GRAND ZIGZAG KNIGHT'S PATHS

JEAN-LUC BARIL, NATHANAËL HASSLER, SERGEY KIRGIZOV AND JOSÉ L. RAMÍREZ

Abstract. We study the enumeration of different classes of grand knight's paths in the plane. In particular, we focus on the subsets of zigzag knight's paths subject to constraints. These constraints include ending at ordinate 0, bounded by a horizontal line, confined within a tube, among other considerations. We present our results using generating functions or direct closed-form expressions. We derive asymptotic results, finding approximations for quantities such as the probability that a zigzag knight's path stays in some area of the plane, or for the average of the final height of such a path. Additionally, we exhibit some bijections between grand zigzag knight's paths and some pairs of compositions.

1. introduction

In combinatorics, the enumeration of different classes of lattice paths according to several parameters is often studied. One of the motivation is to exhibit new one-to-one correspondences between these classes and various objects from other domains, such as computer science, biology, and physics [27]. For instance, they have close connections with RNA structures, pattern-avoiding permutations, directed animals, and other related topics [10, 19, 27]. We refer to [7, 8, 9, 14, 17, 21, 22, 24, 25], and the references therein, for such studies.

Labelle and Yeh [20] investigate knight's paths, that are lattice paths in $\mathbb{N}^2$ starting at the origin and consisting of steps $N=(1,2)$, $\bar{N}=(1,-2)$, $E=(2,1)$, and $\bar{E}=(2,-1)$. Notice that these steps correspond to the right moves of a knight on a chessboard (a right-move of a knight is a jump from left to right from one corner of any two-by-three rectangle to the opposite corner on a chessboard). Recently, in [11], the authors focus on zigzag knight's paths, i.e., knight's paths where the direction of the steps alternate up and down, or equivalently knight's paths avoiding the consecutive patterns NN, NE, EN, EE, $\bar{N}\bar{N}$, $\bar{N}\bar{E}$, $\bar{E}\bar{N}$, $\bar{E}\bar{E}$. They prove that such paths ending on the x-axis with a given number of steps are enumerated by the well-known Catalan numbers (see A000108 in Sloane's On-line Encyclopedia of Integer Sequences [26]), and they exhibit a bijection between these paths and Dyck paths (lattice paths in $\mathbb{N}^2$ starting at the origin, ending on the x-axis and made of steps $U=(1,1)$ and $D=(1,-1)$).

In this work, we extend the aforementioned study by allowing knight's paths to go below the x-axis.

Definition 1.1. A *grand knight's path* is a lattice path in $\mathbb{Z}^2$ starting at the origin consisting of steps $N=(1,2)$, $\bar{N}=(1,-2)$, $E=(2,1)$, and $\bar{E}=(2,-1)$.

The *size* of such a path is the abscissa of its last point. The *empty path* ϵ is a path of size 0. The *height* is the maximal ordinate reached by a point of the path, and the *final height*

Date: February 7, 2024.

Key words and phrases. Knight's path, generating function, kernel method, Fibonacci number, composition of an integer.

is the ordinate of the last point of the path. We also define grand zigzag knight's paths as follow.

Definition 1.2. A *grand zigzag knight's path* is a grand knight's path with the additional property that two consecutive steps cannot be in the same direction, i.e., two consecutive steps cannot be NN, NE, $\bar{N}\bar{N}$, $\bar{N}\bar{E}$, EE, EN, $\bar{E}\bar{E}$, $\bar{E}\bar{N}$.

See Figure 1 for an illustration of two grand knight's paths, with the second one also being a grand zigzag knight's path.

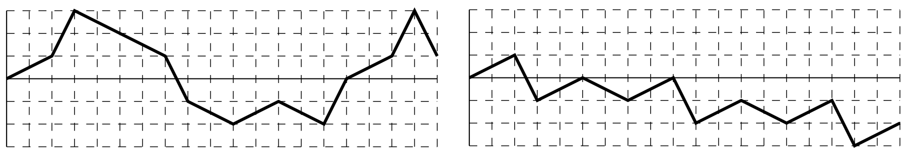

FIGURE 1. On the left: a grand knight's path of size 19, height 3, and final height 1. On the right: a grand zigzag knight's path of size 19, height 1, and final height −2.

The enumeration of grand knight's paths of a given size can be obtained using the work of Banderier and Flajolet [6]. Indeed, it suffices to consider the step polynomial $P(z,u) = zu^2 + \frac{z}{u^2} + \frac{z^2}{u} + z^2u$, where z (resp. u) marks the size (resp. height) of the step, i.e., zu^2, $\frac{z}{u^2}$, $\frac{z^2}{u}$, and z^2u are relative to the steps N, $\bar{N}$, $\bar{E}$, and E, respectively. The Laurent series providing the number of grand knight's paths with respect to the size (marked with z), the number of steps (marked with x), and the final height (marked with u) is given by

$$\frac{1}{1 - xP(z,u)}.$$

Fixing $x = u = 1$, we deduce the generating function $1/(1 - 2z - 2z^2)$ for the number of grand knight's paths with respect to the size. The coefficient of z^n is $\sum_{k=0}^{n+1}\binom{n+1}{2k+1}3^k$, which corresponds to the sequence A002605 in the OEIS [26]. The first values of u_n are

$$1,\ 2,\ 6,\ 16,\ 44,\ 120,\ 328,\ 896,\ 2448,\ 6688,\ 18272.$$

An asymptotic approximation is given by $\sqrt{3}/6 \cdot (\sqrt{3}+1)^{n+1}$.

The generating function for the number of grand knight's paths ending on the x-axis is

$$x\left(\frac{\mathbf{u}_3(x)'}{\mathbf{u}_3(x)} + \frac{\mathbf{u}_4(x)'}{\mathbf{u}_4(x)}\right)\bigg|_{x=1},$$

where $\mathbf{u}_3(x)$ and $\mathbf{u}_4(x)$ are the small roots in u of $1 - xP(u)$ (i.e. $\lim_{x\to 0}\mathbf{u}_i(x) = 0$), and the derivative with respect to x. The bivariate generating function for the number of grand knight's paths of a given size ending at height at least $k \geq 0$ is then

$$W_k(z,u) = \left(\frac{x}{k}\left(\mathbf{u}_3(x)^{-k}\right)' + \frac{x}{k}\left(\mathbf{u}_4(x)^{-k}\right)'\right)\bigg|_{x=1}.$$

In the subsequent sections of the paper, among other things, we re-obtain the aforementioned formulae, although in a different statement (without involving any derivatives).

Recently, Asinowski, Bacher, Banderier, Gittenberger, and Roitner [1, 2, 3, 4, 5] developed a method for solving enumerative problems arising out of paths avoiding consecutive patterns. Grand Zigzag knight's paths could be modeled as grand knight's paths avoiding 8 consecutive patterns of size 2. However, in this paper we present a more specific method, using the point of view of Prodinger [23] based on the kernel method. It consists in studying two generating functions that count the desired paths by distinguishing the direction of the last step (up or down). Then, we provide formulae for different kinds of constraints, give bijections, direct closed-form expressions for the n-th coefficient, and asymptotic results.

Outline of the paper. In Section 2, we apply the kernel method for providing generating function of the number of grand knight's paths with respect to the size and the final height. We deduce asymptotic approximations for the number of grand knight's paths of a given size ending at a non-negative height, and for the expected final height of a grand knight's path ending at positive height. In the following sections, we focus on grand zigzag knight's paths. In Section 3, we give the counterpart of Section 2 for grand zigzag knight's paths. Moreover, we exhibit a bijection between pairs of integer compositions and grand zigzag knight's paths ending at (n, k), where n and k have the same parity. As a byproduct, we obtain a closed form for the number of grand zigzag knight's paths ending at (n, k), and the expected value for the number of steps of a grand zigzag knight's path ending at (n, k); asymptotic approximations are also derived. In Section 4, we consider grand zigzag knight's paths staying in a region delimited by horizontal lines. We make a similar study as for the previous sections, which allows us to estimate asymptotic approximations for the probability that a grand zigzag knight's path chosen uniformly at random among all grand zigzag knight's paths stays above a horizontal line. We end by giving an appendix for the general case of grand zigzag knight's paths staying in a general tube.

