



# The irregularity strength of circulant graphs

J.-L. Baril, H. Kheddouci, O. Togni

*LE2I, UMR CNRS 5158, Université de Bourgogne, BP 47870, 21078 Dijon cedex, France*

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## Abstract

The irregularity strength of a simple graph is the smallest integer  $k$  for which there exists a weighting of the edges with positive integers at most  $k$  such that all the weighted degrees of the vertices are distinct. In this paper we study the irregularity strength of circulant graphs of degree 4. We find the exact value of the strength for a large family of circulant graphs.

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## 1. Introduction and definitions

All the graphs we deal with are undirected, simple and connected.

Let  $G = (V, E)$  be a graph with vertex set  $V$  and edge set  $E$ .

A function  $w : E \rightarrow \mathbb{Z}^+$  is called a *weighting* of  $G$ , and for an edge  $e \in E$ ,  $w(e)$  is called the *weight* of  $e$ . The *strength*  $s(w)$  of  $w$  is defined as  $s(w) = \max_{e \in E} w(e)$ . The *weighted degree* of a vertex  $x \in V$  is the sum of the weights of its incident edges:  $d_w(x) = \sum_{e \ni x} w(e)$ . The *irregularity strength*  $s(G)$  of  $G$  is defined as  $s(G) = \min\{s(w), w \text{ is an irregular weighting of } G\}$ .

The study of  $s(G)$  was initiated by Chartrand et al. [3] and has proven to be difficult in general. There are not many graphs for which the irregularity strength is known. For an overview of the subject, the reader is referred to the survey of Lehel [8] and recent papers [7,6,1,9,5].

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*E-mail addresses:* [barjl@u-bourgogne.fr](mailto:barjl@u-bourgogne.fr) (J.-L. Baril), [kheddouc@u-bourgogne.fr](mailto:kheddouc@u-bourgogne.fr) (H. Kheddouci), [otogni@u-bourgogne.fr](mailto:otogni@u-bourgogne.fr) (O. Togni).

The irregularity strength of regular graphs was considered by several authors. For a regular graph  $G$  of order  $n$  and degree  $r$ , let  $\lambda(G) = \lceil (n + r - 1)/r \rceil$ . A simple counting argument gives  $s(G) \geq \lambda(G)$ .

On the other hand, an upper bound of  $(n/2) + 9$  was given for regular graphs of order  $n$  in [4]. This result was improved recently in [5]. Nevertheless, there is still a great gap between lower and upper bounds on the irregularity strength of regular graphs.

Moreover, for all the connected regular graphs for which the irregularity strength is known, we have  $s(G) \leq \lambda(G) + 1$ .

The following conjecture is due to Jacobson (see [8,4]):

**Conjecture 1.** There exists an absolute constant  $c$  such that for each regular graph  $G$ ,  $s(G) \leq \lambda(G) + c$ .

Circulant graphs are a large family of regular graphs which are studied in the context of interconnection networks [2] and have good properties such as symmetry and vertex-transitivity.

**Definition 1.** Let  $n$  be an integer and let  $s_1, s_2, \dots, s_k$  be a sequence of integers, with  $1 \leq s_1 < s_2 < \dots < s_k \leq n/2$ . The circulant graph  $G = C_n(s_1, s_2, \dots, s_k)$  of order  $n$  is a graph with vertex set  $V(G) = \{0, 1, \dots, n - 1\}$  and edge set  $E(G) = \{(x, x \pm s_i \text{ mod } n), x \in V(G), 1 \leq i \leq k\}$ . It is a Cayley graph on the additive group of integers modulo  $n$ , with set of generators  $\{s_1, s_2, \dots, s_k\}$ .

In this paper, we study the irregularity strength of circulant graphs of degree 4. We find the exact value for the irregularity strength of circulant graphs of the form  $C_n(1, k)$  with  $n \geq 4k + 1$ . These graphs consists in a cycle of length  $n$  plus chords joining vertices at distance  $k$  on the cycle.

We will use the following notations (see Fig. 1): for any  $t$ , the vertices of  $C_{4t+1}(1, k)$  are labeled  $y_0, y_1, \dots, y_{2t}$  going clockwise on the cycle and  $x_0 = y_0, x_1, x_2, \dots, x_{2t}$  going counterclockwise.

