Gray code for derangements

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Abstract

We give a Gray code and constant average time generating algorithm for derangements, i.e., permutations with no fixed points. In our Gray code, each derangement is transformed into its successor either via one or two transpositions or a rotation of three elements. We generalize these results to permutations with number of fixed points bounded between two constants.

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1. Introduction

Various studies have been made on Gray codes and generating algorithms for permutations and their restrictions (with given ups and downs [9,11] or inversions [6,17], involutions, and fixed-point free involutions [18]) or their generalizations (multiset permutations [8,16]). See [7,13] for surveys of permutation generation methods.

A length-\(n\) derangement (or rencontre or coincidence) is a permutation \(\pi \in S_n\) with no fixed points, i.e., \(\pi(i) \neq i\) for all \(i \in [n] = \{1, 2, \ldots, n\}\). If \(D_n\) is the set of all length-\(n\) derangements, then a recurrence relation for \(d_n = \text{card}(D_n)\) is given by

\[ d_n = (n - 1)(d_{n-1} + d_{n-2}) \]

for \(n \geq 2\), with \(d_1 = 0\) and \(d_2 = 1\); see for instance [4, p. 180] or [14, p. 67]. There are sequential [1] and parallel [2, p. 650] algorithms for generating derangements in lexicographic order. However, we know of no published algorithms for derangements in Gray code order. Here, we present such an algorithm which is based on the combinatorial proof of relation (1) above.

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We represent permutations in one-line notation; i.e., \( \pi = (i_1, i_2, \ldots, i_n) \) iff \( \pi(k) = i_k \), and if \( \sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n) \) is a length-\( n \) integer sequence, then \( \sigma \cdot \pi \) is the sequence \( (\sigma_{\pi(1)}, \sigma_{\pi(2)}, \ldots, \sigma_{\pi(n)}) \). As a particular case, when \( \sigma \in S_n \) then \( \sigma \cdot \pi \in S_n \) is their composition (or product).

Let \( i_1, i_2, \ldots, i_k \) be \( k \) different values in \( [n] = \{1, 2, \ldots, n\} \), \( 1 \leq k \leq n \). The cycle \( \gamma = \langle i_1, i_2, \ldots, i_k \rangle \) is the following permutation: \( \gamma(i_1) = i_2, \gamma(i_2) = i_3, \ldots, \gamma(i_{k-1}) = i_k, \gamma(i_k) = i_1 \), and \( \gamma(j) = j \) for all \( j \neq i \), \( 1 \leq l \leq k \). A length-two cycle is a transposition, and each cycle can be written as a product of transpositions: \( \langle i_1, i_2, \ldots, i_k \rangle = \langle i_1, i_k \rangle \cdot \langle i_1, i_{k-1} \rangle \cdots \langle i_1, i_2 \rangle \) for \( k \geq 2 \). Also, the composition of two cycles with disjoint domains is commutative, and each permutation is the product of cycles with disjoint domains.

In a permutation \( \sigma \in S_n \), transposing the positions \( i \) and \( j \) corresponds to the product \( \sigma \cdot \langle i, j \rangle \) and transposing the values \( x \) and \( y \) corresponds to the product \( \langle x, y \rangle \cdot \sigma \).

For a length-\( n \) integer sequence \( z = (z(1), z(2), \ldots, z(n)) \) and a permutation \( \pi \in S_n \), we say that \( \pi \) is the normal form of \( z \) if \( z \) is order-isomorphic to \( \pi \), i.e., \( z(i) < z(j) \) if and only if \( \pi(i) < \pi(j) \) for all \( 1 \leq i, j \leq n \). In this case, all the elements of \( z \) are distinct.

In the Gray code we give in the next section, a derangement is obtained from the previous one via one or two transpositions, and, as a particular case when the domains of the two transpositions are not disjoint, via a length-three cycle. In Section 3 this code is implemented as a generating algorithm and in Section 4 it is extended for permutations with a given number of fixed points and for permutations with the number of fixed points between two bounds.

