

Knight's paths towards Catalan numbers

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Abstract

We provide enumerating results for partial knight's paths of a given size. We prove algebraically that zigzag knight's paths of a given size ending on the x -axis are enumerated by the generalized Catalan numbers, and we give a constructive bijection with peakless Motzkin paths of a given length. After enumerating partial knight's paths of a given length, we prove that zigzag knight's paths of a given length ending on the x -axis are counted by the Catalan numbers. Finally, we give a constructive bijection with Dyck paths of a given length.

Keywords: Catalan number, zigzag knight's path, generating function.

MSC: 05A15, 05A19.

1 Introduction

For several decades lattice paths have been widely studied in the literature. Their links with many problems of various domains as computer science, biology and physics [23], gives them an important place in combinatorics. For instance, they have very tight connections with RNA structures, pattern avoiding permutations, directed animals, and so on [7, 14, 23]. In combinatorics, a classical problem consists in enumerating some special kinds of paths with respect to the length and other statistics (see for instance [3, 5, 6, 11, 13, 17, 18, 20, 21]). Certainly, Dyck paths are the most known. These are lattice paths in \mathbb{N}^2 starting at the origin $(0, 0)$, ending on the x -axis, consisting of steps lying in $\{(1, 1), (1, -1)\}$. They are

enumerated (with respect to the number of steps) by the famous Catalan numbers (see [A000108](#) in Sloane's On-line Encyclopedia of Integer Sequences [22]).

In this work we deal with several classes of knight's paths (see [16]), i.e., lattice paths in \mathbb{N}^2 starting at $(0,0)$, and consisting of right-moves of a knight in the game of chess (a right-move of a knight is a jump from left to right from one corner of any two-by-three rectangle to the opposite corner on a chessboard). Figure 1 shows the four possible right-moves of a knight. It is worth noticing that knight's paths always start at the origin and go to the right at each step. Therefore, this work is not related to the well known mathematical problem about the walk of a knight on a chessboard, namely the knight's tour problem [12], which consists in finding a sequence of moves on a chessboard such that the knight visits every square exactly once.

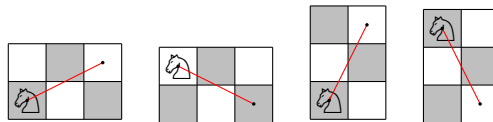


Figure 1: The four possible right-moves of a knight on a chessboard.

More formally, a *knight's path* is defined by a starting point $(0,0)$, it consists of steps lying in $S = \{(1,2), (1,-2), (2,1), (2,-1)\}$, and it never goes below the x -axis. The *length* of a path is the number of its steps, and the *size* of a path is the abscissa of its last point. We denote by ϵ the empty path, i.e., the path of length zero (or equivalently of size 0). The *height* of a point (x,y) of a path is the ordinate y . We refer to Figure 2 for two examples of knight's paths where the heights of the last points are 0 and 2, respectively. Defining $N = (1,2)$, $\bar{N} = (1,-2)$, $E = (2,1)$, and $\bar{E} = (2,-1)$, these two paths can be written as $N\bar{E}N\bar{N}E\bar{E}N\bar{E}N\bar{E}N\bar{E}N\bar{E}N\bar{E}$ and $N\bar{E}N\bar{N}E\bar{E}N\bar{E}N\bar{E}N\bar{E}$. Knight's paths of a given size and ending on the x -axis have been enumerated in [16]. On the other hand, there are enumerative results for knight's paths of a given length which are in one-to-one correspondence with basketball walks in the first quarter plane (see for instance [1, 4, 8, 9]). We will recall some of these results at the beginning of Sections 2 and 3.

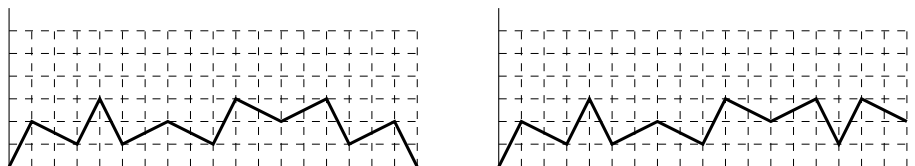


Figure 2: The left drawing shows a knight's path of size 18 which also is a knight's path of length 12. The right drawing shows a knight's path of size 18 (or a knight's path of length 12) ending at height 2.

In this paper, we will focus on a special kind of knight's paths that are enumerated by Catalan and generalized Catalan numbers (see [A000108](#) and [A004148](#) in [22]). More

precisely, we will consider knight's paths with the additional property that two consecutive moves (also called steps) cannot be in the same direction, i.e., two consecutive steps (a, b) and (c, d) lying in S must satisfy $b \cdot d < 0$. A knight's path with this property will be called a *zigzag knight's path*, or *zigzag path* for short. Notice that the two paths on Figure 2 are zigzag paths.

