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Efficient lower and upper bounds of the diagonal-flip distance between triangulations

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Abstract

There remains today an open problem whether the rotation distance between binary trees or equivalently the diagonal-flip distance between triangulations can be computed in polynomial time. We present an efficient algorithm for computing lower and upper bounds of this distance between a pair of triangulations.

Keywords: Triangulations; Diagonal-flip distance; Approximation algorithms

1. Introduction

Culik and Wood defined in 1982 the rotation distance between a pair of binary trees as the minimum number of rotations needed to convert one tree into the other [5]. There exists a well-known explicit bijection between binary trees and triangulations. Thus a system that is isomorphic to binary trees related by rotations is that of triangulations of a convex polygon related by the diagonal-flip transformation. A diagonal-flip transformation is an operation that converts one triangulation into another by removing a diagonal in the triangulation and adding the diagonal that subdivides the resulting quadrilateral in the opposite way. Thus rotation distance of binary trees and diagonal-flip distance of triangulations are equivalent.

An open problem is the complexity status of computing the rotation distance between two binary trees or equivalently the diagonal-flip distance between two triangulations. Lucas has presented a quadratic time algorithm for computing the rotation distance between binary trees of restricted form [8]. But in the general case, there remains the open problem whether these distances can be computed in polynomial time.

Some upper bounds of these distances have been exhibited [6,9,14]. Some authors approach the problem by limiting the reshaping primitive to a restricted version of the general rotation operation [1–4,7,13,16]. Obviously these restricted rotation distances will be bounded below by the usual rotation distance. Another approach for computing an upper bound uses a "flexion" operation on binary trees [12].

In [11], a rough lower bound of the rotation distance is computed using ordinal tools. But to our knowledge, efficient lower bounds are not found in literature.

In this paper we present a polynomial time algorithm for computing lower and upper bounds. Computer ex-

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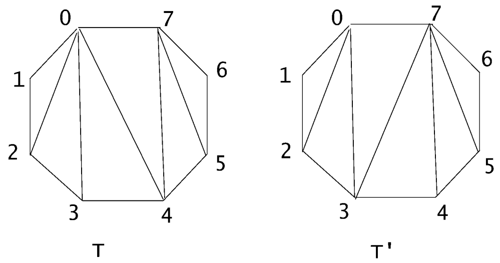

Fig. 1. A diagonal flip in $\mathcal{T}_8$.

periments show that these bounds are efficient. For simplification reasons, we prefer stating our results in terms of triangulations rather than binary trees.

2. Notations and definitions

Let us consider n-gons, i.e., convex polygons with n sides and with a distinguished side as the top. We label the vertices from 0 to $n-1$ counterclockwise such that the top has 0 and $n-1$ as two vertices. Any triangulation of the n-gon has $n-2$ triangles and $n-3$ non-crossing diagonals. Let $\mathcal{T}_n$ denote the set of triangulations of the n-gon. There is an explicit bijection between $\mathcal{T}_n$ and the set of binary trees with $n-2$ internal nodes (and thus $n-1$ leaves) [12,17]. The diagonal-flip operation on $\mathcal{T}_n$ is defined as follows. A diagonal inside the polygon is removed, creating a face with four sides. The opposite diagonal of this quadrilateral is inserted in place of the one removed, restoring the diagram to a triangulation of the polygon [17]. See Fig. 1. If $T, S \in \mathcal{T}_n$, let $\text{dist}(T, S)$ be the minimum number of diagonal-flips needed to transform T into S.

The internal degree $d_i(T)$ of a vertex i of $T \in \mathcal{T}_n$ is the number of diagonals incident to i ($0 \leqslant i \leqslant n-1$). In a pair (T, S) of triangulations of $\mathcal{T}_n$, the composite degree $cd_i(T, S)$ is the total number of diagonals incident to vertex i in *both* triangulations. If $T, S \in \mathcal{T}_n$ have a diagonal $\{i, j\}$ in common, we say that $\{i, j\}$ is a (2, 2)-diagonal if $cd_i(T, S) = cd_j(T, S) = 2$. Given two triangulations of $\mathcal{T}_n$, a flip-to-match diagonal is a diagonal in one of the triangulations which can be flipped to make it match a diagonal in the other [15]. Given $T, S \in \mathcal{T}_n$, we define the type of vertex i by $type_i(T, S) = (k : l)$ where $k = d_i(T)$ and $l = d_i(S)$.

