columns are in one-to-one correspondence with Dyck paths of the corresponding semilength, and since the enumeration of Dyck paths with respect to various statistics has been widely studied in the literature (see, for instance, [14]), we now turn to Stanley polyominoes enumerated by semiperimeter and area.

2.2. Enumeration with respect to the semiperimeter. As noted in the introduction, the semiperimeter of a Stanley polyomino is the sum of its numbers of rows and columns. In this subsection, we therefore set $x = y$ and $z = p = q = 1$ in (1), and write $G(u) = F(x, x, 1, 1, 1; u)$. Then we obtain the following equation

$$(3) \quad G(u) \left(1 + \frac{x^2 u}{(xu - 1)(u - 1)} \right) = \frac{x^2 u}{1 - xu} + \frac{G(1) u^2 x^2}{(xu - 1)(u - 1)}.$$

As in the previous subsection, we use the kernel method (see [2, 21]) to determine $G(1)$. The kernel is

$$K(u) = 1 + \frac{x^2 u}{(xu - 1)(u - 1)},$$

which can be written as

$$K(u) = \frac{x(u - r)(u - s)}{(xu - 1)(u - 1)},$$

where

$$r = \frac{1 + x - x^2 - \sqrt{x^4 - 2x^3 - x^2 - 2x + 1}}{2x}, \quad s = \frac{1 + x - x^2 + \sqrt{x^4 - 2x^3 - x^2 - 2x + 1}}{2x}.$$

Hence, we obtain

$$G(1) = \frac{r - 1}{r} = \frac{1}{2} \left(1 - x + x^2 - \sqrt{1 - 2x - x^2 - 2x^3 + x^4} \right),$$

which is the generating function of the generalized Catalan numbers, up to a shift in the index. This sequence also counts secondary structures of RNA molecules (see [A004148](#)) and peakless Motzkin paths, that is, lattice paths in $\mathbb{N}^2$ starting at the origin, ending on the x -axis, consisting of up-steps $u = (1, 1)$, down-steps $d = (1, -1)$, and horizontal steps $f = (1, 0)$, and avoiding the peaks of the form ud . Let $\mathcal{M}_n$ be the set of peakless Motzkin paths with n steps, and let $\mathcal{M} = \cup_{n \geq 0} \mathcal{M}_n$. This suggests a direct combinatorial interpretation in terms of peakless Motzkin paths.

Let us define recursively a map χ from $\mathcal{M}$ to the set of Stanley polyominoes. If ϵ denotes the empty path, then $\chi(\epsilon)$ is the Stanley polyomino consisting of a single cell. Now let α be a nonempty peakless Motzkin path in $\mathcal{M}$.

- If α starts with a flat step f , say $\alpha = f\beta$ with $\beta \in \mathcal{M}$, then $\chi(\alpha)$ is obtained from $\chi(\beta)$ by adding one cell to the left of the first row.
- If $\alpha = u\beta d\gamma$, where β is nonempty, then $\chi(\alpha)$ is obtained from $P = \chi(\beta\gamma)$ by adding a row with k cells below P , so that the first cell of the new row is shifted one unit to the left of the leftmost cell of the first row of P , where $k - 1$ is the

number of steps of α that end on the x -axis (equivalently, $k - 2$ is the number of steps of γ ending on the x -axis).

See Figure 4 for an illustration of this map on the four peakless Motzkin paths with four steps.

A simple induction shows that, for every $\alpha \in \mathcal{M}_n$, the semiperimeter of $\chi(\alpha)$ is $n + 2$. Indeed, the claim holds for $n = 0$, since $\text{sper}(\chi(\epsilon)) = 2$. Moreover,

$$\text{sper}(\chi(f\beta)) = 1 + \text{sper}(\chi(\beta)), \quad \text{sper}(\chi(u\beta d\gamma)) = 2 + \text{sper}(\chi(\beta\gamma)),$$

and these two relations provide the induction step.

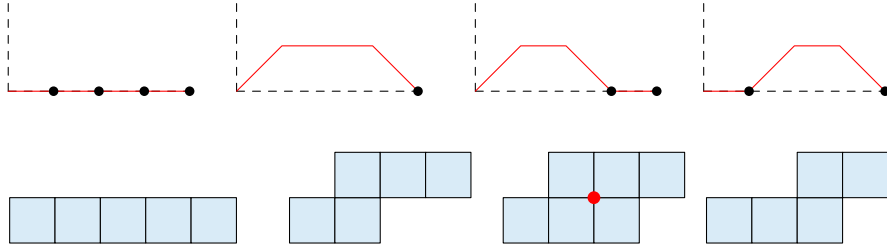


FIGURE 4. The bijection χ maps each of the four peakless Motzkin paths with four steps to the corresponding Stanley polyomino of semiperimeter 6. The number of steps ending on the x -axis corresponds to the number of cells in the first row minus one.

Now, substituting the above expression for $G(1)$ into (3) and cancelling the factor $(u - r)$, we obtain the following result.

Theorem 2.6. *The generating function for Stanley polyominoes with respect to the semiperimeter and the number of cells in the first row is*

$$G(u) = \frac{xu}{r(s-u)} = \frac{(x^2u - xu - u + 2 - \sqrt{x^4 - 2x^3 - x^2 - 2x + 1}u)x^2u}{2(u^2x + x^2u - xu - u + 1)}.$$

Moreover, $[u^k]G(u) = x^2(rx)^{k-1}$, and for $n \geq k + 1$ and $k \geq 2$,

$$\begin{aligned} [x^n u^k]G(u) &= \sum_{j=0}^{n-k-1} \frac{k-1}{2j+k-1} \binom{2j+k-1}{j} (-1)^{n-k-1-j} \\ &\quad \times \sum_{b=0}^{\lfloor \frac{n-k-1-j}{2} \rfloor} \binom{n+j-b-3}{n-k-1-j-b} \binom{n-k-1-j-b}{b}. \end{aligned}$$