Outline of the paper. In Section 2, we provide enumerating results for partial knight's paths (prefixes of knight's paths) of a given size. We make a similar study for partial zigzag paths of a given size. We prove that zigzag paths of a given size ending on the x -axis are enumerated by the generalized Catalan numbers, and the total number of partial zigzag paths of a given size are enumerated as the symmetric secondary structures of RNA molecules with a given number of nucleotides. Next, we give a constructive one-to-one correspondence between zigzag paths of size n and peakless Motzkin paths of length $n + 1$.

In Section 3, we provide enumerating results for partial zigzag knight's paths of a given length. We prove that zigzag paths of a given length ending on the x -axis are enumerated by the Catalan numbers, and the total number of partial zigzag paths of a given length are enumerated by the central binomial coefficient. Finally, we give a constructive one-to-one correspondence between zigzag paths of length n and Dyck paths of semilength $n + 1$.

2 Knight's paths of a given size

In [16], Labelle and Yeh provide generating function for the number of knight's paths of size n ending on the x -axis:

$$A(z) = \frac{1 + 2z + \sqrt{1 - 4z + 4z^2 - 4z^4} - \sqrt{2}\sqrt{1 - 4z^2 - 2z^4 + (2z + 1)\sqrt{1 - 4z + 4z^2 - 4z^4}}}{4z^2}.$$

The first terms of the series expansion are

$$1 + z^2 + 3z^4 + 2z^5 + 12z^6 + 14z^7 + 54z^8 + 86z^9 + 274z^{10} + 528z^{11},$$

and the sequence of coefficients corresponds to [A005220](#) in [22]. The generating function $A(z)$ satisfies the functional equation

$$z^4 A(z)^4 - (2z^3 + z^2)A(z)^3 + (z^4 + 2z^2 + 2z)A(z)^2 - (2z + 1)A(z) + 1 = 0.$$

2.1 Partial knight's paths

In this part, we count partial knight's paths of a given size, i.e., prefixes of knight paths ending at a given abscissa. For $k \geq 0$, we consider the generating function $f_k = f_k(z)$ (resp. $g_k = g_k(z)$), where the coefficient of z^n in the series expansion is the number of partial knight's paths of size n ending at height k with an up-step N or E , (resp. with a down-step

\bar{N} or \bar{E}). So, we easily obtain the following equations:

$$\begin{aligned} f_0 &= 1, \text{ and } f_1 = z^2(f_0 + g_0), \\ f_k &= z^2(f_{k-1} + g_{k-1}) + z(f_{k-2} + g_{k-2}), \quad k \geq 2, \\ g_k &= z^2(f_{k+1} + g_{k+1}) + z(f_{k+2} + g_{k+2}), \quad k \geq 0. \end{aligned} \tag{1}$$

Now, we introduce the bivariate generating functions

$$F(u, z) = \sum_{k \geq 0} u^k f_k(z) \quad \text{and} \quad G(u, z) = \sum_{k \geq 0} u^k g_k(z).$$

For short, we use the notation $F(u)$, $G(u)$, f_k , and g_k for these functions. Summing the recursions in (1), we have:

$$\begin{aligned} F(u) &= 1 + z^2 \sum_{k \geq 1} u^k (f_{k-1} + g_{k-1}) + z \sum_{k \geq 2} u^k (f_{k-2} + g_{k-2}) \\ &= 1 + z^2 u (F(u) + G(u)) + z u^2 (F(u) + G(u)) \\ &= 1 + z u (z + u) (F(u) + G(u)), \\ G(u) &= z^2 \sum_{k \geq 0} u^k (f_{k+1} + g_{k+1}) + z \sum_{k \geq 0} u^k (f_{k+2} + g_{k+2}) \\ &= \frac{z^2}{u} (F(u) - 1 + G(u) - g_0) + \frac{z}{u^2} (F(u) - 1 - u f_1 + G(u) - g_0 - u g_1), \end{aligned}$$

where $g_0 = G(0)$ is the generating function for the number of knight's paths ending on the x -axis with respect to the size, i.e., $g_0 = A(z) - 1$, and g_1 is the generating function for the number of partial knight's paths ending at height 1 with a down-step \bar{N} or \bar{E} . Then, g_1 is the difference between the generating function $A_1(z)$ for the number of knight's paths ending at height 1, and the generating function for the number of knight's paths ending at height 1 with the step $E = (2, 1)$, i.e. $g_1 = A_1(z) - z^2(g_0 + 1)$ where $A_1(z)$ is given in [16] as follows:

$$A_1(z) = z^2 A(z)^2 (1 - z A(z))^{-1}.$$

Solving the above functional equations, we deduce

$$\begin{aligned} F(u) &= \frac{u z^2 A(z) (u^2 z + u z^2 + u + z) + u^2 z^2 A_1(z) (u + z) + u z^2 - u^2 + z}{u^4 z + u^3 z^2 + u z^2 - u^2 + z}, \\ G(u) &= - \frac{(A(z) (u^3 z^2 + u^2 z^3 + u^2 z + u z^2 - u z - 1) + A_1(z) (u^3 z + u^2 z^2 - u) + z u + 1) z}{u^4 z + u^3 z^2 + u z^2 - u^2 + z}, \\ F(u) + G(u) &= \frac{z (u z + 1) A(z) + u z A_1(z) - u^2}{u^4 z + u^3 z^2 + u z^2 - u^2 + z}. \end{aligned}$$

