Adjacent Vertex Distinguishing Edge-Colorings of Meshes

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Abstract

This paper studies the minimum number of colors for an edge-coloring of a graph such that adjacent vertices are distinguished by their color sets (colors of edges that are incident to them). Such a coloring is called adjacent vertex distinguishing. We find optimal adjacent vertex distinguishing colorings of the multidimensional mesh (toroidal or not) and of the hypercube. We show that for both graphs, this number of colors is equal to the maximum degree of the graph plus one.

Keywords : graph, edge-coloring, adjacent-vertex-distinguishing, observability, mesh, hypercube.

1 Introduction

All graphs we deal with are undirected, simple and connected.

Let $G = (V, E)$ be a graph with vertex set V and edge set E.

A *proper edge coloring* c is a mapping from E to $\mathbb{N}$ satisfying $c(xy) \neq c(yz)$, for any $xy, yz \in E$. For any vertex $x \in V$, let $S(x)$ denote the set of the colors of all edges incident to x.

As defined in [BHBLW99, BRS03], a proper edge coloring c is said to be *vertex distinguishing* if $S(x) \neq S(y)$, for any $x, y \in V$, $x \neq y$. The minimum number of colors among all vertex distinguishing colorings of a graph is called the *observability* of the graph in [ČHS96].

In this note, we study a relaxed version of this parameter in which only adjacent vertices have to be distinguished by their color sets (see [BGLS, ZLW02]). Formally, an *adjacent vertex distinguishing* edge coloring is a proper edge-coloring satisfying $S(x) \neq S(y)$, for any x, y with $xy \in E$. Let $\chi'_a(G)$ denote the minimum number of colors of any AVD-coloring of G.

Few is known about AVD-colorings. In [BGLS], the authors prove that $\chi'_a(G) \leq 5$ for graphs of maximum degree 3 and $\chi'_a(G) \leq \Delta(G) + 2$ for bipartite graphs.

The following conjecture was made in [ZLW02]:

Conjecture 1 *Let G be a connected graph of maximum degree Δ and different from C_5 and K_2. Then*

$$\Delta \leq \chi'_a(G) \leq \Delta + 2.$$

Notice that $\chi'_a(G)$ is obviously greater than Δ and if G has two adjacent vertices of degree Δ then $\chi'_a(G) \geq \Delta + 1$.

The purpose of this paper is to find AVD-colorings of (toroidal or simple) multidimensional meshes and of hypercubes, with as few colors as possible. In fact we shall present stronger results that bound the number of colors needed for an AVD-coloring of the Cartesian product of a graph by a path or by a cycle. These results allow us to compute the exact value of the parameter χ'_a.

The *d-dimensional mesh* $M_{n_1,n_2,\ldots,n_d}$ is the Cartesian product of d paths: $M_{l_1,l_2,\ldots,l_d} = P_{l_1} \times P_{l_2} \times \ldots \times P_{l_d}$.

The *d-dimensional toroidal mesh* $TM_{l_1,l_2,\ldots,l_d}$ is the Cartesian product of d cycles: $TM_{l_1,l_2,\ldots,l_d} = C_{l_1} \times C_{l_2} \times \ldots \times C_{l_d}$.

The *d-dimensional hypercube* H_d is the Cartesian product of K_2 by itself d times: $H_d = H_{d-1} \times K_2$, $H_1 = K_2$.

We will use the following notation all along the paper: The 2-dimensional mesh $M_{m,n}$ has vertex set $V = \{x_{i,j}, 0 \leq i \leq m-1, 0 \leq j \leq n-1\}$ and edge set $E = \{x_{i,j}x_{i,j+1}, 0 \leq i \leq m-1, 0 \leq j \leq n-2\} \cup \{x_{i,j}x_{i+1,j}, 0 \leq i \leq m-2, 0 \leq j \leq n-1\}$.

If G is the mesh $M_{m,n}$ then we denote by $\mathcal{V}_k$ the sequence of vertical edges $\mathcal{V}_k = (x_{k-1,0}x_{k,0}, \ldots, x_{k-1,n-1}x_{k,n-1})$ with $1 \leq k \leq m-1$ and for an edge-coloring c of G, we will write $c(\mathcal{V}_k) = (c_1, c_2, \ldots, c_{n-1})$ to mean that the first edge of $\mathcal{V}_k$ is assigned the color c_1, the second the color c_2, and so on.