Proof. To extract the coefficients of $x^n u^k$, we set $t = 1 + x - x^2$. Since

$$xr = \frac{t - \sqrt{t^2 - 4x}}{2} = \frac{x}{t} C\left(\frac{x}{t^2}\right),$$

where $C(z) = (1 - \sqrt{1 - 4z})/2z$ is the generating function of the Catalan numbers, it follows that, for $k \geq 2$,

$$[u^k]G(u) = x^2 \left(\frac{x}{t} C\left(\frac{x}{t^2}\right) \right)^{k-1} = x^{k+1} t^{-(k-1)} C\left(\frac{x}{t^2}\right)^{k-1}.$$

We now use the classical identity (cf. [24])

$$C(z)^m = \sum_{j \geq 0} \frac{m}{2j+m} \binom{2j+m}{j} z^j \quad (m \geq 1).$$

Applying this with $m = k - 1$, we obtain

$$[u^k]G(u) = \sum_{j \geq 0} \frac{k-1}{2j+k-1} \binom{2j+k-1}{j} x^{k+1+j} t^{-(2j+k-1)}.$$

Therefore, for $k \geq 2$,

$$[x^n u^k]G(u) = \sum_{j=0}^{n-k-1} \frac{k-1}{2j+k-1} \binom{2j+k-1}{j} [x^{n-k-1-j}] (1+x-x^2)^{-(2j+k-1)}.$$

Finally, for any integer $m \geq 1$,

$$(1+x-x^2)^{-m} = \sum_{a \geq 0} \binom{m+a-1}{a} (-1)^a x^a (1-x)^a,$$

and hence

$$[x^\ell] (1+x-x^2)^{-m} = (-1)^\ell \sum_{b=0}^{\lfloor \ell/2 \rfloor} \binom{m+\ell-b-1}{\ell-b} \binom{\ell-b}{b}.$$

Substituting $m = 2j + k - 1$ and $\ell = n - k - 1 - j$, we obtain, for $k \geq 2$,

$$\begin{aligned} [x^n u^k]G(u) &= \sum_{j=0}^{n-k-1} \frac{k-1}{2j+k-1} \binom{2j+k-1}{j} (-1)^{n-k-1-j} \\ &\quad \times \sum_{b=0}^{\lfloor \frac{n-k-1-j}{2} \rfloor} \binom{n+j-b-3}{n-k-1-j-b} \binom{n-k-1-j-b}{b}. \end{aligned}$$

This completes the proof. $\square$

The first few terms of the expansion of $G(u)$ as a power series in x are

$$\begin{aligned} x^2 u + u^2 x^3 + u^3 x^4 + (u^4 + u^2) x^5 + u^2 (u^3 + 2u + 1) x^6 + \mathbf{u^2(u^4 + 3u^2 + 2u + 2)x^7} + \\ u^2 (u^5 + 4u^3 + 3u^2 + 5u + 4) x^8 + O(x^9). \end{aligned}$$

The bold coefficient of x^7 corresponds to the Stanley polyominoes shown in Figure 5.

Note that the coefficients of $G(u)$, arranged by powers of x and u , coincide with sequence [A162986](#) in [20], which counts Dyck paths avoiding uuu and ddd according to semilength

and the number of peaks ud starting at level 0 (also called hills). This suggests a second bijective interpretation of Stanley polyominoes with fixed semiperimeter:

Let $\mathcal{D}$ denote the set of Dyck paths avoiding uuu and ddd and let us recursively define a map χ' from $\mathcal{D}$ to the set of Stanley polyominoes. Note that the definition of χ' is very similar to that of χ .

If ϵ denotes the empty path, then $\chi'(\epsilon)$ is the Stanley polyomino consisting of a single row with two cells. Now let α be a nonempty Dyck path in $\mathcal{D}$.

- If α starts with ud , say $\alpha = ud\beta$ with $\beta \in \mathcal{D}$, then $\chi'(\alpha)$ is obtained from $\chi'(\beta)$ by adding one cell on the left of the first row.
- Otherwise, α can be written as $\alpha = u\beta udd\gamma$, where $\beta, \gamma \in \mathcal{D}$. Then $\chi'(\alpha)$ is obtained from $P = \chi'(\beta\gamma)$ by adding a row with k cells below P , so that the first cell of the new row is shifted one unit to the left of the leftmost cell of the first row of P , where $k - 2$ is the number of peaks ud ending on the x -axis in γ .

A simple induction shows that, for every $\alpha \in \mathcal{D}$, the semiperimeter of $\chi'(\alpha)$ is the semilength of α plus 3. Moreover, we have

$$\mathbf{first}(\chi'(ud\beta)) = 1 + \mathbf{first}(\chi'(\beta)), \quad \mathbf{first}(\chi'(u\beta udd\gamma)) = \mathbf{first}(\chi'(\gamma)),$$

and these two relations (with an induction) imply that $\mathbf{first}(\chi'(\alpha))$ equals the number of hills in α plus 2.

See Figure 5 for an illustration of this map.

