Aequat. Math. (2026) 100:29

https://doi.org/10.1007/s00010-026-01275-9

Aequationes Mathematicae

Some distributions on increasing and flattened permutations

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Abstract. We examine the distribution and popularity of different parameters (such as the number of descents, runs, valleys, peaks, right-to-left minima, and more) on the sets of increasing and flattened permutations. For each parameter, we provide an exponential generating function for its corresponding distribution and popularity. Additionally, we present one-to-one correspondences between these permutations and some classes of simpler combinatorial objects.

Mathematics Subject Classification. Primary 05A15, 05A19.

Keywords. Increasing permutation, flattened permutation, generating function.

1. Introduction and notation

Let $\mathfrak{S}_n$ denote the set of permutations of $[n] := \{1, 2, \dots, n\}$. For convenience, we define $\mathfrak{S}_0$ as the singleton set consisting of the empty permutation ϵ, and we set $\mathfrak{S} = \bigcup_{n\geq 0} \mathfrak{S}_n$. A permutation $\pi \in \mathfrak{S}$ can be written in *one-line notation* as the word $\pi = \pi_1\pi_2 \cdots \pi_n$, where $\pi_i = \pi(i)$ for $1 \leq i \leq n$, and it is represented by the empty word (also denoted ϵ) if $\pi \in \mathfrak{S}_0$. We denote the length of π by $|\pi|$. Given a sequence of distinct integers $a = a_1a_2 \cdots a_n$, the *reduction* a, denoted $\mathtt{red}(a)$, is the unique permutation in $\mathfrak{S}_n$ order-isomorphic to a, and the *domain* of a, denoted $\mathsf{dom}(a)$, is the set $\{a_1, a_2, \dots, a_n\}$. For instance, if $a = 3\ 6\ 1\ 4$ then $\mathtt{red}(a) = 2\ 4\ 1\ 3$ and $\mathsf{dom}(a) = \{1, 3, 4, 6\}$.

Let $\pi = \pi_1\pi_2 \cdots \pi_n \in \mathfrak{S}_n$. A *valley* in π is an index ℓ, where $2 \leq \ell \leq n-1$, such that $\pi_{\ell-1} > \pi_\ell < \pi_{\ell+1}$. The *height* of a valley is given by the image of its index, that is π_ℓ. The number of valleys in π is denoted by $\mathsf{val}(\pi)$. Analogously, we define the peak statistic. A *peak* in π is an index ℓ, where $2 \leq \ell \leq n-1$, such that $\pi_{\ell-1} < \pi_\ell > \pi_{\ell+1}$. The number of peaks in π is denoted by $\mathsf{peak}(\pi)$. We say that π has an *ascent* (resp. *descent*) at position ℓ if $\pi_\ell < \pi_{\ell+1}$ (resp. $\pi_\ell > \pi_{\ell+1}$), where $\ell \in [n]$. The number of ascents and descents of π is denoted by $\mathsf{asc}(\pi)$ and $\mathsf{des}(\pi)$, respectively. Clearly, if $\pi \in \mathfrak{S}_n$

Table 1. Permutation statistics.

$\mathsf{val}(\pi)$	number of valleys of π	$\mathsf{run}(\pi)$	number of runs of π
$\mathsf{des}(\pi)$	number of descents of π	$\mathsf{asc}(\pi)$	number of ascents of π
$\mathsf{rlm}(\pi)$	number of right-to-left minima of π	$\mathsf{peak}(\pi)$	number of peaks of π

then we have $\mathsf{des}(\pi) = n - 1 - \mathsf{asc}(\pi)$. A *run* in a permutation π is a subword $\pi_\ell \pi_{\ell+1} \cdots \pi_{\ell+s} \pi_{\ell+s+1}$, where $\ell, \ell+1, \ldots, \ell+s$ are consecutive ascents. The number of runs of π is denoted by $\mathsf{run}(\pi)$. A *right-to-left minimum* of π is an entry π_i such that if $j > i$, then $\pi_i < \pi_j$. The number of right-to-left minima of π is denoted by $\mathsf{rlm}(\pi)$. All these parameters are summarized in Table 1.

In this paper, we focus on the distribution of the aforementioned permutation statistics within two families of permutations: *increasing* and *flattened* permutations.

A permutation π is called *increasing* if the heights of its valleys form an increasing sequence from left to right. For example, in Figure 1, we illustrate an increasing permutation of length 9, with the valleys marked in red. We denote by $\mathcal{I}$ the set of all increasing permutations, and by $\mathcal{I}_n$ the set of those of length n. This concept has also been studied in the context of other combinatorial structures. One of the earliest works was by Barcucci et al. [2] in the context of Dyck path. For example, they prove that the number of Dyck paths of length $2n$, where the heights of their valleys form a non-decreasing sequence, is given by the Fibonacci number F_{2n-1} (see also [9,13]). This concept was later extended by Flórez and Ramírez [14] for Motzkin paths. Flórez at al. [15] also studied this concept in the context of integer compositions, showing connections with unimodal compositions.

A permutation π is called *flattened* if it consists of runs arranged from left to right, with the first entries of each run in increasing order. We denote by $\mathcal{F}$ the set of all flattened permutations and by $\mathcal{F}_n$ the set of those of length n, and we clearly have $\mathcal{F}_n \subset \mathcal{I}_n$ for $n \geq 1$. For example, the permutation $\pi = 1\ 5\ 2\ 4\ 3\ 6\ 9\ 7\ 8$ is flattened, with runs $1\ 5, 2\ 4, 3\ 6\ 9, 7\ 8$. This concept was introduced by Callan [8], who enumerated the number of flattened partitions of length n that avoid a pattern of length 3. Since then, many authors have investigated this concept. For instance, Nabawanda et al. [21] studied the run distribution of flattened permutations and provided a bijection between flattened partitions over $[n+1]$ and set partitions of $[n]$. Mansour et al. [18,19] examined the avoidance of subword patterns in flattened partitions (see also [17]) and the distribution of consecutive patterns on flattened permutations (see [16]). Notably, Mansour et al. [18] also studied some statistics such as descents, ascents, peaks, and valleys in a different context of flattened permutations, specifically focusing on the cycle notation of permutations. Recently, this concept has been explored in other combinatorial objects. For example,

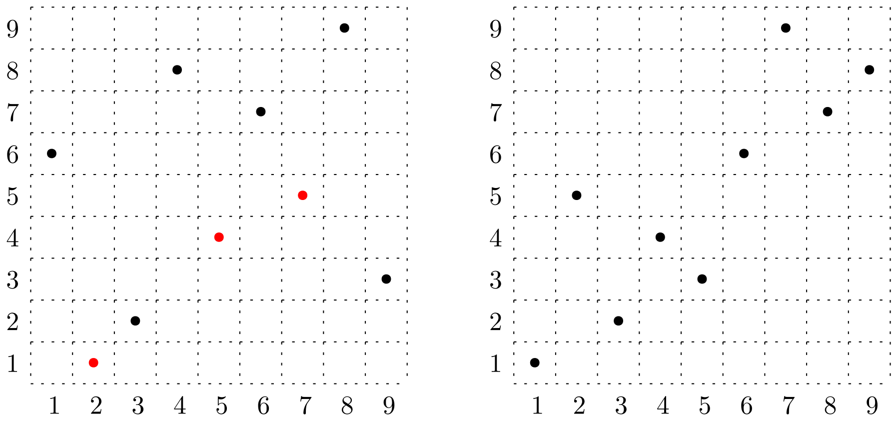

FIGURE 1. Increasing permutation $\pi = 6\ 1\ 2\ 8\ 4\ 7\ 5\ 9\ 3$, and flattened permutation $\pi = 1\ 5\ 2\ 4\ 3\ 6\ 9\ 7\ 8$.

Buck et al. [6] introduced the notion of Stirling permutations, Elder et al. [10] defined and enumerated flattened parking functions, and Baril et al. [3] investigated several statistics on flattened Catalan words.

