Order
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The Phagocyte Lattice of Dyck Words

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Abstract We introduce a new lattice structure on Dyck words. We exhibit efficient algorithms to compute meets and joins of Dyck words.

Key words lattice · Dyck words · noncrossing partitions

1. Introduction

A large number of various classes of combinatorial objects are enumerated by the well-known Catalan numbers. It is the case, among others, of ballot sequences, planar trees, Young tableaux, nonassociative products, stack sortable permutations, and so on. A list of over 60 types of such combinatorial classes of independent interest has been compiled by Stanley [26]. A certain number of explicit bijections between these Catalan classes can also be found in the literature.

From an order theoretic point of view, there are some partial ordering relations on Catalan sets which endow them with a lattice structure. Of much interest are the following lattices. First, the so-called Tamari lattice can be obtained equivalently in three different ways. The coverings correspond to elementary transformations in three Catalan classes, namely to reparenthesizations of letters products [6], to rotations on binary trees [17, 24], and to diagonal flips in triangulations [10, 24]. See references in [18]. Second, the lattice of noncrossing partitions is equipped with the refinement order [11]. Under this partial ordering $\leq$, two partitions π and π' satisfy $\pi \leq \pi'$ if every block of π is a subset of some block of π'. See [3, 4, 20, 22] and numerous references in the exhaustive survey [23]. More recently, a link between

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Dyck paths, noncrossing partitions and 321-avoiding permutations allows to define a new distributive lattice structure on these Catalan sets [1, 5].

In this paper, we introduce a new lattice structure on Dyck words (also called well-formed parentheses strings). We define a 'phagocyte' mutation on Dyck words in the following way. When in a Dyck word a Dyck subword w occurs immediately at the left of a word of the form $((\cdots(())\cdots))$, w is inserted at the center of this word. Thus we obtain $((\cdots((w))\cdots))$. Thereby, when $w((\cdots(())\cdots))$ occurs in a Dyck word, then w is 'phagocyted' by $((\cdots(())\cdots))$, and so is changed to $((\cdots((w))\cdots))$. w has been 'gobbled' by a series of nested parentheses. This elementary mutation endows the Catalan set of Dyck words with a lattice structure. We show how to compute meets and joins of Dyck words efficiently. After Grätzer, which cites Tamari lattices "to dispel the impression that it is always easy to prove that a poset is a lattice" [9, p. 18], we should like to say that the currently studied lattice falls into the same category.

2. Dyck Words and Noncrossing Partitions

The set D of Dyck words over $\{(,)\}$ is the language generated by the grammar $S \to \lambda|(S)|SS$, i.e. the set of legal strings of matched parentheses. Let us denote D_n the set of Dyck words with n open and n close parentheses. The cardinality of D_n is the *n-th* Catalan number: $card(D_n) = (2n)!/(n!(n+1)!)$. The open (resp. close) parentheses of $w \in D_n$ are numbered from 1 to n by traversing w from left to right.

A partition $b_1/b_2/\cdots/b_k$ of $[1,n] = \{1, 2, \cdots, n\}$ into blocks b_i is called noncrossing if there do not exist four numbers $p < q < r < s$ such that $p, r \in b_i$ and $q, s \in b_j$ with $i \neq j$. Thus 146/23/5 is a noncrossing partition of $[1, 6]$ (ncp in short) while 135/2/46 is crossing. Let us denote NC_n the set of all ncp of $[1, n]$. We have $card(NC_n) = card(D_n)$. A ncp will be written by listing the elements in each block in increasing order, and the blocks in increasing order of their minima. There are various ways to represent a ncp. For example, in the linear representation, $[1, n]$ appears as usual on the real line and successive elements in the same block are joined by an arc in the first quadrant. In this paper, we will use what we call the canonical representation. Given a ncp $\pi \in NC_n$, let us define the relation R_π on $[1, n]$ by $i\, R_\pi\, j$ iff i is the smallest element of the block of π which contains j $(i \leq j)$. The canonical representation of π is obtained by drawing the directed graph of R_π on the first quadrant. In these two representations, the noncrossing property of the partition corresponds to the fact that arcs do not intersect.

