Statistics-preserving bijections between classical and cyclic permutations

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1. Introduction and notation

Let $S_n$ be the set of permutations of length $n$, i.e., all one-to-one correspondences from $[n] = \{1, 2, \ldots, n\}$ into itself. We represent a permutation $\sigma \in S_n$ in one-line notation, $\sigma = \sigma_1 \sigma_2 \ldots \sigma_n$, where $\sigma_i = \sigma(i)$, $1 \leq i \leq n$. Moreover, if $\gamma = \gamma(1) \gamma(2) \ldots \gamma(n)$ is a length $n$ permutation then the product $\gamma \cdot \sigma$ is the permutation $\gamma(\sigma_1) \gamma(\sigma_2) \ldots \gamma(\sigma_n)$. In $S_n$, a $k$-cycle $\sigma = (i_1, i_2, \ldots, i_k)$ is a length $n$ permutation verifying $\sigma(i_1) = i_2$, $\sigma(i_2) = i_3$, $\ldots$, $\sigma(i_{k-1}) = i_k$, $\sigma(i_k) = i_1$ and $\sigma(j) = j$ for $j \in [n]\setminus\{i_1, \ldots, i_k\}$. In particular, a 2-cycle is called a transposition. Let $C_n \subset S_n$ be the set of $n$-cycles. The elements of $C_n$ will be called cyclic permutations (or cycles for short). Obviously $C_{n+1}$ and $S_n$ have the same cardinality.

Any permutation $\sigma \in S_n$ is uniquely decomposed as a product of transpositions of the following form:

$$\sigma = \langle p_1, 1 \rangle \cdot \langle p_2, 2 \rangle \cdot \langle p_3, 3 \rangle \cdot \ldots \cdot \langle p_n, n \rangle = \prod_{i=1}^{n} \langle p_i, i \rangle, \quad (1)$$

where $p_i$ are some integers such that $1 \leq p_i \leq i \leq n$. Conversely, any such decomposition provides a permutation in $S_n$. Therefore, (1) yields a bijection from $S_n$ to the product set $T_n = [1] \times [2] \times \cdots \times [n]$. Then we have another way to represent a permutation:

**Definition 1.** The transposition array of a permutation $\sigma = \prod_{i=1}^{n} \langle p_i, i \rangle \in S_n$ is defined by $p_1 p_2 \ldots p_n \in T_n$.

For example, if $\sigma = 1 \ 4 \ 5 \ 6 \ 3 \ 2$ then its decomposition into transpositions is $\langle 1, 1 \rangle \cdot \langle 2, 2 \rangle \cdot \langle 3, 3 \rangle \cdot \langle 2, 4 \rangle \cdot \langle 3, 5 \rangle \cdot \langle 4, 6 \rangle$, and its corresponding transposition array is $1 \ 2 \ 3 \ 2 \ 3 \ 4$. Notice that this decomposition is used in [1] in order to obtain Gray codes for restricted classes of length $n$ permutations.

Let $\sigma$ be a permutation in $S_n$. A descent of $\sigma$ is a position $i$, $1 \leq i \leq n-1$, such that $\sigma(i) > \sigma(i+1)$. Let $D(\sigma)$ be the set of descents of $\sigma$. An excedance (resp.
For instance, descents appear in the theory of lattice path cedances and descents. In the literature, descents of a permutation \( \pi \) are studied. In Chapter 3, we give a constructive bijection from \( \text{weak excedances} \) using the transposition array representation. Section 2 shows how one can characterize special points of a permutation (left-to-right maximum, fixed point, excedance, ...) using the transposition array representation. They will be used throughout the paper. Before this, we state the two straightforward claims:

**Claim 1.** For every \( j \geq 1 \), the permutation \( \prod_{i=1}^{n} (p_i, i) \) fixes all values strictly larger than \( j \).

**Claim 2.** If \( k < \ell \) and if for all \( i \geq \ell \) we have \( p_i \neq k \), then the permutation \( \prod_{i=\ell}^{n} (p_i, i) \) fixes \( k \).

In [1], it is shown how an \( n \)-cycle can be characterized with its transposition array representation.