Similarly, we denote by $\mathcal{H}_k$ the sequence of horizontal edges $\mathcal{H}_k = (x_{k-1,0}x_{k-1,1}, \ldots, x_{k-1,n-2}x_{k-1,n-1})$ with $1 \leq k \leq m$ and for an edge-coloring c of G, we will write $c(\mathcal{H}_k) = (c_1, c_2, \ldots, c_{n-1})$ to denote the colors of the edges of $\mathcal{H}_k$.

We define the sequences of edges of the toroidal mesh $TM_{m,n}$ as we did for the mesh $M_{m,n}$ but in that case, we have the sequence $\mathcal{V}_0 = (x_{m-1,0}x_{0,0}, \ldots, x_{m-1,n-1}x_{0,n-1})$ and one edge more at the beginning of each sequence $\mathcal{H}_k$: $\mathcal{H}_k = (x_{k-1,n-1}x_{k-1,0}, \ldots)$.

2 AVD-colorings of 2-dimensional meshes

Let's begin by the following lemmas :

Lemma 1 *Let $M_{m,n}$ be a 2-dimensional mesh, with $n, m \geq 3$, $n+m \geq 7$. Then* $\chi'_a(M_{m,n}) = 5$.

Proof : We define the coloring c of $\mathcal{H}_k$ $(1 \leq k \leq m)$ of the mesh $M_{m,n}$ by:

$$\begin{cases} c(\mathcal{H}_1) & = (4,3,4,3,\ldots), \\ c(\mathcal{H}_i) & = (c(\mathcal{H}_{i-1})+1) \bmod 5 \text{ for } 2 \leq i \leq m, \end{cases}$$

and the coloring of $\mathcal{V}_k$ $(1 \leq k \leq m-1)$ by:

$$\begin{cases} c(\mathcal{V}_1) & = (1,2,1,2,\ldots), \\ c(\mathcal{V}_i) & = (c(\mathcal{V}_{i-1})+1) \bmod 5 \text{ for } 2 \le i \le m-1. \end{cases}$$

See Figure 1 for an illustration of this coloring.

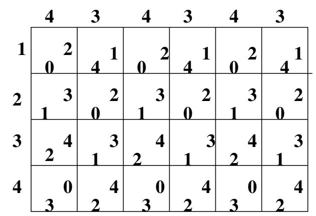

Figure 1: An AVD-coloring for the mesh $M_{5,7}$.

First, one can see that, by construction, c is a proper coloring. It remains to show that c is an AVD-coloring. Notice that for any two vertices x and y, if $d(x) \neq d(y)$ then $S(x) \neq S(y)$. Thus, we only have to compare the sets of colors of adjacent vertices of the same degree.

- **Case 1**: degree 2 vertices. Since there is only four vertices of degree 2 and since they form an independent set in $M_{m,n}$, there is no problem with these vertices.

- **Case 2**: degree 3 vertices. Remark that for any i, $1 \le i \le m - 3, S(x_{i,0}) = \{\alpha, \alpha+1, \beta\}$ and $S(x_{i+1,0}) = \{\alpha+1, \alpha+2, \beta+1\}$ for some $\alpha, \beta \in \{0,1,2,3,4\}$, where addition is taken modulo 5. This proves that $S(x_{i,0}) \neq S(x_{i+1,0})$. A similar argument yields $S(x_{i,n-1}) \neq S(x_{i+1,n-1})$. For two adjacent vertices u and v on the horizontal border, the two horizontal edges incident to u have the same color as the two ones incident to v. But the colors of the verticals edges of u and v are different. Thus the sets of colors are different.

- **Case 3**: degree 4 vertices. By construction, we have $S(x_{i,j}) = \{\alpha, \alpha+1, \beta, \beta+1\}$ for some $\alpha, \beta \in \{0,1,2,3,4\}$, where addition is taken modulo 5. Moreover, $S(x_{i,j+1}) = \{\alpha+1, \alpha+2, \beta, \beta+1\}$ if j is even and $S(x_{i,j+1}) = \{\alpha-1, \alpha, \beta, \beta+1\}$ if j is odd. Therefore, $S(x_{i,j}) \neq S(x_{i,j+1})$.

  On the other hand, $S(x_{i+1,j}) = \{\alpha+1, \alpha+2, \beta+1, \beta+2\}$. Thus, $S(x_{i,j}) = S(x_{i+1,j})$ if $\{\alpha, \beta\} = \{\alpha+2, \beta+2\}$ which is impossible in $\mathbb{Z}_5$.

Hence c is an AVD-coloring.

□

In the two following propositions, we give the exact values of $\chi'_a(TM_{m,n})$ for any m and n.

