Gray codes for p-ary Lucas strings

Jean-Luc Baril, Vincent Vajnovszki*
LE2I– CNRS FRE 2309, Université de Bourgogne
B.P. 47 870, 21078 DIJON-Cedex France
e-mail: {barjl}{vvajnov}@u-bourgogne.fr

June 15, 2003

Abstract

We give the first Gray code for the set of p order, length-n Lucas strings, i.e., the set of length-n binary strings with no p consecutive 1s, where strings are regarded circularly. Our Gray code proves also that the p order, n-ary Lucas cube is Hamiltonian iff n is not a multiple of $p+1$, and its second power is always Hamiltonian.

Key words: Gray code, Fibonacci and Lucas string, Lucas cube, Hamiltonian path

1 Introduction

A *k-Gray code* for a set of binary strings $B \subset \{0,1\}^n$ is an *ordered* list $\mathcal{B}$ for B, such that the Hamming distance between any two consecutive strings in $\mathcal{B}$ is at most k.

The set $F_{n,p}$, of p order length-n Fibonacci strings, is the set of length-n binary strings such that there are no p consecutive 1s. The set $L_{n,p}$, of p order length-n Lucas strings, is defined similarly, but the entries of each string in $L_{n,p}$ is regarded circularly, i.e., in $L_{n,p}$ belong all strings in $F_{n,p}$ which do not begin by 1^ℓ and end by 1^m with $\ell + m \geq p$.

A number of papers concerning the Fibonacci and Lucas strings have been published [2, 6, 7, 8, 9, 12, 15]. In the present one we introduce an order relation on $\{0,1\}^n$ which induces a 1-Gray code order on $L_{n,p}$ if $(p+1)$ does not divide n, and a 2-Gray code order if $(p+1)$ divides n. We show also that in the last case there do not exist more restrictive Gray code. Note that this order relation yields a 1-Gray code on the set $F_{n,p}$, of p order length-n Fibonacci strings, see for instance [12].

The remaining of this paper is organized as follows. In Section 2 we prove that a 1-Gray code for $L_{n,p}$ is possible iff $(p+1)$ does not divide n and in Section 3 we give such a Gray code. In Section 4 the interpretation of this results in terms of Lucas cube are presented in and in Section 5 some algorithmic considerations are given.

2 Parity difference relation

For a binary string set B we denote by B' (resp. B'') the subset of B of strings with an odd (resp. even) number of 1s.

*Corresponding author.

Let $G(L_{n,p})$ be the p order n-ary Lucas cube, i.e. the vertices of the graph $G(L_{n,p})$ are the strings of $L_{n,p}$ and two vertices are connected by an edge if their Hamming distance is one. $G(L_{n,p})$ is bipartite, and with the notations above $\{L'_{n,p}, L''_{n,p}\}$ is a bipartition for $L_{n,p}$. No Hamiltonian path is possible in $G(L_{n,p})$ (or equivalently, 1-Gray code for $L_{n,p}$) if $|card(L'_{n,p}) - card(L''_{n,p})| > 1$, i.e., the number of vertices in the two bipartitions differs by more than one.

The main result of this section is Theorem 1. In the following we suppose $p > 1$ and more often we will omit the subscript p for the sets $F_{n,p}$, $L_{n,p}$ and other related thinks.

Lemma 1 *Let $\{\phi_n\}_{n\geq 0}$ $\{\lambda_n\}_{n\geq 0}$ be the parity difference integer sequences corresponding to Fibonacci and Lucas strings defined by*

- $\phi_n = card(F'_n) - card(F''_n)$, *and*
- $\lambda_n = card(L'_n) - card(L''_n)$.

1. ϕ_n satisfies

$$\phi_n = \phi_{n-1} - \phi_{n-2} + \ldots + (-1)^{p+1}\phi_{n-p}, \text{ for } n \geq p+1, \tag{1}$$

2. λ_n is related to ϕ_n by

$$\lambda_n = \phi_{n-2} - 2\cdot\phi_{n-3} + \ldots + (-1)^{p+1}p\cdot\phi_{n-p-1}, \text{ for } n \geq p+2. \tag{2}$$

Proof of 1. The recursive definition [12]

$$F_n = 0\cdot F_{n-1} \cup 10\cdot F_{n-2} \cup 110\cdot F_{n-3} \cup \ldots \cup 1^{p-1}0\cdot F_{n-p}, \text{ for } n \geq p+1$$

can be expanded as

$$F'_n = 0\cdot F'_{n-1} \cup 10\cdot F''_{n-2} \cup 110\cdot F'_{n-3} \cup \ldots,$$

and

$$F''_n = 0\cdot F''_{n-1} \cup 10\cdot F'_{n-2} \cup 110\cdot F''_{n-3} \cup \ldots,$$

and (1) holds.

