MATCHINGS IN THREE CATALAN LATTICES

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March 19, 2004

Abstract

In this note we consider a series of lattices that are enumerated by the well-known Catalan numbers. For each of these lattices, we exhibit a matching in a constructive way.

KEY WORDS: Lattices, matchings, well-formed parentheses strings, binary trees.

C.R. CATEGORIES: G.2.1. G.2.2.

1 INTRODUCTION

Given a lattice L, we denote the zero (resp. unit) by $\mathbf{0}$ (resp. $\mathbf{1}$) if it exists. The meet and the join of (x, y) are denoted respectively $x \wedge y$ and $x \vee y$. $x \in L$ is a join (resp. meet)-irreducible element if $x = a \vee b$ (resp. $x = a \wedge b$) implies $x = a$ or $x = b$. In other words, join (resp. meet)-irreducible elements have a unique lower (resp. upper) cover. Given a finite lattice L, let $J(L)$ (resp. $M(L)$) denote the set of nonzero join-irreducible (resp. nonunit meet-irreducible) elements of L. We say that L has a matching σ if σ is a map of $J(L) \cup \{\mathbf{0}\}$ to $M(L) \cup \{\mathbf{1}\}$ which is one-to-one and verifies $j \leq \sigma(j)$ for each join-irreducible j [2, 8, 16]. Kung has proved that every consistent lattice has a matching [8]. In this note, we exhibit matchings for three Catalan lattices, i.e. lattices which are enumerated by the well-known Catalan numbers. The three sets of combinatorial objects which are endowed with a lattice structure are respectively the well-formed parentheses strings, the binary trees and the noncrossing partitions. We adopt a *constructive* point of view. Indeed we build *explicit* matchings by giving precise constructions.

For the first two lattices often used in computer science [4, 13, 22], the main idea follows [6, p. 83]. Let C be a particular maximal chain in the lattice L. Let assume that the length of C is equal to $|J(L)|$. For $j \in J(L)$, let $s(j)$ be the smallest member of C such that $j \leq s(j)$. We thereby build a one-to-one map s of $J(L)$ to $C - \{\mathbf{0}\}$. For $m \in M(L)$, let $t(m)$ be the greatest member of C such that $t(m) \leq m$. We thereby build a one-to-one map of $M(L)$ to $C - \{\mathbf{1}\}$. Using the above bijections, a matching in L can be constructed.

2 MATCHINGS IN LATTICES OF WELL-FORMED PARENTHESES STRINGS

A well-formed parentheses string (w.f.p. string in short) is a word on the alphabet $\{(,)\}$ generated by the grammar $S \to (S)|SS|\lambda$ where λ is the empty word. We denote by P_n the set of w.p.f. strings with n open and n close parentheses. It is well-known that $|P_n| = c_n$ where $c_n = \binom{2n}{n}/(n+1)$ is the nth Catalan number. Let $\to$ denote the adjacent parentheses interchange, i.e. we write $w \to w'$ ($w, w' \in P_n$) if there exist x and $y \in \{(,)\}^+$ such that $w = x)(y$ and $w' = x()y$. Thus we obtain w' from w by interchanging two adjacent parentheses. Let $\overset{*}{\to}$ be the reflexive transitive closure of $\to$.
In order to characterize $\overset{*}{\to}$, we use the following coding introduced in [15]. Let define the P-sequence of $w \in P_n$ as the integer sequence $(p_w(1), \ldots, p_w(n))$ where $p_w(i)$ is the number of open parentheses written before the ith close parenthesis of w. For example, if $w = (((()()))(()))() \in P_8$ then $p_w = (4, 5, 5, 5, 7, 7, 7, 8)$.
An n-integer sequence p is the P-sequence of a w.f.p. string of P_n iff $p(n) = n$ and for all $i \in [1, n-1]$: $i \le p(i) \le p(i+1)$ [15]. We have shown in [4] the following characterization:

Theorem 1 *For all $w, w' \in P_n$ we have $w \overset{*}{\to} w'$ iff for all $i \in [1, n] : p_w(i) \le p_{w'}(i)$.*

