Equivalence classes of permutations avoiding generalized patterns modulo left-to-right maxima

Jean-Luc Baril and Armen Petrossian

Laboratory LE2I – CNRS UMR 6306 – University of Burgundy – Dijon

The 13th International Permutation Patterns edition, London, UK, 15-19 June

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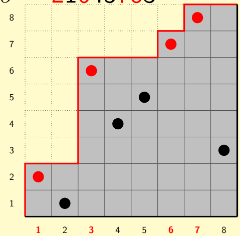
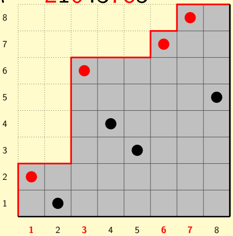
Definition 1 (Equivalence relation modulo left-to-right maxima)

$\boldsymbol{\sigma} \sim_\ell \boldsymbol{\pi}$ iff $L(\sigma) = L(\pi)$ and $\sigma_i = \pi_i$ for all $i \in L(\sigma)$

$\sigma = 21645783$



$\pi = 21643785$

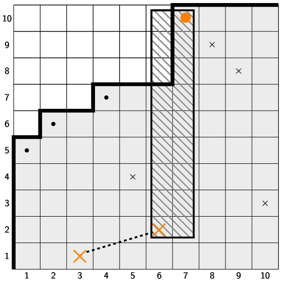


$$L(\sigma) = \{1, 3, 6, 7\} = L(\pi)$$

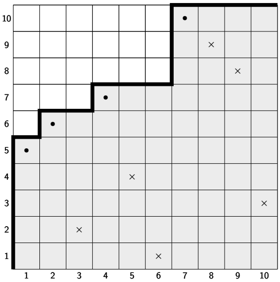
$\boldsymbol{S_n^{\sim_\ell}}$ = {classes of permutations of length n modulo $\sim_\ell$}

Definition 2 (Generalized patterns)

A generalized pattern is a classical pattern where two entries can be separated by a dash, (e.g. ab-c, a-bc)
σ avoids a generalized pattern π if σ does not contain any subsequence order isomorphic to π, where two entries not separated by a dash in π correspond to two consecutive elements in σ.



5 6 1 7 4 2 10 9 8 3 $\notin S(1\text{-}23)$



5 6 2 7 4 1 10 9 8 3 $\in S(1\text{-}23)$

Introduction : notations and our precedent results

Set	Sequence	Sloane	$a_n, 1 \leq n \leq 9$
Permutations	Catalan	A000108	1, 2, 5, 14, 42, 132, 429, 1430, 4862
Cycles	Catalan	A000108	1,1,2, 5, 14, 42, 132, 429, 1430
Involutions	Motzkin	A001006	1, 2, 4, 9, 21, 51, 127, 323, 835
Derangements	Fine	A000957	0, 1, 2, 6, 18, 57, 186, 622, 2120

Table: Number of equivalence classes for classical subsets of permutations.

Pattern	Sequence	Sloane	$a_n, 1 \leq n \leq 9$
{1-2-3}	Central polygonal	A000124	1, 2, 4, 7, 11, 16, 22, 29, 37
{3-1-2}, {3-2-1}	Catalan	A000108	1, 2, 5, 14, 42, 132, 429, 1430, 4862
{1-3-2}, {2-1-3}, {2-3-1}	Power of 2	A000079	1, 2, 4, 8, 16, 32, 64, 128, 256

Table: Number of equivalence classes for permutations avoiding one classical pattern.

Results relative to permutations avoiding generalized patterns of length 3 modulo left-to-right maxima

Pattern	Sequence	Sloane	$a_n, 1 \leq n \leq 9$
{312}, {321}, {3-12}, {31-2}, {3-21}, {32-1}	Catalan	A000108	1, 2, 5, 14, 42, 132, 429, 1430, 4862
{1-23}	Generalized Catalan	A004148	1, 2, 4, 8, 17, 37, 82, 185, 423
{123}, {23-1}	Motzkin	A000124	1, 2, 4, 9, 21, 51, 127, 323, 835
{12-3}	Central polygonal	A000124	1, 2, 4, 7, 11, 16, 22, 29, 37
{213}	Number of Dyck paths with no UDDU	A135307	1, 2, 4, 9, 23, 63, 178, 514, 1515
{1-32}	Number of Dyck paths with no UUDU	A105633	1, 2, 4, 9, 22, 57, 154, 429, 1223
{13-2}, {2-13}, {21-3}, {2-31}	Power of 2	A000079	1, 2, 4, 8, 16, 32, 64, 128, 256
{132}	New		1, 2, 4, 10, 28, 84, 265, 864, 2888
{231}	New		1, 2, 4, 10, 26, 74, 217, 662, 2059

Table: Number of equivalence classes for permutations avoiding one generalized pattern.