A list \( \mathcal{L} \) for a set \( L \) of integer sequences is an ordered list of the elements of \( L \). \( \text{first}(\mathcal{L}) \) is the first element and \( \text{last}(\mathcal{L}) \) the last element on the list \( \mathcal{L} \); \( \overline{\mathcal{L}} \) is the list obtained by reversing \( \mathcal{L} \), and obviously \( \text{first}(\mathcal{L}) = \text{last}(\overline{\mathcal{L}}) \) and \( \text{first}(\overline{\mathcal{L}}) = \text{last}(\mathcal{L}) \); \( \mathcal{L}^{(i)} \) is the list \( \mathcal{L} \) if \( i \) is even, and \( \overline{\mathcal{L}} \) if \( i \) is odd; if \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) are two lists, then \( \mathcal{L}_1 \circ \mathcal{L}_2 \) is their concatenation, and generally \( \bigcirc_{i=1}^{n} \mathcal{L}_i \) is the list \( \mathcal{L}_1 \circ \mathcal{L}_2 \circ \cdots \circ \mathcal{L}_n \).

2. The Gray code

In this section, first we show how the set \( D_n \) can by recursively constructed from \( D_{n-1} \) and \( D_{n-2} \), and then we extend this construction to lists of derangements in order to obtain a Gray code.

Let \( \tau \) be a length-(\( n - 1 \)) derangement, \( n \geq 3 \), and let \( i \) be an integer such that \( 1 \leq i \leq n - 1 \). If we denote by \( \sigma \) the permutation in \( S_n \) obtained from \( \tau \) by replacing the entry with value \( i \) by \( n \) and appending \( i \) in the last position, then \( \sigma \) is a length-\( n \) derangement with \( n \) not belonging to a transposition.

Similarly, let \( \tau \) be a length-(\( n - 2 \)) derangement, \( n \geq 4 \), and let \( i \) be an integer such that \( 1 \leq i \leq n - 1 \). If \( \sigma \) denotes the permutation in \( S_n \) obtained from \( \tau \) by: (1) adding one to each entry greater than or equal to \( i \), (2) inserting \( n \) in position \( i \), and finally, (3) appending \( i \) in the last position, then \( \sigma \) is a length-\( n \) derangement with \( n \) belonging to a transposition (the transposition \( \langle i, n \rangle \)). Moreover, each length-\( n \) derangement, \( n \geq 4 \), can be uniquely obtained by one of these constructions.
More formally, for \( n > 0 \), let \( D'_n \) be the set of length-\( n \) derangements where \( n \) does not belong to a transposition and \( D''_n \) its complement, i.e., \( n \) belongs to a transposition. Clearly, \( D'_1 = D''_1 = D'_2 = \emptyset \), and \( D_2 = D''_2 \) is the single derangement list \((2, 1)\); and \( D'_n \cup D''_n \) is a two-partition of the set \( D_n \) of all length-\( n \) derangements. The functions \( \phi \) and \( \psi \) defined below give a bijection between \([n-1] \times D_{n-1}\) and \( D'_n \) on the one hand and between \([n-1] \times D_{n-2}\) and \( D''_n \) on the other.

**Definition 1.** (1) For \( n \geq 3 \), an integer \( i \in [n-1] \) and a derangement \( \tau \in D_{n-1} \), we define a length-\( n \) permutation \( \sigma = \phi_n(i, \tau) \) by

\[
\sigma(j) = \begin{cases} 
  n & \text{if } \tau(j) = i, \\
  i & \text{if } j = n, \\
  \tau(j) & \text{otherwise.}
\end{cases}
\]

(2) For \( n \geq 4 \), an integer \( i \in [n-1] \) and a derangement \( \tau \in D_{n-2} \), we define a length-\( n \) permutation \( \sigma = \psi_n(i, \tau) \) by

\[
\sigma(j) = \begin{cases} 
  i & \text{if } j = n, \\
  n & \text{if } j = i, \\
  \tau(j) & \text{if } j < i \text{ and } \tau(j) < i, \\
  \tau(j) + 1 & \text{if } j < i \text{ and } \tau(j) \geq i, \\
  \tau(j - 1) & \text{if } j > i \text{ and } \tau(j - 1) < i, \\
  \tau(j - 1) + 1 & \text{if } j > i \text{ and } \tau(j - 1) \geq i.
\end{cases}
\]

With \( i \) and \( \tau \) as above, it is easy to see that

- \( \phi_n(i, \tau) \in D'_n \) and \( \phi_n : [n-1] \times D_{n-1} \to D'_n \) is a bijection; and
- \( \psi_n(i, \tau) \in D''_n \) and \( \psi_n : [n-1] \times D_{n-2} \to D''_n \) is a bijection.