$(P_n, \overset{*}{\to})$ is a distributive lattice with **0** and **1** for all n, which is graded by the rank function $r(w) = \sum_{i=1}^{n} p_w(i)$. It is well-known that every distributive lattice has a matching [2].
We have $p_{\mathbf{0}} = (1, 2, 3, \ldots, n)$ and $p_{\mathbf{1}} = (n, n, n, \ldots, n)$. The P-sequences of the meet and the join of w and $w' \in P_n$ are respectively computed by $p_{w \wedge w'}(i) = min(p_w(i), p_{w'}(i))$ and $p_{w \vee w'}(i) = max(p_w(i), p_{w'}(i))$ for all $i \in [1, n]$. In short, we write $p_{w \wedge w'} = min(p_w, p_{w'})$ and $p_{w \vee w'} = max(p_w, p_{w'})$.

Theorem 2 *w is a non-zero join-irreductible element of P_n iff there exist $k \in [1, n]$ and $l \in [2, n]$ such that $p_w = (1, 2, \ldots, k-1, \underbrace{k+l-1, \ldots, k+l-1}_{l}, k+l, k+l+1, \ldots, n)$.*

Proof. Let $w \neq \mathbf{0}$ be a join-irreducible element of P_n. Thus we have $\{i \in [1, n] | p_w(i) > i\} \neq \emptyset$ since $p_w \neq (1, 2, 3, \ldots, n)$. Let denote $i_1 = min\{i \in [1, n] | p_w(i) > i\}$ and $i_2 = max\{i \in [1, n] | p_w(i) > i\}$. Suppose that $p_w(i_1) < p_w(i_2)$. Let us denote $i = max\{j \in [i_1, i_2[| p_w(j) < p_w(i_2)\}$. Then we obtain the following decomposition: $p_w = max(p', p'')$ with $p' = (p_w(1), \ldots, p_w(i_1) - 1, \ldots, p_w(i_2), \ldots, p_w(n))$ and $p'' = (p_w(1), \ldots, p_w(i_1), \ldots, p_w(i), p_w(i_2) - 1, \ldots, p_w(i_2) - 1, p_w(i_2 + 1), \ldots, p_w(n))$ which contradicts the fact that w has a unique lower cover. Therefore $p_w(i_1) = p_w(i_2) = i_2 + 1$ and we can write $p_w = (1, \ldots, i_1 - 1, i_2 + 1, \ldots, i_2 + 1, \ldots, n)$. The result holds with $i_1 = k$ and $l = i_2 - i_1 + 2$. Conversely, if the P-sequence of w is $p_w = (1, \ldots, k-1, k+l-1, \ldots, k+l-1, k+l, \ldots, n)$, only one w' satisfies $r(w') = r(w) - 1$. This w' verifies $p_{w'} = (1, \ldots, k-1, k+l-2, k+l-1, \ldots, k+l-1, k+l, \ldots, n)$.

Theorem 3 *w is a non-unit meet-irreducible element of P_n iff there exist k and l with $1 \le l \le k \le n-1$ such that $p_w = (\underbrace{k, k, \ldots, k}_{l}, n, \ldots, n)$.*

Proof. Let $w \neq \mathbf{1}$ be a meet-irreducible element of P_n. Thus we have $\{i \in [1, n-1] | p_w(i) < p_w(i+1)\} \neq \emptyset$ since $p_w \neq (n, n, \ldots, n)$. Let us denote $i_1 = min\{i \in [1, n-1] | p_w(i) < p_w(i+1)\}$ and $i_2 = max\{i \in [1, n-1] | p_w(i) < p_w(i+1)\}$. In the case where $i_1 < i_2$, p_w could be written as $p_w = min(p', p'')$ with $p' = (p_w(1), \ldots, p_w(i_1 - 1), p_w(i_1) + 1, \ldots, p_w(i_2), \ldots, p_w(n))$ and $p'' = (p_w(1), \ldots, p_w(i_1), \ldots, p_w(i_2 - 1), p_w(i_2) + 1, p_w(i_2 + 1), \ldots, p_w(n))$, contradicting the existence of a unique upper cover. Thus $i_1 = i_2$ holds and denoting $p_w(i_2) = k$, we have $p_w = (k, \dots, k, n, \dots, n)$. Conversely if the P-sequence of w is $p_w = (k, k, \dots, k, n, n, \dots, n)$, only one w' satisfies $r(w') = r(w) + 1$. Thus w' verifies $p_{w'} = (k, \dots, k, k+1, n, \dots, n)$.

