More restrictive Gray codes for some classes of pattern avoiding permutations

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Abstract

In a recent article [11], Dukes, Flanagan, Mansour and Vajnovszki present Gray codes for several families of pattern avoiding permutations. In their Gray codes two consecutive objects differ in at most four or five positions, which is not optimal. In this paper, we present a unified construction in order to refine their results (or to find other Gray codes). In particular, we obtain more restrictive Gray codes for the two Wilf classes of Catalan permutations of length n; two consecutive objects differ in at most two or three positions which is *optimal* for n odd. Other refinements have been found for permutation sets enumerated by the numbers of Schröder, Pell, even index Fibonacci numbers and the central binomial coefficients. A general efficient generating algorithm is also given.

Key words: Gray code, pattern avoiding permutation, generating algorithm, Catalan numbers, Schröder numbers, central binomial coefficients.

1 Introduction

Let S_n be the set of all permutations of length n ($n \geq 1$). We represent permutations in one-line notation, *i.e.* if $i_1,\ i_2, \ldots,\ i_n$ are n distinct values in $[n] = \{1,\ 2, \ldots,\ n\}$, we denote the permutation $\sigma \in S_n$ by the sequence $i_1 i_2 \ldots i_n$ if $\sigma(k) = i_k$ for $1 \leq k \leq n$. For instance, the identity permutation of length n, id_n, will be written $12\ldots(n-1)n$. Moreover, if $\gamma = \gamma(1)\gamma(2)\ldots\gamma(n)$ is an n-length permutation then the *composition* (*or product*) $\gamma \cdot \sigma$ is the permutation $\gamma(\sigma(1))\gamma(\sigma(2))\ldots\gamma(\sigma(n))$. In S_n, a *k-cycle* $\sigma = \langle i_1,\ i_2, \ldots,\ i_k\rangle$ is an n-length permutation satisfying $\sigma(i_1) = i_2,\ \sigma(i_2) = i_3, \ldots,\ \sigma(i_{k-1}) = i_k,\ \sigma(i_k) = i_1$ and $\sigma(j) = j$ for $j \in [n]\backslash\{i_1, \ldots, i_k\}$; in particular, a *transposition* is a 2-cycle. A permutation $\sigma = \sigma_1\sigma_2\ldots\sigma_n \in S_n$ contains the pattern $\gamma \in S_k$ ($k \geq 2$) if and only if a sequence of indices $1 \leq i_1 < i_2 < \ldots < i_k \leq n$ exists such that

$\sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_k}$ is ordered as γ. We denote by $S_n(\gamma)$ the set of permutations in S_n avoiding the pattern γ. For example, $\mathbf{25314} \notin S_5(123)$ but $43152 \in S_5(123)$. Several algorithms for generating permutation classes have been published (generated in lexicographic order or in constant time per permutation, in the amortized sense). For example, efficient algorithms are known for derangements [1], up-down permutations [7,19], permutations with a fixed number of inversions [12] and for permutations avoiding a pattern [9]. On the other hand, algorithms for generating permutation classes in Gray code order are given for permutations [14,22] and their restrictions [17,19,20], derangements [3,16], with a fixed number of cycles [2], involutions and fixed-point free involutions [25] or their generalizations (multiset permutations [24]). Juarna and Vajnovszki [15] presented Gray codes and generating algorithms for two classes of pattern avoiding permutations: $S_n(123, 132)$, $S_n(123, 132, p(p-1)\ldots 1(p+1))$. In a recent paper, Dukes et al. [11] give Gray codes and generating algorithms for classes of pattern avoiding permutations enumerated by Catalan, Schröder, Pell, even index Fibonacci numbers and the central binomial coefficients: two consecutive elements in their codes differ in at most four or five positions. Inspired by their article, we give here Gray codes for Catalan permutations and for some other classes: two consecutive elements differ in at most three (or two) positions. This induces optimal Gray codes for some of these classes: $S_n(321)$ and $S_n(312)$ for n odd. Notice that in [8] or [23], Gray codes are given for some classes previously mentioned, but they do not provide them in term of classes of pattern avoiding permutations. Despite the fact that some well known bijections exist between these classes they do not conserve the Gray code property.
This paper is organized as follow. In section 2, we give the construction for our Gray codes based on the ECO method [4]. We obtain sufficient conditions showing that our construction produces Gray codes. Thus we investigate several classes of avoiding pattern permutation sets in Table 2. In the third and last section, we shortly explain our generating algorithm to list our Gray codes. We present an implementation of our algorithm, and we discuss its complexity.

