BIJECTIONS BETWEEN DIRECTED-COLUMN CONVEX POLYOMINOES AND RESTRICTED COMPOSITIONS

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Abstract. A bijection is given between the set of directed column-convex polyominoes on triangular and honeycomb lattices of area n and some families of restricted compositions. This is an analogous result to one given by Deutsch and Prodinger for polyominoes over square lattices. As a byproduct, we deduce new close forms for the number of hexagonal and triangular directed column-convex polyominoes of area n with k columns.

1. Introduction

A *polyomino* is a connected set of n unit cells on a lattice structure. In the literature, polyominoes are widely studied in the domain of combinatorics. Generally, the studies consist in the enumeration of some special classes of polyominoes with respect to the type of lattice and some given values of parameters (area, height, number of columns, perimeter, ...). We refer to the survey of Viennot [15], the book edited by Guttmann [12], and the papers [2, 3, 4, 5, 6, 7]. In this paper, we will consider polyominoes in the square (resp. triangular, resp. honeycomb) lattice, where the unit cell is a square (resp. hexagon, resp. triangle). See Figure 1 for an illustration of these lattices and the associated unit cells. Notice that the unit cell for the honeycomb lattice is an equilateral triangle that can be oriented in two ways (triangle pointing upwards and downwards).

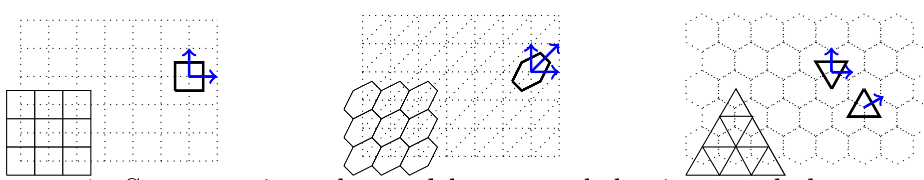

Figure 1. Square, triangular and honeycomb lattices, and the associated unit cells (square, hexagon, and triangle). For the honeycomb lattice, there are two kinds of cells: triangles pointing upwards and downwards.

For each lattice, we consider a set of directions (North/East for the square lattice, North/North-East/East for the triangular and honeycomb lattices). A polyomino P is *directed* if there exists a cell S, called the *source* of P, such that any cell C of P can be obtained by repeatedly joining cells from C using the predetermined set of directions.

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A polyomino P is said *column convex* when any column of P is a connected set, where a *column* of P is defined as the set of cells of P whose centers intersect a fixed line L (vertical line for the square and triangular lattices, and oblique lines of slope $\frac{\pi}{3}$ for the honeycomb lattice).

Definition 1.1. A *dcc-polyomino* consists of a set of unit cells satisfying the three key properties: the set of cells is connected, directed, and column convex.

We refer to Figure 2 for three examples of dcc-polyominoes in the three kinds of lattices. The source cell is located at the bottom left corner and each dcc-polyomino is constructed by attaching unit cells in the allowed directions of the lattice, by taking into account the property of *directed column convexity*.

Let P be a dcc-polyomino. The *area* of P, denoted by $a(P)$, is the number of unit cells of P. We denote the number of columns of P by $c(P)$. The *height* $h(P)$ of P is the length (number of cells) of a longest path from the source of P to any of the cells in P.

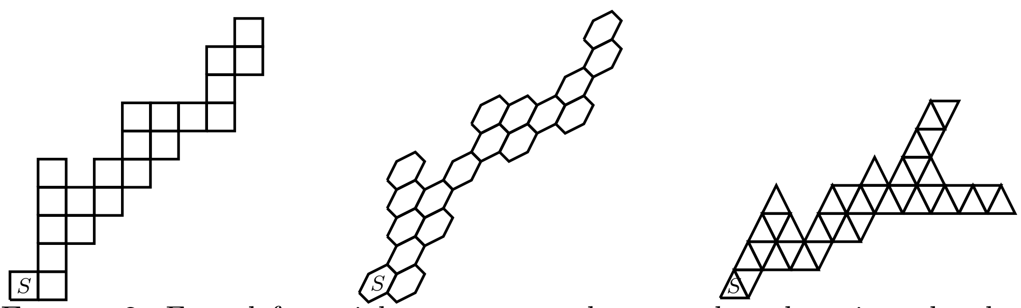


FIGURE 2. From left to right, a square, a hexagonal, and a triangular dcc-polyominoes of areas 21, 18 and 37, respectively. All these polyominoes have 9 columns.

For each kind of lattice described above, Barcucci et al. [3] gave multivariate generating functions for the number of dcc-polyominoes with respect to the area, the number of columns, and the height. The method used consists in giving a recursive description of the set of dcc-polyominoes which induces a functional equation for the multivariate generating function. They also deduce (for each lattice) the average height of dcc-polyominoes and its asymptotic behaviors when the area tends to infinity. In a second study [2], Barcucci, Pinzani, and Sprugnoli use a traditional recurrence relation approach in order to count the number of dcc-polyominoes in the square lattice with area n and with k columns. They prove that this number is given by the binomial coeficcient

$$\binom{n+k-2}{n-k},$$

and they deduce that the number of square dcc-polyominoes of area n is the Fibonacci number F_{2n+1}, where $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Moreover, Deutsch and Prodinger [5] exhibit a constructive bijection between these polyominoes of area n and ordered trees of height at most three with n edges, that transports the number of columns into one plus the number of nodes at level 2. They also give a one-to-one correspondence with nondecreasing Dyck paths that transports the number of columns into the number of peaks, knowing that a nondecreasing Dyck paths is a Dyck path having a nondecreasing sequence of the heights of its valleys (see [1] for an introduction of nondecreasing paths and [9, 10] for some generalizations of these paths).

