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Symmetries in Dyck paths with air pockets

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Abstract. The main objective of this paper is to analyze symmetric and asymmetric peaks in *Dyck paths with air pockets* (DAPs). These paths are formed by combining each maximal run of down-steps in ordinary Dyck paths into a larger, single down-step. To achieve this, we present a trivariate generating function that counts the number of DAPs based on their length and the number of symmetric and asymmetric peaks they contain. We determine the total numbers of symmetric and asymmetric peaks across all DAPs, providing an asymptotic for the ratio of these two quantities. Recursive relations and closed formulas are provided for the number of DAPs of length n, as well as for the total number of symmetric peaks, weight of symmetric peaks, and height of symmetric peaks. Furthermore, a recursive relation is established for the overall number of DAPs, similar to that for classic Dyck paths. A DAP is said to be *non-decreasing* if the sequence of ordinates of all local minima forms a non-decreasing sequence. In the last section, we focus on the sets of non-decreasing DAPs and examine their symmetric and asymmetric peaks.

Mathematics Subject Classification. 05A15, 05A19.

Keywords. Dyck path with air pockets, Non-decreasing Dyck path, Symmetric peak, Popularity.

1. Introduction

In their paper on lattice paths, Baril et al. [3] introduced a new type of lattice paths in the first quadrant of $\mathbb{Z}^2$, called *Dyck paths with air pockets* (DAPs). These paths start at the origin, end on the x-axis, and consist of up-steps $U = (1, 1)$ and down-steps $D_k = (1, -k)$, where $k \geq 1$, and no two down-steps can be consecutive. The *length* of a path P is the number of its steps, denoted by $|P|$. DAPs can be seen as a variation of ordinary Dyck paths where maximal runs of down-steps are replaced by one large down-step. (As remarked in [3], DAPs also correspond to a stack evolution with (partial) reset operations that cannot be consecutive, see [20].) The authors enumerate these paths and their prefixes with respect to the length, the type (up or down) of

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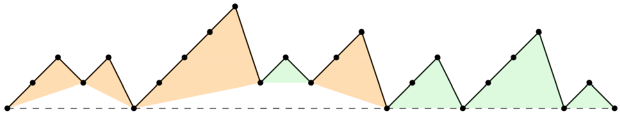

Figure 1. Symmetric and asymmetric peaks of a Dyck path with air pockets

the last step, and the ordinate of the endpoint. Furthermore, they establish a one-to-one correspondence between DAPs of length n and peak-less Motzkin paths of length $n-1$. In a subsequent paper, Baril et al. [4] generalized these paths by allowing them to go below the x-axis, calling them *grand Dyck paths with air pockets* (GDAP). They also provided enumerative results for these paths based on their length and various restrictions on their minimum and maximum ordinates. More recently, the definition of DAPs was extended to include horizontal steps under certain conditions, as described in [5].

Let $\mathcal{D}$ be the set of all DAPs and $\mathcal{D}_n$ be the set of DAPs of length n. Additionally, a special subset of these paths, namely those that are non-decreasing, is considered. A DAP is *non-decreasing* if, as the path is read from left to right, the sequence of the minimal ordinates of the valleys (where a *valley* is an occurrence of D_kU, $k \geq 1$) is non-decreasing. This concept has been previously studied in the literature (see, for example, [2,8,9]). The set of non-decreasing DAPs is denoted by $\mathcal{ND}$.

In this paper, the focus is on analyzing the distribution of symmetric and asymmetric peaks in DAPs and non-decreasing DAPs. A *maximal peak* is defined as an occurrence of U^kD_ℓ, where $k, \ell \geq 1$, that cannot be extended to $U^{k+1}D_\ell$. For the sake of simplicity, the term "peak" is used instead of "maximal peak". (In literature, this type of concept may also be referred to as a *maximal pyramid*.)

On the other hand, a *symmetric peak* is a peak (again, maximal) of the form $\Delta_k := U^kD_k$. An *asymmetric peak* is simply a peak that is not symmetric. We use $\mathsf{peak}(P)$, $\mathsf{sp}(P)$, and $\mathsf{ap}(P)$ to denote the total number of peaks, symmetric peaks, and asymmetric peaks, respectively, in a given path P.

In addition to these parameters, the height and weight of each peak are considered. The *height* of a peak is the maximum ordinate of its points, and the *weight* of a peak is the difference between the maximum and minimum ordinates. For a peak of the form U^kD_ℓ, the weight is simply the $\max\{k, \ell\}$. We use $\mathsf{sumh}(P)$ and $\mathsf{symw}(P)$ to denote the sum of heights and weights of all symmetric peaks in P, respectively.

For example, the path shown in Fig. 1, which has a length of 24, contains four symmetric peaks and four asymmetric peaks. In this particular instance, the height and weight of the symmetric peaks are 2, 3, and 1, corresponding to the height and weight of the last three peaks. It is worth noting that in this example, both the height and weight are equal for each peak. However, in a more general case, the height and weight can be distinct.

In recent years, there has been a significant amount of research on the concept of symmetric peaks. In 2018, Asakly [1] introduced this concept for words. Later, Flórez and Ramírez [18] extended the concept of symmetric and asymmetric peaks to Dyck paths. The concept was further extended to non-decreasing Dyck paths by Elizalde et al. [14] and Flórez et al. [17], to Motzkin paths by Flórez and Ramírez [16], and to partial Dyck paths by Sun et al. [26]. Elizalde also provided other symmetric results on Dyck paths [13]. In some of these papers, the authors refer to these objects as symmetric pyramids instead of symmetric peaks.

This paper explores various characteristics of symmetric and asymmetric peaks in both DAPs and non-decreasing DAPs. To provide statistics and examine the asymptotic behavior of the different features, generating functions (g.f.) are used. These functions are defined using parameters such as path length, the number of symmetric and asymmetric peaks, and the height of symmetric peaks. Specifically, the total number of symmetric and asymmetric peaks, as well as the height of symmetric peaks, are presented. Additionally, we analyze the asymptotic behavior of the ratio between the number of symmetric peaks and the total number of peaks, and the ratio between the number of asymmetric peaks and the total number of peaks.

Recursive relations for the features examined in this paper are provided, and constructive proofs that rely on combinatorial arguments are offered. These proofs aim to provide intuitive insights into the behavior of symmetric peaks and their weight. For example, we present a recursive relation for the total number of DAPs that is reminiscent of the recursive relation for the total number of classic Dyck paths.

Using the generating functions and recursive relations, we derive closed formulas for the statistics presented in this paper. Most of these formulas use binomial coefficients or Fibonacci numbers. Notably, the total number of DAPs is counted by the generalized Catalan number (see A004148 in [23]), which among other things, counts also the peakless Motzkin paths.

Finally, a combinatorial interpretation of DAPs in terms of binary trees is provided. Thus, we establish a constructive bijection between the trees and DAPs.

2. Symmetric and asymmetric peaks in DAP

In this section, a generating function with three variables is introduced to represent the length of the path, the number of symmetric peaks, and the number of asymmetric peaks, as defined in the introduction. Using this generating function, we obtain the number of paths in $\mathcal{D}_n$ that avoid symmetric peaks as a corollary. Additionally, the total number of paths in $\mathcal{D}_n$ that avoid asymmetric peaks—peaks located in paths with pyramids at the ground level— is determined, aligning with the Fibonacci numbers F_n. Furthermore, another corollary of the generating function provides insights into the asymptotic behavior of the ratio between the number of symmetric peaks and the total number of peaks, as well as the ratio between the number of asymmetric peaks and the total number of peaks.

We provide a recursive function that simultaneously counts the total number of paths in $\mathcal{D}_n$ and coincides with the generalized Catalan number. Furthermore, we present recursive relations for the number of symmetric peaks and the total number of peaks. We also derive a closed formula using binomial coefficients for the total number of peaks.

Finally, we establish a combinatorial interpretation of DAPs in terms of binary trees. We define a constructive bijection between the trees and DAPs and provide an example to illustrate how the bijection works.