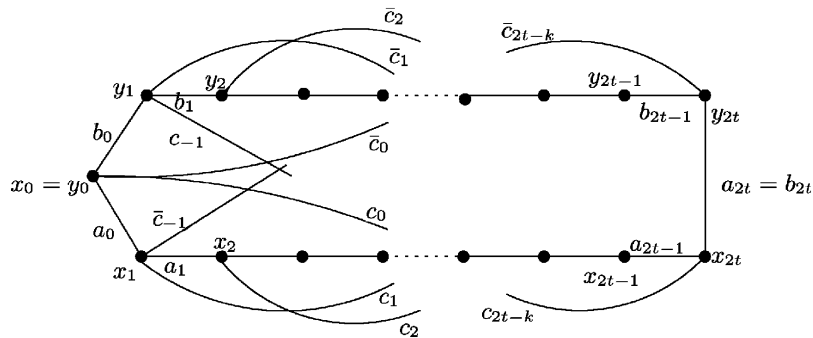


Fig. 1. Notations.

For a weighting  $w$  on  $C_{4t+1}(1, k)$ , denote by  $a_i$  (resp.  $b_i$ ) the weight of the edge  $(x_i, x_{i+1})$  (resp.  $(y_i, y_{i+1})$ ) for  $0 \leq i \leq 2t - 1$ , and by  $a_{2t} = b_{2t}$  the weight of the edge  $(x_{2t}, y_{2t})$ , and let

$$\begin{cases} c_i = w(x_i, x_{i+k}) & 0 \leq i \leq 2t - k, \\ \bar{c}_i = w(y_i, y_{i+k}) & 0 \leq i \leq 2t - k, \\ c_i = w(x_i, y_{4t-i-k+1}) & 2t - k + 1 \leq i \leq 2t, \\ \bar{c}_i = w(y_i, x_{4t-i-k+1}) & 2t - k + 1 \leq i \leq 2t, \\ c_{-i} = w(y_i, x_{k-i}) & 1 \leq i \leq k - 1, \\ \bar{c}_{-i} = w(x_i, y_{k-i}) & 1 \leq i \leq k - 1. \end{cases}$$

Let us give a construction of  $C_{4(t+1)+1}(1, k)$  from  $C_{4t+1}(1, k)$  useful for the proof of main results. Let  $G = C_{4t+1}(1, k)$  and let  $P_4 = (x_{2t+1}, x_{2t+2}, y_{2t+2}, y_{2t+1})$  be a path of length 3 where  $t \geq k \geq 2$ . We define the graph  $G.P_4$  as follows:

$$\begin{aligned} V(G.P_4) &= V(G) \cup V(P_4), \\ E(G.P_4) &= E(G) \setminus \left( \bigcup_{i=1}^k \{(x_{2t-k+i}, y_{2t-i+1})\} \cup \{(x_{2t}, y_{2t})\} \right) \\ &\quad \cup E(P_4) \bigcup_{i=1}^k \{(x_{2t-k+2+i}, y_{2t+3-i})\} \\ &\quad \cup \{(x_{2t-k+1}, x_{2t+1}), (x_{2t-k+2}, x_{2t+2}), (y_{2t-k+1}, y_{2t+1}), \\ &\quad (y_{2t-k+2}, y_{2t+2})\}. \end{aligned}$$

Observe that  $G.P_4 = C_{4t+5}(1, k)$ . Thus for any  $t \geq k \geq 2$ ,  $C_{4t+1}(1, k)$  can be obtained by iterating  $(t - k)$  times this composition starting from  $C_{4k+1}(1, k)$ .

## 2. Preliminaries

**Definition 2.** For  $n = 4t + 1$ , with  $t \geq k \geq 2$ ,  $C_n(1, k)$  verifies the property  $\mathcal{P}_t$  if it admits an irregular weighting of strength  $s = \lambda(C_n(1, k)) = t + 1$  such that:

- (i)  $a_{2t} = b_{2t} = a_{2t-1} = b_{2t-1} + 1 = s$ ,
- (ii)  $c_{2t-i} = \bar{c}_{2t-i} = s$ , for each  $i, 0 \leq i \leq k$ ,
- (iii)  $a_{2t-2i+1} \geq b_{2t-2i+1} \geq \lceil k/2 \rceil + 2 - i$ , for each  $i, 0 \leq i \leq \lceil k/2 \rceil - 1$ ,
- (iv) when  $k$  is even,  $a_{2t-2k+2i+1} \geq i + 1$ , for each  $i, 0 \leq i \leq k - 1$ .