Since $card(D_n) = card(NC_n)$, let us exhibit an explicit bijection between D_n and NC_n [7, 8, 22, 23]. Given a ncp $\pi \in NC_n$ and its canonical representation, for each visited element $i \in [1, n]$ (for i going from 1 to n), we write r open parentheses $(r \geq 0)$ if there are r arcs starting in i and next we write one close parenthesis since there is always exactly one arc ending in i. Figure 1 illustrates this bijection.

$$w = ((()(()))(()())())$$

$$\pi_w = 149/23/57/6/8$$

$$c_w = (1, 2, 2, 1, 5, 6, 5, 8, 1)$$

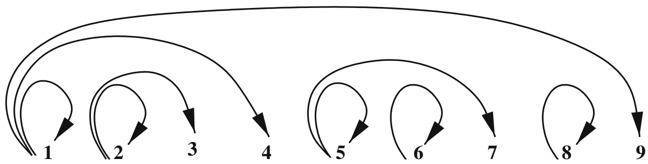

Figure 1 The canonical representation of a Dyck word w of D_9.

We denote by π_w the ncp corresponding to the Dyck word w. Let us remark that the number $nbl(\pi_w)$ of blocks of π_w is equal to the number of occurrences of '()' in w. Given $w \in D_n$, we define the vector c_w where $c_w[i]$ is the smallest element of the block of π_w which contains i $(1 \leq i \leq n)$.

In the sequel, the strict (respectively, large) inclusion relation will be denoted by the symbol $\subset$ (respectively, $\subseteq$).

3. The Phagocyte Transformation

DEFINITION 1. *The phagocyte transformation* $\longrightarrow$ *on* D_n *is defined by* $w \longrightarrow w'$ *if we have* $w = u'v(^n)^n u''$ *and* $w' = u'(^n v)^n u''$, *where* v *is a non-empty Dyck word,* $n \geq 1$ *and* $u', u'' \in \{(,)\}^*$. *Let* $\xrightarrow{*}$ *denote the reflexive transitive closure of* $\longrightarrow$.

LEMMA 1. *For all* $w, w' \in D_n$, *we have* $w \longrightarrow w'$ *iff* π_w *and* $\pi_{w'}$ *hold:*

- *one among the blocks of* $\pi_{w'}$ *is the union of two blocks* b_1 and b_2 of π_w where $\min(b_1) < \min(b_2)$ and $b_2 = [\min(b_2), max(b_2)]$ *is an interval,*
- *every block of* π_w *different from* b_1 *and* b_2 *is also a block of* $\pi_{w'}$.

Proof. This result is a simple application of Definiton 1. Remark that in Figure 2, the covering $1/24/3 \rightarrow 124/3$ is in NC_4 but is not in D_4 since 24 is not an interval. □

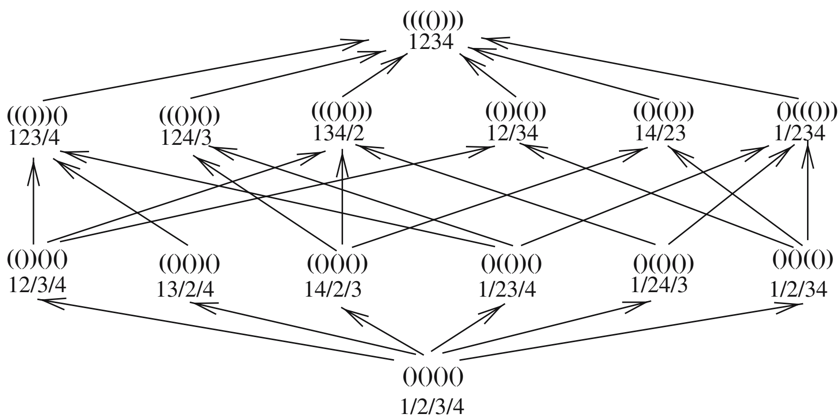

Figure 2 The phagocyte lattice D_4.