Proof of 2. When $n \geq p+2$ the set L_n is the union of the sets in the table below, where the strings with prefix $1^{i-1}0$ are in the ith line and those with the suffix 01^{j-1} are in the jth column, $1 \leq i, j \leq p$.

$0\cdot F_{n-2}\cdot 0,$	$0\cdot F_{n-3}\cdot 01,$	$\cdots$	$0\cdot F_{n-p}\cdot 01^{p-2},$	$0\cdot F_{n-p-1}\cdot 01^{p-1}$
$10\cdot F_{n-3}\cdot 0,$	$10\cdot F_{n-p-4}\cdot 01,$	$\cdots$	$10\cdot F_{n-p-1}\cdot 01^{p-2}$	
$\vdots$	$\vdots$	$\vdots$		
$1^{p-2}0\cdot F_{n-p}\cdot 0,$	$1^{p-2}0\cdot F_{n-p-1}\cdot 01$			
$1^{p-1}0\cdot F_{n-p-1}\cdot 0$				

In this decomposition, each F_{n-k-1} occurs exactly k times, and reading this table diagonally one has

$$card(L_n) = card(F_{n-2}) + 2 \cdot card(F_{n-3}) + 3 \cdot card(F_{n-4}) \dots ,$$

and F_{n-k-1} appears as $1^s 0 \cdot F_{n-k-1} \cdot 01^t$, with $s + t = k - 1$, thus

$$card(L'_n) = card(F'_{n-2}) - 2 \cdot card(F''_{n-3}) + 3 \cdot card(F'_{n-4}) \dots ,$$

and

$$card(L''_n) = card(F''_{n-2}) - 2 \cdot card(F'_{n-3}) + 3 \cdot card(F''_{n-4}) \dots ,$$

so (2) holds.□

The next proposition gives the generating function for the sequences $\{\phi_n\}_{n\geq 0}$ (except its first term) and $\{\lambda_n\}_{n\geq 0}$ (except its $p+2$ first terms).

Proposition 1 *If $\phi(z)$ and $\lambda(z)$ denote the generating functions for the sequences $\{\phi_n\}_{n\geq 0}$ and $\{\lambda_n\}_{n\geq 0}$ then*

1.

$$\phi(z) = \phi_0 + z \cdot (-z)^{p-1} \cdot \frac{1+z}{1-(-z)^{p+1}}, \tag{3}$$

2.

$$\lambda(z) = \sum_{j=0}^{p+1} \lambda_j z^j + z^{p+2} \cdot (-1)^{p+1} \cdot \frac{1-(p+1)(-z)^p + p(-z)^{p+1}}{(1-(-z)^{p+1}) \cdot (1+z)} \tag{4}$$

Proof of 1. For $i = 1, 2, \dots, p-1$ all strings in $\{0,1\}^i$ are in F_i, and half of them are in F'_i and other half are in F''_i, so $\phi_1 = \phi_2 = \dots = \phi_{p-1} = 0$. If $i = p$ then, a single string in $\{0,1\}^p$ does not belong to F_p, namely 1^p, which has p 1s, so $\phi_p = (-1)^{p+1}$. By the relation (1) we have

$$\phi(z) = \phi_0 + z \cdot \frac{f(z)}{\frac{1-(-z)^{p+1}}{1+z}}.$$

where $f(z) = (-z)^{p-1}$ is given by the values of $\phi_1, \phi_2, \dots, \phi_p$. See for instance page 79 of Flajolet's and Sedgewick's seminal book [3].
Proof of 2. If $\lambda^*(z) = \lambda(z) - \sum_{j=0}^{p+1} \lambda_j z^j$, then the relation (2) gives (see again [3])

$$\lambda^*(z) = (\phi(z) - \phi_0) \cdot (z^2 - 2z^3 + \dots + (-1)^{p+1} p z^{p+1})$$

and finally

$$\lambda^*(z) = z^{p+2} \cdot (-1)^{p+1} \cdot \frac{1-(p+1)(-z)^p + p(-z)^{p+1}}{(1-(-z)^{p+1}) \cdot (1+z)} \tag{5}$$

□

For $n \geq p+2$, λ_n is given by the coefficient of z^n in (4). The next corollary shows that the sequence $\{\lambda_n\}_{n\geq 0}$ is periodic up to some n and gives its generating function if $\{\lambda_n\}_{n\geq 0}$ are extended by periodicity to $\lambda_0, \lambda_1, \dots \lambda_{p+1}$.