Proposition 1 *Let $m, n \in \mathbb{Z}$, $m, n \ge 3$. If m or n is even, then $\chi'_a(TM_{m,n}) = 5$.*

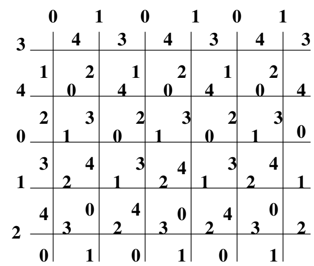

Figure 2: An AVD-coloring for $TM_{5,6}$.

Proof : Without loss of generality, we assume that n is even. We discuss five cases following values of m:

- **Case 1**: $m \equiv 0 \bmod 5$. Let us construct the following edge-coloring for any $TM_{m,n}$ (see Figure 2 for $TM_{5,6}$) :

  $c(\mathcal{H}_i) = (i+2, i+3, i+2, i+3, \ldots, i+3, i+2)$, for each $1 \leq i \leq m$, where $i+j$ is taken modulo 5 with $2 \leq j \leq 3$.

  $c(\mathcal{V}_i) = (i, i+1, i, i+1, \ldots, i, i+1)$, for each $0 \leq i \leq m-1$, where $i+j$ is taken modulo 5 with $0 \leq j \leq 1$.

  Observe that in a such coloring, for each $uv \in E(TM_{m,n})$, there exists always one color $j \in S(v)$ such that $j \notin S(u)$. Therefore c is an AVD-coloring.

- **Case 2**: $m \equiv 3 \bmod 5$. An AVD-coloring of $TM_{3,n}$ is illustrated in B_2 of Figure 3. For $m > 3$, by Case 1, we obtain an AVD-coloring of $TM_{m-3,n}$ such that $\chi'_a(TM_{m-3,n}) = 5$. To construct an AVD-coloring of $TM_{m,n}$, we extend the AVD-coloring of $TM_{m-3,n}$ by joining to it the AVD-coloring of $TM_{3,n}$ (see Figure 3).

- **Case 3**: $m \equiv 4 \bmod 5$. An AVD-coloring of $TM_{4,n}$ is illustrated in Figure 4. For $m > 4$, by Case 1, we obtain an AVD-coloring of $TM_{m-4,n}$ such that $\chi'_a(TM_{m-4,n}) = 5$. To construct an AVD-coloring of $TM_{m,n}$, we extend the AVD-coloring of $TM_{m-4,n}$ by joining to it the AVD-coloring of $TM_{4,n}$ (it is obtained by replacing in Figure 3 the block B_2 by the coloring illustrated in Figure 4).

- **Case 4**: $m \equiv 1 \bmod 5$. An AVD-coloring of $TM_{6,n}$ is given in Figure 5. For $m > 6$, by Case 1, we obtain an AVD-coloring of $TM_{m-6,n}$ such that $\chi'_a(TM_{m-6,n}) = 5$. To construct an AVD-coloring of $TM_{m,n}$, we extend the AVD-coloring of $TM_{m-6,n}$ by joining to it (as it is done in above cases) the AVD-coloring of $TM_{6,n}$.

- **Case 5**: $m \equiv 2 \bmod 5$. Let $m = m' + m''$ such that $m' \equiv 3 \bmod 5$ and $m'' \equiv 4 \bmod 5$. By Cases 2 and 3, we obtain the AVD-colorings of $TM_{m',n}$

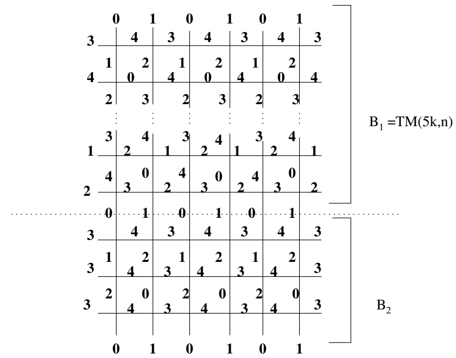

Figure 3: An AVD-coloring for $TM_{5k+3,6}$, with $k \geq 1$.

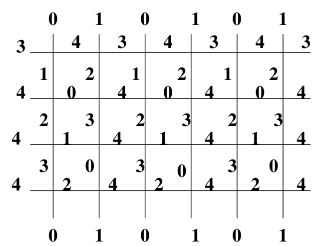

Figure 4: An AVD-coloring for $TM_{4,6}$.

and $TM_{m'',n}$ respectively such that $\chi'_a(TM_{m',n}) = 5$ et $\chi'_a(TM_{m'',n}) = 5$. As in precedent cases, one can construct an AVD-coloring for $TM_{m,n}$ by joining together those of $TM_{m',n}$ and $TM_{m'',n}$.