Motivation: To our knowledge, the literature does not mention any one-to-one correspondence between hexagonal (resp. triangular) dcc-polyominoes of a given area with other classical combinatorial objects so that the number of columns is transported into a natural statistic. The objective of this note is to remedy this shortcoming by exhibiting a unified combinatorial class of objects which is in bijection with the other two kinds of dcc-polyominoes (hexagonal and triangular). As a byproduct, we will deduce new close forms for the number of these dcc-polyominoes of area n with k columns.

Outline of the paper: In Section 2, we exhibit a one-to-one correspondence between hexagonal dcc-polyominoes of area n with k columns and compositions of the integer $n-1$ in which three different types of ones are allowed 1_N, 1_D, and 1_E, and such that $k-1$ parts are different from 1_N. By counting these compositions and using this bijection, we deduce a new close form for the number $p_{n,k}$ of hexagonal dcc-polyominoes of area n with k column. We end the section by proving that the matrix $(p_{n,k})_{n,k\geq 0}$ is a Riordan array. We also give a one-to-one correspondence between these polyominoes and the set of order-consecutive partitions of $\{1, 2, \ldots, n\}$ that transports the number of columns into the number of parts in the partition. Section 3 presents a similar study for triangular dcc-polyominoes in the honeycomb lattice. We exhibit a bijection between these polyominoes of area n with k columns and compositions of the integer $n-1$ in which only parts of the form 2^i, $i \geq 0$, are allowed, and such that $k-1$ parts are different from 1. As previous, this bijection allows us to deduce a new close form for the number $t_{n,k}$ of triangular dcc-polyominoes of area n with k column. We also give a one-to-one correspondence between these polyominoes and the set of consecutive partitions of $\{1, 2, \ldots, n\}$ that transports the number of columns into the number of parts with at least two elements.

We end this section by fixing some definitions about compositions of an integer n. Also, we give some notations used in this note. A *composition* of a positive integer n is a sequence of positive integers $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_\ell)$ such that $\sigma_1 + \sigma_2 + \cdots + \sigma_\ell = n$. The summands σ_i are called *parts* of the composition and n is referred to the *weight* of σ. For example, the compositions of 4 are

$$(4), \quad (3,1), \quad (1,3), \quad (2,2), \quad (2,1,1), \quad (1,2,1), \quad (1,1,2), \quad (1,1,1,1).$$

It is well known [18] that the number of compositions of n with k parts is $\binom{n-1}{k-1}$, and the total number of compositions of n is 2^{n-1}. Throughout this note, we will use the following notations. The composition of the integer 0 will be denoted $()$, and if $c = (m_1, m_2, \ldots, m_k)$ is a composition of $n \geq 0$ with k parts, then $\overline{c}$ corresponds to the sequence $m_1, m_2, \ldots, m_k$, and for an integer $a \geq 1$, the notation $(a, \overline{c})$ corresponds to the composition $(a, m_1, m_2, \ldots, m_k)$ of the integer $n + a$. In particular, if $c = ()$ then we have $(a, \overline{c}) = (a)$.

2. Hexagonal dcc-polyominoes

Barcucci et al. [3] proved that the number of hexagonal dcc-polyominoes having area n, denoted by H_n, is equal to

$$H_n = \frac{1}{4}(\theta_1^n + \theta_2^n) = \sum_{k=0}^{n} \binom{n}{2k} 2^{n-k-1} \quad (n \geq 1),$$

where $\theta_1 = 2 + \sqrt{2}$ and $\theta_2 = 2 - \sqrt{2}$. Moreover, the authors give the generating function of the sequence

$$H(x) := \sum_{n \geq 1} H_n x^n = \frac{x(1-x)}{1-4x+2x^2}.$$

The first few values for $n \geq 1$ of H_n are

$$1, \quad 3, \quad 10, \quad 34, \quad 116, \quad 396, \quad 1352, \quad 4616, \quad 15760, \dots$$

Notice that H_n corresponds with the sequence A007052 in [17]. Among the objects counted by this sequence are the compositions of an integer in which there are three different types of ones, denoted by $1_N, 1_D$, and 1_E, respectively. Let a_n be the number of these compositions of weight n. For example, $a_2 = 10$ and the corresponding compositions are

$$(1_N, 1_N), \quad (1_N, 1_D), \quad (1_N, 1_E), \quad (1_D, 1_N), \quad (1_D, 1_D), \quad (1_D, 1_E),$$
$$(1_E, 1_N), \quad (1_E, 1_D), \quad (1_E, 1_E), \quad (2).$$

Theorem 2.1. *For all $n \geq 0$, we have the equality $a_n = H_{n+1}$.*

Proof. Let $\mathcal{C}$ denote the family (combinatorial class) of compositions in which three different types of ones are allowed, then we can write the symbolic equation:

$$\mathcal{C} = \mathrm{SEQ}(\{1_N, 1_D, 1_E, 2, 3, 4, \dots\}),$$

where SEQ denotes the sequence combinatorial class (the previous equation simply rephrases that every element of $\mathcal{C}$ is a sequence whose terms belong to $\{1_N, 1_D, 1_E, 2, 3, 4, \dots\}$). For a general background about the symbolic method see the book [8]. In terms of generating functions, the last equation translates into

$$A(x) := \sum_{n \geq 0} a_n x^n = \frac{1}{1 - \left(3x + \sum_{\ell \geq 2} x^\ell\right)} = \frac{1}{1 - \frac{3x-2x^2}{1-x}} = \frac{1-x}{1-4x+2x^2},$$

and we obtain that $H(x) = xA(x)$, which means that $a_n = H_{n+1}$. $\square$

As already mentioned in [3], any hexagonal dcc-polyomino P of area $n \geq 1$ can be uniquely decomposed in one of the following forms (see Figure 3):

- (i) P consists of one hexagonal cell;
- (ii) P is obtained by attaching a dcc-polyomino Q of area $n-1$ to the north side of a hexagonal cell which becomes the source of P;
- (iii) P is obtained by attaching a dcc-polyomino Q of area $n-1$ to the north-east side of a hexagonal cell which becomes the source of P;

(iv) P is obtained by attaching a column C of $k \geq 1$ unit cells so that the most southern cell of C is attached (by its east side) to a dcc-polyomino Q of area $n-k$.