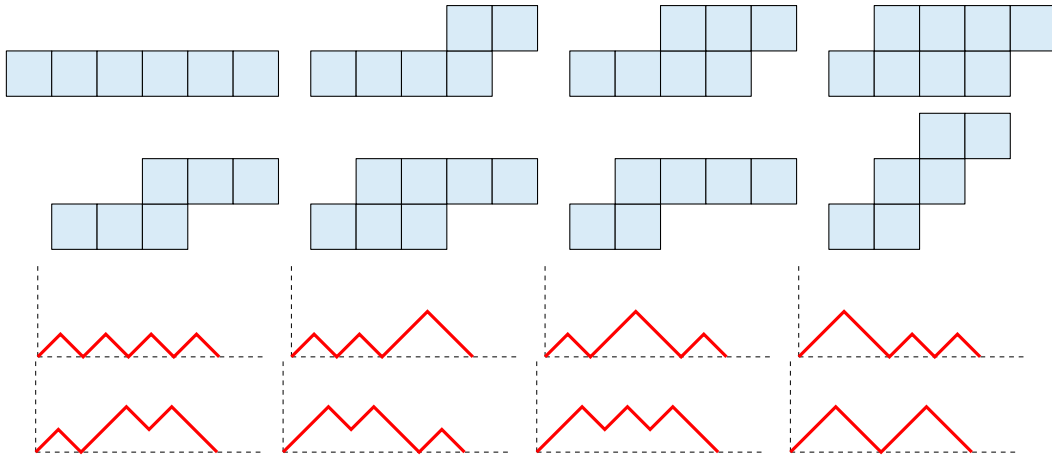


FIGURE 5. The eight Stanley polyominoes with semiperimeter 7. Their contributions to $G(u)$ are, from left to right, u^6x^7 , u^4x^7 , u^4x^7 , u^4x^7 , u^3x^7 , u^3x^7 , u^2x^7 , and u^2x^7 . The corresponding preimages under χ' are shown below, in the same order.

By calculating $\partial_u(G(u))|_{u=1}$ we obtain the following:

Corollary 2.7. *The generating function, with respect to the semiperimeter, for the total number of cells in the first row over all Stanley polyominoes is*

$$\frac{xs}{r(s-1)^2} = \frac{1 - 2x + x^2 - 2x^3 + x^4 - (1 - x + x^2)\sqrt{1 - 2x - x^2 - 2x^3 + x^4}}{2x^2}.$$

Its coefficients are given by sequence [A089735](#), the self-convolution of the RNA secondary structure numbers ([A004148](#)).

Using an argument similar to that of Corollary 2.4, we obtain the following.

Corollary 2.8. *The generating function, with respect to the semiperimeter, for Stanley polyominoes with no strictly internal edges (that is, satisfying $\text{edgint}(P) = 0$) is*

$$\frac{x^3 - x}{x^2 + x - 1}.$$

Moreover, the coefficient of x^n in the series is the Fibonacci number F_{n-1} for $n \geq 2$.

2.3. Enumeration with respect to the area. The variable z tracks the area. Setting $x = y = p = q = 1$ in Theorem 2.1, we obtain the following.

Theorem 2.9. *The generating function for Stanley polyominoes with respect to the area is*

$$F(1, 1, z, 1, 1) = \frac{\sum_{\ell \geq 0} \frac{(-1)^\ell z^{\frac{(\ell+2)(\ell+1)}{2}}}{(z, z)_\ell^2 (1 - z^{\ell+1})}}{1 - \sum_{\ell \geq 0} \frac{(-1)^\ell z^{\frac{(\ell+4)(\ell+1)}{2}}}{(z, z)_{\ell+1}^2}},$$

where $(a, b)_k$ denotes the Pochhammer symbol defined by

$$(a, b)_k = \prod_{j=0}^{k-1} (1 - ab^j).$$

The first few terms in the expansion of $F(1, 1, z, 1, 1)$ as a power series in z are

$$z + z^2 + z^3 + 2z^4 + 3z^5 + \mathbf{6}z^6 + 10z^7 + 19z^8 + 34z^9 + 63z^{10} + 115z^{11} + O(z^{12}).$$

The bold coefficient of z^6 corresponds to the Stanley polyominoes shown in Figure 6.

The coefficient of z^n is the $(n-1)$ st term of the sequence [A227309](#). This sequence is obtained by summing along the falling diagonals of the array [A161492](#), whose entry in the row n and column m counts the number of coin fountains with exactly n coins in the even-numbered rows and $n-m$ coins in the odd-numbered rows of the fountain (see [1]). Therefore, for $n \geq 1$, there is a bijection between the set of Stanley polyominoes of area n and the set of coin fountains C such that $2e(C) - o(C) = n$, where $e(C)$ and $o(C)$ denote the number of coins in the even-numbered and odd-numbered rows of C , respectively.

Let $\mathcal{C}_n$ (resp. $\mathcal{S}_n$) denote the set coin fountains with $n \geq 1$ north-east diagonals (resp. the set of Stanley polyominoes with $n \geq 2$ columns), and let $\mathcal{C} = \cup_{n \geq 1} \mathcal{C}_n$, $\mathcal{S} = \cup_{n \geq 2} \mathcal{S}_n$.

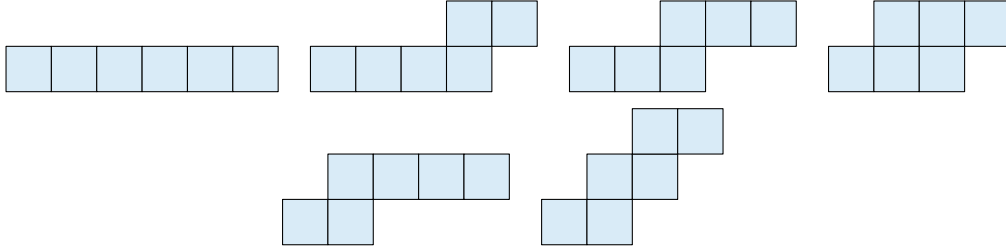


FIGURE 6. The six Stanley polyominoes with area 6.

We now define recursively a map f from $\mathcal{C}$ to $\mathcal{S}$ that sends $\mathcal{C}_n$ into $\mathcal{S}_{n+1}$, and we will prove that f sends the statistic $2e(C) - o(C)$ into the area.

Definition 2.10. If C consists of a single coin (that is, if $n = 1$), then $f(C)$ is the unique Stanley polyomino in $\mathcal{S}_2$ consisting of a single row with two cells.

Now let C be a coin fountain with at least two coins, and let $\text{diag}(C) = k \geq 1$ be the number of coins in the first north-east diagonal of C . Let C' be the coin fountain obtained from C by deleting the first north-east diagonal.