Considering the trivariate generating function:

$$F_{\mathsf{sp},\mathsf{ap}}(x,y,z) = \sum_{P\in\mathcal{D}} x^{|P|} y^{\mathsf{sp}(P)} z^{\mathsf{ap}(P)},$$

which represents the number of Dyck paths with air pockets of length n having k symmetric peaks and ℓ asymmetric peaks. Specifically, the coefficient of $x^n y^k z^\ell$ counts the number of such paths.

The following theorem presents the generating function $F_{\mathsf{sp},\mathsf{ap}}(x,y,z)$ in terms of the length and the numbers of symmetric and asymmetric peaks:

Theorem 2.1. *The generating function $F_{\mathsf{sp},\mathsf{ap}}(x,y,z)$ for the number of DAPs with respect to the length and the numbers of symmetric and asymmetric peaks is*

$$\frac{(1-x)\left(1-x^2-x^2y-x^3y+2x^3z-\sqrt{(1-x^2(1+y+xy-2xz))^2-4xp(x,y,z)}\right)}{2xp(x,y,z)},$$

where $p(x,y,z) = (1-x-x^2(y-z))^2$.

Proof. For short, we set $F = F_{\mathsf{sp},\mathsf{ap}}(x,y,z)$. Let P be a nonempty DAP. Then, we distinguish the following cases.

(1) If $P = U^a D_a Q$, $a \geq 1$, where Q is a DAP, then the g.f. for such paths is $\frac{x^2}{1-x} yF$.

(2) If $P = UU^a D_a Q U^b D_{b+1} R$, $a, b \geq 1$, where Q, R are some DAP, then the g.f. for these paths is

$$x\frac{x^2}{1-x}zF\frac{x^2}{1-x}zF = \frac{x^5}{(1-x)^2}z^2F^2.$$

(3) If $P = UU^aD_a\bar{Q}R$, $a \geq 1$, where Q, R are DAPs (Q non-empty and R possibly empty) and $\bar{Q}$ is obtained from Q after increasing by one the size of the last down-step, and Q does not end with a symmetric peak, then the g.f. for these paths is

$$x\frac{x^2}{1-x}zBF,$$

where B is the g.f. for the number of nonempty DAPs not ending with a symmetric peak. Considering the complement, we obtain easily $B = F - 1 - \frac{x^2}{1-x}yF$.

(4) If $P = UQU^aD_{a+1}R$, $a \geq 1$, where Q, R are some DAPs (Q non-empty and R possibly empty), and Q does not start with a symmetric peak, then the g.f. for these paths is

$$xB'\frac{x^2}{1-x}zF,$$

where B' is the g.f. for the number of nonempty DAPs not starting with a symmetric peak. Using the complement as above, we easily have $B' = B = F - 1 - \frac{x^2}{1-x}yF$.

(5) If $P = U\bar{Q}R$ where Q, R are DAPs (Q non-empty and R possibly empty) such that Q does not start and end with a symmetric peak, and $\bar{Q}$ is obtained from Q after increasing by one the size of the last down-step. The g.f. for this case is xCF where C is the g.f. for the number of nonempty DAPs that do not start and end with a symmetric peak. Considering the complement, we deduce easily $C = B - \frac{x^2y}{1-x}B$.

Summing up all these cases, we obtain the following functional equation:

$$F = 1 + \frac{x^2}{1-x}yF + \frac{x^5}{(1-x)^2}z^2F^2 + 2\frac{x^3}{1-x}zF\left(F - 1 - \frac{x^2}{1-x}yF\right)$$
$$+xF\left(F - 1 - \frac{x^2}{1-x}yF\right)(1 - \frac{x^2y}{1-x}),$$

which gives us the desired result. □

The first terms of the Taylor expansion are as follows:

$$1 + x^2y + x^3y + (y^2+y)x^4 + (2y^2+z^2+y)x^5$$
$$+\boldsymbol{(y^3 + 3y^2 + 3z^2 + y)x^6} + O(x^7),$$

where the weights of the DAPs of length 6 are shown in boldface in the expansion. Figure 5 displays these eight DAPs and their corresponding contributions to $F_{\mathsf{sp,ap}}(x, y, z)$.

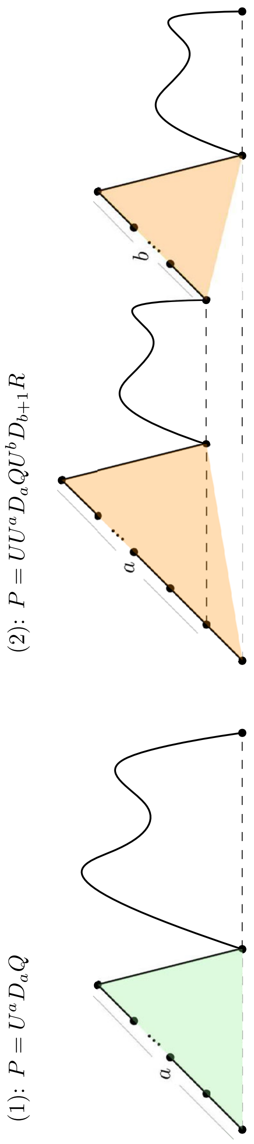

Figure 2. Decomposition of cases (1) and (2)

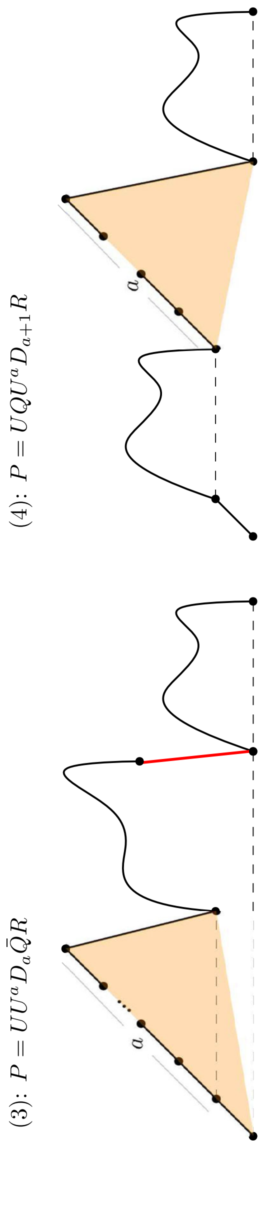

Figure 3. Decomposition of cases (3) and (4)

(5): $P = UQR$

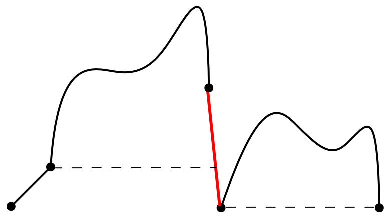

FIGURE 4. Decomposition of case (5)

Corollary 2.2. *The generating function for the number of DAPs avoiding symmetric peaks is given by:*

$$F(x,0,1) = \frac{(1-x)\left(1-x^2+2x^3-\sqrt{(1-x)(1-3x+3x^2-5x^3+4x^4-4x^5)}\right)}{2x(1-x+x^2)^2}.$$

The Taylor expansion of this generating function is

$$1 + x^5 + \boldsymbol{3x^6} + 6x^7 + 12x^8 + 25x^9 + 53x^{10} + 115x^{11} + O(x^{12}),$$

where the sequence of coefficients does not appear in [23]. The bold coefficient in the above series can be verified in Fig. 5.

Here, we use the standard notation F_n to denote the n-th Fibonacci number. Specifically, we have $F_n = F_{n-1}+F_{n-2}$ $(n \geq 2)$, with the initial conditions $F_0 = 0$ and $F_1 = 1$.

Corollary 2.3. *The generating function for the number of DAPs avoiding asymmetric peaks is given by:*

$$F(x,1,0) = \frac{1-x}{1-x-x^2}.$$

That is, the number of DAPs of length n without asymmetric peaks is given by the F_{n-1}.

It is worth noting that the Fibonacci sequence counts the compositions of n into parts of size at least 2. In fact, a non-empty DAP of length n without asymmetric peaks necessarily takes the form

$$U^{a_1}D_{a_1}U^{a_2}D_{a_2}\dots U^{a_k}D_{a_k}, \quad \text{with } k \geq 1,\ a_i \geq 1, \quad \text{and}$$
$$n = k + a_1 + a_2 + \cdots + a_k,$$

and it can be associated with the composition of n: $(a_1{+}1), (a_2{+}1), \dots, (a_k{+}1)$, see for instance Fig. 6.

