Equipopularity of descent-equivalent patterns over descent-equivalence classes of words and permutations

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Graph Theory
Combinatorics and Applications
Al-Ain, U.A.E., October 2019

Outline

- words, patterns, patterns, statistics, popularity
- previous works
- main results statement
- proof preliminaries
  - f-equivalence
  - bijection ψ
  - pattern trace
- sketch of the proofs, examples

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Words

The underlying alphabet of a word is the set of symbols occurring in the word.
The reduction of a word w, denoted $red(w)$, is the word order isomorphic with w on the smallest arity alphabet.
A descent in a word $w_1 w_2 \dots w_n$ is an i with $w_i > w_{i+1}$ and the descent set of w is the set of all such i.

word w	underlying alphabet	$red(w)$	$\mathrm{Des}(w)$
41341	$\{1,3,4\}$	31231	{1,4}
11334	$\{1,3,4\}$	11223	$\varnothing$
42324	$\{2,3,4\}$	31213	{1,3}

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Two same length words are *d*-equivalent if they have the same descent set and the same underlying alphabet.

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Patterns

A pattern is a word with the property that if i occurs in it, then so does j, for any j with $1 \leq j \leq i$.

Equivalently, π is a pattern iff $\pi = red(\pi)$.

The word w contains the pattern π if w has a (not necessarily contiguous) subword whose terms are order isomorphic to (i.e., have same relative ordering as) π

pattern π	word w	$(\pi)w$
212	43143	3
123	231434	3

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Patterns, statistics, popularity

• A combinatorial statistic over the set S is simply an integer valued function on S.
The number of occurrences of a pattern becomes a statistic over the set S of words :

$$|\{w : (\pi)w = p\}|.$$

The popularity of a pattern is the overall number of copies of the pattern within the words of the set

$$\text{the popularity of } \pi \text{ in } S = \sum_{w \in S} (\pi)w,$$

Example

S be the set of 5-permutations with descent set $\{1, 4\}$.

$w \in S$	$(213)w$
21354	3
21453	3
31254	4
31452	2
32451	2
41253	2
41352	2
42351	2
...	0
popularity	20

Previous works

- M. Bóna. Surprising symmetries in objects counted by Catalan numbers. *The Electronic J. of Comb.*, 19(1) :P62, 2012.
- C. Homberger. Expected patterns in permutation class. *The Electronic J. of Comb.*, 19(3) :P43, 2012.
- K. Rudolph. Pattern popularity in 132-avoiding permutations. *The Electronic J. of Comb.*, 20(1) :P8, 2013.
- M. Albert, C. Homberger, and J. Pantone. Equipopularity classes in the separable permutations. *The Electronic J. of Comb.*, 22(2) :P2.2, 2015.

Theorem

For two ptterns π and σ the following statements are equivalent

- π *and* σ *are* d*-equivalent*
- π *and* σ *have the same popularity on any class of* d*-equivalent words*

Example

$w \in S$	$(213)w$	$(312)w$
21354	3	0
21453	3	0
31254	4	1
31452	2	1
32451	2	0
41253	2	3
41352	2	2
42351	2	1
51243	0	5
51342	0	4
52341	0	3
...	0	0
popularity	20	20

Let π and σ be two d-equivalent patterns. σ is an *f*-transformation of π if :

Informally : π and σ *differ little from each other*.

Formally : σ can be obtained from π by either

- increasing or decreasing by 1 an entry in π, or
- interchanging in π two entries with consecutive values.

Example

- $\pi = 211$ and $\sigma = 212$ are f-equivalent
- $\pi = 1332$ and $\sigma = 2331$ are f-equivalent

f-transformation is a symmetric binary relation on a set of d-equivalent patterns and two patterns are said *f*-equivalent if they belong to the same equivalence class.

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Two patterns are f-equivalent if and only if they are d-equivalent.

Example 5124332 and 3115241 are *d*-equivalent

5124332
4125332
4125342
3125342
2125342
2115342
2115341
3115241

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Example 5124**3**32 and **3**11**52**4**1** are *d*-equivalent

5124332
4125332
4125342
3125342
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2115342
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Preliminaries : bijection ψ

We need a bijection ψ from $[q]^n$ into itself satisfying :

(a) ψ preserves the underlying alphabet, and

(b) ψ transforms descent set into ascent set : for any $w \in [q]^n$ $\mathrm{Des}\, w = \mathrm{Asc}\, \psi(w)$.