In the following proposition, we study the 2-dimensional toroidal mesh when m and n are odd.

Proposition 2 *Let $m, n \in \mathbb{Z}$, $m, n \geq 3$. If m and n are odd then* $\chi'_a(TM_{m,n}) = 5$.

Proof : See Figure 6 for $m = n = 3$. The case $m = 9$ and $n = 3$ is obtained by joining three copies of $TM_{3,3}$ colored according to Figure 6. The case $m = n = 9$ is obtained by joining three copies of $TM_{9,3}$.

Now, we decompose the rest of the proof into five cases:

- **Case 1:** $m \equiv 0 \bmod 5$. The AVD-coloring of $TM_{m,n}$ is obtained as follows, for $1 \leq i \leq m-1$ and $1 \leq j \leq m$:

$$\begin{cases} c(\mathcal{V}_0) & = (0,1,0,1,0,\ldots,1,0) \\ c(\mathcal{V}_i) & = (c(\mathcal{V}_{i-1})+1) \bmod 5 \\ c(\mathcal{H}_j) & = (j+1, j+3, j+2, \ldots, j+3, j+2) \bmod 5 \end{cases}$$

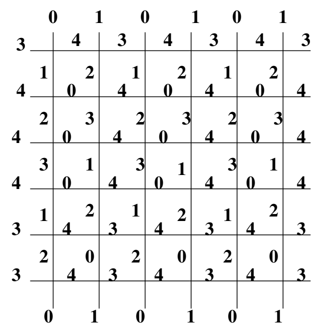

Figure 5: An AVD-coloring for $TM_{6,6}$.

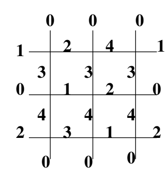

Figure 6: An AVD-coloring for $TM_{3,3}$.

See Figure 7 for an example.

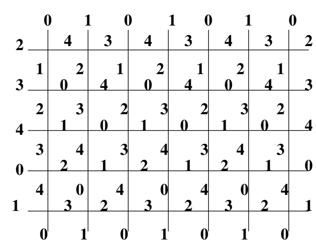

Figure 7: An AVD-coloring of $TM_{5,7}$.

- **Case 2:** $m \equiv 1 \bmod 5$. For $m > 6$, the AVD-coloring of $TM_{m,n}$ results from joining one copy of $TM_{m-6,n}$ given by Case 1 with one copy of $TM_{6,n}$ colored according to Figure 8.
- **Case 3:** $m \equiv 2 \bmod 5$. For $m = 7$, an AVD-coloring is illustrated in Figure 9. For $m > 7$, we join one copy $TM_{m-7,n}$ with the coloring given by Case 1 with one copy of $TM_{7,n}$ colored according to Figure 9.
- **Case 4:** $m \equiv 3 \bmod 5$. The cases $m = n = 3$ and $m = 3$, $n = 9$ are solved in the beginning of the present proof. The case $m = 3$, $n = 5$ can

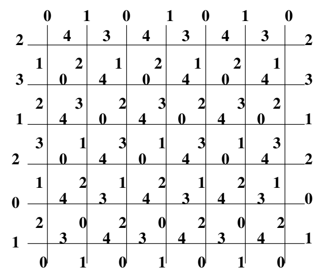

Figure 8: An AVD-coloring of $TM_{6,7}$.

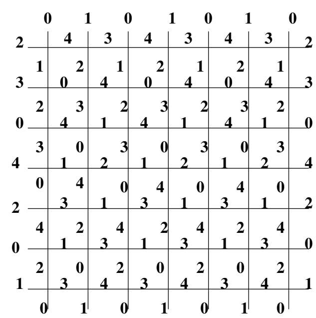

Figure 9: An AVD-coloring of $TM_{7,7}$.

be solved with Case 1 if we take $m = 5$ and $n = 3$. The case $m = 3$, $n = 7$ is similar to Case 3 after permuting m and n. Similarly, when $m = 3$, $n = 11$ by permuting m and n, we solve the case as it is given in Case 2.

So, for $m \geq 13$, we join one copy of $TM_{m-13,n}$ (Case 1) with one copy of $TM_{6,n}$ (Case 2) and one copy of $TM_{7,n}$ (Case 3).