Remark. The number of join-irreductibles is equal to the number of meet-irreductibles, namely $\frac{n(n-1)}{2}$. Furthermore, G. Gratzer [6, p. 83] shows that in a distributive lattice, any maximal chain has length $|J(L)|$. The tool of this proof is using for constructing a matching of P_n.

Theorem 4 *The map $\sigma : J(L) \cup \{\mathbf{0}\} \longrightarrow M(L) \cup \{\mathbf{1}\}$ defined by:*

for $k \in [1, n]$: $\sigma((1, 2, \cdots, k-1, n, \cdots, n)) = (\underbrace{k-1, \cdots, k-1}_{k-1}, n, \cdots, n)$

and if $k + l - 1 < n$ with $l \in [2, n]$:

$$\sigma((1, 2, \dots, k-1, \underbrace{k+l-1, \dots, k+l-1}_{l}, k+l, \dots, n)) = (\underbrace{k+l-1, \dots, k+l-1}_{k}, \underbrace{n, \dots, n}_{n-k})$$

is a matching in L.

Proof. The construction method consists in choosing a maximal chain C of P_n, then in constructing a non-decreasing one-to-one map of $J(L) \cup \{\mathbf{0}\}$ to C and finally in constructing a non-increasing one-to-one map of $M(L) \cup \{\mathbf{1}\}$ to C.
We choose a maximal chain $C = \{\mathbf{0} = c_0 \to c_1 \to \dots \to c_{n(n-1)/2} = \mathbf{1}\}$ in P_n in the following way. We start with $c_0 = \mathbf{0}$ and for each w.f.p. string w of C, we apply the adjacent parentheses interchange)($\to$ () on the rightmost occurrence)(in w. For example in P_5: ()()()()() $\to$ ()()()(()) $\to$ ()()(()()) $\to$ ()()((())) $\to$ ()(()(())) $\to$ ()((()())) $\to$ ()(((()))) $\to$ (()((()))) $\to$ ((()(()))) $\to$ (((()()))) $\to$ ((((())))). See also the bold path in Figure 1.
A non-unit w.f.p. string of this chain C is characterized by a P-sequence of the form $(1, 2, \dots, k-1, p_k, n, \dots, n)$ with $1 \le k \le n-1$ and $k \le p_k \le n-1$.
For $j \in J(L) \cup \{\mathbf{0}\}$, let $s(j)$ be the smallest member of C such that $j \le s(j)$. Thus $s(\mathbf{0}) = \mathbf{0}$ and $s((1, 2, \dots, k-1, \underbrace{k+l-1, \dots, k+l-1}_{l}, k+l, \dots, n)) = (1, 2, \dots, k-1, k+l-1, n, \dots, n)$. s is obviously injective and thus is a one-to-one map. For $m \in M(L) \cup \{\mathbf{1}\}$, let $t(m)$ be the greatest member of C such that $t(m) \le m$. Thus $t(\mathbf{1}) = \mathbf{1}$ and if $k \ne n$, $l \ge 1$ then
$t((\underbrace{k, k, \dots, k}_{l}, n, \dots, n)) = (1, \dots, l-1, k, \underbrace{n, \dots, n}_{n-l})$. t is also injective and thus is a one-to-one map. t^{-1} is defined by $t^{-1}(\mathbf{1}) = \mathbf{1}$ and if $k + l - 1 \ne n$, $t^{-1}((1, 2, \dots, k-1, k+l-1, n, \dots, n)) = (\underbrace{k+l-1, \dots, k+l-1}_{k}, n, \dots, n)$ and $t^{-1}((1, 2, \cdots, k-1, n, \cdots, n)) = (\underbrace{k-1, \cdots, k-1}_{k-1}, n, \cdots, n)$.
Therefore the map $\sigma = t^{-1} \circ s$ defined previously is a matching in P_n.