Definition 1. [17] Given $T \in \mathcal{T}_n$, we define the normalized triangulation $N_T(i, j)$ (respectively $N'_T(i, j)$) with respect to the diagonal $\{i, j\}$ as follows:

(1) $N_T(i, j)$ and $N'_T(i, j)$ contain the diagonal $\{i, j\}$;
(2) $N_T(i, j)$ and $N'_T(i, j)$ contain every diagonal of T that does not cross the diagonal $\{i, j\}$;
(3) if T contains a diagonal $\{a, b\}$ that crosses the diagonal $\{i, j\}$, then $N_T(i, j)$ (respectively $N'_T(i, j)$) contains the diagonals $\{a, j\}$ and $\{b, j\}$ (respectively $\{a, i\}$ and $\{b, i\}$).

Definition 2. In case where $j = i + 2 \bmod n$, we say that $N_T(i, i+2)$ (respectively $N'_T(i, i+2)$) is the counterclockwise (respectively clockwise) normalization with respect to the vertex $i+1$ and we denote $N_T(i, i+2) = N_T(i+1)$ (respectively $N'_T(i, i+2) = N'_T(i+1)$).

Definition 3. Let $\{i, j\}$ be a diagonal of $T \in \mathcal{T}_n$ such that $d_i(T) = d_j(T) = 2$. Then T can be double-normalized with respect to the diagonal $\{i, j\}$ to create a new triangulation $N''_T(i, j)$ as follows:

(1) we flip in T the diagonal $\{i, j\}$,
(2) then we flip the diagonal adjacent to vertex i,
(3) and we flip the diagonal adjacent to vertex j.

Definition 4. Let i be a vertex of $T \in \mathcal{T}_n$ such that $d_i(T) = 3$. We denote $\{a, i\}$, $\{b, i\}$ and $\{c, i\}$ the three diagonals incident to vertex i with the vertices a, b, c in clockwise order. Then T can be pseudo-normalized with respect to vertex i to create a new triangulation $N'''_T(i)$ as follows: we flip the two diagonals $\{a, i\}$ and $\{c, i\}$, then we flip the middle diagonal $\{b, i\}$.

It is worth noting that each of these normalizations creates at least one diagonal of the form $\{i-1, i+1\} \bmod n$. Then the edges $\{i-1, i\}$ and $\{i, i+1\}$ will be "nibbled" away by the algorithm of Section 8.

3. Preliminaries

Lemma 1. [17] *Given $T, S \in \mathcal{T}_n$, if it is possible to flip one diagonal of T creating T_1 so that T_1 has one more diagonal in common with S than does T, then there exists a shortest path from T to S in which the first flip creates T_1.*

Lemma 2. [17] *Given $T, S \in \mathcal{T}_n$, if T and S have a diagonal in common, then a shortest path from T to S never flips this diagonal.*

Lemma 3. [8,17] *If $T, S \in \mathcal{T}_n$ share a common diagonal, this diagonal splits T (respectively S) into two subtriangulations T' and T'' (respectively S' and S'')*

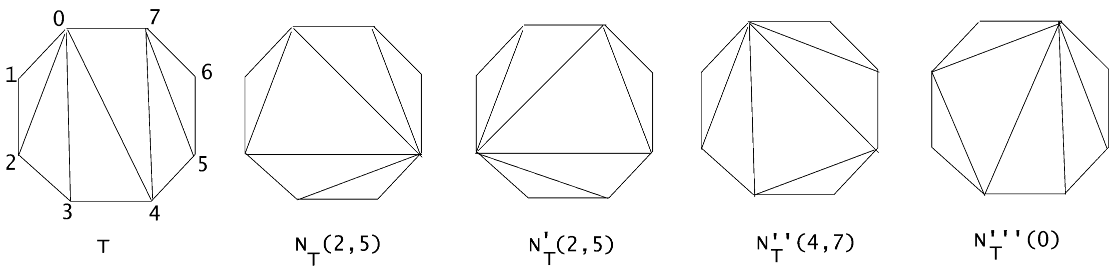

Fig. 2. Four normalizations of $T \in \mathcal{T}_6$.