**Lemma 1.** (See [1, Remark 16].) Let \( \sigma \) be a permutation in \( S_n \) and \( p = p_1 p_2 \ldots p_n \) be its transposition array. Then \( \sigma \) is an \( n \)-cycle if and only if its transposition array contains only one fixed point, i.e., \( p_1 = 1 \) and \( p_i \neq 1 \) for \( i \geq 2 \). More generally, the number of cycles of \( \sigma \) is \( n - \ell \) where \( \ell \) is the number of indices \( 1 \leq i \leq n \), such that \( p_i \neq 1 \).

**Proof.** A simple induction on \( j \) starting at \( j = 1 \) shows that the number of cycles (relatively to the classical decomposition into disjoint cycles) in \( \prod_{i=1}^{n} (p_i, i) \) is \( n - \ell \), where \( \ell \) is the number of indices \( 1 \leq i \leq j \), such that \( p_i \neq 1 \). \( \square \)

**Lemma 2.** Let \( \sigma \) be a permutation in \( S_n \) and \( p_1 p_2 \ldots p_n \) be its transposition array. Then \( \sigma \) is a left-to-right maximum of \( \sigma \) if and only if there exists \( i \geq k \) such that (a) \( p_i = k \), and (b) \( p_j > k \) for \( j > i \).

**Proof.** Assume that \( k \) is a left-to-right maximum of \( \sigma \). We have \( \sigma(j) \leq \sigma(k) \) for \( j \leq k \) and in particular \( \sigma(k) \geq k \). Since \( \sigma = (p_1, 1) \cdot (p_2, 2) \cdot \ldots (p_n, n) \) with \( 1 \leq p_1 \leq \ldots \leq n \), and using Claims 1 and 2, the fact \( \sigma(k) \geq k \) implies the existence of \( i \geq k \) such that \( p_i = k \). We choose the rightmost \( i \geq k \) verifying this property. So, Claims 1 and 2 give \( \sigma(k) = i \). Assume for a contradiction that there is \( j > i \), such that \( p_j < p_i = k \); we choose the rightmost \( j \). This implies \( \sigma(j) = j \geq i = \sigma(k) \) with \( p_j < k \), which contradicts the fact that \( k \) is a left-to-right maximum. So (b) is verified. Using Claims 1 and 2, the converse becomes straightforward. \( \square \)

**Lemma 3.** Let \( \sigma \) be a permutation in \( S_n \) and \( p_1 p_2 \ldots p_n \) be its transposition array. Then \( \sigma \) is a fixed point (resp. quasi-fixed) of \( \sigma \) if and only if (a) \( p_k = k \) (resp. \( p_{k+1} = k \)), and (b) there does not exist \( i > k \) such that \( p_i = k \) (resp. \( i > k + 1 \) such that \( p_i = k \)).

**Proof.** Assume that \( k \) is a fixed point of \( \sigma = (p_1, 1) \cdot (p_2, 2) \cdot \ldots (p_n, n) \) with \( 1 \leq p_1 \leq \ldots \leq n \), i.e., \( \sigma(k) = k \). Suppose for a contradiction there exists \( i > k \) such that \( p_i = k \); we take the rightmost \( i \) verifying this property. Claims 1 and 2 give \( \sigma(k) = i > k \) which contradicts our hypothesis.
Thus $\sigma$ verifies (b). Claim 2 induces $\sigma(k) = \sigma'(k)$ where $\sigma' = (p_1, 1) \cdot (p_2, 2) \cdot \cdots \cdot (p_k, k)$. Since $\sigma(k) = k$, Claim 1 induces $p_k = k$. The converse is straightforward. The proof is obtained mutatis mutandis whenever we replace fixed points with quasi-fixed points.

**Lemma 4.** Let $\sigma$ be a permutation in $S_n$ and $p_1 p_2 \ldots p_n$ be its transposition array. Then $k$ is an excedance (resp. a weak excedance) of $\sigma$ if and only if there exists $i > k$ (resp. $i \geq k$) such that $p_i = k$.

