Information Processing Letters 113 (2013) 17–22

Contents lists available at SciVerse ScienceDirect

Information Processing Letters

www.elsevier.com/locate/ipl

Statistics-preserving bijections between classical and cyclic permutations

Jean-Luc Baril

LE2I UMR-CNRS 5158, Université de Bourgogne, B.P. 47 870, 21078 Dijon Cedex, France

ARTICLE INFO

Article history:
Received 3 January 2012
Received in revised form 28 September 2012
Accepted 15 October 2012
Available online 16 October 2012
Communicated by A. Muscholl

Keywords:
Combinatorial problems
Bijections
Statistics on permutations
Excedance
Left-to-right maximum
Descent
Quasi-fixed point
Cyclic permutation
Derangement

ABSTRACT

Recently, Elizalde (2011) [2] has presented a bijection between the set C_{n+1} of cyclic permutations on $\{1, 2, \ldots, n+1\}$ and the set of permutations on $\{1, 2, \ldots, n\}$ that preserves the descent set of the first n entries and the set of weak excedances. In this paper, we construct a bijection from C_{n+1} to S_n that preserves the weak excedance set and that transfers quasi-fixed points into fixed points and left-to-right maxima into themselves. This induces a bijection from the set D_n of derangements to the set C^q_{n+1} of cycles without quasi-fixed points that preserves the weak excedance set. Moreover, we exhibit a kind of discrete continuity between C_{n+1} and S_n that preserves at each step the set of weak excedances. Finally, some consequences and open problems are presented.

1. Introduction and notation

Let S_n be the set of permutations of length n, *i.e.*, all one-to-one correspondences from $[n] = \{1, 2, \ldots, n\}$ into itself. We represent a permutation $\sigma \in S_n$ in one-line notation, $\sigma = \sigma_1\sigma_2\ldots\sigma_n$ where $\sigma_i = \sigma(i)$, $1 \leqslant i \leqslant n$. Moreover, if $\gamma = \gamma(1)\gamma(2)\ldots\gamma(n)$ is a length n permutation then the *product* $\gamma \cdot \sigma$ is the permutation $\gamma(\sigma_1)\gamma(\sigma_2)\ldots\gamma(\sigma_n)$. In S_n, a *k-cycle* $\sigma = \langle i_1, i_2, \ldots, i_k\rangle$ is a length n permutation verifying $\sigma(i_1) = i_2$, $\sigma(i_2) = i_3, \ldots, \sigma(i_{k-1}) = i_k$, $\sigma(i_k) = i_1$ and $\sigma(j) = j$ for $j \in [n]\backslash\{i_1, \ldots, i_k\}$. In particular, a 2-cycle is called a *transposition*. Let $C_n \subset S_n$ be the set of n-cycles. The elements of C_n will be called cyclic permutations (or cycles for short). Obviously C_{n+1} and S_n have the same cardinality.

Any permutation $\sigma \in S_n$ is uniquely decomposed as a product of transpositions of the following form:

$$\sigma = \langle p_1, 1\rangle \cdot \langle p_2, 2\rangle \cdot \langle p_3, 3\rangle \cdots \langle p_n, n\rangle = \prod_{i=1}^{n} \langle p_i, i\rangle, \qquad (1)$$

where p_i are some integers such that $1 \leqslant p_i \leqslant i \leqslant n$. Conversely, any such decomposition provides a permutation in S_n. Therefore, (1) yields a bijection from S_n to the product set $T_n = [1] \times [2] \times \cdots \times [n]$. Then we have another way to represent a permutation:

Definition 1. The transposition array of a permutation $\sigma = \prod_{i=1}^{n} \langle p_i, i\rangle \in S_n$ is defined by $p_1 p_2 \ldots p_n \in T_n$.

For example, if $\sigma = 1\ 4\ 5\ 6\ 3\ 2$ then its decomposition into transpositions is $\langle \mathbf{1}, 1\rangle \cdot \langle \mathbf{2}, 2\rangle \cdot \langle \mathbf{3}, 3\rangle \cdot \langle \mathbf{2}, 4\rangle \cdot \langle \mathbf{3}, 5\rangle \cdot \langle \mathbf{4}, 6\rangle$, and its corresponding transposition array is 1 2 3 2 3 4. Notice that this decomposition is used in [1] in order to obtain Gray codes for restricted classes of length n permutations.

Let σ be a permutation in S_n. A *descent* of σ is a position i, $1 \leqslant i \leqslant n-1$, such that $\sigma(i) > \sigma(i+1)$. Let $D(\sigma)$ be the set of descents of σ. An *excedance* (resp.

E-mail address: barjl@u-bourgogne.fr.