2 Construction for our Gray codes

In this part, we develop a general construction that provides Gray codes for several classes of pattern avoiding permutations. This construction is a refinement of that given in [11] and, in some cases, yields codes that are optimal. The construction is based on the method ECO [4]. It consists in determining a recursive construction for the considered class based on local expansion performed on the objects. Each object is produced exactly once by an object of lower size.
Let $\sigma \in S_n$; the *sites* of σ are the positions between two consecutives en-

tries, before the first and after the last entry. We suppose that the sites are numbered, from right to left, from 1 to $n+1$. For example, the 3rd site of the permutation $\sigma = 463512$ is between the entries 5 and 1. Moreover, let $\sigma \in S_n(T)$, where T is a set of forbidden patterns, the ith site of σ is said *active* if the permutation γ obtained from σ by inserting $(n+1)$ into this site belongs to $S_{n+1}(T)$. We will denote this permutation γ by $\phi(i,\sigma)$ and we will say that γ is a *successor* (or *son*) of σ. The active sites of a permutation $\sigma \in S_n(T)$ are *right-justified* (see [11]) if all sites to the right of the leftmost active site are also active. We denote by $\chi_T(i,\sigma)$ the number of active sites of $\phi(i,\sigma)$. A set of patterns T is called *regular* [11] if for any $n \geq 1$ and $\gamma \in S_n(T)$,

- γ has at least two active sites, and the active sites of γ are right-justified,
- there exist $\sigma \in S_{n-1}(T)$ and an integer i, $1 \leq i \leq n$, such that $\gamma = \phi(i,\sigma)$,
- $\chi_T(i,\sigma)$ does not depend on σ but only on the number k of active sites of σ; in this case, we will denote $\chi_T(i,\sigma)$ by $\chi_T(i,k)$ if σ has k active sites.

For the remainder of this paper, we only consider regular set T of patterns. Now, for each $n \geq 1$ and for any permutation $\sigma \in S_n(T)$, we associate:

- a *direction*: *up* or *down*; a permutation σ with the direction *up* (*down* respectively) will be denoted by σ^1 (σ^0 respectively). Such a permutation will be called a *directed* permutation.
- a list of its sons considered with their directions, *i.e.* a list consisting of some $\phi(i,\sigma)^j$ where $j \in \{0,1\}$ and i are the active sites of σ in an order that we will describe below.

The list of successors of σ^0 will be obtained by reversing the list of successors of σ^1 and by reversing the direction of each element of the list.
Let σ^1 be a directed permutation in $S_n(T)$ with k active sites and let L be a unimodal sequence of integers where $L(i)$ denotes the ith $(1 \leq i \leq k)$ integer of the following sequence L:

$$L = \begin{cases} 3,5,\ldots,(k-3),(k-1),k,(k-2),(k-4),\ldots,4,2,1 & \text{if } k \text{ is even,} \\ 3,5,\ldots,(k-4),(k-2),k,(k-1),(k-3),\ldots,4,2,1 & \text{if } k \text{ is odd.} \end{cases}$$

We also consider the sequence L' defined as follows:

$$L' = \begin{cases} 2,4,\ldots,(k-4),(k-2),k,(k-1),(k-3),\ldots,5,3,1 & \text{if } k \text{ is even,} \\ 2,4,\ldots,(k-3),(k-1),k,(k-2),(k-4),\ldots,5,3,1 & \text{if } k \text{ is odd.} \end{cases}$$

Let $\Pi(\sigma^1)$ be the list of successors of σ^1 consisting of k directed permutations in $S_{n+1}(T)$ such that: the jth element is $\phi(L(j),\sigma)^{(i-1) \mod 2}$ for $1 \leq j \leq k-1$ and the kth element is $\phi(L(k),\sigma)^1$. Moreover, if we replace the sequence L by L' in this definition, we obtain a list that we denote $\Pi'(\sigma^1)$. Figures 1 and 2 illustrate the different cases for the lists $\Pi(\sigma)$ and $\Pi'(\sigma)$ for σ having $k=7$ or $k=8$ active sites.

