Pizza-cutter's problem and Hamiltonian path

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Introduction and notations

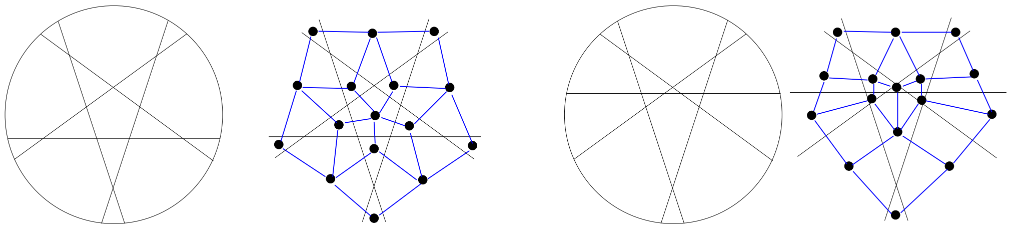
The *pizza-cutter's problem* was introduced and solved by Steiner in 1826 (see [**15**]), and it is considered as the doorstep of the well known Euler's formula $v+f-e=2$ where v is the number of vertices, e the number of edges and f the number of faces in a connected planar graph. The goal is to maximize the number of pieces we can make with n straight cuts through a circular pizza, regardless of size and shape. It is also the maximum number of regions formed by n lines in the plane, appearing in literature as the *Steiner's plane-cutting problem* [**1, 2, 17, 18**]. If ℓ_n denotes this number then it verifies the recurrence relation $\ell_n = \ell_{n-1} + n$ for $n \geq 1$ anchored by $\ell_0 = 1$, which induces the closed form $\ell_n = \frac{n(n+1)}{2} + 1$ (see A000124 in [**14**]). Indeed, from a solution to the problem with $n-1$ lines that forms ℓ_{n-1} regions, we add an nth line that is not parallel to any of the others, and such that $n-1$ new intersection points are created. Then, this line crosses n different regions, and each of them is divided into two regions which induces the above recursive formula.

Historically, the problem of line arrangements in the plane is studied by considering oriented matroids, more specifically known as non degenerate dissection types (see [**4, 5, 8, 12**] for the literature and [**6, 9**] for some databases). In this paper, we consider this problem from the point of view of graph theory. We will called *S-solution* a solution of the Steiner's plane-cutting problem. For each S-solution, we consider the *associated graph* $G=(V,E)$ where vertex set V and edge set E are defined as follows:

- V is the set of regions;
- $(p,q) \in E$ if and only if the two regions p and q are *adjacent*, *i.e.*, if they share a common boundary that is not a corner, where corners are the points shared by three or more regions.

Of course there are many ways to cut the plane into a maximal number of regions with n lines, but G always verifies $\mathrm{card}(V) = \ell_n$ and $\mathrm{card}(E) = n^2$. In the case where two solutions produce two isomorphic graphs, we say that these solutions are *isomorphic*; otherwise they are called *non-isomorphic*. See Figure 1 for an illustration of two non-isomorphic S-solutions with their corresponding graphs. Finding the number of classes of non-isomorphic solutions for the plane-cutting problem still remains an open problem for $n \geq 10$. For $1 \leq n \leq 9$, it is known that these numbers are given by the sequence A090338 in [**14**]: $1, 1, 1, 1, 6, 43, 922, 38609, 3111341$ (see [**9**]).

A graph will be called *traceable* whenever it contains a *Hamiltonian path*, *i.e.*, if there is a path that visits each vertex exactly once. This concept was introduced in 1856 in [**16**], where the author studies whether a polyhedron contains a path that reaches each vertex once and only once. More generally, the problem of determining whether a graph is traceable is NP-complete and has many applications (see [**10**]). In particular, this problem appears in network theory where it is crucial to connect points so that the total length of connecting lines is minimum. On the other hand, finding the traceability can be a simple and efficient method to prove that two graphs are not isomorphic. Then it becomes natural to ask the following question. *For* $n \geq 1$, *does an S-solution*