So, for \( d_n = \text{card}(D_n) \) we have \( d_n = \text{card}(D'_n) + \text{card}(D''_n) = (n-1)d_{n-1} + (n-1)d_{n-2} \)
which is a combinatorial proof of (1).

Conversely, we have.

**Remark 2.** If \( \sigma \in D_n \), \( n \geq 4 \), and \( i = \sigma(n) \), then

1. if \( \sigma(i) \neq n \) (\( n \) is not in a transposition in \( \sigma \)) then \( \sigma = \phi_n(i, \tau) \) with \( \tau \) the permutation represented by the first \( n-1 \) entries of \( \langle i, n \rangle \cdot \sigma \);
2. if \( \sigma(i) = n \) (\( n \) is in a transposition in \( \sigma \)) then \( \sigma = \psi_n(i, \tau) \) with \( \tau \) the permutation represented by the normal form of the sequence \( \langle \sigma(1), \sigma(2), \ldots, \sigma(i-1), \sigma(i+1), \ldots, \sigma(n-1) \rangle \).

In the following we will omit the subscript \( n \) for the functions \( \phi \) and \( \psi \), and it should be clear by context. Also, we extend the functions \( \phi \) and \( \psi \) in a natural way to sets and lists of derangements. For \( i \in [n-1] \) and \( \mathscr{L} \) a list of length-(\( n-1 \)) derangements we
Fig. 1. The list \( D_n \): (a) \( n \) is even, (b) \( n \) is odd.

Similar results hold for the function \( \psi \).

Let \( D_n \) be the list for the set \( D_n \) defined by

\[
D_n = \phi(1, D_{n-1}) \circ \psi(1, D_{n-2}),
\]
\[
\circ \psi(2, D_{n-2}) \circ \phi(2, D_{n-1}),
\]
\[
\circ \phi(3, D_{n-1}) \circ \psi(3, D_{n-2}),
\]
\[
\vdots
\]
\[
= \bigcirc_{i=1}^{n-1} (\phi(i, D_{n-1}) \circ \psi(i, D_{n-2}))^{(i+1)}
\]

for \( n \geq 3 \), anchored by \( D_1 = \psi(1, \emptyset) = \psi(2, \emptyset) = \emptyset \) and \( D_2 = (2, 1) \).

In Fig. 1 below, the list \( D_n \) is illustrated for even and odd \( n \) by a path, where going down means generating a sublist in direct order and going up means generating it in reverse order.

Let \( f_n \) denote the first derangement in the list \( D_n \) and \( \ell_n \) denote the last one. The following lemma evaluates \( f_n \) and \( \ell_n \) for all \( n \).

Lemma 3. If \( n \geq 3 \) then

1. \( f_n = (2, 3, \ldots, n - 1, n, 1) \);
2. \( \ell_n = \begin{cases} 
(2, 3, \ldots, n - 2, n, 1, n - 1) & \text{if } n \text{ is odd}, \\
(2, 3, \ldots, n - 2, 1, n, n - 1) & \text{if } n \text{ is even}. 
\end{cases} \)

Proof. 1. \( f_n = \phi(1, f_{n-1}) \), and by the induction hypothesis, \( f_n = \phi(1, (2, 3, \ldots, n - 1, 1)) = (2, 3, \ldots, n - 1, n, 1) \).

2. The proof for \( \ell_n \) follows a similar induction argument.
(2) If \( n \) is odd, then
\[
\ell_n = \text{last}(\phi(n - 1, D_{n-1}) \circ \psi(n - 1, \overline{D}_{n-2}))
\]
\[
= \text{first}(\phi(n - 1, D_{n-1}))
\]
\[
= \phi(n - 1, f_{n-1})
\]
\[
= (2, 3, \ldots, n - 2, n, 1, n - 1).
\]
If \( n \) is even, then
\[
\ell_n = \text{last}(\phi(n - 1, D_{n-1}) \circ \psi(n - 1, \overline{D}_{n-2}))
\]
\[
= \text{last}(\psi(n - 1, \overline{D}_{n-2}))
\]
\[
= \psi(n - 1, f_{n-2})
\]
\[
= (2, 3, \ldots, n - 2, 1, n, n - 1). \quad \Box
\]

Note that \( f_n(j) = \ell_n(j) = j + 1 \) for all \( j = 1, 2, \ldots, n - 3 \).