Permutations are labeled combinatorial objects, and so it is natural to employ exponential generating functions (egf's) for their enumerative analysis. We provide the bivariate exponential generating functions for increasing and flattened permutations with respect to the length and the parameters listed in Table 1, that is, the coefficient of $\frac{x^n y^k}{n!}$ in the series expansion counts the number of permutations (either increasing or flattened) of length n with the parameter equal to k. We also give the egf for the popularity of each parameter whose coefficient of $\frac{x^n}{n!}$ in its series expansion is the sum of the respective parameter's values taken over all members of either $\mathcal{I}_n$ or $\mathcal{F}_n$.

Our method consists in constructing recursively the combinatorial class in question from smaller classes, $\mathcal{A}_1$ and $\mathcal{A}_2$, using the usual labeled product $\mathcal{A}_1 \star \mathcal{A}_2$ and the boxed product $\mathcal{A}_1^{\Box} \star \mathcal{A}_2$. The boxed product $\mathcal{A}_1^{\Box} \star \mathcal{A}_2$ is a subset of $\mathcal{A}_1 \star \mathcal{A}_2$ where the smallest label appears in the $\mathcal{A}_1$ component. The exponential generating function of $\mathcal{A}_1^{\Box} \star \mathcal{A}_2$ is given by

$$\int_0^x \partial_t A_1(t) \cdot A_2(t) dt$$

where $A_1(t)$ and $A_2(t)$ are the exponential generating functions of $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively. For more details on labeled combinatorial structures and the boxed product, we refer the reader to the textbook of Flajolet and Sedgewick [12] as well as [4] for additional applications.

2. Enumeration of increasing permutations

In this section, we study the bivariate generating function

$$I^s(x,y) := \sum_{\pi\in\mathcal{I}} \frac{x^{|\pi|}}{|\pi|!} y^{\mathsf{s}(\pi)} = \sum_{n\geq 0} \frac{x^n}{n!} \sum_{\pi\in\mathcal{I}_n} y^{\mathsf{s}(\pi)} \tag{1}$$

whenever the statistic s is val, $\mathsf{run} = 1+\mathsf{des}$, and rlm. The coefficient of $\frac{x^n y^k}{n!}$ in $I^s(x,y)$ is the number of increasing permutations π of length n satisfying $s(\pi)=k$.

2.1. For the number of valleys: $\mathsf{s} = \mathsf{val}$

We enumerate increasing permutations according to their length and the number of valleys, $\mathsf{s} = \mathsf{val}$.

Elizalde and Noy [11] have shown that the egf $P(x,y)$ for the number of permutations with respect to the length and number of valleys is given by

$$P(x,y) = \sum_{n\geq 0} \frac{x^n}{n!} \sum_{\pi\in\mathfrak{S}_n} y^{\mathsf{val}(\pi)} = \frac{\sqrt{1-y}}{\sqrt{1-y}-\tanh(x\sqrt{1-y})}. \tag{2}$$

The coefficient of $\frac{x^n y^k}{n!}$ in the series expansion of $P(x,y)$ corresponds to the array A008303 in the OEIS [24]. This counting problem has been extensively studied, with many related papers addressing various aspects of it, see for example [7,20,27].

We denote by $\mathcal{V}$ the set of *valleyless permutations*, which are permutations free of valleys. Rieper and Zeleke [23] established a bijection between valleyless permutations and integer compositions. Consequently, the egf for valleyless permutations is given by

$$V(x) = \frac{1}{2}\left(e^{2x}+1\right) = 1 + e^x \sinh x.$$

This result will be used in the following proof.

Theorem 2.1. *The generating function for increasing permutations with respect to the length and the number of valleys is*

$$I^{\mathsf{val}}(x,y) = e^{2x-\frac{y}{4}+\frac{1}{4}e^{2x}y-\frac{xy}{2}}\left(1+\frac{e^{y/4}}{2}\int_0^x e^{\frac{-e^{2t}}{4}y-2t+\frac{yt}{2}}(-2+y-e^{2t}y)dt\right).$$

Proof. Let π be a nonempty increasing permutation. We decompose π according to the position of the entry 1. Specifically, π can be expressed in one of the following forms: $\pi=\pi_1 1$, $\pi = 1\pi_2$, or $\pi=\pi' 1\pi_2$, where $\pi_2\neq\epsilon$ and π' is free of valleys. From this decomposition, we derive the following symbolic equation:

$$\mathcal{I} = \epsilon + \mathcal{I}\star\{x\}^{\square} + \{x\}^{\square}\star(\mathcal{I}-\epsilon) + (\mathcal{V}-\epsilon)\star\{x\}^{\square}\star(\mathcal{I}-\epsilon).$$

Here, $\mathcal{A}^{\Box} \star \mathcal{B}$ denotes the boxed product of the combinatorial classes $\mathcal{A}$ and $\mathcal{B}$. From the symbolic equation, and after multiplying by y whenever a valley is created, we obtain the functional equation

$$I^{\mathsf{val}}(x,y) = 1+\int_0^x I^{\mathsf{val}}(t,y)dt+\int_0^x (I^{\mathsf{val}}(t,y)-1)dt+y\int_0^x e^t \sinh t(I^{\mathsf{val}}(t,y)-1)dt.$$

Differentiating this functional equation yields the following differential equation:

$$\frac{d}{dx}I^{\mathsf{val}}(x,y) = 2I^{\mathsf{val}}(x,y) - 1 + ye^x \sinh x(I^{\mathsf{val}}(x,y) - 1), \quad I^{\mathsf{val}}(0,y) = 1.$$

The solution of this differential equation provides the desired result. □

Notice that the solution $I^{\mathsf{val}}(x,y)$ can be expressed in a closed form without integrals, depending on the WhittakerM function [26]. However, this form is not particularly elegant, so we do not include it here. We solve the differential equations of this paper by using symbolic software.

The first few terms of the formal power series are

$$\begin{aligned}I^{\mathsf{val}}(x,y) &= \frac{1}{2}\left(e^{2x}+1\right)+\frac{1}{8}\left(-4e^{2x}x+e^{4x}-1\right)y\\ &\quad+\frac{1}{64}\left(e^{2x}\left(8x^2-4x-1\right)+e^{4x}(2-8x)+e^{6x}-2\right)y^2+O(y^3).\end{aligned}$$

Let $\mathsf{inc}^{\mathsf{val}}(n,k)$ denote the number of increasing permutations in $\mathfrak{S}_n$ with exactly k valleys. The first few values of this sequence are

$$I^{\mathsf{val}} := [\mathsf{inc}^{\mathsf{val}}(n,k)]_{n,k\geq 0} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \cdots \\ 1 & 0 & 0 & 0 & 0 \cdots \\ 2 & 0 & 0 & 0 & 0 \cdots \\ 4 & 2 & 0 & 0 & 0 \cdots \\ 8 & \boxed{\mathbf{16}} & 0 & 0 & 0 \cdots \\ 16 & 88 & 8 & 0 & 0 \cdots \\ 32 & 416 & 136 & 0 & 0 \cdots \\ 64 & 1824 & 1440 & 48 & 0 \cdots \\ 128 & 7680 & 12288 & 1384 & 0 \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \ddots \end{pmatrix}.$$

For example, $\mathsf{inc}^{\mathsf{val}}(4,1) = 16$, which corresponds to the boxed entry in the array I^{val} above. The 16 increasing permutations corresponding to this value are

1324, 1423, 2134, 2143, 2314, 2413, 3124, 3142,
3214, 3241, 3412, 4123, 4132, 4213, 4231, 4312.

This sequence satisfies $\mathsf{inc}^{\mathsf{val}}(n,1) = 2^{n-2}(2^{n-1}-n)$, for $n \geq 3$, which appears as A000431 in the OEIS [24].

Due to the functional equation involving $I^{\mathsf{val}}(x,y)$, and after observing that

$$\mathcal{V}(x) = 1 + \sum_{n\geq 1} 2^{n-1}\frac{x^n}{n!},$$

a straightforward series calculation leads to the following result.