**Proof.** Assume that $k$ is an excedance of $\sigma = (p_1, 1) \cdot (p_2, 2) \cdots (p_n, n)$ with $1 \leq p_i \leq i \leq n$, i.e., $\sigma(k) > k$. Suppose for a contradiction that there is no $i > k$ such that $p_i = k$. Claim 2 induces $\sigma(k) = \sigma'(k)$ with $\sigma' = (p_1, 1) \cdot (p_2, 2) \cdots (p_k, k)$. We deduce $\sigma(k) \leq k$ which contradicts the fact that $k$ is an excedance. The converse is straightforward using Claims 1 and 2, and by taking $i$ to be the rightmost $i$ such that $p_i = k$. The reasoning is similar for the case where $k$ is a weak excedance.

In order to illustrate these lemmas, we present an example for each of them. For instance, if we set $\sigma = 2413$, then $\sigma$ is a 4-cycle and its transposition array $p_1 p_2 p_3 p_4 = 1122$ contains only one fixed point ($p_1 = 1$). Moreover, $k = 2$ is a left-to-right maximum of $\sigma$ and there is $i = 4 \geq 2$ such that $p_i = 2$ and $p_j > 2$ for $j > 4$. $k = 2$ is an excedance and 2 appears on the right of $p_2$ in the transposition array 1122 of $\sigma$. For $\sigma = 2431$, $k = 3$ is a fixed point, its transposition array is 1132 and we have $p_k = 3$ and there does not exist $i > 3$ such that $p_i = 3$. Finally, $k = 1$ is a quasi-fixed point, we have $p_1 = 1$ and there does not exist $i > 2$ such that $p_i = 1$.

3. Fixed points, weak excedances and left-to-right maxima

Let $\phi$ be the map from $S_n$ to $S_{n+1}$ defined, for every $\sigma \in S_n$, by

$$\phi(\sigma) = (1, 1) \cdot (p_1, 2) \cdot (p_2, 3) \cdots (p_n, n+1),$$

where the transposition array of $\sigma$ is $p_1 p_2 \cdots p_n$.

For example, $\phi(321) = \phi((1, 1) \cdot (2, 2) \cdot (1, 3) = (1, 1) \cdot (1, 2) \cdot (2, 3) \cdot (1, 4) = 4312$. By construction the transposition array $1 p_1 p_2 p_3 p_4 \cdots p_n$ of $\phi(\sigma)$ has only one fixed point, that is on the first position. With Lemma 1, the permutation $\phi(\sigma)$ is a cyclic permutation and then belongs to $C_{n+1}$. Therefore $\phi$ is a bijection from $S_n$ to $C_{n+1}$ (see Table 1 for $n = 3$). Obviously, this construction allows to go back from $\phi(\sigma)$ to $\sigma$.

Now we prove that the bijection $\phi$ transforms the set of weak excedances of $\sigma \in S_n$ into the excedance set of $\phi(\sigma) \in C_{n+1}$.

**Remark 1.** If $\sigma \in S_n$ has no fixed points, then $E(\sigma) = WE(\sigma)$. This holds in particular when $\sigma$ is a cycle of length at least 2.

**Theorem 2.** The bijection $\phi : S_n \rightarrow C_{n+1}$ satisfies for any $\sigma \in S_n$,

$$\sigma(\phi(\sigma)) = \sigma'(k).$$

**Proof.** Let $k$ be a left-to-right maximum of $\sigma$, i.e., $\sigma(j) \leq \sigma(k)$ for $j < k$. Since it also is a weak excedance, Theorem 2 induces $\sigma'(k) = \sigma(k) + 1$ where $\sigma' = \phi(\sigma)$. Consider $j < k$. If $j$ is a weak excedance of $\sigma$, then we have $\sigma(j) < \sigma(k)$ and with Theorem 2, $\sigma'(j) = \sigma(j) + 1 < \sigma(k) + 1 = \sigma'(k)$. Otherwise $\sigma(j) < j$ and then Theorem 2 ensures that $j$ is not a weak excedance of $\sigma'$ and $\sigma'(j) < j < \sigma'(k) = \sigma(k) + 1$. Thus we have $k \in L(\phi(\sigma))$. The converse is obtained similarly.