Two results using ECO method

Theorem 1 (Enumeration of $S(123)^{\sim_\ell}$)

The set $S(123)$ modulo left-to-right maxima is enumerated by the Motzkin sequence (A000124).

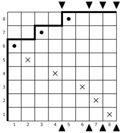
Theorem 2 (Enumeration of $S(1\text{-}23)^{\sim_\ell}$)

The set $S(1\text{-}23)$ modulo left-to-right maxima is enumerated by the Generalized Catalan sequence (A004148).

In order to calculate the number of classes for permutations avoiding 123 or 1-23 we will

- exhibit one-to-one correspondence between the sets of equivalence classes and some subsets of permutations (the set of representatives)
- exhibit a recursive relation via ECO method which defines the set of classes
- exhibit the generating function which defines the wished sequence

The ECO method applied to a set of representatives of $S(123)^{\sim_\ell}$

6 5 7 4 8 3 2 1 

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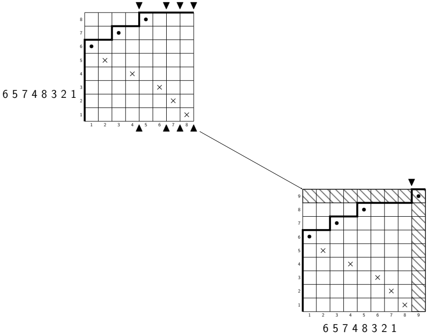

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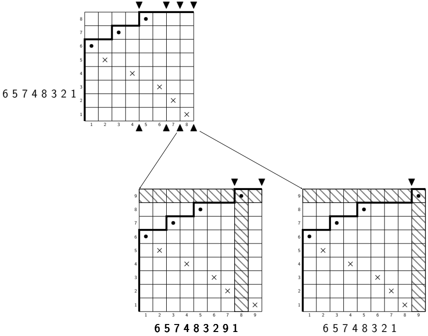

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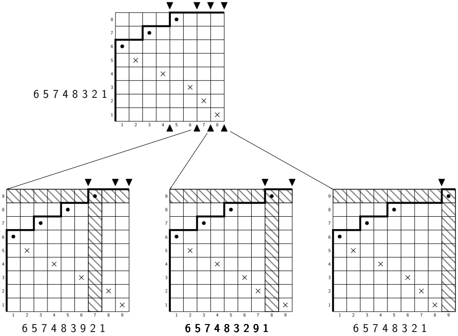

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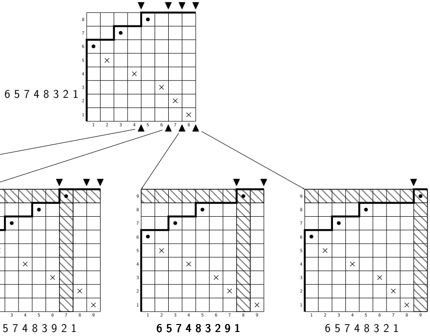

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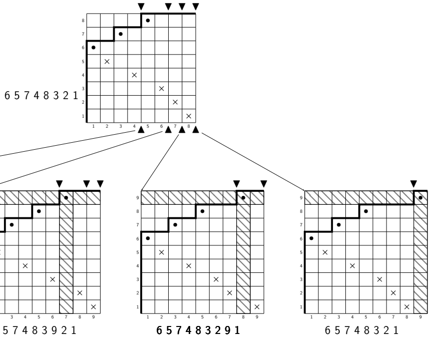

provides the succession rule of the Motzkin sequence

$$\begin{cases} (1) \\ (k) \rightsquigarrow (1)\cdots(k-1)(k+1) \end{cases}$$

$S(1\text{-}23)^{\sim_\ell}$: A set of representatives

Definition 3 (Set of representatives)

Let $\mathcal{A}$ be the set of permutations $\sigma_1 B_1 \sigma_{i_2} \cdots B_{s-1} \sigma_{i_s} B_s$ (only B_1 can be empty)

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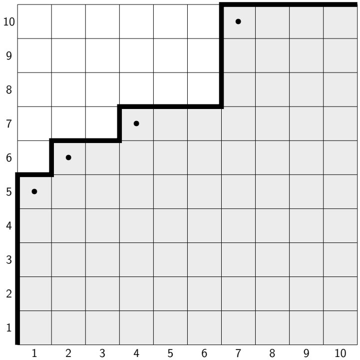
- $\sigma_1 \sigma_{i_2} \cdots \sigma_{i_s}$ list of left-to-right maxima



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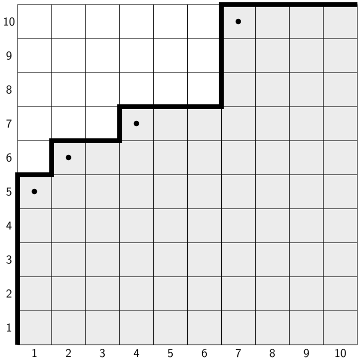
- $[B_1] B_2 \cdots B_s$ decreasing sub-sequences