0020-0190/$ – see front matter © 2012 Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.ipl.2012.10.003

weak excedance) of σ is a position i, $1 \leqslant i \leqslant n$, such that $\sigma(i) > i$ (resp. $\sigma(i) \geqslant i$). The set of excedances (resp. weak excedances) of σ will be denoted $E(\sigma)$ (resp. $WE(\sigma)$). A *left-to-right maximum* is a position i, $1 \leqslant i \leqslant n$, such that $\sigma(i) > \sigma(j)$ for all $j < i$. The set of left-to-right maxima of σ will be denoted $L(\sigma)$. A *fixed point* of σ is a position i such that $\sigma(i) = i$. Let $F(\sigma)$ be the set of fixed points of σ. A *quasi-fixed point* of σ is a position i such that $\sigma(i) = i + 1$. Let $QF(\sigma)$ be the set of quasi-fixed points. For instance, if $\sigma = 1\ 3\ 5\ 2\ 4\ 6$ then $D(\sigma) = \{3\}$, $E(\sigma) = \{2, 3\}$, $WE(\sigma) = \{1, 2, 3, 6\}$, $L(\sigma) = \{1, 2, 3, 6\}$, $F(\sigma) = \{1, 6\}$ and $QF(\sigma) = \{2\}$. Let D_n be the set of length n *derangements* (permutations without fixed points).

A combinatorial statistics on S_n is a map $f : S_n \to \mathbb{N}$. The distribution of f is the sequence $(a_i)_{i\in\mathbb{N}}$ where a_i is the cardinality of $f^{-1}(i)$. Many statistics on S_n have been widely studied (number of descents, inversions and excedances, major index, ...), see [4,10,7,8,11] for instance. Descent and excedance statistics were first studied by MacMahon [10] and can be considered as mirror images of each other. Indeed, the numbers of descents and excedances have the same distribution on the length n permutations. They are called Eulerian statistics. Foata [6] presents a bijection from S_n to itself which exchanges excedances and descents. In the literature, descents of a permutation are mostly studied for their many applications. For instance, descents appears in theory of lattice path enumeration (see Gessel and Viennot [9]).

More recently, Elizalde [2] constructs a bijection from S_n to C_{n+1} that preserves the descent set on the first $(n-1)$ positions.

Theorem 1. *(See [2].) For every n, there is a bijection $\varphi : C_{n+1} \to S_n$ such that if $\pi \in C_{n+1}$ and $\sigma = \varphi(\pi)$, then*

$$D(\pi) \cap [n-1] = D(\sigma).$$

Moreover, this bijection also preserves the weak excedance set:

$$E(\pi) = WE(\pi) = WE(\sigma).$$

Inspired by this theorem, we present similar results for other statistics. Section 2 shows how one can characterize n-cycles, left-to-right maxima, (quasi-)fixed points, and (weak) excedances using the transposition array representation. In Section 3, we give a constructive bijection from S_n to C_{n+1} that preserves the weak excedance set and that transfers fixed points into quasi-fixed points and left-to-right maxima into themselves. As a consequence, we deduce a bijection from the set D_n of derangements (permutations without fixed points) to the set C_{n+1}^q of length $n+1$ cycles without quasi-fixed points that preserves the set of excedances. In Section 4, we exhibit a kind of discrete continuity between S_n and C_{n+1} that preserves at each step the set of weak excedances. Finally, we deduce a bijection (preserving the excedance set) between S_n and the set $\mathcal{T}_n^0$ of elements in C_n in which one entry has been replaced with 0.

2. Preliminaries

In this section, we give several elementary lemmas in order to characterize special points of a permutation (left-to-right maximum, fixed point, excedance, ...) using the transposition array representation. They will be used throughout the paper. Before this, we state the two straightforward claims:

Claim 1. *For every $j \geqslant 1$, the permutation $\prod_{i=1}^{j} \langle p_i, i\rangle$ fixes all values strictly larger than j.*

Claim 2. *If $k < \ell$ and if for all $i \geqslant \ell$ we have $p_i \neq k$, then the permutation $\prod_{i=\ell}^{n} \langle p_i, i\rangle$ fixes k.*

In [1], it is shown how an n-cycle can be characterized with its transposition array representation.

Lemma 1. *(See [1, Remark 16].) Let σ be a permutation in S_n and $p = p_1p_2\dots p_n$ be its transposition array. Then σ is an n-cycle if and only if its transposition array contains only one fixed point, i.e., $p_1 = 1$ and $p_i \neq i$ for $i \geqslant 2$. More generally, the number of cycles of σ is $n - \ell$ where ℓ is the number of indices i, $1 \leqslant i \leqslant n$, such that $p_i \neq i$.*

Proof. A simple induction on j starting at $j = 1$ shows that the number of cycles (relatively to the classical decomposition into disjoint cycles) in $\prod_{i=1}^{j} \langle p_i, i\rangle$ is $(n - \ell)$, where ℓ is the number of i, $1 \leqslant i \leqslant j$, such that $p_i \neq i$. □

Lemma 2. *Let σ be a permutation in S_n and $p_1p_2\dots p_n$ be its transposition array. Then k is a left-to-right maximum of σ if and only if there exists $i \geqslant k$ such that* (a) *$p_i = k$, and* (b) *$p_j > k$ for $j > i$.*