We refer to [3] for the expression of the bivariate generating function for DAPs with respect to the length and the number of peaks (symmetric and asymmetric), which is denoted as $F(x,1,1)$.

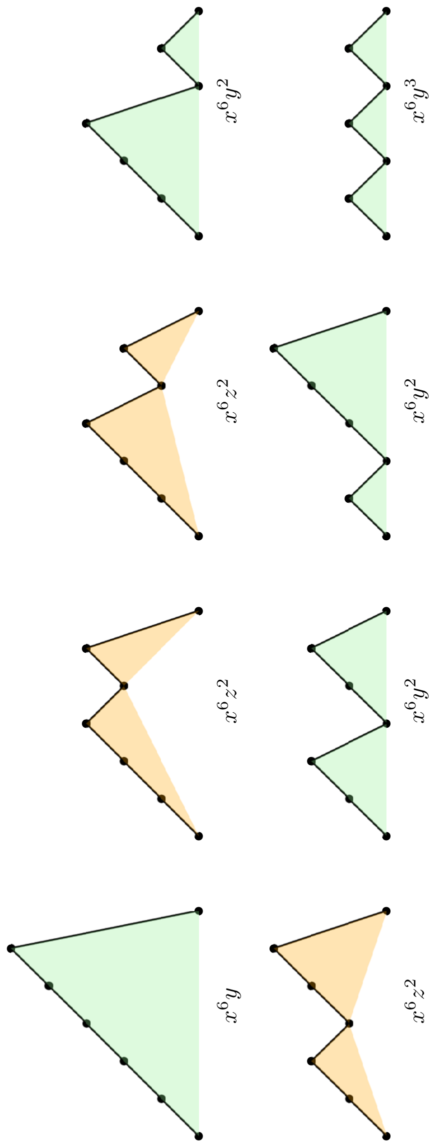

FIGURE 5. The eight DAPs of length 6 and their contribution in $F_{\mathsf{sp},\mathsf{ap}}(x, y, z)$

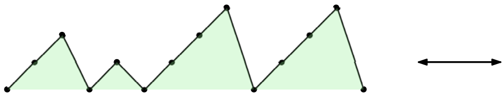

FIGURE 6. Illustration of the bijection between length n DAPs without asymmetric peaks and compositions of n

By calculating $\partial_y(F(x,y,1))|_{y=1}$ and $\partial_z(F(x,1,z))|_{y=1}$, we obtain the following two corollaries. The asymptotic approximations of the coefficient of z^n is obtained using classical methods presented in [15,22].

Corollary 2.4. *The generating function for the total number of symmetric peaks in all DAPs is given by:*

$$\frac{x\left(-1+2x+2x^2+3x^3-2x^4+(1+2x)(1-x)\sqrt{(1-3x+x^2)(1+x+x^2)}\right)}{2(1-x)\sqrt{(1-3x+x^2)(1+x+x^2)}},$$

and the asymptotic for the n-th coefficient is

$$\frac{9-4\sqrt{5}}{\sqrt{\pi n}\sqrt{14\sqrt{5}-30}}\left(\frac{1+\sqrt{5}}{2}\right)^{2n}.$$

The Taylor expansion is $x^2+x^3+3x^4+5x^5+\boldsymbol{10x^6}+21x^7+45x^8+101x^9+O(x^{10})$, where the sequence of coefficients does not appear in [23]. In Fig. 5 it is possible to verify that there are ten symmetric peaks in all DAPs of length 6. Notice that the asymptotic depends on the golden ratio $\frac{1+\sqrt{5}}{2}$ which is a root of the polynomial x^2-x-1. In Theorem 2.9 we give a recurrence relation to calculate the number of symmetric peaks in all DAP.

Corollary 2.5. *The g.f. for the total number of asymmetric peaks in all DAPs is*

$$\frac{x\left(1-x-2x^2-x^3+x^4-(1-x^2)\sqrt{(1-3x+x^2)(1+x+x^2)}\right)}{(1-x)\sqrt{(1-3x+x^2)(1+x+x^2)}},$$

and an asymptotic for the n-th coefficient is

$$\frac{5\sqrt{5}-11}{\sqrt{\pi n}\sqrt{14\sqrt{5}-30}}\left(\frac{1+\sqrt{5}}{2}\right)^{2n}.$$

The Taylor expansion of the generating function for the total number of asymmetric peaks in all DAPs is $2x^5+\boldsymbol{6x^6}+18x^7+50x^8+132x^9+O(x^{10})$. Note that the sequence of coefficients does not appear in [23]. Figure 5 confirms that there are 6 asymmetric peaks in all DAPs of length 6.

We denote the total number of peaks, symmetric peaks, and asymmetric peaks in all DAPs by $p(n)$, $s(n)$, and $t(n)$, respectively. It is worth noting that from Corollary 10 of [3], we have

$$p(n) \sim \frac{\sqrt{5}-2}{\sqrt{\pi n}\sqrt{14\sqrt{5}-30}}\left(\frac{1+\sqrt{5}}{2}\right)^{2n}.$$

In this corollary we present asymptotic ratios for various peak counts in all DAPs.

Corollary 2.6. *An asymptotic expression for the ratio between the number of symmetric peaks and the total number of peaks in all DAPs:*

$$\lim_{n\to\infty}\frac{s(n)}{p(n)} = \sqrt{5}-2 \sim 0.236067977.$$

An asymptotic expression for the ratio between the number of asymmetric peaks and the total number of all peaks in all DAPs:

$$\lim_{n\to\infty}\frac{t(n)}{p(n)} = 3-\sqrt{5} \sim 0.763932023.$$

An asymptotic expression for the ratio between the numbers of asymmetric and the total number symmetric peaks in all DAPs:

$$\lim_{n\to\infty}\frac{t(n)}{s(n)} = \sqrt{5}+1 \sim 3.236068475.$$

2.1. A recursive relation for the number of DAPs

We use $g(n)$ to denoted the number of DAPs in $\mathcal{D}_n$. The classic proof for the number of Dyck paths can be adapted to obtain a recurrence relation for $g(n)$. It is formally given in the Theorem 2.7.

Let $\mathcal{B}_n \subset \mathcal{D}_n$ the set of all DAPs without valleys at a ground level, i.e., DAPs with no occurrence D_kU, $k \geq 1$, touching the x-axis. There is a bijection between $\mathcal{B}_n$ and $\mathcal{D}_{n-1}$ by deleting the first North-East step (U-step) and replacing the last South-East step of length a (D_a-step) with a step D_{a-1} in all paths in $\mathcal{B}_n$. Notice that $\mathcal{B}_3 = \{U^2D_2\}$ maps bijectively to $\mathcal{D}_2 = \{UD_1\}$, while $\mathcal{B}_2 = \{UD_1\}$ is in bijection with $\mathcal{D}_1 := \{\}$ (the empty set).

Theorem 2.7. *For $n > 3$, we have*

$$g(n) = g(n-1) + g(n-2) + \sum_{k=2}^{n-3} g(k)g(n-k-1),$$

anchored with the initial values $g(1)=0$ and $g(2)=g(3)=1$. Furthermore, we have the closed form formula:

$$g(n)=\sum_{k=1}^{n-1}\frac{1}{n-k}\binom{n-k}{k}\binom{n-k}{k-1},\quad n\geq 2. \tag{1}$$

Proof. Let us consider $P\in\mathcal{D}_n$. Then P can be decomposed as either $Q\in\mathcal{B}_n$, or UD_1R with $R\in\mathcal{D}_{n-2}$, or QR with $Q\in\mathcal{B}_{k+1}$ and $R\in\mathcal{D}_{n-k-1}$ with $2\leq k\leq n-3$ (this last decomposition is as in Fig. 4). Due to the above bijection between $\mathcal{D}_{n-1}$ and $\mathcal{B}_n$, the DAPs satisfying the first two cases are enumerated by $g(n-1)+g(n-2)$, and for a fixed k, $2\leq k\leq n-3$, the DAPs satisfying the third case are enumerated by $g(k)g(n-k-1)$. Varying k in the set $\{2,3,\ldots,n-3\}$, we obtain the desired result.