PROPOSITION 1. *$(D_n, \stackrel{*}{\longrightarrow})$ is a poset with zero $\mathbf{0} = ()()\ldots()$ and unit $\mathbf{1} = ((\ldots()\ldots))$, which is graded by the rank function $r(w) = n - nbl(\pi_w)$.*

Proof. The integer valued function r defined by $r(w) = n - nbl(\pi_w)$ verifies obviously $r(w') = r(w) + 1$ if $w \longrightarrow w'$. □

See the diagram of D_4 in Figure 2. In order to prove that the poset $(D_n, \stackrel{*}{\longrightarrow})$ is a lattice, we first give a Lemma on the structure of noncrossing partitions, and then we characterize the reflexive transitive closure of the phagocyte transformation.

LEMMA 2. *Let $\pi \in NC_n$. If b is a block of π which is not an interval, then there exists a block b' of π verifying the following two conditions:*

– $b' \subset [\min(b), \max(b)]$
– *b' is an interval possibly reduced to a singleton.*

Proof. Given a block b of π, let us consider in $[\min(b), \max(b)]$ the leftmost largest interval $I = [x, y]$ such that $I \cap b = \emptyset$. If I is a block of π, the result holds. If I is not a block of π, then let us consider the block b_1 of π such that $x = \min(b_1)$. Remark that b_1 exists by the noncrossing property. We have $b_1 \subset I$ and we repeat this process with the new block b_1 of π. So we construct a finite sequence of non-empty blocks b_i $(1 \leq i \leq r)$ such that $b_{i+1} \subset [\min(b_i), \max(b_i)]$ for each i. The last block b_r of the sequence is an interval possibly reduced to a singleton. Moreover b_r verifies the condition: $b_r \subset [\min(b), \max(b)]$. □

THEOREM 1. *For all $w, w' \in D_n$, we have $w \stackrel{*}{\longrightarrow} w'$ iff the following two conditions $(C1)$ and $(C2)$ hold:*

– *$(C1)$ every block of π_w is a subset of some block of $\pi_{w'}$,*
– *$(C2)$ for all blocks $b_1, b_2 \in \pi_w$, $b' \in \pi_{w'}$ such that $b_1 \cup b_2 \subset b'$ and $\min(b_1) < \min(b_2)$ then $[\min(b_2), \max(b_2)] \subset b'$.*

Remark 1. Condition (C1) is the well-known refinement order on partitions [11, 23]. Condition (C2) means that any block $b' \in \pi_{w'}$ is made up of one block $b_i \in \pi_w$ and possibly disjoint intervals $[k, l]$ for which one block $b_j \in \pi_w$ $(j > i)$ exists such that $k = \min(b_j)$ and $l = \max(b_j)$.

Proof. First let us suppose that $w \stackrel{*}{\longrightarrow} w'$ and let us apply the phagocyte transformation on the Dyck subword $v((\ldots()\ldots))$ of $w \in D_n$ in order to obtain the Dyck word w_1. Let $i \geq 0$ be the number of close parentheses of w before the first open parenthesis of v and let $j > i$ be the number of close parentheses of w before the first open parenthesis of the subword $((\ldots()\ldots))$ defined above. So there are in π_w a block b containing $i+1$ and a block $[j+1, j+k]$ which is an interval with $k \geq 1$. The result is a Dyck word w_1 which contains the Dyck subword $((\ldots(v)\ldots))$. Thus