**Lemma 1.** For  $k \geq 2$ ,  $C_{4k+1}(1, k)$  verifies the property  $\mathcal{P}_k$ .

**Proof.** In order to prove that  $C_{4k+1}(1, k)$  verifies the property  $\mathcal{P}_k$ , we construct an irregular weighting  $w$  as follows:

$$\begin{cases} a_0 = b_0 = 1, \\ a_i = \left\lfloor \frac{i+1}{2} \right\rfloor + 1 & \text{for each } i, \quad 1 \leq i \leq 2k, \\ b_i = a_{i-1} & \text{for each } i, \quad 1 \leq i \leq 2k, \\ c_i = \bar{c}_i = i + 1 & \text{for each } i, \quad 0 \leq i \leq k - 1, \\ c_i = \bar{c}_i = k + 1 & \text{for each } i, \quad k \leq i \leq 2k, \\ c_i = \bar{c}_i = 1 & \text{for each } i, \quad 1 - k \leq i \leq -1. \end{cases}$$

We can easily verify that we have  $d_w(x_0) = d_w(y_0) = 4$ , and for each  $1 \leq i \leq 2k$ :

$$\begin{cases} d_w(x_i) = a_i + a_{i-1} + c_i + c_{i-k} = 4 + 2i, \\ d_w(y_i) = b_i + b_{i-1} + c_i + c_{i-k} = 3 + 2i. \end{cases}$$

Consequently,  $w$  is irregular and of strength equal to  $\lambda(C_{4k+1}(1, k)) = k + 1$  (i.e., statements (i) and (ii) are verified). Moreover, by construction the remaining statements of property  $\mathcal{P}_k$  are also verified: the third statement is verified since  $a_{2k-2i+1} \geq b_{2k-2i+1} = \lfloor (2k - 2i + 1)/2 \rfloor + 1 \geq \lceil k/2 \rceil - i + 2$ . The fourth statement is also verified since  $a_{2i+1} = \lfloor (2i + 2)/2 \rfloor + 1 = i + 2 \geq i + 1$ . See part A of Appendix for an example when  $k = 4$ .  $\square$

### 3. Results

In this section, we present two propositions. The first one construct an irregular weighting for  $C_{4t+1}(1, k)$  for any  $t \geq k$ . The second proposition allows to give an irregular weighting for the remaining graphs  $C_n(1, k)$ , with  $n = 4t + 2, 4t + 3, 4t + 4$  for each  $t \geq k$ .

**Proposition 1.** *For any  $t$  with  $t \geq k \geq 2$ , there exists an irregular weighting of  $C_{4t+1}(1, k)$  which verifies the property  $\mathcal{P}_t$ .*

**Proof.** The proof is done by induction on  $t$ . For  $t = k$ , the result holds by Lemma 1.

Assume that there exists an irregular weighting of  $C_{4t+1}(1, k)$ , which verifies the property  $\mathcal{P}_t$ . We will construct a weighting of  $C_{4t+5}(1, k)$  which verifies property  $\mathcal{P}_{t+1}$ .

Let  $w$  be an irregular weighting of  $G = C_{4t+1}(1, k)$  which verifies the property  $\mathcal{P}_t$ . Let  $G' = G.P_4 = C_{4t+5}(1, k)$  and  $w'$  be the weighting of  $G'$  constructed from the weighting  $w$  of  $G$  as follows:

$$w'(e) = \begin{cases} w(e) & \text{for } e \in E(G) \setminus M, \\ w(e) - 1 & \text{for } e \in M, \\ t + 2 & \text{for } e \in A \cup B, \\ t + 1 & \text{for } e \in C, \end{cases}$$

where

$$\begin{aligned}
 M &= \bigcup_{i=1}^{\lceil k/2 \rceil - 1} \{(x_{2t-2i+1}, x_{2t-2i+2}), (y_{2t-2i+1}, y_{2t-2i+2})\}, \\
 A &= \bigcup_{i=1}^k \{(x_{2t-k+i}, y_{2t-i+3})\}, \\
 B &= \{(x_{2t+2}, y_{2t+2}), (y_{2t-k+2}, y_{2t+2}), (x_{2t-k+2}, x_{2t+2}), (x_{2t+1}, x_{2t+2})\}, \\
 C &= \{(x_{2t}, x_{2t+1}), (y_{2t}, y_{2t+1}), (x_{2t-k+1}, x_{2t+1}), (y_{2t+1}, y_{2t+2}), (y_{2t-k+1}, y_{2t+1})\}.
 \end{aligned}$$