2. Unrestricted grand knight's paths

In this section, we count grand knight's paths starting at the origin $(0, 0)$ and ending at (n, k) for $n \geq 0$, $k \in \mathbb{Z}$. Let $\mathcal{H}_{n,k}$ be the set of such paths and let $h_{n,k}$ be its cardinality. Using the symmetry with respect to the x-axis, we obviously have $h_{n,k} = h_{n,-k}$, and then, we focus on non-negative k. For $k \geq 0$, let $h_k = \sum_{n\geq 0} h_{n,k} z^n$ be the generating function for the number of grand knight's paths ending at height k with respect to the size. A non-empty grand knight's path of final height k and size $n \geq 1$ is of one of the following forms (a) $P\bar{N}$ with $P \in \mathcal{H}_{n-1,k+2}$, (b) $P\bar{E}$ with $P \in \mathcal{H}_{n-2,k+1}$, (c) PE with $P \in \mathcal{H}_{n-2,k-1}$, (d) PN with $P \in \mathcal{H}_{n-1,k-2}$. Therefore we easily obtain the following equations after considering the symmetry $h_{-k} = h_k$:

$$h_0 = 1 + 2zh_2 + 2z^2 h_1, \tag{1}$$

$$h_1 = z^2(h_0 + h_2) + z(h_1 + h_3), \tag{2}$$

$$h_k = z^2(h_{k-1} + h_{k+1}) + z(h_{k-2} + h_{k+2}), \text{ for } k \geq 2. \tag{3}$$

Let $H(u, z) = \sum_{k\geq 0} h_k(z)u^k$ be the bivariate generating function for the number of grand knight's paths with respect to the size and the final height. We will write $H(u)$ for short.

Thus, by multiplying (3) by u^k, summing over $k \geq 2$, with $h_0 + h_1 u$ we obtain

$$\begin{aligned} H(u) &= h_0 + h_1 u + z^2 \sum_{k\geq 2}(h_{k-1} + h_{k+1})u^k + z\sum_{k\geq 2}(h_{k-2} + h_{k+2})u^k \\ &= h_0 + h_1 u + z^2\left(u(H(u) - h_0) + \frac{H(u) - h_0 - h_1 u - h_2 u^2}{u}\right) \\ &\quad + z\left(u^2 H(u) + \frac{H(u) - h_0 - h_1 u - h_2 u^2 - h_3 u^3}{u^2}\right). \end{aligned}$$

From (1) and (2) we get $h_2 = -\frac{1}{2z} + \frac{h_0}{2z} - zh_1$ and $h_3 = \frac{1}{2} + (z^2 + \frac{1}{z} - 1)h_1 - (\frac{1}{2} + z)h_0$. By replacing h_2 and h_3 by those expressions in the above equation, and after simplifying, we eventually have

$$H(u)\left(u^2 - zu^4 - z - z^2u - z^2u^3\right) = \frac{u^2}{2} + h_0\left(\frac{u^2}{2} - z - z^2 u\right) + zuh_1(u^2 - 1). \tag{4}$$

We use the kernel method (see [23]) to determine h_0 and h_1. Let $K(u) = u^2 - zu^4 - z - z^2u - z^2u^3$ be the *kernel* of Equation (4), and let u_1 and u_2 be two of the roots of $K(u)$:

$$u_1 = \frac{-z^2 + \sqrt{z^4 + 8z^2 + 4z}}{4z} + \frac{1}{2\sqrt{2}}\sqrt{\frac{z^3 - z\sqrt{z^4 + 8z^2 + 4z} - 4z + 2}{z}}$$

and

$$u_2 = \frac{-z^2 - \sqrt{z^4 + 8z^2 + 4z}}{4z} - \frac{1}{2\sqrt{2}}\sqrt{\frac{z^3 + z\sqrt{z^4 + 8z^2 + 4z} - 4z + 2}{z}}.$$

Notice that $u_1 = \mathbf{u}_1(1)$ and $u_2 = \mathbf{u}_2(1)$ where $\mathbf{u}_1(x)$ and $\mathbf{u}_2(x)$ are the two other roots of the step polynomial $1 - xP(z,u)$ (see Introduction), which implies that $1/(u - u_1)$ and $1/(u - u_2)$ have no power series expansions around $(u, z) = (0, 0)$. This implies that $(u - u_1)(u - u_2)$ must be a factor of the numerator. Then by (4), we have

$$\frac{u_i^2}{2} + h_0\left(\frac{u_i^2}{2} - z - z^2u_i\right) + zu_ih_1(u_i^2 - 1) = 0, \quad \text{for } i = 1, 2.$$

After solving for h_0 and h_1, we obtain

$$h_0 = \frac{1 + u_1u_2}{1 + u_1u_2 - 2z^2(u_1 + u_2) - 2z + 2z(1 - u_1^2 - u_2^2)u_1^{-1}u_2^{-1}}$$

and

$$h_1 = \frac{u_1 + u_2}{1 + u_1u_2}h_0.$$

Then we can deduce $H(u)$ from (4), which provides the following.

Theorem 2.1. *The bivariate generating function for grand knight's paths with respect to the size and the final height is*

$$H(u) = \frac{-h_1u + h_0u_1u_2}{(u - u_1)(u - u_2)}.$$

By decomposing $H(u)$ into simple elements, we deduce a close form for the coefficient $[u^k]H(u)$ of u^k in $H(u)$.

Corollary 2.2. *The generating function for grand knight's paths ending at height k with respect to the size is*

$$[u^k]H(u) = \frac{h_0u_1 - h_1}{u_1 - u_2}u_2^{-k} - \frac{h_0u_2 - h_1}{u_1 - u_2}u_1^{-k}.$$

Here is the matrix $(h_{n,k})_{0\leq k,n\leq 9}$:

$$\left(\begin{array}{ccccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
2 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 2 & 3 & 2 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\
\mathbf{8} & 6 & 1 & 3 & 4 & 3 & 0 & 0 & 1 & 0 & \cdots \\
6 & 12 & 16 & 12 & 3 & 4 & 5 & 4 & 0 & 0 & \cdots \\
44 & 33 & 18 & 21 & 27 & 20 & 6 & 5 & 6 & 5 & \cdots \\
60 & 76 & 95 & 72 & 40 & 34 & 41 & 30 & 10 & 6 & \cdots \\
256 & 210 & 154 & 155 & 177 & 135 & 75 & 52 & 58 & 42 & \cdots \\
460 & 520 & 581 & 480 & 335 & 288 & 299 & 228 & 126 & 76 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right).$$

This corresponds to the table A096608 in [26]. We refer to Figure 2 for the illustration of the 8 grand knight's paths of size four ending on the x-axis.

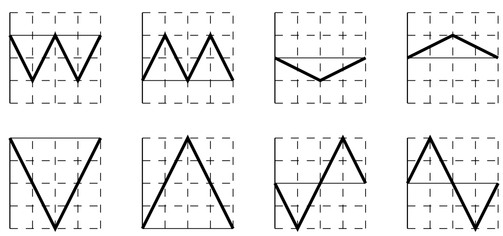

FIGURE 2. The 8 grand knight's paths of size 4 ending on the x-axis.

Corollary 2.3. *The generating function for the total number t_n of grand knight's paths ending at a non-negative height with respect to the size is*

$$H(1) = \frac{h_0u_1u_2 - h_1}{(1-u_1)(1-u_2)}.$$

An asymptotic approximation of t_n is

$$\frac{\sqrt{3}}{12}\left(1+\sqrt{3}\right)^{n+1}.$$

The first terms of t_n, $0 \leq n \leq 12$, are

1, 1, 4, 8, 26, 63, 186, 478, 1352, 3574, 9927, 26640, 73354,

and this sequence does not appear in the OEIS.

Corollary 2.4. *The generating function for the total sum s_n of the final heights in all grand knight's paths of size n ending at a non-negative height is*

$$\frac{\partial}{\partial u}H(u)\mid_{u=1} = \frac{h_0u_1^2u_2 + u_1u_2(h_0u_2 - 2h_0 - h_1) + h_1}{(1-u_1)^2(1-u_2)^2}.$$

An asymptotic approximation of s_n is

$$\frac{(4\sqrt{3}+7)\sqrt{2}\sqrt{137\sqrt{3}-237}}{6}\sqrt{\frac{n}{\pi}}\left(1+\sqrt{3}\right)^n.$$

The first terms of s_n, $0 \leq n \leq 12$, are

$$0,\ 2,\ 5,\ 20,\ 56,\ 180,\ 516,\ 1552,\ 4452,\ 13000,\ 37120,\ 106684,\ 303090,$$

and this sequence does not appear [26].