We distinguish two cases:

- If $k = 2\ell + 1$ with $\ell \geq 0$, then $f(C)$ is obtained from $f(C')$ by adding below $f(C')$ a row with $\ell + 2$ cells, in such a way that its leftmost cell lies one unit to the left of the leftmost cell of the second row.
- If $k = 2\ell$ with $\ell \geq 1$, then $f(C)$ is obtained from $f(C')$ by adding one cell to the beginning of each of the first ℓ rows of $f(C')$.

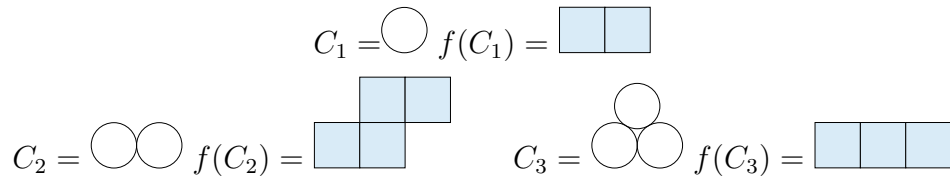
The next lemma shows that f is well defined, that is, that the image of any coin fountain under f is a Stanley polyomino. To this end, we introduce two statistics on Stanley polyominoes: $\text{firstD}(P)$, the number of cells in the first north-east diagonal of P , and $\text{firstR}(P)$, the number of cells in the first row of P .

Lemma 2.11. *Let C be a coin fountain with d north-east diagonals, and let $k = \text{diag}(C)$ be the number of coins in its first north-east diagonal. Then the polyomino $P = f(C)$ has $d + 1$ columns, and it satisfies the following:*

- if $k = 2\ell$ with $\ell \geq 1$, then $\text{firstD}(P) = \ell$ and $\text{firstR}(P) \geq \ell + 2$;
- If $k = 2\ell + 1$ with $\ell \geq 0$, then $\text{firstD}(P) \geq \ell + 1$ and $\text{firstR}(P) = \ell + 2$.

Proof. We proceed by induction on the number d of north-east diagonals of C .

For the initial cases $d = 1$ and $d = 2$, the claim is verified directly from the examples C_1 , C_2 , and C_3 shown below. For instance, $\text{diag}(C_3) = 2$, so $\ell = 1$, and indeed $\text{firstD}(f(C_3)) = 1 = \ell$ and $\text{firstR}(f(C_3)) = 3 \geq \ell + 2$.



Now assume that the statement holds for all coin fountains with at most $d-1$ north-east diagonals, and let C be a coin fountain with d north-east diagonals. Let $k = \text{diag}(C)$, let C' be the coin fountain obtained from C by deleting its first north-east diagonal, and let $k' = \text{diag}(C')$. Applying the induction hypothesis, we assume that $P' = f(C')$ satisfies the statements.

We distinguish two cases.

- Suppose that the first diagonal of C contains $k = 2\ell \geq 2$ coins. If k' is odd, then $k' \geq 2\ell - 1 = 2(\ell - 1) + 1$, and by the induction hypothesis applied to C' we have

$$\text{firstD}(P') \geq \ell \quad \text{and} \quad \text{firstR}(P') = (\ell - 1) + 2 = \ell + 1.$$

Hence, we can add ℓ cells at the beginning of the first ℓ rows of C' in order to obtain the Stanley polyomino $f(C)$, and it yields

$$\text{firstD}(f(C)) = \ell \quad \text{and} \quad \text{firstR}(f(C)) \geq \ell + 2,$$

which satisfies the statement.

If k' is even, then $k' \geq 2\ell$, and the induction hypothesis gives

$$\text{firstD}(P') \geq \ell \quad \text{and} \quad \text{firstR}(P') \geq \ell + 2.$$

Again, we can add ℓ cells at the beginning of the first ℓ rows of C' in order to obtain the Stanley polyomino $f(C)$, and we have

$$\text{firstD}(P) = \ell \quad \text{and} \quad \text{firstR}(P) \geq \ell + 3,$$

which satisfies the statements. See the left part of Figure 7 for an illustration of this case.

- Suppose that the first diagonal of C contains $k = 2\ell + 1 \geq 1$ coins. If k' is odd, then $k' \geq 2\ell + 1$, and the induction hypothesis gives

$$\text{firstD}(P') \geq \ell + 1 \quad \text{and} \quad \text{firstR}(P') \geq \ell + 2.$$

Hence, we can add a row of $\ell + 2$ cells below C' (such that this row starts one cell before the first row of P') in order to obtain the Stanley polyomino $f(C)$, and we have

$$\text{firstD}(P) \geq \ell + 2 \quad \text{and} \quad \text{firstR}(P) = \ell + 2,$$

which satisfies the statements.

If k' is even, then $k' \geq 2\ell$, and the induction hypothesis gives

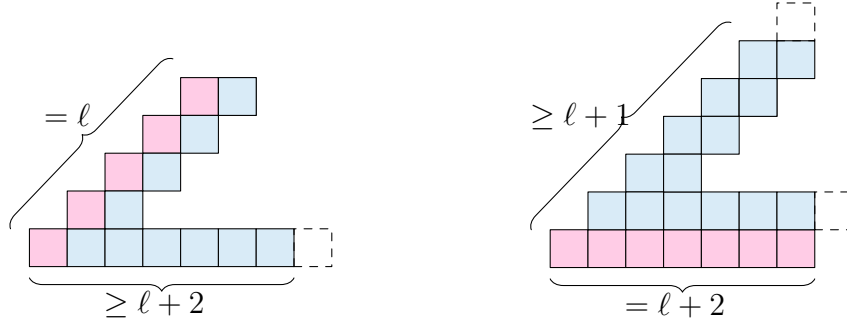
$$\text{firstD}(P') \geq \ell \quad \text{and} \quad \text{firstR}(P') \geq \ell + 2.$$

Hence, we can add $\ell + 2$ cells below C' (such that this row starts one cell before the first row of P') in order to obtain the Stanley polyomino $f(C)$, and we have

$$\text{firstD}(P) \geq \ell + 1 \quad \text{and} \quad \text{firstR}(P) = \ell + 2,$$

which satisfies the statements. See the right part of Figure 7 for an illustration of this case.