Corollary 1 *For $n \geq p+2$ the sequence $\{\lambda_n\}_n$ has the period $2(p+1)$. In addition, if one defines $\lambda_n = \lambda_{n+2(p+1)}$ for all $n = 0, 1, \ldots, p+1$ then its generating function becomes*

$$\lambda(z) = \frac{(-z)^{p+1} + (p+1)z + p}{(1-(-z)^{p+1}) \cdot (1+z)}. \tag{6}$$

Proof. For $\lambda^*(z)$ given by relation (5) it is easy to show that $\frac{\lambda^*(z)}{z^{p+2}}(1 - z^{2(p+1)})$ is a polynomial of degree less than $2p+2$. Thus $\frac{\lambda^*(z)}{z^{p+2}}$ is the generating function for a periodic integer sequence with the period $2(p+1)$, and so is $\lambda(z)$ assuming it is extended by periodicity.

For (6) it is enough to find a polynomial $g(z)$ of degree less than $p+2$ such that $\lambda^*(z) - g(z)$ is divisible by $z^{2(p+1)}$, and in this case $\lambda(z) = \frac{\lambda^*(z)-g(z)}{z^{2(p+1)}}$. It is not hard to check that $g(z) = \frac{(-z)^{p+2}((-z)^p-1)}{z+1}$ satisfies this and we obtain (6).□

Corollary 2 *The parity difference integer sequence corresponding to Lucas strings satisfies*

$$\lambda_{n,p} = \begin{cases} (-1)^{n+1} & \text{if } (p+1) \not| n, \\ (-1)^n \cdot p & \text{if } (p+1) | n. \end{cases} \tag{7}$$

Proof. $\lambda(z)$ in the relation (6) can be expressed as

$$\begin{aligned} \lambda(z) &= \frac{p+1}{1-(-z)^{p+1}} - \frac{1}{1+z} \\ &= (p+1) \cdot \sum_{k \geq 0} (-z)^{k(p+1)} - \sum_{n \geq 0} (-z)^n \end{aligned}$$

and $\lambda_{n,p}$ is the coefficient of z^n in $\lambda(z)$.□

Remark that the choice of $\lambda_1, \lambda_2, \ldots, \lambda_{p+1}$ in Corollary 1 seems arbitrary. In fact, for $i = 1, 2, \ldots, p$ all the 2^i strings in $\{0,1\}^i$ are in $L_{i,p}$, except 1^i which contains p consecutive 1s if strings are regarded circularly. In this case $\lambda_{i,p} = (-1)^{i+1}$, and similarly, $\lambda_{p+1,p} = (-1)^{p+1}$; which agrees with (7).

Theorem 1 *If $L_{n,p}$ is Hamiltonian then $(p+1) \not| n$.*

Proof. If $L_{n,p}$ is Hamiltonian then $|\lambda_{n,p}| \leq 1$ so $(p+1) \not| n$.□

The Hamiltonism of $L_{n,p}$, when $(p+1) \not| n$, is shown constructively in the next section.

3 The Gray codes

Here we adopt the convention that lower case bold letters represent length-n binary strings, e.g., $\boldsymbol{x} = x_1 x_2 \ldots x_n$; and we use the same group of letters to denote a set A and an *ordered list* $\mathcal{A}$ for a set A. A list $\mathcal{A}$ for the set $A \subset \{0,1\}^n$ is equivalent to an order relation on A: $\boldsymbol{x} < \boldsymbol{y}$ iff $\boldsymbol{x}$ precedes $\boldsymbol{y}$ in $\mathcal{A}$. For example, if $\boldsymbol{x} \neq \boldsymbol{y} \in \{0,1\}^n$ and i is the leftmost position with $x_i \neq y_i$ then:

- the lexicographic order is given by: $\boldsymbol{x} < \boldsymbol{y}$ iff x_i is even $(= 0)$, and obviously y_i is odd $(= 1)$,

- the *reflected Gray code order* due to Frank Gray in 1953 [4] is given by: $\boldsymbol{x} < \boldsymbol{y}$ iff $\sum_{j=1}^{i} x_j$ is even (and $\sum_{j=1}^{i} y_j$ is odd).