The first terms of its series expansion are

$$1 + u^2z + (1 + u + u^4)z^2 + (u + 2u^2 + 2u^3 + u^6)z^3 + (3 + u + u^2 + 2u^3 + 3u^4 + 3u^5 + u^8)z^4 \\ + (2 + 4u + 9u^2 + 8u^3 + 3u^4 + 3u^5 + 4u^6 + 4u^7 + u^{10})z^5.$$

Notice that the array $[z^n u^k](F(u) + G(u))$ corresponds to [A096587](#). Figure 3 shows the knight's path of length 5 ending at height 1.

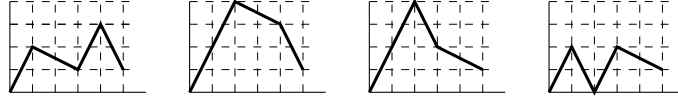


Figure 3: Knight's path of size 5 ending at height 1.

Naturally, we retrieve the generating function for the knight's paths ending on the x -axis $F(0) + G(0) = A(z)$. Moreover, the generating function for the total number of partial knight's paths with respect to the size is

$$F(1) + G(1) = \frac{z(z+1)A(z) + zA_1(z) - 1}{2z^2 + 2z - 1}.$$

The first terms of its series expansion are

$$1 + z + 3z^2 + 6z^3 + 16z^4 + 38z^5 + 99z^6 + 248z^7 + 646z^8 + 1659z^9 + 4342z^{10},$$

and the sequence of coefficients corresponds to [A096588](#) in [22].

Since it seems difficult to obtain closed forms for f_k and g_k , we only give the first terms of the series expansions of $[u^k](F(u) + G(u))$ for three values of k ($k = 0, 1, 2$):

- $1 + z^2 + 3z^4 + 2z^5 + 12z^6 + 14z^7 + 54z^8 + 86z^9$,
- $z^2 + z^3 + 3z^4 + 4z^5 + 12z^6 + 22z^7 + 61z^8 + 128z^9$,
- $z + 2z^3 + z^4 + 9z^5 + 10z^6 + 42z^7 + 64z^8 + 213z^9$,

which correspond respectively to [A005220](#) and [A005221](#) in [22] for the first two. The third sequence does not appear in [22].

2.2 Zigzag knight's paths

In this part, we count partial zigzag knight's paths of a given size. We keep the notation of the previous section in the context of zigzag paths. Then, we easily have the following equations:

$$\begin{aligned} f_0 &= 1, & f_1 &= z^2(f_0 + g_0), & \text{and } f_2 &= z(f_0 + g_0 + zg_1), \\ f_k &= z^2g_{k-1} + zg_{k-2}, & k &\geq 3, \\ g_k &= z^2f_{k+1} + zf_{k+2}, & k &\geq 0, \end{aligned} \tag{2}$$

Summing these recursions, we obtain:

$$F(u) = 1 + zu(z + u)(1 + G(u)),$$

$$G(u) = \frac{z^2}{u}(F(u) - 1) + \frac{z}{u^2}(F(u) - 1 - uz^2(1 + g_0)),$$

where $g_0 = G(0)$ is the generating function for the number of non-empty zigzag paths ending on the x -axis with respect to the size.

Solving these functional equations, we deduce

$$F(u) = \frac{g_0 u^2 z^4 + g_0 u z^5 + u^2 z^4 + u z^5 + u^2 z^3 + u z^4 - u^3 z - u^2 z^2 + u z^2 + z^3 - u}{u^2 z^3 + u z^4 + u z^2 + z^3 - u}$$

and

$$G(u) = \frac{z^2(-u^2 z - u z^2 + g_0 z - u)}{u^2 z^3 + u z^4 + u z^2 + z^3 - u}.$$

In order to compute g_0 , we use the kernel method on $G(u)$ (see [2]). We have

$$G(u) = \frac{z^2(-u^2 z - u z^2 + g_0 z - u)}{z^3(u - s)(u - r)},$$

where

$$r = \frac{1 - z^4 - z^2 - \sqrt{z^8 - 2z^6 - z^4 - 2z^2 + 1}}{2z^3}, \quad (3)$$

and

$$s = \frac{1 - z^4 - z^2 + \sqrt{z^8 - 2z^6 - z^4 - 2z^2 + 1}}{2z^3}.$$

It suffices to plug $u = r$ in the numerator of $G(u)$. Then, g_0 satisfies $-r^2 z - r z^2 + g_0 z - r = 0$, which implies that

$$g_0 = \frac{r}{z^3} - 1.$$

After this, and using $sr = 1$, we simplify of both, numerators and denominators, in $F(u)$ and $G(u)$ by factorizing them with $(u - r)$.