in such a way that T' *and* S' (*and thus* T'' *and* S'') *have the same vertices. Then we have the following formula*:

$$\mathrm{dist}(T, S) = \mathrm{dist}(T', S') + \mathrm{dist}(T'', S'').$$

Lemma 4. *If* $T, S \in \mathcal{T}_n$, *then there exists a vertex* i *such that* $cd_i(T, S) \leqslant 3$.

Proof. Let us consider the graph which is obtained by matching T and S. We have $\sum_{i=1}^{n} cd_i(T, S) = 4(n-3)$. Thus the average verifies the inequality

$$\frac{1}{n}\sum_{i=1}^{n} cd_i(T, S) = \frac{4(n-3)}{n} < 4$$

and there exists at least a vertex i such that $cd_i(T, S) \leqslant 3$. □

According to Lemma 4, we study in the sequel the different types of vertices which can occur, i.e. $(2:0)$, $(1:1)$, $(3:0)$ and $(2:1)$. The case $(1:0)$ is a particular case of Lemma 1.

It is important to notice that, in Sections 4–7 below, we assume that the hypotheses of Lemmas 1 and 2 are not verified.

4. Properties of vertices of type (2 : 0)

Lemma 5. *Given* $T, S \in \mathcal{T}_n$ *with a vertex* i *such that* $type_i(T, S) = (2:0)$, *there exists a shortest path from* T *to* S *in which the two first flips create either* $N_T(i)$ *or* $N'_T(i)$.

Proof. Let $\mathcal{P}: T = T_0, T_1, T_2, \ldots, T_p = S$ be a shortest path connecting T and S in which T_l and T_{l+1} differ only by one flip. Since $d_i(S) = 0$, S contains the diagonal $\{i-1, i+1\} \bmod n$. Necessarily there exist in $\mathcal{P}$ two pairs (T_j, T_{j+1}) and (T_k, T_{k+1}) such that $d_i(T_j) = 2$, $d_i(T_{j+1}) = 1$, $d_i(T_k) = 1$ and $d_i(T_{k+1}) = 0$. There are two cases to consider:

Case 1: $d_{i+1}(T_{j+1}) > d_{i+1}(T_j)$. Then we have $N_{T_j}(i) = N_{T_{j+1}}(i)$ and $N_{T_k}(i) = N_{T_{k+1}}(i)$. Consider the sequence of triangulations: $N_{\mathcal{P}}(i) : N_T(i) = N_{T_0}(i), N_{T_1}(i), N_{T_2}(i), \ldots, N_{T_p}(i) = N_S(i)$. Successive triangulations of $N_{\mathcal{P}}(i)$ are either identical or differ only by one flip. The length of $N_{\mathcal{P}}(i)$ is at most $p-2$. Since $\mathrm{dist}(T, N_T(i)) = 2$, we have built a new shortest path from T to S which contains $N_T(i)$.

Case 2: $d_{i-1}(T_{j+1}) > d_{i-1}(T_j)$. Then we have $N'_{T_j}(i) = N'_{T_{j+1}}(i)$ and $N'_{T_k}(i) = N'_{T_{k+1}}(i)$. The proof concludes *mutatis mutandis*. □

Theorem 1. *Given* $T, S \in \mathcal{T}_n$ *with a vertex* i *such that* $type_i(T, S) = (2:0)$, *then we have*:

$$\mathrm{dist}(T, S) = \min\big(\mathrm{dist}\big(N_T(i), S\big), \mathrm{dist}\big(N'_T(i), S\big)\big) + 2$$

and

$$\mathrm{dist}(T, S) \geqslant \max\big(\mathrm{dist}\big(N_T(i), S\big), \mathrm{dist}\big(N'_T(i), S\big)\big) + 1.$$