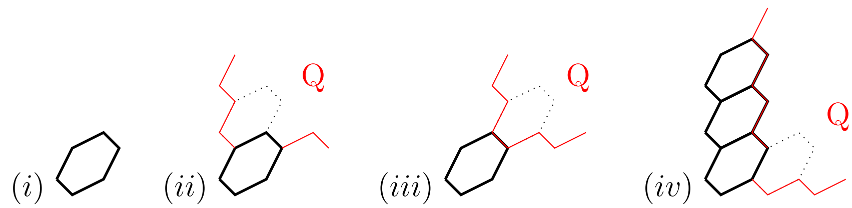
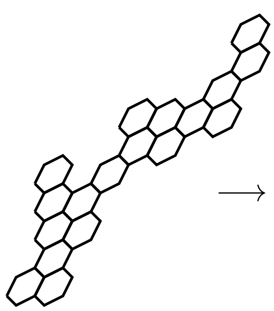


FIGURE 3. Recursive decomposition of a hexagonal dcc-polyomino P.

According to this decomposition, we define recursively a map ϕ from the set of hexagonal dcc-polyominoes of area $n+1$ and the set $\mathcal{C}_n^3$ of compositions of n having parts in $\{1_N, 1_D, 1_E, 2, 3, 4, \dots\}$ (the part one can be take three colors).

- If P belongs to the case (i), then we set $\phi(P) = ()$ (empty composition);
- If P belongs to the case (ii), then we set $\phi(P) = (1_N, \overline{\phi(Q)})$;
- If P belongs to the case (iii), then we set $\phi(P) = (1_D, \overline{\phi(Q)})$;
- If P belongs to the case (iv), then we distinguish two cases:
  - If $k = 1$ (k is the number of cells in the first column of P), then we set $\phi(P) = (1_E, \overline{\phi(Q)})$;
  - Otherwise we have $k \geq 2$, and we set $\phi(P) = (k, \overline{\phi(Q)})$.

See Figure 4 for an illustration of the map ϕ on a hexagonal dcc-polyomino.



$\longrightarrow (1_E, 1_N, 1_N, 3, 1_N, 1_D, 1_D, 2, 1_N, 1_E, 1_E, 1_N, 1_D, 1_N)$

FIGURE 4. A hexagonal dcc-polyomino of area 18 with 9 columns and its image by ϕ, which is a composition of 17 with parts in $\{1_N, 1_D, 1_E, 2, 3, 4, \dots\}$. The number of parts different from 1_N equals 8, which also is the number of columns minus one.

Theorem 2.2. *For all $n \geq 0$, ϕ is a bijection between the set of dcc-polyominoes of area $n+1$ and the set of compositions of n where the parts belong to $\{1_N, 1_D, 1_E, 2, 3, 4, \dots\}$. Moreover, ϕ transports the number of columns minus one into the number of parts different from 1_N in the composition.*

Proof. We can easily observe that the image by ϕ of a hexagonal dcc-polymomino of area $n+1$ is a composition in $\mathcal{C}_n^3$. Moreover, if this polyomino P has $k+1$ columns, then $\phi(P)$

has exactly k parts lying in $\{1_D, 1_E, 2, 3, 4, \dots\}$. Conversely, any composition in $\mathcal{C}_n^3$ with k parts different from 1_N can be uniquely decomposed into one of the following forms:

(i) the empty composition () whenever $n = 0$;

(ii) $(1_N, c_1, \dots, c_\ell)$, $\ell \geq 0$, where $(c_1, \dots, c_\ell) \in \mathcal{C}_{n-1}^3$ with k parts different from 1_N;

(iii) $(1_D, c_1, \dots, c_\ell)$, $\ell \geq 0$, where $(c_1, \dots, c_\ell) \in \mathcal{C}_{n-1}^3$ with $k-1$ parts different from 1_N;

(iv_a) $(1_E, c_1, \dots, c_\ell)$, $\ell \geq 0$, where $(c_1, \dots, c_\ell) \in \mathcal{C}_{n-1}^3$ with $k-1$ parts different from 1_N;

(iv_b) $(a, c_1, \dots, c_\ell)$, $\ell \geq 0$, where $a \geq 2$ and $(c_1, \dots, c_\ell) \in \mathcal{C}_{n-a}^3$ with $k-1$ parts different from 1_N.

Therefore, the set of hexagonal dcc-polyominoes of area $n+1$ and the set $\mathcal{C}_n^3$ have the same recursive description, which ensures that ϕ is a bijection that transports the number of columns minus one into the number of parts different from 1_N. $\square$

Let $p_{n,k}$ be the number of hexagonal dcc-polyominoes of area n with exactly k columns. Notice that an immediate consequence of the recursive decomposition of a dcc-polyomino is the recursive formula $p_{n,1} = 1$ for $n \geq 1$, and for $n \geq 2$, $k \geq 2$,

$$p_{n,k} = p_{n-1,k} + p_{n-1,k-1} + \sum_{\ell=1}^{n-1} p_{n-\ell,k-1}.$$

As a byproduct of the bijection ϕ given in Theorem 2.2 we deduce a close form for $p_{n,k}$.