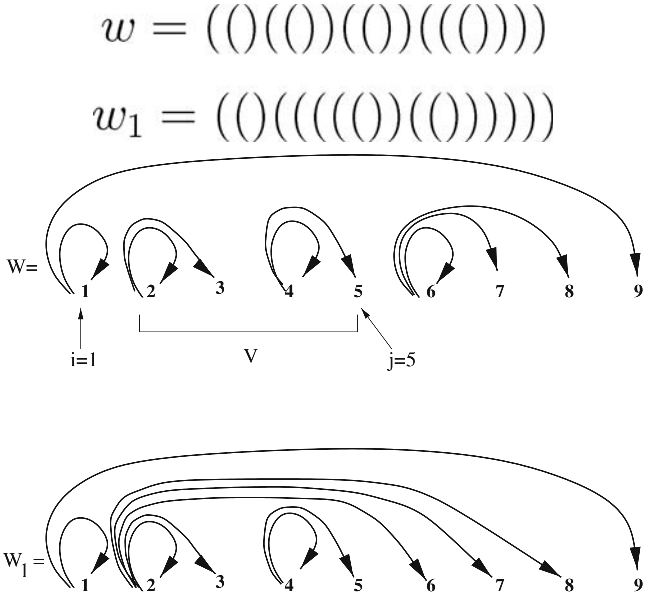

Figure 3 A phagocyte mutation $w \longrightarrow w_1$; $\pi_w = 19/23/45/678$ and $\pi_{w_1} = 19/23678/45$.

π_{w_1} has a block which contains b and $[j+1, j+k]$. See Figure 3. So conditions $(C1)$ and $(C2)$ are satisfied for w and w_1 such that $w \longrightarrow w_1$. By applying the phagocyte mutation several times, conditions $(C1)$ and $(C2)$ are also satisfied for w and w' such that $w \overset{*}{\longrightarrow} w'$. □

Conversely, suppose that w and w' verify the two conditions $(C1)$ and $(C2)$, and denote it by $w \prec w'$. Let us consider $w, w' \in D_n$ such that $w \neq w'$ and $w \prec w'$. Consider also $\pi_w = b_1/b_2/\ldots$ and $\pi_{w'} = b'_1/b'_2/\ldots$ with $\min(b_i) < \min(b_{i+1})$ and $\min(b'_i) < \min(b'_{i+1})$ for all $i \geq 1$. Let us denote by m the smallest integer such that $b_m \neq b'_m$. Since $b_1 = b'_1, \ldots, b_{m-1} = b'_{m-1}$, we have $\min(b_m) = \min(b'_m) = i_0$. By condition $(C1)$, we have $b_m \subset b'_m$ since $i_0 \in b_m \cap b'_m$. Now let j_0 be the smallest integer such that $j_0 \notin b_m$ and $j_0 \in b'_m$. Let b_p $(p \geq m+1)$ be the block of π_w which contains j_0. Since $j_0 \in b_p \cap b'_m$, we have $b_p \subseteq b'_m$ and by the condition $(C2)$ we have $[j_0 = \min(b_p), \max(b_p)] \subseteq b'_m$. Then there are two cases to consider:

- (a): b_p is an interval.
  Let denote $j_0 = \min(b_p)$. Let v be the largest Dyck subword of w, the last close parenthesis of which has the number $j_0 - 1$ and the first open parenthesis of which has the number $y \geq i_0$. So there exists in w a Dyck subword $v(\ldots()\ldots)$ such that the first open (resp. last close) parenthesis of $(\ldots()\ldots)$ has the number $j_0 = \min(b_p)$ (resp. $\max(b_p)$). If we apply on w the phagocyte transformation $v(\ldots()\ldots) \longrightarrow (\ldots(v)\ldots)$, we obtain a Dyck word w_1. We have $w_1 \prec w'$. Indeed π_w and π_{w_1} have the same blocks except one block which is the union of b_m and b_p.

- (b): b_p is not an interval, i.e. $b_p \subset [\min(b_p), \max(b_p)]$. Let denote $j_0 = \min(b_p)$. According to Lemma 2, there exists a block b_l $(l \geq p)$ of π_w which is an interval $b_l = [r, s]$ included in $[\min(b_p), \max(b_p)]$. Observe that $r = s$ can occur. We choose the rightmost block b_l. Let b be the block of π_w which contains $r-1$ and $x = min(b)$. Let v be the largest Dyck subword of w the last close parenthesis of which has the number $r-1$ and the first open parenthesis of which has the number $y \geq x$. Then, we apply on w the phagocyte mutation $v(\ldots()\ldots) \longrightarrow (\ldots(v)\ldots)$ where the first open (resp. last close) parentheses of the nested parenthesis $((\ldots()\ldots))$ has number r (resp. s). We obtain the Dyck word w_1 which verifies $w \longrightarrow w_1$ and $w_1 \prec w'$ as above.