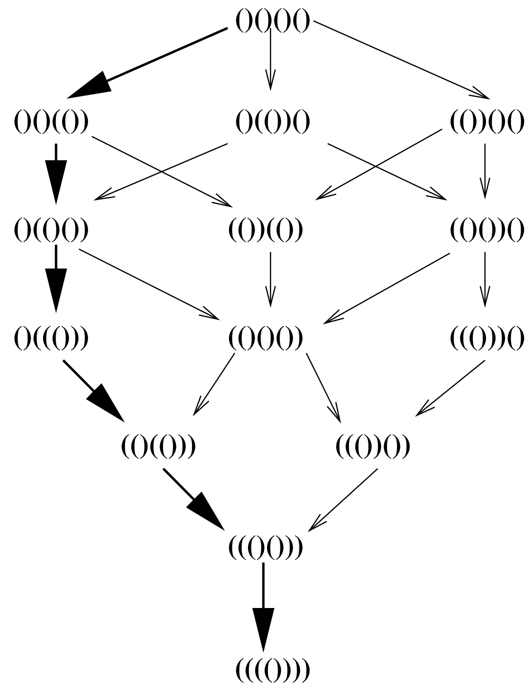

Figure 1: The distributive lattice P_4

3 MATCHINGS IN TAMARI LATTICES

The so-called Tamari lattices are orderings of w.f.p. words that were introduced by Tamari fifty years ago [20] and later shown to be lattices [1, 5, 10, 11]. The Tamari lattices can be described in many ways via the known bijections between families of Catalan combinatorial objects [3]. A system that is isomorphic to Tamari lattices is that of triangulations of a polygon related by the diagonal flip operation [19]. Another system that is isomorphic to Tamari lattices too is that of binary trees related by rotations [11, 12, 14].

A binary tree is a rooted, ordered tree in which every internal node $\bigcirc$ has exactly two sons. External nodes or leaves have no children and are denoted $\Box$. We denote by B the set of binary trees. We denote by T_L and T_R the left and right subtrees of $T \in B$ if $T \neq \Box$. In Polish notation, we can write $T = \bigcirc T_L T_R$.

We denote by B_n the set of binary trees with n internal nodes (and thus with $n+1$ leaves). It is well-known that $|B_n| = c_n$.

The leaves of a binary tree T are numbered by a preorder traversal of T (i.e. from left to right). The weight $|T|$ of a tree T is the number of leaves of T. Given $T \in B_n$, the weight sequence of T is the integer sequence $w_T = (w_T(1), \ldots, w_T(n))$ where $w_T(i)$ is the weight of the largest subtree of T whose last leaf is i [11, 14]. Rotation is a transformation $\rightarrow$ on B_n such that a subtree $\bigcirc T_1 \bigcirc T_2 T_3$ of a tree of B_n is replaced by the subtree $\bigcirc \bigcirc T_1 T_2 T_3$ where $T_1, T_2, T_3 \in B$. We have proved in [11] the following characterization:

Theorem 5 *Given T and $T' \in B_n$, we have $T \xrightarrow{*} T'$ iff $w_T(i) \leq w_{T'}(i)$ for all $i \in [1, n]$.*

$(B_n, \xrightarrow{*})$ is a semidistributive lattice for all n, called Tamari lattice, with $\mathbf{0}$ and $\mathbf{1}$ [1, 5, 10, 11]. Thus it is a consistent lattice [21] and therefore it has a matching [8]. We have $w_{\mathbf{0}} = (1, 1, \ldots, 1)$ and $w_{\mathbf{1}} = (1, 2, 3, \ldots, n)$. The weight sequence of the meet of T and T' is easy to compute: $w_{T \wedge T'}(i) = min(w_T(i), w_{T'}(i))$. See [13] for computing the join. The following characterization can be shown easily:

Theorem 6 *A n-integer sequence w is the weight sequence of a tree of B_n iff for all $i \in [1, n]$:*

(i) $1 \leq w_i \leq i$ *and*

(ii) if $j \in [i - w_i + 1, i]$ *then* $i - w_i \leq j - w_j$.

Theorem 7 *T is a join-irreducible element of B_n iff there exist i and $k \in [2, n]$ such that $w_T(i) = k$ with $k \leq i$ and $w_T(j) = 1$ for all $j \neq i$.*

Proof. All elements of a weight sequence of a tree T are equal to 1 except one iff there exists a unique occurrence ◯◯ of two consecutive internal nodes in the Polish notation of T. Thus T has a unique lower cover.