Theorem 2.3. *If $n \geq k \geq 2$, then*

$$p_{n,k} = \sum_{i=0}^{k-1} \binom{k-1}{i} \binom{n+i-1}{n-k}.$$

Proof. Due to the bijection ϕ defined above, $p_{n,k}$ corresponds to the number of compositions of $n-1$ with parts in $\{1_N, 1_D, 1_E, 2, 3, 4, \dots\}$, and where exactly $k-1$ parts are different from 1_N. Such a composition c can be uniquely obtained from a composition of r, $k-1 \leq r \leq n-1$, with $k-1$ parts and where all parts lie in $\{1_D, 1_E, 2, 3, 4, \dots\}$, by adding $(n-1-r)$ parts 1_N in the right places. Since there are $\binom{n-r-1+k-1}{k-1} = \binom{n-r+k-2}{k-1}$ ways for adding these parts into k places (this is the number of ways to choose $k-1$ parts among $n-1-r+k-1$ parts), we obtain

$$p_{n,k} = \sum_{r=k-1}^{n-1} \binom{n-2-r+k}{k-1} a_{r,k-1},$$

where $a_{r,k-1}$ is the number of compositions of r with $(k-1)$ parts lying into $\{1_D, 1_E, 2, 3, 4, \dots\}$.

From the definition of the sequence $a_{r,s}$ and for a given s, we obtain the following expression for its generating function:

$$\sum_{r\geq 0} a_{r,s} x^r = (2x + x^2 + x^3 + \cdots)^s = x^s \left(1 + \frac{1}{1-x}\right)^s = x^s \sum_{i=0}^{s} \binom{s}{i} \frac{1}{(1-x)^i}.$$

From the equality $1/(1-x)^{m+1} = \sum_{\ell=0}^{\infty} \binom{m+\ell}{\ell} x^{\ell}$, we have

$$\sum_{r\geq 0} a_{r,s}x^r = x^s + \sum_{i=1}^{s} \binom{s}{i} \sum_{\ell=0}^{\infty} \binom{i+\ell-1}{\ell} x^{\ell+s} = x^s + \sum_{i=1}^{s}\sum_{\ell=0}^{\infty} \binom{s}{i}\binom{i+\ell-1}{\ell} x^{\ell+s}.$$

By setting $\ell = 0$, the coefficient of x^s in this expression is

$$a_{s,s} = 1 + \sum_{i=1}^{s} \binom{s}{i}\binom{i-1}{0} = 1 + \sum_{i=1}^{s}\binom{s}{i} = 2^s.$$

For $r > s$, setting $\ell = r - s$ yields

$$a_{r,s} = \sum_{i=1}^{s}\binom{s}{i}\binom{i+r-s-1}{r-s} = \sum_{i=1}^{s}\binom{s}{i}\binom{r-s+i-1}{r-s}.$$

Therefore, by considering the previous value of $a_{r,s}$ for $s = k-1$, we obtain

$$\begin{aligned}
p_{n,k} &= a_{k-1,k-1}\binom{n-1}{k-1} + \sum_{j=k}^{n-1}\binom{n-2-j+k}{k-1} a_{j,k-1} \\
&= 2^{k-1}\binom{n-1}{k-1} + \sum_{j=k}^{n-1}\binom{n-2-j+k}{k-1}\sum_{i=1}^{k-1}\binom{k-1}{i}\binom{j-k+i}{j-k+1} \\
&= 2^{k-1}\binom{n-1}{k-1} + \sum_{i=1}^{k-1}\sum_{j=k}^{n-1}\binom{n-2-j+k}{k-1}\binom{k-1}{i}\binom{j-k+i}{j-k+1} \\
&= 2^{k-1}\binom{n-1}{k-1} + \sum_{i=1}^{k-1}\binom{k-1}{i}\sum_{j=0}^{n-1-k}\binom{j+i}{j+1}\binom{n-j-2}{k-1} \\
&= 2^{k-1}\binom{n-1}{k-1} + \sum_{i=1}^{k-1}\binom{k-1}{i}\sum_{j=1}^{n-k}\binom{(i-1)+j}{j}\binom{n-j-1}{n-j-k} \\
&= 2^{k-1}\binom{n-1}{k-1} + \sum_{i=1}^{k-1}\binom{k-1}{i}\sum_{j=1}^{n-k}\binom{(i-1)+j}{j}\binom{(k-1)+(n-k-j)}{n-j-k}.
\end{aligned}$$

The last sum can be simplified by means of the following identity [11, (identity (3.2)]

$$\sum_{j=0}^{m}\binom{x+j}{j}\binom{y+m-j}{m-j} = \binom{x+y+m+1}{m}$$

by setting $x = i-1, y = k-1$, and $m = n-k$. Indeed,

$$\sum_{j=1}^{n-k}\binom{(i-1)+j}{j}\binom{(k-1)+(n-k-j)}{n-j-k} = \binom{n+i-1}{n-k} - \binom{n-1}{n-k}.$$