Proof. Assume that k is a left-to-right maximum of σ. We have $\sigma(j) \leqslant \sigma(k)$ for $j \leqslant k$ and in particular $\sigma(k) \geqslant k$. Since $\sigma = \langle p_1, 1\rangle \cdot \langle p_2, 2\rangle \cdots \langle p_n, n\rangle$ with $1 \leqslant p_i \leqslant i \leqslant n$, and using Claims 1 and 2, the fact $\sigma(k) \geqslant k$ induces the existence of $i \geqslant k$ such that $p_i = k$. We choose the rightmost $i \geqslant k$ verifying this property. So, Claims 1 and 2 give $\sigma(k) = i$. Assume for a contradiction that there is j, $j > i$, such that $p_j < p_i = k$; we choose the rightmost j. This implies $\sigma(p_j) = j > i = \sigma(k)$ with $p_j < k$, which contradicts the fact that k is a left-to-right maximum. So (b) is verified. Using Claims 1 and 2, the converse becomes straightforward. □

Lemma 3. *Let σ be a permutation in S_n and $p_1p_2\dots p_n$ be its transposition array. Then k is a fixed* (*resp. quasi-fixed*) *point of σ if and only if* (a) *$p_k = k$* (*resp. $p_{k+1} = k$*), *and* (b) *there does not exist $i > k$ such that $p_i = k$* (*resp. $i > k+1$ such that $p_i = k$*).

Proof. Assume that k is a fixed point of $\sigma = \langle p_1, 1\rangle \cdot \langle p_2, 2\rangle \cdots \langle p_n, n\rangle$ with $1 \leqslant p_i \leqslant i \leqslant n$, *i.e.*, $\sigma(k) = k$. Suppose for a contradiction there exists $i > k$ such that $p_i = k$; we take the rightmost i verifying this property. Claims 1 and 2 give $\sigma(k) = i > k$ which contradicts our hypothesis.

Thus σ verifies (b). Claim 2 induces $\sigma(k) = \sigma'(k)$ where $\sigma' = \langle p_1, 1\rangle \cdot \langle p_2, 2\rangle \cdots \langle p_k, k\rangle$. Since $\sigma(k) = k$, Claim 1 induces $p_k = k$. The converse is straightforward. The proof is obtained *mutatis mutandis* whenever we replace fixed points with quasi-fixed points. □

Lemma 4. *Let σ be a permutation in S_n and $p_1 p_2 \ldots p_n$ be its transposition array. Then k is an excedance (resp. a weak excedance) of σ if and only if there exists $i > k$ (resp. $i \geqslant k$) such that $p_i = k$.*

Proof. Assume that k is an excedance of $\sigma = \langle p_1, 1\rangle \cdot \langle p_2, 2\rangle \cdots \langle p_n, n\rangle$ with $1 \leqslant p_i \leqslant i \leqslant n$, *i.e.*, $\sigma(k) > k$. Suppose for a contradiction that there is no $i > k$ such that $p_i = k$. Claim 2 induces $\sigma(k) = \sigma'(k)$ with $\sigma' = \langle p_1, 1\rangle \cdot \langle p_2, 2\rangle \cdots \langle p_k, k\rangle$. We deduce $\sigma(k) \leqslant k$ which contradicts the fact that k is an excedance. The converse is straightforward using Claims 1 and 2, and by taking i to be the rightmost i such that $p_i = k$. The reasoning is similar for the case where k is a weak excedance. □

In order to illustrate these lemmas, we present an example for each of them. For instance, if we set $\sigma = 2\mathbf{4}13$ then σ is a 4-cycle and its transposition array $p_1p_2p_3p_4 = 1122$ contains only one fixed point ($p_1 = 1$). Moreover, $k = 2$ is a left-to-right maximum of σ and there is $i = 4 \geqslant 2$ such that $p_i = 2$ and $p_j > 2$ for $j > 4$. $k = 2$ is an excedance and 2 appears on the right of p_2 in the transposition array 1122 of σ. For $\sigma = 2431$, $k = 3$ is a fixed point, its transposition array is 1132 and we have $p_3 = 3$ and there does not exist $i > 3$ such that $p_i = 3$. Finally, $k = 1$ is a quasi-fixed point, we have $p_2 = 1$ and there does not exist $i > 2$ such that $p_i = 1$.

3. Fixed points, weak excedances and left-to-right maxima

Let ϕ be the map from S_n to S_{n+1} defined, for every $\sigma \in S_n$, by

$$\phi(\sigma) = \langle 1, 1\rangle \cdot \langle p_1, 2\rangle \cdot \langle p_2, 3\rangle \cdots \langle p_n, n+1\rangle,$$

where the transposition array of σ is $p_1 p_2 \ldots p_n$.

For example, $\phi(321) = \phi(\langle 1, 1\rangle \cdot \langle 2, 2\rangle \cdot \langle 1, 3\rangle) = \langle 1, 1\rangle \cdot \langle 1, 2\rangle \cdot \langle 2, 3\rangle \cdot \langle 1, 4\rangle = 4312$. By construction the transposition array $1p_1p_2 \ldots p_n$ of $\phi(\sigma)$ has only one fixed point, that is on the first position. With Lemma 1, the permutation $\phi(\sigma)$ is a cyclic permutation and then belongs to C_{n+1}. Therefore ϕ is a bijection from S_n to C_{n+1} (see Table 1 for $n = 3$). Obviously, this construction allows to go back from $\phi(\sigma)$ to σ.