On the other hand, let $G(x)$ be the generating function of the sequence $g(n)$. We have

$$G(x)=F_{\mathsf{sp,ap}}(x,1,1)=\frac{1-x-x^2-\sqrt{1-2x-x^2-2x^3+x^4}}{2x}.$$

If $U(x)=G(x)/x$, then $U(x)=x(1+(1+x)U(x)+xU^2(x))$. Consider the auxiliary function $f(x,t)$ defined by $f(x,t)=x(1+(1+t)f(x,t)+tf^2(x,t))=x(\Phi(x,t))$, where $\Phi(u)=(1+u)(1+tu)$. From the Lagrange inversion, (see [21] for instance), we have that

$$\begin{aligned}[x^n]f(x,t)&=\frac{1}{n}[u^{n-1}]\Phi(u)^n=\frac{1}{n}[u^{n-1}](1+u)^n(1+tu)^n\\&=\frac{1}{n}\sum_{i=0}^{n-1}\binom{n}{n-1-i}\binom{n}{i}t^i,\quad n\geq 1.\end{aligned}$$

Then $g(n)=[x^n]G(x)=[x^{n-1}]U(x)=[x^{n-1}]f(x,x)$. Comparing coefficients we obtain that

$$\begin{aligned}g(n)&=[x^{n-1}]\sum_{n\geq 1}\frac{1}{n}\sum_{i=0}^{n-1}\binom{n}{n-1-i}\binom{n}{i}x^{n+i}\\&=[x^{n-1}]\sum_{i\geq 0}\sum_{n\geq 0}\frac{1}{n+i+1}\binom{n+i+1}{n}\binom{n+i+1}{i}x^{n+2i+1}.\end{aligned}$$

Setting $h=n+2i+1$, this implies

$$\begin{aligned}g(n)&=[x^{n-1}]\sum_{i\geq 0}\sum_{h\geq 2i+1}\frac{1}{h-i}\binom{h-i}{h-2i-1}\binom{h-i}{i}x^h\\&=\sum_{i=0}^{\lfloor\frac{n-2}{2}\rfloor}\frac{1}{n-1-i}\binom{n-1-i}{i+1}\binom{n-1-i}{i}.\end{aligned}$$

$\square$

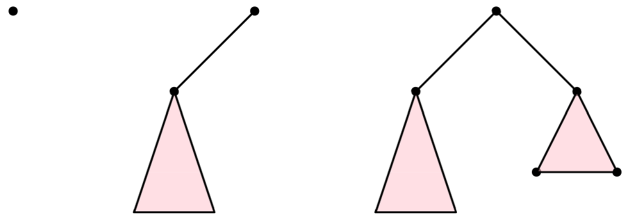

FIGURE 7. Decomposition of the trees in $\mathcal{T}$

The first ten values of the sequence $g(n)$ for $n = 2, \ldots, 11$ are as follows:

$$1, \quad 1, \quad 2, \quad 4, \quad 8, \quad 17, \quad 37, \quad 82, \quad 185, \quad 423.$$

The sequence $g(n)$ has connections to various combinatorial structures, such as generalized bracketings [12], peakless Motzkin paths [7], and zigzag knight's paths [6]. Additionally, recurrence relations of this type have been studied and generalized by Stein and Waterman [25].

We give a probably new combinatorial interpretation in terms of binary trees. An *ordered binary tree* is a rooted tree where each node is either a leaf (with no children), or an internal node (with one or two left/right children). For instance, there are five ordered binary trees with three nodes, and it is well known that these trees with n nodes are counted by the n-th Catalan number (see A000108). Let $\mathcal{T}_n$ be the set of ordered binary trees with n nodes where (i) any right node has a left child, and (ii) any right node is sibling to a left node. Let $\mathcal{T}$ be the set $\bigcup_{n\geq 1} \mathcal{T}_n$. Any tree $T \in \mathcal{T}$ is either a simple root, or a root with a left subtree $T_\ell \in \mathcal{T}$, or a root with a left subtree $T_\ell \in \mathcal{T}$ paired with a right subtree T_r with at least two nodes, see Fig. 7.

From this description of $\mathcal{T}$, we easily deduce that $\mathcal{T}_n$ has the same cardinality as $\mathcal{D}_n$. Indeed, if $T := T(x)$ is the generating function for the cardinality of $\mathcal{T}$ with respect to the number of nodes, then the description induces $T = x + xT + xT(T - x)$.

Let us now exhibit the bijection between $\mathcal{D}_n$ and $\mathcal{T}_n$. Given a path $P \in \mathcal{D}_n$, we can express it as a concatenation of subpaths of the form $U^t D_d$, $t, d \geq 1$. Initiating the process, let us create a root node r. Starting at r as the current node, for the first U-step in P (reading from left to right), we add a new left-child to the current node and move to the newly created leaf, which becomes the new current node. We repeat this procedure for the t consecutive U-steps. For each D_d-step, we create a right-child to the $(d + k)$-th ancestor (i.e., we go back $(d + k)$ times towards the root) where k is the number of ancestors of the current node with a right child, or equivalently the number of ancestors of out-degree 2). We then move to the newly created leaf (which becomes the new current node). We repeat this procedure until we traverse entirely the path P, except the last down-step.

The fact that a DAP does not contain two consecutive down-steps implies that any right node has a left child. Due to the construction, any right node is sibling of a left node, which ensures that the generated tree belongs to $\mathcal{T}$. With the above cardinality argument, this map is a bijection. See Fig. 8 for an illustration of the bijection applied to all DAPs of length 6.

2.2. The total number of symmetric peaks

In this section, a recursive relation for the total number of symmetric peaks in $\mathcal{D}_n$ is given. Let us denote by $s^*(n)$ the total number of all first peaks that are symmetric, excluding the symmetric peak $U^{n-1}D_{n-1}$ of length n (i.e, the path formed by a single peak). Additionally, let us use Δ_k to represent the symmetric peak U^kD_k.

Using the symmetry construction of paths in $\mathcal{D}_n$, we can show that the total number of last symmetric peaks at ground level in $\mathcal{D}_n$ is equal to the total number of first symmetric peaks in $\mathcal{D}_n$, except for the peak $U^{n-1}D_{n-1}$ of length n. If we include the path formed by a single peak, the total count is given by $s^*(n)+1$, for $n \geq 2$. We state this fact formally in the following lemma.

Lemma 2.8. *If $n > 1$, then $s^*(n)$ is given by*

$$s^*(n) = \sum_{i=2}^{n-2}\sum_{k=1}^{i-1} \frac{1}{i-k}\binom{i-k}{k}\binom{i-k}{k-1}.$$

Proof. The total number of the first (resp., last) peaks that are symmetric is counted by the total number of first (resp., last) peaks at a ground level. Thus, this is counted by the total number of paths of the form $\Delta_i P$ (resp., $P\Delta_i$) where P is a DAP of length $n-i-1$ for $1 \leq i \leq n-3$. Clearly, for i fixed it is counted by the total number of paths in $\mathcal{D}_{n-i-1}$; this is given by $g(n-i-1)$ (see Theorem 2.7). Varying i from 1 to $n-3$, we obtain the recurrence relation $s^*(n) = \sum_{i=2}^{n-2} g(i)$ for $n > 4$, with the initial values $s^*(4) = 1$ and $s^*(n) = 0$ for $n < 4$. This, together with Theorem 2.7, implies the desired formula. $\square$

As stated in the introduction, our goal is to derive recurrence relations using combinatorial arguments. Thus, we now present a recurrence relation and its proof, which help us achieve this objective to some extent.

Theorem 2.9. *The sequence $s(n)$ satisfies the following recurrence relation for $n > 4$*

$$s(n) = s(n-1) + s(n-2) - 2s^*(n-1) + g(n-2) + \sum_{k=2}^{n-3} 2g(n-k-1)s'(k),$$

where $s'(k) = s(k) - s^*(k)$, and with initial values $s(2) = s(3) = 1$ and $s(4) = 3$.