(a) (b)

Figure 1: Two non-isomorphic S-solutions for $n = 5$ with their associated graphs drawn in blue color.

exist such that its corresponding graph is traceable (respectively not traceable)? This amounts to finding a solution to the pizza-cutter's problem in which we can *eat up all pieces of the pizza such that two consecutive eaten pieces are adjacent.*

In Section 2, from an S-solution we show how we can label each region with a binary string. This induces a graph where the vertex set is the set of labels, and two binary strings are adjacent if their Hamming distance is one. Then, we prove that the traceability of the associated graph is equivalent to that of the graph on labels. In Section 3, we construct an S-solution where the associated graph is not traceable for all $n \geq 5$. In Section 4, we adapt this construction in order to obtain an S-solution for all n such that the graph is traceable. To our knowledge, no such precise constructions have previously been published. Finally, we formulate some open problems.

Binary string interpretation

A *binary string* s of *length* n is a word $s_1 s_2 \dots s_n$ on the alphabet $\{0, 1\}$. The value s_i, $1 \leq i \leq n$, will be called the ith *digit* of s. A *substring* t of s is a word made up with some consecutive digits of s. A *run of* 1*'s* in s is a maximal substring of s of the form 1^k where $k \geq 1$, *i.e.*, a substring constituted of 1's that cannot be extended in a larger substring of 1's in s. For a binary string set B, we denote by B' (resp. B'') the subset of B of strings with an odd (resp. even) number of 1's.

The *Hamming distance* between two n-length binary strings s and t is the number of i, $1 \leq i \leq n$, such that s_i is different from t_i. A *Gray code* for a set of binary strings $B \subseteq \{0, 1\}^n$ is an *ordered list* $\mathcal{B}$ for B, such that the Hamming distance between any two consecutive strings in $\mathcal{B}$ is exactly one. Obviously, a Gray code $\mathcal{B}$ for the set B can be viewed as a Hamiltonian path in the restriction of the hypercube Q_n to the set B. Note that if B' (resp. B'') is the subset of B consisting of strings with an odd (resp. even) number of 1's, then no Gray code is possible for B whenever $|\operatorname{card}(B') - \operatorname{card}(B'')| > 1$.

Now, let us consider an S-solution with n lines numbered from 1 to n. We label each region with a binary string of length n where the ith digit is either 0 or 1 depending on whether the region is on one side or the other of the ith line. See Figure 2 for three illustrations of such a labeling. Of course, there are $n!$ possible ways to label n lines from 1 to n, and for each line we have two half-planes delimited by this line. Therefore, for an S-solution there are at most $2^n \cdot n!$ possible sets of labels. In the following, such a set will be called *admissible* for a given S-solution.

Lemma 1. Let us consider an S-solution for $n \geq 1$, $G = (V, E)$ its associated graph and W an admissible set of binary strings for this solution. Let $H = (W, F)$ be the graph where the vertex set is W and such that two elements are adjacent in H if and only if their Hamming distance is one. Then G and H are isomorphic; and thus, G is traceable if and only if H is traceable.

Proof. Indeed, it is straightforward to see that the two following assertions are equivalent: (1) two regions r and s are adjacent; and (2) the Hamming distance of the binary strings labeling r and s is one. □

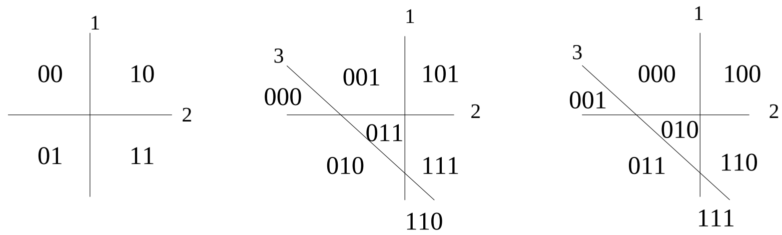


Figure 2: Regions labeled using admissible sets of binary strings. The leftmost and rightmost labelings provide the sets L_2 and L_3, while the central one does not generate L_3.