Corollary 2.2. *For $n,k \geq 1$, we have the recursive formula*

$$\mathit{inc}^{\mathit{val}}(n+1,k) = 2\,\mathit{inc}^{\mathit{val}}(n,k) + \sum_{i=1}^{n-1}\binom{n}{i}2^{i-1}\mathit{inc}^{\mathit{val}}(n-i,k-1),$$

with the initial condition $\mathit{inc}^{\mathit{val}}(0,0)=1$ and $\mathit{inc}^{\mathit{val}}(n,0) = 2^{n-1}$ for $n\geq 1$.

Corollary 2.3. *The generating function $I(x)$ for total number of increasing permutations is*

$$I(x) := I^{\mathit{val}}(x,1) = e^{\frac{3x}{2}-\frac{1}{4}+\frac{1}{4}e^{2x}}\left(1-\frac{e^{1/4}}{2}\int_0^x e^{\frac{-e^{2t}}{4}-\frac{3t}{2}}(1+e^{2t})dt\right)$$
$$= 1+\frac{x}{1!}+2\frac{x^2}{2!}+6\frac{x^3}{3!}+24\frac{x^4}{4!}+112\frac{x^5}{5!}+584\frac{x^6}{6!}++3376\frac{x^7}{7!}+O(x^8).$$

Moreover, if $\mathit{inc}(n)$ is the number of increasing permutations of length n, then

$$\mathit{inc}(n+1) = 2\,\mathit{inc}(n) + \sum_{i=1}^{n-1}\binom{n}{i}2^{i-1}\mathit{inc}(n-i), \quad n\geq 1$$

with the initial conditions $\mathit{inc}(0) = 1 = \mathit{inc}(1)$.

Corollary 2.4. *The generating function for total number of valleys in all increasing permutations of a given length is*

$$\mathit{Tot}^{\mathit{val}}(x) = \partial_y(I^{\mathit{val}}(x,y))|_{y=1}$$
$$= 2\frac{x^3}{3!}+16\frac{x^4}{4!}+104\frac{x^5}{5!}+688\frac{x^6}{6!}+4848\frac{x^7}{7!}+O(x^8).$$

The sequences of coefficients of $\frac{x^n}{n!}$ from the two series expansions in the previous corollaries do not appear in the OEIS.

2.2. For the number of runs: s=run=1+des=n-asc

In this part, we focus on the enumeration of increasing permutations with respect to the length and the statistic $\mathsf{s} = \mathsf{run}$ that counts the number of runs. It is worth noticing that, in a nonempty permutation, the number of descents equals the number of runs minus one, i.e., $\mathsf{des} = \mathsf{run} - 1$. Therefore, we have

$$I^{\mathsf{run}}(x,y) = 1 + y(I^{\mathsf{des}}(x,y)-1) \quad \text{and} \quad I^{\mathsf{asc}}(x,y) = 1+\frac{1}{y}(I^{\mathsf{des}}(xy,1/y)-1).$$

For the classical permutations, the number of permutations of length n with exactly k descents is given by the Eulerian numbers (sequence A008292). It is well-known that (cf. [4,5,22])

$$E(x,y) = \sum_{\pi \in \mathfrak{S}} \frac{x^{|\pi|}}{|\pi|!} y^{\mathsf{des}(\pi)} = \frac{1-y}{1-ye^{x(1-y)}}.$$

Theorem 2.5. *The generating function for increasing permutations with respect to the number of runs is*

$$I^{run}(x,y) = \exp\left(\frac{x(y+1)\left(y^2+y+1\right)+y\left(e^{x(y+1)}-1\right)}{(y+1)^2}\right)\left(\frac{e^{y/(y+1)^2}}{y+1}\int_0^x \left(-ye^{(y+1)t}-1\right)\exp\left(-\frac{(y+1)\left(y^2+y+1\right)t+ye^{(y+1)t}}{(y+1)^2}\right)dt+1\right).$$

Proof. Let π be a nonempty increasing permutation. We consider the same decomposition of π based on the position of 1, as in the proof of Theorem 2.1. From this decomposition, we obtain the symbolic equation:

$$\mathcal{I} = \epsilon + \{x\} + (\mathcal{I} - \epsilon) \star \{x\}^{\Box} + \{x\}^{\Box} \star (\mathcal{I} - \epsilon) + (\mathcal{V} - \epsilon) \star \{x\}^{\Box} \star (\mathcal{I} - \epsilon), \tag{3}$$

where $\mathcal{V}$ is the set of valleyless permutations.

Now let us give the egf for the set $\mathcal{V}$ with respect to the length and the number of runs. Using the following recursive decomposition of $\mathcal{V}$,

$$\mathcal{V} = \epsilon + \{x\} + \{x\}^{\Box} \star (\mathcal{V} - \epsilon) + (\mathcal{V} - \epsilon) \star \{x\}^{\Box},$$

we deduce the functional equation (we multiply by y whenever a run is created)

$$V(x,y) = 1 + xy + \int_0^x (V(t,y)-1)dt + y\int_0^x (V(t,y)-1)dt.$$

Then we obtain

$$V(x,y) = \frac{e^{(y+1)x}y+1}{y+1} = 1 + \sum_{n\geq 1}\sum_{j=0}^{n-1}\binom{n-1}{j} y^{j+1}\frac{x^n}{n!}.$$

From (3), and after multiplying by y whenever a run is created, we obtain the functional equation

$$I^{\mathsf{run}}(x,y) = 1+xy+(1+y)\int_0^x (I^{\mathsf{run}}(t,y)-1)dt + \int_0^x (V(t,y)-1)(I^{\mathsf{run}}(t,y)-1)dt.$$

Differentiating the previous equation, we obtain the following differential equation:

$$\frac{d}{dx}I^{\mathsf{run}}(x,y) = y+(1+y)(I^{\mathsf{run}}(x,y)-1)+(V(x,y)-1)(I^{\mathsf{run}}(x,y)-1), \quad I^{\mathsf{run}}(0,y) = 1.$$

The solution of this differential equation is the desired result. $\Box$

The first few terms of the formal power series are

$$I^{\mathsf{run}}(x,y) = 1 + (e^x - 1)\,y + e^x\,(-x + e^x - 1)\,y^2 + \frac{1}{2}\left(e^x\left(-x^2 + 4x + 1\right) - 4e^{2x} + e^{3x} + 2\right)y^3 + O\left(y^4\right).$$

Let $\mathsf{inc}^{\mathsf{run}}(n,k)$ the number of increasing permutations in $\mathfrak{S}_n$ with exactly k runs. Then, we have the following

$$I^{\mathsf{run}} := [\mathsf{inc}^{\mathsf{run}}(n,k)]_{n,k\geq 0} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 4 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & \boxed{\mathbf{11}} & 11 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 26 & 58 & 26 & 1 & 0 & \cdots \\ 0 & 1 & 57 & 234 & 234 & 57 & 1 & \cdots \\ 0 & 1 & 120 & 831 & 1472 & 831 & 120 & \cdots \\ 0 & 1 & 247 & 2757 & 7735 & 7735 & 2757 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

For example, $\mathsf{inc}^{\mathsf{run}}(4,2) = 11$, the entry boxed in the array I^{run} above, and the corresponding increasing permutations are

$$\begin{array}{cccccc} 1243, & 1324, & 1342, & 1423, & 2134, & 2314, \\ 2341, & 2413, & 3124, & 3412, & 4123. & \end{array} \tag{4}$$

Note that the values $\mathsf{inc}^{\mathsf{run}}(n,2)$, for $n \geq 2$, match the Eulerian numbers A000295, which count the permutations with exactly one descent, that is, increasing permutations with two runs.

A simple calculation on series induces the following result.

