Neighbor-distinguishing k-tuple edge-colorings of graphs

Jean-Luc Baril and Olivier Togni [1]

LE2I UMR-CNRS 5158, Université de Bourgogne
B.P. 47 870, 21078 DIJON-Cedex France
e-mail: {barjl,olivier.togni}@u-bourgogne.fr

Abstract

This paper studies proper k-tuple edge-colorings of graphs that distinguish neighboring vertices by their sets of colors. Minimum number of colors for such colorings are determined for cycles, complete graphs and complete bipartite graphs. A variation in which the color sets assigned to edges have to form cyclic intervals is also studied and similar results are given.

Key words: Graph, k-tuple edge-coloring, Neighbor-distinguishing, Adjacent vertex-distinguishing, Fractional coloring, Circular coloring.

1 Introduction

The graphs considered in this paper are undirected and without loops. For a graph $G = (V, E)$ with vertex set V and edge set E, let $\Delta(G)$ (or simply Δ for short) be its maximum degree.

A *proper k-tuple edge-coloring* ϕ of a graph G is a mapping from E to $\mathbb{N}^k$ such that edges incident with the same vertex receive disjoint color sets. The k-tuple version of the vertex-coloring problem was first considered by Stahl [12]. For any vertex x of G, let $S_\phi(x)$ denote the union of the color sets of all edges incident to x (we will omit the subscript ϕ if no confusion is possible). A proper k-tuple edge coloring is said to be *neighbor distinguishing* (*ND*) if $S(x) \neq S(y)$ $\forall xy \in E$. The least number of colors needed for a k-tuple ND-coloring of a graph G will be called its *k-tuple ND-chromatic index* and will be denoted by $\chi'_a(G; k)$. Notice that a graph with an isolated edge does not

[1] Corresponding author.

admit a k-tuple ND-coloring for any k. Consequently, even if not specified, all graphs we deal with are assumed to have no single edge as a component.

The study of 1-tuple ND-colorings of graphs was initiated in [16] under the name of adjacent strong edge-colorings and with the notation $\chi'_a(G) = \chi'_a(G;1)$. In [1], Balister et al. proved that $\chi'_a(G) \leq 5$ for graphs of maximum degree 3 and $\chi'_a(G) \leq \Delta + 2$ for bipartite graphs. In [2], the 1-tuple ND-chromatic index of multidimensional meshes was determined.

The bound $\chi'_a(G) \geq \Delta$ is trivial. Moreover if G contains two adjacent vertices of degree Δ then $\chi'_a(G) \geq \Delta + 1$. The following conjecture was made in [16]:

Conjecture 1 ([16]) *Let $G \neq C_5$ be a connected graph of maximum degree Δ, then*

$$\Delta \leq \chi'_a(G) \leq \Delta + 2.$$

In relation with this conjecture, Hatami [7] has shown that $\chi'_a(G) \leq \Delta + 300$ for any graph G of maximum degree $\Delta > 10^{20}$. Greenhill and Ruciński [5] prove the conjecture for almost all 4-regular graphs. Edwards et al. [4] showed that $\Delta + 1$ colors are sufficient for planar bipartite graphs of maximum degree $\Delta \geq 12$.

Some extensions and variations were also considered: total ND-colorings [15,3], ND-colorings from lists [8] and non proper ND-colorings [6]. For other related distinguishing coloring parameters, see [13].

Not surprisly, Conjecture 1 cannot be extended to k-tuple edge-colorings, in the light of Shannon's well-known chromatic index theorem. Hence, the aim of this paper is to study k-tuple ND-colorings of graphs. We also study a variation of k-tuple ND-coloring where the set of colors assigned to each edge has to form a cyclic interval: a k-tuple ND-coloring ϕ of a graph G with colors from $\{0, \ldots, N-1\}$ is said to be a *cyclic k-tuple ND-coloring* if the color set $\phi(xy)$ of the edge between x and y is an interval modulo N (of size k) which will be denoted by $\phi(xy) = [i, i+k-1]_N$ for some i, $0 \leq i \leq N-1$, where $[a,b]_N = [a \mod N, (a+1) \mod N, \ldots, b \mod N]$. The least number of colors needed for a cyclic k-tuple ND-coloring of a graph G will be called its *cyclic k-tuple ND-chromatic index* and will be denoted by $\chi'_{ac}(G;k)$.

Observe that for any graph G and any $k \geq 1$, $\chi'_{ac}(G;k) \geq \chi'_a(G;k)$, and, as we will show later, there exist several classes of graphs for which $\chi'_{ac}(G;k) > \chi'_a(G;k)$.

Without the ND constraint, k-tuple edge-colorings lead to the definition of the fractional chromatic index $\chi'_f(G)$ which can be defined by $\chi'_f(G) = \inf_k \frac{\chi'(G;k)}{k} = \min_k \frac{\chi'(G;k)}{k}$ [14]. In the same vein, cyclic k-tuple edge-colorings of graphs lead to the definition of the circular chromatic index [10,11]: the circular chromatic index of G is the ratio minimum N/k for which there exists an edge coloring of G by cyclic (or circular) intervals modulo N of size k. Notice that, as can be seen in the remainder of the paper, there are graphs G for which $\chi'_a(G;k)/k$ is a strictly decreasing function of k, thus the infimum is never reached (which is not the case without the ND constraint).

The paper is organized as follows: Section 2 presents some general simple results about the k-tuple ND- and cyclic k-tuple ND-chromatic indices of graphs. In Section 3, we determine the k-tuple ND- and cyclic k-tuple ND-chromatic indices of paths and cycles. In Section 4, k-tuple ND- and cyclic k-tuple ND-chromatic indices of complete and complete bipartite graphs are determined. Section 5 concludes the paper by presenting a conjecture about the ND-chromatic index of a multigraph.

2 General observations

We begin by some simple observations that will be used throughout the rest of the paper.

Let $I^N_{j,k}$ be the cyclic interval $[j, j+k-1]_N$. If N and k are clear from the context, we will permit ourselves to write I_j instead of $I^N_{j,k}$. Let also $\mathcal{I}^N_k = \{I^N_{j,k}, 0 \le j \le N-1\}$.

Observation 1 *For any graph G,*

$$\chi'_{ac}(G;k) \le k\chi'_a(G).$$

Proof : Starting from an ND-coloring of G with $\chi'_a(G)$ colors, a cyclic k-tuple ND-coloring of G with $N = k\chi'_a(G)$ colors can be obtained simply by replacing each color i by the color interval I_{ki}. □

Observation 2 *For any graph G and any integers k and k' with $1 \le k' < k$,*

$$\chi'_a(G;k) \le \chi'_a(G;k') + \chi'(G;k-k').$$

In particular, $\chi'_a(G;k) \le \chi'_a(G) + (k-1)\chi'(G)$.

Remember that a graph is *class 1* if $\chi'(G) = \Delta(G)$ and *class 2* if $\chi'(G) = \Delta(G)+1$. Thus, derived from the above observation, we have the following:

Observation 3 *For any class 1 graph G,*

$$\chi'_a(G;k) \le \chi'_a(G) + (k-1)\Delta(G).$$

As a corollary, we easily obtain the next proposition:

Proposition 2 *For a class 1 graph G,*

- *if G has two adjacent vertices of maximum degree and if $\chi'_a(G) = \Delta(G)+1$, then $\chi'_a(G;k) = k\Delta(G)+1$;*
- *if $\chi'_a(G) = \Delta(G)$, then $\chi'_a(G;k) = k\Delta(G)$.*

For instance, the cycle C_{6p} is class 1 and $\chi'_a(C_{6p}) = 3$, thus $\chi'_a(C_{6p};k) = 2k$.