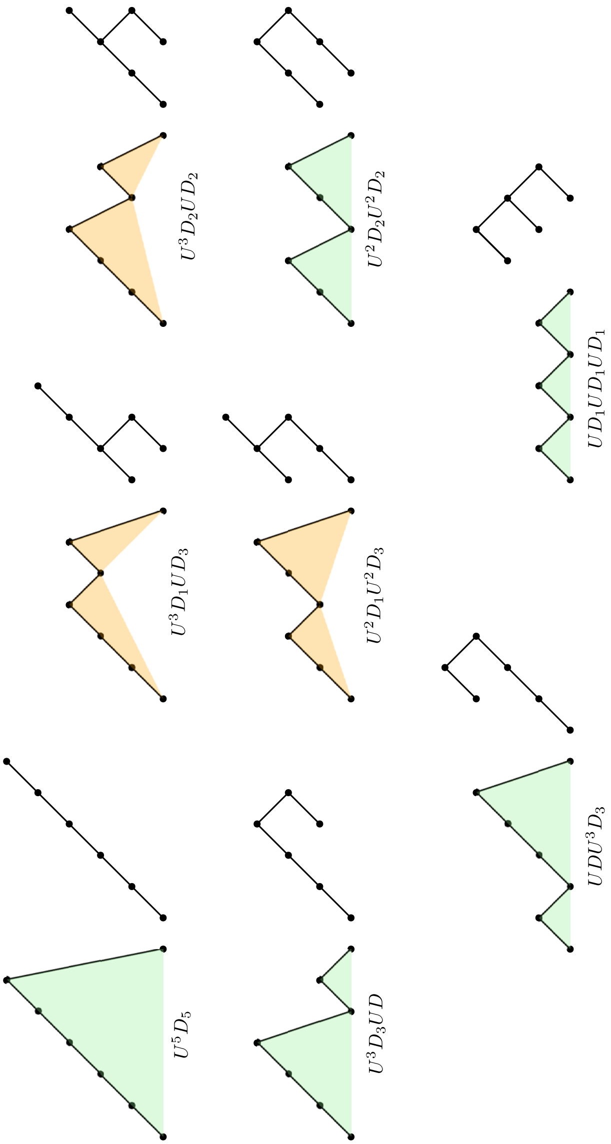

Figure 8. Bijection between $\mathcal{D}_6$ and $\mathcal{T}_6$

Proof. Notice that any non-empty path in $\mathcal{D}_n$ can be decomposed as QR where $Q \in \mathcal{B}_{k+1}$ and $R \in \mathcal{D}_{n-k-1}$ for $k \geq 1$ (see Fig. 4).

To count the number of symmetric peaks in a path of the form QR, we observe that it is given by the sum of the number of symmetric peaks in Q and the number of symmetric peaks in R. The first is found multiplying the number of symmetric peaks in $\mathcal{B}_{k+1}$ by the total number of path in $\mathcal{D}_{n-k-1}$. We distinguish four cases.

(1) If $k = 1$, then we have $Q = \Delta_1 \in \mathcal{B}_2$. The total number of peaks derived from $Q = \Delta_1$ is $g(n-2)$ (equal to the number of paths in $\mathcal{D}_{n-2}$). Therefore, adding $g(n-2)$ with the total number of symmetric peaks counted over all paths $R \in \mathcal{D}_{n-2}$, we obtain that there are $s(n-2) + g(n-2)$ symmetric peaks in the paths lying in this case.

(2) If $k = 2$, then using a similar argument as for $k = 1$, there are $s(n-3) + g(n-3)$ symmetric peaks counted over all paths lying in this case.

(3) If $k = n-1$, then the number of symmetric peaks derived from $Q \in \mathcal{B}_{n-1+1}$ can be counted using $s(n-1)$ (by the bijection given above). However the first and the last peaks that are symmetric in all paths in $\mathcal{D}_{n-1}$ are not symmetric peaks in $\mathcal{B}_n$, so we have to subtract them from $s(n-1)$. We can use $s^*(n-1)$ for this purpose. Thus, the total number of symmetric peaks counted over all paths in this case is $s(n-1) - 2s^*(n-1)$.

(4) If $2 < k < n-2$, then the number of symmetric peaks derived from $Q \in \mathcal{B}_{k+1}$ can be counted using $s(k) - 2s^*(k)$. (The shaded peaks in Fig. 3 shows the part that we need to subtract by $2s^*(n-1)$.) Therefore, the total number of symmetric peaks counted over all paths of the form QR, $Q \in \mathcal{B}_{k+1}$, $R \in \mathcal{D}_{n-k-1}$, is given by $(s(k) - 2s^*(k))g(n-k-1) + s(n-k-1)g(k)$. By varying k from 3 to $n-3$ and adding the special three cases, we obtain the desired result.

This completes the proof. □

Let us define $p(n)$ as the total number of peaks in $\mathcal{D}_n$. Using a binomial expression, we can write this sequence as:

$$p(n) = \sum_{k=0}^{n} \binom{k-1}{2k-n} \binom{k}{2k-n+1}.$$

A proof of this result, using generating functions, can be found in [3, p. 10]. Alternatively, by modifying the proof of Theorem 2.9 —by removing the use of $s^*(k)$— we can derive a recurrence relation that counts the total number of peaks in $\mathcal{D}_n$, that is, for $n \geq 5$

$$p(n) = p(n-1) + p(n-2) + g(n-2) + 2\sum_{k=2}^{n-3} p(k)g(n-k-1),$$

with the initial values $p(2) = p(3) = 1$ and $p(4) = 3$. It is important to note that a recurrence relation for the number of asymmetric peaks in $\mathcal{D}_n$ can be obtained by subtracting $s(n)$ from $p(n)$.

3. Counting symmetric weight

Recall that the *weight* of a peak is the difference between the maximum and minimum ordinates, which is also equal to the $\max\{k, \ell\}$ whenever the peak is $U^k D_\ell$. For a symmetric peak $U^k D_k$, its weight is k. Let $\mathsf{symw}(P)$, called the *sum of symmetric weights*, be the sum of weights of all symmetric peaks in P. As an illustration, the symmetric peaks of the path depicted in Fig. 1 have weights $1, 2, 3$, and 1, respectively. Thus, $\mathsf{symw}(P) = 7$.

In this section, we present several results related to the sum of weights of symmetric peaks in $\mathcal{D}_n$. Firstly, we derive a generating function in two variables that counts the sum of weights of all symmetric peaks in $\mathcal{D}_n$. Secondly, we establish a recursive relation for the sum of weights of all symmetric peaks in $\mathcal{D}_n$. We start by introducing the generating function $W(x, q)$ of all DAPs with respect to their length and sum of symmetric weights, defined as follows:

$$W(x, q) = \sum_{P \in \mathcal{D}} x^{|P|} q^{\mathsf{symw}(P)}.$$

Using the same decomposition as in the proof of Theorem 2.1, we derive the following result.

Theorem 3.1. *The generating function $W(x, q)$ for the number of DAPs with respect to the length and the sum of symmetric weights is*

$$W(x, q) = \frac{(1-x)(1-qx)\left(1 - qx - (1+q)x^2 + (2+q)x^3 - qx^4 - \sqrt{w(x,q)}\right)}{2x(1 - x(1+q) + x^2)^2},$$

where $w(x, q)$ is defined as:

$$\begin{aligned} w(x, q) = (1-x)(1 - (3+2q)x + (1+q)(3+q)x^2 \\ -(5+q^2)x^3 + 2(2+2q-q^2)x^4 - 2(2+q+q^2)x^5 + q(4+q)x^6 - q^2x^7). \end{aligned}$$

The first terms of the Taylor expansion are as follows:

$$\begin{aligned} 1 + qx^2 + q^2x^3 + (q^2 + q^3)x^4 + (1 + 2q^3 + q^4)x^5 \\ + \boldsymbol{(3 + q^3 + 3q^4 + q^5)x^6} + O(x^7). \end{aligned}$$

All DAPs of length 6 are displayed in Fig. 9, with their corresponding symmetric weights highlighted in boldface in the previous expansion.

The *total symmetric weight* of $\mathcal{D}_n$ is defined as the sum of the symmetric weights over all paths in $\mathcal{D}_n$, and it is denoted by $w(n)$. By calculating $\partial_q(W(x, q))|_{q=1}$, we can obtain the following two corollaries.