As a consequence of Theorem 2 we deduce:

**Corollary 1.** The bijection $\phi : S_n \rightarrow C_{n+1}$ satisfies for any $\sigma \in S_n$,

$$L(\sigma) = L(\phi(\sigma)) \quad \text{and},$$

if $k$ is a left-to-right maximum of $\sigma$ then $\sigma'(k) = \sigma(k) + 1$ where $\sigma' = \phi(\sigma)$.

**Proof.** Let $k$ be a left-to-right maximum of $\sigma$, i.e., $\sigma(j) \leq \sigma(k)$ for $j \leq k$. Since it also is a weak excedance, Theorem 2 induces $\sigma'(k) = \sigma(k) + 1$ where $\sigma' = \phi(\sigma)$. Consider $j < k$. If $j$ is a weak excedance of $\sigma$, then we have $\sigma(j) < \sigma(k)$ and with Theorem 2, $\sigma'(j) = \sigma(j) + 1 < \sigma(k) + 1 = \sigma'(k)$. Otherwise $\sigma(j) < j$ and then Theorem 2 ensures that $j$ is not a weak excedance of $\sigma'$ and $\sigma'(j) < j < \sigma'(k) = \sigma(k) + 1$. Thus we have $k \in L(\phi(\sigma))$. The converse is obtained similarly.

See Table 1 for an illustration of Theorem 2 and Corollary 1 for $n = 3$.

As the bijection $\phi$ moves fixed points into quasi-fixed points, Theorem 2 and Remark 1 induce the following result:

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>$T(\sigma)$</th>
<th>$T(\phi(\sigma))$</th>
<th>$L(\sigma)$</th>
<th>$F(\sigma)$</th>
<th>$WE(\sigma)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>123</td>
<td>123</td>
<td>2341</td>
<td>1, 2, 3</td>
<td>1, 2, 3</td>
<td>1, 2, 3</td>
</tr>
<tr>
<td>132</td>
<td>122</td>
<td>2413</td>
<td>1, 2</td>
<td>1</td>
<td>1, 2</td>
</tr>
<tr>
<td>213</td>
<td>113</td>
<td>3142</td>
<td>1, 3</td>
<td>3</td>
<td>1, 3</td>
</tr>
<tr>
<td>231</td>
<td>112</td>
<td>3421</td>
<td>1, 2</td>
<td>0</td>
<td>1, 2</td>
</tr>
<tr>
<td>312</td>
<td>111</td>
<td>4123</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>321</td>
<td>121</td>
<td>4312</td>
<td>1, 2</td>
<td>2</td>
<td>1, 2</td>
</tr>
</tbody>
</table>

The bijection $\phi$ from $S_3$ to $C_4$. Column $T(\sigma)$ (resp. $T(\phi(\sigma))$) gives the transposition array of $\sigma$ (resp. $\phi(\sigma)$). The weak excedances are illustrated in boldface. The last three columns give respectively the sets of left-to-right maxima, fixed points and weak excedances of $\sigma$. 

Moreover if $k$ is a weak excedance of $\sigma$ then $\sigma'(k) = \sigma(k) + 1$ where $\sigma' = \phi(\sigma)$, and we have

$$F(\sigma) = Q F(\phi(\sigma)).$$
In the case where $\sigma \in S_n$, $\pi$ is reduced to the cycle $c = \langle n, c_1, c_2, \ldots, c_r \rangle$. Observe that $\sigma$ is obtained from $\pi$ by gluing $c$ into disjoint cycles. So, this implies that $h_k(\sigma) \in S_n(k-1)$.

**Theorem 3.** For $k \in [2, \ldots, n]$, $h_k$ is a bijection.

**Proof.** Let us prove that $h_k$ is a bijection. In order to show the injectivity we take $\sigma$ and $\pi$ in $S_n(k)$ such that $h_k(\sigma) = h_k(\pi)$. We distinguish three cases.