Theorem 1 *The bivariate generating functions $F(u)$, $G(u)$, and $F(u) + G(u)$ are given by*

$$F(u) = \frac{ru(u + z)}{z^2(1 - ru)} + 1, \quad G(u) = \frac{r}{(1 - ru)z^3} - 1, \quad \text{and} \quad F(u) + G(u) = \frac{ru^2 z + ru z^2 + r}{z^3(1 - ru)}.$$

Moreover, we have

$$f_0 = [u^0]F(u) = 1, \quad (4)$$

$$f_k = [u^k]F(u) = \frac{rz + 1}{z^2} r^{k-1} - [k = 1] \frac{1}{z^2}, \quad k \geq 1, \quad (5)$$

$$g_0 = [u^0]G(u) = \frac{r}{z^3} - 1, \quad (6)$$

$$g_k = [u^k]G(u) = \frac{r^{k+1}}{z^3}, \quad k \geq 1. \quad (7)$$

The first terms of the series expansion of $F(u) + G(u)$ are

$$1 + u^2z + (1 + u)z^2 + (u + u^2)z^3 + (2 + u + u^3)z^4 + (2u + 3u^2)z^5 \\ + (4 + 2u + u^2 + 2u^3)z^6 + (5u + 6u^2 + u^4)z^7.$$

Figure 4 shows the zigzag knight's paths of length 5 ending at height 2.

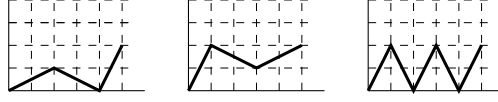


Figure 4: Zigzag knight's path of size 5 ending at height 2.

Corollary 1 *The generating function for the number of zigzag knight's paths of a given size ending on the x-axis is*

$$F(0) + G(0) = \frac{r}{z^3},$$

and the sequence of coefficients in the series expansion corresponds to [A004148](#) in [22] (the non-zero coefficients are the generalized Catalan numbers). The first terms of the series expansion are

$$1 + z^2 + 2z^4 + 4z^6 + 8z^8 + 17z^{10} + 37z^{12} + 82z^{14} + 185z^{16} + 423z^{18} + 978z^{20}.$$

Corollary 2 *The generating function for the number of partial zigzag knight's paths of a given size is*

$$F(1) + G(1) = \frac{rz^2 + rz + r}{z^3(1-r)},$$

and the sequence of coefficients of z^n in the series expansion corresponds to [A088518](#) in [22], which counts also symmetric secondary structures of RNA molecules with n nucleotides. For $n \geq 0$, the first terms of the series expansions are

$$1 + z^2 + 2z^3 + 2z^4 + 4z^5 + 5z^6 + 9z^7 + 12z^8 + 21z^9 + 29z^{10} + 50z^{11}.$$

Theorem 2 *We have $[z^0]f_0 = 1$, and for $n \geq 1$, $k \geq 1$,*

$$[z^{2n}]f_{2k+1} = a(n-k, 2k+1) + a(n-k+1, 2k), \\ [z^{2n-1}]f_{2k} = a(n-k, 2k) + a(n-k+1, 2k-1), \\ [z^{2n}]g_{2k} = a(n-k+1, 2k+1), \\ [z^{2n-1}]g_{2k-1} = a(n-k+1, 2k),$$

where

$$a(n, k) = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{k}{n-i} \binom{n-i}{i+k} \binom{n-i}{i}.$$

All other coefficients are equal to 0.

Proof. Let \hat{r} be the generating function $z^{-1/2}r(z^{1/2})$, where $r(z)$ is defined in (3). Then

$$\hat{r} = \frac{1 - z - z^2 - \sqrt{1 - 2z - z^2 - 2z^3 + z^4}}{2z^2}.$$

The generating function \hat{r} satisfies the functional equation $\hat{r} = z(1 + (1+z)\hat{r} + z\hat{r}^2)$. Consider the auxiliary function $q(z, t)$ defined by

$$q(z, t) = z(1 + (1+t)q(z, t) + tq(z, t)^2) = z\Phi(q(z, t)),$$

where $\Phi(u) = (1+u)(1+tu)$. From the Lagrange inversion, (see [19] for instance), we have that

$$\begin{aligned} [z^n]q(z, t)^k &= \frac{k}{n}[u^{n-k}]\Phi(u)^n = \frac{k}{n}[u^{n-k}](u+1)^n(tu+1)^n \\ &= \frac{k}{n} \sum_{i=0}^{n-k} \binom{n}{n-k-i} \binom{n}{i} t^i, \quad n \geq 1. \end{aligned}$$

So, we obtain

$$\begin{aligned} [z^n]\hat{r}^k &= [z^n]q(z, z)^k = [z^n] \sum_{n \geq 1} \frac{k}{n} \sum_{i=0}^{n-k} \binom{n}{n-k-i} \binom{n}{i} z^{n+i} \\ &= [z^n] \sum_{i \geq 0} \sum_{n \geq 0} \frac{k}{n+i+1} \binom{n+i+1}{n+1-k} \binom{n+i+1}{i} z^{n+2i+1}. \end{aligned}$$