Now we prove that the bijection ϕ transforms the set of weak excedances of $\sigma \in S_n$ into the excedance set of $\phi(\sigma) \in C_{n+1}$.

Remark 1. If $\sigma \in S_n$ has no fixed points, then $E(\sigma) = WE(\sigma)$. This holds in particular when σ is a cycle of length at least 2.

Theorem 2. *The bijection $\phi : S_n \to C_{n+1}$ satisfies for any $\sigma \in S_n$,*

$$WE(\sigma) = E\big(\phi(\sigma)\big) = WE\big(\phi(\sigma)\big).$$

Moreover if k is a weak excedance of σ then $\sigma'(k) = \sigma(k) + 1$ where $\sigma' = \phi(\sigma)$, and we have

$$F(\sigma) = QF\big(\phi(\sigma)\big).$$

Proof. Let σ be a permutation in S_n and $p_1 p_2 \ldots p_n$ its transposition array, *i.e.*, $\sigma = \langle p_1, 1\rangle \cdot \langle p_2, 2\rangle \cdots \langle p_n, n\rangle$. The transposition array of $\phi(\sigma)$ is $q_1 q_2 \ldots q_{n+1} = 1 p_1 p_2 \ldots p_n$. By Lemma 4, k is a weak excedance of σ if and only if there exists $i \geqslant k$ such that $p_i = k$ which is equivalent to the existence of $j \geqslant k + 1$ such that $q_j = k$. Thus, by Lemma 4 again, k is a weak excedance of σ if and only if k is an excedance of $\phi(\sigma)$, and $WE(\sigma) = E(\phi(\sigma))$. Moreover, for $k \in WE(\sigma)$, let i_0 be the rightmost $i \geqslant k$ such that $p_i = k$. Claims 1 and 2 give $\sigma(k) = i_0$, and since $q_{i_0+1} = p_{i_0} = k$, we obtain $\sigma'(k) = i_0 + 1$ where $\sigma' = \phi(\sigma)$. Thus we have $F(\sigma) \subseteq QF(\phi(\sigma))$. For $k \in WE(\phi(\sigma))$, Remark 1 induces that $k \in E(\phi(\sigma))$, and Lemma 4 ensures the existence of $i > k$ such that $q_i = k$. Let i_0 be the rightmost i. Claims 1 and 2 give $\sigma'(k) = i_0$, and since $q_{i_0} = p_{i_0 - 1} = k$, we obtain $\sigma(k) = i_0 - 1$ which allows to conclude. □

Table 1
The bijection ϕ from S_3 to C_4. Column $T(\sigma)$ (resp. $T(\phi(\sigma))$) gives the transposition array of σ (resp. $\phi(\sigma)$). The weak excedances are illustrated in **boldface**. The last three columns give respectively the sets of left-to-right maxima, fixed points and weak excedances of σ.

σ	$T(\sigma)$	$\phi(\sigma)$	$T(\phi(\sigma))$	$L(\sigma)$	$F(\sigma)$	$WE(\sigma)$
123	123	**234**1	1123	{1, 2, 3}	{1, 2, 3}	{1, 2, 3}
132	122	**24**13	1122	{1, 2}	{1}	{1, 2}
21**3**	113	**3**1**4**2	1113	{1, 3}	{3}	{1, 3}
231	112	**34**21	1112	{1, 2}	∅	{1, 2}
312	111	**4**123	1111	{1}	∅	{1}
321	121	**43**12	1121	{1}	{2}	{1, 2}

As a consequence of Theorem 2 we deduce:

Corollary 1. *The bijection $\phi : S_n \to C_{n+1}$ satisfies for any $\sigma \in S_n$,*

$$L(\sigma) = L\big(\phi(\sigma)\big) \quad \textit{and,}$$

if k is a left-to-right maximum of σ then $\sigma'(k) = \sigma(k) + 1$ where $\sigma' = \phi(\sigma)$.

Proof. Let k be a left-to-right maximum of σ, *i.e.*, $\sigma(j) \leqslant \sigma(k)$ for $j \leqslant k$. Since it also is a weak excedance, Theorem 2 induces $\sigma'(k) = \sigma(k) + 1$ where $\sigma' = \phi(\sigma)$. Consider $j < k$. If j is a weak excedance of σ, then we have $\sigma(j) < \sigma(k)$ and with Theorem 2, $\sigma'(j) = \sigma(j) + 1 < \sigma(k) + 1 = \sigma'(k)$. Otherwise $\sigma(j) < j$ and then Theorem 2 ensures that j is not a weak excedance of σ' and $\sigma'(j) < j < k < \sigma'(k) = \sigma(k) + 1$. Thus we have $k \in L(\phi(\sigma))$. The converse is obtained similarly. □

See Table 1 for an illustration of Theorem 2 and Corollary 1 for $n = 3$.