1. If $\pi$ and $\sigma$ belong to $S_n(k-1)$ then $h_k(\pi) = \pi$ and $h_k(\sigma) = \sigma$, and $\pi = \sigma$.

2. If $\pi$ and $\sigma$ do not belong to $S_n(k-1)$ then we have $\sigma \cdot (j, n) = \pi \cdot (\ell, n)$ where $j = \sigma^{-1}(k-1)$ and $\ell = \pi^{-1}(k-1)$. Let $c = \langle n, c_1, c_2, \ldots, c_r \rangle$ (resp. $c' = \langle n, c_1', c_2', \ldots, c_r' \rangle$) be the cycle containing $n$ in $\pi$ (resp. $\sigma$), and $d = \langle k-1, d_1, d_2, \ldots, d_s \rangle$ (resp. $d' = \langle k-1, d'_1, d'_2, \ldots, d'_s \rangle$) be the cycle of $\pi$ (resp. $\sigma$) containing $k$. We have $c \neq d$, $c' \neq d'$, $d = \ell$, $d' = j$. $h_k(\pi)$ (resp. $h_k(\sigma)$) is obtained by gluing $c$ and $d$ (resp. $c'$ and $d'$) as explained just after Definition 2. Because $h_k(\pi) = h_k(\sigma)$, we deduce that $\pi \in S_n(k-1)$ and $\sigma \in S_n(k-1)$ does not occur since the last entry of $h_k(\sigma)$ is at least $k$ while the last value of $h_k(\pi)$ is $k-1$.

Therefore $h_k$ is injective.

Now let us consider $\pi \in S_n(k-1)$. If $\pi$ also belongs to $S_n(k)$ then $\pi$ is the image of $\pi$ by $h_k$. If $\pi$ does not belong to $S_n(k)$ then we necessarily have $\pi(n) = k-1$. Let $c = \langle n, k-1, c_1, \ldots, c_r \rangle$ be the cycle of $\pi$ containing $n$ and $l_0$ be the smallest $i$ such that $c_i \geq k$. Then the permutation $\sigma$ obtained from $\pi$ by splitting $c$ into the two cycles $c' = \langle n, c_0, c_{l_0+1}, \ldots, c_r \rangle$ and $d = \langle k-1, c_1, \ldots, c_{l_0} \rangle$, belongs to $S_n(k)$ and satisfies $h_k(\sigma) = \pi$. Thus $h_k$ is surjective. □

A consequence of Theorem 3 is that the cardinality of the sets $S_{n+1}(k)$ is $n!$ for each $k \in [1, \ldots, n+1]$. A simple combinatorial argument proves the following remarkable equality.

**Remark 2.** For all $k \in [1, \ldots, n]$, we have

$$n! = (n-k+1) \cdot \sum_{i=0}^{k-1} {k-1 \choose i} (n-k+i)! (k-i-1)!.$$
Both sides of the above equality count the cardinality of \( S_{n+1}(k) \). Indeed, a permutation \( \sigma \) belongs to \( S_{n+1}(k) \) if and only if \( \sigma(n+1) \geq k \) and each integer in \([k, n+1]\) lies into the same cycle \( c = (n+1, c_1, c_2, c_3, \ldots) \) of \( \sigma \). So, there are \((n-k+1)\) choices for \( c_1 = \sigma(n+1) \in [k, n] \). In order to complete \( c \) we choose a set \( I \) of \( i \) values among \([1, 2, \ldots, k-1]\), \( 0 \leq i \leq k-1 \) and we consider all arrangements of elements of the set \( I \cup [k, n]\setminus\{c_1\} \). Thus, for a given \( i \), \( 0 \leq i \leq k-1 \), there are \((n-k+1) \cdot (k-1) \cdot \ldots \cdot (k-i)!\) possible cycles \( c \) and \((k-i)\) choices for the remaining values. Moving \( i \) from 0 to \( k-1 \), we obtain the above formula.

**Theorem 4.** For \( k \in \{2, \ldots, n\} \), the bijection \( h_k : S_n(k) \to S_n(k-1) \) satisfies for any \( \sigma \in S_n(k) \),

\[
W E(\sigma) \cap \{1, 2, \ldots, n-1\} = W E(h_k(\sigma)).
\]

**Proof.** Let \( \sigma \in S_n(k) \). Notice that \( n \) is never a weak excedance of \( h_k(\sigma) \).

The case \( h_k(\sigma) = \sigma \) is trivial. Now let us assume that \( \sigma' = h_k(\sigma) = \sigma \cdot (j, n) \) where \( j = \sigma^{-1}(k-1) \). We necessarily have \( j \leq k-1 \) and \( \sigma(n) \geq k \).