Note that, by statement (iii) of property  $\mathcal{P}_t$  of  $G$ ,  $w(e) \geq 2$  if  $e \in M$ . Thus,  $w'(e) \geq 1$  for each  $e \in E(G')$ . Moreover, the degrees of the vertices given by the weighting  $w'$  are as follows:

$$\begin{cases} d_{w'}(x_i) = d_w(x_i) = 2i + 4 & \text{for } 0 \leq i \leq 2t - k + 1, \\ d_{w'}(y_i) = d_w(y_i) = 2i + 3 & \text{for } 1 \leq i \leq 2t - k + 1. \end{cases}$$

Observe that for each of the following vertices, the weight of one adjacent edge has been decreased by one, and the weight of another adjacent edge has been increased by one. So the weighted degrees are unchanged: for  $0 \leq i \leq \lceil k/2 \rceil - 2$ ,

$$\begin{cases} d_{w'}(x_{2t-2i}) = (a_{2t-2i-1} - 1) + a_{2t-2i} + c_{2t-2i-k} + (c_{2t-2i} + 1) = d_w(x_{2t-2i}), \\ d_{w'}(y_{2t-2i}) = (b_{2t-2i-1} - 1) + b_{2t-2i} + \bar{c}_{2t-2i-k} + (\bar{c}_{2t-2i} + 1) = d_w(y_{2t-2i}), \\ d_{w'}(x_{2t-2i-1}) = a_{2t-2i-1} + (a_{2t-2i} - 1) + c_{2t-2i-k} + (c_{2t-2i} + 1) \\ \qquad \qquad \qquad = d_w(x_{2t-2i-1}), \\ d_{w'}(y_{2t-2i-1}) = b_{2t-2i-1} + (b_{2t-2i} - 1) + \bar{c}_{2t-2i-k} + (\bar{c}_{2t-2i} + 1) \\ \qquad \qquad \qquad = d_w(y_{2t-2i-1}). \end{cases}$$

Moreover, when  $k$  is even, the weighted degrees of vertices  $x_{2t-k+2}$  and  $y_{2t-k+2}$  are not given above. Their weighted degrees are:

$$\begin{cases} d_{w'}(x_{2t-k+2}) = a_{2t-k+1} + a_{2t-k+2} + c_{2t-2k+2} + c_{2t-k+1} = d_w(x_{2t-k+2}) + 1, \\ d_{w'}(y_{2t-k+2}) = b_{2t-k+1} + b_{2t-k+2} + c_{2t-2k+2} + c_{2t-k+1} = d_w(y_{2t-k+2}) + 1. \end{cases}$$

We can see that, in this case,  $d_{w'}(x_{2t-k+2}) = d_{w'}(y_{2t-k+3})$  and there does not exist a vertex  $u \in V(G')$  such that  $d_{w'}(u) = d_w(y_{2t-k+2})$  (see part B of Appendix).

For the vertices of  $P_4$  we have

$$\begin{cases} d_{w'}(x_{2t+1}) = 4t + 6, \\ d_{w'}(x_{2t+2}) = 4t + 8, \\ d_{w'}(y_{2t+1}) = 4t + 5, \\ d_{w'}(y_{2t+2}) = 4t + 7. \end{cases}$$

Let  $a'_i, b'_i, c'_i$  and  $\bar{c}'_i$  be the values of edges of  $G'$  given by the weighting  $w'$  as defined in Section 1. We distinguish two cases depending on the parity of  $k$ .

Case 1:  $k$  is odd.

In order to verify the property  $\mathcal{P}_{t+1}$  for  $G'$ , one can see that  $1 \leq w'(e) \leq t + 2$  for each  $e \in E(G')$  and for each  $u, v \in V(G')$ ,  $d_{w'}(u) \neq d_{w'}(v)$  when  $u \neq v$ .

Thus  $w'$  is irregular and the statements (i) and (ii) hold.