Setting $h = n + 2i + 1$, this implies

$$\begin{aligned} [z^n]\hat{r}^k &= [z^n] \sum_{i \geq 0} \sum_{h \geq 2i+1} \frac{k}{h-i} \binom{h-i}{h-2i-k} \binom{h-i}{i} z^h \\ &= \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{k}{n-i} \binom{n-i}{i+k} \binom{n-i}{i} = a(n, k). \end{aligned}$$

Then, for all positive integers n and k we have

$$[z^{2n}]r^{2k} = a(n-k, 2k) \quad \text{and} \quad [z^{2n-1}]r^{2k-1} = a(n-k, 2k-1).$$

From Theorem 1, we conclude

$$\begin{aligned} [z^{2n}]f_{2k+1} &= [z^{2n}] \left(\frac{r^{2k+1}}{z} + \frac{r^{2k}}{z^2} \right) = [z^{2n+1}]r^{2k+1} + [z^{2n+2}]r^{2k} \\ &= a(n-k, 2k+1) + a(n-k+1, 2k). \end{aligned}$$

Analogously we obtain the remaining identities. \square

Here are other examples of the first terms of the series expansions of $[u^k](F(u) + G(u))$ for $k = 1, 2, 3$:

- $z^2 + z^3 + z^4 + 2z^5 + 2z^6 + 5z^7 + 4z^8 + 12z^9 + 8z^{10} + 28z^{11} + 17z^{12}$,
- $z + z^3 + 3z^5 + z^6 + 6z^7 + 3z^8 + 13z^9 + 9z^{10} + 29z^{11} + 25z^{12} + 65z^{13}$,
- $z^4 + 2z^6 + 6z^8 + z^9 + 15z^{10} + 4z^{11} + 37z^{12} + 14z^{13} + 91z^{14} + 44z^{15}$,

which do not appear in [22].

2.3 A bijective approach

Corollary 1 proves that there is a bijection between the set of zigzag knight's paths of size n ending on the x -axis, and the set of peakless Motzkin paths of length $n+1$, i.e., lattice paths in \mathbb{N}^2 , starting at the origin, ending at $(n+1, 0)$, consisting of steps $U = (1, 1)$, $D = (1, -1)$, $F = (1, 0)$, and avoiding the pattern UD (see for instance [10]). Any non-empty peakless Motzkin path is either of the form (1) FQ , or (2) $URDS$, or (3) URD where Q (resp. R, S) is a possible empty (resp. non-empty) peakless Motzkin path. Let \mathcal{M}_n be the set of peakless Motzkin paths of length n , and $\mathcal{M} = \bigcup_{n \geq 0} \mathcal{M}_n$.

On the other hand, any zigzag knight's path is either of the form (1) $N\bar{N}\beta$, (2) $E\bar{E}\beta$, or (3) $N\bar{E}\beta\bar{N}\gamma$, where β, γ are zigzag knight's paths, and $N = (1, 2)$, $\bar{N} = (1, -2)$, $E = (2, 1)$, $\bar{E} = (2, -1)$. We refer to the top of Figure 5 for an illustration of these different forms. Let \mathcal{R}_n be the set of zigzag paths of semi-size n ending on the x -axis, and $\mathcal{R} = \bigcup_{n \geq 0} \mathcal{R}_n$.

Definition 1 We recursively define the map ψ from \mathcal{R} to \mathcal{M} as follows. For $\alpha \in \mathcal{R}$, we set:

$$\psi(\alpha) = \begin{cases} F & \text{if } \alpha = \epsilon, & (i) \\ U\psi(\beta)D & \text{if } \alpha = E\bar{E}\beta \text{ with } \beta \in \mathcal{R}, & (ii) \\ F\psi(\beta) & \text{if } \alpha = N\bar{N}\beta \text{ with } \beta \in \mathcal{R}, & (iii) \\ U\psi(\beta)D\psi(\gamma) & \text{if } \alpha = N\bar{E}\beta E\bar{N}\gamma \text{ with } \beta, \gamma \in \mathcal{R}. & (iv) \end{cases}$$

Due to the recursive definition, the image by ψ of a zigzag knight's path of semi-size n is a peakless Motzkin path of length $n+1$. For instance, the images of $N\bar{N}$, $E\bar{E}$, $N\bar{N}N\bar{N}$ are respectively FF , FFF , and UFD . We refer to Figure 5 for an illustration of this mapping.

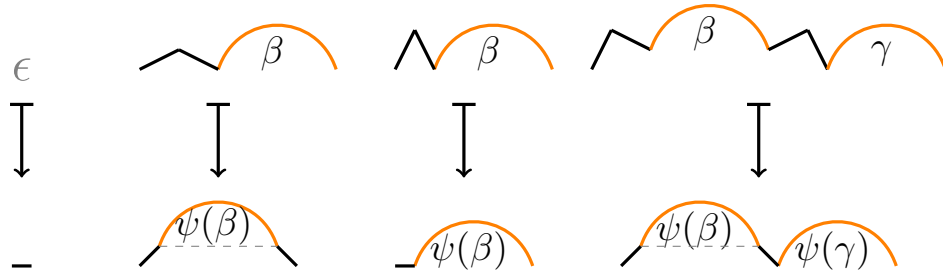


Figure 5: Illustration of the map ψ according to Definition 1.