Therefore,

$$\begin{aligned}
p_{n,k} &= 2^{k-1}\binom{n-1}{k-1} + \sum_{i=1}^{k-1}\binom{k-1}{i}\left[\binom{n+i-1}{n-k} - \binom{n-1}{n-k}\right] \\
&= 2^{k-1}\binom{n-1}{k-1} - \binom{n-1}{n-k}\sum_{i=1}^{k-1}\binom{k-1}{i} + \sum_{i=1}^{k-1}\binom{k-1}{i}\binom{n+i-1}{n-k} \\
&= 2^{k-1}\binom{n-1}{k-1} - \binom{n-1}{n-k}(2^{k-1}-1) + \sum_{i=1}^{k-1}\binom{k-1}{i}\binom{n+i-1}{n-k} \\
&= \sum_{i=0}^{k-1}\binom{k-1}{i}\binom{n+i-1}{n-k}. \qquad \square
\end{aligned}$$

Using the same decomposition as previously for defining the bijection with compositions having parts into $\{1_N, 1_D, 1_E, 2, 3, 4, \dots\}$, we can easily exhibit another bijection between dcc-polyominoes of area n with k columns and order-consecutive partitions of $\{1, 2, \dots, n\}$ with k parts, knowing that an *ordered partition* of $\{1, 2, \dots, n\}$ with p parts is a p-uplet $(S_1, S_2, \dots, S_p)$ of subsets such that $S_i \cap S_j = \emptyset$ if $i \neq j$, and $\bigcup_{i=1}^{p} S_i = \{1, 2, \dots, n\}$. An *order-consecutive partition* of $\{1, 2, \dots, n\}$ is an ordered partition satisfying the property: for $j = 1, \dots, p$, $\bigcup_{i=1}^{j} S_i$ is an interval.

So, we define recursively a map ψ from the set of hexagonal polyominoes of area $n+1$ and the set $\mathcal{OCP}_n$ of order-consecutive partitions of $\{1, 2, \dots, n\}$.

- If P belongs to the case (i), then we set $\psi(P) = \{1\}$;
- If P belongs to the case (ii), then $\psi(P)$ is obtained from $\psi(Q)$ by inserting n in the last part; for instance, if $f(Q) = \{3, 4\}\{2\}\{1\}$, then $f(P) = \{3, 4\}\{2\}\{1, 5\}$;
- If P belongs to the case (iii), then $\psi(P)$ is obtained from $\psi(Q)$ by adding the part $\{n\}$ on the right; for instance, if $f(Q) = \{3, 4\}\{2\}\{1\}$, then $f(P) = \{3, 4\}\{2\}\{1\}\{5\}$;
- If P belongs to the case (iv), then $\psi(P)$ is obtained from $\psi(Q)$ by increasing by $k \geq 1$ all values in $\psi(Q)$, and by adding the part $\{1, 2, \dots, k\}$ on the right; for instance, if $f(Q) = \{3, 4\}\{2\}\{1\}$ and $k = 4$, then $f(P) = \{7, 8\}\{6\}\{5\}\{1, 2, 3, 4\}$.

With a same argument as the proof of Theorem 2.2, we can easily prove that ψ is a bijection that transports the number of columns into the number of parts. The image of the polyomino represented in Figure 4 is

$$\{9\}\{8, 10, 11\}\{5, 6, 7, 12\}\{13\}\{14\}\{3, 4, 15\}\{2\}\{1, 16\}\{17, 18\}.$$

As a consequence of this bijection and using Theorem 2.3 and Theorem 6 in [13], we deduce another close form for $p_{n,k}$.

Corollary 2.4. *The number of hexagonal dcc-polyominoes of area n with k columns is*

$$p_{n,k} = \sum_{i=0}^{k-1} (-1)^{k-1-i} \binom{k-1}{i} \binom{2k-i-2}{i}.$$

2.1. **A relation with Riordan arrays.** Let $\mathcal{P}$ be the matrix defined by $\mathcal{P} = [p_{n,k}]_{n,k\geq 1}$. The first few rows of the matrix $\mathcal{P}$ are

$$\mathcal{P} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 5 & 4 & 0 & 0 & 0 & 0 & 0 \\ 1 & 9 & 16 & 8 & 0 & 0 & 0 & 0 \\ 1 & 14 & 41 & 44 & 16 & 0 & 0 & 0 \\ 1 & 20 & 85 & 146 & 112 & 32 & 0 & 0 \\ 1 & 27 & 155 & 377 & 456 & 272 & 64 & 0 \\ 1 & 35 & 259 & 833 & 1408 & 1312 & 640 & 128 \\ \vdots & & \vdots & & \vdots & & \vdots & \end{pmatrix}.$$

The matrix $\mathcal{P}$ correspond to the array A056242 and it is a Riordan array.

We now give a short background for Riordan arrays [16]. An infinite lower triangular matrix is called a *Riordan array* if its kth column satisfies the generating function $g(x)\left(f(x)\right)^k$ for $k \geq 0$, where $g(x)$ and $f(x)$ are formal power series with $g(0) \neq 0$, $f(0) = 0$ and $f'(0) \neq 0$ (where $f'(x)$ is the formal derivative of $f(x)$). The matrix corresponding to the pair $f(x), g(x)$ is denoted by $(g(x), f(x))$. If we multiply (g, f) by a column vector $(c_0, c_1, \dots)^T$ with the generating function $h(x)$, then the resulting column vector has a generating function $gh(f)$. This property is known as the fundamental theorem of Riordan arrays or summation property.

The product of two Riordan arrays $(g(x), f(x))$ and $(h(x), l(x))$ is defined by

$$(g(x), f(x)) * (h(x), l(x)) = \left(g(x)h\left(f(x)\right), l\left(f(x)\right)\right).$$

The set of all Riordan arrays is a group under the operator "$*$" [16]. The identity element is $I = (1, x)$, and the inverse of $(g(x), f(x))$ is

$$(g(x), f(x))^{-1} = \left(1/\left(g \circ \overline{f}\right)(x), \overline{f}(x)\right), \tag{1}$$

where $\overline{f}(x)$ is the compositional inverse of $f(x)$.