We say that an order relation $<$ on a set of strings induces a k-*Gray code* if the set listed in $<$ order yields a k-Gray code, i.e., successive strings differ in at most k positions. So, the reflected Gray code order above induces a 1-Gray code on $\{0,1\}^n$; and its restriction to the strings with fixed density (i.e. strings in $\{0,1\}^n$ with a constant number of 1s) induces a 2-Gray code, called *revolving door code* by Nijenhuis and Wilf [10].

The next definition gives an other order relation on binary strings and all of those presented here are particular cases of *genlex order* [14], that is, any set of strings listed in such an order has the property that strings with a common prefix are contiguous.

Definition 1 *[12] We say that* $\boldsymbol{x}$ *is less then* $\boldsymbol{y}$ *in* local reflected order, *denoted by* $\boldsymbol{x} \prec \boldsymbol{y}$, *if* $\sum_{j=1}^{i}(1-x_j)$ *is odd (and obviously,* $\sum_{j=1}^{i}(1-y_j)$ *is even).*

Remark 1

1. *As the reflected Gray code order, the local reflected order* $\prec$ *induces a* 1*-Gray code on* $\{0,1\}^n$ *and a* 2*-Gray code on length-*n *binary strings with fixed density,*
2. *In [12] is shown that, unlike the reflected Gray code order, the local reflected order* $\prec$ *induces a* 1*-Gray code on the set* $F_{n,p}$*, of* p *order length-*n *Fibonacci strings.*

Let $\mathcal{F}_{n,p}$ and $\mathcal{L}_{n,p}$ be the lists obtained by ordering the sets $F_{n,p}$ and $L_{n,p}$, respectively, by the relation $\prec$. $\mathcal{F}_{n,p}$ is a 1-Gray code and the aim of this section is to prove constructively that $\mathcal{L}_{n,p}$ is also a Gray code. More precisely the relation $\prec$ defined above yields a

- 1-Gray code order for $L_{n,p}$ if $(p+1) \not| n$,
- 2-Gray code order for $L_{n,p}$ if $(p+1)|n$.

If α is a length-k string, we denote by $\alpha^{\frac{n}{k}}$ the length-n prefix of the infinite string $\alpha\alpha\alpha\ldots$, or equivalently,

$$\alpha^{\frac{n}{k}} = \underbrace{\alpha\alpha\ldots\alpha}_{\lfloor\frac{n}{k}\rfloor}\alpha_r,$$

where $r = n \mod k$, and α_r is the length-r prefix of α. In the following, the length-$(p+1)$ binary string

$$\chi = \underbrace{1\ldots1}_{p-1}00$$

plays a central role for our purposes.

In [12] is proved that the first and the last strings of $\mathcal{F}_{n,p}$ are $first(\mathcal{F}_{n,p}) = 0\chi^{\frac{n-1}{p+1}}$ and $last(\mathcal{F}_{n,p}) = \chi^{\frac{n}{p+1}}$. The next lemma and Corollary 3 gives similar results for $\mathcal{L}_{n,p}$.

Lemma 2

1. $first(\mathcal{L}_{n,p}) = 0\chi^{\frac{n-1}{p+1}}$
2. $last(\mathcal{L}_{n,p}) = \chi^{\frac{n-1}{p+1}}0.$

Proof. 1. Let $f_1 f_2 \dots f_n = 0\chi^{\frac{n-1}{p+1}}$ and $1 \le j \le n$ such that $\sum_{i=1}^{j}(1-f_i)$ is even, then: (1) $f_j = 0$ and (2) f_{j-1} is the rightmost 1 bit in a contiguous 1s sequence of length $p-1$, so, by Definition 1, f has no predecessor in $L_{n,p}$ in $\prec$ order.
2. Analogously, the string $\chi^{\frac{n-1}{p+1}}0$ has no successor in $\mathcal{L}_{n,p}$. □