Finally, an induction on the number d of diagonals completes the proof. $\square$


 FIGURE 7. The two cases in the recursive construction of f .

The previous lemma shows that f is a well-defined map from $\mathcal{C}$ to $\mathcal{S}$, and that $f(C)$ has $d + 1$ columns whenever C is a coin fountain with d diagonals. A simple induction shows that f is injective: indeed, there do not exist integers ℓ_1 and ℓ_2 such that both

$$\text{firstD}(P) = \ell_1, \quad \text{firstR}(P) \geq \ell_1 + 2,$$

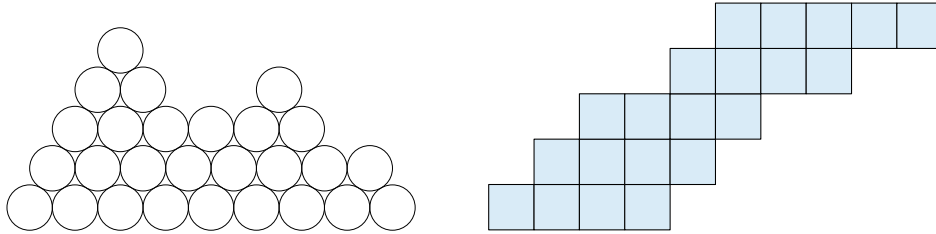
and

$$\text{firstD}(P) \geq \ell_2 + 1, \quad \text{firstR}(P) = \ell_2 + 2$$

hold simultaneously. Since $\mathcal{C}_n$ and $\mathcal{S}_{n+1}$ have the same cardinality, it follows that f is a bijection. Finally, by the definition of f and a straightforward induction on the number of diagonals, we have $2e(C) - o(C) = \text{area}(f(C))$. This proves the following theorem.

Theorem 2.12. *The map f from $\mathcal{C}$ to $\mathcal{S}$ is a bijection such that, for every $C \in \mathcal{C}_n$, we have $f(C) \in \mathcal{S}_{n+1}$ and $2e(C) - o(C) = \text{area}(f(C))$.*

We refer to Figure 8 for an illustration of the bijection f on a concrete example.


 FIGURE 8. A coin fountain C such that $2e(C) - o(C) = 21$, together with its image under f .

As a byproduct, this bijection allows us to address an open problem posed by Bala (see [1]), namely the construction of an explicit bijection between coin fountains and skew Ferrers diagrams [12] (equivalently, parallelogram polyominoes).

Let $\mathcal{P}_n$ be the set of parallelogram polyominoes of area n and let $\mathcal{C}_n$ be the set of coin fountains with n coins in the even-numbered rows. We define the map ψ from $\mathcal{P}_n$ to $\mathcal{C}_n$ as follows. For $P \in \mathcal{P}_n$, we set

$$\psi(P) = f^{-1}(\phi^{-1}(h(P))),$$

where ϕ and f are the maps defined in this paper, and h is the map from parallelogram polyominoes to Dyck paths originally introduced in [10] and later used in [6]. See Figure 9 for an illustration of the map ψ .

Theorem 2.13. *The map ψ is a bijection from the set of parallelogram polyominoes of area n with k columns to the set of coin fountains with n coins in the even-numbered rows and $n - k$ coins in the odd-numbered rows.*

Proof. Let P be a parallelogram polyomino of area n with k columns. By the definition of the map h in [6], we have

$$\text{hpeak}(h(P)) = \text{area}(P) = n \quad \text{and} \quad \text{nbp}(h(P)) = \text{col}(P) = k.$$

By the definition of ϕ , it follows that

$$\text{area}(\phi^{-1}(h(P))) - \text{row}(\phi^{-1}(h(P))) = \text{area}(P) \quad \text{and} \quad \text{row}(\phi^{-1}(h(P))) = \text{col}(P).$$

Finally, we have

$$e(\psi(P)) = \text{area}(P) \quad \text{and} \quad e(\psi(P)) - o(\psi(P)) = \text{col}(P),$$

which means that

$$e(\psi(P)) = \text{area}(P) = n \quad \text{and} \quad o(\psi(P)) = \text{area}(P) - \text{col}(P) = n - k.$$

□

Corollary 2.14. *The number of Stanley polyominoes with area $n \geq 2$ and r rows is equal to the number of parallelogram polyominoes with area $n - r$ and r columns. Equivalently, it is equal to the number of coin fountains with $n - r$ coins in the even-numbered rows and $n - 2r$ coins in the odd-numbered rows.*

Proof. Let P be a parallelogram polyomino of area N with k columns, and set $Q = \phi^{-1}(h(P))$. By the proof of Theorem 2.13, we have

$$\text{area}(Q) - \text{row}(Q) = N, \quad \text{row}(Q) = k.$$

Hence

$$\text{area}(Q) = N + k, \quad \text{row}(Q) = k.$$

Therefore the map $\phi^{-1} \circ h$ is a bijection from parallelogram polyominoes of area N with k columns to Stanley polyominoes of area $N + k$ with k rows. Now write $n = N + k$ and $r = k$. Then $N = n - r$, and we obtain the first statement.