The third one (iii) is also verified since

$$a'_{2(t+1)-2i+1} \geq b'_{2(t+1)-2i+1} = t + 1 \geq \left\lceil \frac{k}{2} \right\rceil + 2 - i \quad \text{for } i = 1 \text{ or } 2$$

and

$$\begin{aligned} a'_{2(t+1)-2i+1} &\geq b'_{2(t+1)-2i+1} \geq b_{2t-2(i-1)+1} - 1 \geq \left\lceil \frac{k}{2} \right\rceil + 2 - (i - 1) - 1 \\ &\geq \left\lceil \frac{k}{2} \right\rceil + 2 - i \quad \text{for each } i, \quad 3 \leq i \leq \left\lceil \frac{k}{2} \right\rceil - 1. \end{aligned}$$

Case 2:  $k$  is even (a complete example is given in Appendix). We use a cycle  $C$  induced by the following vertices  $C = (x_{2t-k+2}, x_{2t-k+1}, \dots, x_{2t-2k+2})$ . In order for the weighting to be irregular, it suffices to decrease by two the weighted degree of  $x_{2t-k+2}$  in  $w'$  (i.e., to obtain  $d_{w'}(x_{2t-k+2}) = d_w(y_{2t-k+2})$ ) without changing degrees of other vertices.

To do this, we decrease and increase alternately by one the weight of the edges of  $C$  starting from  $x_{2t-k+2}$ . As  $C$  is odd, the two edges of  $C$  which are adjacent to  $x_{2t-k+2}$  are both decreased by one. Thus the degree of  $x_{2t-k+2}$  is decreased by two. Moreover, for any other vertex of  $C$ , the degree is unchanged.

As  $G$  verified statement (iv) of property  $\mathcal{P}_t$ , then the weights of edges of  $C$  which are decreased are greater or equal to one. In other hand, as the strength of  $G$  is equal to  $t + 1$  and edges of  $C$  are in  $G$ , so by increasing some edges of  $C$  by one, the strength of  $w'$  remains  $t + 2 = \lambda(G')$ .

Observe that statements (i)–(iii) of property  $\mathcal{P}_{t+1}$  are proved similarly than in case  $k$  odd. So we now prove statement (iv) for  $G'$ . Note that

$$\begin{aligned} a'_{2(t+1)-2k+2i+1} &= a'_{2t-2k+2(i+1)+1} = a_{2t-2k+2(i+1)+1} - 1 \geq i + 1 \\ &\text{for } 1 \leq i \leq k - 2. \end{aligned}$$

Moreover, for  $i = k - 1$ ,  $a'_{2(t+1)-2k+2i+1} = t + 2 \geq k - 1 + 1 = k$ .  $\square$

The following proposition extends the irregular weighting to cases  $n = 4t + 2, 4t + 3$  and  $4t + 4$ , for  $t \geq k$ .

**Proposition 2.** *If  $C_{4t+1}(1, k)$  verifies property  $\mathcal{P}_t$  then there exists an irregular weighting of  $C_{4t+1+i}(1, k)$ , for any  $i, 1 \leq i \leq 3$ , of strength  $s' = \lambda(C_{4t+1+i}(1, k)) = t + 2$ .*

**Proof.** Observe that  $\lambda(C_{4t+1+i}(1, k)) = \lceil (4t + 1 + i + 3)/4 \rceil = t + 2$ . We start from  $G = C_{4t+1}(1, k)$  with an irregular weighting verifying property  $\mathcal{P}_t$  and we add  $i$  vertices,

$1 \leq i \leq 3$ , by subdividing the edge  $(x_{2t}, y_{2t})$  in  $i + 1$  edges (see Figs. 2–4, where  $\alpha_1, \alpha_2, \beta_1$  and  $\beta_2$  are the weights of some edges of the irregular weighting of  $C_{4t+1}(1, k)$ ).

The new edges and the edges whose endpoints have been modified are weighted as shown below:

Case  $i = 1$ : a weighting is obtained simply by subdividing the edge  $(x_{2t}, y_{2t})$  in two and then by giving the weight  $t + 2$  to the two created edges (see Fig. 2).

Case  $i = 2$ : we distinguish between the case  $k$  odd and  $k$  even. The process for both cases is given in Fig. 3, where dashed edges have weight  $t + 1$  and bold edges are edges whose weights have been modified.