Corollary 2.5. *The expected final height of a grand knight's path of size n ending at positive height is asymptotically*

$$(5+3\sqrt{3})\sqrt{137\sqrt{3}-237}\sqrt{\frac{2n}{3\pi}}.$$

3. Unrestricted grand zigzag knight's paths

A *grand zigzag knight's path* is a grand knight's path avoiding the consecutive patterns $EN, NE, EE, NN, \bar{E}\bar{N}, \bar{N}\bar{E}, \bar{E}\bar{E}, \bar{N}\bar{N}$.

Let $\mathcal{Z}_{n,k}$ be the set of grand zigzag knight's paths of size n ending at height k, and $\mathcal{Z}^+_{n,k}$ (resp. $\mathcal{Z}^-_{n,k}$) be the subset of $\mathcal{Z}_{n,k}$ of paths starting with E or N (resp. $\bar{E}$ or $\bar{N}$). Note that the symmetry with respect to the x-axis provides a direct bijection between $\mathcal{Z}_{n,k}$ and $\mathcal{Z}_{n,-k}$, and also between $\mathcal{Z}^+_{n,k}$ and $\mathcal{Z}^-_{n,-k}$.

3.1. Enumeration using algebraic method.

Let $\hat{F}(x,u,z)$ be the generating function for the number of grand zigzag knight's paths ending with E or N with respect to the number of steps (marked by x), the final height (marked by u, the power of u is negative for paths ending below x-axis) and the size (final abscissa, marked by z). Denote by $\hat{G}(x,u,z)$ the generating function for grand zigzag knight's paths ending with $\bar{E}$ or $\bar{N}$ and by $\hat{Z}(x,u,z)$ the generating function for all grand zigzag knight paths.

We have the following system of equations:

$$\begin{cases} \hat{F}(x,u,z) &= (1+\hat{G}(x,u,z))(xzu^2+xz^2u), \\ \hat{G}(x,u,z) &= (1+\hat{F}(x,u,z))(xzu^{-2}+xz^2u^{-1}), \\ \hat{Z}(x,u,z) &= 1+\hat{F}(x,u,z)+\hat{G}(x,u,z). \end{cases} \tag{5}$$

The solutions of (5) are rational:

$$\hat{F}(x,u,z) = -\frac{xz\,(u+z)\left(u^2+uxz^2+xz\right)}{u^2x^2z^3+ux^2z^4+ux^2z^2-u+x^2z^3},$$

$$\hat{G}(x,u,z) = -\frac{xz\,(uz+1)\left(u^2xz+uxz^2+1\right)}{u\left(u^2x^2z^3+ux^2z^4+ux^2z^2-u+x^2z^3\right)},$$

$$\hat{Z}(x,u,z) = -\frac{\left(u^2+uxz^2+xz\right)\left(u^2xz+uxz^2+1\right)}{u\left(u^2x^2z^3+ux^2z^4+ux^2z^2-u+x^2z^3\right)}.$$

Since they are Laurent series with infinitely many non zero coefficients for negative exponents of u, there is no simple explicit formula for the coefficient of u^k. This is why we consider only positive heights in what follows.

Fixing $u = 1$ and $x = 1$ in the third equation, we obtain the generating function for all grand zigzag knight's paths of a given size:

$$\hat{Z}(1,1,z) = \frac{1+z+z^2}{1-z-z^2}.$$

Here are the first coefficients of z^n for $0 \le n \le 16$:

$$1, 2, 4, 6, 10, 16, 26, 42, 68, 110, 178, 288, 466, 754, 1220, 1974, 3194.$$

This corresponds to the sequence A128588 in [26], where the n-th term (except the first term which is 1) is twice the Fibonacci number F_n defined by $F_n = F_{n-1} + F_{n-2}$ for $n \ge 2$, with $F_0 = 0$ and $F_1 = 1$.

For $k \in \mathbb{Z}$, let f_k (resp. g_k) be the generating function of the number of grand zigzag knight's paths ending at height k with E or N (resp. $\bar{E}$ or $\bar{N}$), with respect to the size. Due to the symmetry with respect to the x-axis, we have $f_{-k} = g_k$ for $k \neq 0$, and $f_0 = 1 + g_0$. So, we will focus on positive final heights. Let us consider $F(u,z) = \sum_{k\ge 0} f_k(z)u^k$ and $G(u,z) = \sum_{k\ge 0} g_k(z)u^k$ (for short, we will write $F(u)$ and $G(u)$). We easily obtain the following equations, with the convention that the empty path is counted in f_0:

$$f_1 = z^2 + zg_{-1} + z^2 g_0 = z^2 + zf_1 + z^2 g_0 = zf_1 + z^2 f_0, \tag{6}$$

$$f_2 = z + zg_0 + z^2 g_1, \tag{7}$$

$$f_k = zg_{k-2} + z^2 g_{k-1} \quad \text{for } k \ge 3, \tag{8}$$

$$g_k = zf_{k+2} + z^2 f_{k+1} \quad \text{for } k \ge 0. \tag{9}$$

From equations (6-9) we deduce the following functional equations:

$$F(u) = \left(1 + \frac{z^3 u}{1-z}\right) f_0 + zu(z+u)(G(u)+1),$$

$$G(u) = -\left(\frac{z}{u^2} + \frac{z^2}{u} + \frac{z^3}{u(1-z)}\right) f_0 + \left(\frac{z}{u^2} + \frac{z^2}{u}\right) F(u),$$

which provides

$$F(u) = \frac{f_0(-uz^4 + uz^3 + z^4 - uz^2 - z^3 - uz + u) - u^3 z^2 - u^2 z^3 + u^3 z + u^2 z^2}{(-1+z)(u^2 z^3 + uz^4 + z^2 u + z^3 - u)},$$

$$G(u) = \frac{\left(f_0(uz^3 + z^2 - z) - u^2 z^2 - z^3 u + u^2 z + z^2 u - zu - z^2 + u + z\right) z^2}{(-1+z)(u^2 z^3 + uz^4 + z^2 u + z^3 - u)}.$$

Let

$$r := \frac{1 - z^4 - z^2 - \sqrt{z^8 - 2z^6 - z^4 - 2z^2 + 1}}{2z^3},$$

$$s := \frac{1 - z^4 - z^2 + \sqrt{z^8 - 2z^6 - z^4 - 2z^2 + 1}}{2z^3}$$

be the two roots in u of the kernel $u^2z^3 + uz^4 + z^2u + z^3 - u$. Applying the kernel method, we obtain

$$f_0 = \frac{r(z-1)}{z^3(rz^2 + z - 1)} = \frac{(1-z)(-1 + z^2 + z^4 + \sqrt{1 - 2z^2 - z^4 - 2z^6 + z^8})}{z^5(1 - 2z + z^2 - z^4 - \sqrt{1 - 2z^2 - z^4 - 2z^6 + z^8})},$$

and we can state the following after simplifying by the factor $(u - r)$.

Theorem 3.1. *The generating functions $F(u)$, $G(u)$ are given by:*

$$F(u) = \frac{-u^2 - u(r+z) - z^2 f_0 s}{z^2(u-s)} \quad \text{and} \quad G(u) = \frac{-z^2 u - z^2 f_0 s + z^2 s}{z^2(u-s)}.$$

The bivariate generating function $H(u) := F(u)+G(u)$ for the number of grand zigzag knight's paths (ending at a non-negative ordinate) with respect to the size and the final height is

$$H(u) = -\frac{u^2 + u(r+z+z^2) + z^2 s(2f_0 - 1)}{z^2(u-s)}.$$

The generating function for the total number of grand zigzag knight's paths of size n ending at a non negative ordinate is

$$H(1) = -\frac{1 + r + z + z^2 + z^2 s(2f_0 - 1)}{z^2(1-s)}.$$

Here are the first terms of $H(1)$ for $0 \le n \le 16$:

1, 1, 3, 3, 7, 9, 18, 24, 45, 63, 115, 166, 296, 435, 763, 1138, 1973.

Notice that the generating function for the total number of grand zigzag knight's paths is thus

$$2H(1) - (2f_0 - 1) = \frac{1+z+z^2}{1-z-z^2}.$$

As expected (see the beginning of this section), the coefficient of z^n in $2H(1)-(2f_0-1)$ is twice the n-th Fibonacci number (see A128588 in [26]). Indeed, if we look at grand zigzag knight's paths of size n starting with an up-step (so half of the total number of grand zigzag knight's paths of size n), it can start either with E, and then followed by a grand zigzag knight's path of size $n-2$ starting with a down-step, or it can start with N, and then followed by a grand zigzag knight's path of size $n-1$ starting with a down-step. See Table 1 for more values of $|\mathcal{Z}_{n,k}|$. Note that the sequence $(|\mathcal{Z}^+_{2n,0}|)_{n\geq 0}$ corresponds to A051286 in [26].