Proof. The equality is a straightforward consequence of Lemma 5. It is worth noting that $\mathrm{dist}(N_T(i), N'_T(i)) = 1$. Thus

$$\begin{aligned}&\mathrm{dist}(T, S)\\ &\quad \geqslant \min\big(\mathrm{dist}\big(N_T(i), S\big), \mathrm{dist}\big(N_T(i), S\big) - 1\big) + 2\\ &\quad = \mathrm{dist}\big(N_T(i), S\big) + 1\end{aligned}$$

and

$$\mathrm{dist}(T, S) \geqslant \mathrm{dist}\big(N'_T(i), S\big) + 1$$

by symmetry. □

Theorem 2. *Given* $T, S \in \mathcal{T}_n$ *with a* $(2,2)$-*diagonal* $\{i, j\}$ *such that* $d_T(i) = 2$, $d_S(i) = 0$, $d_T(j) = 2$ *and* $d_S(j) = 0$, *we have*:

$$\mathrm{dist}(T, S) = \mathrm{dist}\big(N''_T(i, j), S\big) + 3.$$

Proof. According to Theorem 1: $\mathrm{dist}(T, S) = 2 + \min(\mathrm{dist}(N_T(i), S), \mathrm{dist}(N'_T(i), S))$. Thus there are two cases to consider (see Fig. 3):

Case 1: $d_j(N_T(i)) = 1$. Then we have: $\mathrm{dist}(N_T(i), S) = \mathrm{dist}(N''_T(i, j), S) + 1$ and $\mathrm{dist}(N'_T(i), S) \geqslant \mathrm{dist}(N''_T(i,$

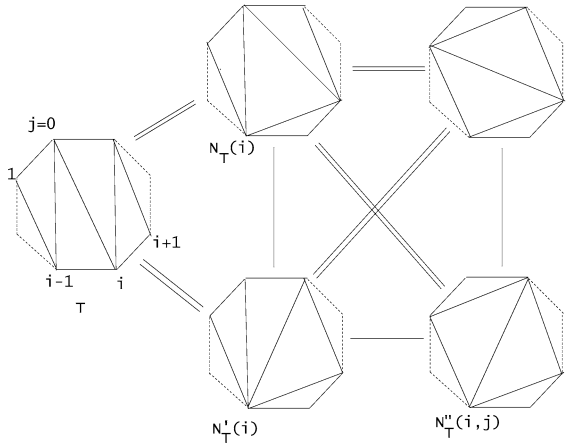

Fig. 3. Illustration of the proof of Theorem 2. A simple line is for distance one and a double line for distance two.

$j), S) + 1$. Thus there is a shortest path from T to S which contains $N''_T(i, j)$.

Case 2: $d_j(N'_T(i)) = 1$. Then we have $\operatorname{dist}(N'_T(i), S) = \operatorname{dist}(N''_T(i, j), S) + 1$ and $\operatorname{dist}(N_T(i), S) \geqslant \operatorname{dist}(N''_T(i, j), S) + 1$ proving that there exists a shortest path from T to S which contains $N''_T(i, j)$.

The proof concludes with $\operatorname{dist}(T, N''_T(i, j)) = 3$. $\square$

5. Properties of vertices of type (1 : 1)

Theorem 3. *Given $T, S \in \mathcal{T}_n$ with a vertex i such that $type_i(T, S) = (1 : 1)$ then the following inequalities hold*:

$$\operatorname{dist}\big(N_T(i), N_S(i)\big) + 1 \leqslant \operatorname{dist}(T, S) \leqslant \operatorname{dist}\big(N_T(i), N_S(i)\big) + 2.$$

Proof. Let $\mathcal{P} : T = T_0, T_1, T_2, \ldots, T_p = S$ be a shortest path connecting T and S in which T_l and T_{l+1} differ only by one flip. Let us consider the path: $T = T_0, N_{T_0}(i), N_{T_1}(i), N_{T_2}(i), \ldots, N_{T_p}(i), T_p = S$ of length $p + 2$. Two successive trees differ only by one flip, proving the right inequality. Since $type_i(T, S) = (1 : 1)$, $d_i(T) = d_i(S) = 1$, there are two cases to consider:

Case 1: there exists in $\mathcal{P}$ a tree T_q such that $d_i(T_q) = 0$. Thus $\{i - 1, i + 1\} \bmod n$ is a diagonal of T_q. By Lemma 1, there exists a shortest path connecting T and T_q which contains $N_T(i)$. Similarly there exists a shortest path connecting T_q and S which contains $N_S(i)$. We obtain $\operatorname{dist}(T, S) = \operatorname{dist}(N_T(i), N_S(i)) + 2 > \operatorname{dist}(N_T(i), N_S(i)) + 1$.

