Equivalence classes of permutations avoiding generalized patterns modulo left-to-right maxima

Jean-Luc Baril and Armen Petrossian

Laboratory LE2I – CNRS UMR 6306 – University of Burgundy – Dijon

The 13th International Permutation Patterns edition, London, UK, 15-19 June
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Definition 1 (Equivalence relation modulo left-to-right maxima)

\[ \sigma \sim_\ell \pi \text{ iff } L(\sigma) = L(\pi) \text{ and } \sigma_i = \pi_i \text{ for all } i \in L(\sigma) \]

\[ L(\sigma) = \{1, 3, 6, 7\} = L(\pi) \]

\[ S_{n}^{\sim_\ell} = \{ \text{classes of permutations of length } n \text{ modulo } \sim_\ell \} \]
Definition 2 (Generalized patterns)

A generalized pattern is a classical pattern where two entries can be separated by a dash, (e.g. \(ab-c\), \(a-bc\)).

\(\sigma\) avoids a generalized pattern \(\pi\) if \(\sigma\) does not contain any subsequence order isomorphic to \(\pi\), where two entries not separated by a dash in \(\pi\) correspond to two consecutive elements in \(\sigma\).
Introduction: notations and our precedent results

<table>
<thead>
<tr>
<th>Set</th>
<th>Sequence</th>
<th>Sloane</th>
<th>$a_n, 1 \leq n \leq 9$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Permutations</td>
<td>Catalan</td>
<td>A000108</td>
<td>1, 2, 5, 14, 42, 132, 429, 1430, 4862</td>
</tr>
<tr>
<td>Cycles</td>
<td>Catalan</td>
<td>A000108</td>
<td>1, 2, 5, 14, 42, 132, 429, 1430</td>
</tr>
<tr>
<td>Involutions</td>
<td>Motzkin</td>
<td>A001006</td>
<td>1, 2, 4, 9, 21, 51, 127, 323, 835</td>
</tr>
<tr>
<td>Derangements</td>
<td>Fine</td>
<td>A000957</td>
<td>0, 1, 2, 6, 18, 57, 186, 622, 2120</td>
</tr>
</tbody>
</table>

**Table:** Number of equivalence classes for classical subsets of permutations.

<table>
<thead>
<tr>
<th>Pattern</th>
<th>Sequence</th>
<th>Sloane</th>
<th>$a_n, 1 \leq n \leq 9$</th>
</tr>
</thead>
<tbody>
<tr>
<td>{1-2-3}</td>
<td>Central polygon</td>
<td>A000124</td>
<td>1, 2, 4, 7, 11, 16, 22, 29, 37</td>
</tr>
<tr>
<td>{3-1-2}, {3-2-1}</td>
<td>Catalan</td>
<td>A000108</td>
<td>1, 2, 5, 14, 42, 132, 429, 1430, 4862</td>
</tr>
<tr>
<td>{1-3-2}, {2-1-3}, {2-3-1}</td>
<td>Power of 2</td>
<td>A000079</td>
<td>1, 2, 4, 8, 16, 32, 64, 128, 256</td>
</tr>
</tbody>
</table>

**Table:** Number of equivalence classes for permutations avoiding one classical pattern.
Results relative to permutations avoiding generalized patterns of length 3 modulo left-to-right maxima