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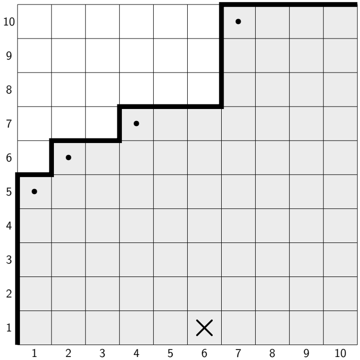
- $[B_1] B_2 \cdots B_s$ decreasing sub-sequences
  1. We insert 1 before σ_{i_s}



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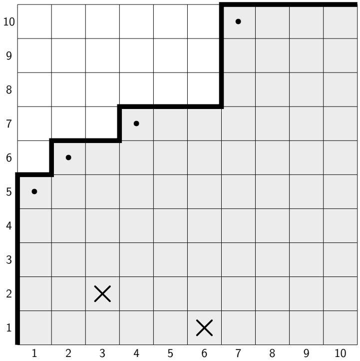
- $\sigma_1 \sigma_{i_2} \cdots \sigma_{i_s}$ list of left-to-right maxima
- $[B_1] B_2 \cdots B_s$ decreasing sub-sequences
  1. We insert $[(s-1)](s-2), \cdots, 1$ before $[\sigma_{i_2}]\sigma_{i_3} \cdots \sigma_{i_s}$ in this order



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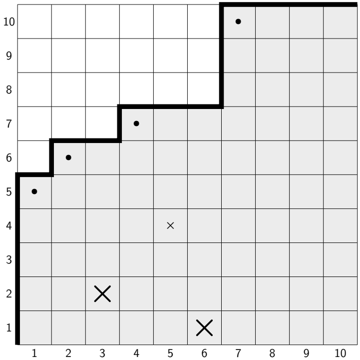
  1. We insert $[(s-1)](s-2), \cdots, 1$ before $[\sigma_{i_2}]\sigma_{i_3} \cdots \sigma_{i_s}$ in this order
  2. From left-to-right all other elements are chosen as large as possible.



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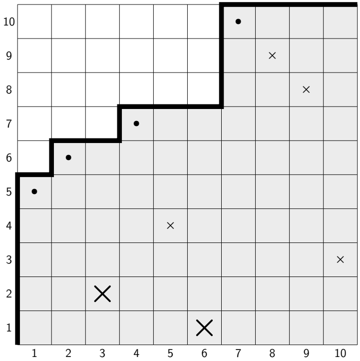

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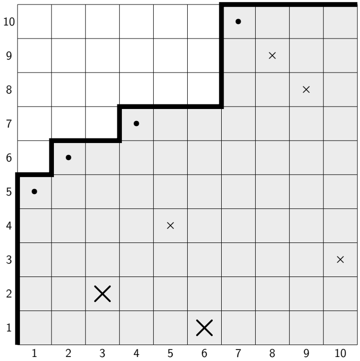
    2. From left-to-right all other elements are chosen as large as possible.

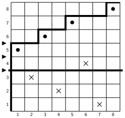
Lemma 1 (Bijection between set of representatives and $S(1\text{-}23)^{\sim\ell}$)

There is a bijection between $\mathcal{A}$ and $S(1\text{-}23)$ modulo left-to-right maxima.



$S(1\text{-}23)^{\sim_\ell}$: Succession rule

$\sigma = \sigma_1 B_1 \sigma_{i_2} \cdots B_{s-1} \sigma_{i_{s-1}} B_s$ (a representative)



5 4 7 3 8 2 9 6 1 10;

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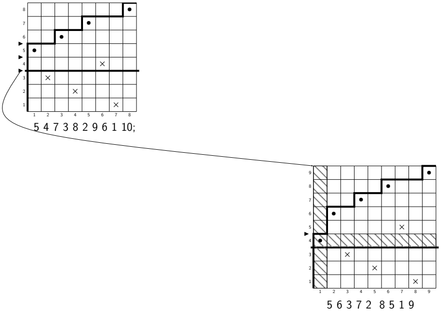

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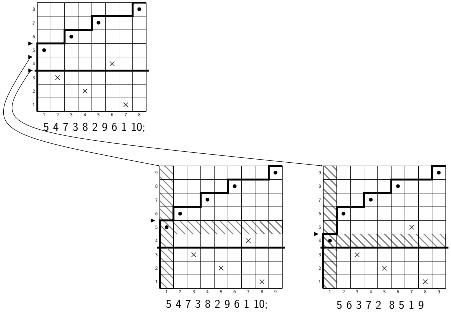