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The bijection ϕ in satisfies for any word w :

(i) $\phi(w)$ is a rearrangement of the symbols of w,

(ii) $\mathrm{Des}\, w = \{n - i \ : \ i \in \mathrm{Des}\, \phi(w)\}$, and

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Trace

Let $\pi = \pi_1 \pi_2 \dots \pi_k$ be a pattern and $t = t_1 t_2 \dots t_k$ be a length k word over $[q] \cup \{\square\}$, $q \geq 1$.
We say that t is a trace of π if t_i and t_j have the same relative order ($<$, $=$, or $>$) as π_i and π_j whenever $t_i, t_j \neq \square$.

Example

- $t = \square\mathbf{44}\square$ is a trace of the pattern $\pi = 1\mathbf{33}2$
- in $w = \underline{1}5\underline{4}15\underline{4}3\underline{2}$
  1443 is an occurrence of $\pi = 1\mathbf{33}2$ with trace $t = \square\mathbf{44}\square$ in at $A = \{3, 6\}$
- in $\underline{1}\underline{\mathbf{5}}41\underline{\mathbf{5}}43\underline{2}$
  1552 is an occurrence of $\pi = 1\mathbf{33}2$ with trace $t = \square\mathbf{55}\square$ at $A = \{2, 5\}$

We denote by $(t, A, \pi)w$ the number of occurrences of π in a word w with trace t at A.

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We denote by $(t, A, \pi)w$ the number of occurrences of π in a word w with trace t at A.

What happens if we fix the trace t and the position set A?

For σ is an f-transformation of π, then

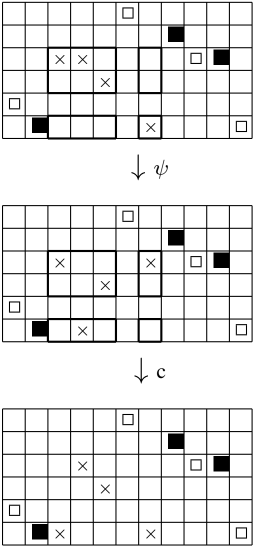
$$(t, A, \pi) \text{ and } (t, A, \sigma)$$

have the same distribution on any d-equivalence class.

	$A = \{3,6\}$, $t = \square\mathbf{44}\square$	
$w \in S$	$\pi = 1\mathbf{33}2$ $(t, A, \pi)w$	$\sigma = 2\mathbf{33}1$ $(t, A, \sigma)w$
15**4**15**4**32	2	0
15**4**25**4**31	1	0
15**4**25**4**32	2	0
15**4**35**4**21	1	0
15**4**35**4**32	2	0
25**4**15**4**31	1	1
25**4**15**4**32	1	0
25**4**25**4**31	1	1
25**4**35**4**21	0	1
25**4**35**4**31	1	1
35**4**15**4**21	0	2
35**4**15**4**32	0	1
35**4**25**4**21	0	2
35**4**25**4**31	0	1
35**4**35**4**21	0	2
...	0	0

Lemma

Let $\pi = \pi_1\pi_2\ldots\pi_k$ and $\sigma = \sigma_1\sigma_2\ldots\sigma_k$ be two d-equivalent patterns with $\pi_\ell = \sigma_\ell$ for any ℓ, except $\pi_i = \sigma_i + 1$ for some i. Let also t be a trace of both π and σ with one $\square$ symbol and A be a subset of $\{1, 2, \ldots, n\}$ of cardinality $k - 1$. Then on any d-equivalent class (t, A, π) and (t, A, σ) have the same distribution.

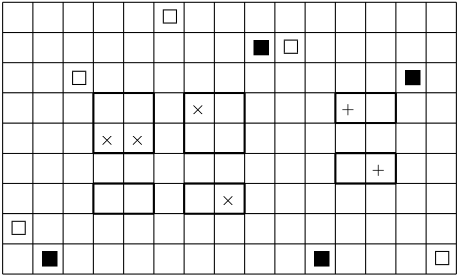

ψ
c

Lemma

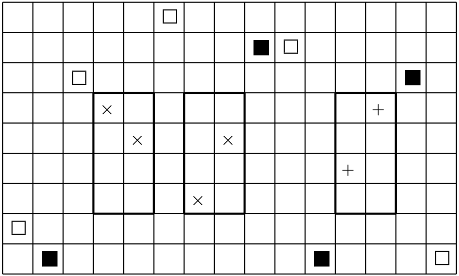
Let $\pi = \pi_1\pi_2 \dots \pi_k$ *and* $\sigma = \sigma_1\sigma_2 \dots \sigma_k$ *be two d-equivalent patterns such that there are i and j,* $i < j$*, with*