3 The path P_n and cycle C_n

Let P_n be the path of order n, with vertex set $V = \{0, 1, \ldots, n-1\}$ and edge set $E = \{e_i = i(i+1), 0 \le i \le n-2\}$ and let C_n be the cycle of order n with vertex set V and edge set $E = \{e_i = i(i+1) \bmod n, 0 \le i \le n-1\}$.

For the path, we easily obtain the following theorem:

Theorem 3 *For $n \ge 4$ and $k \ge 1$, $\chi'_{ac}(P_n;k) = \chi'_a(P_n;k) = 2k+1$.*

Proof : The theorem directly follows from Proposition 2 since P_n is class 1 and $\chi'_a(P_n) = 3$. However, a cyclic k-tuple ND-coloring ϕ can be simply constructed by setting $\phi(e_i) = I^N_{ki,k}$ for $0 \le i \le n-1$ and $N = 2k+1$. □

In order to determine the cyclic k-tuple ND-chromatic index of the cycle, we need some notation. Let the *gap* $\mathrm{gap}(I^N_{a,k}, I^N_{b,k})$ be the number of integers between the last element of $I^N_{a,k}$ and the first of $I^N_{b,k}$ considered modulo N, i.e. $\mathrm{gap}(I^N_{a,k}, I^N_{b,k}) = b-a-k \mod N$. Notice that this definition is not symmetrical since there exist many cases such that $\mathrm{gap}(I^N_{a,k}, I^N_{b,k}) \ne \mathrm{gap}(I^N_{b,k}, I^N_{a,k})$.

Any cyclic k-tuple coloring of the cycle C_n can naturally be associated with the sequence of gaps between intervals of consecutive edges. For instance, for the cycle C_9 with $k = 4$ and $N = 10$ colors, the sequence $S = (2,1,0,0,0,0,0,0,1)$ corresponds with intervals $(I_0, I_6, I_1, I_5, I_9, I_3, I_7, I_1, I_5)$ along the edges $(e_0, e_1, \ldots, e_{n-1})$ of the cycle.

Notice also that in order for a sequence $S = (s_1, s_2, \ldots, s_n)$ of gaps to correspond with a cyclic k-tuple coloring of the cycle C_n with N colors, we must have $\sum_{i=1}^n s_i + kn \equiv 0 \mod N$ (because the gap between the last interval and the first one is completely determined by the $n-1$ other gaps).

Lemma 4 *For $k \ge 1$ and $n \ge 4$ even, $\chi'_{ac}(C_n;k) \le 2k+2$.*

Proof : We define a cyclic k-tuple ND-coloring of C_n with $N = 2k+2$ colors by giving its sequence of gaps S:

$$S = (\underbrace{2, 2, \ldots, 2}_{\frac{n}{2}-1}, 1, \underbrace{0, 0, \ldots, 0}_{\frac{n}{2}-1}, 1).$$

The associated coloring is clearly proper since S does not contain any gap of size at least three. The coloring is also ND since the sequence does not contain any subsequence of length two of the form $0, 2$; $2, 0$ or $1, 1$ that are the only cases which prevent the coloring from being ND when $N = 2k + 2$. □

Lemma 5 *For $k \geq 2$ and n odd, $n \geq 2k + 3$, $\chi'_{ac}(C_n; k) \leq 2k + 2$.*

Proof : We define a cyclic k-tuple ND-coloring of C_n with $2k + 2$ colors by giving its sequence of gaps S:

$$S = \begin{cases} (\underbrace{2, 2, \ldots, 2}_{\frac{n-k-3}{2}}, 1, 0, 0, 0, \ldots, 0, 1) & \text{if } k \text{ is even,} \\ (\underbrace{2, 2, \ldots, 2}_{\frac{n-k-4}{2}}, 1, 0, 1, 0, 0, \ldots, 0, 1) & \text{otherwise.} \end{cases}$$

As above, it can be shown that the associated coloring is ND. In addition, we just need to verify that $\frac{n-k-3}{2} \geq 1$ for k even, and $\frac{n-k-4}{2} \geq 1$ otherwise. Indeed, $n - k - 3 \geq 2k + 3 - k - 3 \geq k \geq 2$ when k is even; and $n - k - 4 \geq 2k + 3 - k - 4 = k - 1 \geq 2$ for k odd and $k \geq 2$. □

Theorem 6 *For $n \geq 3$ and $k \geq 1$,*

$$\chi'_{ac}(C_n; k) = \begin{cases} 5 & \textit{if } (n, k) = (5, 1), \\ 2k + \lceil \frac{2k}{n-1} \rceil & \textit{if } n < 2k + 1 \textit{ is odd}, \\ 2k + 1 & \textit{if } n \equiv 0 \bmod (2k + 1), \\ 2k + 2 & \textit{otherwise} . \end{cases}$$

Proof : The case $(n, k) = (5, 1)$ is trivial.

We shall now show that $\chi'_{ac}(C_n; k) = 2k+1$ if and only if $n \equiv 0 \bmod (2k + 1)$.

Let $N = 2k+1$ and $n \equiv 0 \bmod (2k + 1)$. The k-tuple coloring ϕ of C_n defined by $\phi(e_j) = I_{kj}, 0 \leq j \leq n-1$ or equivalently by the gap sequence $S = (0, \ldots, 0)$ is clearly proper and ND since $\phi(e_{n-1}) = I_{2k^2} = I_{k+1} \neq \phi(e_1) = I_k$. This coloring is exhibited in Figure 1 for $(n, k) = (7, 3)$.

On the other hand, it is easily seen that any cyclic k-tuple ND-coloring of C_n with $2k + 1$ colors is isomorphic to ϕ up to a renumbering of the colors since in that case the interval of colors that can be assigned to an edge e_i

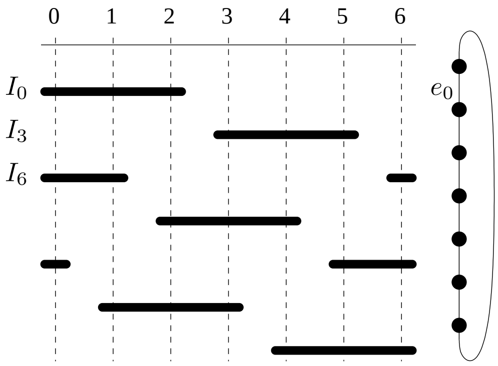

Fig. 1. A cyclic 3-tuple ND-coloring of C_7 with $N = 7$ colors.

is completely determined by the intervals assigned to its two preceding edges e_{i-1} and e_{i-2}. Thus a cyclic k-tuple ND-coloring of C_n with $2k+1$ colors exists only if $n \equiv 0 \bmod (2k+1)$.

We now show that if $n < 2k+1$ and n is odd then $\chi'_{ac}(C_n; k) = 2k + \lceil \frac{2k}{n-1} \rceil$. It is sufficient to show that $\chi'_{ac}(C_n; k) \leq 2k + \lceil \frac{2k}{n-1} \rceil$ since $\chi'_{ac}(C_{2p+1}; k) \geq \chi'(C_{2p+1}; k) \geq k\frac{2p+1}{p} = k(2 + \frac{1}{p})$ (because at most p edges of C_{2p+1} can be given the same color).