<table>
<thead>
<tr>
<th>Pattern</th>
<th>Sequence</th>
<th>Sloane</th>
<th>$a_n, 1 \leq n \leq 9$</th>
</tr>
</thead>
<tbody>
<tr>
<td>{312}, {321}, {3-12},</td>
<td>Catalan</td>
<td>A000108</td>
<td>1, 2, 5, 14, 42, 132, 429, 1430, 4862</td>
</tr>
<tr>
<td>{31-2}, {3-21}, {32-1}</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>{1-23}</td>
<td>Generalized Catalan</td>
<td>A004148</td>
<td>1, 2, 4, 8, 17, 37, 82, 185, 423</td>
</tr>
<tr>
<td>{123}, {23-1}</td>
<td>Motzkin</td>
<td>A000124</td>
<td>1, 2, 4, 9, 21, 51, 127, 323, 835</td>
</tr>
<tr>
<td>{12-3}</td>
<td>Central polygon</td>
<td>A000124</td>
<td>1, 2, 4, 7, 11, 16, 22, 29, 37</td>
</tr>
<tr>
<td>{213}</td>
<td>Number of Dyck paths with no UDDU</td>
<td>A135307</td>
<td>1, 2, 4, 9, 23, 63, 178, 514, 1515</td>
</tr>
<tr>
<td>{1-32}</td>
<td>Number of Dyck paths with no UUDU</td>
<td>A105633</td>
<td>1, 2, 4, 9, 22, 57, 154, 429, 1223</td>
</tr>
<tr>
<td>{13-2}, {2-13}, {21-3},</td>
<td>Power of 2</td>
<td>A000079</td>
<td>1, 2, 4, 8, 16, 32, 64, 128, 256</td>
</tr>
<tr>
<td>{231}</td>
<td>New</td>
<td></td>
<td>1, 2, 4, 10, 28, 84, 265, 864, 2888</td>
</tr>
<tr>
<td>{132}</td>
<td>New</td>
<td></td>
<td>1, 2, 4, 10, 26, 74, 217, 662, 2059</td>
</tr>
</tbody>
</table>

**Table:** Number of equivalence classes for permutations avoiding one generalized pattern.
Two results using ECO method

Theorem 1 (Enumeration of $S(123)^{\sim\ell}$)

The set $S(123)$ modulo left-to-right maxima is enumerated by the Motzkin sequence (A000124).

Theorem 2 (Enumeration of $S(1-23)^{\sim\ell}$)

The set $S(1-23)$ modulo left-to-right maxima is enumerated by the Generalized Catalan sequence (A004148).

In order to calculate the number of classes for permutations avoiding 123 or 1-23 we will

- exhibit one-to-one correspondence between the sets of equivalence classes and some subsets of permutations (the set of representatives)
- exhibit a recursive relation via ECO method which defines the set of classes
- exhibit the generating function which defines the wished sequence
The ECO method applied to a set of representatives of $S(123)\sim\ell$
The ECO method applied to a set of representatives of $S(123)^{\sim_l}$
The ECO method applied to a set of representatives of $S(123)^{\sim \ell}$
The ECO method applied to a set of representatives of \( S(123) \sim^\ell \)
The ECO method applied to a set of representatives of $S(123)^{\sim\ell}$
The ECO method applied to a set of representatives of $S(123)^{\sim_{\ell}}$

provides the succession rule of the Motzkin sequence

$$
\begin{cases}
(1) \\
(k) \sim (1) \cdots (k-1)(k+1)
\end{cases}
$$
Definition 3 (Set of representatives)

Let \( \mathcal{A} \) be the set of permutations
\[
\sigma_1 B_1 \sigma_2 \cdots B_{s-1} \sigma_s B_s \quad \text{(only } B_1 \text{ can be empty)}
\]
Definition 3 (Set of representatives)

Let $A$ be the set of permutations $\sigma_1 B_1 \sigma_2 \cdots B_{s-1} \sigma_s B_s$ (only $B_1$ can be empty)

- $\sigma_1 \sigma_2 \cdots \sigma_s$ list of left-to-right maxima

We insert $[(s-1)(s-2)\cdots 1]$ before $\sigma_2 \sigma_3 \cdots$ in this order.