Remark 1. With the hypotheses of Lemma 1, a necessary condition for the traceability of the graph G is that the cardinalities of W' and W'' differ by at most one.

We end this section by introducing a set that will be crucial in the following. Let L_n be the set of binary strings of length n containing at most one run of 1's. Any string $s_1 s_2 \dots s_n \in L_n, n \geq 1$, can be written either $s = 0s_2 \dots s_n$ where $s_2 \dots s_n \in L_{n-1}$, or $s = 1^k 0^{n-k}$ with $1 \leq k \leq n$. So, we have $\mathrm{card}(L_n) = \mathrm{card}(L_{n-1}) + n$ which induces $\mathrm{card}(L_n) = \ell_n$. Now, we denote by L'_n (respectively L''_n) the subset of L_n constituted of strings in L_n with an odd (respectively even) number of 1's. For instance, $L_3 = \{000, 001, 010, 100, 110, 011, 111\}$, $L'_3 = \{001, 010, 100, 111\}$ and $L''_3 = \{000, 110, 011\}$.

An S-solution where $G = (V, E)$ is not traceable

In this part, we construct an S-solution such that for each $n \geq 5$, its associated graph $G = (V, E)$ is not traceable. For this, we prove that the set L_n of n-length binary strings with at most one run of 1's is admissible for this solution and that the cardinality of their two subsets L'_n and L''_n differ by at least 2. Using Lemma 1 and Remark 1, we conclude that G is not traceable.

Lemma 2. For $n \geq 1$, there is an S-solution such that the set L_n of binary strings of length n with at most one run of 1's is admissible.

Proof. We proceed by induction on the number n of lines. The case $n = 1$ is trivial since we label the two half-planes by 0 and 1 and $L_1 = \{0, 1\}$.

Assume now that there is an S-solution of $n - 1$ lines such that the regions can be labeled with the binary strings of the set L_{n-1}. Since there are exactly $n - 1$ binary strings ending in a one in L_{n-1}, the $(n - 1)$th line splits the plane into two half-planes such that one of them contains exactly the $n - 1$ regions labeled

$0^{n-2}1, 0^{n-3}1^2, \ldots, 01^{n-2}, 1^{n-1}$. Then, we necessarily have the leftmost configuration illustrated in Figure 3 where all previous binary strings appear on the same half-plane defined by the line $n-1$ (line 5 in Figure 3). Now, it suffices to place the nth line (line 6 in Figure 3) such that: it crosses the region 0^{n-1} and all other regions labeled 0^k1^{n-1-k} for $0 \leq k \leq n-2$ (the process is illustrated in Figure 3). Note that we can always add this line since it can be obtained from the $(n-1)$th line by a rotation centered on a point placed on the border between the regions 0^{n-1} and $0^{n-2}1$, and with an angle small enough to allow the nth line to intersect the first $(n-2)$ lines (as the $(n-1)$th line). Then, the labels of the new created regions are obtained by adding 1 to the right of 0^{n-1} and 0^k1^{n-1-k} for $0 \leq k \leq n-2$, and adding 0 to the right of all other labels in L_{n-1}. Finally, the set of the obtained labels is exactly the set L_n of binary strings of length n with at most one run of 1's, and the proof is obtained by induction. □

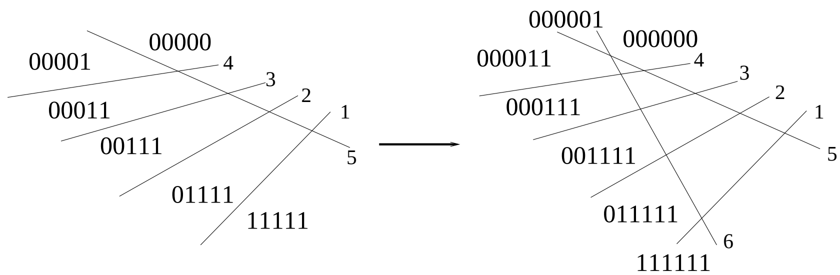


Figure 3: An illustration for the induction in the proof of Lemma 2.