Corollary 3.2. *The generating function for the number of grand zigzag knight's paths ending at ordinate k with respect to the size is given by:*

$$[u^0]H(u) = 2f_0 - 1, \quad [u^1]H(u) = \frac{r+z+2z^2 f_0}{z^2 s},$$
$$[u^k]H(u) = \frac{r^{k-1}}{z^2}(1 + r(r+z) + 2rz^2 f_0) \quad \text{for } k \ge 2.$$

Corollary 3.3. *An asymptotic approximation for the expected final height of a grand zigzag knight's path of size n ending on or above the x-axis is*

$$\frac{2(\sqrt{5}-2)}{\sqrt{7\sqrt{5}-15}}\sqrt{\frac{n}{\pi}}.$$

This is also the variance (divided by 2) of the final height of a grand zigzag knight's path (not necessarily ending on or above the x-axis) of size n (the expected value of the final height being simply 0 for those paths).

Proof. The generating function for the total number of grand zigzag knight's paths of size n is $H(1)$, and the one for the total sum of final heights over all grand zigzag knight's paths of size n is $\frac{\partial}{\partial u}H(u)\mid_{u=1}$. We compute asymptotics of the coefficients of both those generating functions using singularity analysis (see [15]), and then we take the ratio to obtain the stated result. □

Remark 3.4. From [11, Theorem 1], we deduce that the expected final height of a grand zigzag knight's path of size n and staying above the x-axis is equivalent as $n \to \infty$ to

$$\frac{(5+\sqrt{5})\sqrt{7\sqrt{5}-15}}{20}\sqrt{\pi n}.$$

Thus, the final height of a zigzag knight's path (i.e. a grand zigzag knight's path in $\mathbb{N}^2$), is, in average, ≈ 1.57079633 times the final height of a grand zigzag knight's path in $\mathbb{Z}^2$ ending at non-negative height.

$k \backslash n$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	1	0	2	0	4	2	10	6	22	16	52	44	126	116	306	302
1	0	0	1	2	2	4	4	10	11	26	28	64	71	160	183	402
2	0	1	0	1	0	3	2	7	6	16	18	40	52	100	142	252
3	0	0	0	0	1	0	2	0	6	2	16	8	41	28	107	90
4	0	0	0	0	0	0	0	1	0	3	0	10	2	30	10	85

Table 1. The number of grand zigzag knight's paths from $(0,0)$ to (n,k) for $(n,k) \in [\![0,15]\!] \times [\![0,4]\!]$.

We refer to Figure 3 for the illustration of the 6 grand zigzag knight's paths of size on the x-axis.

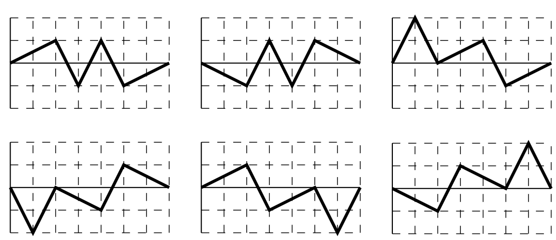

Figure 3. The 6 grand zigzag knight's paths of size 7 ending on the x-axis.

Now, we focus on grand zigzag knight's paths ending on the x-axis with no steps (except the last) ending on the x-axis. Let $\mathcal{B}$ the the set of such paths.

Corollary 3.5. *The generating function $B(z)$ for the number b_n of grand zigzag knight's paths in $\mathcal{B}$ with respect to the size is*

$$B(z) = \frac{2z^5r + 2z^4 - 2z^3 - 3rz + 3r}{r(1-z)}.$$

Proof. Let $\Gamma = 2f_0 - 1$ be the generating function for the grand zigzag knight's paths ending on the x-axis. Each element counted by Γ is either empty or can be uniquely decomposed into a juxtaposition $B_1 \cdots B_k$ of non-empty paths $B_i \in \mathcal{B}$ so that B_iB_{i+1} forms a zigzag, which forces B_{i+1} to start with an up step (resp. down step) if B_i ends with a down step (resp. up step). Therefore, the generating function for B_1 is $B(z)-1$, and the generating function for B_i, $i \geq 2$, is $(B(z)-1)/2$. Thus, we have

$$\Gamma = 1 + (B(z)-1)\sum_{k\geq 0}\left(\frac{B(z)-1}{2}\right)^k = \frac{1+B(z)}{3-B(z)}.$$

We deduce $B(z) = (3\Gamma-1)/(\Gamma+1)$ and the expression of $B(z)$ follows from Corollary 3.2. □

The first terms of b_n for $0 \leq n \leq 22$ are

1, 0, 2, 0, 2, 2, 4, 2, 4, 2, 6, 2, 10, 2, 18, 2, 36, 2, 76, 2, 166, 2, 372.

All the odd terms equal 2 for $2n+1 \geq 5$, because the only paths of size $2n+1$ in $\mathcal{B}$ are $E\bar{N}N\cdots N\bar{N}E$ (or $E\bar{N}N\cdots\bar{N}N\bar{E}$) and its symmetric. Figure 4 shows the 6 grand zigzag knight paths of size 10 in $\mathcal{B}$.

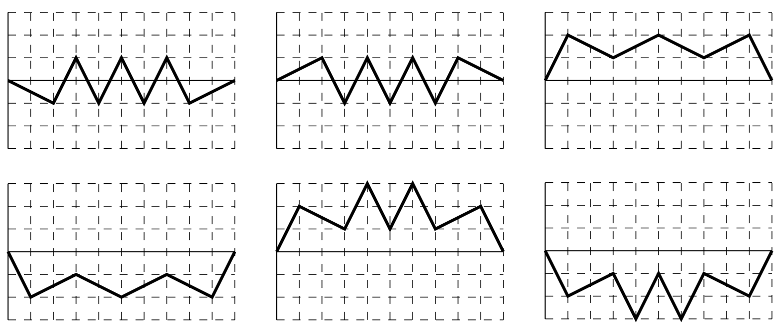

FIGURE 4. The 6 grand zigzag knight's paths of size 10 having exactly the starting and the ending point on the x-axis.

3.2. **A bijective approach.** In this part, we exhibit a bijection between pairs of integer compositions and grand zigzag knight's paths ending at (n,k) where n and k have the same parity.

In [12], Bóna and Knopfmacher provide the following bijection φ between pairs of compositions of n with parts in $\{1,2\}$ that have the same number of parts, and lattice paths starting at $(0,0)$ with size n and steps among $\{(1,0),(2,0),(2,1),(1,-1)\}$. Let $X = (x_1,\ldots,x_k)$ and $Y = (y_1,\ldots,y_k)$ be compositions of n with k parts. The map is defined by $\varphi(X,Y) = \varphi(x_1,y_1)\cdots\varphi(x_k,y_k)$ with

$$\varphi(x_i,y_i) = \begin{cases} (1,0), & \text{if } x_i = y_i = 1,\\ (2,0), & \text{if } x_i = y_i = 2,\\ (2,1), & \text{if } x_i = 2 \text{ and } y_i = 1,\\ (1,-1), & \text{if } x_i = 1 \text{ and } y_i = 2.\end{cases}$$

Here we construct a similar bijection with grand zigzag knight's paths ending on a point whose coordinates have same parity.

Let $\mathcal{C}_{n,m}$ be the set of ordered pairs (X,Y) of compositions of n and m, respectively, into parts equal to 1 or 2, such that X and Y have the same number of parts. Note that $\mathcal{C}_{n,m} = \varnothing$ if $n \leq \lceil m/2\rceil$ or $m \leq \lceil n/2\rceil$.