Case 2: there exists in $\mathcal{P}$ a tree T_r such that $d_i(T_r) = 1$ and $d_i(T_{r+1}) = 2$. We necessarily have either $N_{T_r}(i) = N_{T_{r+1}}(i)$ or $N'_{T_r}(i) = N'_{T_{r+1}}(i)$. In the first case, the length of the path $N_{\mathcal{P}}(i) : N_T(i), N_{T_1}(i), N_{T_2}(i), \ldots, N_S(i)$ connecting $N_T(i)$ and $N_S(i)$ is $p - 1$, proving $\operatorname{dist}(N_T(i), N_S(i)) \leqslant \operatorname{dist}(T, S) - 1$. The second case is similar, thus completing the proof of the left inequality. $\square$

6. Properties of vertices of type (3 : 0)

Theorem 4. *Given $T, S \in \mathcal{T}_n$ with a vertex i such that $type_i(T, S) = (3 : 0)$ then the following inequalities hold*:

$$\operatorname{dist}\big(N'''_T(i), S\big) + 1 \leqslant \operatorname{dist}(T, S) \leqslant \operatorname{dist}\big(N'''_T(i), S\big) + 3.$$

Proof. The right inequality follows trivially from $\operatorname{dist}(T, N'''_T(i)) = 3$. Let $\mathcal{P} : T = T_0, T_1, T_2, \ldots, T_p = S$ be a shortest path connecting T and S in which T_l and T_{l+1} differ only by one flip. Since $d_i(T) = 3$ and $d_i(S) = 0$, necessarily there exist in $\mathcal{P}$ three pairs of trees (T_q, T_{q+1}), (T_r, T_{r+1}), (T_s, T_{s+1}) such that $d_i(T_q) = 3$, $d_i(T_{q+1}) = 2$, $d_i(T_r) = 2$, $d_i(T_{r+1}) = 1$, $d_i(T_s) = 1$, $d_i(T_{s+1}) = 0$. We have either $N_{T_r}(i) = N_{T_{r+1}}(i)$ or $N'_{T_r}(i) = N'_{T_{r+1}}(i)$. But we have $N_{T_s}(i) = N_{T_{s+1}}(i)$ and $N'_{T_s}(i) = N'_{T_{s+1}}(i)$. There are two cases to consider:

Case 1: there exist $u, v \in \{q, r, s\}$ with $u \neq v$ such that $N_{T_u}(i) = N_{T_{u+1}}(i)$ and $N_{T_v}(i) = N_{T_{v+1}}(i)$. Let us consider the path $N_{\mathcal{P}}(i) : N_T(i) = N_{T_0}(i), N_{T_1}(i), N_{T_2}(i), \ldots, N_{T_p}(i) = N_S(i) = S$ connecting $N_T(i)$ and S. Indeed $N_S(i) = S$ since $d_i(S) = 0$. The length of $N_{\mathcal{P}}(i)$ is $p - 2$ proving $\operatorname{dist}(N_T(i), S) + 2 \leqslant \operatorname{dist}(T, S)$.

Case 2: there exist $u', v' \in \{q, r, s\}$ with $u' \neq v'$ such that $N'_{T_{u'}}(i) = N'_{T_{u'+1}}(i)$ and $N'_{T_{v'}}(i) = N'_{T_{v'+1}}(i)$. Similarly $\operatorname{dist}(N'_T(i), S) + 2 \leqslant \operatorname{dist}(T, S)$ holds.