Let $\{\phi_n\}_n \geq 0$ be the parity difference integer sequence corresponding to the binary strings with at most one run of 1's, *i.e.*, $\phi_n = \mathrm{card}(L'_n) - \mathrm{card}(L''_n)$ for $n \geq 0$.

Lemma 3. For $n \geq 1$, we have $\phi_n = \lfloor \frac{n-1}{2} \rfloor$.

Proof. For $1 \leq i \leq n$, we denote by L_n^i the subsets of L_n made of strings with exactly i ones. Thus, it follows trivially that $\mathrm{card}(L_n^i) = n - i + 1$ for $1 \leq i \leq n$, and $\mathrm{card}(L_n^0) = 1$. Moreover, for i odd, $1 \leq i \leq n-1$, we have $\mathrm{card}(L_n^i) - \mathrm{card}(L_n^{i+1}) = 1$. Since $L'_n = \bigcup_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} L_n^{2i-1}$ and $L''_n = \bigcup_{i=0}^{\lfloor \frac{n}{2} \rfloor} L_n^{2i}$, we distinguish two cases. If n odd then $\phi_n = \mathrm{card}(L'_n) - \mathrm{card}(L''_n) = \mathrm{card}(L_n^n) - \mathrm{card}(L_n^0) + \bigcup_{i=1}^{\lfloor \frac{n-1}{2} \rfloor} (\mathrm{card}(L_n^{2i-1}) - \mathrm{card}(L_n^{2i})) = \lfloor \frac{n-1}{2} \rfloor$. If n is even then $\phi_n = \bigcup_{i=1}^{\lfloor \frac{n}{2} \rfloor} (\mathrm{card}(L_n^{2i-1}) - \mathrm{card}(L_n^{2i})) - \mathrm{card}(L_n^0) = \lfloor \frac{n-1}{2} \rfloor$. □

Theorem 1. For each $n \geq 5$, there exists an S-solution such that its associated graph is not traceable. See Figure 4.

Proof. For $n \geq 5$, Lemma 3 induces $\phi_n = \lfloor \frac{n-1}{2} \rfloor \geq 2$. Remark 1 and Lemma 2 allow to conclude. □

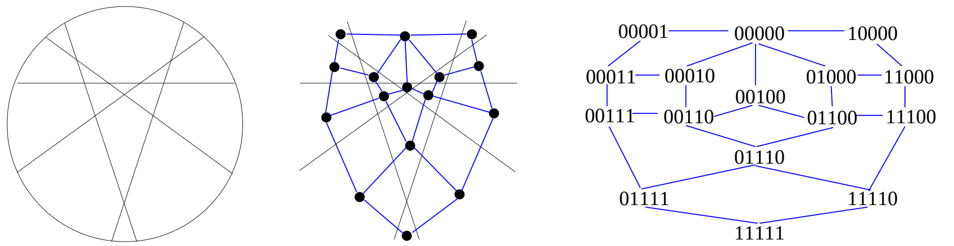


Figure 4: An S-solution where its associated graph is not traceable.

An S-solution where $G = (V, E)$ is traceable

In this part, for each $n \geq 1$ we construct an S-solution such that its associated graph is traceable.

From the set L_n defined at the end of Section 2, we define the set K_n by replacing all strings $0^{4i}00100^{n-4(i+1)} \in L_n$ with $0^{4i}01010^{n-4(i+1)}$ for $0 \leq i \leq \lfloor \frac{n}{4} \rfloor - 1$. For instance, we obtain K_5 (resp. K_8) from L_5 (resp. L_8) by replacing 00100 (resp. 00100000 and 00000010) with 01010 (resp. 01010000 and 00000101).