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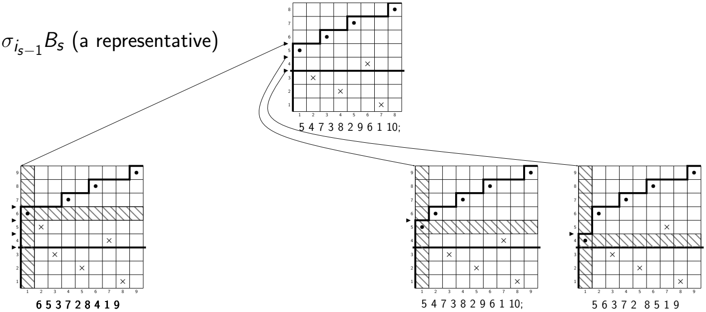

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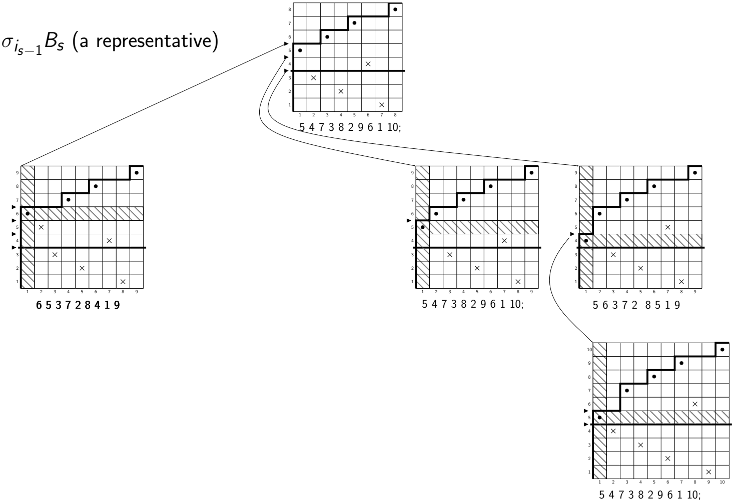

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When π becomes π_ℓ if the elements of π larger than ℓ are replaced by their successors.

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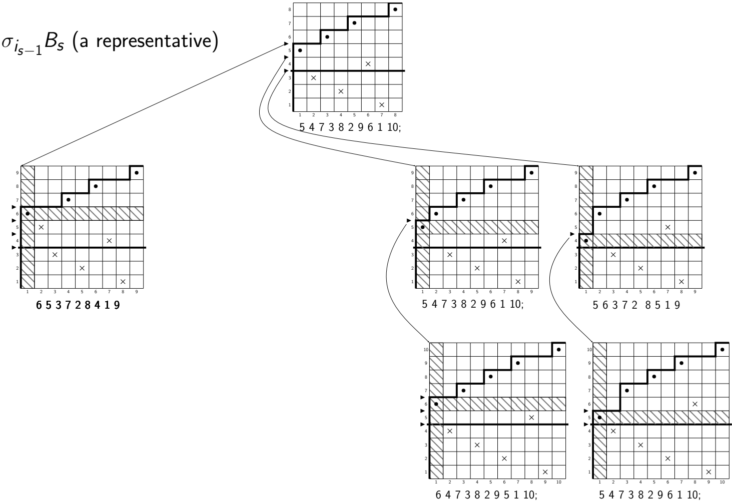

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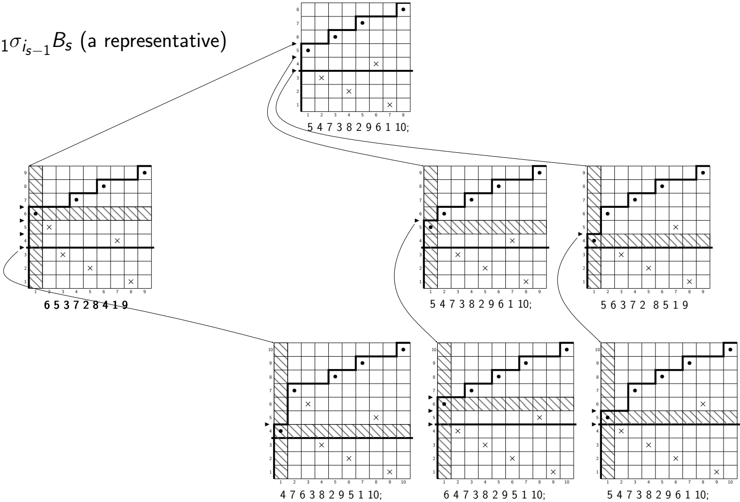