By repeating this process, we find a finite sequence of Dyck words w_k such that $w_k \prec w'$ for all k and $w \longrightarrow w_1 \longrightarrow w_2 \ldots \longrightarrow w_l = w'$. By transitivity $w \xrightarrow{*} w'$ follows.

Remark 2. This proof is constructive since we exhibit a minimal length path between w and w' if $w \xrightarrow{*} w'$.

THEOREM 2. *For all n, the poset* $(D_n, \xrightarrow{*})$ *is a lattice.*

Proof. It suffices to show that any two elements of D_n have a greatest lower bound. The existence of least upper bounds then follows automatically since D_n is finite. Given $w, w' \in D_n$ with $w \neq w'$, let us consider $b \in D_n$ such that $b \xrightarrow{*} w$, $b \xrightarrow{*} w'$ and b maximal, i.e. there does not exist $b' \in D_n$ $(b' \neq b)$ such that $b \xrightarrow{*} b'$, $b' \xrightarrow{*} w$, $b' \xrightarrow{*} w'$. Assume that there exists $a \in D_n$ maximal such that $a \xrightarrow{*} w$, $a \xrightarrow{*} w'$ and $a \neq b$. We denote:

$$\pi_a = A_1/A_2/\ldots,$$

$$\pi_b = B_1/B_2/\ldots,$$

$$\pi_w = W_1/W_2/\ldots,$$

$$\pi_{w'} = W'_1/W'_2/\ldots.$$

Let m be the smallest integer such that $A_m \neq B_m$. Obviously there exists x such that $x \in A_m$ and $x \notin B_m$ or $x \notin A_m$ and $x \in B_m$. Let us define W_k (respectively, W'_l) as the block of π_w (respectively, $\pi_{w'}$) which contains x. We have $A_m \subseteq W_k$ and $A_m \subseteq W'_l$. Moreover, $B_m \subset W_k$ and $B_m \subset W'_l$ since $A_m \cap B_m \neq \emptyset$ and $\min(A_m) = \min(B_m)$. By Remark 1 we have:

$$W_k = A_{i_0} \cup \bigcup_{i \in I} [r_i, \sim s_i] \text{ and}$$

$$W_k = B_{i_0} \cup \bigcup_{j \in J} [u_j, \sim v_j]$$

where $i_0 \leq m$, and for each $i \in I$ (resp. $j \in J$) r_i and s_i (resp. u_j and v_j) are the minimum and the maximum of the same block of π_a (resp. π_b). We have as well: $W'_l = A_{i'_0} \cup \bigcup_{i' \in I'} [r'_{i'}, \sim s'_{i'}]$ and $W'_l = B_{i'_0} \cup \bigcup_{j' \in J'} [u'_{j'}, \sim v'_{j'}]$.

Since $x \notin B_m$ and $A_{i_0} = B_{i_0}$ for all $i_0 < m$, x belongs to an interval $[u_{j_0}, v_{j_0}]$, $j_0 \in J$, where u_{j_0} and v_{j_0} are in a same block B_x of π_b ($x \in B_x$). Similarly x belongs to an interval $[u'_{j'_0}, v'_{j'_0}]$ with $j'_0 \in J'$.

Since b verifies the noncrossing property, we have necessarily

$$[u_{j_0}, v_{j_0}] \subseteq [u'_{j'_0}, v'_{j'_0}] \text{ or}$$

$$[u'_{j'_0}, v'_{j'_0}] \subseteq [u_{j_0}, v_{j_0}]$$

Assume that the first case holds.

If $[u'_{j'_0}, v'_{j'_0}] \neq [u_{j_0}, v_{j_0}]$ then $[u'_{j'_0}, v'_{j'_0}] \setminus [u_{j_0}, v_{j_0}]$ has some elements below and above the interval $[u_{j_0}, v_{j_0}]$. Let us remark that the block containing $u'_{j'_0}$ surrounds the block B_x.