Corollary 2.6. *We have the following recursive formula for $n,k \geq 1$*

$$\mathit{inc}^{\mathit{run}}(n+1,k) = \mathit{inc}^{\mathit{run}}(n,k) + \mathit{inc}^{\mathit{run}}(n,k-1) + \sum_{i=1}^{n-1}\sum_{j=1}^{k-1}\binom{n}{i}\binom{i-1}{k-j-1}\mathit{inc}^{\mathit{run}}(n-i,j),$$

with the initial condition $\mathit{inc}^{\mathit{run}}(0,0) = 1$, $\mathit{inc}^{\mathit{run}}(n,0) = 0$ for $n \geq 1$.

Observing the array I^{run}, we can hope for the existence of an involution ϕ on $\mathcal{I}_n$ such that $\mathsf{run}(\phi(\pi)) = n+1-\mathsf{run}(\pi)$. Then, let us define recursively the map ϕ on $\mathcal{I}$ as follows:

For $\pi \in \mathcal{I}$, we set:

$$\phi(\pi) = \begin{cases} \epsilon & \text{if } \pi = \epsilon & (i) \\ (1+\phi(\pi'))1 & \text{if } \pi = 1(1+\pi'), \quad \pi' \in \mathcal{I} & (ii) \\ 1(1+\phi(\pi')) & \text{if } \pi = (1+\pi')1, \quad \pi' \in \mathcal{I}, \pi' \neq \epsilon & (iii) \\ \pi_1^r 1\chi(\phi(\mathtt{red}(\pi_2)), \mathtt{dom}(\pi_2)) & \text{if } \pi = \pi_1 1 \pi_2, \quad \pi_1, \pi_2 \neq \epsilon & (iv) \end{cases}$$

where π_1^r is the reverse of π_1 and $\mathtt{dom}(\pi_2)$ is the set of values in π_2 and $\chi(\pi, S)$ (with $|\pi| = |S|$) is the unique sequence taking values in S such that $\mathtt{red}(\sigma) = \pi$.

For instance, if $\pi = 4132$ and $S = \{2,6,7,9\}$, then $\chi(\pi, S) = 9276$ since $\mathtt{red}(9276) = 4132$.

Let us give an example. Easily, we can see that $\phi(1) = 1$, $\phi(12) = 21$ and $\phi(21) = 12$. Now, let $\pi = 23154$ be an increasing permutation with 3 runs. Then, we have $\pi = \pi_1 1 \pi 2$ with $\pi_1 = 23$, $\pi_2 = 54$ and $\mathtt{dom}(\pi_2) = \{4,5\}$. Therefore, we obtain

$$\chi(\phi(\mathtt{red}(\pi_2), \mathtt{dom}(\pi_2)) = \chi(\phi(21), \{4,5\}) = \chi(12, \{4,5\}) = 45.$$

Finally, we have $\phi(23154) = 32145$ and the number of runs in 32145 is 3 which also is $5+1$ minus the number of runs of $\pi = 23154$.

For example, the images by ϕ (keeping the order) of the permutations in (4) are the increasing permutations with 3 runs, respectively:

$$\begin{array}{cccccc} 3421, & 3241, & 2431, & 4231, & 2143, & 3214, \\ 1432, & 4213, & 3142, & 4312, & 4132. & \end{array}$$

Corollary 2.7. *The generating function for total number of runs in all increasing permutations of a given length is*

$$\begin{aligned} \mathsf{Tot}^{\mathsf{run}}(x) &= \partial_y (I^{\mathsf{run}}(x,y))|_{y=1} \\ &= x + 3\frac{x^2}{2!} + 12\frac{x^3}{3!} + 60\frac{x^4}{4!} + 336\frac{x^5}{5!} + 2044\frac{x^6}{6!} + 13504\frac{x^7}{7!} + O(x^8). \end{aligned}$$

The sequence of coefficients of $\frac{x^n}{n!}$ in this series expansion does not appear in the OEIS.

Let $\mathcal{I}_n^{\mathsf{val=des}}$ be the set of increasing permutations π of length n such that $\mathsf{val}(\pi) = \mathsf{des}(\pi)$. We set $\mathcal{I}^{\mathsf{val=des}} = \bigcup_{n\geq 0} \mathcal{I}_n^{\mathsf{val=des}}$.

Theorem 2.8. *The cardinality $\mathsf{inc}^{\mathsf{val=des}}(n)$ of $\mathcal{I}_n^{\mathsf{val=des}}$ is given by the n-th Gould number (see A040027 in [24]):*

$$\mathsf{inc}^{\mathsf{val=des}}(n) = \sum_{k=0}^{n-1} \sum_{j=k}^{n-1} \left\{ {j \atop k} \right\} k^{n-1-j},$$

where $\left\{ {j \atop k} \right\}$ is the Stirling number of second kind.

Proof. Let π be a permutation in $\mathcal{I}_n^{\mathsf{val=des}}$. Then π has a unique decomposition as follows: $\pi = \pi_1 a_1 \pi_2 a_2 \cdots \pi_k a_k$, where $a_1, a_2, \ldots, a_k$ are the right-to-left minima of π. Due to the equality $\mathsf{val}(\pi) = \mathsf{des}(\pi)$, $\pi_i a_i$ is either reduced to a_i whenever $\pi_i = \epsilon$, or $\pi_i a_i$ has only one descent and this descent involves a_i. Moreover, π_k is necessarily empty. Therefore, we construct the partition $B = B_1/B_2/\cdots/B_k$ of $[n]$ where $B_i = \mathtt{dom}(\pi_i a_i)$ for $1 \leq i \leq k$. Since the a_i are right-to-left minima of π, then we have $a_1 = 1 < a_2 < \cdots < a_k$, and then $\min(B_1) < \min(B_2) < \cdots < \min(B_k) = a_k$. Thus, B is a partition ordered by the smallest element of the blocks such that the last block is reduced to a singleton.

Conversely, let us consider such a partition $B = B_1/B_2/\cdots/B_k$. Then we construct an permutation π as follows: $\pi = \pi_1 a_1 \pi_2 a_2 \cdots \pi_k a_k$, where $a_i = \min(B_i)$ and π_i consists of elements of $B_i \backslash \{a_i\}$ sorted in increasing order. A simple observation proves that $\pi \in \mathcal{I}_n^{\mathsf{val}=\mathsf{des}}$, which establishes a bijection between $\mathcal{I}_n^{\mathsf{val}=\mathsf{des}}$ and the set of partitions of $[n]$ having the last part reduced to a singleton. Since these partitions are counted by the nth Gould number (see A040027 in [24]), we obtain the desired result. $\square$

2.3. For the number of right-to-left minima: $\mathsf{s}=\mathsf{rlm}$

In this part, we focus on the enumeration of increasing permutations with respect to the length and the statistic $\mathsf{s} = \mathsf{rlm}$ that counts the number of right-to-left minima.

For the classical permutations, the number of permutations of length n with exactly k right-to-left minima is the (unsigned) Stirling numbers of the first kind (sequence A132393). It is well-known that (cf. [5]) the corresponding bivariate generating function is

$$S(x,y) = \sum_{\pi \in \mathfrak{S}} \frac{x^{|\pi|}}{|\pi|!} y^{\mathsf{rlm}(\pi)} = \frac{1}{(1-x)^y}.$$

Theorem 2.9. *The generating function for increasing permutations with respect to the number of right-to-left minima is*

$$I^{\mathsf{rlm}}(x,y) = e^{\frac{y}{4}\left(2\,x+e^{2x}\right)}\left(\frac{y}{2}\int_0^x \left(2\,I\left(t\right)-1-e^{2t}\right) e^{-\frac{y}{4}\left(2t+e^{2t}\right)}dt + e^{-\frac{y}{4}}\right),$$

where $I(z)$ is given in Corollary 2.3.