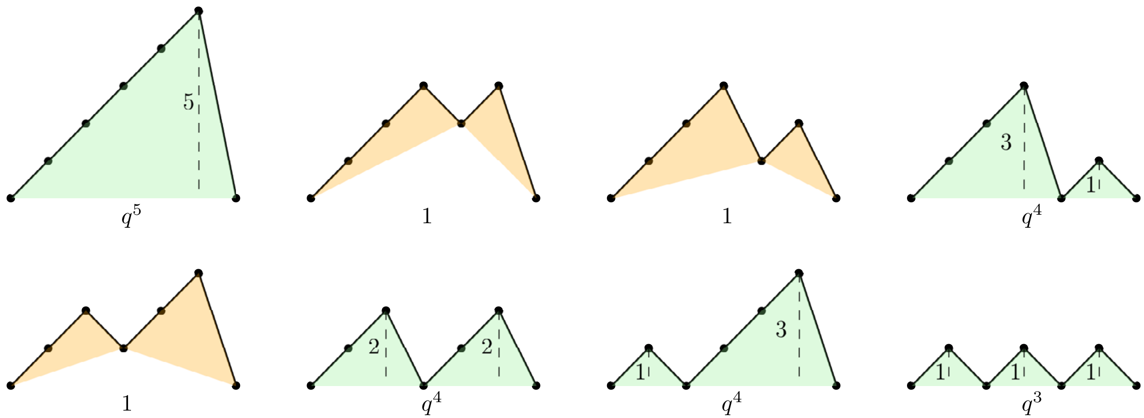

Figure 9. Symmetric weight of all DAPs of length 6

Corollary 3.2. *The generating function for the total symmetric weight in all DAPs is given by*

$$\frac{x\left(-1+2x+2x^2+3x^3-2x^4+(1-x)(1+2x)\sqrt{1-2x-x^2-2x^3+x^4}\right)}{2(1-x)^2\sqrt{1-2x-x^2-2x^3+x^4}},$$

and an asymptotic expression for the n-th coefficient is

$$\frac{5\sqrt{10}-11\sqrt{2}}{4\sqrt{\pi n}\,\sqrt{-15+7\sqrt{5}}}\cdot\left(\frac{1+\sqrt{5}}{2}\right)^{2n}.$$

The Taylor expansion of the generating function is

$$x^2+2x^3+5x^4+10x^5+20x^6+41x^7+86x^8+187x^9+O(x^{10}).$$

It worth to mention that the sequence of coefficients does not appear yet in [23].

Let us use $w^*(n)$ to denote the total sum of symmetric weights of all first peaks that are symmetric over all paths in $\mathcal{D}_n$, except for the path U^nD_n. The weight sum of all first peaks that are symmetric, including the weight of the single peak of length n, is given by $w^*(n)+(n-1)$. Using the symmetry construction of paths in $\mathcal{D}_n$, we can observe that the total weight of the first peaks that are symmetric in $\mathcal{D}_n$ is equal to the total weight of the last peak at a ground level that is symmetric in $\mathcal{D}_n$. It is worth noting that $w^*(n)$ also counts the total height of all first peaks that are symmetric in $\mathcal{D}_n$, except for the path formed by the single peak. Recall that the height of an occurrence U^kD_ℓ in a path is the maximal ordinate of its points.

Lemma 3.3. *If $n>1$, then $w^*(n)$ is given by*

$$w^*(n)=\sum_{i=2}^{n-2}\sum_{k=1}^{n-i-1}\frac{i-1}{n-i-k}\binom{n-i-k}{k}\binom{n-i-k}{k-1}.$$

Proof. For a given $i\geq 1$, the sum of the weights (over all DAPs in $\mathcal{D}_n$) of all first (resp., last) peaks Δ_i is i multiplied by the number of DAPs of length $n-i-1$, which is $i\cdot g(n-i-1)$. Varying i in the set $\{1,2,\ldots,n-3\}$, we obtain the recurrence relation $w^*(n)=\sum_{i=2}^{n-2}(i-1)g(n-i)$ for $n>4$, with the initial values $w^*(4)=1$ and $w^*(n)=0$ for $n<4$.

Using Theorem 2.7, we have the binomial expression for $w^*(n)$ given in the statement of this lemma. $\square$

Theorem 3.4. *For $n\geq 4$, we have*

$$w(n)=w(n-1)+w(n-2)-2w^*(n-1)+1+\sum_{k=1}^{n-3}2w'(k)g(n-k-1),$$

where $w'(k)=w(k)-w^*(k)+\frac{1}{2}$, and the initial values are $w(1)=0$, $w(2)=1$, and $w(3)=2$.

Proof. Consider any non-empty path in $\mathcal{D}_n$, which can be decomposed as QR where $Q \in \mathcal{B}_{k+1}$ and $R \in \mathcal{D}_{n-k-1}$ for $k \geq 1$ (it can be seen in Fig. 4). We consider four cases.

(1) If $k = 1$, then $Q = \Delta_1$ and the total symmetric weight of all symmetric peaks (over $\mathcal{D}_n$) derived from $Q = \Delta_1$ is $g(n-2)$. Adding this to the total symmetric weight of all paths of the form $R \in \mathcal{D}_{n-2}$, we obtain that $w(n-2)+g(n-2)$ is the total symmetric weight of all paths in this case.

(2) If $k = 2$, then, with the same argument as for the previous case, the total symmetric weight of all paths in this case is $w(n-3) + 2g(n-3)$.

(3) If $k = n-1$, then the total symmetric weight of $Q \in \mathcal{B}_{n-1+1}$ is given by the total symmetric weight of $\mathcal{D}_{n-1}$ which is $w(n-1)$. However, there are two sub-cases to consider in this counting: firstly, the weight of the single peak in $\mathcal{D}_{n-1}$ is $n-1$ but in $\mathcal{B}_n$ it is n, so we must add one to the counting; secondly, the first and the last peaks that are symmetric in all paths in $\mathcal{D}_{n-1}$ cannot be counted, as they are not symmetric peaks in $\mathcal{B}_n$, so we have to subtract them from $w(n-1)$. We use $w^*(n-1)$ to adjust for these differences. Thus, the total symmetric weight of $\mathcal{B}_n$ is $w(n-1) - 2w^*(n-1) + 1$.

(4) If $3 \leq k \leq n-3$, then the total symmetric weight derived from $Q \in \mathcal{B}_{k+1}$ can be counted using $w(k) - 2w^*(k) + 1$. Therefore, the total symmetric weight for QR (see Fig. 3) is given by $(w(k) - 2w^*(k) + 1)g(n-k-1) + w(n-k-1)g(k)$. By varying k from 3 to $n-3$ and adding the special three cases, we obtain the desired result.

This completes the proof. □

4. Counting symmetric height

In this section, we present a method for counting the sum of heights of peaks in $\mathcal{D}_n$, using a generating function expressed as a continuous fraction. Additionally, we provide recursive relations to compute both the sum of heights of all peaks and the sum of heights of symmetric peaks in $\mathcal{D}_n$.

Let $h(n)$ and $h_s(n)$ denote the sums of heights of all peaks and all symmetric peaks, respectively, in the set $\mathcal{D}_n$ of DAPs of length n. The height of a peak is defined as the y-coordinate of its highest point measured from the ground level. We use $\mathsf{sumh}(P)$ to denote the sum of heights of all peaks in the path P, and $\mathsf{peak}(P)$ to denote the number of peaks in the path P.