Fig. 1. If $k = 7$ then $\Pi(\sigma^1)$ is the list $\phi(3,\sigma)^0, \phi(5,\sigma)^1, \phi(7,\sigma)^0, \phi(6,\sigma)^1, \phi(4,\sigma)^0, \phi(2,\sigma)^1, \phi(1,\sigma)^1$; if $k = 8$, $\Pi(\sigma^1)$ is $\phi(3,\sigma)^0, \phi(5,\sigma)^1, \phi(7,\sigma)^0, \phi(8,\sigma)^1, \phi(6,\sigma)^0, \phi(4,\sigma)^1, \phi(2,\sigma)^0, \phi(1,\sigma)^1$. Each point (resp. encircled point) numbered by $\ell \in [1..k]$ represents $\phi(\ell,\sigma)^1$ (resp. $\phi(\ell,\sigma)^0$).

Fig. 2. If $k = 7$ then $\Pi'(\sigma^1)$ is the list $\phi(2,\sigma)^0, \phi(4,\sigma)^1, \phi(6,\sigma)^0, \phi(7,\sigma)^1, \phi(5,\sigma)^0, \phi(3,\sigma)^1, \phi(1,\sigma)^1$; if $k = 8$, $\Pi'(\sigma^1)$ is $\phi(2,\sigma)^0, \phi(4,\sigma)^1, \phi(6,\sigma)^0, \phi(8,\sigma)^1, \phi(7,\sigma)^0, \phi(5,\sigma)^1, \phi(3,\sigma)^0, \phi(1,\sigma)^1$. Each point (resp. encircled point) numbered by $\ell \in [1..k]$ represents $\phi(\ell,\sigma)^1$ (resp. $\phi(\ell,\sigma)^0$).

Now, we set $d_n = \text{card}(S_n(T))$ and we recursively define our Gray code $\mathcal{S}_n(T)$ for $S_n(T)$ as follows:

$$\mathcal{S}_n(T) = \begin{cases} 1^0 & \text{if } n = 1 \\ \bigoplus_{i=1}^{d_{n-1}} \tilde{\Pi}(\sigma[i]) & \text{otherwise,} \end{cases}$$

where

- $\sigma[i]$ is the ith directed permutation of the list $\mathcal{S}_{n-1}(T)$,
- $\tilde{\Pi}(\sigma[i]) = \Pi'(\sigma[i])$ if: the direction of $\sigma[i]$ is 0 and the direction of $\sigma[i+1]$ is 1; $\sigma[i]$ and $\sigma[i+1]$ differ in three positions and have respectively at least three sons and exactly two sons,
- $\tilde{\Pi}(\sigma[i]) = \Pi'(\sigma[i])$ if: the direction of $\sigma[i]$ is 1 and the direction of $\sigma[i-1]$ is 0; $\sigma[i]$ and $\sigma[i-1]$ differ in three positions and have respectively at least three sons and exactly two sons,
- $\tilde{\Pi}(\sigma[i]) = \Pi(\sigma[i])$ otherwise.

Table 1 illustrates our construction for $\mathcal{S}_5(321)$ and $S_5(312)$. Notice that the lists Π and Π' appear in the construction of the Gray code $\mathcal{S}_5(321)$; but only the lists of the form Π are used for $\mathcal{S}_5(312)$.
For the remainder of this paper, we study several properties that allow us to prove this definition gives almost all Gray codes such that two consecutive elements in the list differ in at most three positions. If $\mathcal{L}$ is a list, we denote by $\text{first}(\mathcal{L})$ and $\text{last}(\mathcal{L})$ the first and last element of $\mathcal{L}$.