Proof. Let π be a nonempty increasing permutation. We consider the decomposition of π with respect to the position of the entry 1. From this decomposition we obtain the symbolic equation:

$$\mathcal{I} = \epsilon + \mathcal{I} \star \{x\}^{\square} + \{x\}^{\square} \star (\mathcal{I}-\epsilon) + (\mathcal{V}-\epsilon) \star \{x\}^{\square} \star (\mathcal{I}-\epsilon).$$

From the symbolic method (cf. [12]) we obtain the functional equation

$$I^{\mathsf{rlm}}(x,y) = 1 + y\int_0^x I^{\mathsf{rlm}}(t,1)dt + y\int_0^x (I^{\mathsf{rlm}}(t,y)-1)dt + y\int_0^x e^t \sinh t (I^{\mathsf{rlm}}(t,y)-1)dt,$$

where $I^{\mathsf{rlm}}(t,1) = I(t)$ is the egf for increasing permutations with respect to the length (see Corollary 2.3).

Differentiating the previous equation, we obtain the following differential equation:

$$\frac{d}{dx} I^{\mathsf{rlm}}(x,y) = yI^{\mathsf{rlm}}(x,1) + yI^{\mathsf{rlm}}(x,y) - y + ye^x \sinh x (I^{\mathsf{rlm}}(x,y)-1), \quad I^{\mathsf{rlm}}(0,y) = 1.$$

The solution of this equation is the desired result. $\square$

Let $\mathsf{inc}^{\mathsf{rlm}}(n,k)$ the number of increasing permutations in $\mathfrak{S}_n$ with exactly k right-to-left minima. Then, we have the following

$$I^{\mathsf{rlm}} := [\mathsf{inc}^{\mathsf{rlm}}(n,k)]_{n,k\geq 0} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 2 & 3 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 6 & \boxed{\mathbf{11}} & 6 & 1 & 0 & 0 & \cdots \\ 0 & 24 & 42 & 35 & 10 & 1 & 0 & \cdots \\ 0 & 112 & 174 & 197 & 85 & 15 & 1 & \cdots \\ 0 & 584 & 812 & 1116 & 667 & 175 & 21 & \cdots \\ 0 & 3376 & 4836 & 6510 & 5085 & 1822 & 322 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

For example, $\mathsf{inc}^{\mathsf{rlm}}(4,2) = 11$, the entry boxed in the array I^{rlm} above, and the corresponding increasing permutations are

$$\begin{array}{cccccc} 1342, & 1432, & 2143, & 2314, & 2413, & 3142, \\ 3214, & 3412, & 4132, & 4213, & 4312. & \end{array}$$

Due to the functional equation involving $I^{\mathsf{rlm}}(x,y)$ a simple calculation on series induces the following corollary.

Corollary 2.10. *We have the following recursive formula for $n,k\geq 1$:*

$$\mathit{inc}^{\mathit{rlm}}(n+1,k) = i^{\mathit{rlm}}(n,k-1) + \sum_{i=1}^{n-1}\binom{n}{i}2^{i-1}\mathit{inc}^{\mathit{rlm}}(n-i,k-1),$$

with the initial condition $\mathit{inc}^{\mathit{rlm}}(0,0) = 1$, $\mathit{inc}^{\mathit{rlm}}(n,0) = 0$ for $n\geq 1$, and $\mathit{inc}^{\mathit{rlm}}(n,1) = \mathit{inc}(n-1)$ for $n\geq 1$.

Corollary 2.11. *The generating function for total number of right-to-left minima in all increasing permutations of a given length is*

$$\begin{aligned} \mathit{Tot}^{\mathit{rlm}}(x) &= \partial_y(I^{\mathit{rlm}}(x,y))|_{y=1} \\ &= x + 3\frac{x^2}{2!} + 11\frac{x^3}{3!} + 50\frac{x^4}{4!} + 258\frac{x^5}{5!} + 1472\frac{x^6}{6!} + 9232\frac{x^7}{7!} + O(x^8). \end{aligned}$$

This sequence is not currently listed in the OEIS.

2.4. For the number of peaks: s=peak

In this part, we focus on the enumeration of increasing permutations with respect to the length and the statistic $\mathsf{s} = \mathsf{peak}$ that counts the number of peaks. Let $I^{\mathsf{peak}}(x,y)$ be the bivariate generating function for this. We will consider four sets $\mathcal{I}^{a,b}$, $a,b\in\{0,1\}$, consisting of increasing permutations π such that π does not start with a descent if and only if $a=0$, and does not end with an ascent if and only if $b=0$. For $a,b\in\{0,1\}$, $I^{a,b}(x,y)$ is the generating function for the number of permutations $\pi \in \mathcal{I}^{a,b}$ with respect to the length and the number of valleys.

Theorem 2.12. *The generating function for increasing permutations with respect to the number of peaks is*

$$I^{\mathsf{peak}}(x,y) = 1 + x + y(I^{0,0}(x,y) - 1 - x) + I^{0,1}(x,y) + \frac{1}{y}I^{1,1}(x,y) + I^{1,0}(x,y),$$

where

$$I^{0,1}(x,y) = e^{\frac{ye^{2x}}{4} - \frac{xy}{2} + x}\int_0^x e^{-\frac{ye^{2t}}{4} + \frac{ty}{2} - t}dt - \\ y\int_0^x \left(e^{\frac{ye^{2t}}{4} - \frac{ty}{2} + t}\left(e^t - 1\right)\int_0^t e^{-\frac{ye^{2u}}{4} + \frac{yu}{2} - u}du\right)dt - x,$$

$$I^{1,1}(x,y) = y\int_0^x \left(e^{-\frac{ty}{2} + \frac{ye^{2t}}{4} + t}\left(e^t - 1\right)\int_0^t e^{-\frac{ye^{2u}}{4} - u + \frac{yu}{2}}du\right)dt,$$

$$I^{1,0}(x,y) = y\int_0^x (e^t - 1)(I^{0,0}(t,y) + I^{1,0}(t,y) - t - 1)dt \\ + \int_0^x (I^{1,0}(t,y) + I^{1,1}(t,y) + t)dt,$$

and

$$I^{0,0}(x,y) = 1 + x + \frac{y}{2}\int_0^x \left((e^t - 1)^2(I^{0,0}(t,y) + I^{1,0}(t,y) - t - 1)\right)dt + \\ + \int_0^x (I^{0,0}(t,y) + I^{1,0}(t,y) - t - 1)dt + \int_0^x (I^{0,0}(t,y) + I^{0,1}(t,y) - t - 1)dt.$$

Proof. Let π be a nonempty increasing permutation. We consider the decomposition of π with respect to the position of the entry 1. From this

$$\begin{aligned}
\mathcal{I}^{1,1} &= \left(\mathcal{V}^1 \cup \{x\}\right) \star \{x\}^{\Box} \star \left(\mathcal{I}^{0,1} \cup \mathcal{I}^{1,1} \cup \{x\}\right)\\
\mathcal{I}^{0,1} &= \left(\mathcal{V}^0 \backslash \{\epsilon, x\}\right) \star \{x\}^{\Box} \star \left(\mathcal{I}^{0,1} \cup \mathcal{I}^{1,1} \cup \{x\}\right) + \{x\}^{\Box} \star \left(\mathcal{I}^{0,1} \cup \mathcal{I}^{1,1} \cup \{x\}\right)\\
\mathcal{I}^{1,0} &= \left(\mathcal{V}^1 \cup \{x\}\right) \star \{x\}^{\Box} \star \left(\mathcal{I}^{0,0} \cup \mathcal{I}^{1,0} \backslash \{\epsilon, x\}\right) + \left(\mathcal{I}^{1,0} \cup \mathcal{I}^{1,1} \cup \{x\}\right) \star \{x\}^{\Box}\\
\mathcal{I}^{0,0} &= \epsilon + \{x\} + \left(\mathcal{V}^0 \backslash \{\epsilon, x\}\right) \star \{x\}^{\Box} \star \left(\mathcal{I}^{0,0} \cup \mathcal{I}^{1,0} \backslash \{\epsilon, x\}\right) + \{x\}^{\Box} \star \left(\mathcal{I}^{0,0} \cup \mathcal{I}^{1,0} \backslash \{\epsilon, x\}\right)\\
&\quad + \left(\mathcal{I}^{0,0} \cup \mathcal{I}^{0,1} \backslash \{\epsilon, x\}\right) \star \{x\}^{\Box},
\end{aligned}$$

where $\mathcal{V}^a$, $a \in \{0,1\}$, is the set of valleyless permutations π such that π does not start with a descent if and only if $a = 0$. From the symbolic method (cf. [12]) we obtain a system of functional equations which gives the desired result for $I^{a,b}(x,y)$, $a,b \in \{0,1\}$. We obtain the desired result for $I^{\mathsf{peak}}(x,y)$ by observing that: the number of peaks and the number of valleys are equal in a permutation $\pi \in \mathcal{I}^{0,1} \cup \mathcal{I}^{1,0}$; the number of peaks equals to the number of valleys plus one in a permutation $\pi \in \mathcal{I}^{0,0} \backslash \{\epsilon, 1\}$; and the number of peaks equals to the number of valleys minus one in a permutation $\pi \in \mathcal{I}^{1,1}$. $\square$

We were unable to obtain solutions that could be presented in the paper. Thus, we leave as an open question the problem of finding a simpler expression for $I^{\mathsf{peak}}(x,y)$.