Let us take \( i \in \{1, 2, \ldots, n-1\} \). In the case where \( i \neq j \) and \( i \neq n \), we obtain \( \sigma'(i) = \sigma(i) \); then \( i \) is a weak excedance of \( \sigma \) if and only if \( i \) is also one for \( \sigma' = h_k(\sigma) \).

If \( i = j \) then \( \sigma'(i) = \sigma(n) \). As \( \sigma \in S_n(k) \), we have \( \sigma(n) \geq n \) and then \( \sigma(n) > k-1 \). We obtain \( j = i \leq k-1 \) and \( \sigma(i) = \sigma(j) = k-1 \), and hence \( i \) is a weak excedance of \( \sigma \). Moreover, we have \( \sigma'(i) = \sigma(n) > k-1 \) so that \( i \) is a weak excedance of \( \sigma' \). Finally, we have \( W E(\sigma) \cap \{1, 2, \ldots, n-1\} = W E(h_k(\sigma)) \). \( \square \)

**Corollary 3.** There is a bijection \( h \) from \( S_n \) to \( C_n+1 \) that preserves the set of all weak excedances.

**Proof.** We set \( h = h_2 \circ h_3 \circ \cdots \circ h_{n-1} \circ h_n \circ h_{n+1} \) from \( S_{n+1}(n+1) \) to \( S_{n+1}(1) = C_{n+1} \). Since \( S_{n+1}(n+1) \) is the set of permutations in \( S_n \) after adding \( n+1 \) on the right, we have \( W E(\sigma) = W E((n+1) \cap \{1, 2, \ldots, n\} \). Theorem 4 allows to conclude that \( W E(\sigma) = W E(h(\sigma)) \). \( \square \)

Notice that the bijection \( h \) does not transform fixed points into quasi-fixed points (see for instance \( h(132) = 3421 \)), unlike the bijection \( \phi \) in Section 3.

5. Consequences and open problems

In this section we give some direct consequences of our study and we propose two open problems concerning the descent sets on cyclic permutations and derangements.

Following the notation from [2,3], let \( T_n^0 \) be the set whose elements are \( n \)-cycles in one-line notation in which one entry has been removed with 0. For example, \( T_3^0 = \{031, 201, 230, 012, 302, 310\} \). Obviously the cardinality of \( T_n^0 \) is \( n! \). Let \( \sigma \) be an element of \( T_n^0 \); we say that \( k, 1 \leq k \leq n, \) is an excedance of \( \sigma \) whenever \( \sigma(k) > k \) and \( E(\sigma) \) will denote the set of excedances of \( \sigma \). Theorem 5 is a counterpart of Elizalde’s result [2] for the set of excedances on \( T_n^0 \).

If \( E \) is a subset of \([n]\) then we define the set \( E^+ \subseteq \{2, 3, \ldots, n+1\}\) by:

\[
E^+ = \{e + 1 \mid e \in E \}.
\]

Moreover, we define the involution \( \chi_n \) from \( S_n \) into itself as follows: for any \( n \geq 1 \) and \( \sigma \in S_n \), \( \chi_n(\sigma)(i) = n + 1 - \sigma(n + 1 - i) \) for \( i \leq n \). For instance, we have \( \chi_4(4132) = 3241 \). Less formally, \( \chi_n(\sigma) \) is obtained from \( \sigma \in S_n \) by reading the complement of \( \sigma \) from right to left. In the following we will omit the subscript \( n \) for \( \chi \); it should be clear from the context.

Using the map \( \phi : S_n \to C_{n+1} \) presented in Section 3, we consider the bijection \( \psi \) from \( S_n \) to \( C_{n+1} \) defined by:

\[
\psi(\sigma) = \chi(\phi(\chi(\sigma))).
\]

For example, \( \psi(312) = \chi(\phi(231)) = \chi(3421) = 4312 \); see Table 3 for an illustration of this map when \( n = 3 \).

The following corollary appears as a direct consequence of Theorem 2.