The next lemma ensures a smooth transition between the sublists in relation (2), namely between: (i) the list \( \psi(i, \overline{D}_{n-2}) \) and \( \psi(i + 1, D_{n-2}) \), with \( i \) odd; (ii) the list \( \phi(i, D_{n-1}) \) and \( \phi(i + 1, D_{n-1}) \) with \( i \) even; and (iii) the list \( \phi(i, D_{n-1}) \) and \( \psi(i, \overline{D}_{n-2}) \), or equivalently, the list \( \psi(i, \overline{D}_{n-2}) \) and \( \phi(i, \overline{D}_{n-1}) \). More precisely, successive derangements in \( D_n \) differ either by one or two transpositions or by a circular shift of three elements (See Table 1 for two examples).

**Lemma 4.** (i) If \( n \geq 4 \) then
\[
\psi(i + 1, f_{n-2}) = \begin{cases} 
\psi(i, f_{n-2}) \cdot (i - 1, n) \cdot (i, i + 1) & \text{if } 1 < i \leq n - 2, \\
\psi(i, f_{n-2}) \cdot (1, 2) \cdot (n - 1, n) & \text{if } i = 1
\end{cases}
\]

<table>
<thead>
<tr>
<th>Table 1</th>
</tr>
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<tbody>
<tr>
<td>The lists ( D_4 ) and ( D_5 )</td>
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<td>2143</td>
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<td>10</td>
<td>54231</td>
</tr>
<tr>
<td>11</td>
<td>53421</td>
</tr>
</tbody>
</table>

In \( D_5 \) the sublists \( \phi(i, D_4) \) and \( \psi(i, D_3) \), in direct or reverse order, for some \( i, 1 \leq i \leq 4 \), are in bold-face and italic, respectively.
or, conversely, by reading this relation left-to-right and replacing \( i \) by \( i - 1 \),

\[
\psi(i - 1, f_{n-2}) = \begin{cases} 
\psi(i, f_{n-2}) \cdot \langle i - 2, n \rangle \cdot \langle i - 1, i \rangle & \text{if } 2 < i \leq n - 1, \\
\psi(i, f_{n-2}) \cdot \langle 1, 2 \rangle \cdot \langle n - 1, n \rangle & \text{if } i = 2.
\end{cases}
\]  

(4)

(ii) If \( n \geq 3 \) then

\[
\phi(i + 1, f_{n-1}) = \begin{cases} 
\phi(i, f_{n-1}) \cdot \langle n, i, i - 1 \rangle & \text{if } 1 < i \leq n - 2, \\
\phi(i, f_{n-1}) \cdot \langle 1, n - 1, n \rangle & \text{if } i = 1
\end{cases}
\]  

(5)

or, conversely and by replacing \( i \) by \( i - 1 \),

\[
\phi(i - 1, f_{n-1}) = \begin{cases} 
\phi(i, f_{n-1}) \cdot \langle i - 2, i - 1, n \rangle & \text{if } 2 < i \leq n - 1, \\
\phi(i, f_{n-1}) \cdot \langle n, n - 1, 1 \rangle & \text{if } i = 2.
\end{cases}
\]  

(6)

(iii) If \( n = 4 \)

\[
\psi(i, \ell_2) = \begin{cases} 
\phi(i, \ell_3) \cdot \langle i, i + 1 \rangle & \text{if } i \neq 3, \\
\phi(i, \ell_3) \cdot \langle 1, 3 \rangle & \text{if } i = 3.
\end{cases}
\]  

(7)

If \( n \geq 5 \)

\[
\psi(i, \ell_{n-2}) = \begin{cases} 
\phi(i, \ell_{n-1}) \cdot \langle n - 2, n - 3, n - 1 \rangle & \text{if } i = 1, \text{ n even}, \\
\phi(i, \ell_{n-1}) \cdot \langle 1, n - 3, n - 2 \rangle & \text{if } i = 1, \text{ n odd}, \\
\phi(i, \ell_{n-1}) \cdot \langle i - 1, i \rangle \cdot \langle n - 3, n - 2 \rangle & \text{if } 2 \leq i \leq n - 4, \\
\phi(i, \ell_{n-1}) \cdot \langle n - 2, n - 3, n - 4 \rangle & \text{if } i = n - 3, \\
\phi(i, \ell_{n-1}) \cdot \langle n - 4, n - 3, n - 1 \rangle & \text{if } i = n - 2, \\
\phi(i, \ell_{n-1}) \cdot \langle n - 4, n - 2 \rangle \cdot \langle n - 3, n - 1 \rangle & \text{if } i = n - 1, \text{ n even}, \\
\phi(i, \ell_{n-1}) \cdot \langle n - 1, n - 2, n - 4 \rangle & \text{if } i = n - 1, \text{ n odd,}
\end{cases}
\]  

(8)

or conversely

\[
\phi(i, \ell_{n-1}) = \psi(i, \ell_{n-2}) \cdot \pi^{-1}
\]  

(9)

with \( \pi \) the permutation which occurs in the corresponding case in relation (7) or (8).