Let $n = 2p + 1$, with $p \leq k - 1$ and let $s = p\lceil \frac{k}{p} \rceil - k$.

Consider the k-tuple coloring ϕ of C_{2p+1} with $N = 2k + \lceil \frac{2k}{n-1} \rceil$ colors defined by the following sequence of gaps:

$$S = (\underbrace{1, 0, 1, 0, \ldots, 1, 0}_{2s}, 0, 0, \ldots, 0, 0).$$

Remark that $0 \leq s \leq p\frac{k+p}{p} - k = p$.

As S does not contain (a) any gap of size at least 2, or (b) any subsequence of the form $1, 1$, and since the sum of gaps plus nk equals 0 modulo $2k + \lceil \frac{2k}{n-1} \rceil$, then ϕ is a k-tuple ND-coloring of C_n with $2k + \lceil \frac{2k}{n-1} \rceil$ colors. An illustration is given in Figure 2 for $(n, k) = (9, 10)$ (thus $s = 2$).

We end the proof by showing that $\chi'_{ac}(C_n; k) = 2k+2$ for the remaining cases. By the above, we know that $\chi'_{ac}(C_n; k) > 2k + 1$ if $n \not\equiv 0 \bmod (2k + 1)$. Thus, by virtue of Lemma 4 and Lemma 5, we have that $\chi'_{ac}(C_n; k) \leq 2k + 2$ for these cases, which completes the proof. $\square$

Theorem 7 *For $n \geq 3$ and $k \geq 1$,*

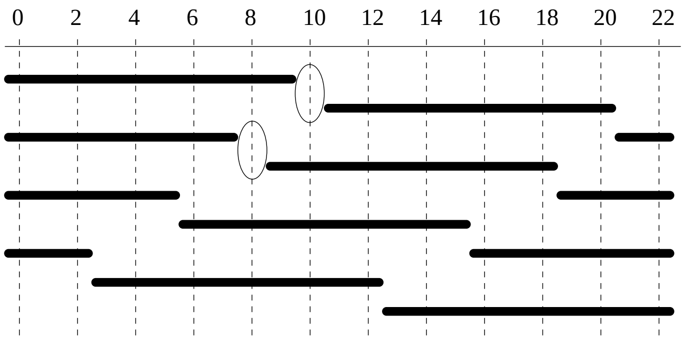

Fig. 2. A cyclic 10-tuple ND-coloring of C_9 with $N = 23$ colors and is associated sequence of gaps $(1, 0, 1, 0, 0, 0, 0, 0, 0)$.

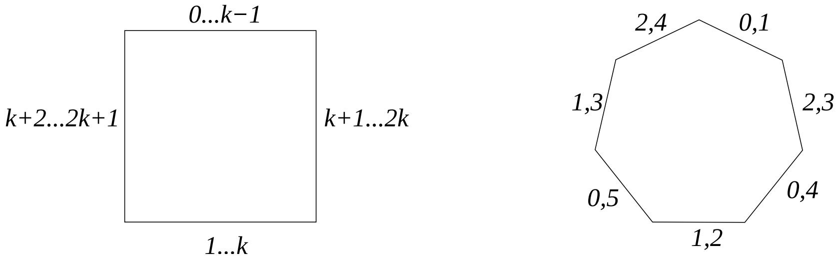

Fig. 3. The cases $n = 4$ and $(n, k) = (7, 2)$.

$$\chi'_a(C_n; k) = \begin{cases} 5 & \textit{if } (n,k) = (5,1), \\ 2k + \lceil \frac{2k}{n-1} \rceil & \textit{if } n < 2k+1 \textit{ is odd}, \\ 2k+2 & \textit{if } n = 4 \textit{ or } (n,k) = (7,2), \\ 2k+1 & \textit{otherwise.} \end{cases}$$

Proof : Recall that for any graph G and any $k \geq 1$, $\chi'_a(G; k) \leq \chi'_{ac}(G; k)$. Then, with Theorem 6, it remains to treat the following cases:

• $n = 4$ or $(n, k) = (7, 2)$. It is easily seen that a k-tuple ND-coloring with $2k + 1$ colors does not exist in these cases and a k-tuple ND-coloring with $2k + 2$ colors is illustrated in Figure 3.

• When n is odd, we have that $\chi'_a(C_{2p+1}; k) \geq \chi'(C_{2p+1}; k) \geq k\frac{2p+1}{p} = k(2 + \frac{1}{p}) = 2k + \lceil \frac{2k}{n-1} \rceil$ (since at most p edges of C_{2p+1} can be given the same color).

• $k = 2$, $n \geq 5$, $n \neq 7$. In this part we consider five subcases depending on the residues of n modulo 5:
Subcase 0: $n \equiv 0 \mod 5$ and $n \geq 5$. We color the cycle by using the sequence of color sets $S_1 = (\{0, 1\}, \{2, 3\}, \{0, 4\}, \{1, 2\}, \{3, 4\})$ repetitively on each group of five consecutive edges of C_n.

Subcase 1: $n \equiv 1 \mod 5$ and $n \geq 6$. For $n = 5q + 1$, we color the cycle by using $(q - 1)$ times the sequence S_1 and one time the sequence $S_2 = (\{0, 1\}, \{2, 3\}, \{0, 4\}, \{1, 3\}, \{0, 2\}, \{3, 4\})$.

Subcase 2: $n \equiv 2 \mod 5$ and $n \geq 12$. for $n = 5q + 2$ with $q \geq 2$, we color the cycle by using $(q-2)$ times the sequence S_1 and two times the sequence S_2.

Subcase 3: $n \equiv 3 \mod 5$ and $n \geq 8$. for $n = 5q + 3$, we color the cycle by using $(q-1)$ times the sequence S_1 and one time the sequence $S_3 = (\{0,1\}, \{2,3\}, \{0,4\}, \{1,2\}, \{3,4\}, \{0,2\}, \{1,3\}, \{2,4\})$.

Subcase 4: $n \equiv 4 \mod 5$ and $n \geq 9$. For $n = 5q + 3$, we color the cycle by using $(q-1)$ times the sequence S_1 and one time the sequence $S_4 = (\{0,1\}, \{2,3\}, \{0,4\}, \{1,2\}, \{0,3\}, \{1,4\}, \{0,2\}, \{1,3\}, \{2,4\})$.

In each case we obtain a 2-tuple ND-coloring of C_n with $5 = 2k + 1$ colors.

- $k \geq 3$ and $n = 2p$ is even. By Observation 2, we have $\chi'_a(C_{2p}; k) \leq \chi'_a(C_{2p}; 2) + \chi'(C_{2p}; k-2) = 5 + 2(k-2) = 2k + 1$.

- $k \geq 3$ and n odd, $n \geq 2k + 1 \geq 7$. We set $N = 2k + 1$, $n = qN + r$ with $0 \leq r \leq N - 1$ and $q \geq 1$.

Subcase 1: $n \equiv 0 \mod N$. We color the cycle by using the periodic coloring E defined by $E_i = I^N_{ik,k}$, for $i \geq 0$.

Subcase 2: $n \not\equiv 0 \mod N$. We provide a k-tuple ND-coloring for each value of the residue of r modulo 6 by giving the sequence of color sets to assign to the edges in a consecutive manner (see Figure 4 for an illustration of the construction).