Let $\mathcal{R}$ be the set of all classical Dyck paths. We define the generating function $R(x,p,q)$ over all classical Dyck paths

$$R(x,p,q) = \sum_{P \in \mathcal{R}} x^{|P|/2} p^{\mathsf{peak}(P)} q^{\mathsf{sumh}(P)}.$$

Deutsch [11] proved that this generating function satisfies the functional equation

$$R(x,p,q) = 1 + x(R(x,qp,q) - 1 + pq)R(x,p,q). \tag{2}$$

Let $Q(x,q)$ be the generating functions of all DAPs with respect to the length and the sum of the heights. That is,

$$Q(x,q) = \sum_{P\in\mathcal{D}} x^{|P|} q^{\mathsf{sumh}(P)}.$$

Theorem 4.1. *An expression for generating function of $Q(x,q)$ is given by the continued fraction*

$$Q(x,q) = \cfrac{1}{1+x-qx^2-\cfrac{x}{1+x-q^2x^2-\cfrac{x}{1+x-q^3x^2-\cfrac{x}{\ddots}}}}.$$

Proof. Notice that each peak in a Dyck path has to be counted as a down step in a DAP. Therefore, we have the relation $Q(x,q) = R(x,x,q)$. From the functional equation (2), we obtain

$$Q(x,q) = R(x,x,q) = 1 + x(R(x,qx,q) - 1 + qx)R(x,x,q).$$

Therefore, we have

$$Q(x,q) = \frac{1}{1 - x(R(x,qx,q) - 1 + qx)} = \frac{1}{1 + x - qx^2 - xR(x,qx,q)}.$$

Iterating this expression yields the desired result. □

The first terms of the continued fraction are as follows:

$$\begin{aligned} &1 + qx^2 + q^2x^3 + (q^2+q^3)x^4 + (2q^3+2q^4)x^5 \\ &+(q^3+3q^4+3q^5+q^6)x^6 + (3q^4+6q^5+5q^6+2q^7+q^8)x^7 + O(x^8). \end{aligned}$$

Theorem 4.2. *The generating function for the sum of the heights of all peaks in $\mathcal{D}_n$ is*

$$H(x) = \sum_{n\geq 0} h(n)x^n = \frac{x^2}{(1-3x+x^2)(1+x+x^2)}.$$

Moreover,

$$h(n) = \sum_{k=0}^{n} \frac{1}{2}\binom{2n-2k}{2k-1},$$

and an asymptotic for the n-th coefficient is

$$\frac{\sqrt{5}}{20} \cdot \left(\frac{1+\sqrt{5}}{2}\right)^{2n}.$$

Proof. We can use Theorem 4.1 to derive an expression for the generating function $H(x)$ as follows:

$$H(x) = \partial_q(Q(x,q))|_{q=1} = \sum_{\ell \geq 1} \frac{\ell x^{\ell+1}}{M^{2\ell}(x)} = \frac{x^2 M^2(x)}{(x - M^2(x))^2}, \tag{3}$$

where

$$M(x) = 1 + x - x^2 - \cfrac{x}{1 + x - x^2 - \cfrac{x}{1 + x - x^2 - \cfrac{x}{\ddots}}} = 1 + x - x^2 - \frac{x}{M(x)}.$$

Note that we can express $M(x)$ as

$$M(x) = 1/2\left(1 + x - x^2 - \sqrt{1 - 2x - x^2 - 2x^3 + x^4}\right).$$

Substituting this expression for $M(x)$ into (3), we obtain the desired expression for $H(x)$.

To obtain the combinatorial sum, we can manipulate the generating function using standard techniques. □

Upon comparing this generating function with the sequence defined in A182890, it is evident that they are identical.

Theorem 4.3. *The sequence $h(n)$ satisfies the following recurrence relation for $n \geq 4$:*

$$h(n) = h(n-2) + h(n-1) + p(n-1) + g(n-2) + \sum_{k=2}^{n-3} (2h(k) + p(k))\, g(n-k-1),$$

with initial values $h(2) = 1$ and $h(3) = 2$.

Proof. Consider any non-empty path P in $\mathcal{D}_n$, which can be decomposed as $P = QR$ where $Q \in \mathcal{B}_{k+1}$ and $R \in \mathcal{D}_{n-k-1}$ for some $k \geq 1$. We distinguish three cases.

(1) If $k = 1$, then $Q = \Delta_1$, and the sum of the heights of these peaks (over all paths in $\mathcal{D}_n$) is the cardinality of $\mathcal{D}_{n-2}$, which is $g(n-2)$. Adding this to the sum of the heights of all peaks in paths of the form $R \in \mathcal{D}_{n-2}$ gives us that $h(n-2) + g(n-2)$ is the sum of the heights of peaks of all paths in this case.

(2) If $k = n - 1$, then the sum of the heights of all peaks in $\mathcal{B}_{n-1+1} = \mathcal{B}_n$ is given by the sum of the heights of all peaks in $\mathcal{D}_{n-1}$ plus the total number of its peaks. This gives us that the sum of the heights of all peaks of all paths in this case is $h(n-1) + p(n-1)$.

(3) If $2 \le k \le n-3$, then the sum of the heights of all peaks in $\mathcal{B}_{k+1}$ can be counted using $g(n-k-1)(h(k) + p(k))$. Therefore, the sum of the heights of all peaks of all paths in this case (with k fixed) is given by $h(n-k)g(k) + g(n-k-1)(h(k) + p(k))$. By varying k from 2 to $n-3$, we obtain the recurrence relation in the statement of the theorem.

This completes the proof. □

The first ten values of the sequence $h(n)$ for $n = 2, \dots, 11$ are as follows:

$$1, \quad 2, \quad 5, \quad 14, \quad 36, \quad 94, \quad 247, \quad 646, \quad 1691, \quad 4428.$$

We use $h_s(n)$ to represent the sum of the heights of all symmetric peaks of $\mathcal{D}_n$.

Theorem 4.4. *The sequence $h_s(n)$, which represents the sum of the heights of all symmetric peaks in $\mathcal{D}_n$, can be described by the following recurrence relation:*

$$h_s(n) = h_s(n-2) - h_s(n-1) + g(n-2) + 2h'_s(n-1) + 2\sum_{k=2}^{n-3} h'(k)g(n-k-1),$$

where $h'_s(k) = h_s(k) + \frac{1}{2}s(k) - s^*(k) - w^*(k)$ for $n \ge 4$, with initial values $h_s(2) = 1$ and $h_s(3) = 2$.

Proof. The proof follows similar steps as the proof of Theorem 3.4, but we need to take into account the height of the peak measured from the ground.

Consider any non-empty path in $\mathcal{D}_n$, which can be decomposed as QR where $Q \in \mathcal{B}_{k+1}$ and $R \in \mathcal{D}_{n-k-1}$ for $k \ge 1$. We distinguish three cases.

(1) If $k = 1$, then the sum of the heights of all symmetric peaks in all these paths is $h_s(n-2) + g(n-2)$.

(2) If $k = n-1$, then the sum of the heights of all symmetric peaks in all these paths is given by $h_s(n-1) - 2w^*(n-1) + s(n-1) - 2s^*(n-1)$.

(3) If $3 \le k \le n-3$, then the sum of the heights of all symmetric peaks in all paths in $\mathcal{B}_{k+1}$ is $(h_s(k) - 2w^*(k) + s(k) - 2s^*(k))g(n-k-1)$. Therefore, the sum of the heights of all symmetric peaks in all paths in this case (with k fixed) is given by $(h_s(k) - 2w^*(k) + s(k) - 2s^*(k))g(n-k-1) + w(n-k-1)g(k)$. By varying k from 3 to $n-3$ and adding the special two cases, we obtain the desired result.

This completes the proof. □

The first ten values of the sequence $h_s(n)$ for $n = 2, \dots, 11$ are as follows:

$$1, \quad 2, \quad 5, \quad 10, \quad 20, \quad 42, \quad 91, \quad 206, \quad 485, \quad 1174.$$

5. Symmetric and asymmetric peaks in non-decreasing DAP

In 1997, Barcucci et al. [2] introduced the concept of non-decreasing Dyck paths. Later in 2001, Prodinger [24] studied non-decreasing Dyck paths in relation to Elena trees. In 2003, Deutsch and Prodinger [10] provided bijections between non-decreasing Dyck paths, directed column-convex polyominoes, Elena trees, and ordered trees of height at most three. In 2015, Flórez et al. [8] used generating functions to count the number of peaks, pyramid weight, and number of valleys in all non-decreasing Dyck paths of a given length. Recently, there has been significant research on the concept of non-decreasing paths, see for example, [3,9,14,17].

In this section, we first focus on the set $\mathcal{ND}$ of non-decreasing DAPs, that is, DAPs where the sequence of the minimal ordinates of the valleys D_kU, $k \geq 1$, (taken from left to right) is non-decreasing. We consider the trivariate generating function

$$G_{\mathsf{sp},\mathsf{ap}}(x, y, z) = \sum_{P \in \mathcal{ND}} x^{|P|} y^{\mathsf{sp}(P)} z^{\mathsf{ap}(P)},$$

where the coefficient of $x^n y^k z^\ell$ is the number of non-decreasing Dyck paths with air pockets of length n with k symmetric peaks and ℓ asymmetric peaks. Corollary 2.3 counts the total number of DAPs formed by symmetric peaks (with no asymmetric peaks). The same result holds here. That is, the total number of non-decreasing DAPs that consist solely of paths with peaks located at a ground level F_{n-1} for $n > 1$.