In order to transform b into w, we must merge the block B_x with the block B_{i_0}. So the only possibility is either that the block which contains $u'_{j'_0}$ is in B_{i_0} or that we also must merge B_x with B_{i_0}. This proves that the interval $[u'_{j'_0}, v'_{j'_0}]$ is included into W_k.

Hence, each block included in $[u'_{j'_0}, v'_{j'_0}]$ is also included into W_k and W'_l, which is trivially verified if $[u'_{j'_0}, v'_{j'_0}] = [u_{j_0}, v_{j_0}]$. We can built $b' \in D_n$ such that $\pi_{b'}$ and π_b have the same blocks except the block containing $u'_{j'_0}$ which is replaced by its union with each block of π_b included in $[u_{j_0}, v_{j_0}]$. So $b' \in D_n$ verifies $b \neq b'$ and $b \overset{*}{\longrightarrow} b'$. By construction $b' \overset{*}{\longrightarrow} w$ and $b' \overset{*}{\longrightarrow} w'$, which contradicts the maximality of b. □

In the following propositions, we enumerate the join- and meet-irreducible elements of D_n. Recall that $x \in D_n$ is a join (resp. meet)-irreducible element if $x = a \vee b$ (resp. $x = a \wedge b$) implies $x = a$ or $x = b$. In other words, join (resp. meet)-irreducible elements are the elements that have an unique lower (resp. upper) cover. The proofs are easily obtained.

PROPOSITION 2. *The number of join-irreducible elements in the phagocyte lattice D_n is $n(n-1)/2$.*

Proof. It is enough to count all ncps consisting of one block with two elements and all other blocks having one element. □

PROPOSITION 3. *The number of meet-irreducible elements in the phagocyte lattice D_n is $2^{n-1} - 1$.*

Proof. Let denote $X = \{(^p)^p, p \geq 1\}$ and $Y = \{(^q, q \geq 1\}$. Then elements we are interested in are exactly alternative sequences of element in X and Y followed by two elements in X and some), i.e. the sequences of the form $x_1 y_2 x_3 \ldots y_{k-2} x_{k-1} x_k)^r$ where $x_i \in X$, $y_i \in Y$ and $r \geq 0$. Thus the number of these sequences equals to the number of non-trivial compositions of n i.e. $2^{n-1} - 1$. □

4. Computing Meets and Joins

Using Theorem 1, we exhibit algorithms for computing the meet $w \wedge w'$ and the join $w \vee w'$ of two Dyck words $w, w' \in D_n$.

Meet algorithm:
for $i := 1$ **to** n **do** $c_{w\wedge w'}[i] := 0$ **enddo**
for $i := 1$ **to** n **do**
  if $c_{w\wedge w'}[i] = 0$ **then**
    if $c_w[i] < i$ **or** $c_{w'}[i] < i$ **then**
      $j_0 := i$;
      for $j := i$ **to** n **do**
        if $c_{w\wedge w'}[j] = 0$ **and** $c_w[j] = c_w[i]$ **and** $c_{w'}[j] = c_{w'}[i]$ **then**
          $c_{w\wedge w'}[j] := j_0$;
        else $j_0 := j$; **endif**
      endo
    else
      for $j := i$ **to** n **do**
        if $c_w[j] = i$ **and** $c_{w'}[j] = i$ **then**
          $c_{w\wedge w'}[j] := i$; **endif**
      enddo
    endif
  endif
enddo

Proof and analysis of the meet algorithm. Suppose the current element $c_{w\wedge w'}[i]$ has been computed for all $i < i_0$. For computing $c_{w\wedge w'}[i_0]$ there are two cases to consider:

- (1): $c_w[i_0] = i_0$ and $c_{w'}[i_0] = i_0$.
  So there exist two blocks $b \in \pi_w$ and $b' \in \pi_{w'}$ such that $i_0 = \min(b) = \min(b')$. We put $c_{w\wedge w'}[i_0] = i_0$. Now for $j \geq i_0$, let $b_j \in \pi_w$ (respectively, $b'_j \in \pi_{w'}$) be the block of π_w (respectively, $\pi_{w'}$) which contains j. If we have $i_0 \in b_j$ and $i_0 \in b'_j$ then we put also $c_{w\wedge w'}[j] = i_0$. We repeat this process for all $j \geq i_0$.
- (2): $c_w[i_0] < i_0$ or $c_{w'}[i_0] < i_0$.
  Assume that $c_w[i_0] < i_0$. Thus there exists a block $b \in \pi_w$ such that $\min(b) = c_w[i_0] < i_0$ and $i_0 \in b$. Since $c_{w\wedge w'}[i_0] = 0$ is not determined by the algorithm, i_0 does not belong to the block b' of $\pi_{w\wedge w'}$ which contains $c_w[i_0] < i_0$. To construct a path between $w \wedge w'$ and w, we must merge the block b'' of $\pi_{w\wedge w'}$ which contains i_0 with the block b' since i_0 and $c_w[i_0]$ are in the same block $b \in \pi_w$. Using Theorem 1, we obtain $[\min(b''), \max(b'')] \subseteq b$. So b'' is the greatest interval $b'' = [i_0, k]$ with $k \geq i_0$ verifying $b'' \subseteq b$ and b'' is also included in a block of $\pi_{w'}$. We repeat this process for all integers greater than i_0.

Let σ be the noncrossing partition corresponding to the vector $c_{w\wedge w'}$ obtained by the above algorithm. Then σ verifies the followings properties:

First by construction, each block of σ is included into a block of π_w and a block of $\pi_{w'}$.

Second if there exist a block b_σ in σ and a block b_π in π_w (respectively, $\pi_{w'}$) such that $\min(b_\sigma) = \min(b_\pi)$, then $\pi_w \backslash b_\sigma$ is by construction an union of intervals I verifying (i) I is also a block of σ and $\min(I) \geq \min(\sigma)$, (ii) I is included into a block of π_w (respectively, $\pi_{w'}$). From Theorem 1, we obtain $\sigma \xrightarrow{*} \pi_w$ and $\sigma \xrightarrow{*} \pi_{w'}$.

Moreover, we determine at each step of algorithm the greatest block verifying the two previous properties. This means that σ is the greatest noncrossing partition such that $\sigma \xrightarrow{*} \pi_w$ and $\sigma \xrightarrow{*} \pi_{w'}$. Indeed in the case (1), we consider the greatest set containing i_0 which is included into a block of π_w and $\pi_{w'}$ and in the case (2), we consider the greatest interval containing i_0 which is also included into a block of π_w and $\pi_{w'}$. So, σ corresponds to the noncrossing partition of $w \wedge w'$.

The space complexity is $O(n)$. The time complexity is $O(n^2)$ because computing $nbl(\pi_w)$ requires $O(n)$ time and the meet algorithm requires two nested loops.

We can easily define a recursive algorithm to compute the join $\pi \sqcup \pi'$ of two ncp π and π' in the classical lattice of noncrossing partitions. So we can suppose that $c_{w \sqcup w'}$ has been computed for all $w, w' \in D_n$. Assume that c_w and $c_{w'}$ are determined. Let d_w (respectively, $d_{w'}$) be the vector defined by $d_w[i]$ (respectively, $d_{w'}[i]$) is the greatest element of the block of π_w (respectively, $\pi_{w'}$) which contains i.

Join algorithm.

```
for i := 1 to n do
    if c_{w∨w'}[i] = 0 then
        for j := i to n if
            if c_{w⊔w'}[j] = c_{w⊔w'}[i] then
                c_{w∨w'}[j] := c_{w⊔w'}[i];
                if c_w[j] ≠ c_w[i] then
                    for k := j + 1 to d_w[j] do
                        c_{w∨w'}[k] := c_{w⊔w'}[i]; enddo
                endif
                if c_{w'}[j] ≠ c_{w'}[i] then
                    for k := j + 1 to d_{w'}[j] do
                        c_{w∨w'}[k] := c_{w⊔w'}[i]; enddo
                endif
            endif
        enddo
    endif
enddo
```