- (i) $r \equiv 0 \mod 6$. Starting from the qN-first color sets of E, we append the r-first color sets of the following periodic coloring F:
  $F_0 = \{0, 1, 3, 4, \ldots, k\}$, $F_1 = \{2, k+1, \ldots, 2k-1\}$, $F_2 = \{1, 3, 4, \ldots, k, 2k\}$, $F_3 = \{0, k+1, \ldots, 2k-1\}$, $F_4 = I^N_1$, $F_5 = I^N_{k+1}$ with $F_i = F_{i-6}$ for $i \geq 6$.
- (ii) $r \equiv 1 \mod 6$. We discuss on the parity of k.

  For k odd, starting from the $((q-1)N + k)$-first color sets of E, we append the $k + 1$ color sets defined for $0 \leq i \leq \lceil \frac{k}{2} \rceil - 2$ by:

  $$\begin{cases} \{\lceil \frac{k}{2} \rceil + i, \ldots, 2k - \lceil \frac{k}{2} \rceil, 2k + 1 - i, \ldots, 2k\} \\ \{0, 1, 2, \ldots, \lceil \frac{k}{2} \rceil - 1 + i, 2k + 1 - \lceil \frac{k}{2} \rceil, \ldots, 2k - 1 - i\} \end{cases} \text{ and by}$$

  $$\begin{cases} \{k-1, k+1, \ldots, 2k - \lceil \frac{k}{2} \rceil, 2k + 2 - \lceil \frac{k}{2} \rceil, \ldots, 2k\} \\ \{0, 1, \ldots, k-2, k\} \end{cases},$$

  where a color set of the form $\{a, \ldots, b, c, \ldots, d\}$ with $c > d$ has to be understood as $\{a, \ldots, b\}$.

  Then, we complete this coloring by using the r-first color sets of G defined at the end of this subcase.

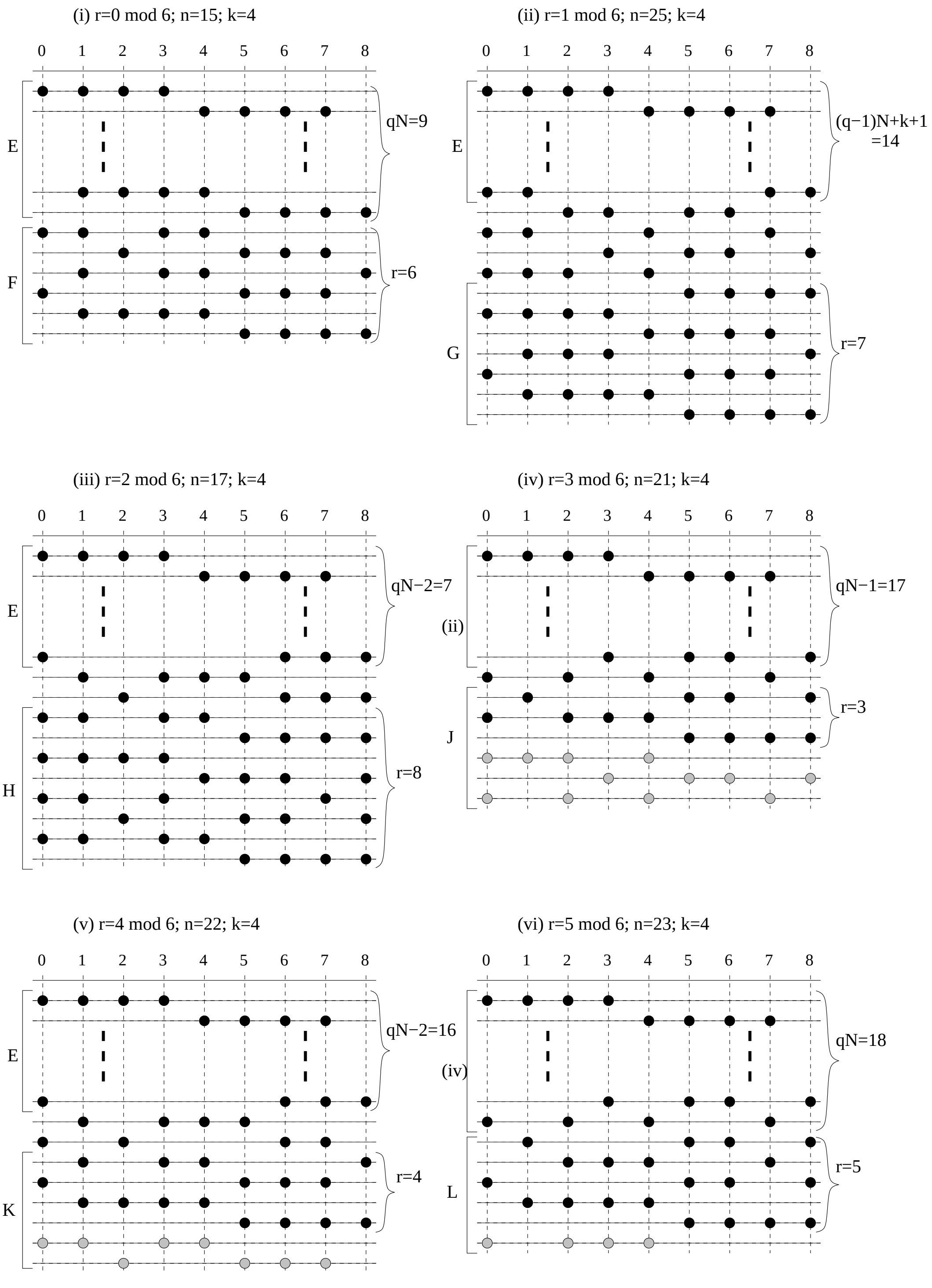

Fig. 4. Structure of the k-tuple ND-colorings of C_n when $n \not\equiv 0 \mod (2k+1)$, n odd.

For k even, we start by the $((q-1)N+k+1)$-first color sets of E; we append the k color sets defined for $0 \leq i \leq \frac{k}{2}-1$ by:

$$\begin{cases} \{\frac{k}{2}+i, \ldots, k-1, k+1, \ldots, 2k-\frac{k}{2}, 2k-i+1, \ldots, 2k\} \\ \{0, 1, \ldots, \frac{k}{2}-1+i, k, 2k+1-\frac{k}{2}, \ldots, 2k-i-1\} \end{cases};$$

and we complete this coloring by using the r-first color sets of the periodic coloring G defined by:
$G_0 = I^N_{k+1}$, $G_1 = I^N_0$, $G_2 = I^N_k$, $G_3 = \{1, 2, \ldots, k-1, 2k\}$, $G_4 = \{0, k+1, k+2, \ldots, 2k-1\}$, $G_5 = I^N_1$ with $G_i = G_{i-6}$ for $i \geq 6$.