- **Case 5:** $m \equiv 4 \bmod 5$. The beginning of the present proof gives the case $m = n = 9$ and the other cases when $m = 9$ are solved in the precedent cases. So, for $m \geq 14$, we join one copy of $TM_{m-14,n}$ (Case 1) with one copy of $TM_{14,n}$ (Case 3).

□

3 AVD-colorings of n-dimensional meshes and hypercubes

To find the minimum number of colors for an AVD-coloring of multidimensional meshes and hypercubes, we shall prove more general results concerning AVD-colorings of the Cartesian product of a graph by a path or a cycle.

Theorem 1 *Let $d \geq 2$ be an integer and let G be a graph of maximum degree $\Delta \leq d-1$. If there exists an AVD-coloring of G with d colors, then $\chi'_a(G \times P_2) \leq d+1$.*

Proof : Let $d = \chi'_a(G)$. The graph $G' = G \times P_2$ consists of two copies G_0 and G_1 of G and a perfect matching between them. Take an AVD-coloring c of G with colors $0, 1, \ldots, d-1$. For an edge e (a vertex u, respectively) of G, let e_i (u_i respectively) be the corresponding edge (vertex respectively) of G_i for $i = 0, 1$. A coloring c' of G' is defined as follows:

for each $e \in E(G)$,

$$c'(e_0) = c(e) \text{ and } c'(e_1) = (c(e) + 1) \bmod d,$$

and for each $u \in V(G)$,

$$c'(u_0 u_1) = d.$$

It is routine to see that c' is an AVD-coloring. □

Theorem 2 *Let $d \geq 2$ be an integer and let G be a graph of maximum degree $\Delta \leq d-1$ and of minimum degree $\delta \geq 2$. If there exists an AVD-coloring of G with d colors, then $\chi'_a(G \times C_k) \leq d+2$.*

Proof : Let c be an AVD-coloring of G with the d colors $0, 1, \ldots, d-1$. Let $G' = G \times C_k$. The graph G' consists in k copies of G, say $G_0, G_1, \ldots, G_{k-1}$; with for each $i, 0 \leq i \leq k-1$, a perfect matching linking each vertex of copy G_i with the corresponding vertex of G_{i+1} (addition modulo k). For an edge e (a vertex u, respectively) of G, let e_i ($u_i$respectively) be the corresponding edge (vertex respectively) in G_i.

Case 1: k is even. The coloring c' of G' is described below:

for each $e \in E(G)$, for each $i, 0 \leq i \leq \frac{k-2}{2}$, set

$$\begin{cases} c'(e_{2i}) = c(e), \\ c'(e_{2i+1}) = (c(e) + 1) \bmod d. \end{cases}$$

For the edges of the perfect matchings, we add two new colors d and $d+1$: for each $u \in V(G)$, for each $j, 0 \leq j \leq \frac{k-2}{2}$, set

$$\begin{cases} c'(u_{2j} u_{2j+1}) = d+1, \\ c'(u_{2j+1} u_{2j+2}) = d. \end{cases}$$

An illustration is given in Figure 10.

Now, let us prove that c' is an AVD-coloring: First notice that as c is an AVD-coloring of G, c' is an AVD-coloring of each copy G_i. Next, let $j \in \{0, 1, \ldots d-1\}$ be a color such that $j \notin S(u_{2i})$ and $(j-1) \bmod d \in S(u_{2i})$ (there always exists one). Then, by the definition of the coloring c', we have $j \in S(u_{2i+1})$, which proves that these two vertices have distinct sets of colors.

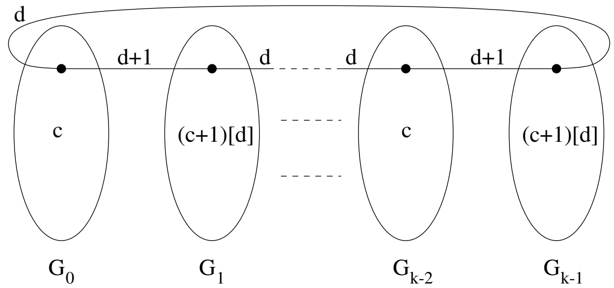

Figure 10: The coloring c' of G' when k is even.