Theorem 8 *T is a meet-irreducible element of B_n iff there exist $k, l \in [1, n]$ such that* $w_T = (1, 2, \ldots, k, 1, 2, \ldots, l, k + l + 1, \ldots, n)$.

Proof. A tree T has a unique upper cover iff there exists a unique occurrence □◯ of a leaf followed by an internal node in the Polish notation of T. If k denotes the number of this leaf, then $w_T(i) = i$ for $1 \leq i \leq k$. This internal node is the root of a subtree T'. If $l = |T'_L|$, then $w_T(i) = i - k$ for $k + 1 \leq i \leq k + l$.

Remark. The number of join-irreductibles is equal to the number of the meet-irreductibles [1, 10], namely $\frac{n(n-1)}{2}$. Furthermore, B_n has a maximal chain of length $|J(L)|$ (see [10, 13]). The previous tool is now applied once again.

Theorem 9 *The map $\sigma : J(L) \cup \{\mathbf{0}\} \longrightarrow M(L) \cup \{\mathbf{1}\}$ defined by :*

$$\sigma(0) = h(1, \cdots, 1) = (1, 2, \cdots, n - 1, 1)$$

for $i \geq 2$*:*

$$\sigma((\underbrace{1, 1, \cdots, 1}_{i-1}, 2, 1, \cdots, 1)) = (1, 2, 3, \cdots, i - 2, 1, 2, \cdots, n - i + 2)$$

and if $3 \leq k \leq i$*:*

$$\sigma((\underbrace{1, 1, \cdots, 1}_{i-1}, k, 1, \cdots, 1)) = (1, 2, 3, \cdots, i - k + 1, 1, 2, \cdots, k - 2, i, i + 1, \cdots, n)$$

is a matching in L.

Proof. First we choose a chain C of maximal length ($|J(L)|$). The smallest element of C is $\mathbf{0}$. In order to obtain the successor of a tree T, we apply the rotation transformation on the rightmost occurrence of $\square\bigcirc$ in the Polish notation of T. For example in B_5, we obtain the maximal chain: $11111 \to 11112 \to 11113 \to 11123 \to 11124 \to 11134 \to 11234 \to 11235 \to 11245 \to 11345 \to 12345$. See also the bold path in Figure 2. The non-unit trees of this maximal chain can be characterized by a weight sequence of the form:

$$c_{k,l} = (\underbrace{1, \cdots, 1}_{k}, \underbrace{1, 2, \cdots, l}_{l}, l+2, l+3, \cdots, n-k+1) \tag{1}$$

with $k \geq 1$, $l \geq 1$ *and* $k + l \leq n$. Let denote $c_{0,n} = (1, 2, \cdots, n) = w_{\mathbf{1}}$.

As in the case of the previous distributive lattice, we construct a non-decreasing one-to-one map f between $J(L) \cup \{\mathbf{0}\}$ and the chain C and then a non-increasing one-to-one map g between $M(L) \cup \{\mathbf{1}\}$ and this chain C. For the first bijection s, we associate to an element j of $J(L) \cup \{\mathbf{0}\}$ the smallest element of the chain such that $j \leq s(j)$. This allows us to define f between $J(L) \cup \{\mathbf{0}\}$ and C by $f(\mathbf{0}) = \mathbf{w_0} = \mathbf{c_{n-1,1}}$, if $2 < k \leq i$ then $f((\underbrace{1, 1, \cdots, 1, 1}_{i-1}, k, 1, 1, \cdots, 1)) = (\underbrace{1, \cdots, 1}_{i-k+1}, \underbrace{1, 2, \cdots, k-2}_{k-2}, k, k+1, \cdots, n-i+k) = c_{i-k+1,k-2}$

and $f((\underbrace{1, 1, \cdots, 1}_{i-1}, 2, 1, \cdots, 1)) = (\underbrace{1, 1, \cdots, 1}_{i-1}, 2, 3, \cdots, n-i+2) = c_{i-2,n-i+2}$

Similarly, we associate to an element of $M(L) \cup \{\mathbf{1}\}$ the greatest element of the chain which is lower than it. The second non-increasing bijection g is therefore defined between $M(L) \cup \{\mathbf{1}\}$ and C by:

$g(\mathbf{1}) = \mathbf{1}$ and if $k \geq 1$, $l \geq 1$ $(k + l \leq n)$ then $g((1, 2, \cdots, k, 1, 2, \cdots, l, k+l+1, \cdots, n)) = (\underbrace{1, \cdots, 1}_{k}, \underbrace{1, 2, \cdots, l}_{l}, l+2, l+3, \cdots, n-k+1) = c_{k,l}$.