Proposition 1 *Let T be a regular set of patterns. The list $\mathcal{S}_n(T)$ contains exactly once all permutations in $S_n(T)$. The first element of the list is $123\dots(n-1)n$ and the last element is $2134\dots(n-1)n$.*

Proof. This holds for $n = 1$ and $n = 2$ since $\mathcal{S}_1(T) = \{1\}$ and $\mathcal{S}_2(T) = \{12, 21\}$ where T is a regular pattern set that does not contain the pattern 21. By induction, let us suppose that this is true for all k, $2 \le k < n$. Then,
first($\mathcal{S}_n(T)$)= ϕ(1,first($\mathcal{S}_{n-1}(T)$))= $1234\dots(n-1)n$, and
last($\mathcal{S}_n(T)$)= ϕ(1,last($\mathcal{S}_{n-1}(T)$))= $2134\dots(n-1)n$. The fact that the list $\mathcal{S}_n(T)$ contains each of the permutations exactly once is deduced directly from the above recursive definition of $\mathcal{S}_n(T)$. □

Lemma 2 *Let T be a regular set of patterns. Let σ be a permutation in $\mathcal{S}_n(T)$, and let $r \in \{0, 1\}$ be its direction, then two successive permutations in $\Pi(\sigma^r)$ differ in at most three positions.*

Proof. Let α and β the two successive permutations in $\Pi(\sigma^r)$. Thus if k is the number of active sites of σ, there exists i and j, $i \ne j$, $1 \le i, j \le k$ such that $\alpha = \phi(i, \sigma)$ and $\beta = \phi(j, \sigma)$. Considering the definitions of the sequences L or L', we distinguish two cases: (a) if $i = j - 1$ then $\alpha = \beta \cdot \langle i, j\rangle$ which induces the result; (b) if $i = j - 2$ then $\alpha = \beta \cdot \langle i, j, i + 1\rangle$ and the two permutations α and β differ in three positions. The other cases, (*i.e.* $j = i - 1$ and $j = i - 2$), are similar. □

Lemma 3 *Let T be a regular set of patterns. Let σ^1 and γ^0 be two consecutive directed permutations in the list $\mathcal{S}_n(T)$. Then $d(\sigma, \gamma) = d(\mathit{last}(\Pi(\sigma^1)), \mathit{first}(\Pi(\gamma^0)))$.*

Proof. Indeed, let us assume that σ has k sons and γ has ℓ sons. Then we have:

$$\begin{aligned} d(\text{last}(\Pi(\sigma^1)), \text{first}(\Pi(\gamma^0))) &= d(\phi(L(k), \sigma), \phi(L(\ell), \gamma)) \\ &= d(\phi(1, \sigma), \phi(1, \gamma)) = d(\sigma, \gamma). \end{aligned}$$

□

Lemma 4 *Let T be a regular set of patterns. Let σ^0 and γ^1 be two consecutive directed permutations in the list $\mathcal{S}_n(T)$. If both permutations σ and γ have at least three successors then we have: $d(\sigma, \gamma) = d(\mathit{last}(\Pi(\sigma^0)), \mathit{first}(\Pi(\gamma^1)))$.*

Proof. Since each permutation σ and γ has at least three sons, their corresponding sequences L satisfy $L(1) = 3$. Thus, we have:

$$\begin{aligned} d(\text{last}(\Pi(\sigma^0)), \text{first}(\Pi(\gamma^1))) &= d(\phi(L(1), \sigma), \phi(L(1), \gamma)) \\ &= d(\phi(3, \sigma), \phi(3, \gamma)) = d(\sigma, \gamma). \end{aligned}$$

□

Lemma 5 *Let T be a regular set of patterns. Let σ^0 and γ^1 be two consecutive directed permutations in the list $\mathcal{S}_n(T)$. If both permutations σ and γ have two successors then we have: $d(\sigma, \gamma) = d(\mathit{last}(\Pi(\sigma^0)), \mathit{first}(\Pi(\gamma^1)))$.*

Proof. Since each permutation σ and γ has two sons, their corresponding sequences L verify $L(1) = 2$. Thus, we have:

$$\begin{aligned} d(\text{last}(\Pi(\sigma^0)), \text{first}(\Pi(\gamma^1))) &= d(\phi(L(1), \sigma), \phi(L(1), \gamma)) \\ &= d(\phi(2, \sigma), \phi(2, \gamma)) = d(\sigma, \gamma). \end{aligned}$$ □