Even though rows and columns of Riordan arrays are indexed starting at 0, the elements of $\mathcal{P}$ are shifted so that the entry in row 0 and column 0 is in fact $p_{1,1}$.

Theorem 2.5. *The matrix $\mathcal{P}$ is the Riordan array*

$$\left(\frac{1}{1-x}, \frac{x(2-x)}{(1-x)^2}\right).$$

Proof. Due to the recursive decomposition of a hexagonal dcc-polyomino (see Figure 3), we deduce that the bivariate generating function $H(u, x)$, where the coefficient of $x^n u^k$ is the number of hexagonal dcc-polyominoes of area n and with k columns, we have the following functional equation:

$$H(u,x) = xu + xH(u,x) + uxH(u,x) + \frac{ux}{1-x}H(u,x).$$

We deduce

$$H(u,x) = \frac{ux\,(1-x)}{ux^2 - 2\,ux + x^2 - 2\,x + 1}.$$

A simple calculation allows us to check that

$$H(u,x) = \frac{x}{1-x}\left(\frac{1}{1-\frac{xu(2-x)}{x^2-2x+1}}\right),$$

which ensures that the matrix $\mathcal{P}$ is the Riordan array $\left(\frac{1}{1-x}, \frac{x(2-x)}{(1-x)^2}\right)$. □

Every element $d_{n+1,k+1}$ of a Riordan array (not belonging to row 0 or column 0) can be expressed as a linear combination of the elements in the preceding row. The coefficients of the linear combination are called the *A-sequence*. Additionally, the elements in column 0, except for the element $d_{0,0}$, can also be expressed as a linear combination of the preceding row. In this case the coefficients of the linear combination are called the *Z-sequence* [14]. Therefore, the A-sequence, Z-sequence, and the element $d_{0,0}$ give a complete characterization of a Riordan array. Summarizing these comments: let $n, k \in \mathbb{Z}_{\geq 0}$. An infinite lower triangular array $\mathcal{D} = [d_{n,k}]$ is a Riordan array if and only if $d_{0,0} \neq 0$ and there are two sequences $A = (a_0 \neq 0, a_1, a_2, \dots)$ and $Z = (z_0, z_1, z_2, \dots)$ such that

$$\begin{aligned} d_{n+1,k+1} &= a_0 d_{n,k} + a_1 d_{n,k+1} + a_2 d_{n,k+2} + \cdots, \\ d_{n+1,0} &= z_0 d_{n,0} + z_1 d_{n,1} + z_2 d_{n,2} + \cdots. \end{aligned}$$

Moreover, if $\mathcal{D} = (g(x), f(x))$ is a Riordan array with inverse $\mathcal{D}^{-1} = (d(x), h(x))$, then the A-sequence and Z-sequence of $\mathcal{D}$ are given by

$$A(x) = \frac{x}{h(x)}; \qquad Z(x) = \frac{1}{h(x)}\left(1 - d_{0,0}d(x)\right).$$

Since the inverse Riordan array $\mathcal{P}$ is given by

$$\mathcal{P}^{-1} = \left(\frac{1}{\sqrt{1+x}}, \frac{1+x-\sqrt{1+x}}{1+x}\right),$$

we have the generating function of the A-sequence and Z-sequence of $\mathcal{P}$. Thus,

$$\begin{aligned} A(x) &= 1 + x + \sqrt{1+x} = 1 + x + \sum_{n\geq 2} \frac{1}{1-2n}\binom{2n}{n}\left(\frac{-1}{4}\right)^n x^n, \\ Z(x) &= 1. \end{aligned}$$

Therefore, we have the following curious relation. If $n, k \geq 1$, then

$$\begin{aligned} p_{n+1,k+1} &= 2p_{n,k} + \frac{3}{2}p_{n,k+1} + \sum_{\ell\geq 2} \frac{1}{1-2\ell}\binom{2\ell}{\ell}\left(\frac{-1}{4}\right)^{\ell} p_{n,k+\ell} \\ &= 2p_{n,k} + \frac{3}{2}p_{n,k+1} + \sum_{\ell\geq 2} C_{\ell-1}\frac{(-1)^{\ell-1}}{2^{2\ell-1}}p_{n,k+\ell}, \end{aligned}$$

where C_ℓ is the ℓ-th Catalan number.

3. Triangular dcc-polyominoes

Let $\mathcal{T}$ be the set of triangular dcc-polyominoes. Barcucci et al. [3] proved that the generating function for the number of triangular dcc-polyominoes having area n, denoted by T_n, is given by

$$T(x) := \sum_{n\geq 1} T_n x^n = \frac{x(1-x^2)}{1-x-2x^2+x^3}.$$

The first few values for $n \geq 1$ of T_n are

$$1, \quad 1, \quad 2, \quad 3, \quad 6, \quad 10, \quad 19, \quad 33, \quad 61, \quad 108, \dots$$

Notice that T_n corresponds with the sequence A028495 in [17]. From the expression of $T(x)$ follows that T_n satisfies the recurrence relation

$$T_n = T_{n-1} + 2T_{n-2} - T_{n-3} \quad (n \geq 4),$$

with initial conditions $T_1 = 1, T_2 = 1$, and $T_3 = 2$. This relation can be applied repeatedly in the following manner:

$$\begin{aligned} T_n - T_{n-1} &= 2T_{n-2} - T_{n-3} \\ &= T_{n-2} + (T_{n-2} - T_{n-3}) \\ &= T_{n-2} + 2T_{n-4} - T_{n-5} \\ &= T_{n-2} + T_{n-4} + (T_{n-4} - T_{n-5}) \\ &\ \vdots \\ &= \begin{cases} T_{n-2} + T_{n-4} + \cdots + T_2 + T_0, & \text{if } n \text{ is even;} \\ T_{n-2} + T_{n-4} + \cdots + T_3 + T_1, & \text{if } n \text{ is odd;} \end{cases} \end{aligned}$$