The inverse function is therefore a non-decreasing one-to-one map and is defined by : $g^{-1}(\mathbf{1}) = \mathbf{1}$ and $g^{-1}(c_{k,l}) = (\underbrace{1, 2, \cdots, k-1, k}_{k}, \underbrace{1, 2, \cdots, l}_{l}, k+l-1, \cdots, n)$. Thus $\sigma = g^{-1} \circ f$ constitutes a matching in L.

4 MATCHINGS IN LATTICES OF NONCROSSING PARTITIONS

A partition $B_1/B_2/\ldots/B_k$ of $\{1, 2, \ldots, n\}$ is called noncrossing if there do not exist four numbers $a < b < c < d$ such that $a, c \in B_i$ and $b, d \in B_j$ with $i \neq j$. For example 12579/34/6/8 is a noncrossing partition of $\{1, 2, \ldots, 9\}$ (ncp in short) while 13568/2479 is crossing. We denote NC_n the set of all ncp of $\{1, 2, \ldots, n\}$. We have $|NC_n| = c_n$. The refinement order $\leq$ is defined on NC_n in the following manner: two ncp p and p' satisfy $p \leq p'$ if every block of p is a subset of some block of p'. NC_n is a graded lattice under refinement with $\mathbf{0}$ and $\mathbf{1}$ [3, 7, 17, 18]. We have $\mathbf{0} = 1/2/3/\ldots/n$ and $\mathbf{1} = 123\ldots n$. The following characterizations can be proved easily:

Theorem 10 *p is a join-irreducible element of NC_n iff $p = B_1/B_2/\ldots/B_{n-1}$ with $|B_1| = 2$ and $|B_i| = 1$ for $2 \leq i \leq n-1$. p is a meet-irreducible element of NC_n iff $p = B_1/B_2$.*

Theorem 11 *The map $\sigma : J(L) \longrightarrow M(L)$ defined by :*

$\sigma(\{i, n\} \underset{j \notin \{i,n\}}{/} \{j\}) = \{i+1\}/\{j, j \neq i+1\}$ *for* $i \in [1, n-2]$