Case  $i = 3$ : we distinguish between the case  $k$  odd and  $k$  even. The process for both cases is given in Fig. 4.  $\square$

From Propositions 1 and 2, we deduce the following result:

**Theorem 1.** For  $n \geq 4k + 1$  and  $k \geq 2$ ,

$$s(C_n(1, k)) = \left\lceil \frac{n + 3}{4} \right\rceil.$$

**Corollary 1.** For  $n \geq 4s_k + 1$ ,

$$s(C_n(1, s_1, s_2, \dots, s_k)) \leq \left\lceil \frac{n + 3}{4} \right\rceil.$$

**Proof.** Starting from an irregular weighting of  $C_n(1, s_k)$  given by Theorem 1, assign the weight 1 to the other edges of  $C_n(1, s_1, s_2, \dots, s_k)$  and the weighting will remain irregular.  $\square$

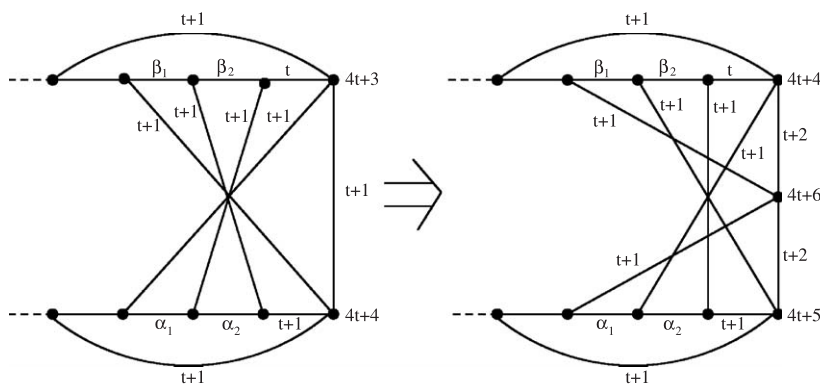


Fig. 2. An irregular weighting of  $C_{4t+2}(1, k)$  on the right is obtained from the irregular weighting of  $C_{4t+1}(1, k)$  on the left.

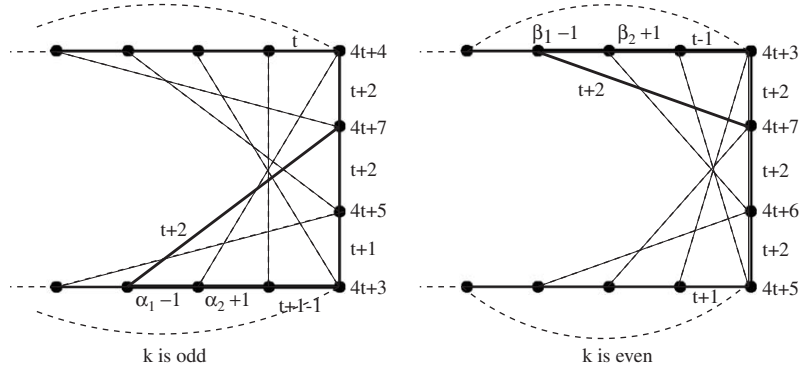


Fig. 3. An irregular weighting of  $C_{4t+3}(1, k)$ , with  $k$  odd ( $k = 5$ ) on the left and  $k$  even ( $k = 4$ ) on the right.

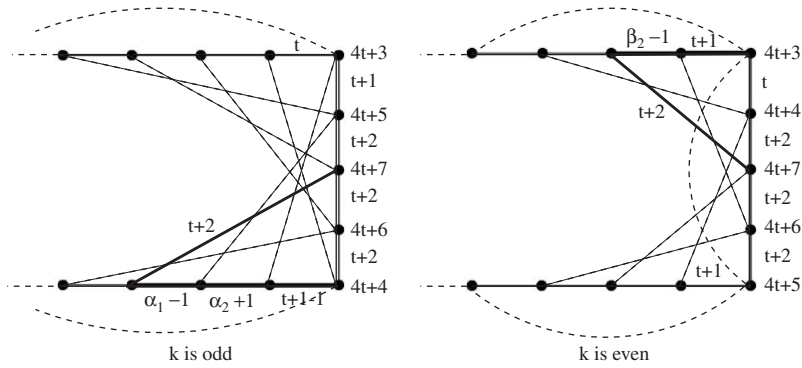


Fig. 4. An irregular weighting of  $C_{4t+4}(1, k)$ , with  $k$  odd ( $k = 5$ ) on the left and  $k$  even ( $k = 4$ ) on the right.