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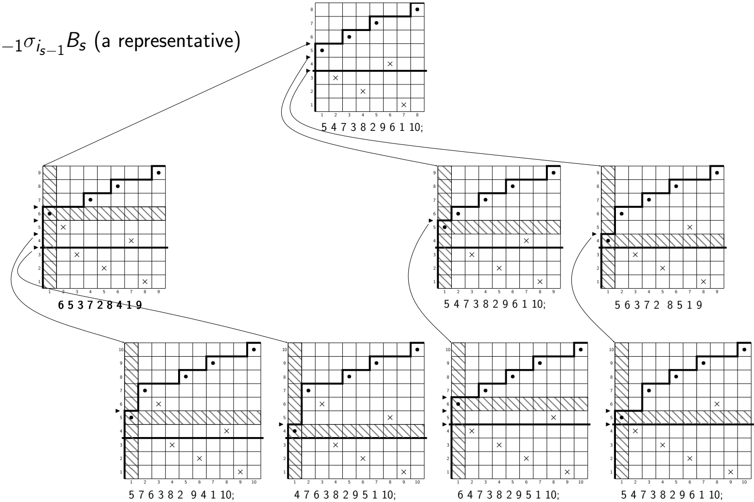

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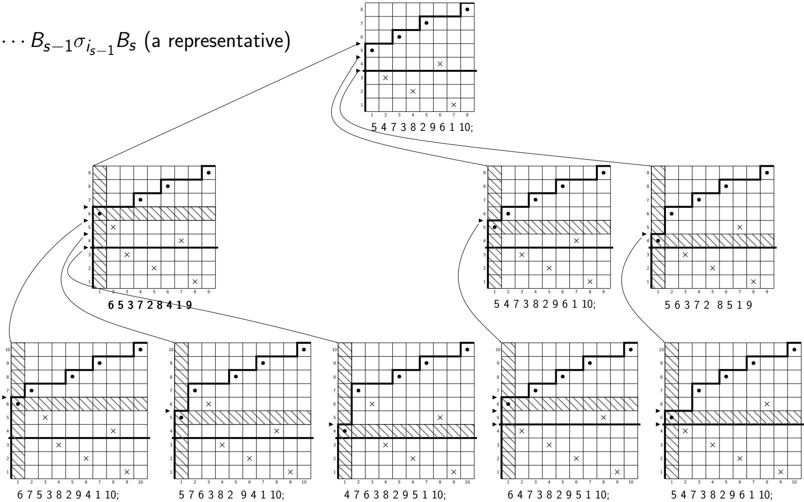

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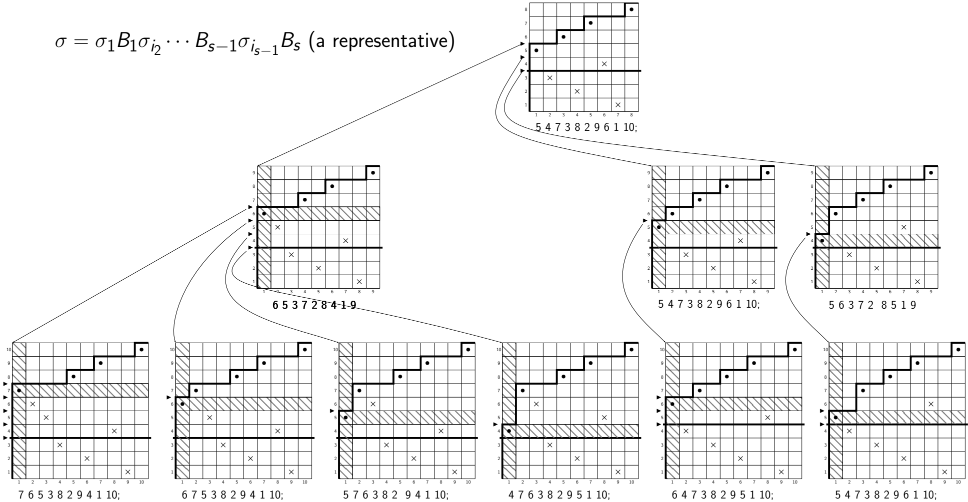

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When π becomes π_ℓ if the elements of π larger than ℓ are replaced by their successors.

s the following

Eco rule for Generalized Catalan

succession rule :