Proof of the join algorithm. Suppose the current element $c_{w \vee w'}[i]$ has been computed for all $i < i_0$. So we put $c_{w \vee w'}[i_0] = i_0$ and for all $j > i_0$ such that j and i_0 are in the same block in $\pi_{w \sqcup w'}$, we put $c_{w \vee w'}[j] = c_{w \sqcup w'}[i_0]$. We discuss the two following cases (1) and (2):

– (1): If $c_w[j] \neq c_w[i_0]$: i_0 and j do not belong to the same block in π_w. To construct a path between w and $w \vee w'$, we must include the interval $\left[j+1, d_w[j]\right]$ in the block of $\pi_{w \vee w'}$ which contains i_0. So if $k \in \left[j+1, d_w[j]\right]$, we put $c_{w \vee w'}[k] = c_{w \sqcup w'}[i_0]$.
– (2): If $c_{w'}[j] \neq c_{w'}[i_0]$. The proof is similar to the proof of (1).

Let σ (respectively, τ) be the noncrossing partition obtained by $c_{w\vee w'}$ (respectively, $c_{w\sqcup w'}$). By construction, every block of σ is included into a block of τ, so every block of π_w and $\pi_{w'}$ is included into a block of σ.

Moreover, if there exist a block b_τ in τ and a block b_π in π_w (respectively, $\pi_{w'}$) such that $\min(b_\tau) = \min(b_\pi)$ then $b_\tau \backslash b_\pi$ is by construction an union of blocks b in π_w (respectively, $\pi_{w'}$) with $\min(b) \geq \min(b_\tau)$. In order to obtain σ from τ, it is necessary to merge the block b_π with the intervals $[\min(b), \max(b)]$ (by Theorem 1). So at the end of the algorithm, σ verifies the conditions (C_1) and (C_2) of Theorem 1. Thus we obtain $\pi_w \stackrel{*}{\longrightarrow} \sigma$ and $\pi_{w'} \stackrel{*}{\longrightarrow} \sigma$.

The minimality of σ holds since at each step of this algorithm we run only the necessary operations. Thus $\sigma = \pi_{w\vee w'}$.

For example, if $\pi_w = 149/23/57/6/8$ and $\pi_{w'} = 147/23/56/89$, the algorithms give $\pi_{w\wedge w'} = 14/23/5/6/7/8/9$ and $\pi_{w\vee w'} = 1456789/23$.

5. Conclusion

In this paper, a new lattice structure has been defined on the Catalan sets of Dyck words via a natural transformation. The simple and natural definition of the phagocyte transformation is unfortunately at odds with the rather complex theorem which characterizes this transformation.

The greedy transformation $v() \to (v)$ defined in [7] is a particular case of the phagocyte transformation. However the poset obtained by the greedy transformation is not an effective lattice. Nevertheless, this poset is a graded lower semi-modular meet-semilattice. This property allows to compute the corresponding shortest path metric [12].

Some problems remain to be solved.

Is there an algorithm to compute the Möbius function of the phagocyte lattice of Dyck words as in [16]?

Computer experiments show that the number of coverings $cov(n)$ of D_n is equal to $3(2n+2)!/((n+1)!(n+4)!)$ for small values of n ($n \leq 6$). We obtain the first terms of the sequence **A003517**=$[1, 6, 27, 110, \cdots]$ included in the Sloane Encyclopedia [25]. This sequence enumerates two combinatorial classes of objects: (a) The set PIS_n of permutations on $[n]$ with exactly one increasing subsequence of length three [13] and (b) the set NIP_n of pairs of non-intersecting paths of length n and distance three [21]. In order to prove the equality $cov(n) = card(PIS_{n+1}) = card(NIP_{n+1})$, can we exhibit two explicit bijections between the coverings of D_n and two above combinatorial classes? Is there a polynomial time algorithm to compute the minimal path length distance between Dyck words in the phagocyte lattice [12]? If so, a new shortest-path-type metric could be obtained, and could be added to the existing metrics on Catalan sets [2, 7, 14, 15, 17, 19, 24]. Let us recall that we still do not know if the rotation distance on binary trees can be computed in polynomial time.

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