We summarize our results in a table at the end of this section.

Theorem 5.1. *The generating function $G_{\mathsf{sp},\mathsf{ap}}(x, y, z)$ for the number of non-decreasing DAPs with respect to the length and the numbers of symmetric and asymmetric peaks is*

$$\frac{1 - 3x + (3 - y)x^2 - (1 - 2y + z)x^3 - (y - z)x^4 + x^5 z^2}{(1 - x - x^2 y)(1 - 2x - x^2(-1 + y) + x^3(y - z))}.$$

Proof. To simplify notation, we set $G = G_{\mathsf{sp},\mathsf{ap}}(x, y, z)$. Let P be a nonempty non-decreasing DAPs. We distinguish the following cases.

(1) If $P = U^a D_a Q$, $a \geq 1$, where Q is a DAP (possibly empty), then the g.f. is $\frac{x^2}{1-x} yG$.

(2) If $P = UU^a D_a Q U^b D_{b+1}$, for $a, b \geq 1$, where Q is a DAP, then necessarily Q is empty or of the form $Q = U^{a_1} D_{a_1} U^{a_2} D_{a_2} \cdots U^{a_k} D_{a_k}$, for some $k \geq 1$, and $a_i \geq 1$ for $i \leq k$. See Fig. 10 for an illustration. The generating function for this case is given by

$$x \frac{x^2}{1-x} z \frac{1}{1 - \frac{x^2}{1-x} y} \frac{x^2}{1-x} z = \frac{z^2 x^5}{(1 - x - x^2 y)(1 - x)}.$$

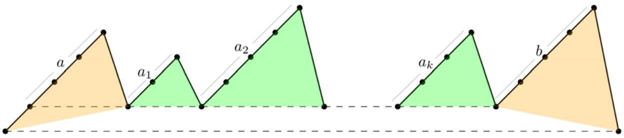

FIGURE 10. Decomposition of case (2)

(3) If $P = UU^aD_a\bar{Q}$, for $a \geq 1$, where Q is not empty and does not start or end with a symmetric peak, and $\bar{Q}$ is obtained from Q by increasing by one the size of its last down-step. The g.f. for this case is $xz\frac{x^2}{1-x}B$, where B is the generating function for non-empty non-decreasing DAPs that do not end with a symmetric peak. Considering the complement, we easily have $B = G - \frac{1}{1-\frac{x^2}{1-x}y}$.

(4) If $P = U\bar{Q}$ where Q is not empty and does not end and start with a symmetric peak, and $\bar{Q}$ is obtained from Q by increasing by one the size of its last down-step. The g.f. for this case is xC, where C is the generating function for non-empty non-decreasing DAPs that do not start or end with a symmetric peak. We easily have $C = B \cdot (1 - \frac{x^2}{1-x}y)$.

Summarizing all these cases, we obtain the following functional equation:

$$G = 1 + \frac{x^2}{1-x}yG + \frac{z^2x^5}{(1-x-x^2y)\,(1-x)}$$
$$+\frac{x^3z}{1-x}\left(G - \frac{1}{1-\frac{x^2y}{1-x}}\right) + x\left(G - \frac{1}{1-\frac{x^2y}{1-x}}\right)\left(1 - \frac{x^2y}{1-x}\right),$$

which induces the result. □

5.1. Increasing DAPs

Let $\mathcal{I}$ be the set of strictly increasing DAPs, which are non-decreasing DAPs where consecutive valleys cannot have the same ordinate. Notice that any path in $\mathcal{I}$ has one of the following three forms: (i) a non-decreasing DAP without symmetric peaks, (ii) a symmetric peak followed by a non-decreasing DAP without symmetric peaks, or (iii) a symmetric peak followed by another symmetric peak.

Now, let us consider the trivariate generating function

$$H_{\mathsf{sp},\mathsf{ap}}(x,y,z) = \sum_{P\in\mathcal{I}} x^{|P|}y^{\mathsf{sp}(P)}z^{\mathsf{ap}(P)},$$

where the coefficient of $x^ny^kz^\ell$ is the number of increasing Dyck paths with air pockets of length n having k symmetric peaks and ℓ asymmetric peaks.

TABLE 1. Some statistics for non-decreasing DAPs

	g.f.	Formula
Non-decreasing DAPs avoiding symmetric peaks	$\frac{1-3x+3x^2-2x^3+x^4+x^5}{(1-x)(1-2x+x^2-x^3)}$	A077855
Non-decreasing DAPs avoiding asymmetric peaks	$\frac{1-x}{1-x-x^2}$	F_{n-1}
Total number of symmetric peaks	$\frac{x^2(1-4x+5x^2-3x^3+2x^4+x^5-x^6)}{(1-2x)^2(1-x-x^2)}$	$(n-1)2^{n-6}+F_{n-1}$ $(n \geq 5)$
Total number of asymmetric peaks	$\frac{x^5\left(2-4x+x^3\right)}{(1-x-x^2)(1-2x)^2}$	$(n+5)2^{n-6}+F_{n-1}$ $(n \geq 5)$

TABLE 2. Some statistics for increasing DAPs

	g.f.	Formula/asymptotic
Increasing DAPs avoiding symmetric peaks	$\frac{1-3x+3x^2-2x^3+x^4+x^5}{(1-x)(1-2x+x^2-x^3)}$	A077855
Increasing DAPs avoiding asymmetric peaks	$\frac{1-2x+2x^2-x^3+x^4}{(1-x)^2}$	Sequence of natural numbers (modulo a shift)
Total number of symmetric peaks	$\frac{x^2\left(1-3x+5x^2-6x^3+3x^4-x^5\right)}{(1-x)^2(1-2x+x^2-x^3)}$	Asymptotic for the n-th term is $-\frac{a\left(a^5-3a^4+6a^3-5a^2+3a-1\right)(a^2-a+2)^n}{(3a^2-2a+2)(a-1)^2}$, where $a \sim 0.5698402912$
Total number of asymmetric peaks	$\frac{x^5(1-x+x^2)(2-4x+2x^2-x^3)}{(1-x)^2(1-2x+x^2-x^3)^2}$,	$-\frac{a^3\left(a^3-2a^2+4a-2\right)\left(a^2-a+1\right)(a^2-a+2)^n}{(3a^2-2a+2)^2(a-1)^2}n$, where a is defined above

Based on the three previous forms of an increasing DAP, we can deduce the following.

Theorem 5.2. *The generating function $H_{sp,ap}(x,y,z)$ for the number of increasing DAPs with respect to the length and the numbers of symmetric and asymmetric peaks is*

$$H_{sp,ap}(x,y,z) = G(x,0,z)\left(1+\frac{x^2y}{1-x}\right)+\left(\frac{x^2y}{1-x}\right)^2.$$

The first terms of the Taylor expansion are

$$1+x^2y+x^3y+(y^2+y)x^4+(2y^2+z^2+y)x^5+(3y^2+3z^2+y)x^6+O(x^7).$$

The g.f. for the number of increasing DAPs is

$$H(x,1,1)=\frac{(1-x)^2(1+x^2)}{1-2x+x^2-x^3}.$$

The Taylor expansion of the given expression is

$$1+x^2+x^3+2x^4+4x^5+7x^6+12x^7+21x^8+37x^9+O(x^{10}),$$

where the sequence of coefficients of x^n corresponds to A005251 in [23]. Notice that the increasing DAPs of length n are in bijection with the Dyck path of semi-length $n-1$, subject to the condition that the vector of valleys $(\nu_1,\nu_2,\ldots,\nu_k)$ satisfies $\nu_{i+1}-\nu_i \geq 2$, as shown in [19].

We can adapt the same proofs used in the previous sections to non-decreasing DAPs. Therefore, we will only provide a summary of the results without including the proofs.

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Declarations

Conflict of interest The authors declare no competing interests.

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