Lemma 6 *Let T be a regular set of patterns. Let σ^0 and γ^1 be two consecutive directed permutations in the list $\mathcal{S}_n(T)$ such that $\sigma = \gamma \cdot \langle n-1, n\rangle$. If σ has at least three sons and γ has two sons then we have: $d(\mathit{last}(\Pi(\sigma^0)), \mathit{first}(\Pi(\gamma^1))) = 3$.*

Proof. Since the permutation σ has at least three sons, it has a corresponding sequence L such that $L(1) = 3$; as γ has two sons its corresponding sequence L is such that $L(1) = 2$. Thus, we have:

$$\begin{aligned} d(\text{last}(\Pi(\sigma^0)), \text{first}(\Pi(\gamma^1))) &= d(\phi(L(1), \sigma), \phi(L(1), \gamma)) \\ &= d(\phi(3, \sigma), \phi(2, \gamma)) \\ &= d(\phi(3, \gamma \cdot \langle n-1, n\rangle), \phi(2, \gamma)) \\ &= d(\phi(2, \gamma \cdot \langle n-1, n\rangle) \cdot \langle n-1, n\rangle, \phi(2, \gamma)) \\ &= d(\phi(2, \gamma) \cdot \langle n-1, n, n+1\rangle), \phi(2, \gamma)) = 3. \end{aligned}$$ □

Lemma 7 *Let T be a regular set of patterns. Let σ^0 and γ^1 be two consecutive directed permutations in the list $\mathcal{S}_n(T)$ such that σ and γ differ in three positions. If σ has two sons (resp. at least three sons) and γ has at least three sons (resp. exactly two sons), then we have: $d(\mathit{last}(\Pi(\sigma^0)), \mathit{first}(\Pi(\gamma^1))) = 3$.*

Proof. Since the permutation σ has two sons, it has a corresponding sequence L such that $L(1) = 2$; as γ has at least three sons its corresponding sequence L' is such that $L'(1) = 2$. Thus, we have:

$$\begin{aligned} d(\text{last}(\Pi(\sigma^0)), \text{first}(\Pi(\gamma^1))) &= d(\phi(L(1), \sigma), \phi(L'(1), \gamma)) \\ &= d(\phi(2, \sigma), \phi(2, \gamma)) = 3. \end{aligned}$$ □

Lemma 8 *Let T be a regular set of patterns. Let σ^1 and γ^1 be two consecutive directed permutations in the list $\mathcal{S}_n(T)$ such that $\sigma = \gamma \cdot \langle n-1, n\rangle$. If γ has at least three sons, then we have: $d(\mathit{last}(\Pi(\sigma^1)), \mathit{first}(\Pi(\gamma^1))) = 2$. On the other hand if γ has two sons then we have: $d(\mathit{last}(\Pi(\sigma^1)), \mathit{first}(\Pi(\gamma^1))) = 3$.*

Proof. Let us assume that the permutation σ has k sons. Thus its corresponding sequence L is such that $L(k) = 1$. Let us assume that γ has at least three sons. It corresponding sequence L is such that $L(1) = 3$. Thus, we have:

$$\begin{aligned} d(\text{last}(\Pi(\sigma^1)), \text{first}(\Pi(\gamma^1))) &= d(\phi(L(k), \sigma), \phi(L(1), \gamma)) \\ &= d(\phi(1, \sigma), \phi(3, \gamma)) \\ &= d(\phi(1, \gamma \cdot \langle n-1, n\rangle), \phi(3, \gamma)) \\ &= d(\phi(1, \gamma) \cdot \langle n-1, n\rangle, \phi(3, \gamma)) \\ &= d(\phi(1, \gamma) \cdot \langle n-1, n\rangle, \phi(1, \gamma) \cdot \langle n-1, n\rangle\langle n, n+1\rangle) \\ &= d(\phi(1, \gamma), \phi(1, \gamma) \cdot \langle n-1, n+1\rangle) = 2. \end{aligned}$$

The proof for γ with two sons is similar. □

Remark that:
- if $\chi_T(i,k) = 2$ for all i and k, the list $\mathcal{S}_n(T)$ defined above is a cyclic Gray code for $S_n(T)$ such that two consecutive elements in the list differ in two positions. Indeed, it suffices to apply Lemmas 2, 3, and 5.
- If $\chi_T(i,k) \geq 3$ for all i and k, then the list $\mathcal{S}_n(T)$ defined above is a cyclic Gray code for $S_n(T)$. Two consecutive elements in the list differ in at most three positions. It suffices to apply Lemmas 2, 3, 4 and 8.