From left-to-right all other elements are chosen as large as possible.
Definition 3 (Set of representatives)

Let $\mathcal{A}$ be the set of permutations $\sigma_1 B_1 \sigma_2 \cdots B_{s-1} \sigma_s B_s$ (only $B_1$ can be empty)

- $\sigma_1 \sigma_2 \cdots \sigma_s$ list of left-to-right maxima
- $[B_1] B_2 \cdots B_s$ decreasing sub-sequences
Definition 3 (Set of representatives)

Let $\mathcal{A}$ be the set of permutations $\sigma_1 B_1 \sigma_2 \cdots B_{s-1} \sigma_s B_s$ (only $B_1$ can be empty)

- $\sigma_1 \sigma_2 \cdots \sigma_s$ list of left-to-right maxima
- $[B_1]B_2 \cdots B_s$ decreasing sub-sequences

1. We insert before $\sigma_s$
Definition 3 (Set of representatives)

Let $\mathcal{A}$ be the set of permutations

$\sigma_1 B_1 \sigma_2 \cdots B_{s-1} \sigma_i B_s$ (only $B_1$ can be empty)

- $\sigma_1 \sigma_2 \cdots \sigma_i$ list of left-to-right maxima
- $[B_1] B_2 \cdots B_s$ decreasing sub-sequences

1. We insert $[(s - 1)](s - 2), \cdots, 1$ before $[\sigma_i] \sigma_{i+1} \cdots \sigma_i$ in this order.
Definition 3 (Set of representatives)

Let $\mathcal{A}$ be the set of permutations $\sigma_1 B_1 \sigma_2 \cdots B_{s-1} \sigma_s B_s$ (only $B_1$ can be empty)

- $\sigma_1 \sigma_2 \cdots \sigma_s$ list of left-to-right maxima
- $[B_1] B_2 \cdots B_s$ decreasing sub-sequences

1. We insert $[(s-1)](s-2), \cdots, 1$ before $[\sigma_2] \sigma_3 \cdots \sigma_s$ in this order
2. From left-to-right all other elements are chosen as large as possible.
Definition 3 (Set of representatives)

Let $\mathcal{A}$ be the set of permutations

$\sigma_1 B_1 \sigma_{i_2} \cdots B_{s-1} \sigma_{i_s} B_s$ (only $B_1$ can be empty)

- $\sigma_1 \sigma_{i_2} \cdots \sigma_{i_s}$ list of left-to-right maxima
- $[B_1] B_2 \cdots B_s$ decreasing sub-sequences

1. We insert $[(s-1)](s-2), \cdots, 1$ before $[\sigma_{i_2}] \sigma_{i_3} \cdots \sigma_{i_s}$ in this order
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**Definition 3 (Set of representatives)**

Let $\mathcal{A}$ be the set of permutations

$$\sigma_1 B_1 \sigma_2 \cdots B_{s-1} \sigma_s B_s$$

(only $B_1$ can be empty)

- $\sigma_1 \sigma_2 \cdots \sigma_s$ list of left-to-right maxima
- $[B_1]B_2 \cdots B_s$ decreasing sub-sequences

1. We insert $[(s-1)](s-2), \cdots, 1$ before $[\sigma_2]\sigma_3 \cdots \sigma_s$ in this order
2. From left-to-right all other elements are chosen as large as possible.

**Lemma 1 (Bijection between set of representatives and $S(1-23) \sim \ell$)**

There is a bijection between $\mathcal{A}$ and $S(1-23)$ modulo left-to-right maxima.
$S(1-23)^{\sim \ell}$: Succession rule

$\sigma = \sigma_1 B_1 \sigma_2 \cdots B_{s-1} \sigma_{i_{s-1}} B_s$ (a representative)
\( S(1-23)^{\sim_\ell} : \) Succession rule

\[ \sigma = \sigma_1 B_1 \sigma_2 \cdots B_{s-1} \sigma_{s-1} B_s \text{ (a representative)} \]

If \( \sigma_1 > \sigma_2 \), \( \sigma \) admits \([s, \sigma_1 + 1]\) as a set of active sites.
$S(1-23)^{\sim \ell} :$ Succession rule

$\sigma = \sigma_1 B_1 \sigma_2 \cdots B_{s-1} \sigma_{s-1} B_s$ (a representative)

If $\sigma_1 > \sigma_2$, $\sigma$ admits $[s, \sigma_1 + 1]$ as a set of active sites.
If $\sigma_1 < \sigma_2$, $\sigma$ admits $\sigma_1 + 1$ as only one active site and $[s + 1, \sigma_1 + 2]$
\( S(1-23)^{\sim \ell} : \) Succession rule

\[
\sigma = \sigma_1 B_1 \sigma_2 \cdots B_{s-1} \sigma_{i-1} B_s \quad \text{(a representative)}
\]

If \( \sigma_1 > \sigma_2 \), \( \sigma \) admits \([s, \sigma_1 + 1]\) as a set of active sites.