Case 2: k is odd. The coloring c' of G' is constructed first from the coloring given in Case 1 for the copies $G_0, G_1, \ldots, G_{k-3}$, i-e:
for each $e \in E(G)$, set

$$\begin{cases} c'(e_{2i}) = c(e), & 0 \le i \le \frac{k-3}{2} \\ c'(e_{2i+1}) = (c(e)+1) \bmod d, & 0 \le i \le \frac{k-5}{2}, \end{cases}$$

and for each $u \in V(G)$, set

$$\begin{cases} c'(u_{2j}u_{2j+1}) = d+1, & 0 \le j \le \frac{k-3}{2} \\ c'(u_{2j+1}u_{2j+2}) = d, & 0 \le j \le \frac{k-5}{2}. \end{cases}$$

Then for the edges of G_{k-2}, set

$$\begin{cases} c'(e_{k-2}) = c(e)+1 \text{ (without modulo) for } d \text{ even} \\ c'(e_{k-2}) = \tau(c(e)) \text{ for } d \text{ odd} \end{cases}$$

where τ is a bijection from $\{0, 1, \ldots d-1\}$ onto $\{1, 2, \ldots d-1, d\}$ defined by:

$$\tau(i) = \begin{cases} i-1, & \text{for } 4 \le i \le d-3 \text{ or } i = 2 \\ d-2, & \text{for } i = 0 \\ d-1, & \text{for } i = 1 \\ d-3, & \text{for } i = 3 \\ 2, & \text{for } i = d-2 \\ d, & \text{for } i = d-1 \end{cases} \qquad \text{for } d \ge 5,$$

and $\tau(0) = 2$, $\tau(1) = 3$, $\tau(2) = 1$ for $d = 3$.
For G_{k-1}, set

$$c'(e_{k-1}) = \sigma(c(e)),$$

where σ is a bijection from $\{0, 1, \ldots d-1\}$ onto $\{1, 2, \ldots d-1, d+1\}$ defined by:

$$\sigma(i) = \begin{cases} i+2, & \text{for } 0 \le i \le d-3 \\ 1, & \text{for } i = d-2 \\ d+1, & \text{for } i = d-1 \end{cases} \qquad \text{for } d \text{ even and,}$$

$$\sigma(i) = \begin{cases} i+1, & \text{for } 1 \le i \le d-5 \text{ or } i = d-3 \\ d-3, & \text{for } i = 0 \\ d-1, & \text{for } i = d-4 \\ d+1, & \text{for } i = d-1 \\ 1, & \text{for } i = d-2 \end{cases} \qquad \text{for } d \text{ odd, } d \neq 3,$$

and $\sigma(0) = 1$, $\sigma(1) = 2$, $\sigma(2) = 4$ for $d = 3$.
For each edge between copies G_{k-2}, G_{k-1} and G_0, set

$$\begin{cases} c'(u_{k-2}u_{k-1}) = 0, \\ c'(u_{k-1}u_0) = d. \end{cases}$$

An illustration is given in Figure 11 when d is even.

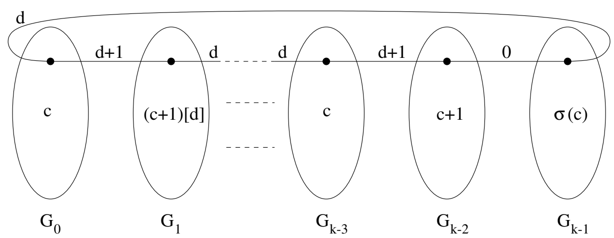

Figure 11: The coloring c' of G' when k is odd and d is even.

Now, let us prove that c' is an AVD-coloring when d is even. First, as for the previous case, it is not difficult to see that since c is an AVD-coloring of G, then c' is an AVD-coloring of each copy G_i (in particular for the copy G_{k-1}, as σ is a bijection). Next, notice that a vertex u_i of copy G_i is distinguished from a vertex u_{i+1} of copy G_{i+1}, for $0 \le i \le k-4$ since the coloring for this part of the graph G' is similar to the one defined for even k. Thus we only have to check that two adjacent vertices from copies G_{k-3}, G_{k-2}, G_{k-1} and G_0 receive distinct sets of colors: let u be a vertex of G and let $S = S_c(u)$ denote its set of colors with the coloring c.

Let $j \in \{0, 1, \ldots d-1\}$ be a color such that $j \notin S(u_{k-3})$ and $j+1 \bmod d \in S(u_{k-3})$. Then, by the definition of the coloring c' we have, $S(u_{k-2}) \neq S(u_{k-3})$ since $j+1 \bmod d \in S(u_{k-3})$ but $j+1 \notin S(u_{k-2})$.