So, for example, if \( n \geq 5 \), \( i = n - 1 \) then:

\[
\bullet \ \phi(i, \ell_{n-1}) = \psi(i, \ell_{n-2}) \cdot \langle n - 4, n - 2 \rangle \cdot \langle n - 3, n - 1 \rangle \text{ if } n \text{ is even (in this case } \pi = \pi^{-1}),
\]

\[
\bullet \ \phi(i, \ell_{n-1}) = \psi(i, \ell_{n-2}) \cdot \langle n - 1, n - 4, n - 2 \rangle \text{ if } n \text{ is odd.}
\]
Proof. The proof is direct, and consists essentially of checking each case. For brevity, we do not give the proof of (iii) which is similar to the first two cases.

(i) If \( 1 < i \leq n - 2 \) then

\[
\begin{align*}
\psi(i + 1, f_{n-2}) &= (2, \ldots, i - 1, \ i, \ i + 2, \ n, \ i + 3, \ldots, 1, \ i + 1), \\
\psi(i, f_{n-2}) &= (2, \ldots, i - 1, \ i + 1, \ n, \ i + 2, \ i + 3, \ldots, 1, \ i)
\end{align*}
\]

and if \( i = 1 \) then

\[
\begin{align*}
\psi(2, f_{n-2}) &= (3, n, 4, \ldots, n - 1, 1, 2), \\
\psi(1, f_{n-2}) &= (n, 3, 4, \ldots, n - 1, 2, 1),
\end{align*}
\]

where the little numbers are indices of array elements.

(ii) If \( 1 < i \leq n - 2 \) then

\[
\begin{align*}
\phi(i + 1, f_{n-1}) &= (2, \ldots, i - 1, \ i, \ n, \ i + 2, \ldots, 1, \ i + 1), \\
\phi(i, f_{n-1}) &= (2, \ldots, i - 1, \ n, \ i + 1, \ i + 2, \ldots, 1, \ i)
\end{align*}
\]

and if \( i = 1 \) then

\[
\begin{align*}
\phi(2, f_{n-1}) &= (n, 3, \ldots, n - 1, 1, 2), \\
\phi(1, f_{n-1}) &= (2, 3, \ldots, n - 1, n, 1).
\end{align*}
\]

Corollary 5. Successive derangements in \( D_n \) differ at most in four positions.

Note that \( D_n \) is a cyclic Gray code.

3. Generating algorithm

The definition given by (2) says that \( D_n \) is the concatenation of many lists, which are all similar in some sense to \( D_{n-1} \) or \( D_{n-2} \). This result is formalized in Lemma 9 below, and our generating algorithm for \( D_n \) is based on it. Now we give some technical definitions.

Two lists are isomorphic if, in the first list, a sequence is transformed into its successor via the same permutation as the corresponding sequence in the second list is transformed into its successor; and two lists are similar if after erasing the constant entries in the first list, and possibly reversing it, the lists become isomorphic. More formally.

Definition 6. Let \( L \) and \( \mathcal{F} \), respectively, be a list of length-\( n \) integer sequences and a list of permutations in \( S_n \). We say that \( L \) is isomorphic to \( \mathcal{F} \) if:

(1) the lists contain the same number of sequences, say \( p \),
Table 2

\( A \) is isomorphic to \( D_4 \) and \( B = \phi(2, D_4) \) is \( \{1, 2, 3, 4\}\)-similar to \( D_4 \).