**Corollary 4.** The bijection \( \psi : S_n \to C_{n+1} \) satisfies for any \( \sigma \in S_n \),

\[
E^+(\sigma) \cup \{1\} = E(\psi(\sigma)).
\]

**Proof.** It is obvious that an \( n \)-cycle has 1 as excedance; thus 1 \( \in E(\psi(\sigma)) \). Now if \( k \notin E^+(\sigma) \cup \{1\} \) then we have \( k > 1, \sigma(k-1) \leq k-1 \) and \( n+1 - (k-1) = n + 2 - k \) is a weak excedance of \( \sigma \), and we have \( \chi(\sigma)(n+2-k) = n + 1 - \sigma(k-1) \). With Theorem 2, \( n+2-k \) is an excedance of \( \phi(\chi(\sigma)) \), and we have \( \chi(\phi(\chi(\sigma)))(n+2-k) = n+1-\sigma(k-1) \). Thus, \( \chi(\phi(\chi(\sigma)))(n+2-k) = n+2-(n+1-\sigma(k-1)) \), i.e., \( \chi(\phi(\chi(\sigma)))(k) = \sigma(k-1) \leq k-1 \) which induces that \( k \) is not an excedance of \( \psi(\sigma) \). Each implication of this reasoning can be also viewed as an equivalence which achieves the proof. \( \square \)
Theorem 5. Let n be a positive integer. There is a bijection $f$ between $\mathcal{T}_n^0$ and $S_n$ such that if $\sigma \in \mathcal{T}_n^0$, then
\[ E(\sigma) = E(f(\sigma)). \]

Proof. Let $\sigma$ be an element of $\mathcal{T}_n^0$. $\sigma$ is obtained from a cycle $\pi$ by replacing $\pi(k)$ with $0$ for some $k$, $1 \leq k \leq n$. Assume that the cyclic representation of $\pi$ is $\langle a_1, a_2, \ldots, a_{n-1}, k \rangle$ where $a_i \in [n]\setminus\{k\}$ for all $i \leq n - 1$. Let us consider $\pi'$ defined by the cyclic representation $\pi' = \langle a_1 + 1, a_2 + 1, \ldots, a_{n-1} + 1, k + 1, 1 \rangle$. Obviously, we have $E(\sigma) = E(\pi')$. Now, we set $f(\sigma) = \psi^{-1}(\pi')$ where $\psi$ is the bijection defined above. We immediately have:
\[ E(\sigma) = E(f(\sigma)), \]
which means $E(\sigma) = E(f(\sigma)).$ \qed

Thus we can directly deduce:

Corollary 5. For any $n$ and any $I \subseteq [n - 1]$, \[ |\{ \tau \in \mathcal{T}_n^0, E(\tau) = I \}| = |\{ \sigma \in S_n, E(\sigma) = I \}|. \]

This last result appears as a counterpart of the following Elizalde’s result for descents:

Theorem 6. (See [2].) For any $n$ and any $I \subseteq [n - 1]$, \[ |\{ \tau \in \mathcal{T}_n^0, D(\tau) = I \}| = |\{ \sigma \in S_n, D(\sigma) = I \}|. \]

We conclude this paper by giving two open problems about descent statistics (see [5] for some other open problems about descent statistics). Experimental investigations allow us to think that the answer to these two problems is positive. Problem 1 appears as a counterpart of Corollary 2 for descent statistics. Problem 2 would be a generalization of the Elizalde’s result (Theorem 1, p. 2) and appears as a counterpart of Theorem 4 for descent statistics.

Problem 1. Is it true that there exists a bijection from the set of derangements $D_n$ to the set $\mathcal{C}_n^{n+1}$ of $(n + 1)$-cycles without quasi-fixed points that preserves the set of descents in $\{1, 2, \ldots, n - 1\}$?

For $1 \leq k \leq n$, we denote $S_n(k)$ the set of permutations $\sigma \in S_n$ such that the cycle of $\sigma$ that contains $n$ is of length $k$.

Problem 2. For $k \in \{2, \ldots, n\}$, is it true that there is a bijection $h_k^i$ from $S_n(k)$ to $S_n(k - 1)$ that preserves the set of descents in $\{1, 2, \ldots, n - 2\}$, i.e., \[ D(\sigma) \cap \{1, 2, \ldots, n - 2\} = D(h_k^i(\sigma)) \cap \{1, 2, \ldots, n - 2\}? \]

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References