Theorem 3 For all $n \geq 2$, the map ψ induces a bijection between \mathcal{R}_n and \mathcal{M}_{n+1} .

Proof. In Section 2.2, we have proved that the cardinality of \mathcal{R}_n is given by the n -th term of generalized Catalan number. So it suffices to prove the injectivity of ψ . We proceed by induction on n . The case $n = 0$ is obvious since $\mathcal{R}_0 = \{\epsilon\}$ and $\mathcal{M}_1 = \{F\}$. For all $k \leq n$, we assume that ψ is an injection from \mathcal{R}_k to \mathcal{M}_{k+1} , and we prove the result for $k = n+1$.

According to Definition 1, if α and β in \mathcal{R}_{n+1} satisfy $\psi(\alpha) = \psi(\beta)$, then α and β necessarily come from the same case among (i) – (iv). Using the induction hypothesis, we conclude directly that $\alpha = \beta$, which completes the induction. Thus ψ is injective and so bijective. \square

3 Knight's paths of a given length

Knight's paths of a given length n are clearly in one-to-one correspondence with basketball walks (see [1, 4, 8, 9]), i.e., lattices paths in \mathbb{N}^2 , starting at the origin, and consisting of n steps lying in $S' = \{(1, 1), (1, -1), (1, 2), (1, -2)\}$. In [4], Banderier et al. (see Prop. 3.7) prove that the generating function $E(z) = \sum_{n \geq 0} e_n z^n$ for the number of these paths with respect to the length satisfies

$$z^4 E^4 - (2z^3 + z^2) E^3 + (3z^2 + 2z) E^2 - (2z + 1) E + 1 = 0,$$

and they obtain

$$e_n = \frac{1}{n+1} \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{2n+2}{i} \binom{n-i-1}{n-2i}.$$

The first values of e_n , $n \geq 0$, are 1, 0, 2, 2, 11, 24, 93, 272, 971, 3194, 11293 (sequence [A187430](#)).

3.1 Partial knight's paths

Partial knight's paths are also handled by Banderier et al. in [4]. In their Prop. 3.2, they prove that the generating function for basketball walks with respect to the number of steps, starting at the origin, ending at height k , is given by

$$\frac{u_1^{k+1}(z) - u_2^{k+1}(z)}{u_1(z) - u_2(z)},$$

where $u_1(z)$ and $u_2(z)$ are the small roots of

$$1 - z(u^{-2} + u^{-1} + u + u^2) = 0.$$

For instance (see [4]), the generating function for basketball walks with respect to the number of steps, starting at the origin, ending at height k , $k = 1, 2$, are respectively

$$\frac{1}{2} \left(1 + \sqrt{\frac{2 - 3z - 2\sqrt{1 - 4z}}{z}} \right) \quad \text{and} \quad \frac{3 - \sqrt{1 - 4z} - \sqrt{2 + 12z + 2\sqrt{1 - 4z}}}{4z}.$$

The first terms of the series expansions are respectively $z + z^2 + 3z^3 + 7z^4 + 22z^5 + 65z^6 + 213z^7$ (sequence [A166135](#)) and $z + z^2 + 4z^3 + 9z^4 + 31z^5 + 91z^6 + 309z^7$ (sequence [A111160](#)).

3.2 Zigzag knight's paths

In this part, we count partial zigzag knight's paths of a given length. We adopt the notation used in Section 2.2 for zigzag paths of given length. Then, we easily have the following equations:

$$\begin{aligned} f_0 &= 1, f_1 = z(f_0 + g_0), \text{ and } f_2 = z(f_0 + g_0 + g_1), \\ f_k &= zg_{k-1} + zg_{k-2}, \quad k \geq 3, \\ g_k &= zf_{k+1} + zf_{k+2}, \quad k \geq 0, \end{aligned} \tag{8}$$

Summing the recursions in (8), we have:

$$\begin{aligned} F(u) &= 1 + u(z + g_0z) + z(f_0 + g_0 + g_1)u^2 + z \sum_{k \geq 3} g_{k-1}u^k + z \sum_{k \geq 3} g_{k-2}u^k, \\ &= 1 + u(z + g_0z) + z(f_0 + g_0 + g_1)u^2 + zu(G(u) - g_0 - g_1u) + zu^2(G(u) - g_0), \\ &= 1 + zu(1 + u)(1 + G(u)), \\ G(u) &= z \sum_{k \geq 0} f_{k+1}u^k + z \sum_{k \geq 0} f_{k+2}u^k \\ &= \frac{z}{u}(F(u) - 1) + \frac{z}{u^2}(F(u) - 1 - uz(1 + g_0)), \end{aligned}$$

where $g_0 = G(0)$ is the generating function with respect to the length for the number of non-empty zigzag knight paths ending on the x -axis.