It is enough to observe that $\operatorname{dist}(N_T(i), N'''_T(i)) = 1$ and $\operatorname{dist}(N'_T(i), N'''_T(i)) = 1$. In particular we deduce $\operatorname{dist}(N'''_T(i), S) \leqslant 1 + \operatorname{dist}(N_T(i), S) \leqslant 1 + \operatorname{dist}(T, S) - 2$ proving the left inequality. $\square$

7. Properties of vertices of type (2 : 1)

Theorem 5. *Given $T, S \in \mathcal{T}_n$ with a vertex i such that $type_i(T, S) = (2 : 1)$ then the following inequalities hold*:

$$\operatorname{dist}(T, S) \geqslant \max\big(\operatorname{dist}\big(N_T(i), N_S(i)\big), \operatorname{dist}\big(N'_T(i), N'_S(i)\big)\big) + 1,$$

$$\operatorname{dist}(T, S) \leqslant \min\big(\operatorname{dist}\big(N_T(i), N_S(i)\big), \operatorname{dist}\big(N'_T(i), N'_S(i)\big)\big) + 3.$$

Proof. Since $d_i(T) = 2$ and $d_i(S) = 1$, we have $\mathrm{dist}(T, N_T(i)) = 2$ and $\mathrm{dist}(S, N_S(i)) = 1$ proving the second inequality. For proving the first inequality, let $\mathcal{P}: T = T_0, T_1, T_2, \ldots, T_p = S$ be a shortest path connecting T and S in which two consecutive trees differ only by one flip. We denote $\{i, a\}$ and $\{i, b\}$ the two diagonals of T incident to vertex i with a and b in clockwise order. Now let $f \geqslant 1$ be the smallest $s \in [1, p]$ such that the diagonal $\{i, a\}$ is not the first diagonal (in clockwise order) adjacent to i in T_s. Similarly let $\ell \geqslant 1$ be the smallest $r \in [1, p]$ such that the diagonal $\{i, b\}$ is not the last diagonal (in clockwise order) adjacent to i in T_r. Thus we necessarily have $N_{T_{f-1}}(i) = N_{T_f}(i)$ and $N'_{T_{\ell-1}}(i) = N'_{T_\ell}(i)$. Therefore we obtain $\mathrm{dist}(N_T(i), N_S(i)) \leqslant \mathrm{dist}(T, S) - 1$ and $\mathrm{dist}(N'_T(i), N'_S(i)) \leqslant \mathrm{dist}(T, S) - 1$ proving the first inequality. □

8. Computing lower and upper bounds

The goal of the following algorithm is to "nibble" the edges (and consequently the vertices) of the two triangulations by applying Lemma 1 and then Theorems 1–5. It is necessary to check whether there is a flip-to-match diagonal first. In this case, the corresponding flip is carried out. Otherwise, the existence of type (1 : 0), (2 : 0), (1 : 1), (3 : 0) and (2 : 1) diagonals (in this precise order) should be checked. In fact, whatever the order, the algorithm is correct. However we choose this precise order for obvious statistical reasons. It will lead statistically to minimum number of operations.

Now we provide a recursive algorithm (according to Lemma 4) to find lower and upper bounds (in our algorithm *low* and *up*) for the rotation distance between two triangulations $T, S \in \mathcal{T}_n$. It should be pointed out that, in this algorithm, trees T, S and therefore types $(i : j)$, $(j : i)$ play the same role.