For the second statement, apply Theorem 2.13: the same parallelogram polyominoes are in bijection with coin fountains having N coins in the even-numbered rows and $N - k$ coins in the odd-numbered rows. Substituting $N = n - r$ and $k = r$ establishes the claim. □

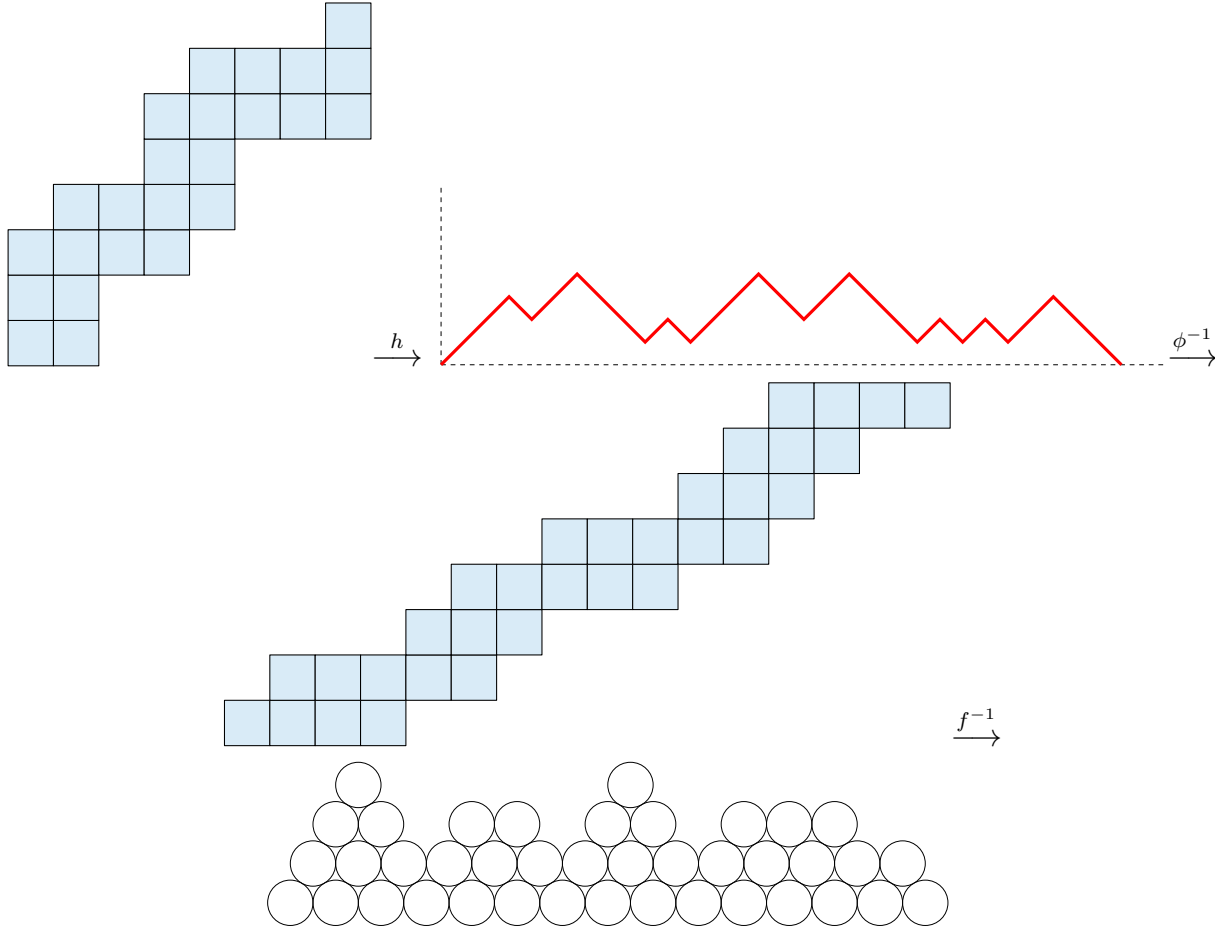


FIGURE 9. A parallelogram polyomino of area 24 and the construction of its image by ψ .

3. CONTINUED FRACTION POINT OF VIEW

The expression obtained in Theorem 2.9 can also be written as a continued fraction by means of the bijection ϕ between Stanley polyominoes with $n + 1$ columns and Dyck paths of semilength n (see the introduction). For background on the connection between lattice paths and continued fractions, see [16]. Since ϕ sends the area of a Stanley polyomino to the sum of peak heights plus the number of peaks of the corresponding Dyck path, we study Dyck paths according to the three statistics: the number of peaks, marked by p , the sum of peak heights, marked by q , and the sum of valley heights, marked by v .

Let $\mathcal{D}$ denote the set of non-empty Dyck paths. We consider the generating function

$$A(p, q, v) = \sum_{D \in \mathcal{D}} p^{\text{nbp}(D)} q^{\text{sump}(D)} v^{\text{sumv}(D)},$$

where $\text{nbp}(D)$ denotes the number of peaks of D , $\text{sump}(D)$ and $\text{sumv}(D)$ denote the sum of peak heights and the sum of the valley heights of D , respectively.

Theorem 3.1. *Let $A(p, q, v)$ be the generating function for Dyck paths where p marks the number of peaks, q marks the sum of peak heights, and v marks the sum of valley heights. Equivalently, $A(p, q, v)$ is the generating function for Stanley polyominoes with respect to the number of rows, the area minus the number of rows, and the number of interior points. Then*

$$A(p, q, v) = -1 + \frac{v}{1 + v - pqv - \frac{v}{1 + v - pq^2v^2 - \frac{v}{1 + v - pq^3v^3 - \frac{v}{1 + v - \dots}}}}.$$

Proof. By the bijection ϕ and the remark preceding the theorem, it is sufficient to work with Dyck paths. Let D be a nonempty Dyck path, and write its first-return decomposition as $D = uQdR$, where Q and R are Dyck paths, possibly empty.