Now we describe the succession rules in the lists $\mathcal{F}_{n,p}$ and $\mathcal{L}_{n,p}$. Since in the list $\mathcal{F}_{n,p}$ and $\mathcal{L}_{n,p}$ strings with a common prefix are contiguous, the successor of $\boldsymbol{x} \in \mathcal{F}_{n,p}$ (resp. of $\boldsymbol{x} \in \mathcal{L}_{n,p}$) is given by changing the rightmost bit in $\boldsymbol{x}$ such that the obtained string remains in $\mathcal{F}_{n,p}$ (resp. in $\mathcal{L}_{n,p}$), and it is greater than $\boldsymbol{x}$ in $\prec$ orders. More formally we have

Lemma 3 *Let $\boldsymbol{x} \ne last(\mathcal{F}_{n,p})$ and $s(\boldsymbol{x})$ its successor in $\mathcal{F}_{n,p}$ then either 1 or 2 below holds:*

1. *$\boldsymbol{x}$ contains an odd number of 0s and it has not a suffix of the form $1^{p-1}0$. In this case $s(\boldsymbol{x}) = x_1 \dots x_{n-1}(1-x_n)$.*
2. *$\boldsymbol{x}$ contains an even number of 0s or it ends by $1^{p-1}0$ and let $x_1 x_2 \dots x_{k-1} x_k$ be the length minimal prefix of $\boldsymbol{x}$ with an odd number of 0s and such that $x = x_1 x_2 \dots x_{k-1} x_k 0 \chi^{\frac{n-k-1}{p+1}}$. In this case $s(\boldsymbol{x}) = x_1 x_2 \dots x_{k-1}(x_k - 1)0\chi^{\frac{n-k-1}{p+1}}$.*

Note that if $\boldsymbol{x} \ne last(\mathcal{F}_{n,p})$ then the prefix required by the point 2 above always exists. For example, in $\mathcal{F}_{6,3}$, by point 1 above we have: $s(01001\mathbf{1}) = 01001\mathbf{0}$ and $s(10110\mathbf{0}) = 10110\mathbf{1}$; and by point 2: $s(\mathbf{0}00110) = \mathbf{1}00110$ and $s(0\mathbf{1}0110) = 0\mathbf{0}0110$. In [12] similar ideas is used to compute, in constant delay, the successor $s(\boldsymbol{x})$ of a Fibonacci string $\boldsymbol{x}$.

Obviously, if $\boldsymbol{x}$ is a Lucas string then $s(\boldsymbol{x})$ is a Fibonacci but not necessarily a Lucas string too. The next proposition says that if we denote by $succ(x)$ the successor of $x \in L_{n,p}$ in the list $\mathcal{L}_{n,p}$ then $succ(x)$ is $s(x)$ or $s(s(x))$ or $s(s(s(x)))$.

Proposition 2 *Let $\boldsymbol{x} \in L_{n,p}$, $\boldsymbol{x} \ne last(L_{n,p})$ and $s(\boldsymbol{x}) \notin L_{n,p}$ then either 1 or 2 below holds:*

1. *$(p+1)|n$ and $\boldsymbol{x} = 1^k 00\chi^{\frac{n-k-2}{p+1}}$ with $0 \le k < p-1$. In this case*

$$\begin{aligned} succ(\boldsymbol{x}) &= 1^{k+1}0\chi^{\frac{n-k-3}{p+1}}0 \\ &= s^2(\boldsymbol{x}). \end{aligned}$$

2. *$\boldsymbol{x} = 1^k \alpha 1^\ell 0$ with $0 < k, \ell < p-1$ and $k + \ell \ge p-1$. In this case*

$$\begin{aligned} succ(\boldsymbol{x}) &= 1^k s(\alpha) 1^\ell 0 \\ &= s^3(\boldsymbol{x}). \end{aligned}$$

Remark 2

1. *If $(p+1)|n$ then there exist exactly $p-1$ Lucas string as in pont 1 of Proposition 2, one for each k, $0 \le k < p-1$. In addition, if $\boldsymbol{x}$ is such a string and d denotes the Hamming distance on length-n strings then*

   (a)

$$\begin{aligned} d(\boldsymbol{x}, succ(\boldsymbol{x})) &= d(\boldsymbol{x}, s^2(\boldsymbol{x})) \\ &= 2, \end{aligned}$$

Table 1: The lists $\mathcal{L}_{4,2}$ and $\mathcal{L}_{4,3}$. Changed bits are in bold-face.