Solving these functional equations, we deduce

$$F(u) = \frac{g_0u^2z^3 + g_0uz^3 + u^2z^3 - u^3z + u^2z^2 + uz^3 - u^2z + 2uz^2 + z^2 - u}{u^2z^2 + 2uz^2 + z^2 - u},$$

and

$$G(u) = \frac{z^2(-u^2 + g_0 - 2u)}{u^2z^2 + 2uz^2 + z^2 - u}.$$

In order to compute g_0 , we use the kernel method on $G(u)$. We have

$$G(u) = \frac{z^2(-u^2 + g_0 - 2u)}{z^2(u - r)(u - s)},$$

where

$$r = \frac{1 - 2z^2 - \sqrt{1 - 4z^2}}{2z^2} \tag{9}$$

and

$$s = \frac{1 - 2z^2 + \sqrt{1 - 4z^2}}{2z^2}.$$

Plugging $u = r$ in the numerator of $G(u)$, we obtain $-r^2 + g_0 - 2r = 0$, which implies that $g_0 = r(2 + r)$. After this, and using $sr = 1$, we simplify of both, numerators and denominators, in $F(u)$ and $G(u)$ by factorizing them with $(u - r)$.

Theorem 4 *The bivariate generating functions $F(u)$, $G(u)$, and $F(u) + G(u)$ are given by*

$$F(u) = \frac{ru^2 + ur(1-z) + z}{(1-ru)z}, \quad G(u) = \frac{r(u+r+2)}{1-ru},$$

and

$$F(u) + G(u) = \frac{r^2z + ru^2 + ru + 2rz + z}{(1-ru)z}.$$

Moreover, we have

$$f_0 = [u^0]F(u) = 1, \tag{10}$$

$$f_k = [u^k]F(u) = \frac{1+r}{z}r^{k-1} - [k=1]\frac{1}{z}, \quad k \geq 1, \tag{11}$$

$$g_0 = [u^0]G(u) = r(2+r), \tag{12}$$

$$g_k = [u^k]G(u) = \frac{r^{k+1}}{z^2}, \quad k \geq 1. \tag{13}$$

The first terms of the series expansion of $F(u) + G(u)$ are

$$1 + (u + u^2)z + (2 + u)z^2 + (2u + 3u^2 + u^3)z^3 + (5 + 4u + u^2)z^4 \\ + (5u + 9u^2 + 5u^3 + u^4)z^5 + (14 + 14u + 6u^2 + u^3)z^6.$$

Figure 6 shows the zigzag knight's paths of length 5 ending at height 1.

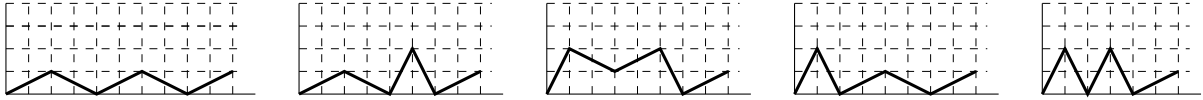


Figure 6: Partial zigzag knight's path of length 5 ending at height 1.

Corollary 3 *The generating function for the number of zigzag knight paths of a given length ending on the x -axis is*

$$F(0) + G(0) = 1 + r(2 + r),$$

and the sequence of coefficients in the series expansion corresponds to the Catalan numbers ([A000108](#) in [22]). The first terms of the series expansion are

$$1 + 2z^2 + 5z^4 + 14z^6 + 42z^8 + 132z^{10} + 429z^{12} + 1430z^{14} + 4862z^{16} + 16796z^{18} + 58786z^{20}.$$

Corollary 4 *The generating function for the number of partial zigzag knight's paths of a given length is*

$$F(1) + G(1) = \frac{2rz + r}{(1-r)z^2} = \frac{4z + 2}{1 - 4z^2 + \sqrt{1 - 4z^2}},$$

and the sequence of coefficients in the series expansion corresponds to [A210736](#) in [22]. The first terms of the series expansion are

$$1 + 2z + 3z^2 + 6z^3 + 10z^4 + 20z^5 + 35z^6 + 70z^7 + 126z^8 + 252z^9 + 462z^{10}.$$

Corollary 5 We have $[z^0]f_0 = 1$, and for $n \geq 1$,

$$\begin{aligned} [z^{2n-1}]f_k &= \frac{2k-1}{n+k} \binom{2n}{n-k+1}, \quad k \geq 1, \\ [z^{2n}]g_0 &= \frac{1}{n+2} \binom{2n+2}{n+1}, \\ [z^{2n}]g_k &= \frac{k+1}{n+1} \binom{2n+2}{n-k}, \quad k \geq 1. \end{aligned}$$

All other coefficients are equal to 0.