If Q is empty, then $D = udR$. The initial peak contributes a factor pq , and the suffix R contributes a factor $1 + A(p, q, v)$. Hence the contribution of this case is

$$pq(1 + A(p, q, v)).$$

Now assume that Q is nonempty. In the decomposition each peak of Q increases its height by 1, so the sum of peak heights increases by $\text{nbp}(Q)$. Similarly, each valley of Q increases its height by 1, so the sum of valley heights increases by the number of valleys of Q . Since Q is a nonempty Dyck path, the number of valleys of Q is $\text{nbp}(Q) - 1$. Hence the weight of the elevated copy of Q is

$$p^{\text{nbp}(Q)} q^{\text{sump}(Q) + \text{nbp}(Q)} v^{\text{sumv}(Q) + \text{nbp}(Q) - 1} = \frac{1}{v} (pqv)^{\text{nbp}(Q)} q^{\text{sump}(Q)} v^{\text{sumv}(Q)}.$$

Therefore the contribution of this case is

$$\frac{1}{v} A(pqv, q, v) (1 + A(p, q, v)).$$

Combining the two cases, we obtain

$$A(p, q, v) = pq + pqA(p, q, v) + \frac{1}{v} A(pqv, q, v) (A(p, q, v) + 1),$$

which can be written

$$A(p, q, v) = -1 + \frac{v}{v - pqv - A(pqv, q, v)}.$$

Substituting this into the recursion and iterating yields the continued fraction. $\square$

The first few terms of the series expansion of $A(p, q, v)$ in powers of q are

$$\begin{aligned} & qp + p(p+1)q^2 + p(p^2 + 2p + 1)q^3 + (p + 3p^2 + 3p^3 + p^4 + p^2v)q^4 + \\ & p(p^4 + 4p^3 + 2p^2v + 6p^2 + 2pv + 4p + 1)q^5 + \\ & p(p^5 + 5p^4 + 3p^3v + p^2v^2 + 10p^3 + 6p^2v + pv^2 + 10p^2 + 3pv + 5p + 1)q^6 + O(q^7). \end{aligned}$$

The bold coefficient of q^4 corresponds to the Dyck paths shown in Figure 10.

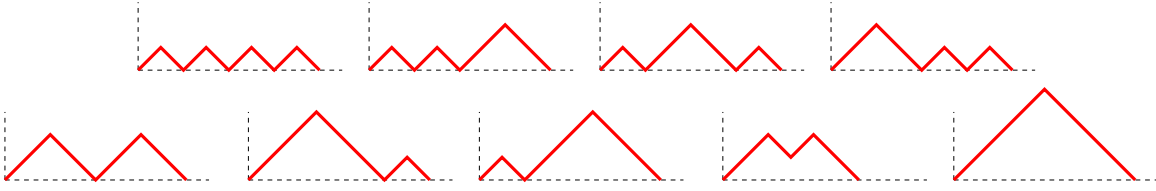


FIGURE 10. The nine Dyck paths with sum of peak heights equal to 4. Their contributions to $A(p, q, v)$ are, from left to right, p^4 , p^3 , p^3 , p^3 , p^2 , p^2 , p^2 , p^2v , and p .

Setting $p = q$ and $v = 1$ in $A(p, q, v)$, we obtain $A(q, q, 1) = F(1, 1, q, 1, 1)$. Hence the following corollary gives an alternative expression for $F(1, 1, q, 1, 1)$ as a continued fraction.

Corollary 3.2. *The generating function $F(1, 1, q, 1, 1)$ for the number of Stanley polyominoes with respect to the area is*

$$F(1, 1, q, 1, 1) = -1 + \frac{1}{2 - q^2 - \frac{1}{2 - q^3 - \frac{1}{2 - q^4 - \frac{1}{2 - q^5 - \dots}}}}.$$

Corollary 3.3. *Let $A(p, p, 0)$ be the generating function for Dyck paths all of whose valleys are at height 0, where p marks the sum of the peak heights plus the number of peaks. Equivalently, $A(p, p, 0)$ is the generating function for Stanley polyominoes with no interior points, where p marks the area. Then*

$$A(p, p, 0) = \frac{p^2}{1 - p - p^2}.$$

Moreover, $[p^n]A(p, p, 0) = F_{n-1}$ for $n \geq 2$ where F_n is the n th Fibonacci number.

Proof. A Dyck path all of whose valleys are at height 0 is a concatenation of pyramids of the form $u^k d^k$, with $k \geq 1$. A pyramid $u^k d^k$ has exactly one peak, at height k , so its contribution is p^{k+1} . Therefore the generating function for a single pyramid is

$$\sum_{k \geq 1} p^{k+1} = \frac{p^2}{1 - p}.$$

Hence the generating function for a nonempty sequence of such pyramids is

$$A(p, p, 0) = \frac{p^2}{1 - p} (1 + A(p, p, 0)).$$

Solving for $A(p, p, 0)$ gives

$$A(p, p, 0) = \frac{p^2}{1 - p - p^2}.$$

The coefficient formula follows immediately from the classical generating function of the Fibonacci numbers. $\square$

For example, $[p^6]A(p, p, 0) = F_5 = 5$, which corresponds to the Dyck paths shown in Figure 11.

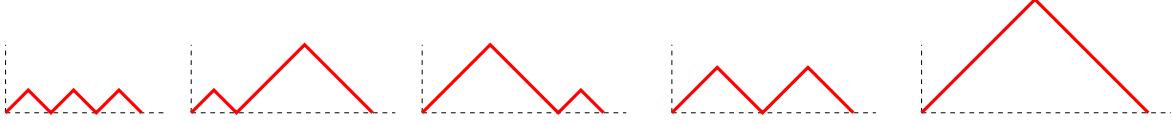


FIGURE 11. The five Dyck paths with sum of peak heights plus number of peaks equal to 6.

Corollary 3.4. *Let $A(1, q, 1)$ be the generating function for Dyck paths, where q marks the sum of the peak heights. Equivalently, $A(1, q, 1)$ is the generating function for Stanley polyominoes with respect to the area minus the number of rows. Then*

$$A(1, q, 1) = -1 + \frac{1}{2 - q - \frac{1}{2 - q^2 - \frac{1}{2 - q^3 - \frac{1}{2 - q^4 - \dots}}}}.$$

The first few terms of the series expansion of $A(1, q, 1)$ are

$$q + 2q^2 + 4q^3 + 9q^4 + 20q^5 + 46q^6 + 105q^7 + 242q^8 + 557q^9 + O(q^{10}).$$

The sequence of coefficients is a shift of the sequence [A006958](#), whose n th term counts the number of parallelogram polyominoes with n cells. Indeed, by Corollary 2.14, to every parallelogram polyomino P of area n there corresponds a Stanley polyomino $Q = \phi^{-1}(h(P))$ such that $n = \text{area}(Q) - \text{row}(Q)$.