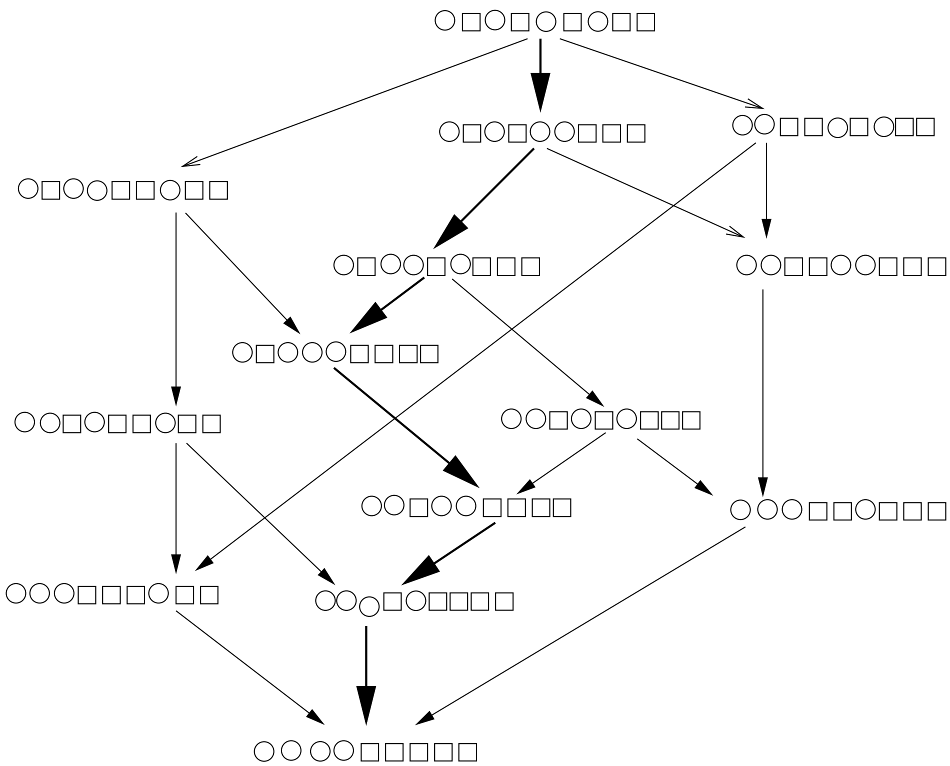

Figure 2: The Tamari lattice B_4

$\sigma(\{n-1,n\} \underset{j\notin\{n-1,n\}}{/} \{j\}) = \{1\}/\{j, j\neq 1\}$
$\sigma(\{i,j\} \underset{k\notin\{i,j\}}{/} \{k\}) = \{i, i+1, \cdots, j\}/\{k, k\notin [i,j]\}$ *for* $1 \leq i < j \leq n-1$
is a matching in NC_n*.*

Proof. We just need to verify the surjectivity of σ. Let $A = A_1/A_2$ be a ncp in $M(L)$. If there exists $i \in \{1,2\}$ such that $|A_i| = 1$, by definition of σ, A is in the range of σ. Let us suppose now that there does not exist $i \in \{1,2\}$ such that $|A_i| = 1$ and that n is in the block A_2. Let $i = \min\{k \in A_1\}$ and $j = \max\{k \in A_1\}$. Thus, we obtain $i < j < n$. Now, according to the definition of a ncp, there is no element c in A_2 such that $i < c < j < n$. This implies that all integers between i and j are in A_1. So, we can write $A_1 = \{i, i+1, \cdots, j\}$. Therefore, we have $\sigma(\{i,j\} \underset{k\notin\{i,j\}}{/} \{k\}) = \{i, i+1, \cdots, j\}/\{k, k \notin [i,j]\}$ and thus σ is a bijection.

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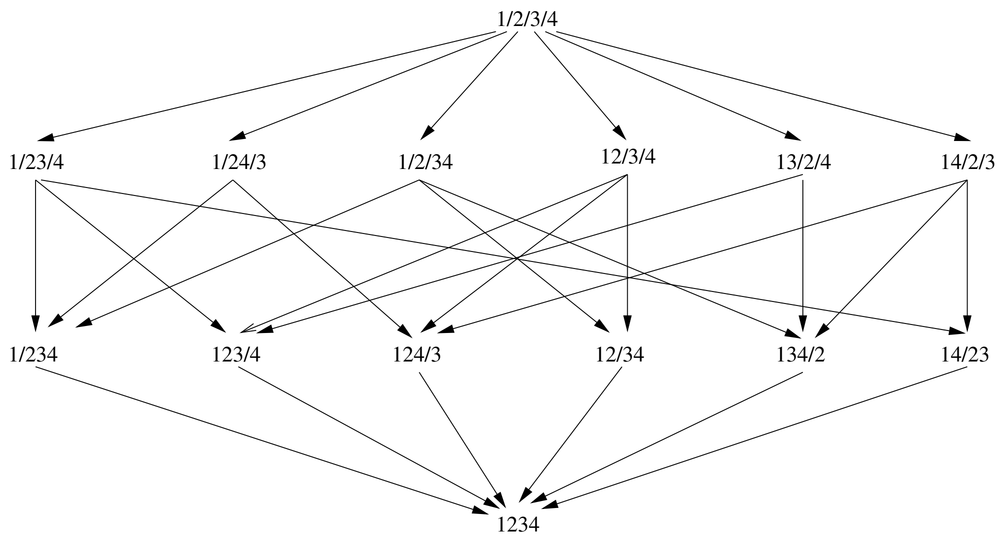

Figure 3: The lattice NC_4

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