Lemma 3.6. *If $n \le m$, then*

$$|\mathcal{C}_{m,n}| = |\mathcal{C}_{n,m}| = \sum_{i=0}^{n-\lceil m/2\rceil} \binom{n-i}{i}\binom{n-i}{m-n+i}.$$

Proof. Let us count the compositions of n into parts equal to 1 or 2 with i parts ($n \ge i \ge \lceil n/2\rceil$). Since there are i parts, there are necessarily $n-i$ parts equal to 2, and $2i-n$ parts equal to 1. Then it remains to choose the places of the 1's among those i parts, which can be done in $\binom{i}{2i-n} = \binom{i}{n-i}$ ways. Therefore, if we assume $n \le m$,

$$|\mathcal{C}_{n,m}| = \sum_{i=\lceil m/2\rceil}^{n} \binom{i}{n-i}\binom{i}{m-i} = \sum_{i=0}^{n-\lceil m/2\rceil} \binom{n-i}{i}\binom{n-i}{m-n+i}. \qquad \square$$

Lemma 3.7. *If $n, k \in \mathbb{N}$ with $n = k \pmod 2$, then there is a bijection ϕ between $\mathcal{C}_{\frac{n-k}{2},\frac{n+k}{2}}$ and $\mathcal{Z}^+_{n,k}$, and a bijection ψ between $\mathcal{C}_{\frac{n-k}{2},\frac{n+k}{2}}$ and $\mathcal{Z}^-_{n,k}$.*

Proof. Let $(X,Y) \in \mathcal{C}_{\frac{n-k}{2},\frac{n+k}{2}}$. Let i be the number of parts of X and Y, so that $X = (x_1,\ldots,x_i)$ and $Y = (y_1,\ldots,y_i)$, with $x_j, y_j \in \{1,2\}$. We define $\phi(X,Y)$ as the path $\phi(x_1,y_1)\cdots\phi(x_k,y_k)$, where

$$\phi(x_j,y_j) = \begin{cases} E\bar{E}, & \text{if } x_j = y_j = 2,\\ N\bar{N}, & \text{if } x_j = y_j = 1,\\ N\bar{E}, & \text{if } x_j = 1 \text{ and } y_j = 2,\\ E\bar{N}, & \text{if } x_j = 2 \text{ and } y_j = 1.\end{cases}$$

Note that the size of $\phi(x_j,y_j)$ is x_j+y_j, so the size of $\phi(X,Y)$ is $\sum_{j=1}^{i}(x_j+y_j) = n$. The final height of $\phi(x_j,y_j)$ is $y_j - x_j$, and consequently, the final height of $\phi(X,Y)$ is $\sum_{j=0}^{i}(y_j-x_j) = k$. Therefore, ϕ maps into $\mathcal{Z}^+_{n,k}$. The inverse of ϕ is easy to obtain, since each path of $\mathcal{Z}^+_{n,k}$ can be seen as a path of steps belonging to $\{E\bar{E}, N\bar{N}, N\bar{E}, E\bar{N}\}$. Thus, ϕ is a bijection between $\mathcal{C}_{n,m}$ and $\mathcal{Z}^+_{n,k}$. The bijection ψ can be constructed similarly since each path of $\mathcal{Z}^-_{n,k}$ can be seen as a path of steps belonging to $\{\bar{E}E, \bar{N}N, \bar{E}N, \bar{N}E\}$. $\square$

Example 3.8. If we consider the compositions

$$X = (2,2,2,1,1,1,1,2,1) \quad \text{and} \quad Y = (1,2,1,2,2,1,2,1,2),$$

then $(X,Y) \in \mathcal{C}_{13,14}$ is mapped to the path $E\bar{N}E\bar{E}E\bar{N}N\bar{E}N\bar{E}N\bar{N}N\bar{E}E\bar{N}N\bar{E} \in \mathcal{Z}^+_{27,1}$, see Figure 5.

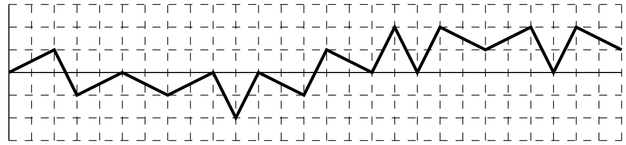

FIGURE 5. The path $\phi(X,Y)$ is a grand zigzag knight's path of size 27 and final height 1.

Theorem 3.9. *If* $n = k \pmod 2$*, with* $(n,k) \neq (0,0)$*, then*

$$|\mathcal{Z}_{n,k}| = 2\sum_{i=0}^{\frac{n-|k|}{2}} \binom{\frac{n-|k|}{2} - i}{i}\binom{\frac{n-|k|}{2} - i}{|k| + i}.$$

If $n \neq k \pmod 2$*, then*

$$|\mathcal{Z}_{n,k}| = |\mathcal{Z}^+_{n-1,k-2}| + |\mathcal{Z}^+_{n-2,k-1}| + |\mathcal{Z}^-_{n-1,k+2}| + |\mathcal{Z}^-_{n-2,k+1}|,$$

where each of those four terms can be expressed with the above formula (without the factor 2).

Proof. By Lemma 3.7, if $n = k \pmod 2$, then we have $|\mathcal{Z}_{n,k}| = |\mathcal{Z}^+_{n,k}| + |\mathcal{Z}^-_{n,k}| = 2|\mathcal{C}_{\frac{n-k}{2},\frac{n+k}{2}}|$. We conclude using Lemma 3.6. If $n \neq k \pmod 2$, then we have

$$\mathcal{Z}_{n,k} = \mathcal{Z}_{n-1,k-2} \cdot N \cup \mathcal{Z}_{n-2,k-1} \cdot E \cup \mathcal{Z}_{n-1,k+2} \cdot \bar{N} \cup \mathcal{Z}_{n-2,k+1} \cdot \bar{E},$$

which completes the proof. □

Remark 3.10. When $n = k \pmod 2$, $|\mathcal{Z}_{n,k}|$ is even (except for $(n,k) = (0,0)$). Indeed, since each path of $\mathcal{Z}_{n,k}$ has an even number of steps, we have a bijection from $\mathcal{Z}^+_{n,k}$ to $\mathcal{Z}^-_{n,k}$, induced by

$$\begin{array}{ccc} N\bar{N} & \to & \bar{N}N \\ E\bar{E} & \to & \bar{E}E \\ N\bar{E} & \to & \bar{E}N \\ E\bar{N} & \to & \bar{N}E. \end{array}$$

3.3. **Step number of a grand zigzag knight's path.** For $n/2 \leq i \leq n$, let $\mathcal{Z}^i_{n,k}$ be the subset of $\mathcal{Z}_{n,k}$ consisting of paths with i steps (see Section 3 for a definition of $\mathcal{Z}_{n,k}$). We define similarly $\mathcal{Z}^{i,+}_{n,k}$ and $\mathcal{Z}^{i,-}_{n,k}$. Note that $|\mathcal{Z}^i_{n,k}| = 0$ if $i \neq n-k \pmod 2$.

Theorem 3.11. *For* $n = k \pmod 2$ *and* i *even,*

$$|\mathcal{Z}^{i,+}_{n,k}| = |\mathcal{Z}^{i,-}_{n,k}| = \binom{i/2}{\frac{n-i-k}{2}}\binom{i/2}{\frac{n-i+k}{2}}.$$

For $n \neq k \pmod 2$ *and* i *odd,*

$$|\mathcal{Z}^{i,+}_{n,k}| = \binom{\frac{i+1}{2}}{\frac{n-k-i}{2} + 1}\binom{\frac{i-1}{2}}{\frac{n+k-i}{2} - 1},$$

$$|\mathcal{Z}^{i,-}_{n,k}| = \binom{\frac{i-1}{2}}{\frac{n-k-i}{2} - 1}\binom{\frac{i+1}{2}}{\frac{n+k-i}{2} + 1}.$$

Proof. Let us first assume $n = k \pmod 2$ and i is even. Let $\mathbf{e}, \mathbf{n}, \bar{\mathbf{e}}, \bar{\mathbf{n}}$ be the numbers of steps $E, N, \bar{E}, \bar{N}$ of a path in $\mathcal{Z}^{i,+}_{n,k}$ (or $\mathcal{Z}^{i,-}_{n,k}$). Then we have the following equations:

$$\begin{cases} \mathbf{e} + \bar{\mathbf{e}} & = & n - i, \\ \mathbf{n} + \bar{\mathbf{n}} & = & 2i - n, \\ \mathbf{e} + \mathbf{n} & = & i/2, \\ \bar{\mathbf{e}} + \bar{\mathbf{n}} & = & i/2, \\ \mathbf{e} + 2\mathbf{n} - \bar{\mathbf{e}} - 2\bar{\mathbf{n}} & = & k. \end{cases}$$

The associated matrix has rank 4, thus we deduce the unique solution

$$(\mathbf{e},\mathbf{n},\bar{\mathbf{e}},\bar{\mathbf{n}}) = \left(\frac{n-i-k}{2}, \frac{2i-n+k}{2}, \frac{n-i+k}{2}, \frac{2i-n-k}{2}\right).$$