By contradiction, we are going to show that a vertex u_{k-2} is distinguished from its neighbor u_{k-1}. Let $S' = \{s+1 | \; s \in S\}$ and $S_\sigma = \{\sigma(s) | \; s \in S\}$. Assume that $S(u_{k-2}) = S(u_{k-1})$, i-e. $S' \cup \{0, d+1\} = S_\sigma \cup \{0, d\}$. Then we have $d+1 \in S_\sigma$ and thus $d-1 \in S$ i-e. $d \in S'$. Now, as $\delta(G) \ge 2$ by hypothesis, there exists $j, 0 \le j \le d-2$ such that $j \in S$. If $j = d-2$ then $1 \in S_\sigma$ and

$1 \in S_\sigma \Rightarrow 1 \in S' \Rightarrow 0 \in S \Rightarrow \sigma(0) = 2 \in S_\sigma \Rightarrow 2 \in S' \Rightarrow \ldots \Rightarrow d-1 \in S$.

We obtain $S = \{0, 1, \ldots, d-1\}$, which contradicts the hypothesis that $\Delta(G) \le d-1$. If $j < d-2$, the argument is similar.

To show that a vertex u_{k-1} is distinguished from its neighbor u_0, we proceed in a similar way. Assume that $S(u_{k-1}) = S(u_0)$, i-e. that $S_\sigma \cup \{0, d\} = S \cup \{d, d+1\}$. This implies $0 \in S$ and when d is even $0 \in S \Rightarrow \sigma(0) = 2 \in S \Rightarrow \sigma(2) = 4 \in S \Rightarrow \ldots \Rightarrow \sigma(d-4) = d-2 \in S \Rightarrow \sigma(d-2) = 1 \in S \Rightarrow \sigma(1) = 3 \in S \Rightarrow \ldots \Rightarrow \sigma(d-3) = d-1 \in S$.

Thus $S = \{0, 1, \ldots, d-1\}$, a contradiction.

If d is odd the argument is similar.

In both cases, we have that c' is an AVD-coloring of G' with $d+2$ colors. □

Theorem 3 *Let $d \geq 2$ be an integer and let G be a graph of maximum degree $\Delta \leq d-1$. If there exists an AVD-coloring of G with d colors, then $\chi'_a(G \times P_k) \leq d+2$.*

Proof : It is easy to see that the coloring c' given in the proof of Theorem 2 for even k also works for $G \times P_{k'}$ ($k' \in \{k, k-1\}$) since vertices are distinguished in each copy G_i, and $G \times P_{k'}$ is obtained from $G \times C_k$ by deleting the edges (which have the same color) of the perfect matching between copies G_0 and G_{k-1}; and if k' is odd we also delete the copy G_{k-1} with all incident edges. □

Corollary 1 *For the hypercube H_d of dimension $d \geq 3$, we have*

$$\chi'_a(H_d) = d+1.$$

Proof : Figure 12 gives an AVD-coloring of H_3 with 4 colors. As $H_d = H_{d-1} \times P_2$, then by Theorem 1 we obtain the result for $d \geq 4$. □

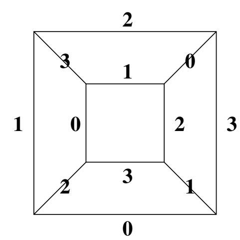

Figure 12: An AVD-coloring of H_3.

If we start with the AVD-coloring of the 2-dimensional toroidal mesh given by Proposition 1 or Proposition 2 and if we apply inductively Theorem 2, we obtain the following result for the toroidal mesh.

Corollary 2 *For the k-dimensional toroidal mesh $TM_{m_1,m_2,\ldots,m_k}$, with $k \geq 2$ and $m_i \geq 3$, we have*

$$\chi'_a(TM_{m_1,m_2,\ldots,m_k}) = 2k+1.$$

Remark 1 *Observe that $\chi'_a(C_{3m}) = 3$. Therefore, in this case, one can start the induction from only the cycle (instead of the 2-dimensional toroidal mesh).*

Theorem 3 together with Lemma 1 give the following result for the mesh.

Corollary 3 *For the k-dimensional mesh $M_{m_1,m_2,\ldots,m_k}$, with $k \geq 2$ and $m_i > 3$, we have*

$$\chi'_a(M_{m_1,m_2,\ldots,m_k}) = 2k + 1.$$

Finally one can give a more general result from the above Theorems.

Theorem 4 *Let $G = G_1 \times G_2 \times \ldots \times G_k$ be a general mesh with $k \geq 2$, where each G_i is either a path or a cycle. Then*
(i) $\chi'_a(G) = \Delta(G)$ iff $G_i = P_3$ for $i = 1, 2, \ldots, k$;
(ii) $\chi'_a(G) = \Delta(G) + 2$ iff $k = 2$ and $(G_1, G_2) \in \{(C_5, P_2), (P_2, P_2)\}$;
(iii) $\chi'_a(G) = \Delta(G) + 1$ in all remaining cases.