Let $\mathsf{inc}^{\mathsf{peak}}(n,k)$ the number of increasing permutations in $\mathfrak{S}_n$ with exactly k peaks. Then, we have the following

$$I^{\mathsf{peak}} := [\mathsf{inc}^{\mathsf{peak}}(n,k)]_{n,k\geq 0} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \cdots \\ 1 & 0 & 0 & 0 & 0 \cdots \\ 2 & 0 & 0 & 0 & 0 \cdots \\ 4 & 2 & 0 & 0 & 0 \cdots \\ 8 & \boxed{\mathbf{16}} & 0 & 0 & 0 \cdots \\ 16 & 80 & 16 & 0 & 0 \cdots \\ 32 & 341 & 211 & 0 & 0 \cdots \\ 64 & 1361 & 1815 & 136 & 0 \cdots \\ 128 & 5286 & 12988 & 3078 & 0 \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \ddots \end{pmatrix}.$$

For example, $\mathsf{inc}^{\mathsf{peak}}(4,1) = 16$, the entry boxed in the array I^{peak} above, and the corresponding increasing permutations are

$$1243,\ 1324,\ 1342,\ 1423,\ 1432,\ 2143,\ 2314,\ 2341,$$
$$2413,\ 2431,\ 3142,\ 3241,\ 3412,\ 3421,\ 4132,\ 4231.$$

Corollary 2.13. *The generating function for total number of peaks in all increasing permutations of a given length is*

$$\begin{aligned} \mathit{Tot}^{\mathit{peak}}(x) &= \partial_y(I^{\mathit{peak}}(x,y))|_{y=1} \\ &= 2\frac{x^3}{3!} + 16\frac{x^4}{4!} + 112\frac{x^5}{5!} + 763\frac{x^6}{6!} + 5399\frac{x^7}{7!} + 40496\frac{x^8}{8!} + O(x^9). \end{aligned}$$

The sequence of coefficients of $\frac{x^n}{n!}$ in this series expansion does not appear in the OEIS.

2.5. Alternating increasing permutations

A permutation $\pi = \pi_1\pi_2\cdots\pi_n$ is *alternating* if its entries satisfy $\pi_1 > \pi_2 < \pi_3 > \cdots$. The exponential generating function for the number of alternating permutations is a classical result of André [1], and is given by

$$\sum_{n\geq 0} E_n \frac{x^n}{n!} = \sec x + \tan x,$$

where E_n corresponds to the Euler number (see A000111 in [24]). For a comprehensive overview, see the survey by Stanley [25].

Theorem 2.14. *The generating function $A(x)$ for the total number of increasing alternating permutations is*

$$\begin{aligned} A(x) &= e^{\frac{x^2}{2}}(x+1)\int_0^x e^{-t^2}dt - e^{\frac{x^2}{2}} + 2 \\ &= 1 + \frac{x}{1!} + \frac{x^2}{2!} + 2\frac{x^3}{3!} + 5\frac{x^4}{4!} + 8\frac{x^5}{5!} + 33\frac{x^6}{6!} + 48\frac{x^7}{7!} + 279\frac{x^8}{8!} + O(x^8). \end{aligned}$$

Proof. Let $\mathcal{A}_E$ (resp. $\mathcal{A}_O$) be the set of increasing alternating permutations of even (resp. odd) length, respectively. From the decomposition with respect to the position of the entry 1, we obtain the symbolic equations:

$$\begin{aligned} \mathcal{A}_O &= \{x\} + \{x\} * \{x\}^{\square} * \mathcal{A}_O, \\ \mathcal{A}_E &= \epsilon + \mathcal{A}_O * \{x\}^{\square} + \{x\} * \{x\}^{\square} * (\mathcal{A}_E - \epsilon). \end{aligned}$$

Therefore, we obtain the differential equations

$$\begin{aligned} \frac{d}{dx}A_O(x) &= 1 + xA_O(x), \quad A_O(0) = 0, \\ \frac{d}{dx}A_E(x) &= A_O(x) + x(A_E(x) - 1), \quad A_E(0) = 1. \end{aligned}$$

The solutions of these differential equations are

$$A_O(x) = e^{x^2/2}\int_0^x e^{-t^2/2}dt \quad \text{and} \quad A_E(x) = 2 - e^{x^2/2} + xe^{x^2/2}\int_0^x e^{-t^2/2}dt.$$

Since $A(x) = A_O(x) + A_E(x)$, we obtain the desired result. $\square$

Let a_n denote the sequence of coefficients of $\frac{x^n}{n!}$ in $A(x)$. Notice that $a_{2n+1} = (2n)!!$ and $a_{2n+2} = (2n+2)!! - (2n+1)!!$.

3. Flattened permutations

In this section, we study the bivariate exponential generating function

$$F^s(x,y) = \sum_{\pi\in\mathcal{F}} \frac{x^{|\pi|}}{|\pi|!}y^{\mathsf{s}(\pi)} = \sum_{n\geq 0}\frac{x^n}{n!}\sum_{\pi\in\mathcal{F}_n} y^{\mathsf{s}(\pi)},$$

where $\mathcal{F}$ is the set of flattened permutations, i.e. permutations consisting of runs arranged from left to right, with the first entries of each run in increasing order. We consider the statistics $\mathsf{s} = \mathsf{val}$, $\mathsf{run} = 1+\mathsf{des} = n-\mathsf{asc} = 1+\mathsf{peak}$, and rlm. The coefficient of $\frac{x^n y^k}{n!}$ in $F^s(x,y)$ is the number of flattened permutations π of length n satisfying $s(\pi) = k$. Note that the statistic run has already been investigate by Nabawanda et al. [21] using combinatorial arguments and recurrence relations.

3.1. For the number of valleys: $\mathsf{s} = \mathsf{val}$

We begin with the enumeration of flattened permutations by length and number of valleys.

Theorem 3.1. *The generating function for flattened permutations with respect to the length and the number of valleys is*

$$F^{val}(x,y) = 1 + \int_0^x B^{val}(t,y)dt,$$

where

$$B^{val}(x,y) = e^{-xy+e^x y+x-y}\left(e^y \int_0^x e^{-e^t y+ty-t}\left(-e^t y+y+e^t-1\right)dt+1\right).$$

Proof. We first define the subset $\mathcal{B}$ of permutations $\pi' \in \mathfrak{S}$ having one of the following forms: (i) $\pi' = \epsilon$, or (ii) $\pi' = 1(1+\pi_1)$ where $\pi_1 \in \mathcal{B}$, or (iii) $\pi' = 23\cdots k1$ where $k \geq 2$, or (iv) $\pi = \pi_2 1 \pi_1$ where $\texttt{red}(\pi_2) = 12\cdots k$, $k \geq 1$ and $\texttt{red}(\pi_1) \in \mathcal{B}\backslash\{\epsilon\}$. Considering this definition, a nonempty flattened permutation π is necessary of the form $\pi = 1(1+\pi')$ where $\pi' \in \mathcal{B}$. So, we first give the egf of the set $\mathcal{B}$ with respect to the length and the number of valleys.