Then, it remains to choose the positions of the N's among the up-steps, and the positions of the $\bar{N}$'s among the down-steps. This eventually yields the desired formula. When $n \neq k \pmod 2$ and i is odd, the third and the fourth equations become

$$\begin{cases} \mathbf{e}+\mathbf{n} &= (i+1)/2, \\ \bar{\mathbf{e}}+\bar{\mathbf{n}} &= (i-1)/2, \end{cases}$$

for $\mathcal{Z}_{n,k}^{i,+}$, and

$$\begin{cases} \mathbf{e}+\mathbf{n} &= (i-1)/2, \\ \bar{\mathbf{e}}+\bar{\mathbf{n}} &= (i+1)/2, \end{cases}$$

for $\mathcal{Z}_{n,k}^{i,-}$. We then conclude similarly. □

Corollary 3.12. *If $n = k \pmod 2$ with $(n,k) \neq (0,0)$,*

$$|\mathcal{Z}_{n,k}| = 2\sum_{\substack{i=0\\ i\ even}}^{n-k} \binom{i/2}{\frac{n-i-k}{2}}\binom{i/2}{\frac{n-i+k}{2}}.$$

If $n \neq k \pmod 2$,

$$|\mathcal{Z}_{n,k}| = \sum_{\substack{i=0\\ i\ odd}}^{n-k+1} \left[\binom{\frac{i+1}{2}}{\frac{n-k-i}{2}+1}\binom{\frac{i-1}{2}}{\frac{n+k-i}{2}-1} + \binom{\frac{i-1}{2}}{\frac{n-k-i}{2}-1}\binom{\frac{i+1}{2}}{\frac{n+k-i}{2}+1}\right].$$

Remark 3.13. Note that we obtain the same formula as the one in Theorem 3.9. Also, we now have a formula for $|\mathcal{Z}_{n,k}|$ when $n \neq k \pmod 2$ involving only 2 sums, instead of 4 as stated in Theorem 3.9.

Corollary 3.14. *The expected value for the number of steps of a grand zigzag knight's path ending at (n,k) is*

$$\frac{2}{|\mathcal{Z}_{n,k}|}\sum_{\substack{i=0\\ i\ even}}^{n-k} i\binom{i/2}{\frac{n-i-k}{2}}\binom{i/2}{\frac{n-i+k}{2}} \quad \textit{if} \quad n = k \pmod 2 \quad \textit{and}$$

$$\frac{1}{|\mathcal{Z}_{n,k}|}\sum_{\substack{i=0\\ i\ odd}}^{n-k} i\left[\binom{\frac{i+1}{2}}{\frac{n-k-i}{2}+1}\binom{\frac{i-1}{2}}{\frac{n+k-i}{2}-1} + \binom{\frac{i-1}{2}}{\frac{n-k-i}{2}-1}\binom{\frac{i+1}{2}}{\frac{n+k-i}{2}+1}\right],$$

otherwise.

Theorem 3.15. *An asymptotic approximation for the expected number of steps of a grand zigzag knight's path ending on the x-axis of size $2n$ (which is also the expected number of parts of a pair of compositions of n with parts in $\{1,2\}$ and having the same number of parts) is*

$$\frac{1+\sqrt{5}}{2\sqrt{5}}(2n).$$

Proof. By Corollary 3.14 and Theorem 3.9, it suffices to estimate $\sum_{i=0}^{n}\binom{i}{n-i}^2$ and $\sum_{i=0}^{n} i\binom{i}{n-i}^2$ (respectively A051286 and A182879). Their generating functions are respectively $1/\sqrt{1-2z-z^2-z^3+z^4}$ and $z(1+2z^2-z^3)/((1-3z+z^2)(1+z+z^2))^{3/2}$. We conclude using singularity analysis. □

Conjecture 3.16. An asymptotic approximation for the expected number of steps of a grand zigzag knight's path ending on the x-axis of size n is

$$\frac{1+\sqrt{5}}{2\sqrt{5}}n.$$

4. Bounded grand zigzag knight's paths

In this section we focus on grand zigzag knight's paths staying in some regions delimited by horizontal lines.

4.1. Grand zigzag knight's paths staying above a horizontal line.

First, we consider paths that stay above the line $y = -m$ for a given integer $m \geq 0$. By symmetry, this is equivalent to count grand zigzag knight's paths staying below the line $y = +m$. Note that the case $m = 0$ has been done in [11], thus we will assume $m \geq 1$.

Remark 4.1. When $n = k \pmod 2$, there is a bijection between those paths ending at (n, k) and pairs of compositions $(X, Y) \in \mathcal{C}_{\frac{n-k}{2},\frac{n+k}{2}}$ such that for all j,

$$-m \leq \sum_{i=1}^{j}(y_i - x_i)$$

(see the bijections ϕ and ψ used in Lemma 3.7).

Figure 6 shows an example of a grand zigzag knight's path above the line $y = -3$, with size 27, and ending at height -2.

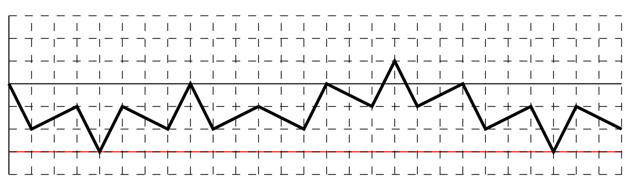

FIGURE 6. A grand zigzag knight's path staying above the line $y = -3$, with size 27, and ending at height -2.

As in the previous section, for $k \geq 0$, let f_{-m+k} (resp. g_{-m+k}) be the generating function of the number of grand zigzag knight's paths staying above the ordinate $-m$ and ending at height $-m + k$ with an up-step N or E (resp. with a down-step $\bar{N}$ or $\bar{E}$). Then we easily obtain the following equations

$$f_{-m} = 0,\ f_{-m+1} = \mathbb{1}_{[m=1]} + z^2 g_{-m}, \tag{10}$$

$$f_{-m+k} = zg_{-m+k-2} + z^2 g_{-m+k-1} + \mathbb{1}_{[k=m]} + \mathbb{1}_{[k=m+1]}z^2 + \mathbb{1}_{[k=m+2]}z \quad \text{for } k \geq 2 \tag{11}$$

and

$$g_{-m+k} = zf_{-m+k+2} + z^2 f_{-m+k+1} \text{ for } k \geq 0, \tag{12}$$

where $\mathbb{1}_{[a=b]}$ is the constant function 1 if $a = b$, and the constant function 0 otherwise.

Setting $F(u) = \sum_{k\geq 0} f_{-m+k}u^k$ and $G(u) = \sum_{k\geq 0} g_{-m+k}u^k$, we obtain

$$F(u) = zu^{m+1}(z+u) + u^m + zu(z+u)G(u),$$
$$G(u) = -\frac{z}{u}f_{-m+1} + \left(\frac{z}{u^2} + \frac{z^2}{u}\right)F(u).$$

Using the kernel method (the roots r and s of the kernel are the same as for Section 3.1), we find $f_{-m+1} = (1+zr)r^{m-1} + \frac{r^{m+1}}{z}$.

Remark 4.2. From equation (10), we get $g_{-m} = \frac{1+zr}{z^2}r^{m-1} + \frac{r^{m+1}}{z^3} - \mathbb{1}_{[m=1]}\frac{1}{z^2}$. This is the same expression as the generating function for the total number of grand zigzag knight's paths ending at height m and staying above the x-axis, see [11, Theorem 1]. Indeed, reading the paths backwards, we get a bijection between those ending at height m and staying above $y = 0$, and those ending at height $-m$ and staying above $y = -m$, see Figure 7.

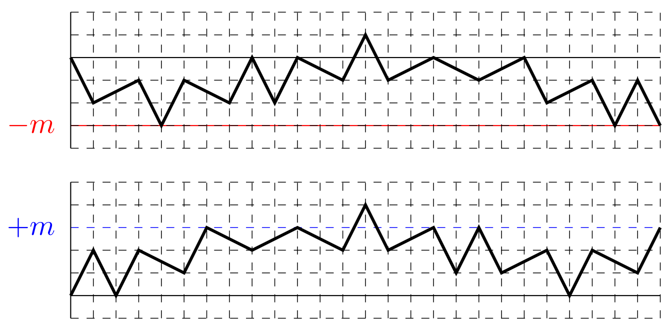

FIGURE 7. There is a bijection between grand zigzag knight's paths staying above $y = -m$ and ending at height $-m$, and grand zigzag knight's paths staying above $y = 0$ and ending at height k.