If \( \sigma_1 < \sigma_2 \), \( \sigma \) admits \( \sigma_1 + 1 \) as only one active site and \([s + 1, \sigma_1 + 2]\)

is the set of active sites of its unique child \( \ell \pi_{\ell} \), with \( \ell = \sigma_1 + 1 \).
$S(1-23)^{\sim\ell}$: Succession rule

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When $\pi$ becomes $\pi_\ell$ if the elements of $\pi$ larger than $\ell$ are replaced by their successors.
\( S(1-23)^\sim_\ell \) : Succession rule

\[ \sigma = \sigma_1 B_1 \sigma_{i_2} \cdots B_{s-1} \sigma_{i_{s-1}} B_s \text{ (a representative)} \]

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When \( \pi \) becomes \( \pi_\ell \) if the elements of \( \pi \) larger than \( \ell \) are replaced by their successors.
$S(1-23) \sim \ell$ : Succession rule

\[ \sigma = \sigma_1 B_1 \sigma_2 \cdots B_{s-1} \sigma_{s-1} B_s \text{ (a representative)} \]

If $\sigma_1 > \sigma_2$, $\sigma$ admits $[s, \sigma_1 + 1]$ as a set of active sites.

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is the set of active sites of its unique child \( \ell \pi_{\ell} \), with \( \ell = \sigma_1 + 1 \).

When \( \pi \) becomes \( \pi_{\ell} \) if the elements of \( \pi \) larger than \( \ell \) are replaced by their successors.
$S(1-23)^{\sim\ell}$ : Succession rule

\[ \sigma = \sigma_1 B_{\ell_1} \sigma_2 \cdots B_{\ell_{s-1}} \sigma_{i_{s-1}} B_s \text{ (a representative)} \]

If $\sigma_1 > \sigma_2$, $\sigma$ admits $[s, \sigma_1 + 1]$ as a set of active sites.
If $\sigma_1 < \sigma_2$, $\sigma$ admits $\sigma_1 + 1$ as only one active site and $[s + 1, \sigma_1 + 2]$ is the set of active sites of its unique child $\ell \pi_{\ell}$, with $\ell = \sigma_1 + 1$.
When $\pi$ becomes $\pi_{\ell}$ if the elements of $\pi$ larger than $\ell$ are replaced by their successors.
$S(1-23)^{\sim \ell} :$ Succession rule

$$\sigma = \sigma_1 B_1 \sigma_{i_2} \cdots B_{s-1} \sigma_{i_{s-1}} B_s \text{ (a representative)}$$

If $\sigma_1 > \sigma_2$, $\sigma$ admits $[s, \sigma_1 + 1]$ as a set of active sites.

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When $\pi$ becomes $\pi_{\ell}$ if the elements of $\pi$ larger than $\ell$ are replaced by their successors.
We deduce from this the following Eco rule for Generalized Catalan succession rule:

\[
\begin{align*}
(2) \\
(k) &\leadsto (1)_2 \cdots (1)_k(k + 1) \\
(1)_k &\leadsto (k)
\end{align*}
\]

Lemma 2

There a bijection between the set of vertices of level \( n \) and the set of vertices labelled by \((1)_2\) of level \( n + 2 \) of the tree induced by the above succession rule.