$\mathcal{L}_{4,2}$	$\mathcal{L}_{4,3}$
0 1 0 **0**	0 1 **1** 0
0 **1** 0 1	0 1 0 **0**
0 0 0 **1**	0 **1** 0 1
0 0 **0** 0	0 0 0 **1**
0 0 1 0	0 0 **0** 0
1 0 **1** 0	0 0 1 **0**
1 0 0 0	**0** 0 1 **1**
	1 0 **1** 0
	1 0 0 **0**
	1 **0** 0 **1**
	1 1 0 0

(b) *the string* $\boldsymbol{v} = 1^k 00\chi^{\frac{n-k-3}{p+1}} 0 \in L_{n,p}$ *is the predecessor of* $\boldsymbol{x}$ *in* $\prec$ *order (i.e.* $succ(\boldsymbol{v}) = \boldsymbol{x}$*) and* $d(\boldsymbol{v}, succ(\boldsymbol{x})) = 1$.

2. *If* $\boldsymbol{x}$ *is a Lucas string as in pont 2 of Proposition 2 then*

$$\begin{aligned} d(\boldsymbol{x}, succ(\boldsymbol{x})) &= d(1^k\alpha 1^\ell 0, 1^k\alpha' 1^\ell 0) \\ &= d(\alpha, \alpha') \\ &= 1. \end{aligned}$$

With this remark we have

Corollary 3

1. *If* $(p+1) \not| n$ *then* $\mathcal{L}_{n,p}$ *is a 1-Gray code for* $L_{n,p}$.
2. *If* $(p+1)|n$ *then* $\mathcal{L}_{n,p}$ *is a 2-Gray code for* $L_{n,p}$ *and there are exactly* $p-1$ *strings with* $d(\boldsymbol{x}, succ(\boldsymbol{x})) = 2$.

See Table 1 and Figure 1.b for the list $\mathcal{L}_{n,p}$.

4 Lucas cubes

In terms of graph, the precedent results can be expressed as follows. Let $B \subset \{0,1\}^n$ and $G(B)$ be the restriction of the hypercube $\{0,1\}^n$ to the set B, i.e., the graph where vertex correspond to the strings in the set B, and two vertices are connected by an edge if their Hamming distance is one. Let also $(G(B))^k$ its kth power, edges connecting those vertices with Hamming distance less then or equal to k. A k-Gray code for B is a Hamiltonian path for $(G(B))^k$, with $(G(B))^1 = G(B)$. So, the list $\mathcal{F}_{n,p}$ is a Hamiltonian path in the Fibonacci cube $G(F_{n,p})$, for all $n, p \geq 1$ [12]. Corollary 3 says that $\mathcal{L}_{n,p}$ is a Hamiltonian path in:

- the Lucas cube $G(L_{n,p})$ iff n is not a multiple of $p+1$, and

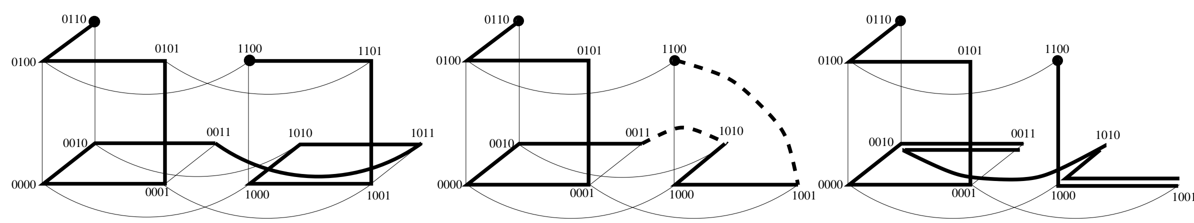
- the second power of the Lucas cube $(G(L_{n,p}))^2$ elsewhere.

In certain situations, we may known that a given graph does not have a Hamiltonian path. In such cases, it may be desirable to visit each vertex but not necessarily once, such that the Hamming distance between two successive vertices is one. Following [11], a graph is in the class $\mathcal{H}(s,t)$ if it has a path that visit every vertex at least s times and at most t times, and such a path is called $\mathcal{H}(s,t)$-path. Thus a graph is in $\mathcal{H}(1,1)$ exactly if it is Hamiltonian. In this context, by Remark 2 one has

Corollary 4 *If* $(p+1)|n$ *then* $G(L_{n,p})$ *is in* $\mathcal{H}(1,2)$.