- (iii) $r \equiv 2 \mod 6$. To the $qN-2$-first sets of E, we add the two sets $\{1, 3, 4, \ldots, k+1\}$ and $\{2, k+2, k+3, \ldots, 2k\}$ and we append the r-first color sets of the following periodic coloring H defined by:
$H_0 = \{0, 1, 3, 4, \ldots, k\}$, $H_1 = I^N_{k+1}$, $H_2 = I^N_0$, $H_3 = \{k, k+1, \ldots, 2k-2, 2k\}$, $H_4 = \{0, 1, 3, 4, \ldots, k-1, 2k-1\}$, $H_5 = \{2, k+1, k+2, \ldots, 2k-2, 2k\}$ with $H_i = H_{i-6}$ for $i \geq 6$.
- (iv) $r \equiv 3 \mod 6$. Starting from the $(qN-1)$-first sets of the coloring found in subcase (ii); we complete with $\{0, 2, \ldots, k-2, k, 2k+1-\lceil\frac{k}{2}\rceil\}$ ($\{0, 3, 5\}$ if $k=3$) and we append the r-first color sets of the following periodic coloring J:
$J_0 = \{1, k+1, \ldots, 2k-\lceil\frac{k}{2}\rceil, 2k+2-\lceil\frac{k}{2}\rceil, \ldots, 2k\}$, $J_1 = \{0, 2, \ldots, k\}$, $J_2 = I^N_{k+1}, k$, $J_3 = \{0, \ldots, k-2, k\}$, $J_4 = \{k-1, k+1, \ldots, 2k-\lceil\frac{k}{2}\rceil, 2k+2-\lceil\frac{k}{2}\rceil, \ldots, 2k\}$, $J_5 = \{0, 2, \ldots, k-2, k, 2k+1-\lceil\frac{k}{2}\rceil\}$ with $J_i = J_{i-6}$ for $i \geq 6$.
- (v) $r \equiv 4 \mod 6$. Starting from the $(qN-2)$-first sets of the coloring E; we add the two sets $\{1, 3, 4, \ldots, k+1\}$ and $\{0, 2, k+2, k+3, \ldots, 2k-1\}$ and we append the r-first color sets of the following periodic coloring K:
$K_0 = \{1, 3, 4, \ldots, k, 2k\}$, $K_1 = \{0, k+1, \ldots, 2k-1\}$, $K_2 = I^N_1$, $K_3 = I^N_{k+1}$, $K_4 = \{0, 1, 3, 4, \ldots, k-1, k\}$, $K_5 = \{2, k+1, k+2, \ldots, 2k-2, 2k-1\}$ with $K_i = K_{i-6}$ for $i \geq 6$.
- (vi) $r \equiv 5 \mod 6$. To the qN-first sets of the coloring found in subcase (iv); we append the r-first color sets of the following periodic coloring L:
$L_0 = \{1, k+1, \ldots, 2k-\lceil\frac{k}{2}\rceil, 2k+2-\lceil\frac{k}{2}\rceil, \ldots, 2k\}$, $L_1 = \{2, \ldots, k, 2k+1-\lceil\frac{k}{2}\rceil\}$, $L_2 = \{0, k+1, \ldots, 2k-\lceil\frac{k}{2}\rceil, 2k+2-\lceil\frac{k}{2}\rceil, \ldots, 2k\}$, $L_3 = I^N_1$, $L_4 = I^N_{k+1}$, $L_5 = \{0, 2, \ldots, k\}$ with $L_i = L_{i-6}$ for $i \geq 6$.

□

4 The complete graph K_n and complete bipartite graph $K_{n,n}$

Let $V(K_n) = \{0, 1, \ldots, n-1\}$ and $E(K_n) = \{e_{ij}, 0 \leq i, j \leq n-1\}$, with $e_{ij} = ij$ and let $V(K_{m,n}) = \{x_0, x_1, \ldots, x_{n-1}\} \cup \{y_0, y_1, \ldots, y_{n-1}\}$ and $E(K_{m,n}) = \{e_{ij}, 0 \leq i \leq m-1, 0 \leq j \leq n-1\}$, with $e_{ij} = x_i y_j$.

Theorem 8 *For any $n \geq 3$ and $k \geq 1$,*

$$\chi'_a(K_n;k) = \chi'_{ac}(K_n;k) = \begin{cases} k(n-1)+2 & \text{if } n \text{ is even,} \\ kn & \text{if } n \text{ is odd.} \end{cases}$$

Proof : First, remark that a k-tuple ND-coloring of K_n with N colors can exist only if there exist n distinct subsets of $\{0,\ldots,N-1\}$ of size $k(n-1)$ such that each number from $\{0,\ldots,N-1\}$ appears an even number of times.

Thanks to this remark, we can see that if n is even, then $\chi'_a(K_n;k) \geq k(n-1)+2$ since in any n sets of $k(n-1)$ elements among $k(n-1)+1$ (in each set, one color is not present and thus appears in the $(n-1)$ other sets), all elements appear an odd number of times, which is impossible.

If n is odd, $n = 2p+1$ for some p, the fact that $\chi'_a(K_{2p+1};k) \geq k(2p+1)$ can be shown by contradiction: assume that $k(2p+1)-1$ colors suffice. Then, among any $2p+1$ sets of $k(2p)$ colors among $k(2p+1)-1$, there is at least one color that belongs to each set (assume that each color is present in at most $2p$ sets, then we would have $2p(k(2p+1)-1) \geq \sum_{i=1}^{2p+1} 2pk = 2pk(2p+1)$, a contradiction), thus an odd number of times, which is impossible. The fact that $\chi'_{ac}(K_{2p+1};k) \leq k(2p+1)$ (and thus that $\chi'_a(K_{2p+1};k) \leq k(2p+1)$) comes directly from Observation 1, as $\chi'_a(K_{2p+1}) = 2p+1$.

If n is even, then K_n is class 1 and $\chi'_a(K_n) = n+1$ [16]. Thus, by Observation 3, we have that $\chi'_a(K_n;k) \leq n+1+(k-1)(n-1) = k(n-1)+2$. It remains to show that $\chi'_{ac}(K_{2p};k) \leq k(2p-1)+2$. In order to do that, we shall start with a cyclic edge coloring of K_{2p} for which all vertices have the same set of colors and modify it by increasing or decreasing by one the color intervals of some edges in order vertices to have sets of colors different from each other.

Let ϕ be the cyclic k-tuple proper coloring of K_n with $N-2 = k(n-1)$ colors defined for $0 \leq i < j \leq n-1$ as follow:

$$\phi(e_{ij}) = \begin{cases} I^{N-2}_{k(i+j),k} & \text{if } 0 \leq i < j \leq n-2, \\ I^{N-2}_{2ki,k} & \text{otherwise.} \end{cases}$$

Notice that each vertex x has color set $S(x) = \{0,1,\ldots,N-3\}$ and that each color interval can also be considered modulo $N = k(n-1)+2$ since $I^{N-2}_{k(i+j),k} = I^{N}_{k(i+j),k}$ and $I^{N-2}_{2ki,k} = I^{N}_{2ki,k}$.

Now we modify this coloring in order to obtain a cyclic k-tuple ND-coloring with N colors (all intervals are modulo N in the rest of the proof). We distinguish two cases depending on the residue of n modulo 4.

Case 1: $n \equiv 0 \bmod 4$. We increase by one each color of the interval $\phi(e_{ij})$, $i < j$, for (i, j) such that

$$\begin{cases} \frac{3n}{4} \le i < j \le n-1 \\ j = n-1 \text{ and } \frac{n}{4} \le i \le \frac{n}{2} - 1 \\ j + i < n-1 \text{ and } 0 \le i \le \frac{n}{4} - 2 \text{ and } \frac{3n}{4} \le j \le n-2 \\ j + i < n-1 \text{ and } \frac{n}{4} \le i < j \le \frac{3n}{4} - 2, \end{cases}$$

and we decrease by one each color of the interval $\phi(e_{ij})$, $i < j$, for (i, j) such that

$$0 \le i \le \tfrac{n}{4} - 1 \text{ and } j = n - 1 - i.$$

This construction is illustrated in Appendix A for the case $n = 12$ and $k = 3$.