We will enumerate the sets of vertices labelled by \((1)_2\) of level \( n + 2 \) for all positive integers \( n \).
We denote $a_{n,n+2-\ell}$, $\ell \geq 2$, the number of vertices $(1)_{\ell}$ of level $n$. 

$$S(1-23)^{\sim_{\ell}}$$
We denote $a_{n,n+2-\ell}$, $\ell \geq 2$, the number of vertices $(1)_\ell$ of level $n$

- $a_{n,n+2-\ell} = a_{n-1,n+1-(\ell-1)} + \sum_{i=\ell}^{n-1} a_{n-2,n-i}$ \quad \forall \ell \geq 3
We denote $a_{n,n+2-\ell}$, $\ell \geq 2$, the number of vertices $(1)_{\ell}$ of level $n$

- $a_{n,n+2-\ell} = a_{n-1,n+1-(\ell-1)} + \sum_{i=\ell}^{n-1} a_{n-2,n-i} \quad \forall \ell \geq 3$

- If the grandfather of $(1)_{\ell}$ is $(i)$ with $i \geq \ell - 1$

  there are two paths $(i) \leadsto (i+1) \leadsto (1)_{\ell}$, $(i) \leadsto (1)_{\ell-1}$
We denote $a_{n,n+2-\ell}$, $\ell \geq 2$, the number of vertices $(1)_\ell$ of level $n$

- $a_{n,n+2-\ell} = a_{n-1,n+1-(\ell-1)} + \sum_{i=\ell}^{n-1} a_{n-2,n-i} \quad \forall \ell \geq 3$

- If the grandfather of $(1)_\ell$ is $(1)_i; i \in [\ell, n-1]$ there is
  a unique path $(1)_i \leadsto (i) \leadsto (1)_\ell$
We denote $a_{n,n+2-\ell}$, $\ell \geq 2$, the number of vertices $(1)_\ell$ of level $n$

- $a_{n,k} = a_{n-1,k} + \sum_{l=1}^{k-2} a_{n-2,l}$, $k \in [n-1]$ (with $k = n + 2 - \ell$)
- $a_{n,n} = a_{n-1,n-1} + \sum_{i=1}^{n-2} a_{n-2,i}$

- If the grandfather of $(1)_2$ is $(k)$ with $k \geq 2$
- there are two paths $(k) \leadsto (k+1) \leadsto (1)_2$, $(k) \leadsto (1)_2$
We denote $a_{n,n+2-\ell}$, $\ell \geq 2$, the number of vertices $(1)_{\ell}$ of level $n$

- $a_{n,k} = a_{n-1,k} + \sum_{\ell=1}^{n-2} a_{n-2,\ell}$, $k \in [n-1]$ (with $k = n+2-\ell$)
- $a_{n,n} = a_{n-1,n-1} + \sum_{\ell=1}^{n-2} a_{n-2,\ell}$

- If the grandfather of $(1)_2$ is $(1)_k$ with $k \geq 2$
- there is a unique path $(1)_k \rightsquigarrow (k) \rightsquigarrow (1)_2$
We denote \( a_{n,n+2-\ell} \), \( \ell \geq 2 \), the number of vertices \((1)_{\ell}\) of level \(n\)

- \( a_{n,k} = a_{n-1,k} + \sum_{l=1}^{k-2} a_{n-2,l} \), \( k \in [n-1] \) (with \( k = n + 2 - \ell \))

- \( a_{n,n} = a_{n-1,n-1} + \sum_{i=1}^{n-2} a_{n-2,i} \)

Let \( b_{n,k} = a_{n,k} \) for \( k \in [n-1] \) and \( b_{n,k} = 0 \) otherwise \((n \geq 0, k \geq 0)\)
We denote $a_{n,n+2-\ell}$, $\ell \geq 2$, the number of vertices $(1)_\ell$ of level $n$

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Let $b_{n,k} = a_{n,k}$ for $k \in [n-1]$ and $b_{n,k} = 0$ otherwise ($n \geq 0$, $k \geq 0$)

Let $c_n = a_{n,n}$ for ($n \geq 1$),
We denote $a_{n,n+2-\ell}$, $\ell \geq 2$, the number of vertices $(1)_{\ell}$ of level $n$