Proof. The generating function $\hat{r} := r(z^{1/2})$ (see eq. (9)) satisfies the functional equation $\hat{r} = x\Phi(\hat{r})$, where $\Phi(u) = u^2 + 2u + 1$. From the Lagrange inversion we have that

$$[z^{2n}]r^k = [z^n]\hat{r}^k = \frac{k}{n}[u^{n-k}]\Phi(u)^n = \frac{k}{n}[u^{n-k}](u+1)^{2n} = \frac{k}{n} \binom{2n}{n-k}, \quad n \geq 1.$$

From eq. (11) we have

$$\begin{aligned} [z^{2n-1}]f_k &= [z^{2n}](r^{k-1} + r^k) = \frac{k-1}{n} \binom{2n}{n-k+1} + \frac{k}{n} \binom{2n}{n-k} \\ &= \frac{k-1}{n} \binom{2n}{n-k+1} + \frac{k(n-k+1)}{n(n+k)} \binom{2n}{n-k+1} \\ &= \frac{2k-1}{n+k} \binom{2n}{n-k+1}. \end{aligned}$$

On the other hand,

$$[z^{2n}]g_0 = [z^{2n}]r(2+r) = \frac{2}{n} \binom{2n}{n-1} + \frac{2}{n} \binom{2n}{n-2} = \frac{1}{n+2} \binom{2n+2}{n+1},$$

and for $k \geq 1$

$$[z^{2n}]g_k = [z^{2n}] \frac{r^{k+1}}{z^2} = [z^{2n+2}]r^{k+1} = \frac{k+1}{n+1} \binom{2n+2}{n-k}.$$

□

Here are other examples of the first terms of the series expansions of $[u^k](F(u) + G(u))$ for $k = 1, 2, 3$:

- $z + z^2 + 2z^3 + 4z^4 + 5z^5 + 14z^6 + 14z^7 + 48z^8 + 42z^9 + 165z^{10}$,
- $z + 3z^3 + z^4 + 9z^5 + 6z^6 + 28z^7 + 27z^8 + 90z^9 + 110z^{10}$,
- $z^3 + 5z^5 + z^6 + 20z^7 + 8z^8 + 75z^9 + 44z^{10} + 275z^{11} + 208z^{12}$, which do not appear in [22].

3.3 A bijective approach

Corollary 3 proves that there is a bijection between the set of zigzag knight's paths of semi-length n ending on the x -axis, and the set of Dyck paths of semi-length n . Any non-empty Dyck path is either of the form (1) UD , (2) UQD , (3) UDQ or (4) $UQDR$ where Q, R are non-empty Dyck paths. Let \mathcal{D}_n (resp. \mathcal{S}_n) be the set of Dyck paths (resp. zigzag paths) of semi-length n , and $\mathcal{D} = \bigcup_{n \geq 0} \mathcal{D}_n$ (resp. $\mathcal{S} = \bigcup_{n \geq 0} \mathcal{S}_n$).

Definition 2 We recursively define the map ϕ from \mathcal{S} to \mathcal{D} as follows. For $\alpha \in \mathcal{S}$, we set:

$$\phi(\alpha) = \begin{cases} UD & \text{if } \alpha = \epsilon, & (i) \\ U\phi(\beta)D & \text{if } \alpha = E\bar{E}\beta \text{ with } \beta \in \mathcal{S}, & (ii) \\ UD\phi(\beta) & \text{if } \alpha = N\bar{N}\beta \text{ with } \beta \in \mathcal{S}, & (iii) \\ U\phi(\beta)D\phi(\gamma) & \text{if } \alpha = N\bar{E}\beta E\bar{N}\gamma \text{ with } \beta, \gamma \in \mathcal{S}. & (iv) \end{cases}$$

Due to the recursive definition, the image by ϕ of zigzag knight's path of semi-length n is a Dyck path of semi-length $n + 1$. For instance, the images of $N\bar{N}$, $E\bar{E}$, $N\bar{N}N\bar{N}$ are respectively $UDUD$, $UUDD$, and $UDUDUD$. We refer to Figure 7 for an illustration of this mapping.

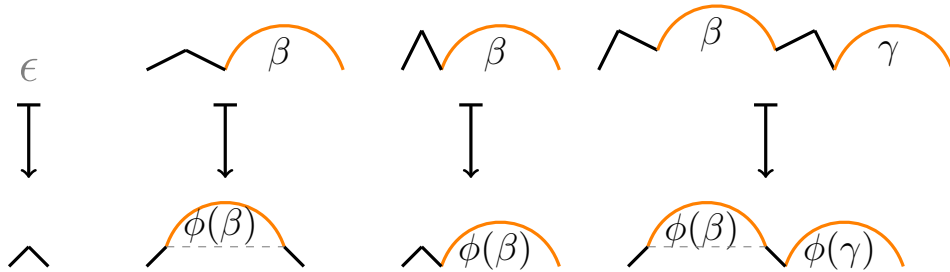


Figure 7: Illustration of the map ϕ according to Definition 2.

The proof of the following theorem is obtain *mutatis mutandis* as for Theorem 2 (see Section 2).

Theorem 5 For all $n \geq 2$, the map ϕ induces a bijection between \mathcal{S}_n and \mathcal{D}_{n+1} .

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