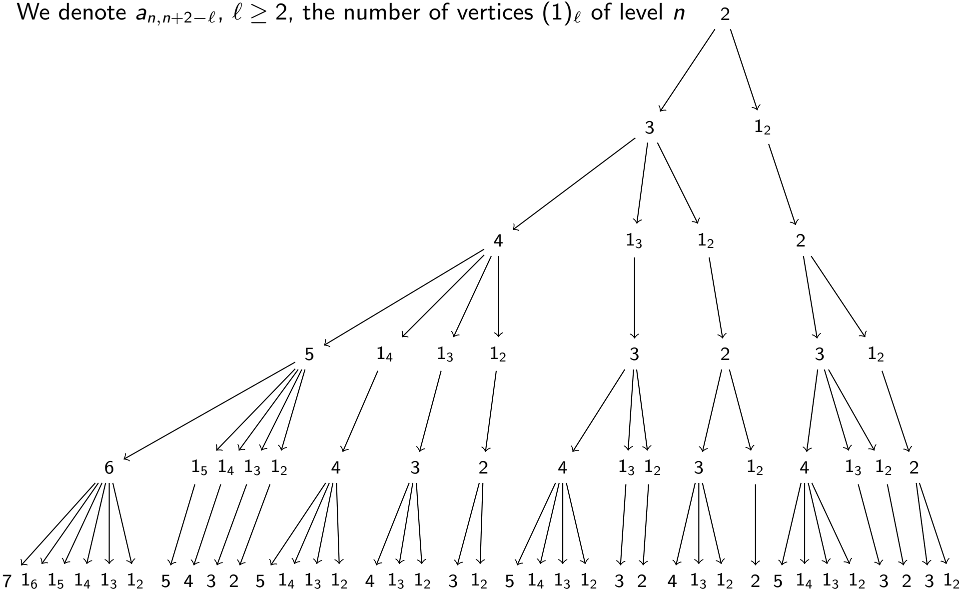
$$\begin{cases} (2) \\ (k) \rightsquigarrow (1)_2 \cdots (1)_k (k+1) \\ (1)_k \rightsquigarrow (k) \end{cases}$$

Lemma 2

There a bijection between the set of vertices of level n and the set of vertices labelled by $(1)_2$ of level $n+2$ of the tree induced by the above succession rule.

We will enumerate the sets of vertices labelled by $(1)_2$ of level $n+2$ for all positive integers n.

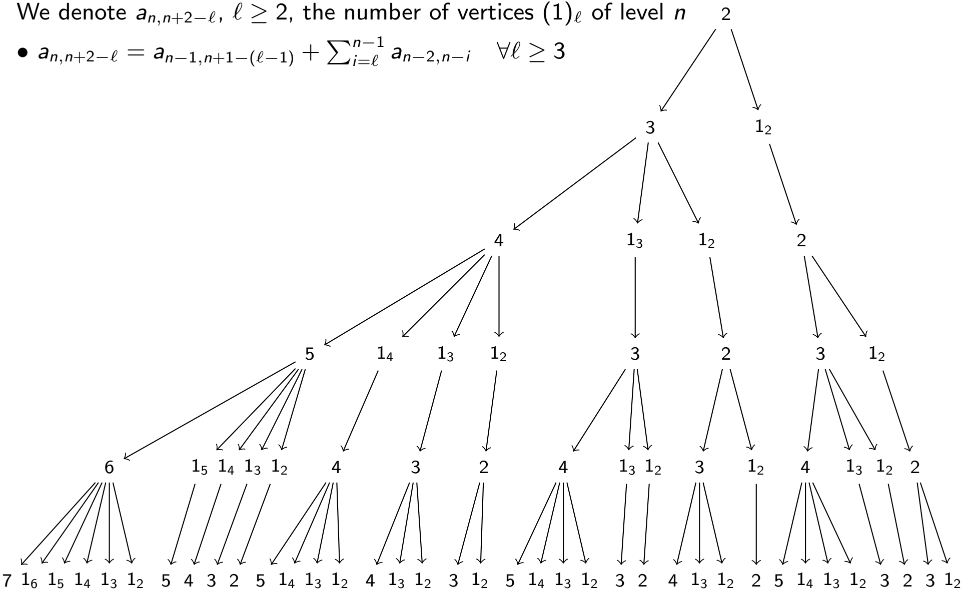
We denote $a_{n,n+2-\ell}$, $\ell \geq 2$, the number of vertices $(1)_\ell$ of level n



$S(1\text{-}23)^{\sim_\ell}$: Recursive and functional equations

We denote $a_{n,n+2-\ell}$, $\ell \geq 2$, the number of vertices $(1)_\ell$ of level n

- $a_{n,n+2-\ell} = a_{n-1,n+1-(\ell-1)} + \sum_{i=\ell}^{n-1} a_{n-2,n-i} \quad \forall \ell \geq 3$