Corollary 3.5. *Let $A(1, q, q)$ be the generating function for Dyck paths, where q marks the sum of the peak heights plus the sum of the valley heights. Equivalently, $A(1, q, q)$ is the generating function for Stanley polyominoes with respect to the area plus the number of interior points minus the number of rows. Then*

$$A(1, q, q) = -1 + \frac{q}{1 + q - q^2 - \frac{q}{1 + q - q^4 - \frac{q}{1 + q - q^6 - \frac{q}{1 + q - q^8 - \dots}}}}.$$

The first terms of the series expansion of $A(1, q, q)$ are

$$q + 2q^2 + 4q^3 + 8q^4 + 17q^5 + 36q^6 + 76q^7 + 162q^8 + 345q^9 + O(q^{10}).$$

The sequence of coefficients is a shift of [A226729](https://oeis.org/A226729). To the best of our knowledge, the above combinatorial interpretation is new. There are, however, some conjectures suggesting that this sequence also counts certain families of restricted compositions.

REFERENCES

- [1] P. Bala. Fountains of coins and skew Ferrers diagrams. (2019), <https://oeis.org/A161492>.
- [2] C. Banderier, M. Bousquet-Mélou, A. Denise, P. Flajolet, D. Gardy, and D. Gouyou-Beauchamps. Generating functions for generating trees. *Discrete Math.* **246** (2002), 29–55.
- [3] J.-L. Baril, S. Kirgizov, J.L. Ramírez, and D. Villamizar. The combinatorics of Motzkin polyominoes. *Discrete Appl. Math.* **364** (2025), 1–15.
- [4] A. Blecher, C. Brennan, A. Knopfmacher, and T. Mansour. The perimeter of words. *Discrete Math.* **340** (10) (2017), 2456–2465.
- [5] A. Blecher, C. Brennan, and A. Knopfmacher. The site-perimeter of compositions. *Discrete Math. Appl.* **32** (2) (2022), 75–89.
- [6] M. Bousquet-Mélou. Une bijection entre les polyominos convexes dirigés et les mots de Dyck bilatères. *RAIRO Theor. Inform. Appl.* **26** (3) (1992), 205–219.
- [7] M. Bousquet-Mélou. q -Énumération de polyominos convexes. *J. Combin. Theory Ser. A* **64** (2) (1993), 265–288.
- [8] M. Bousquet-Mélou and A. Rechnitzer. The site-perimeter of bargraphs. *Adv. Appl. Math.* **31** (2003), 86–112.
- [9] D. Callan, T. Mansour, and J. L. Ramírez. Statistics on bargraphs of Catalan words. *J. Autom. Lang. Comb.* **26** (2021), 177–196.
- [10] M.P. Delest and X. Viennot. Algebraic languages and polyominoes enumeration. *Theoret. Comput. Sci.* **34** (1984), 169–206.
- [11] M.P. Delest, D. Gouyou-Beauchamps, and B. Vauquelin. Enumeration of parallelogram polyominoes with given bond and site perimeter. *Graphs Combin.* **3** (1987), 325–339.
- [12] M.P. Delest and J.M. Fedou. Attribute grammars are useful for combinatorics. *Theoret. Comput. Sci.* **98** (1) (1992), 65–76.
- [13] M.P. Delest and J.M. Fedou. Enumeration of skew ferrer diagrams. *Sém. Lothar. Combin.* **23** B23d (1993), 51–66.
- [14] E. Deutsch. Dyck path enumeration. *Discrete Math.* **204** (1999), 167–202.
- [15] E. Duchi. Polyominoes, permutominoes and permutations. *Habilitation à diriger des recherches*, <https://www.irif.fr/~duchi/hdrDuchi.pdf>, (2018).
- [16] P. Flajolet. Combinatorial aspects of continued fractions. *Discrete Math.* **32** (1980), 125–161.
- [17] S. Feretić. A perimeter enumeration of column-convex polyominoes. *Discrete Math. Theor. Comput. Sci.* **9** (1) (2007), 57–83.
- [18] T. Mansour and A. Sh. Shabani. Enumerations on bargraphs. *Discrete Math. Lett.* **2** (2019), 65–94.
- [19] A. J. Guttmann (Ed.). *Polygons, Polyominoes and Polycubes*. Lecture Notes in Physics 775. Springer, Heidelberg, Germany, 2009.
- [20] OEIS Foundation Inc., The On-Line Encyclopedia of Integer Sequences, <http://oeis.org/>.
- [21] H. Prodinger. The kernel method: a collection of examples. *Sém. Lothar. Combin.* **50** (2004), Paper B50f.
- [22] R.P. Stanley. *Catalan Numbers*. Cambridge University Press, 2015.
- [23] H.N.V. Temperley. Combinatorial problems suggested by the statistical mechanics of domains and of rubber-like molecules. *Phys. Rev.* **103** (1956), 1–16.
- [24] H.S. Wilf. *Generatingfunctionology*. 3rd ed., A K Peters/CRC Press, 2005.

LIB, UNIVERSITÉ BOURGOGNE EUROPE, B.P. 47 870, 21078, DIJON CEDEX, FRANCE

Email address: `barjl@u-bourgogne.fr`

THE JOHN KNOPFMACHER CENTRE FOR APPLICABLE ANALYSIS AND NUMBER THEORY, SCHOOL OF MATHEMATICS, UNIVERSITY OF THE WITWATERSRAND, PRIVATE BAG 3, WITS 2050, JOHANNESBURG, SOUTH AFRICA

Email address: `Aubrey.Blecher@wits.ac.za`

DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD NACIONAL DE COLOMBIA, BOGOTÁ, COLOMBIA

Email address: `jllramirezr@unal.edu.co`