As the bijection ϕ moves fixed points into quasi-fixed points, Theorem 2 and Remark 1 induce the following result:

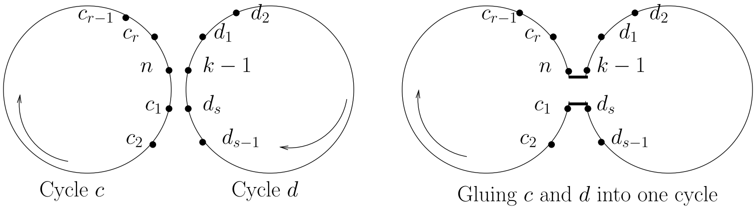

Fig. 1. Gluing of $c = \langle n, c_1, c_2, \ldots, c_r\rangle$ and $d = \langle k-1, d_1, d_2, \ldots, d_s\rangle$.

Table 2
The sets $S_4(k)$ for $k = 1, 2, 3, 4$. The weak excedances on positions $\{1, 2, 3\}$ are illustrated in **boldface**.

$S_4(1)$	$S_4(2)$	$S_4(3)$	$\sigma \in S_4(4) = S_3 \cdot 4$	$WE(\sigma) \cap \{1,2,3\}$
4123	**4**123	**4**123	**3**124	{1}
31**4**2	**3**1**4**2	**2**1**4**3	**2**1**3**4	{1, 3}
4312	**43**12	**42**13	**32**14	{1, 2}
2413	**24**13	**24**13	**23**14	{1, 2}
2341	**134**2	**124**3	**123**4	{1, 2, 3}
3421	**14**23	**14**23	**13**24	{1, 2}

Corollary 2. *The bijection ϕ from the set D_n of derangements* (i.e., *length n permutations without fixed points*) *to the set C^q_{n+1} of length $(n+1)$ cycles without quasi-fixed points is a bijection such that*: *for any $\sigma \in D_n$*,

$$E(\sigma) = E\big(\phi(\sigma)\big).$$

Moreover, if k is an excedance of σ then $\sigma'(k) = \sigma(k) + 1$ where $\sigma' = \phi(\sigma)$.

4. A discrete continuity preserving the set of weak excedances

In this part, we give a kind of discrete continuity between S_n and C_{n+1} that preserves at each step the set of weak excedances. For $1 \leqslant k \leqslant n$, we denote by $S_n(k)$ the set of permutations $\sigma \in S_n$ such that: (a) $\sigma(n) \geqslant k$; and (b) all integers of the interval $[k, n]$ lie in a same cycle of σ (in the decomposition of σ into disjoint cycles). Obviously, $S_{n+1}(1) = C_{n+1}$ and $S_{n+1}(n+1)$ is obtained from S_n by adding $n+1$ to the right of each permutation of S_n. Table 2 shows the different sets $S_4(k)$ for $k = 1, 2, 3, 4$.

Definition 2. For $k \in \{2, \ldots, n\}$, we define $h_k : S_n(k) \to S_n(k-1)$ by:

$$h_k(\sigma) = \begin{cases} \sigma & \text{if } \sigma \in S_n(k-1), \\ \sigma \cdot \langle \sigma^{-1}(k-1), n\rangle & \text{otherwise.} \end{cases}$$

For Definition 2 to be valid, we need to prove that $h_k(\sigma)$ is indeed in $S_n(k-1)$ for any $\sigma \in S_n(k)$. Let σ be a permutation in $S_n(k)$. Its decomposition into disjoint cycles contains a cycle $c = \langle n, c_1, c_2, \ldots, c_r\rangle$ with $r \geqslant 0$, $c_1 \geqslant k$ and such that $[k, n] \subseteq \{n, c_1, c_2, \ldots, c_r\}$. The case $r = 0$ means that c is reduced to the cycle $c = \langle n\rangle$. In the case where $\sigma \notin S_n(k-1)$, $k-1$ does not appear in c and thus, it lies in another cycle $d = \langle k - 1, d_1, d_2, \ldots, d_s\rangle$, $s \geqslant 0$. Therefore, the decomposition of $h_k(\sigma) = \sigma \cdot \langle \sigma^{-1}(k-1), n\rangle = \sigma \cdot \langle d_s, n\rangle$ into disjoint cycles is obtained from that of σ by gluing c and d into the cycle $\langle n, k-1, d_1, \ldots, d_s, c_1, c_2, \ldots, c_r\rangle$. See Fig. 1 for an illustration of the gluing. So, this implies that $h_k(\sigma) \in S_n(k-1)$.

Theorem 3. *For $k \in \{2, \ldots, n\}$, h_k is a bijection.*

Proof. Let us prove that h_k is a bijection. In order to show the injectivity we take σ and π in $S_n(k)$ such that $h_k(\pi) = h_k(\sigma)$. We distinguish three cases.

(1) If π and σ belong to $S_n(k-1)$ then $h_k(\pi) = \pi$ and $h_k(\sigma) = \sigma$, and $\pi = \sigma$.