This can be rewritten as

$$T_n = T_{n-1} + \sum_{k=1}^{\lfloor n/2 \rfloor} T_{n-2k}.$$

The sequence $(T_n)_{n\geq 0}$ enumerates a variety of combinatorial objects, such as all paths of length of n on the path graph P_6 and the compositions of n whose parts belong to the set $\{1, 2, 4, 6, 8, \dots\}$. Let us establish the relation between triangular dcc-polyominoes and this family of compositions. Let b_n be the number of compositions of n into parts from $\{1, 2, 4, 6, 8, \dots\}$.

Proposition 3.1. *For all $n \geq 0$, we have the equality $b_n = T_{n+1}$.*

Proof. We will prove this statement using the symbolic method. Let $\mathcal{B}$ be the family of all compositions whose parts belong to the set $\{1, 2, 4, 6, 8, \dots\}$. Thus $\mathcal{B} = \text{SEQ}(\{1, 2, 4, 6, 8, \dots\})$. In terms of generating functions, the last equation translates into

$$B(x) := \sum_{n\geq 1} b_n x^n = \frac{1}{1 - \left(x + \sum_{\ell\geq 1} x^{2\ell}\right)} = \frac{1}{1 - x - \frac{x^2}{1-x^2}} = \frac{1 - x^2}{1 - x - 2x^2 + x^3}.$$

Therefore, we obtain $T(x) = xB(x)$, which means that $b_n = T_{n+1}$. $\square$

As already mentioned in [3], any triangular dcc-polyomino P of area $n \geq 1$ can be uniquely decomposed in one of the following forms (see Figure 5):

- (i) P consists of one triangular cell (a triangle pointing upwards);
- (ii) P consists of two triangular cells (two triangles pointing upwards and downwards);
- (iii) P is obtained by attaching a dcc-polyomino Q of area $n - 2$ to the north side of two triangular cells where the leftmost cell becomes the source of P;
- (iv) P is obtained by attaching a column C of $k \geq 2$ triangular dcc-polyominoes so that the most southern down-cell of C is attached (by its east side) to a triangular dcc-polyomino Q of area $n - k$.

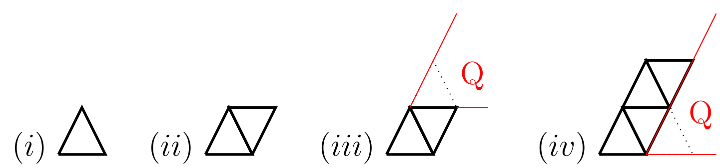


FIGURE 5. Recursive decomposition of a triangular dcc-polyomino P.

According to this decomposition, we define recursively a map χ from the set of triangular dcc-polyominoes of area $n + 1$ and the set $\mathcal{CP}_n$ of compositions of n having parts in $\{1, 2, 4, 6, 8, \dots\}$.

- If P belongs to the case (i), then we set $\chi(P) = ()$ (empty composition);
- If P belongs to the case (ii), then we set $\chi(P) = (1)$;
- If P belongs to the case (iii), then we set $\chi(P) = (1, 1, \overline{\chi(Q)})$;
- If P belongs to the case (iv), then we distinguish two cases:
  - If the number $k \geq 2$ of cells in the first column is odd, then we set $\chi(P) = (1, k - 1, \overline{\chi(Q)})$;
  - Otherwise ($k \geq 2$ is even), we set $\chi(P) = (k, \overline{\chi(Q)})$;

See Figure 6 for an illustration of the map χ on a triangular dcc-polyomino.

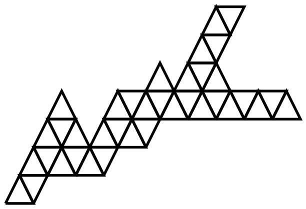
 $\longrightarrow$ $(1,1,1,4,1,2,1,1,4,1,1,1,2,8,1,2,2,2)$

FIGURE 6. A triangular dcc-polyomino of area 37 with 9 columns and its image by χ, which is a composition of 36 with parts in $\{1,2,4,6,8,,\dots\}$ and so that 8 parts are different from 1.

Theorem 3.2. *For all $n \geq 0$, χ is a bijection between the set of triangular dcc-polyominoes of area $n+1$ and the set of compositions of n where the parts belong to $\{1,2,4,6,8,\dots\}$. Moreover, χ transports the number of columns minus one into the number of parts different from 1 in the composition.*

Proof. We can easily observe that the image by χ of a triangular dcc-polymomino of area $n+1$ is a composition in $\mathcal{CP}_n$. Moreover, if this polyomino P has $k+1$ columns, then $\phi(P)$ has exactly k parts different to one. Conversely, any composition in $\mathcal{CP}_n$ with k parts different from 1 can be uniquely decomposed into one of the following forms:

(i) the empty composition () whenever $n=0$;
(ii) the composition (1);
(ii) $(1,1,c_1,\dots,c_\ell)$, $\ell \geq 0$, where $(c_1,\dots,c_\ell) \in \mathcal{CP}_{n-2}$ with k parts different from 1;
(iv_a) $(1,a,c_1,\dots,c_\ell)$, $\ell \geq 0$, $a \geq 2$, where $(c_1,\dots,c_\ell) \in \mathcal{CP}_{n-a-1}$ with $k-1$ parts different from 1;
(iv_b) $(a,c_1,\dots,c_\ell)$, $\ell \geq 0$, $a \geq 2$, where $(c_1,\dots,c_\ell) \in \mathcal{CP}_{n-a}$ with $k-1$ parts different from 1;