- $a_{n,k} = a_{n-1,k} + \sum_{i=1}^{k-2} a_{n-2,i}$, $k \in [n-1]$ (with $k = n+2-\ell$)
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Let $b_{n,k} = a_{n,k}$ for $k \in [n-1]$ and $b_{n,k} = 0$ otherwise ($n \geq 0$, $k \geq 0$)

Let $c_n = a_{n,n}$ for ($n \geq 1$),

\[
\begin{cases}
  c_1 = 1, & b_{2,1} = 1 \\
  b_{n,k} = b_{n-1,k}(1 - \delta_{n-1,k}) + c_{n-1,n-1}\delta_{n-1,k} + \sum_{i=1}^{k-2} b_{n-2,i} - \sum_{1 \leq i \leq n-3 \leq k-3} b_{n-2,i} \\
  c_n = c_{n-1} + c_{n-2} + \sum_{i=1}^{n-3} b_{n-2,i} & (n \geq 2, \ k \geq 1)
\end{cases}
\]
We denote $a_{n,n+2-\ell}$, $\ell \geq 2$, the number of vertices $(1)_\ell$ of level $n$

- $a_{n,k} = a_{n-1,k} + \sum_{i=1}^{k-2} a_{n-2,i}$, $k \in [n-1]$ (with $k = n + 2 - \ell$)
- $a_{n,n} = a_{n-1,n-1} + \sum_{i=1}^{n-2} a_{n-2,i}$

Let $b_{n,k} = a_{n,k}$ for $k \in [n-1]$ and $b_{n,k} = 0$ otherwise ($n \geq 0$, $k \geq 0$)

Let $c_n = a_{n,n}$ for ($n \geq 1$),

\[
\begin{align*}
   c_1 &= 1, \quad b_{2,1} = 1 \\
   b_{n,k} &= b_{n-1,k}(1 - \delta_{n-1,k}) + c_{n-1,n-1}\delta_{n-1,k} + \sum_{i=1}^{k-2} b_{n-2,i} - \sum_{1 \leq i \leq n-3 \leq k-3} b_{n-2,i} \\
   c_n &= c_{n-1} + c_{n-2} + \sum_{i=1}^{n-3} b_{n-2,i} \quad (n \geq 2, \quad k \geq 1)
\end{align*}
\]

\[
\begin{array}{cccccccc}
1 \\
1 & 1 \\
1 & 1 & 2 \\
1 & 1 & 3 & 4 \\
1 & 1 & 4 & 6 & 8 \\
1 & 1 & 5 & 8 & 13 & 17 \\
1 & 1 & 6 & 10 & 19 & 29 & 37 \\
1 & 1 & 7 & 12 & 26 & 44 & 65 & 82
\end{array}
\]
We denote $a_{n,n+2-\ell}$, $\ell \geq 2$, the number of vertices $(1)_\ell$ of level $n$

- $a_{n,k} = a_{n-1,k} + \sum_{i=1}^{k-2} a_{n-2,i}$, $k \in [n-1]$ (with $k = n+2-\ell$)
- $a_{n,n} = a_{n-1,n-1} + \sum_{i=1}^{n-2} a_{n-2,i}$

Let $b_{n,k} = a_{n,k}$ for $k \in [n-1]$ and $b_{n,k} = 0$ otherwise ($n \geq 0$, $k \geq 0$)

Let $c_n = a_{n,n}$ for $(n \geq 1)$,

$$\begin{cases}
  c_1 = 1, \quad b_{2,1} = 1 \\
  b_{n,k} = b_{n-1,k}(1-\delta_{n-1,k}) + c_{n-1,n-1}\delta_{n-1,k} + \sum_{i=1}^{k-2} b_{n-2,i} - \sum_{1\leq i\leq n-3\leq k-3} b_{n-2,i} \\
  c_n = c_{n-1} + c_{n-2} + \sum_{i=1}^{n-3} b_{n-2,i} \quad (n \geq 2, \ k \geq 1)
\end{cases}$$