(2) If π and σ do not belong to $S_n(k-1)$ then we have $\sigma \cdot \langle j, n\rangle = \pi \cdot \langle \ell, n\rangle$ where $j = \sigma^{-1}(k-1)$ and $\ell = \pi^{-1}(k-1)$. Let $c = \langle n, c_1, c_2, \ldots, c_r\rangle$ (resp. $c' = \langle n, c'_1, c'_2, \ldots, c'_{r'}\rangle$) be the cycle containing n in π (resp. σ), and $d = \langle k-1, d_1, d_2, \ldots, d_s\rangle$ (resp. $d' = \langle k-1, d'_1, d'_2, \ldots, d'_{s'}\rangle$) be the cycle of π (resp. σ) containing ℓ (resp. j). We have $c \neq d$, $c' \neq d'$, $d_s = \ell$, $d'_{s'} = j$. $h_k(\pi)$ (resp. $h_k(\sigma)$) is obtained by gluing c and d (resp. c' and d') as explained just after Definition 2. Because $h_k(\pi) = h_k(\sigma)$, we deduce that

$$\begin{aligned} &\langle n, k-1, d_1, \ldots, d_s, c_1, c_2, \ldots, c_r\rangle \\ &\quad = \big\langle n, k-1, d'_1, \ldots, d'_{s'}, c'_1, c'_2, \ldots, c'_{r'}\big\rangle. \end{aligned}$$

Since $c_1 \geqslant k$, $c'_1 \geqslant k$, $[k, n] \subseteq \{c_1, c_2, \ldots, c_r\}$ and $[k, n] \subseteq \{c'_1, c'_2, \ldots, c'_{r'}\}$, we necessarily have $s = s'$, $r = r'$, $c_i = c'_i$ for $i \leqslant r$, and $d_i = d'_i$ for $i \leqslant s$. Thus we obtain $\sigma = \pi$.

(3) The case $\pi \in S_n(k) \backslash S_n(k-1)$ and $\sigma \in S_n(k-1)$ does not occur since the last entry of $h_k(\sigma)$ is at least k while the last value of $h_k(\pi)$ is $k-1$.

Therefore h_k is injective.

Now let us consider $\pi \in S_n(k-1)$. If π also belongs to $S_n(k)$ then π is the image of π by h_k. If π does not belong to $S_n(k)$ then we necessarily have $\pi(n) = k-1$. Let $c = \langle n, k-1, c_1, \ldots, c_r\rangle$ be the cycle of π containing n and i_0 be the smallest i such that $c_i \geqslant k$. Then the permutation σ obtained from π by splitting c into the two cycles $c' = \langle n, c_{i_0}, c_{i_0+1}, \ldots, c_r\rangle$ and $d = \langle k-1, c_1, \ldots, c_{i_0-1}\rangle$, belongs to $S_n(k)$ and satisfies $h_k(\sigma) = \pi$. Thus h_k is surjective. □

A consequence of Theorem 3 is that the cardinality of the sets $S_{n+1}(k)$ is $n!$ for each $k \in \{1, 2, \ldots, n+1\}$. A simple combinatorial argument proves the following remarkable equality.

Remark 2. For all $k \in \{1, 2, \ldots, n\}$, we have

$$n! = (n-k+1) \cdot \sum_{i=0}^{k-1} \binom{k-1}{i} (n-k+i)!(k-i-1)!.$$

Table 3
The bijection f^{-1} from S_3 to $\mathcal{T}_3^0$ and the bijection ψ from S_3 to C_4. Excedances are illustrated in **boldface**.

σ	$f^{-1}(\sigma)$	$\psi(\sigma)$	$E(f^{-1}(\sigma)) = E(\sigma)$	$E^+(\sigma) \cup \{1\} = E(\psi(\sigma))$
123	012	**4**123	$\emptyset$	$\{1\}$
1**3**2	0**3**1	**3**1**4**2	$\{2\}$	$\{1,3\}$
213	**2**01	**24**13	$\{1\}$	$\{1,2\}$
231	**23**0	**234**1	$\{1,2\}$	$\{1,2,3\}$
312	**3**02	**43**12	$\{1\}$	$\{1,2\}$
321	**3**10	**34**21	$\{1\}$	$\{1,2\}$

Both sides of the above equality count the cardinality of $S_{n+1}(k)$. Indeed, a permutation σ belongs to $S_{n+1}(k)$ if and only if $\sigma(n+1) \geqslant k$ and each integer in $[k, n+1]$ lies into the same cycle $c = \langle n+1, c_1, c_2, c_3, \ldots \rangle$ of σ. So, there are $(n-k+1)$ choices for $c_1 = \sigma(n+1) \in [k, n]$. In order to complete c we choose a set I of i values among $\{1, 2, \ldots, k-1\}$, $0 \leqslant i \leqslant k-1$ and we consider all arrangements of elements of the set $I \cup [k, n] \backslash \{c_1\}$. Thus, for a given i, $0 \leqslant i \leqslant k-1$, there are $(n-k+1) \cdot \binom{k-1}{i} \cdot (n-k+i)!$ possible cycles c and $(k-i-1)!$ choices for the remaining values. Moving i from 0 to $k-1$, we obtain the above formula.

Theorem 4. *For $k \in \{2, \ldots, n\}$, the bijection $h_k : S_n(k) \to S_n(k-1)$ satisfies for any $\sigma \in S_n(k)$,*

$$WE(\sigma) \cap \{1, 2, \ldots, n-1\} = WE\big(h_k(\sigma)\big).$$

Proof. Let $\sigma \in S_n(k)$. Notice that n is never a weak excedance of $h_k(\sigma)$.