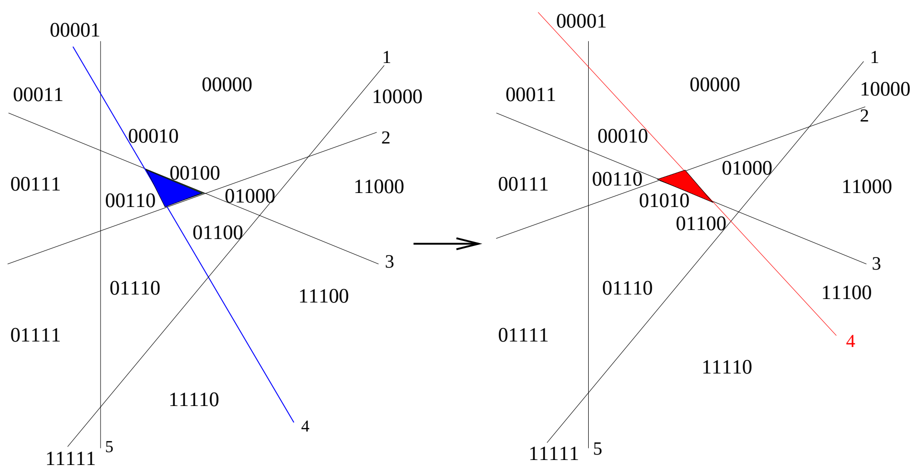


Figure 5: Construction in the proof of Lemma 4.

Lemma 4. For $n \geq 1$, there is an S-solution such that the set K_n is admissible.

Proof. Let us take the S-solution constructed in the proof of Lemma 2. Then we modify the position of each line labeled $4i$, $1 \leq i \leq \lfloor \frac{n}{4} \rfloor$ in the following way. For i from 1 to $\lfloor \frac{n}{4} \rfloor$, the line labeled $4i$ is moved so that in this new position, the half-plane delimited by this line and containing the point of intersection of the lines $4i - 1$ and $4i - 2$ does not contain any other points of intersection between lines from 1 to $4i - 1$. See Figure 5 for an illustration of the process. Then, a simple observation provides that the labels in L_n are preserved up to the labels $0^{4i}00100^{n-4(i+1)}$, $0 \leq i \leq \lfloor \frac{n}{4} \rfloor - 1$, that are replaced with $0^{4i}01010^{n-4(i+1)}$ which transforms the set L_n into the set K_n. Thus K_n is admissible. $\square$

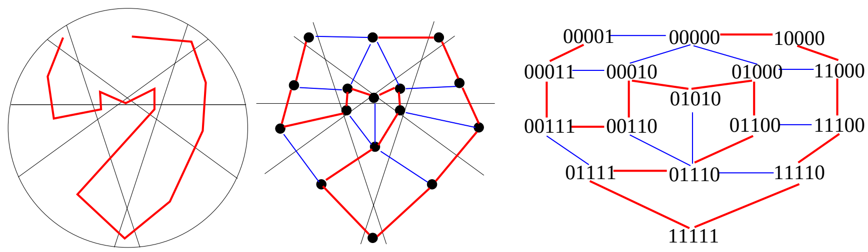


Figure 6: An S-solution where its associated graph is traceable. The red edges constitute a Hamiltonian path.

Theorem 2. For $n \geq 1$, there exists an S-solution such that its associated graph is traceable.

Proof. Due to Lemmas 1 and 4, it suffices to prove that the set K_n can be ordered in a list $\mathcal{K}_n$ such that *two consecutive elements differ by one digit, i.e.,* $\mathcal{K}_n$ is in *Gray code order*. In order to facilitate the reading of the proof (somewhat theoretical), we invite the reader to follow it by setting $n = 7$ or $n = 8$ before referring to Table 1.

Let $\mathcal{S}_n$, $n \geq 0$, be the list of the $n+1$ binary strings defined as follows: *the ith binary element of the list is* $1^{i-1}0^{n-i+1}$, $1 \leq i \leq n+1$. For instance, the list $\mathcal{S}_4$ is $0000, 1000, 1100, 1110, 1111$. For $n = 0$, the list $\mathcal{S}_n$ is reduced to the empty string. Obviously, two consecutive elements of $\mathcal{S}_n$ differ by exactly one digit and the first and last elements of $\mathcal{S}_n$ are respectively 0^n and 1^n.