In Table 2, we give several avoiding pattern sets T for which we can apply our construction in order to obtain Gray codes for $S_n(T)$. Notice that this construction allows us to obtain optimal Gray code for $S_{2n+1}(321)$ and $S_{2n+1}(312)$, $n \geq 0$. Indeed the difference between the number of even and odd permutations in $S_{2n+1}(321)$ (resp. $S_{2n+1}(312)$) is at least two (see [21]). Thus we cannot have a Gray code such that two consecutive permutations differ by a transposition. Since the difference between the number of even and odd permutations in $S_{2n}(321)$ (resp. $S_{2n}(312)$) is zero, we cannot deduce the optimality of our Gray code for $S_{2n}(321)$ (resp. $S_{2n}(312)$) and for other classes. Besides the Gray code for $S_n(321, 312)$ is optimal and it corresponds to the one in [15] which is obtained by another method.

Table 1
The optimal cyclic Gray codes for $\mathcal{S}_5(312)$ and $\mathcal{S}_5(321)$. Two consecutive elements differ in at most three positions.

$\mathcal{S}_5(312)$			$\mathcal{S}_5(321)$		
12345	**321**45	**2**4**3**15	12345	1**3**2**4**5	**231**45
123**54**	321**54**	243**51**	123**54**	132**54**	231**54**
12**543**	32**541**	**2543**1	**1523**4	13**52**4	23**51**4
12**45**3	32**45**1	2**45**31	**51**234	13**452**	23**451**
124**35**	324**15**	**23**541	**125**34	134**25**	234**15**
1**432**5	**432**15	23**45**1	12**453**	**341**25	**2413**5
143**52**	432**51**	234**15**	124**35**	341**52**	241**53**
1**543**2	4**532**1	23**154**	**412**35	34**51**2	24**51**3
1**45**32	**54**321	231**45**	412**53**	**3145**2	**2145**3
1**3**5**4**2	**435**21	**2143**5	4**512**3	314**25**	214**35**
13**45**2	**34**521	214**53**	41**52**3	31**524**	21**534**
134**25**	3**54**21	21**54**3	**14**523	**35**124	**25**134
13**254**	3**425**1	21**354**	14**25**3	**3125**4	**2135**4
132**45**	342**15**	213**45**	142**35**	312**45**	213**45**

3 Algorithmic considerations

In this part, we provide the procedure *gen_up*$(size, k)$ for generating our Gray codes for $S_n(T)$ characterized by the succession functions $\chi_T(k,i)$ given in Table 2. We initialize $\sigma := 123\dots(n-1)n$ and the call *gen_down*$(1,2)$ produces the Gray code $\mathcal{S}_n(T)$ (the procedure *gen_down*$(size, k)$ consists in the statements of *gen_up*$(size, k)$ in reverse order). This algorithm is inspired from the one in [11]. Only the order of the statements are modified according to the two sequences L and L'. Notice that the sequence L' is used if and only if we have the hypotheses of Lemma 7, *i.e.* when the current permutation σ produces a list Π' obtained from L'. This algorithm enables us to ensure that we transform an object into its successor in constant amortized time. Indeed, the degree of each call is not 1; moreover, between two recursive calls we execute at most two transpositions when we update the current permutation σ. Moreover at least one permutation is generated in each recursive call. This means that the total amount of computation divided by the number of objects is bounded by a constant. Thus the complexity of this algorithm is $\mathcal{O}(|S_n(T)|)$. An applet for the generation of Gray codes $\mathcal{S}_n(T)$ for each set T cited in Table 2, is available on the web site http://www.u-bourgogne.fr/ jl.baril/applet.html.

```
procedure gen_up(size, k)
dr := 0;
if size = n then output σ;
else
    for i := 1 to k do
        update σ
        if we have hypotheses of Lemma 7 then
            gen_down(size + 1, χ_T(k, L'(i)));
            dr := 1 − dr;
        else
            update σ
            if dr = 1 then gen_up(size + 1, χ_T(k, L(i)));
            else gen_down(size + 1, χ_T(k, L(i)));
            end if
            dr := 1 − dr;
        end if
end if
end procedure;
```