See Figure 1.c for such a path in $G(L_{4,3})$.

Figure 1: (a) The Hamiltonian path $\mathcal{F}_{4,3}$ in $G(F_{4,3})$. (b) The *'path'* $\mathcal{L}_{4,3}$ in $G(L_{4,3})$, dashed arcs connect distance-2 vertices. (c) An $\mathcal{H}(1,2)$-path in $G(L_{4,3})$.



(a) (b) (c)

5 Algorithmic considerations

In [12] is given an exhaustive generating algorithm for the list $\mathcal{F}_{n,p}$ which runs with constant delay between any two successive Fibonacci strings. Here we show how one can modify this algorithm in order to produce efficiently the list $\mathcal{L}_{n,p}$.

Proposition 2 insures us that at most two Fibonacci strings exist between any two consecutive Lucas string in $\mathcal{L}_{n,p}$; and by point 2 of Lemma 2 the last string in $\mathcal{L}_{n,p}$ is followed by at most one Fibonacci string. So, the algorithm in [12] can be modified to generate the list $\mathcal{L}_{n,p}$ by simply bypassing the Fibonacci strings which are not Lucas strings. The obtained algorithm inherits the constant delay property if one can decide, in constant time, if the current generated Fibonacci string is also a Lucas string (the test $x \in L_{n,p}$ in the algorithm below). Additional variables, as: (1) the length of the contiguous prefix of 1s and (2) the length and the first position of the rightmost contiguous sequence of 1s, can be used to discriminate, in constant time, between Lucas and Fibonacci strings. The call of $succ\_fib(x)$ below produces the successor of x *as a Fibonacci string*. A Java applet generating the list $\mathcal{L}_{n,p}$ is available at the web site of the second author [13].

$x := first(\mathcal{L}_{n,p})$;
while $x \neq last(\mathcal{L}_{n,p})$ **do**
    $x := succ\_fib(x)$;

```
    if x ∈ L_{n,p}
    then Print(x);
enddo
```

Acknowledgments

We thank Olivier Togni who presented us the problem we deal with in this paper.

References

[1] P.J. Chase, Combination generation and graylex ordering, *Congr. Numer.* **69** (1989), 215–242.

[2] E. Ded, D. Torri, N. Zagaglia Salvi, The observability of the Fibonacci and the Lucas cubes, *Combinatorics '98 (Palermo). Discrete Math.* **255**(1-3)(2002), 55 –63.

[3] P. Flajolet, R. Sedgewick, *Introduction à l'analyse des algorithmes*, Thomson Publishing, 1996.

[4] F. Gray, Pulse code communication, U.S. Patent 2632058 (1953).

[5] V.E. Hoggatt, M. Bicknell-Johnson, Generalized Lucas sequences, *Fibonacci Quart.* **15**(2)(1977), 131–139.

[6] W-J. Hsu, Fibonacci cubes – a new interconnection topology, *IEEE Transactions on Parallel and Distributed Systems* **4**(1) (1993), 3–12.

[7] J.C. Lagarias, D.P. Weisser, Fibonacci and Lucas cubes, *Fibonacci Quart.* **19**(1)(1981), 39–43.

[8] J. Liu, W-J. Hsu, M.J. Chung, Generalized Fibonacci cubes are mostly Hamiltonian, *Journal of Graph Theory* **18**(8) (1994), 817–829.

[9] E. Munarini, C. Perelli Cippo, N. Zagaglia Salvi, On the Lucas cubes, *Fibonacci Quart.* **39**(1)(2001), 12 –21.

[10] A. Nijenhuis, H.S. Wilf, *Combinatorial Algorithms for Computers and Calculators*, Academic Press, 1978.

[11] G. Pruesse, F. Ruskey, Generating linear extensions fast, *SIAM J. Comput.*, **23**(1994), 373–386.

[12] V. Vajnovszki, A loopless generation of bitstrings without p consecutive ones, *Discrete Mathematics and Theoretical Computer Science - Springer 2001*, 227–240.

[13] *http://www.u-bourgogne.fr/v.vincent/*

[14] T.R. Walsh, Generating Gray Codes in $O(1)$ Worst-Case Time Per Word, to appear in *DMTCS03, LNCS 2731*, Springer 2003.

[15] J. Wu, Extended Fibonacci Cubes, *IEEE Transactions on Parallel and Distributed Systems* **8**(12)(1997), 1203–1210.