From the above decomposition of $\mathcal{B}$ we obtain the symbolic equation:

$$\mathcal{B} = \epsilon + \{x\}^{\Box} \star \mathcal{B} + (\mathcal{ID} - \epsilon) \star \{x\}^{\Box} + (\mathcal{ID} - \epsilon) \star \{x\}^{\Box} \star (\mathcal{B} - \epsilon),$$

where the set $\mathcal{ID}$ consists of all identity permutations of length $n \geq 0$. Therefore, we obtain the following functional equation:

$$B^{\mathsf{val}}(x,y) = 1 + \int_0^x B^{\mathsf{val}}(t,y)dt + \int_0^x (e^t-1)dt + y\int_0^x (e^t-1)(B^{\mathsf{val}}(t,y)-1)dt.$$

Differentiating the previous equation, we obtain the following differential equation:

$$\frac{d}{dx}B^{\mathsf{val}}(x,y) = B^{\mathsf{val}}(x,y) + e^x - 1 + y(e^x-1)(B^{\mathsf{val}}(x,y)-1), \quad B^{\mathsf{val}}(0,y) = 1.$$

The solution of this equation provides the expected expression of $B(x,y)$. Now, by considering the decomposition $\mathcal{F} = 1 + \{x\}^{\Box} \star B$, we obtain the desired result. $\Box$

Using symbolic computation, we find the first few terms of the series expansion:

$$F^{\mathsf{val}}(x,y) = (2x + e^x(x-1)) + \frac{1}{4}\left(-2e^x(x-2)x + e^{2x}(2x-3)+3\right)y$$
$$+\frac{1}{36}\left(-9e^{2x}\left(2x^2-2x-7\right) + 6e^x\left(2x^3-6x^2-15\right) + 2e^{3x}(6x-11)+49\right)y^2 + O(y^3).$$

Let $f^{\mathsf{val}}(n,k)$ denote the number of flattened permutations in $\mathfrak{S}_n$ with exactly k valleys. The first few values of this sequence are

$$F^{\mathsf{val}} := [f^{\mathsf{val}}(n,k)]_{n,k\geq 0} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \cdots \\ 1 & 0 & 0 & 0 & 0 \cdots \\ 1 & 0 & 0 & 0 & 0 \cdots \\ 2 & 0 & 0 & 0 & 0 \cdots \\ 3 & 2 & 0 & 0 & 0 \cdots \\ 4 & \boxed{\mathbf{11}} & 0 & 0 & 0 \cdots \\ 5 & 39 & 8 & 0 & 0 \cdots \\ 6 & 114 & 83 & 0 & 0 \cdots \\ 7 & 300 & 522 & 48 & 0 \cdots \\ 8 & 741 & 2594 & 797 & 0 \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \ddots \end{pmatrix}.$$

For example, $f^{\mathsf{val}}(5,2) = 11$, the entry boxed in the array F^{val} above, and the corresponding flattened permutations are

$$12435, \quad 12534, \quad 13245, \quad 13254, \quad 13425, \quad 13524,$$
$$14235, \quad 14253, \quad 14523, \quad 15234, \quad 15243.$$

Corollary 3.2. *We have the following recursive formula for $n, k \geq 1$, $f^{val}(n+1,k) = b(n,k)$, with*

$$b(n+1,k) = b(n,k) + \sum_{i=1}^{n-1} \binom{n}{i} b(i,k-1)$$

with the initial conditions $b(0,0) = 1$, $b(n,0) = n$ for $n \geq 1$, and $f^{val}(0,0) = 1$.

Corollary 3.3. *The generating function for flattened permutations without valleys is*

$$\begin{aligned} F^{val}(x,0) &= 2 + x + (x-1)e^x \\ &= 1 + \frac{x}{1!} + \frac{x^2}{2!} + 2\frac{x^3}{3!} + 3\frac{x^4}{4!} + 4\frac{x^5}{5!} + 5\frac{x^6}{6!} + 6\frac{x^7}{7!} + O(x^8), \end{aligned}$$

and the number of flattened permutations of length n without valleys is $n-1$ for $n \geq 2$.

Corollary 3.4. *The generating function $F(x)$ for the total number of flattened permutations is*

$$\begin{aligned} F(x) := F^{val}(x,1) &= 1 + \int_0^x e^{e^t-1} dt \\ &= 1 + \frac{x}{1!} + \frac{x^2}{2!} + 2\frac{x^3}{3!} + 5\frac{x^4}{4!} + 15\frac{x^5}{5!} + 52\frac{x^6}{6!} + 203\frac{x^7}{7!} + O(x^8). \end{aligned}$$

As expected (see [18]), the total number of flattened permutations of length $n \geq 0$ is given by the Bell number B_{n-1} (sequence A000110).

Corollary 3.5. *The generating function for the total number of valleys in all flattened permutations with respect to the length is given by*

$$\begin{aligned} Tot^{\mathsf{val}}(x) &= \partial_y(F^{\mathsf{val}}(x,y))|_{y=1} \\ &= \int_0^x \left(Ei\left((1,1) - Ei\left(1, e^t\right)\right) e^{e^t} - (2+x+e^t)e^{e^t-1} \right) dt, \end{aligned}$$

where

$$Ei(a,z) = \int_1^\infty e^{-tz} t^{-a} dt.$$

The first few terms of the formal power series are

$$Tot^{\mathsf{val}}(x) = 2\frac{x^4}{4!} + 11\frac{x^5}{5!} + 55\frac{x^6}{6!} + 280\frac{x^7}{7!} + 1488\frac{x^8}{8!} + O(x^9).$$

The sequence of coefficients of $\frac{x^n}{n!}$ in this series expansion does not appear in [24].

3.2. For the number of runs: $\mathsf{s} = \mathsf{run} = \mathbf{1} + \mathsf{des} = \mathbf{1} + \mathsf{peak} = \boldsymbol{n} - \mathsf{asc}$

In this part, we focus on the enumeration of flattened permutations with respect to the length and the statistic $\mathsf{s} = \mathsf{run}$ that counts the number of runs. Observe that, for flattened permutations, the number of runs satisfies $\mathsf{run} = 1 + \mathsf{des} = 1 + \mathsf{peak}$.

The next result was originally proved by Nabawanda et al. [21]. Here, we present an alternative proof using the symbolic method.

Theorem 3.6. *The generating function for flattened permutations with respect to the length and the number of runs is*

$$F^{\mathsf{run}}(x,y) = 1 + xy + \int_0^x (B^{\mathsf{run}}(t,y) - 1)dt,$$

where

$$B^{\mathsf{run}}(x,y) = e^{-xy+x-y}\left(ye^{e^x y} + (1-y)e^{x(y-1)+y}\right).$$

Proof. We make a similar proof as for Theorem 5.1 by considering the set $\mathcal{B}$ defined by the symbolic equation:

$$\mathcal{B} = \epsilon + \{x\}^\square \star \mathcal{B} + (\mathcal{ID} - \epsilon) \star \{x\}^\square + (\mathcal{ID} - \epsilon) \star \{x\}^\square \star (\mathcal{B} - \epsilon).$$

Thus we deduce

$$B^{\mathsf{run}}(x,y) = 1+xy+\int_0^x (B^{\mathsf{run}}(t,y)-1)dt+y^2\int_0^x (e^t-1)dt+y\int_0^x (e^t-1)(B^{\mathsf{run}}(t,y)-1)dt.$$

Differentiating the previous equation, we obtain the following differential equation:

$$\frac{d}{dx}B^{\mathsf{run}}(x,y) = y+B^{\mathsf{run}}(x,y)-1+y^2(e^x-1)+y(e^x-1)(B^{\mathsf{run}}(x,y)-1), \quad B^{\mathsf{run}}(0,y) = 1.$$