1 1 1 2 1 1 3 4 1 1 4 6 8 1 1 5 8 13 17 1 1 6 10 19 29 37 1 1 7 12 26 44 65 82
We denote $a_{n,n+2-\ell}$, $\ell \geq 2$, the number of vertices $(1)_\ell$ of level $n$

- $a_{n,k} = a_{n-1,k} + \sum_{i=1}^{n-2} a_{n-2,i}$, $k \in [n-1]$ (with $k = n+2-\ell$)
- $a_{n,n} = a_{n-1,n-1} + \sum_{i=1}^{n-2} a_{n-2,i}$

Let $b_{n,k} = a_{n,k}$ for $k \in [n-1]$ and $b_{n,k} = 0$ otherwise ($n \geq 0$, $k \geq 0$)

Let $c_n = a_{n,n}$ for $(n \geq 1)$,

\[
\begin{cases}
    c_1 = 1, \quad b_{2,1} = 1 \\
    b_{n,k} = b_{n-1,k}(1 - \delta_{n-1,k}) + c_{n-1}\delta_{n-1,k} + \sum_{i=1}^{n-2} b_{n-2,i} - \sum_{1 \leq i \leq n-3 \leq k-3} b_{n-2,i} \\
    c_n = c_{n-1} + c_{n-2} + \sum_{i=1}^{n-3} b_{n-2,i} \quad (n \geq 2, \ k \geq 1)
\end{cases}
\]

\[
\begin{array}{cccccccccccc}
1 & 1 &  &  &  &  &  &  &  &  &  &  \\
1 & 1 & 2 &  &  &  &  &  &  &  &  &  \\
1 & 1 & 3 & 4 &  &  &  &  &  &  &  &  \\
1 & 1 & 4 & 6 & 8 &  &  &  &  &  &  &  \\
1 & 1 & 5 & 8 & 13 & 17 &  &  &  &  &  &  \\
1 & 1 & 6 & 10 & 19 & 29 & 37 &  &  &  &  &  \\
1 & 1 & 7 & 12 & 26 & 44 & 65 & 82 &  &  &  &  \\
\end{array}
\]
We denote $a_{n,n+2-\ell}$, $\ell \geq 2$, the number of vertices $(1)_\ell$ of level $n$

- $a_{n,k} = a_{n-1,k} + \sum_{l=1}^{k-2} a_{n-2,l}$, $k \in [n-1]$ (with $k = n + 2 - \ell$)
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Let $c_n = a_{n,n}$ for ($n \geq 1$),

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\begin{align*}
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    b_{n,k} &= b_{n-1,k}(1 - \delta_{n-1,k}) + c_{n-1,n-1}\delta_{n-1,k} + \sum_{i=1}^{k-2} b_{n-2,i} - \sum_{1 \leq i \leq n-3 \leq k-3} b_{n-2,i} \\
    c_n &= c_{n-1} + c_{n-2} + \sum_{i=1}^{n-3} b_{n-2,i} \quad (n \geq 2)
\end{align*}
\]

\[
\begin{array}{ccccccccc}
1 \\
1 & 1 \\
1 & 1 & 2 \\
1 & 1 & 3 & 4 \\
1 & 1 & 4 & 6 & 8 \\
1 & 1 & 5 & 8 & 13 & 17 \\
1 & 1 & 6 & 10 & 19 & 29 & 37 \\
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  c_n = c_{n-1} + c_{n-2} + \sum_{i=1}^{n-3} b_{n-2,i} \quad (n \geq 2, \quad k \geq 1)
\end{cases}
$$

\[
\begin{array}{cccccccc}
1 \\
1 & 1 \\
1 & 1 & 2 \\
1 & 1 & 3 & 4 \\
1 + 1 + 4 & 6 & 8 \\
1 & 1 & 5 & 8 & +13 & 17 \\
1 & 1 & 6 & 10 & 19 & 29 & 37 \\
1 & 1 & 7 & 12 & 26 & 44 & 65 & 82
\end{array}
\]
We denote $a_{n,n+2-\ell}$, $\ell \geq 2$, the number of vertices $(1)_\ell$ of level $n$