Let ϕ' be this new coloring. The vertices can be classified in four groups depending on their two missing colors that are given by the next table:

Group	Missing colors for i	i
(a)	$k-1$ and $k(\frac{3n}{4}+i)$	$[0, \frac{n}{4}-1]$
(b)	$k(\frac{n}{4}+i)$ and $N-1$	$[\frac{n}{4}, \frac{3n}{4}-1]$
(c)	$k-1$ and $k(i+1-\frac{n}{4})$	$[\frac{3n}{4}, n-2]$
(d)	$k-1$ and $k\frac{n}{2}$	$n-1$

Now let us verify that ϕ' is a cyclic k-tuple ND-coloring for K_n when $n \equiv 0$ mod 4. If we compare missing color sets in the same group, it is straightforward to see that they are different. Notice that a missing color set of the group (b) contains the color $N-1$ that does not belong to the sets in others groups. Moreover, the color $k-1$ belongs to each missing color sets of the groups (a), (c) and (d). It then suffices to remark that the second missing colors of two such sets can not be the same.

Case 2: $n \equiv 2 \bmod 4$. The process is similar with that of Case 1 except that some color intervals are increased by two: we increase by one each color interval $\phi(e_{ij})$, $i < j$, for (i, j) such that

$$\begin{cases} \lfloor \frac{3n}{4} \rfloor \le i < j \le n-1 \\ j = n-1 \text{ and } \lfloor \frac{n}{4} \rfloor \le i \le \frac{n}{2} - 1 \\ j + i < n-1 \text{ and } 0 \le i \le \lfloor \frac{n}{4} \rfloor - 1 \text{ and } \lfloor \frac{3n}{4} \rfloor \le j \le n-2 \\ j + i < n-1 \text{ and } \lfloor \frac{n}{4} \rfloor \le i < j \le \lfloor \frac{3n}{4} \rfloor - 1, \end{cases}$$

we yet increase by one each color interval $\phi(e_{ij})$ for $(i, j) = (\lfloor \frac{n}{4} \rfloor, \lfloor \frac{3n}{4} \rfloor - 1)$ and $(i, j) = (\lfloor \frac{n}{4} \rfloor + 1, \lfloor \frac{3n}{4} \rfloor - 2)$, and we decrease by one each color interval $\phi(e_{ij})$, $i < j$, for (i, j) such that

$$0 \le i \le \lfloor \tfrac{n}{4} \rfloor - 1 \text{ and } j = n - 1 - i.$$

Let ϕ' be this new coloring. This construction is illustrated in Appendix A for the case $n = 14$ and $k = 3$.

We can see that each vertex i, $0 \leq i \leq n-1$, has exactly two missing colors that are given by the next table:

Group	Missing colors for i	i
(a)	$k-1$ and $k(\lfloor\frac{3n}{4}\rfloor + i)$	$[0, \lfloor\frac{n}{4}\rfloor - 1]$
(b)	$k(\lfloor\frac{n}{4}\rfloor + i)$ and $k(n-2)+1$	$[\lfloor\frac{n}{4}\rfloor, \lfloor\frac{n}{4}\rfloor + 1]$
(c)	$k(\lfloor\frac{n}{4}\rfloor + i)$ and $k(n-1)+1$	$[\lfloor\frac{n}{4}\rfloor + 2, \lfloor\frac{3n}{4}\rfloor - 3]$
(d)	$k(\lfloor\frac{n}{4}\rfloor + i)$ and $k(n-2)+1$	$[\lfloor\frac{3n}{4}\rfloor - 2, \lfloor\frac{3n}{4}\rfloor - 1]$
(e)	$k(i - \lfloor\frac{n}{4}\rfloor)$ and $k(n-1)+1$	$\lfloor\frac{3n}{4}\rfloor$
(f)	$k-1$ and $k(i - \lfloor\frac{n}{4}\rfloor)$	$[\lfloor\frac{3n}{4}\rfloor + 1, n-2]$
(g)	$k-1$ and $2k\lfloor\frac{n}{4}\rfloor$	$n-1$

Like for the case $n \equiv 0 \mod 4$, it can be shown that ϕ' is a cyclic k-tuple ND-coloring for K_n when $n \equiv 2 \bmod 4$. □

For complete bipartite graphs $K_{m,n}$, the only "interesting" case is when $m = n$ (if $m \neq n$ then adjacent vertices have different degrees).

Theorem 9 *For any $n \geq 2$ and $k \geq 1$,*

$$\chi'_a(K_{n,n}; k) = \chi'_{ac}(K_{n,n}; k) = kn + 2.$$

Proof : Let X and Y be the two sets of vertices of the bipartition. Firstly, it can be seen that $\chi'_a(K_{n,n}; k) \geq kn+2$. Assume, to the contrary, that a k-tuple ND-coloring of $K_{n,n}$ with $kn+1$ colors exists. Then, as each vertex of $K_{n,n}$ has degree kn, the set of missing colors $\bar{S}(v)$ on each vertex v consists of only one color and as $\{\bar{S}(x), x \in X\} = \{\bar{S}(y), y \in Y\}$, then for each vertex $x \in X$, there exists a vertex $y \in Y$ such that $\bar{S}(x) = \bar{S}(y)$ or equivalently such that $S(x) = S(y)$, a contradiction.

Secondly, we show that $\chi'_{ac}(K_{n,n}; k) \leq kn + 2$ by constructing such a cyclic k-tuple ND-coloring, using a similar argument than the one of the proof of Theorem 8 for the complete graph. Indeed, our coloring distinguishes all vertices of $K_{n,n}$, not only adjacent ones.

As for the complete graph, we start with the cyclic k-tuple proper coloring ϕ of $K_{n,n}$ with $N-2 = kn$ colors defined for $0 \leq i, j \leq n-1$ by:

$$\phi(e_{ij}) = I^{N-2}_{k(n+i-j-1),k}.$$

Notice that each vertex x has color set $S(x) = \{0, 1, \ldots, N-3\}$. Now we modify this coloring in order to obtain a cyclic k-tuple ND-coloring with $N = kn+2$ colors. We distinguish two cases depending on the residue of n modulo 2.

Case 1: $n \equiv 0 \bmod 2$. We increase by one each $\phi(e_{ij})$ for (i,j) such that $0 \leq i \leq j \leq n-2$, and we decrease by one each $\phi(e_{i(i-1)})$ for $i \in [\frac{n}{2}, n-1]$, resulting in a coloring ϕ'. This construction is illustrated in Appendix B for the case $n = 12$ and $k = 3$. It can be seen that the missing color sets on two vertices x_i and y_j are always different since we have

$$\bar{S}_{\phi'}(x_i) = \begin{cases} \{k(i+1), N-1\} & \text{for } i \in [0, \frac{n}{2}-1], \\ \{k(i+1), k-1\} & \text{for } i \in [\frac{n}{2}, n-1]. \end{cases}$$

$$\bar{S}_{\phi'}(y_j) = \begin{cases} \{k(n-j-1), N-1\} & \text{for } j \in [0, \frac{n}{2}-2], \\ \{k(n-j-1), k-1\} & \text{for } j \in [\frac{n}{2}-1, n-2], \\ \{N-2, N-1\} & \text{for } j = n-1. \end{cases}$$

Thus ϕ' is a cyclic k-tuple ND-coloring of $K_{n,n}$ with $N+2 = kn+2$ colors when $n \equiv 0 \bmod 2$.