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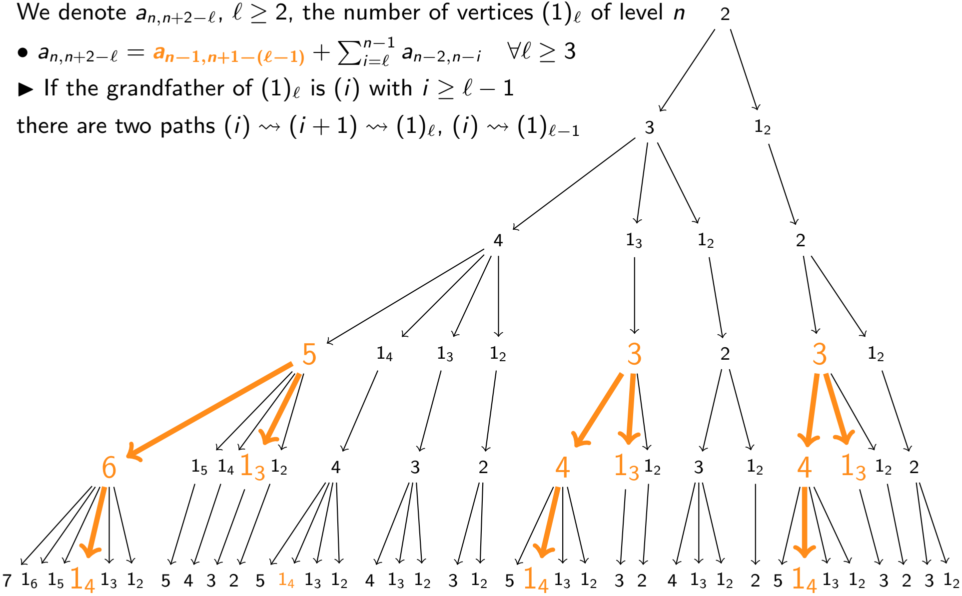
▶ If the grandfather of $(1)_\ell$ is (i) with $i \geq \ell - 1$

there are two paths $(i) \rightsquigarrow (i+1) \rightsquigarrow (1)_\ell$, $(i) \rightsquigarrow (1)_{\ell-1}$



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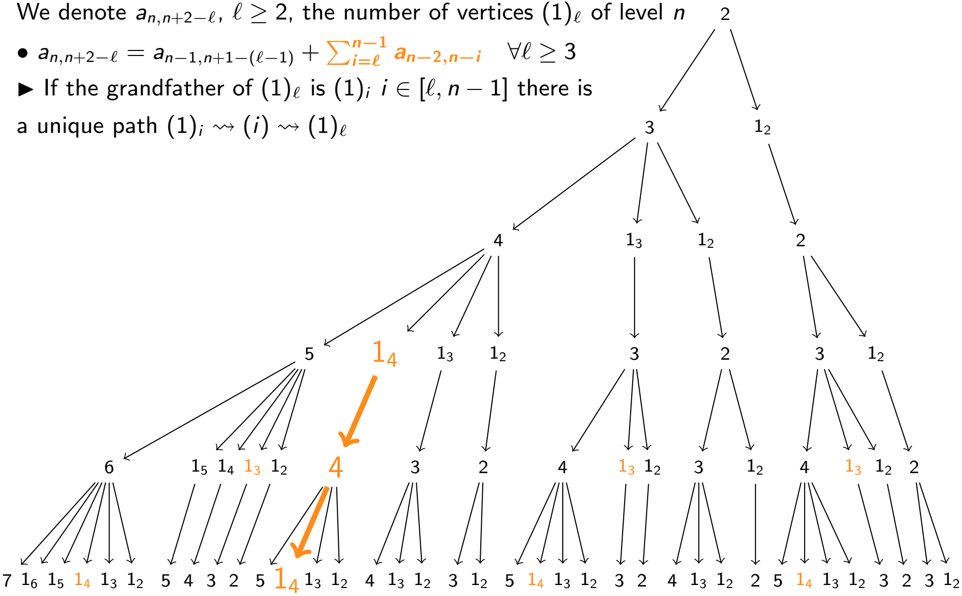
- $a_{n,n+2-\ell} = a_{n-1,n+1-(\ell-1)} + \sum_{i=\ell}^{n-1} a_{n-2,n-i} \quad \forall \ell \geq 3$

▶ If the grandfather of $(1)_\ell$ is $(1)_i$ $i \in [\ell, n-1]$ there is a unique path $(1)_i \rightsquigarrow (i) \rightsquigarrow (1)_\ell$