The case $h_k(\sigma) = \sigma$ is trivial. Now let us assume that $\sigma' = h_k(\sigma) = \sigma \cdot \langle j, n \rangle$ where $j = \sigma^{-1}(k-1)$. We necessarily have $j \leqslant k-1$ and $\sigma(n) \geqslant k$.

Let us take $i \in \{1, 2, \ldots, n-1\}$. In the case where $i \neq j$ and $i \neq n$, we obtain $\sigma'(i) = \sigma(i)$; then i is a weak excedance of σ if and only if it is also one for $\sigma' = h_k(\sigma)$.

If $i = j$ then $\sigma'(i) = \sigma(n)$. As $\sigma \in S_n(k)$, we have $\sigma(n) \geqslant k$ and then $\sigma(n) > k-1$. We obtain $j = i \leqslant k-1$ and $\sigma(i) = \sigma(j) = k-1$, and hence i is a weak excedance of σ. Moreover, we have $\sigma'(i) = \sigma(n) > k-1$ so that i is a weak excedance of σ'. Finally, we have $WE(\sigma) \cap \{1, 2, \ldots, n-1\} = WE(h_k(\sigma))$. □

Corollary 3. *There is a bijection h from S_n to C_{n+1} that preserves the set of all weak excedances.*

Proof. We set $h = h_2 \circ h_3 \circ \cdots \circ h_{n-1} \circ h_n \circ h_{n+1}$ from $S_{n+1}(n+1)$ to $S_{n+1}(1) = C_{n+1}$. Since $S_{n+1}(n+1)$ is the set of permutations in S_n after adding $n+1$ on the right, we have $WE(\sigma) = WE(\sigma \cdot (n+1)) \cap \{1, 2, \ldots, n\}$. Theorem 4 allows to conclude that $WE(\sigma) = WE(h(\sigma))$. □

Notice that the bijection h does not transform fixed points into quasi-fixed points (see for instance $h(132) = 3421$), unlike the bijection ϕ in Section 3.

5. Consequences and open problems

In this section we give some direct consequences of our study and we propose two open problems concerning the descent sets on cyclic permutations and derangements.

Following the notation from [2,3], let $\mathcal{T}_n^0$ be the set whose elements are n-cycles in one-line notation in which one entry has been replaced with 0. For example, $\mathcal{T}_3^0 = \{031, 201, 230, 012, 302, 310\}$. Obviously the cardinality of $\mathcal{T}_n^0$ is $n!$. Let σ be an element of $\mathcal{T}_n^0$, we say that k, $1 \leqslant k \leqslant n$, is an excedance of σ whenever $\sigma(k) > k$ and $E(\sigma)$ will be denote the set of excedances of σ. Theorem 5 is a counterpart of Elizalde's result [2] for the set of excedances on $\mathcal{T}_n^0$.

If E is a subset of $[n]$ then we define the set $E^+ \subseteq \{2, 3, \ldots, n+1\}$ by:

$$E^+ = \{e+1 \mid e \in E\}.$$

Moreover, we define the involution χ_n from S_n into itself as follows: for any $n \geqslant 1$ and $\sigma \in S_n$, $\chi_n(\sigma)(i) = n+1-\sigma(n+1-i)$ for $i \leqslant n$. For instance, we have $\chi_4(4132) = 3241$. Less formally, $\chi_n(\sigma)$ is obtained from $\sigma \in S_n$ by reading the complement of σ from right to left. In the following we will omit the subscript n for χ; it should be clear from the context.

Using the map $\phi : S_n \to C_{n+1}$ presented in Section 3, we consider the bijection ψ from S_n to C_{n+1} defined by:

$$\psi(\sigma) = \chi\big(\phi\big(\chi(\sigma)\big)\big).$$

For example, $\psi(312) = \chi(\phi(231)) = \chi(3421) = 4312$; see Table 3 for an illustration of this map when $n = 3$.

The following corollary appears as a direct consequence of Theorem 2.

Corollary 4. *The bijection $\psi : S_n \to C_{n+1}$ satisfies for any $\sigma \in S_n$,*

$$E^+(\sigma) \cup \{1\} = E\big(\psi(\sigma)\big).$$

Proof. It is obvious that an n-cycle has 1 as excedance; thus $1 \in E(\psi(\sigma))$. Now if $k \notin E^+(\sigma) \cup \{1\}$ then we have $k > 1$, $\sigma(k-1) \leqslant k-1$ and $n+1-(k-1) = n+2-k$ is a weak excedance of $\chi(\sigma)$, and we have $\chi(\sigma)(n+2-k) = n+1-\sigma(k-1)$. With Theorem 2, $n+2-k$ is an excedance of $\phi(\chi(\sigma))$, and we have $\phi(\chi(\sigma))(n+2-k) = n+1-\sigma(k-1)+1$. Thus, $\chi(\phi(\chi(\sigma)))(n+2-(n+2-k)) = n+2-(n+1-\sigma(k-1)+1)$, *i.e.*, $\chi(\phi(\chi(\sigma)))(k) = \sigma(k-1) \leqslant k-1$ which induces that k is not an excedance of $\psi(\sigma)$. Each implication of this reasoning can be also viewed as an equivalence which achieves the proof. □