Theorem 4.3. *The bivariate generating functions for the number of grand zigzag knight's paths staying above $y = -m$ and ending with respectively an up-step and a down-step with respect to the size and the final height are*

$$F(u) = -\frac{u(u^m(1+z^2u+zu^2) - r^{m-1}z(u+z)(z+rz^2+r^2))}{z^3(u-r)(u-s)},$$
$$G(u) = \frac{r^{m-1}(z+r^2+z^2r) - zu^{m-1}(1+zu)(1+z^2u+zu^2)}{z^3(u-r)(u-s)}.$$

Setting $H(u) := F(u) + G(u)$, the generating function for the total number of grand zigzag knight's paths staying above $y = -m$ with respect to the size is

$$H(1) = \frac{zr^{m-1} + z^2r^m + r^{m+1} - z^2 - z - 1}{z^2+z-1}.$$

Here are the first terms of $H(1)$ when $m = 2$ and $0 \leq n \leq 15$:

1, 2, 4, 6, 9, 15, 23, 38, 58, 95, 147, 239, 373, 603, 947, 1525.

Remark 4.4. $H(1)$ converges as $m \to \infty$ to $(1+z+z^2)/(1-z-z^2)$, the generating function for the total number of grand zigzag knight's paths (see the beginning of Section 3). Moreover, the valuation of r is 3, thus we can estimate the rate of convergence:

$$\mathrm{val}\left(H(1) - \frac{1+z+z^2}{1-z-z^2}\right) = 3m-2.$$

In particular, every grand zigzag knight's path of size $\le 3m-3$ stays above $y=-m$, and we know that the path $(\bar{N}E)^{m-1}\bar{N}$ has size $3m-2$ and goes below $y=-m$.

Corollary 4.5. *For $m \ge 0$, the probability that a grand zigzag knight's path chosen uniformly at random among all grand zigzag knight's paths of size n stays above the line $y=-m$ is asymptotically $c_m/\sqrt{n}$, with*

$$c_m = \begin{cases} \frac{2+\sqrt{5}}{2}\sqrt{\frac{7\sqrt{5}-15}{\pi}}, & \text{if } m=0,\\ \frac{4m+3-\sqrt{5}}{4(\sqrt{5}-2)}\sqrt{\frac{7\sqrt{5}-15}{\pi}}, & \text{if } m \ge 1.\end{cases}$$

Proof. Using basic singularity analysis, since the generating function for the total number of grand zigzag knight's paths with respect to the size is $(1+z+z^2)/(1-z-z^2)$, we deduce that the asymptotic for the total number $a(n)$ of grand zigzag knight's paths of size n is $\frac{4}{\sqrt{5}(\sqrt{5}-1)}\alpha^{-n}$, with $\alpha = \frac{\sqrt{5}-1}{2}$. Now we study $H(1)$ (see Theorem 4.3). It turns out that the main singularity of this function (the one with smallest modulus) is also α. Using the following approximation near α:

$$\frac{r^j}{z-\alpha} = \frac{1}{z-\alpha} + \frac{j\sqrt{10-4\sqrt{5}}}{\sqrt{5}-2}(z-\alpha)^{-1/2} + O(1),$$

we deduce

$$H(1) \sim \frac{(2m+1-\alpha)\sqrt{10-4\sqrt{5}}}{\sqrt{5}(\sqrt{5}-2)}(z-\alpha)^{-1/2}$$

as $z \to \alpha$. We thus have the following approximation for the number $a_m(n)$ of grand zigzag knight's paths of size n and staying above $y=-m$ (see for instance [15, Chapter VI]) as $n \to \infty$, when $m \ge 1$:

$$a_m(n) \sim \frac{(2m+1-\alpha)\sqrt{10-4\sqrt{5}}}{\sqrt{5\alpha}(\sqrt{5}-2)}\frac{\alpha^{-n}}{\sqrt{\pi n}}.$$

By taking the quotient, we obtain the asymptotic for the desired probability as $n \to \infty$:

$$\frac{a_m(n)}{a(n)} \sim \frac{4m+3-\sqrt{5}}{4(\sqrt{5}-2)}\sqrt{\frac{7\sqrt{5}-15}{\pi n}}.$$

When $m=0$, we use similarly the generating function from [11, Corollary 2]. □

Remark 4.6. We can deduce from the previous proof (going one order further) that the asymptotic probability that a grand zigzag knight's path of size n chosen uniformly at random has minimal height $-m$ is

$$\frac{a_m(n)-a_{m-1}(n)}{a(n)} \sim \begin{cases} \frac{2+\sqrt{5}}{2}\sqrt{\frac{7\sqrt{5}-15}{\pi n}}, & \text{if } m=0,\\ \frac{5+3\sqrt{5}}{4}\sqrt{\frac{7\sqrt{5}-15}{\pi n}}, & \text{if } m=1,\\ (2+\sqrt{5})\sqrt{\frac{7\sqrt{5}-15}{\pi n}}, & \text{if } m \ge 2.\end{cases}$$

In particular, asymptotically, every minimal height ≤ -2 has same probability. By symmetry, this is exactly the same for maximal heights.

4.2. **Grand zigzag knight's paths staying in a symmetric tube.** Here we count grand zigzag knight's paths that stay between the lines $y = -m$ and $y = +m$, for $m \geq 1$. See [18, 13, 16] for previous studies of lattices paths staying in a tube. Figure 8 shows an example of a grand zigzag knight's path staying between the lines $y = -2$ and $y = 2$, with size 27 and ending at height 0.

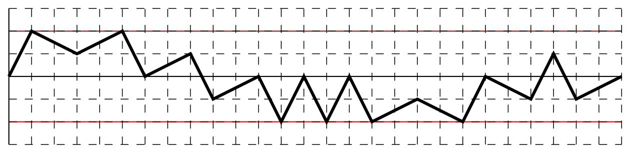

FIGURE 8. A grand zigzag knight's path staying between the lines $y = -2$ and $y = +2$, with size 27 and ending at height 0.

We use the same notation as in the previous subsection and we set $\mathbb{1}_{[a=b]} = 1$ if $a = b$, and 0 otherwise. Note that $f_{-k} = g_k$ for all $-m \leq k \leq m$. We obtain the following equations:

$$f_{-m} = g_m = 0, \; f_{-m+1} = \mathbb{1}_{[m=1]} + z^2 g_{-m}, \; g_{m-1} = z^2 f_m, \tag{13}$$

$$f_{-m+k} = z g_{-m+k-2} + z^2 g_{-m+k-1} + \mathbb{1}_{[k=m]} + \mathbb{1}_{[k=m+1]} z^2 + \mathbb{1}_{[k=m+2]} z \tag{14}$$

for $2 \leq k \leq 2m$, and

$$g_{-m+k} = z f_{-m+k+2} + z^2 f_{-m+k+1} \quad \text{for } 0 \leq k \leq 2m-2. \tag{15}$$

Remark 4.7. We actually have a finite linear system with unknowns f_k, g_k, with polynomial coefficients. Therefore, we expect rational solutions and in particular, the generating function for the total number of grand zigzag knight's paths staying between $y = -m$ and $y = +m$ is rational.

Setting $F(u) = \sum_{k=0}^{2m} f_{-m+k} u^k$ and $G(u) = \sum_{k=0}^{2m} g_{-m+k} u^k$ we obtain

$$F(u) = u^m - z u^{2m+1} (f_{-m+1} - \mathbb{1}_{[m=1]}) + zu(z+u)(G(u) + u^m),$$
$$G(u) = -\frac{z}{u} f_{-m+1} + \left(\frac{z}{u^2} + \frac{z^2}{u} \right) F(u).$$

The kernel method then leads to

$$f_{-m+1} = \frac{r^m (1 + r z^2 + r^2 z (1 + \mathbb{1}_{[m=1]}))}{z(rz + z^2 + r^{2m+1})}.$$

Theorem 4.8. *The generating functions $F(u)$ and $G(u)$ counting the number of grand zigzag knight's paths staying between $y = -m$ and $y = +m$ with respect to the size and the final height are given by:*

$$F(u) = -\frac{(z(\mathbb{1}_{[m=1]} - f_{-m+1}) u^{2m+1} - z^2 (u+z) f_{-m+1} + u^m (1 + z^2 u + z u^2)) u}{z^3 (u-r)(u-s)},$$

$$G(u) = -\frac{z(1+zu)(\mathbb{1}_{[m=1]} - f_{-m+1}) u^{2m+1} - u f_{-m+1} + u^m (1+zu)(1 + z^2 u + z u^2)}{z^2 (u-r)(u-s)}.$$

Setting $H(u) = F(u) + G(u)$, the generating function for the total number of grand zigzag knight's paths staying between $y = -m$ and $y = +m$ is

$$H(1) = \frac{(2f_{-m+1} - \mathbb{1}_{[m=1]})z - z^2 - z - 1}{z^2 + z - 1}.$$

Remark 4.9. As a consequence of Remark 4.7, f_{-m+1} and $H(1)$ are rational. As in Remark 4.4, $H(1)$ converges as $m \to \infty$ to $(1 + z + z^2)/(1 - z - z^2)$, the generating function for the total number of grand zigzag knight's paths (see the beginning of Section 3).