Proof : We prove this theorem by induction on k using Theorems 1, 2 and 3. The starting points of this induction is the Lemma 1, Propositions 1 and 2 and the following facts.

(a) $\chi'_a(M_{3,3}) = \chi'(M_{3,3}) = 4$ since in $M_{3,3}$ only vertices of distincts degrees are adjacent.

(b) $\chi'_a(C_5 \times P_2) = 5$: The inegality $\chi'_a(C_5 \times P_2) \leq 5$ follows from the colorings illustrated in Figure 13.

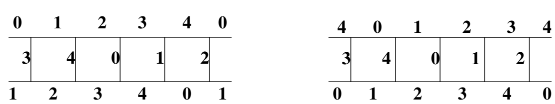

Figure 13: Two AVD-colorings of $C_5 \times P_2$.

Assume that $\chi'_a(C_5 \times P_2) \leq 4$, i.e. $\chi'_a(C_5 \times P_2) = 4$ (the graph $C_5 \times P_2$ is cubic). As $\lfloor | E(C_5 \times P_2) | /4 \rfloor = \lfloor 15/4 \rfloor = 3$, there is a color set C of cardinality at most 3. Observe that the vertex independence number of $C_5 \times P_2$ is 4 and that a 4-vertex independent set is unique (up to an automorphism of $C_5 \times P_2$). If $| C | \leq 2$, the vertex set $W := V(C_5 \times P_2) \backslash V(C)$ is of cardinality at least 6, and so there is an edge $w_1 w_2$ with $w_1, w_2 \in W$. Then, however, w_1 and w_2 are not distinguished by their color sets (the color of the edges of C is missing at both w_1 and w_2), a contradiction. On the other hand, if $| C | = 3$, then $| W | = 4$ and W must be an independent set (otherwise we have a contradiction again).The graph $(C_5 \times P_2) \backslash W$ consists (see the observation above) of two components P_1 and one copmponent P_4; therefore, its vertex set cannot be covered by three independent edges of C, a contradiction.

(c) $\chi'_a(C_m \times P_2) = 4$ for any $m \neq 5$: This can be seen by induction on m. Indeed, for $m = 3, 4, 8$ an appropriate coloring of $C_m \times P_2$ is illustrated in Figure 14, sharing the "block" B of the first three "columns".

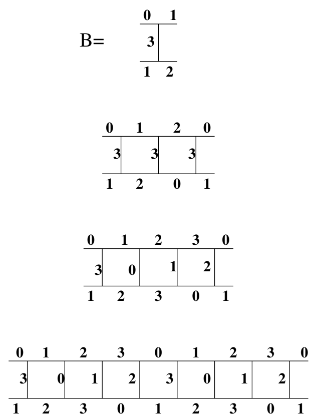

Figure 14: AVD-colorings for $C_m \times P_2$, $m = 3, 4, 8$.

A coloring of $C_{m+3} \times P_2$ is obtained from a coloring of $C_m \times P_2$ (containing B) when replacing B by the "block" illustrated in Figure 15 containing B as a "subblock" on both its left and right end.

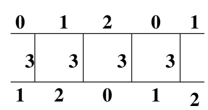

Figure 15: An AVD-coloring for $C_4 \times P_2$.

(d) $\chi'_a(P_2 \times P_2) = \chi'_a(C_4) = 4$ (according to [ZLW02]).
(e) $\chi'_a(C_5 \times P_2 \times P_2) = 5$: Use for the two vertex-disjoint copies of $C_5 \times P_2$ present in $C_5 \times P_2 \times P_2$ as induced subgraphs the two colorings illustrated in Figure 13.

Corresponding pairs of vertices joined by a "vertical" edge of $C_5 \times P_2$ have color sets $\{\alpha, \alpha+1, \alpha+3\}$ (top) and $\{\alpha+1, \alpha+2, \alpha+3\}$ (bottom) in the copy colored by the coloring on the left of Figure 13 with α increasing from 0 to 4 when going from the left to the right. Therefore, "horizontal" edges between corresponding vertices of the two copies of $C_5 \times P_2$ can be colored by $\alpha + 2$ (top) and $\alpha + 4$ (bottom). It is easy to check that an appropriate coloring of $C_5 \times P_2 \times P_2$ is obtained.

□

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4.

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