- $a_{n,k} = a_{n-1,k} + \sum_{i=1}^{k-2} a_{n-2,i}$, $k \in [n-1]$ (with $k = n+2-\ell$)
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  c_n = c_{n-1} + c_{n-2} + \sum_{i=1}^{n-3} b_{n-2,i} & (n \geq 2, \ k \geq 1)
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\[
\begin{array}{ccccccc}
  1 \\
  1 & 1 \\
  1 & 1 & 2 \\
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We denote $a_{n,n+2-\ell}$, $\ell \geq 2$, the number of vertices $(1)_{\ell}$ of level $n$

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\begin{array}{ccccccc}
    1 \\
    1 & 1 \\
    1 & 1 & 2 \\
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c_n &= c_{n-1} + c_{n-2} + \sum_{i=1}^{n-3} b_{n-2,i} \quad (n \geq 2, \ k \geq 1)
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\]

We deduce the following functional equation:

\[
\left\{
\begin{align*}
B(x, y) &= x(B(x, y) + C(xy)) + \frac{x^2y^2}{1-y}(B(x, y) - B(xy, 1)) \\
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We deduce the following functional equation:

- $B(x, y) = x(B(x, y) + C(xy)) + \frac{x^2y^2}{1-y}(B(x, y) - B(xy, 1))$
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    B(x,y) &= \sum_{n,k \geq 0} b_{n,k} x^n y^k, \ C(z) = \sum_{n \geq 1} c_n z^{n-1}
\end{aligned}
\]
We denote $a_{n,n+2−ℓ}$, $ℓ ≥ 2$, the number of vertices $(1)_ℓ$ of level $n$

- $a_{n,k} = a_{n−1,k} + ∑_{i=1}^{k−2} a_{n−2,i}$, $k ∈ [n − 1]$ (with $k = n + 2 − ℓ$)
- $a_{n,n} = a_{n−1,n−1} + ∑_{i=1}^{n−2} a_{n−2,i}$

Let $b_{n,k} = a_{n,k}$ for $k ∈ [n − 1]$ and $b_{n,k} = 0$ otherwise ($n ≥ 0$, $k ≥ 0$)

Let $c_n = a_{n,n}$ for ($n ≥ 1$),

$$
\begin{align*}
    c_1 & = 1, \quad b_{2,1} = 1 \\
    b_{n,k} & = b_{n−1,k}(1 − δ_{n−1,k}) + c_{n−1,n−1}δ_{n−1,k} + ∑_{i=1}^{k−2} b_{n−2,i} − ∑_{1≤i≤n−3≤k−3} b_{n−2,i} \\
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\end{align*}
$$

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$$
\begin{align*}
    B(x, y) & = x(B(x, y) + C(xy)) + \frac{x^2y^2}{1−y}(B(x, y) − B(xy, 1)) \\
    C(xy) & = 1 + xyC(xy) + x^2y^2(C(xy) + B(xy, 1)) \\
    B(x, y) & = ∑_{n,k≥0} b_{n,k}x^ny^k, \quad C(z) = ∑_{n≥1} c_nz^{n−1}
\end{align*}
$$
We deduce from this the following generating functions:

\[ B(x, y) + C(xy) = -\frac{1 - x - xy + x^2y - x^2y^2 - x^3y^2 - (1-x)\sqrt{1 - 2xy - x^2y^2 - 2x^3y^3 + x^4y^4}}{2x^3y^2(1 - x - y + xy - x^2y^2)} \]

\[ C(x) = \frac{1 + x - x^2 + \sqrt{1 - 2x - x^2 - 2x^3 + x^4}}{1 - x - 2x^2 - x^3 + x^4 + (1-x^2)\sqrt{1 - 2x - x^2 - 2x^3 + x^4}} \]

Coefficients of the taylor expansion of the last generating function equal to \( c_n \), which enumerates the vertices \((1)_2\) at level \( n \) for all positive integers \( n \), form the Generalized Catalan sequence.
Thank you for your attention!