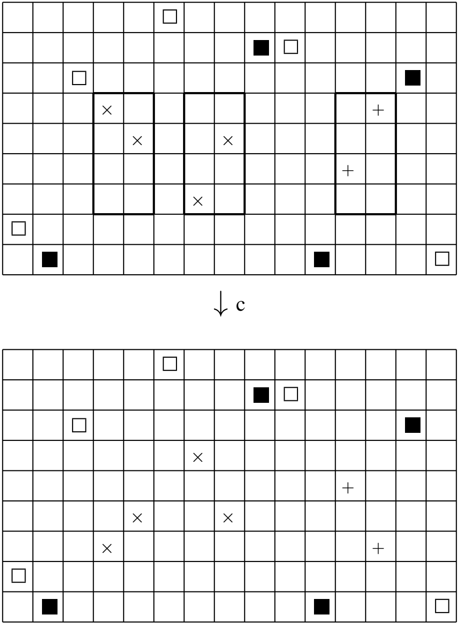
- $\pi_\ell = \sigma_\ell$ *for any* ℓ*, except* $\pi_i = \sigma_j$ *and* $\pi_j = \sigma_i$*,*
- $\pi_i = \pi_j + 1$*,*
- *each of* π_i *and* π_j *occur once in* π *(or, equivalently,* σ_i *and* σ_j *occur once in* σ*).*

Let also $t = t_1 t_2 \dots t_k$ *be a trace of both* π *and* σ *with two* $\square$ *symbols. If A is a subset of* $\{1, 2, \dots, n\}$ *of cardinality* $k - 2$*, then on any d-equivalent class the statistics* (t, A, π) *and* (t, A, σ) *have the same distribution.*



$\downarrow \psi$

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What happens if we fix only the position set A?

For σ is an f-transformation of π, then π and σ have the same popularity on any d-equivalence class.

	$A = \{3,4\}$, $t = \square t_1 t_2 \square$ some common trace	
$w \in S$	$\pi = 2123$ $\sum_t (t, A, \pi) w$	$\sigma = 3122$ $\sum_t (t, A, \sigma) w$
14**24**5	1	0
15**24**4	0	1
24**12**5	1	0
24**14**5	1	0
25**12**4	1	0
25**14**4	0	1
45**12**2	0	2
...	0	0
popularity	$\sum_w \sum_t (t, A, \pi) w =$ 4	$\sum_w \sum_t (t, A, \sigma) w =$ 4

What happens if we fix only the trace t?

For σ is an f-transformation of π, then π and σ have the same popularity on any d-equivalence class.

	$t = \mathbf{42}\square$ common trace	
$w \in S$	$\pi = 211$ $\sum_A (t, A, \pi) w$	$\sigma = 212$ $\sum_A (t, A, \sigma) w$
412321	1	0
412431	0	1
412432	1	1
422321	3	0
422431	1	2
423321	1	0
423421	1	1
423431	0	1
424431	0	2
...	0	0
popularity	$\sum_w \sum_A (t, A, \pi) w =$ 8	$\sum_w \sum_A (t, A, \sigma) w =$ 8

What happens if neither the trace t nor the position are fixed

For σ is an f-transformation of π, then π and σ have the same popularity on any d-equivalence class.

$w \in S$	$\pi = 213$ $\sum_A \sum_t (t, A, \pi) w$	$\sigma = 312$ $\sum_A \sum_t (t, A, \sigma) w$
21354	3	0
21453	3	0
31254	4	1
31452	2	1
32451	2	0
41253	2	3
41352	2	2
42351	2	1
51243	0	5
51342	0	4
52341	0	3
...	0	0
popularity	$\sum_w \sum_A \sum_t (t, A, \pi) w =$ 20	$\sum_w \sum_A \sum_t (t, A, \sigma) w =$ 20

If π is are *f*-transformation of σ, then
π and σ have the same popularity on any *d*-equivalence class.

$$\Downarrow$$

If π and σ are *f*-equivalent, then
π and σ have the same popularity on any *d*-equivalence class.

$$\Downarrow$$

If π and σ are *d*-equivalent, then
π and σ have the same popularity on any *d*-equivalence class.

Thank you for your attention !