Theorem 5. *Let n be a positive integer. There is a bijection f between $\mathcal{T}_n^0$ and S_n such that if $\sigma \in \mathcal{T}_n^0$, then*

$$E(\sigma) = E\big(f(\sigma)\big).$$

Proof. Let σ be an element of $\mathcal{T}_n^0$; σ is obtained from a cycle π by replacing $\pi(k)$ with 0 for some k, $1 \leqslant k \leqslant n$. Assume that the cyclic representation of π is $\langle a_1, a_2, \ldots, a_{n-1}, k\rangle$ where $a_i \in [n]\backslash\{k\}$ for all $i \leqslant n-1$. Let us consider π' defined by the cyclic representation $\pi' = \langle a_1+1, a_2+1, \ldots, a_{n-1}+1, k+1, 1\rangle$. Obviously, we have $E(\sigma)^+ \cup \{1\} = E(\pi')$. Now, we set $f(\sigma) = \psi^{-1}(\pi')$ where ψ is the bijection defined above. We immediately have:

$$\begin{aligned} E(\sigma)^+ \cup \{1\} &= E\big(\pi'\big) \\ &= E\big(f(\sigma)\big)^+ \cup \{1\}, \end{aligned}$$

which means $E(\sigma) = E(f(\sigma))$. □

Thus we can directly deduce:

Corollary 5. *For any n and any $I \subseteq [n-1]$,*

$$\big|\big\{\tau \in \mathcal{T}_n^0,\ E(\tau) = I\big\}\big| = \big|\big\{\sigma \in S_n,\ E(\sigma) = I\big\}\big|.$$

This last result appears as a counterpart of the following Elizalde's result for descents:

Theorem 6. *(See [2].) For any n and any $I \subseteq [n-1]$,*

$$\big|\big\{\tau \in \mathcal{T}_n^0, D(\tau) = I\big\}\big| = \big|\big\{\sigma \in S_n, D(\sigma) = I\big\}\big|.$$

We conclude this paper by giving two open problems about descent statistics (see [5] for some other open problems about descent statistics). Experimental investigations allow us to think that the answer to these two problems is positive. Problem 1 appears as a counterpart of Corollary 2 for descent statistics. Problem 2 would be a generalization of the Elizalde's result (Theorem 1, p. 2) and appears as a counterpart of Theorem 4 for descent statistics.

Problem 1. Is it true that there exists a bijection from the set of derangements D_n to the set C_{n+1}^q of $(n+1)$-cycles without quasi-fixed points that preserves the set of descents in $\{1, 2, \ldots, n-1\}$?

For $1 \leqslant k \leqslant n$, we denote $S'_n(k)$ the set of permutations $\sigma \in S_n$ such that the cycle of σ that contains n is of length k.

Problem 2. For $k \in \{2, \ldots, n\}$, is it true that there is a bijection h'_k from $S'_n(k)$ to $S'_n(k-1)$ that preserves the set of descents in $\{1, 2, \ldots, n-2\}$, *i.e.*,

$$D(\sigma) \cap \{1, 2, \ldots, n-2\} = D\big(h'_k(\sigma)\big) \cap \{1, 2, \ldots, n-2\}?$$

Acknowledgements

I would like to thank the anonymous referees for their very careful reading of this paper and their helpful comments and suggestions.

References

[1] J.-L. Baril, Gray code for permutations with a fixed number of cycles, Discrete Math. 307 (2007) 1559–1571.
[2] S. Elizalde, Descent sets of cyclic permutations, Adv. in Appl. Math. 47 (4) (2011) 688–709.
[3] S. Elizalde, The number of permutations realized by a shift, SIAM J. Discrete Math. 23 (2009) 765–786.
[4] R. Ehrenborg, E. Steingrímsson, The excedance set of a permutation, Adv. in Appl. Math. 24 (2000) 284–299.
[5] N. Eriksen, R. Freij, J. Wastlund, Enumeration of derangements with descents in prescribed positions, Electron. J. Combin. 16 (2009) R32.
[6] D.C. Foata, M.P. Schützengerger, Théorie Géométrique des Polynômes Eulériens, Lecture Notes in Math., vol. 138, Springer-Verlag, Berlin, 1970.
[7] D. Foata, On the Netto inversion number of a sequence, Proc. Amer. Math. Soc. 19 (1968) 236–240.
[8] D. Foata, M.P. Schützenberger, Major index and inversion number of permutations, Math. Nachr. 83 (1978) 143–159.
[9] I. Gessel, G. Viennot, Binomial determinants, paths, and hook length formulae, Adv. Math. 58 (1985) 300–321.
[10] P.A. MacMahon, Combinatory Analysis, vols. 1–2, Cambridge University Press, Cambridge, UK, 1915, reprinted by Chelsea, New York, 1955.
[11] R.P. Stanley, Enumerative Combinatorics, vol. I, Cambridge University Press, Cambridge, 1997.