Using the lists $\mathcal{S}_n$, $n \geq 0$, we define an ordered list $\mathcal{L}_n$ of the set L_n:

$$\mathcal{L}_n = 0^n \odot \bigodot_{i=0}^{n-1} 0^i 1 \cdot \mathcal{S}^i_{n-i-1},$$

where $\odot$ is the concatenation operator of lists, and where $\mathcal{S}^i_n$ is the reverse list of $\mathcal{S}_n$ (*i.e.*, the list $\mathcal{S}_n$ considered from the last to the first element) whenever i is odd, and the list $\mathcal{S}_n$ otherwise. See Table 1 for an illustration of the two lists $\mathcal{L}_7$ and $\mathcal{L}_8$.

In the list $\mathcal{L}_n$, it is straightforward to see that two consecutive elements differ by at most one digit excepted for the transitions between the sublists $0^i1 \cdot \mathcal{S}^i_{n-i-1}$ and $0^{i+1}1 \cdot \mathcal{S}^{i+1}_{n-i-2}$ for i odd and $1 \leq i \leq n-2$. In these cases, the transitions move two digits since (when i is odd) the last element of $0^i1 \cdot \mathcal{S}^i_{n-i-1}$ is $\mathbf{0^i10^{n-i-1}}$ and the first element of $0^{i+1}1 \cdot \mathcal{S}^{i+1}_{n-i-2}$ is $0^{i+1}10^{n-i-2}$. Moreover, the first and last elements of the list $\mathcal{L}_n$ are respectively 0^n and $0^{n-1}1$. Now we modify the list $\mathcal{L}_n$ in order to construct a list $\mathcal{K}_n$ in Gray code order for the set K_n.

For all odd i such that $i = 1 \mod 4$, $1 \leq i \leq n-3$, we replace the string 0^i0100^{n-i-3}, with 0^i1010^{n-i-3} and we change the place of 0^i0010^{n-i-3} by inserting it just after 0^i1010^{n-i-3} and thus just before 0^i0110^{n-i-3}. See Table 1 for an illustration of this process for the lists $\mathcal{K}_7$ and $\mathcal{K}_8$.

By construction, the four binary strings $\mathbf{0^i1000^{n-i-3}}$, 0^i1010^{n-i-3}, 0^i0010^{n-i-3} and 0^i0110^{n-i-3} are consecutive in the list $\mathcal{K}_n$ and the three transitions differ by only one digit.

On the other hand, since we change the position of the binary strings of the form 0^i0010^{n-i-3}, for $i = 1 \mod 4$, $1 \leq i \leq n-3$, we create a new transition between its predecessor 0^i00110^{n-i-4} and its successor 0^i00010^{n-i-4} (if it exists) that moves only one digit. Notice that if $i = n-3$ then the string 0^i0010^{n-i-3} has no successor in the list $\mathcal{L}_n$ and after moving its position, the last element of $\mathcal{K}_n$ becomes $0^{n-2}11$.

- If n is even, then the last transition of two digits in $\mathcal{L}_n$ occurs between $0^{n-3}100$ and $0^{n-2}10$ which means that all transitions of two digits have been treated above, and the list $\mathcal{K}_n$ is in Gray code order. So, first and last elements are respectively 0^n and $0^{n-2}11$ for $n = 0 \mod 4$, and 0^n and $0^{n-1}1$ for $n = 2 \mod 4$.

- If n is odd, then the last transition of two digits in $\mathcal{L}_n$ occurs between $0^{n-2}10$ and $0^{n-1}1$. We distinguish two subcases. If $n \neq 3 \mod 4$, then the string $0^{n-2}10$ is moved by the above process and the obtained list is in Gray code order. So, first and last elements are respectively 0^n and $0^{n-1}1$ (the Gray code is cyclic). Now, if $n = 3 \mod 4$, then we insert the first element 0^n between $0^{n-2}10$ and $0^{n-1}1$ and we obtain a Gray code. First and last elements are respectively 10^{n-1} and $0^{n-1}1$.