4 Acknowledgements

I thank the anonymous referees for their constructive remarks which have greatly improved this paper.

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Table 2
Classes of permutations avoiding patterns in T with their corresponding succession functions $\chi_T(i,k)$. We give also the different Lemmas that are used in order to obtain a Gray code and the maximum numbers such that two consecutive permutations differ in [11] and in our Gray code. A dot in the last column means that there is no Gray code for these classes in the literature.

Classes	T	$\chi_T(i,k)$	Lemmas	[11],B
2^{n-1} [6]	$\{321, 312\}$	2	$2, 3, 5$	$3, 2$
Catalan [26]	$\{312\}$	$i+1$	$2, 3, 4, 6, 8$	$5, 3$
	$\{321\}$	$k+1$ if $i=1$ i otherwise	$2, 3, 4, 6, 7, 8$	$5, 3$
Even index Fibonacci [6]	$\{321, 3412\}$	$k+1$ if $i=1$ 2 otherwise	$2, 3, 5, 6, 8$	$5, 3$
	$\{321, 4123\}$	3 if $i=1$ i otherwise	$2, 3, 4, 5, 6, 8$	$5, 3$
	$\{321, 4321\}$	3 if $i=3$ & $k=3$ $i+1$ otherwise	$2, 3, 4, 6, 8$	$\cdot, 3$
Pell [6]	$\{321, 3412, 4123\}$	3 if $i=1$ 2 otherwise	$2, 3, 5, 6, 8$	$5, 3$
	$\{312, 4321, 3421\}$	3 if $i=2$ 2 otherwise	$2, 3, 5, 6, 8$	$\cdot, 3$
Schröder [13]	$\{4321, 4312\}$	$k+1$ if $i=1$ or $i=2$ i otherwise	$2, 3, 4, 6, 8$	$5, 3$
	$\{4231, 4132\}$	$k+1$ if $i=1$ or $i=k$ $i+1$ otherwise	$2, 3, 4, 6, 8$	$5, 3$
	$\{4123, 4213\}$	$k+1$ if $k-1 \leq i \leq k$ $i+2$ otherwise	$2, 3, 4, 6, 8$	$5, 3$
Central binomial coefficient [13]	$\{4321, 4231, 4312, 4132\}$	$k+1$ if $i=1$ 3 if $i=2$ i otherwise	$2, 3, 4, 6, 8$	$5, 3$
	$\{4231, 4132, 4213, 4123\}$	3 if $i=1$ $i+1$ otherwise	$2, 3, 4, 6, 8$	$5, 3$
Generalized pattern [6,10]	$\{321, (p+1)12\dots p\}$	$k+1$ if $i=1$ & $k<p$ p if $i=1$ & $k=p$ i otherwise	$2, 3, 4, 6, 7, 8$	$5, 3$
	$\{321, 3412, (p+1)12\dots p\}$	$k+1$ if $i=1$ & $k<p$ p if $i=1$ & $k=p$ 2 otherwise	$2, 3, 5, 6, 8$	$5, 3$
Generalized Motzkin [5]	for $p \geq 2$, $\{321, (p+2)\bar{1}(p+3)2\dots(p+1)\}$	$k+1$ if $i=1$ i if $2 \leq i \leq p$ $i-1$ otherwise	$2, 3, 4, 6, 7, 8$	$\cdot, 3$
Generalized Schröder [18]	$\cup_{\tau \in S_{n-1}}\{(p+1)\tau p\}$	$k+1$ if $k \leq p$ or $p+i \geq k+1$ $i+p-1$ otherwise	$2, 3, 4, 6, 8$	$5, 3$