Example 4.10. For $m = 1$, the generating function for the number of grand zigzag knight's paths staying between $y = -1$ and $y = +1$ and ending on the x-axis is $(-z^4+z-1)/(z^4+z-1)$. Its first terms for $0 \le n \le 18$ are:

$$1, 0, 0, 0, 2, 2, 2, 2, 4, 6, 8, 10, 14, 20, 28, 38, 52, 72, 100.$$

The term of order $2n+4$ corresponds to $2\times$A052535(n)= $2\sum_{k=0}^{\lfloor n/2\rfloor}\binom{2n-3k}{k}$. The term of order $2n+3$ corresponds to $2\times$A158943(n).

There are explicit bijections between such paths from $(0, 0)$ to $(2n+4, 0)$ and starting with E and

- tilings of a $2 \times 2n$ rectangle with 1×2 and 4×1 tiles,
- pairs $X = (x_1, \dots, x_k), Y = (y_1, \dots, y_k)$ of compositions of $n + 2$ with same number of parts, with all parts in $\{1, 2\}$ and such that for all $1 \le j \le k$,

$$\left|\sum_{i=1}^{j}(x_i - y_i)\right| \le 1.$$

- compositions of n with parts in $\{2, 1, 3, 5, 9, 11, \dots\}$. Indeed, a grand zigzag knight's path of size $2n + 4$ staying between $y = -1$ and $y = +1$ can be uniquely decomposed as $E\mathbf{P}\bar{E}$ with $\mathbf{P}$ of size $2n$ and having its steps in $\{\bar{E}E\} \cup \bigcup_{k\ge0}\{\bar{N}(E\bar{E})^kN\}$. If $\mathbf{P} = S_1 \cdots S_j$, then we set $\Phi(E\mathbf{P}\bar{E}) = (\Phi(S_1), \dots, \Phi(S_j))$ with $\Phi(\bar{E}E) = 2$ and $\Phi(\bar{N}(E\bar{E})^kN) = 2k + 1$.

Appendix A. Grand zigzag knight's paths staying in a general tube

Here we count grand zigzag knight's paths that stay between the lines $y = -m$ and $y = +M$ for $m, M \ge 0$ with $M \ge m$. In particular we always have $M \ge 1$ (the case $m = M = 0$ being trivial). Figure 9 shows an example of a grand zigzag knight's path staying between the lines $y = -2$ and $y = +3$, with size 27 and ending at height 0.

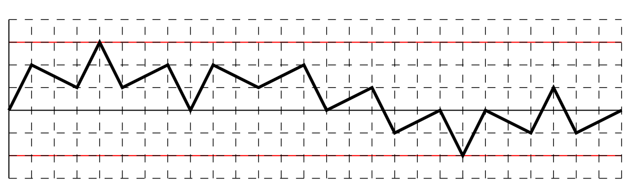

Figure 9. A grand zigzag knight's path staying between the lines $y = -2$ and $y = +3$, with size 27 and ending at height 0.

Remark A.1. When $n = k \pmod 2$, there is a bijection between those paths ending at (n,k) and pairs of compositions $(X,Y) \in \mathcal{C}_{\frac{n-k}{2},\frac{n+k}{2}}$ such that for all $1 \le j \le k$,

$$-m \le \sum_{i=1}^{j}(y_i - x_i) \le M$$

(see bijections ϕ and ψ in Lemma 3.7).

With a similar reasoning as in Section 4, we obtain the following equations:

$$F(u) = u^m - z^3u^{m+M+1}f_M + zu(z+u)(G(u)+u^m),$$
$$G(u) = -\frac{z}{u}f_{-m+1} + \left(\frac{z}{u^2}+\frac{z^2}{u}\right)(F(u) - \mathbb{1}_{[m=0]}).$$

Remark A.2. As in Remark 4.7, the solutions are rational.

Using the kernel method, we find:

$$f_{-m+1} = \frac{(1+sz)\left((\mathbb{1}_{[m=0]} - s^m(1+sz^2+s^2z))r^{m+M+2} + s^{m+M+1}(r^{m+1}(1+rz^2+r^2z) - \mathbb{1}_{[m=0]}z^2(z+r)(1+zr))\right)}{s^{m+M}z^2(r+z(2+sz)) - r^{m+M+1}},$$

$$f_M = -\frac{r^m + z(z+r)\left(r^{m+1} - z\left(s^{m-1}(1+sz)(1+s^2z+sz^2) - \mathbb{1}_{[m=0]}(r-s)\right)\right)}{z^3(z^2(r+z)(1+sz)s^{m+M} - r^{m+M+1})}.$$

Theorem A.3. *The generating functions $F(u)$ and $G(u)$ counting the number of grand zigzag knight's paths staying between $y = -m$ and $y = +M$ with respect to the size and the final height are given by:*

$$F(u) = \frac{z^3u^{m+M+2}f_M + uz^2(u+z)f_{-m+1} + \mathbb{1}_{[m=0]}z^2(u+z)(1+uz) - u^{m+1}(1+uz^2+u^2z)}{z^3(u-r)(u-s)},$$
$$G(u) = \frac{z^3u^{m+M+1}(1+uz)f_M + uf_{-m+1} + \mathbb{1}_{[m=0]}(1+uz) - u^m(1+uz)(1+uz^2+u^2z)}{uz^2(u-r)(u-s)}.$$

We set $H(u) := F(u) + G(u)$.

Corollary A.4. *The generating function for the number of grand zigzag knight's path staying between $y = 0$ and $y = +M$ and ending on the x-axis is $H(0) = \frac{f_1}{z^2}$.*

When $M = 2$, we obtain the first terms, $0 \le n \le 18$,

1, 0, 1, 0, 2, 0, 4, 0, 7, 0, 14, 0, 26, 0, 50, 0, 95, 0, 181,

where the even terms are A052535 (see Example 4.10).

Remark A.5. We have a convergence to the generating function for all grand zigzag knight's path staying above the x-axis and ending on it $A = \frac{r}{z^3}$ (see [11, Corollary 1]) as $M \to \infty$.

Corollary A.6. *The generating function for the total number of grand zigzag knight's paths staying between $y = -m$ and $y = +M$ is*

$$H_{m,M}(z) := H(1) = \frac{(f_{-m+1} + z^2f_M - 1)z + \mathbb{1}_{[m=0]}z(1+z) - 1 - z^2}{z^2+z-1}.$$

As a byproduct and using the inclusion–exclusion principle, the generating function for grand zigzag knight's paths staying between lines $y = -m$ and $y = M$, and reaching them, is $H_{m,M}(z) - H_{m-1,M}(z) - H_{m,M-1}(z) + H_{m-1,M-1}(z)$ when $1 \le m \le M$, and $H_{0,M}(z) - H_{0,M-1}(z)$ when $0 \le M$. Then the generating function grand zigzag knight's paths such that the difference between the maximal and minimal ordinates of a point of the path is exactly k is

$$2\left(H_{0,k}(z)-H_{0,k-1}(z)+\sum_{m=1}^{\lfloor k/2\rfloor}\left(H_{m,k-m}(z)-H_{m-1,k-m}(z)-H_{m,k-m-1}(z)+H_{m-1,k-m-1}(z)\right)\right).$$

Table 2 gives the first coefficients for $k=1,2,3$. The sequences do not appear in [26].

k \ n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	0	0	2	0	2	0	2	0	2	0	2	0	2	0	2	0	2
2	0	2	2	6	6	14	24	30	46	60	88	118	168	228	320	438	610
3	0	0	0	0	2	4	10	16	32	52	94	148	252	392	648	996	1612

TABLE 2. The number of grand zigzag knight's paths such that the difference of the maximal and minimal heights is at most k.

Acknowledgments. This work was supported in part by ANR-22-CE48-0002 funded by l'Agence Nationale de la Recherche and by the project ANER ARTICO funded by Bourgogne-Franche-Comté region (France). José L. Ramírez was partially supported by Universidad Nacional de Colombia, Project No. 57340.

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LIB, Université de Bourgogne, B.P. 47 870, 21078, Dijon Cedex, France
Email address: barjl@u-bourgogne.fr, nathanael.hassler@ens-rennes.fr, sergey.kirgizov@u-bourgogne.fr

Departamento de Matemáticas, Universidad Nacional de Colombia, Bogotá, Colombia
Email address: jlramirezr@unal.edu.co