Finally, for all $n \geq 1$ the constructed list $\mathcal{K}_n$ is in Gray code order. □

Remark 2. For $n = 1, 2 \mod 4$, the Hamming distance between the first and last elements of the list $\mathcal{K}_n$ is one. Thus the associated graph becomes Hamiltonian (see Figure 6).

Going further

This study brings up several open questions. In this paper, we use a constructive method in order to prove that the pizza-cutter's problem admits an S-solution where its associated graph is traceable. Is it possible to provide a similar result using probabilistic method as studied in [**3, 13**]? For a given n, can we find the number of isomorphism classes of S-solutions for $n \geq 10$? How many classes induce a traceable graph? For a given S-solution, can we characterize its corresponding admissible sets? More generally, can we make the same study for the space-cutting problem where dimension of the space is greater than two?

Acknowledgements

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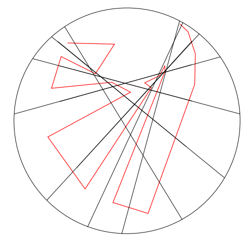
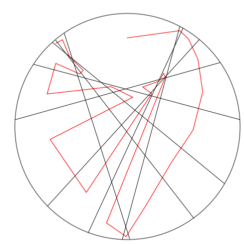
TABLE 1: The Lists $\mathcal{L}_n$ and $\mathcal{K}_n$ for $n = 7, 8$ and the Hamiltonian path on the S-solutions associated to the sets K_7 and K_8.

$\mathcal{L}_7$				$\mathcal{K}_7$	
	0000000	$i=3$	0001111	1000000	0001111
$i=0$	1000000		0001110	1100000	0001110
	1100000		0001100	1110000	0001100
	1110000		0001000	1111000	0000100
	1111000	$i=4$	0000100	1111100	0000110
	1111100		0000110	1111110	0000111
	1111110		0000111	1111111	0000011
	1111111	$i=5$	0000011	0111111	0000010
$i=1$	0111111		0000010	0111110	0000000
	0111110	$i=6$	0000001	0111100	0000001
	0111100			0111000	
	0111000			0110000	
	0110000			**0100000**	
	0100000			*0101000*	
$i=2$	*0010000*			0001000	
	0011000			0011000	
	0011100			0011100	
	0011110			0011110	
	0011111			0011111	



$\mathcal{L}_8$				$\mathcal{K}_8$	
	00000000		00111100	00000000	00111000
$i=0$	10000000		00111110	10000000	00111100
	11000000		00111111	11000000	00111110
	11100000	$i=3$	00011111	11100000	00111111
	11110000		00011110	11110000	00011111
	11111000		00011100	11111000	00011110
	11111100		00011000	11111100	00011100
	11111110		00010000	11111110	00011000
	11111111	$i=4$	00001000	11111111	00001000
$i=1$	01111111		00001100	01111111	00001100
	01111110		00001110	01111110	00001110
	01111100		00001111	01111100	00001111
	01111000	$i=5$	00000111	01111000	00000111
	01110000		00000110	01110000	00000110
	01100000		**00000100**	01100000	**00000100**
	01000000	$i=6$	*00000010*	**01000000**	*00000101*
$i=2$	*00100000*		00000011	*01010000*	00000001
	00110000	$i=7$	00000001	00010000	00000011
	00111000			00110000	



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Summary The pizza-cutter's problem consists in maximizing the number of pieces we can make with n straight cuts regardless of size and shape. For a solution to this problem, we consider the graph $G = (V, E)$ where the vertex set V is the set of pieces and $(p, q) \in E$ if and only if the two pieces p and q are adjacent. We prove that there exists a solution where the graph G contains (respectively does not contain) a Hamiltonian path. Finally we present some open questions.

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