Therefore, the set of triangular dcc-polyominoes of area $n+1$ and the set $\mathcal{CP}_n$ have the same recursive description, which ensures that χ is a bijection that transports the number of columns minus one into the number of parts different from 1. □

Let $t_{n,k}$ be the number of triangular dcc-polyominoes of area n with exactly k columns. Notice that an immediate consequence of the recursive decomposition of a dcc-polyomino is the recursive formula $t_{n,1}=1$ for $n \geq 1$, and for $n \geq 3$, $k \geq 2$,

$$t_{n,k} = t_{n-2,k} + \sum_{\ell=2}^{n-1} t_{n-\ell,k-1}.$$

As a byproduct of the bijection given in Theorem 3.2 we give a closed form for $t_{n,k}$.

Theorem 3.3. *If $n \geq k \geq 1$, then*

$$t_{n,k} = \sum_{r=\lfloor k/2 \rfloor}^{\lfloor (n-1)/2 \rfloor} \binom{n-2-2r+k}{k-1}\binom{r-1}{r-k+1}.$$

Proof. Due to the bijection χ defined previously, $t_{n,k}$ corresponds to the number of compositions of $n-1$ with parts in $\{1,2,4,6,8,\ldots\}$, and where exactly $k-1$ parts are different from 1. Such a composition c can be uniquely obtained from a composition of r, $k-1\leq r\leq n-1$, with $k-1$ parts and where all parts lie in $\{2,4,6,8,\ldots\}$, by adding $(n-1-r)$ parts 1 in the right places. Since there are $\binom{n-r-1+k-1}{k-1}=\binom{n-r+k-2}{k-1}$ ways for adding these parts into k places (this is the number of ways of choosing $k-1$ parts among $n-1-r+k-1$ parts), we obtain

$$t_{n,k}=\sum_{r=k-1}^{n-1}\binom{n-2-r+k}{k-1}b_{r,k-1},$$

where $b_{r,k-1}$ is the number of compositions of r having all its $(k-1)$ parts lying into $\{2,4,6,8,\ldots\}$.

From the definition of the sequence $b_{r,s}$ and for a given s, we obtain the following expression for its generating function :

$$\sum_{r\geq 0}b_{r,s}x^r=(x^2+x^4+x^6+\cdots)^s=x^{2s}\left(\frac{1}{1-x^2}\right)^s.$$

From the equality $1/(1-x)^{m+1}=\sum_{\ell=0}^{\infty}\binom{m+\ell}{\ell}x^{\ell}$, we have

$$\sum_{r\geq 0}b_{r,s}x^r=x^{2s}\sum_{\ell=0}^{\infty}\binom{s+\ell-1}{\ell}x^{2\ell}.$$

We obtain $b_{r,s}=0$ when r is odd and $b_{r,s}=\binom{r/2-1}{r/2-s}$ whenever r is even. Therefore, by considering the previous value of $b_{r,s}$ for $s=k-1$, we obtain

$$\begin{aligned}t_{n,k}&=\sum_{r=k-1}^{n-1}\binom{n-2-r+k}{k-1}b_{r,k-1}\\&=\sum_{r=\lfloor k/2\rfloor}^{\lfloor (n-1)/2\rfloor}\binom{n-2-2r+k}{k-1}\binom{r-1}{r-k+1}.\end{aligned}$$ □

From this bijection, we deduce easily one-to-one correspondence between triangular dcc-polyominoes of area n with k columns and consecutive partitions of n with k parts (i.e. partitions where every subset consists of consecutive elements). Indeed, the consecutive partition p associated to the polyomino P is defined from the composition $\chi(P)=(c_1,c_2,\ldots,c_s)$ as follows:

$$\begin{aligned}p=\{1,\ldots,c_1\}\{c_1+1,\ldots,c_1+c_2\}\{c_1+c_2+1,\ldots,c_1+c_2+c_3\}\cdots\\\cdots\{c_1+c_2+\ldots+c_{s-1}+1,\ldots,c_1+c_2+\ldots+c_s\}.\end{aligned}$$

The image of the polyomino represented in Figure 4 is

$$\begin{aligned}&\{1\}\{2\}\{3\}\{4,5,6,7\}\{8\}\{9,10\}\{11\}\{12\}\{13,14,15,16\}\{17\}\{18\}\\&\{19\}\{20,21\}\{22,23,24,25,26,27,28,29\}\{30\}\{31,32\}\{33,34\}\{35,36\}.\end{aligned}$$

Let $\mathcal{B} = [b(n,k)]$ be the Riordan array defined by

$$\mathcal{B} = \left(\frac{1}{1-x}, \frac{x}{(1-x)(1-x^2)} \right),$$

and let $\mathcal{T}$ be the matrix defined by $\mathcal{T} = [t_{n,k}]_{n,k\geq 1}$ (A060098). The first few rows of the matrix $\mathcal{T}$ are

$$\mathcal{T} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 & 0 & 0 \\ 1 & 6 & 3 & 0 & 0 & 0 & 0 \\ 1 & 9 & 8 & 1 & 0 & 0 & 0 \\ 1 & 12 & 16 & 4 & 0 & 0 & 0 \\ 1 & 16 & 30 & 13 & 1 & 0 & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \end{pmatrix}.$$

Notice that the anti-diagonals of the matrix $\mathcal{B}$ are the rows of the matrix $\mathcal{T}$, that is, $t_{n,k} = b(n-k,k)$.

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