Algorithm (Computing *low* and *up*)
Given $T, S \in \mathcal{T}_n$

procedure low-up(T, S, n)
if $n \geqslant 3$ **then**
if T and S have a common diagonal **then**
  $T := (T_1, T_2)$; $S := (S_1, S_2)$;
  $n_1 := \mathrm{size}(T_1) = \mathrm{size}(S_1)$; $n_2 := \mathrm{size}(T_2) = \mathrm{size}(S_2)$;
  low-up(T_1, S_1, n_1);
  low-up(T_2, S_2, n_2);
else
  if T and S verify Lemma 1 **then**
    flip the diagonal in T or in S
    $low := low + 1$; $up := up + 1$
    low-up(T, S, n);
  else
    if T and S have a vertex i of type (2 : 0) **then**
      if T and S have a (2, 2)-diagonal $\{i, j\}$ **then**
        $T := N''_T(i, j)$ or $S := N''_S(i, j)$
        $low := low + 3$; $up := up + 3$
        low-up$(T, S, n - 2)$;
      else
        $T := N_T(i)$ or $S := N_S(i)$
        $low := low + 1$; $up := up + 2$
        low-up$(T, S, n - 1)$;
    else
      if T and S have a vertex i of type (1 : 1) **then**
        $T := N_T(i)$ and $S := N_S(i)$
        $low := low + 1$; $up := up + 2$
        low-up$(T, S, n - 1)$;
      else
      if T and S have a vertex i of type (3 : 0) **then**
        $T := N'''_T(i)$ or $S := N'''_S(i)$
        $low := low + 1$; $up := up + 3$
        low-up$(T, S, n - 1)$;
      else T and S have a vertex i of type (2 : 1)
        $T := N_T(i)$ and $S := N_S(i)$
        $low := low + 1$; $up := up + 3$
        low-up$(T, S, n - 1)$;
end low-up

Examples. If (T, S) are the two triangulations $\in \mathcal{T}_{23}$ of Rogers [15, p. 88], then the above algorithm provides $20 \leqslant \mathrm{dist}(T, S) \leqslant 29$. The exact rotation distance is 21.

If (T, S) are the two triangulations $\in \mathcal{T}_{20}$ of Lucas [8, p. 261], then the above algorithm provides $17 \leqslant \mathrm{dist}(T, S) \leqslant 25$. The exact rotation distance is 20.

An applet for computing lower and upper bounds is available on the web site: http://www.u-bourgogne.fr/jl.baril/titi.html. This applet will choose the better of the two upper bounds, either the result of this paper, or the result of [12]. Let us remark that the upper bound provided in [12] is often, but not necessarily, better than the upper bound computed in this paper. This applet uses the coding of binary trees (or equivalently triangulations) by weight sequences [10] and represents triangulations by adjacency matrices. Recall that the adjacency matrix M_T of T is such that $M_T(i, j) = 1$ if the vertices i and j are connected in T and $M_T(i, j) = 0$ otherwise. Between two recursive calls we carry out a flip-to-match diagonal and some computations in order to obtain the adjacency matrices of T_1 and T_2 from that of T. This step requires a time complexity $\mathcal{O}(n^2)$. Since the triangulations have $n - 3$ diagonals the time complexity is given by $\sum_{i=1}^{n-3} i^2$ and thus the time complexity is $\mathcal{O}(n^3)$.

9. Computer experiments

We present here some exhaustive and statistical results. In the first array we obtain exhaustive values for $n \leqslant 13$ about the ratio of pairs (T, S) verifying $(up - low) \leqslant k$ with $k = 0, 1, 2, n/2$. In the second array we provide probabilistic results for some $n \in [14, 100]$ by generating a random huge number of pairs (T, S). We notice that our algorithm gives efficient lower and upper bounds for large values of n: 99% of pairs (T, S) have upper and lower bounds verifying $(up - low) \leqslant n/3$.

Exhaustive results

Table 1

n-gons	5	6	7	8	9	10	11	12	13
$Up = Low$	100(%)	96	88	78	67	57	47	38	33
$Up - Low \leqslant 1$	100	100	99	99	95	90	82	73	54
$Up - Low \leqslant 2$	100	100	100	100	100	99	96	91	64
$Up - Low \leqslant n/2$	100	100	100	100	100	100	99.9	99.9	99.9

Statistical results

Table 2

n-gons	14	15	16	17	18	20	30	50	100
$Up - Low \leqslant n/10$	81.5 (%)	77	75	71	69	65	59	45	39
$Up - Low \leqslant n/5$	98	97	97	96	95	95	95	94	93
$Up - Low \leqslant n/3$	99.9	99.9	99.9	99.9	99.8	99.8	99.8	99.7	99.7
$Up - Low \leqslant n/2$	99.9	99.9	99.9	99.9	99.9	99.9	99.9	99.9	99.9

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