Solving this differential equation yields the closed-form for $B^{\mathsf{run}}(x,y)$. Now, by considering the decomposition $\mathcal{F} = 1 + \{x\}^{\square} \star B$, we obtain

$$F^{\mathsf{run}}(x,y) = 1 + xy + \int_0^x (B^{\mathsf{run}}(t,y) - 1)dt,$$

which provides the desired result. $\square$

We can use a symbolic software computation to obtain the first few terms of the formal power series

$$F^{\mathsf{run}}(x,y) = 1 + x + (e^x - 1)\,y + (e^x(\sinh(x) - x))\,y^2 + \frac{1}{12}\left(6e^x\left(x^2+1\right) - 3e^{2x}(2x+1) + 2e^{3x} - 5\right)y^3 + O(y^4).$$

Let $f^{\mathsf{run}}(n,k)$ denote the number of flattened permutations in $\mathfrak{S}_n$ with exactly k runs. The first few values of this sequence are

$$F^{\mathsf{run}} := [f^{\mathsf{run}}(n,k)]_{n,k\geq 0} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 4 & 0 & 0 & \cdots \\ 0 & 1 & \boxed{\mathbf{11}} & 3 & 0 & \cdots \\ 0 & 1 & 26 & 25 & 0 & \cdots \\ 0 & 1 & 57 & 130 & 15 & \cdots \\ 0 & 1 & 120 & 546 & 210 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

For example, $f^{\mathsf{run}}(5,2) = 11$, the entry boxed in the array F^{run} above, and the corresponding flattened permutations are

12354, 12435, 12453, 12534, 13245, 13425,
13452, 13524, 14235, 14523, 15234.

Nabawanda et al. [21] proved the following recurrence relation

$$f^{\mathsf{run}}(n,k) = kf^{\mathsf{run}}(n-1,k) + (n-2)f^{\mathsf{run}}(n-2,k-1).$$

Corollary 3.7. *The generating function for the total number of runs in all flattened permutations with respect to the length is given by*

$$Tot^{\mathsf{run}}(x) = \partial_y(F^{\mathsf{val}}(x,y))|_{y=1} = x + \int_0^x \left(-e^{e^t-1}t + e^{t+e^t-1} - 1\right)dt.$$
$$= x + \frac{x^2}{2!} + 3\frac{x^3}{3!} + 9\frac{x^4}{4!} + 32\frac{x^5}{5!} + 128\frac{x^6}{6!} + 565\frac{x^7}{7!} + 2719\frac{x^8}{8!} + O(x^9).$$

Let $\mathcal{F}_n^{\mathsf{val=des}}$ be the set of flattened permutations π of length n such that $\mathsf{val}(\pi) = \mathsf{des}(\pi)$. We set $\mathcal{F}^{\mathsf{val=des}} = \sum_{n\geq 0} \mathcal{F}_n^{\mathsf{val=des}}$.

Theorem 3.8. *The cardinality of $\mathcal{F}_n^{\mathsf{val=des}}$ is given by the n-th Gould number (see A040027 in [24]):*

$$|\mathcal{F}_n^{\mathsf{val=des}}| = \sum_{k=0}^{n-2}\sum_{j=k}^{n-2} \left\{ j \atop k \right\} k^{n-2-j},$$

where $\left\{ j \atop k \right\}$ is the Stirling number of second kind.

Proof. The proof is obtained *mutatis mutandis* as for Theorem 2.8. □

3.3. For the number of right-to-left minima: $\mathsf{s} = \mathsf{rlm}$

In this part, we focus on the enumeration of flattened permutations with respect to the length and the statistic $\mathsf{s} = \mathsf{rlm}$ that counts the number of right-to-left minima.

Theorem 3.9. *The generating function for flattened permutations with respect to the length and the number of right-to-left minima is*

$$F^{\mathsf{rlm}}(x, y) = 1 + y \int_0^x B^{\mathsf{rlm}}(t, y)dt,$$

where $B^{\mathsf{rlm}}(x, y) = e^{y(e^x-1)}$.

Proof. We make a similar proof as for Theorem 5.1 and Theorem 5.6 by considering the set $\mathcal{B}$ defined by the symbolic equation:

$$\mathcal{B} = \epsilon + \{x\}^{\square} \star \mathcal{B} + (\mathcal{ID} - \epsilon) \star \{x\}^{\square} + (\mathcal{ID} - \epsilon) \star \{x\}^{\square} \star (\mathcal{B} - \epsilon).$$

Thus we deduce the functional equation

$$B^{\mathsf{rlm}}(x, y) = 1+y \int_0^x B^{\mathsf{rlm}}(t, y)dt+y \int_0^x (e^t-1)dt+y \int_0^t (e^t-1)(B^{\mathsf{rlm}}(t, y)-1)dt.$$

Differentiating the previous equation, we obtain the following differential equation:

$$\frac{d}{dx}B^{\mathsf{rlm}}(x, y) = yB^{\mathsf{rlm}}(x, y)+y(e^x-1)+y(e^x-1)(B^{\mathsf{rlm}}(x, y)-1), \quad B^{\mathsf{rlm}}(0, y) = 1.$$

The solution of this equation provides the expected expression of $B^{\mathsf{rlm}}(x, y)$. Now, by considering the decomposition $\mathcal{F} = 1 + \{x\}^{\square} \star B$, we obtain

$$F^{\mathsf{rlm}}(x, y) = 1 + y \int_0^x B^{\mathsf{rlm}}(t, y)dt,$$

which provides the desired result. □

The first few terms of the formal power series are

$$F^{\mathsf{rlm}}(x,y) = 1 + yx + (e^x - x - 1)\,y^2 + \frac{1}{4}\left(2x - 4e^x + e^{2x} + 3\right)y^3 + \frac{1}{36}\left(-6x + 18e^x - 9e^{2x} + 2e^{3x} - 11\right)y^4 + O(y^5).$$

Let $f^{\mathsf{rlm}}(n,k)$ the number of flattened permutations in $\mathfrak{S}_n$ with exactly k right-to-left minima. The first few values of this sequence are

$$F^{\mathsf{rlm}} := [f^{\mathsf{rlm}}(n,k)]_{n,k\geq 0} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 3 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & \boxed{7} & 6 & 1 & 0 & \cdots \\ 0 & 0 & 1 & 15 & 25 & 10 & 1 & \cdots \\ 0 & 0 & 1 & 31 & 90 & 65 & 15 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

For example, $f^{\mathsf{rlm}}(5,3) = 7$, the entry boxed in the array F^{rlm} above, and the corresponding flattened permutations are

$$12453, \quad 13254, \quad 13425, \quad 13524, \quad 14253, \quad 14523, \quad 15243.$$

Corollary 3.10. *For $n,k \geq 1$, we have*

$$f^{\mathsf{rlm}}(n,k) = \left\{ {n-1 \atop k-1} \right\}.$$

Notice that $f^{\mathsf{rlm}}(n,k)$ equals the Stirling number of the second kind $\left\{ {n-1 \atop k-1} \right\}$ that also counts the number of set partitions of $[n-1]$ with $k-1$ blocks. We decompose a flattened permutation $\pi = a_1\pi_2a_2\dots\pi_ka_k$ where $a_1 = 1, a_2, \cdots, a_k$ are the right-to-left minima of π, and we construct the partition $B = B_1/B2/\cdots/B_k$ of $[n-1]$, where $B_i = \mathtt{dom}((\pi_i - 1)(a_i - 1))$ for $2 \leq i \leq k$.

Corollary 3.11. *The generating function for the total number of right-to-left minima in all flattened permutations with respect to the length is given by n-th Bell number.*

Author contributions The two authors wrote the main manuscript text and the two authors reviewed the manuscript.

Funding Open access funding provided by Université Bourgogne Europe.

Data availibility No datasets were generated or analysed during the current study.

Declarations

Competing interests The authors declare no competing interests.

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Received: May 13, 2025
Revised: May 13, 2025
Accepted: February 20, 2026