Case 2: $n \equiv 1 \bmod 2$. We also increase by one each $\phi(e_{ij})$ for (i,j) such that $0 \leq i \leq j \leq n-2$; we decrease by one each $\phi(e_{i(i-1)})$ for $i \in [\frac{n+1}{2}, n-1]$ and we yet increase by one $\phi(e_{ii})$ for $i = \lfloor \frac{n}{2} \rfloor - 1$, resulting in a coloring ϕ'. This construction is illustrated in Appendix B for the case $n = 11$ and $k = 3$. It can be seen that the missing colors on two vertices x_i and y_j are always different since we have

$$\bar{S}_{\phi'}(x_i) = \begin{cases} \{k(i+1), N-1\} & \text{for } i \in [0, \lfloor \frac{n}{2} \rfloor], i \neq \lfloor \frac{n}{2} \rfloor - 1, \\ \{k(i+1), N-k-1\} & \text{for } i = \lfloor \frac{n}{2} \rfloor - 1, \\ \{k(i+1), k-1\} & \text{for } i \in [\lfloor \frac{n}{2} \rfloor + 1, n-1]. \end{cases}$$

$$\bar{S}_{\phi'}(y_j) = \begin{cases} \{k(n-j-1), N-1\} & \text{for } j \in [0, \lfloor \frac{n}{2} \rfloor - 2], \\ \{k(n-j-1), N-k-1\} & \text{for } j = \lfloor \frac{n}{2} \rfloor - 1, \\ \{k(n-j-1), k-1\} & \text{for } j \in [\lfloor \frac{n}{2} \rfloor, n-2], \\ \{N-2, N-1\} & \text{for } j = n-1. \end{cases}$$

Thus ϕ' is a cyclic k-tuple ND-coloring of $K_{n,n}$ with $N = kn+2$ colors when $n \equiv 1 \bmod 2$. □

5 Concluding remarks

We have turned our attention on k-tuple ND-colorings of graphs or equivalently, to ND-colorings of k-uniform multigraphs (multigraphs where each edge has multiplicity k). However, it seems also interesting to study ND-colorings of non-uniform multigraphs. Going in this direction, we propose the following 'Vizing-like' conjecture for the ND-chromatic index of a (not necessarily uniform) multigraph. It extends the one given in [16] for graphs and is similar with the one given for the total chromatic number of a multigraph (see [9], Section 4.9):

Conjecture 10 *For any connected multigraph G of order at least three, $G \neq C_5$, and of multiplicity $\mu(G)$,*

$$\chi'_a(G) \leq \Delta(G) + \mu(G) + 1.$$

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A Cyclic 3-tuple ND-colorings of K_{12} and K_{14}

Contruction of the cyclic 3-tuple ND-coloring ϕ' given in the proof of Theorem 8: below are the matrices of the color intervals on the edges of K_{12} and K_{14} (to simplify, only the first color of each cyclic interval is given). Values increased by one are in bold; values decreased by one are underlined and values increased by two are overlined (and in bold). The column-vectors on the right are the sets of the two missing colors $\bar{S}(i)$ of each vertex i.

$$\phi'(K_{12}) = \begin{bmatrix}
 & 3 & 6 & 9 & 12 & 15 & 18 & 21 & 24 & \mathbf{28} & \mathbf{31} & \underline{34} \\
3 & & 9 & 12 & 15 & 18 & 21 & 24 & 27 & \mathbf{31} & \underline{34} & 6 \\
6 & 9 & & 15 & 18 & 21 & 24 & 27 & 30 & \underline{34} & 3 & 12 \\
9 & 12 & 15 & & \mathbf{22} & \mathbf{25} & \mathbf{28} & \mathbf{31} & 0 & 3 & 6 & \mathbf{19} \\
12 & 15 & 18 & \mathbf{22} & & \mathbf{28} & \mathbf{31} & 0 & 3 & 6 & 9 & \mathbf{25} \\
15 & 18 & 21 & \mathbf{25} & \mathbf{28} & & 0 & 3 & 6 & 9 & 12 & \mathbf{31} \\
18 & 21 & 24 & \mathbf{28} & \mathbf{31} & 0 & & 6 & 9 & 12 & 15 & 3 \\
21 & 24 & 27 & \mathbf{31} & 0 & 3 & 6 & & 12 & 15 & 18 & 9 \\
24 & 27 & 30 & 0 & 3 & 6 & 9 & 12 & & 18 & 21 & 15 \\
\mathbf{28} & \mathbf{31} & \underline{34} & 3 & 6 & 9 & 12 & 15 & 18 & & \mathbf{25} & \mathbf{22} \\
\mathbf{31} & \underline{34} & 3 & 6 & 9 & 12 & 15 & 18 & 21 & \mathbf{25} & & \mathbf{28} \\
\underline{34} & 6 & 12 & \mathbf{19} & \mathbf{25} & \mathbf{31} & 3 & 9 & 15 & \mathbf{22} & \mathbf{28} &
\end{bmatrix}, \quad \begin{bmatrix} \{2,27\} \\ \{2,30\} \\ \{2,33\} \\ \{18,34\} \\ \{21,34\} \\ \{24,34\} \\ \{27,34\} \\ \{30,34\} \\ \{33,34\} \\ \{2,21\} \\ \{2,24\} \\ \{2,18\} \end{bmatrix}$$

$$\phi'(K_{14}) = \begin{bmatrix}
 & 3 & 6 & 9 & 12 & 15 & 18 & 21 & 24 & 27 & \mathbf{31} & \mathbf{34} & \mathbf{37} & \underline{40} \\
3 & & 9 & 12 & 15 & 18 & 21 & 24 & 27 & 30 & \mathbf{34} & \mathbf{37} & \underline{40} & 6 \\
6 & 9 & & 15 & 18 & 21 & 24 & 27 & 30 & 33 & \mathbf{37} & \underline{40} & 3 & 12 \\
9 & 12 & 15 & & \mathbf{22} & \mathbf{25} & \mathbf{28} & \mathbf{31} & \mathbf{34} & \overline{\mathbf{38}} & 0 & 3 & 6 & \mathbf{19} \\
12 & 15 & 18 & \mathbf{22} & & \mathbf{28} & \mathbf{31} & \mathbf{34} & \overline{\mathbf{38}} & 0 & 3 & 6 & 9 & \mathbf{25} \\
15 & 18 & 21 & \mathbf{25} & \mathbf{28} & & \mathbf{34} & \mathbf{37} & 0 & 3 & 6 & 9 & 12 & \mathbf{31} \\
18 & 21 & 24 & \mathbf{28} & \mathbf{31} & \mathbf{34} & & 0 & 3 & 6 & 9 & 12 & 15 & \mathbf{37} \\
21 & 24 & 27 & \mathbf{31} & \mathbf{34} & \mathbf{37} & 0 & & 6 & 9 & 12 & 15 & 18 & 3 \\
24 & 27 & 30 & \mathbf{34} & \overline{\mathbf{38}} & 0 & 3 & 6 & & 12 & 15 & 18 & 21 & 9 \\
27 & 30 & 33 & \overline{\mathbf{38}} & 0 & 3 & 6 & 9 & 12 & & 18 & 21 & 24 & 15 \\
\mathbf{31} & \mathbf{34} & \mathbf{37} & 0 & 3 & 6 & 9 & 12 & 15 & 18 & & \mathbf{25} & \mathbf{28} & \mathbf{22} \\
\mathbf{34} & \mathbf{37} & \underline{40} & 3 & 6 & 9 & 12 & 15 & 18 & 21 & \mathbf{25} & & \mathbf{31} & \mathbf{28} \\
\mathbf{37} & \underline{40} & 3 & 6 & 9 & 12 & 15 & 18 & 21 & 24 & \mathbf{28} & \mathbf{31} & & \mathbf{34} \\
\underline{40} & 6 & 12 & \mathbf{19} & \mathbf{25} & \mathbf{31} & \mathbf{37} & 3 & 9 & 15 & \mathbf{22} & \mathbf{28} & \mathbf{34} &
\end{bmatrix}, \quad \begin{bmatrix} \{2,30\} \\ \{2,33\} \\ \{2,36\} \\ \{18,37\} \\ \{21,37\} \\ \{24,40\} \\ \{27,40\} \\ \{30,40\} \\ \{33,37\} \\ \{36,37\} \\ \{21,40\} \\ \{2,24\} \\ \{2,27\} \\ \{2,18\} \end{bmatrix}$$