$S(1\text{-}23)^{\sim_\ell}$: Recursive and functional equations

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- $a_{n,k} = a_{n-1,k} + \sum_{l=1}^{k-2} a_{n-2,l}$, $k \in [n-1]$ (with $k = n+2-\ell$)
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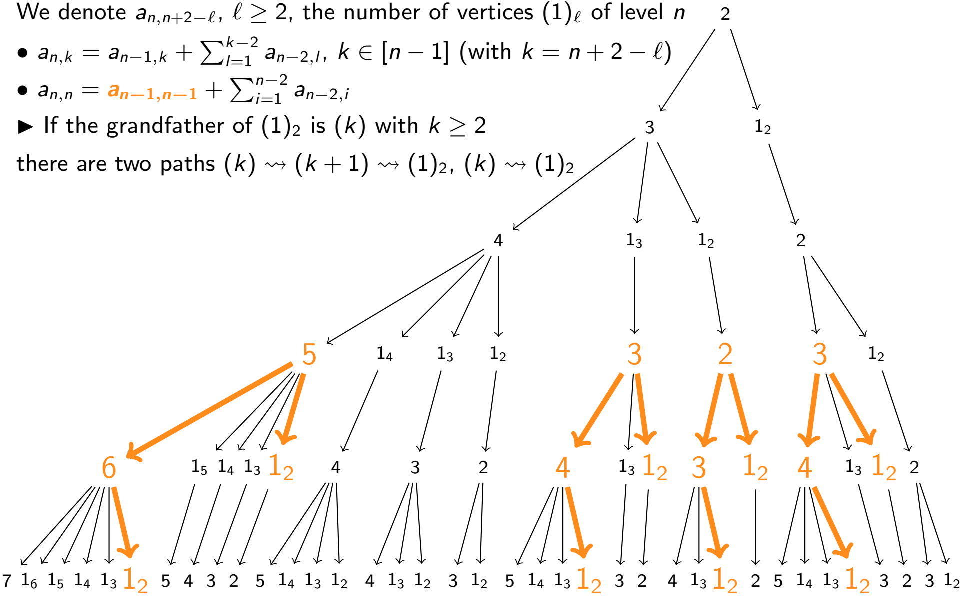
▶ If the grandfather of $(1)_2$ is (k) with $k \geq 2$

there are two paths $(k) \rightsquigarrow (k+1) \rightsquigarrow (1)_2$, $(k) \rightsquigarrow (1)_2$



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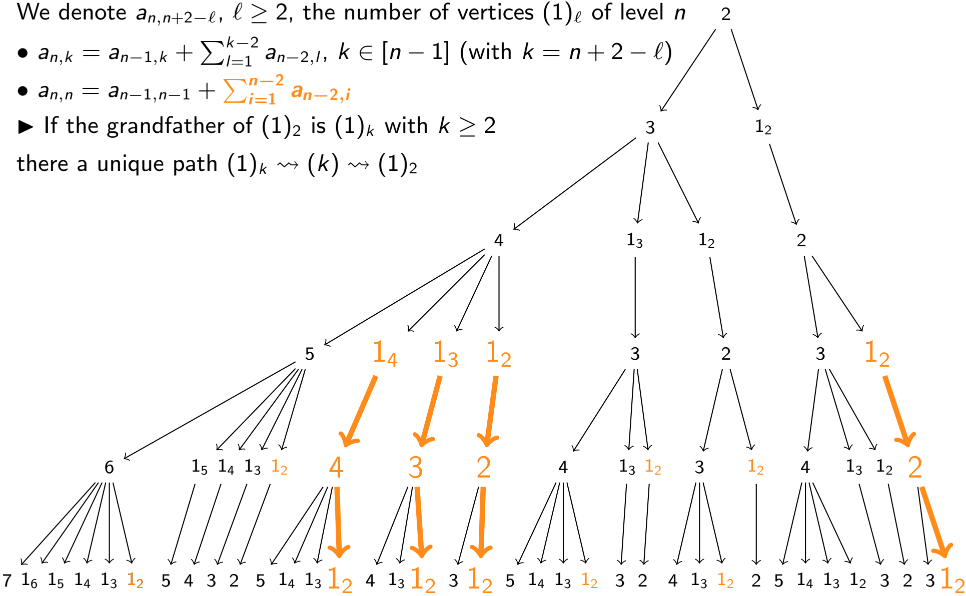

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We deduce the following functional equation:

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$S(1\text{-}23)^{\sim\ell}$: Recursive and functional equations

We denote $a_{n,n+2-\ell}$, $\ell \geq 2$, the number of vertices $(1)_\ell$ of level n

- $a_{n,k} = a_{n-1,k} + \sum_{l=1}^{k-2} a_{n-2,l}$, $k \in [n-1]$ (with $k = n+2-\ell$)
- $a_{n,n} = a_{n-1,n-1} + \sum_{i=1}^{n-2} a_{n-2,i}$

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Generating function

We deduce from this the following generating functions:

$$B(x,y)+C(xy)=-\frac{1-x-xy+x^2y-x^2y^2-x^3y^2-(1-x)\sqrt{1-2\,xy-x^2y^2-2\,x^3y^3+x^4y^4}}{2x^3y^2\,(1-x-y+xy-x^2y^2)}$$

$$C(x)=\frac{1+x-x^2+\sqrt{1-2x-x^2-2x^3+x^4}}{1-x-2x^2-x^3+x^4+(1-x^2)\sqrt{1-2x-x^2-2x^3+x^4}}$$

Coefficients of the taylor expansion of the last generating function equal to c_n, which enumerates the vertices $(1)_2$ at level n for all positive integers n, form the Generalized Catalan sequence.

Thank you for your attention!