**4. Concluding remarks**

We studied the irregularity strength of circulant graphs  $C_n(1, k)$  when  $k \leq (n - 1)/4$  and found the exact value. Our method does not seem to work for  $(n/4) \leq k \leq (n/2)$ , because we have not found an irregular weighting to start the induction process.

But, the isomorphisms of circulant graphs allow us to come back to some cases when  $k \leq (n/4)$ . For instance, the following isomorphisms are easy to see:

$$\begin{cases} C_{2k-1}(1, k) \cong C_{2k-1}(1, 2), \\ C_{3k-1}(1, k) \cong C_{3k-1}(1, 3), \\ C_{4k-1}(1, k) \cong C_{4k-1}(1, 4). \end{cases}$$

More generally, if  $\gcd(n, k) = 1$  then we have  $C_n(1, k) \cong C_n(1, k^{-1} \pmod n)$ .

Moreover, our study is also generalized to some circulant graphs  $C_n(s_1, s_2)$  with  $s_1 \neq 1$  and  $s_2 > s_1$ . If  $\gcd(n, s_1) = 1$  or  $\gcd(n, s_2) = 1$  then  $C_n(s_1, s_2) \cong C_n(1, k)$  for some  $k$ .



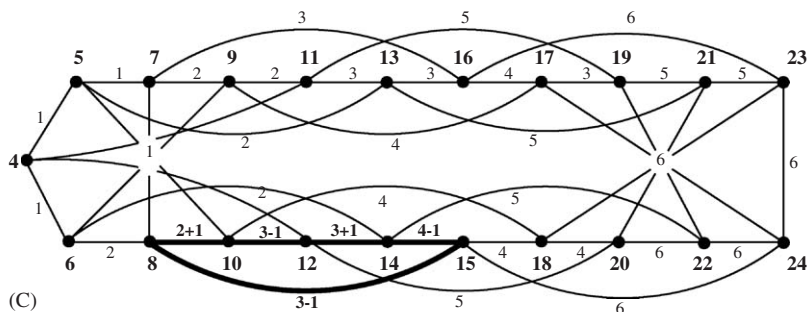
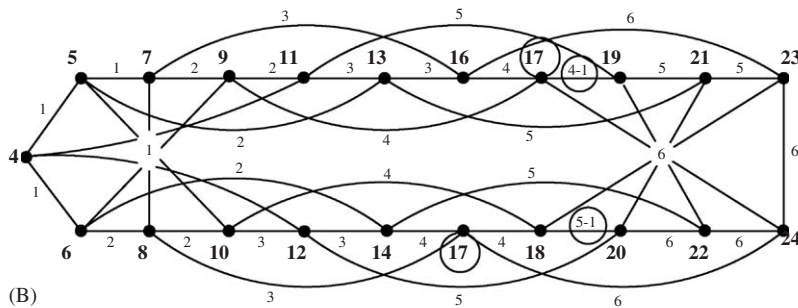
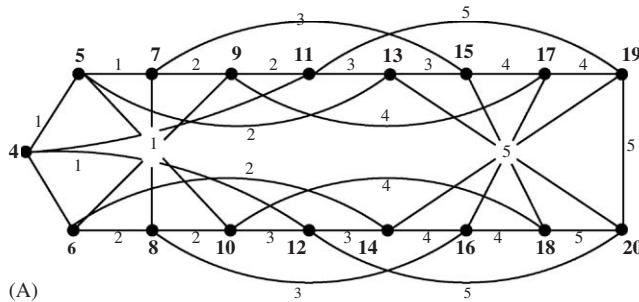
**Appendix: a complete example when  $k = 4$**

Let us give a complete example of the construction of an irregular weighting of  $C_{21}(1, 4)$  from  $C_{17}(1, 4)$ . The values on the edges represent their weights and the bold values on vertices represent their weighted degrees.

**Part A:** an irregular weighting of  $C_{17}(1, 4)$  verifying property  $\mathcal{P}_4$ .

**Part B:** the first step of the extension of the weighting of  $C_{17}(1, 4)$  to the graph  $C_{21}(1, 4)$  by composition with  $P_4$ . Notice that the weighting is not irregular as there is two vertices of degree 17.

**Part C:** the weighting of part B is modified to become irregular by using a cycle of length 5 (in bold on the figure) to decrease the weighted degree of one of the two vertices of degree 17 to 15.



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