B Cyclic 3-tuple ND-colorings of $K_{12,12}$ and $K_{11,11}$

Contruction of the cyclic 3-tuple ND-coloring ϕ' given in the proof of Theorem 9: below are the matrices of the color intervals on the edges of $K_{12,12}$ and $K_{11,11}$ (to simplify, only the first color of each cyclic interval is given). Values increased by one are in bold; values decreased by one are underlined and values increased by two are overlined (and in bold). The two column-vectors on the right of each matrix are the sets of the two missing colors of the vertices x_i and y_j.

$$\phi'(K_{12,12}) = \left[\begin{array}{ccccccccccc|c}
\mathbf{34} & \mathbf{31} & \mathbf{28} & \mathbf{25} & \mathbf{22} & \mathbf{19} & \mathbf{16} & \mathbf{13} & \mathbf{10} & \mathbf{7} & \mathbf{4} & 0 \\
0 & \mathbf{34} & \mathbf{31} & \mathbf{28} & \mathbf{25} & \mathbf{22} & \mathbf{19} & \mathbf{16} & \mathbf{13} & \mathbf{10} & \mathbf{7} & 3 \\
3 & 0 & \mathbf{34} & \mathbf{31} & \mathbf{28} & \mathbf{25} & \mathbf{22} & \mathbf{19} & \mathbf{16} & \mathbf{13} & \mathbf{10} & 6 \\
6 & 3 & 0 & \mathbf{34} & \mathbf{31} & \mathbf{28} & \mathbf{25} & \mathbf{22} & \mathbf{19} & \mathbf{16} & \mathbf{13} & 9 \\
9 & 6 & 3 & 0 & \mathbf{34} & \mathbf{31} & \mathbf{28} & \mathbf{25} & \mathbf{22} & \mathbf{19} & \mathbf{16} & 12 \\
12 & 9 & 6 & 3 & 0 & \mathbf{34} & \mathbf{31} & \mathbf{28} & \mathbf{25} & \mathbf{22} & \mathbf{19} & 15 \\
15 & 12 & 9 & 6 & 3 & \underline{\mathit{37}} & \mathbf{34} & \mathbf{31} & \mathbf{28} & \mathbf{25} & \mathbf{22} & 18 \\
18 & 15 & 12 & 9 & 6 & 3 & \underline{\mathit{37}} & \mathbf{34} & \mathbf{31} & \mathbf{28} & \mathbf{25} & 21 \\
21 & 18 & 15 & 12 & 9 & 6 & 3 & \underline{\mathit{37}} & \mathbf{34} & \mathbf{31} & \mathbf{28} & 24 \\
24 & 21 & 18 & 15 & 12 & 9 & 6 & 3 & \underline{\mathit{37}} & \mathbf{34} & \mathbf{31} & 27 \\
27 & 24 & 21 & 18 & 15 & 12 & 9 & 6 & 3 & \underline{\mathit{37}} & \mathbf{34} & 30 \\
30 & 27 & 24 & 21 & 18 & 15 & 12 & 9 & 6 & 3 & \underline{\mathit{37}} & 33
\end{array}\right], \left[\begin{array}{c}
\{3,37\} \\ \{6,37\} \\ \{9,37\} \\ \{12,37\} \\ \{15,37\} \\ \{18,37\} \\ \{2,21\} \\ \{2,24\} \\ \{2,27\} \\ \{2,30\} \\ \{2,33\} \\ \{2,36\}
\end{array}\right], \left[\begin{array}{c}
\{33,37\} \\ \{30,37\} \\ \{27,37\} \\ \{24,37\} \\ \{21,37\} \\ \{2,18\} \\ \{2,15\} \\ \{2,12\} \\ \{2,9\} \\ \{2,6\} \\ \{2,3\} \\ \{36,37\}
\end{array}\right]$$

$$\phi'(K_{11,11}) = \left[\begin{array}{cccccccccc|c}
\mathbf{31} & \mathbf{28} & \mathbf{25} & \mathbf{22} & \mathbf{19} & \mathbf{16} & \mathbf{13} & \mathbf{10} & \mathbf{7} & \mathbf{4} & 0 \\
0 & \mathbf{31} & \mathbf{28} & \mathbf{25} & \mathbf{22} & \mathbf{19} & \mathbf{16} & \mathbf{13} & \mathbf{10} & \mathbf{7} & 3 \\
3 & 0 & \mathbf{31} & \mathbf{28} & \mathbf{25} & \mathbf{22} & \mathbf{19} & \mathbf{16} & \mathbf{13} & \mathbf{10} & 6 \\
6 & 3 & 0 & \mathbf{31} & \mathbf{28} & \mathbf{25} & \mathbf{22} & \mathbf{19} & \mathbf{16} & \mathbf{13} & 9 \\
9 & 6 & 3 & 0 & \overline{\mathbf{32}} & \mathbf{28} & \mathbf{25} & \mathbf{22} & \mathbf{19} & \mathbf{16} & 12 \\
12 & 9 & 6 & 3 & 0 & \mathbf{31} & \mathbf{28} & \mathbf{25} & \mathbf{22} & \mathbf{19} & 15 \\
15 & 12 & 9 & 6 & 3 & \underline{\mathit{34}} & \mathbf{31} & \mathbf{28} & \mathbf{25} & \mathbf{22} & 18 \\
18 & 15 & 12 & 9 & 6 & 3 & \underline{\mathit{34}} & \mathbf{31} & \mathbf{28} & \mathbf{25} & 21 \\
21 & 18 & 15 & 12 & 9 & 6 & 3 & \underline{\mathit{34}} & \mathbf{31} & \mathbf{28} & 24 \\
24 & 21 & 18 & 15 & 12 & 9 & 6 & 3 & \underline{\mathit{34}} & \mathbf{31} & 27 \\
27 & 24 & 21 & 18 & 15 & 12 & 9 & 6 & 3 & \underline{\mathit{34}} & 30
\end{array}\right], \left[\begin{array}{c}
\{3,34\} \\ \{6,34\} \\ \{9,34\} \\ \{12,34\} \\ \{15,31\} \\ \{18,34\} \\ \{2,21\} \\ \{2,24\} \\ \{2,27\} \\ \{2,30\} \\ \{2,33\}
\end{array}\right], \left[\begin{array}{c}
\{30,34\} \\ \{27,34\} \\ \{24,34\} \\ \{21,34\} \\ \{18,31\} \\ \{2,15\} \\ \{2,12\} \\ \{2,9\} \\ \{2,6\} \\ \{2,3\} \\ \{33,34\}
\end{array}\right]$$