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Note that \( A \) is the reverse of the list obtained by erasing the last entry of each sequence in \( B \).
Lemma 9. Let \( \mathcal{L} \) be a length-\( n \) sequence list, \( T \subseteq [n] \), and \( m = \text{card}(T) \geq 4 \). If \( \mathcal{L} \) is \( T \)-similar to the derangement list \( \mathcal{D}_m \), then \( \mathcal{L} = \mathcal{L}_1 \circ \mathcal{L}_2 \circ \cdots \circ \mathcal{L}_{2(m-1)} \) where each sublist \( \mathcal{L}_j \) is \( U_j \)-similar to the derangement list \( \mathcal{D}_{\text{card}(U_j)} \), with \( U_j \) a \( c \)-subset of \( T \).

Proof. By Lemma 8 above and applying recursively relation (2).

Obviously, \( \mathcal{D}_n \) is \( \{1,2,\ldots,n\} \)-similar to itself and the procedure \textit{gen\_up} in Fig. 2 generates the list \( \mathcal{D}_n \) according to the lemma above: the lists \( \mathcal{L}_j \) are produced iteratively, and each of them is generated recursively. So, each call of this procedure fills up entries with indices in an active set \( T \subseteq [n] \) associated with it, and in a recursive call \( T \) is replaced by a \( c \)-subset of \( T \).

In our algorithm, the set \( T \) of active indices is represented by four global variables: the integers \textit{head} and \textit{tail} and the arrays \textit{succ} and \textit{pred}, defined as follows. If at a computational step \( T = \{i_1,i_2,\ldots,i_k\} \), then \( \text{head} = i_1, \text{tail} = i_k, \text{succ}[i_j] = i_{j+1} \) and \( \text{pred}[i_j] = i_{j-1} \).
If the active set associated with the current call is $T$, then the call of $\text{gen}_\text{up}(j,t,\text{run})$ initiated by the current call generates a sublist that is $U$-similar to $D_j$, where

$$
U = \begin{cases} 
T \setminus \{\text{tail}\} & \text{if } t = \phi, \\
T \setminus \{\text{run},\text{tail}\} & \text{if } t = \psi
\end{cases}
$$

(recall that $\text{tail} = \max(T)$).

Less formally, $\text{gen}_\text{up}(j,\phi,\text{run})$ produces a ‘$\phi(j,D_j)$-like’ list, and $\text{gen}_\text{up}(j,\psi,\text{run})$ produces a ‘$\psi(j,D_j)$-like’ list (see relation (2)), where $\text{run}$ is the $i$th element in the set $T$. The call of $\text{gen}_\text{down}$ works as $\text{gen}_\text{up}$ except $D_j$ is replaced by $\overline{D_j}$.

For a simpler expression of the generating algorithm we consider initially the active set $T = [n+1]$, and each call begins by removing $\text{tail}$, the largest element in $T$. Thus, before the first call of the generating procedure, the variables which correspond to $T$ are: $\text{head} = 1$, $\text{tail} = n + 1$, $\text{succ}[i] = i + 1$ for $1 \leq i \leq n$, and $\text{pred}[i] = i - 1$ for $2 \leq i \leq n + 1$. The current derangement is stored in a global variable $d$, initialized by $d = \text{first}(D_n)$.

The main call $\text{gen}_\text{up}(n,\phi,0)$ produces the list $D_n$, $n \geq 3$, and the value $r = 0$ is for convenience; in fact when $t = \phi$ the value of $r$ is not required. The procedure which generates the reverse list $\overline{D_n}$ is called $\text{gen}_\text{down}$, shown in the appendix, and essentially executes the statements of $\text{gen}_\text{up}$ in reverse order and replaces the calls of $\text{gen}_\text{up}$ by $\text{gen}_\text{down}$ and vice versa. Procedures $\text{remove}(r)$ and $\text{append}(r)$, also shown in the appendix, remove and append $r$ in the current active set (given by the variables $\text{head}$, $\text{tail}$, $\text{succ}$ and $\text{pred}$).

Between any successive calls at least one update statement is performed, and after each update statement (including the case $n = 3$) a new derangement is produced and printed out. The current derangement $d$ is transformed into its successor according to relations (3), (5), (7), (8) or (9) in Lemma 4. More precisely, the current derangement is subject to the transformation given in the appropriate case of Lemma 4, and it acts on the active indices.

For example, in our algorithm relation (5) becomes:

```latex
\begin{align*}
\text{if } i &= 1 \\
\text{then } d &:= d \cdot \langle \text{head}, \text{pred}[\text{tail}], \text{tail} \rangle; \\
\text{else } d &:= d \cdot \langle \text{tail}, \text{run}, \text{pred}[\text{run}] \rangle;
\end{align*}
```

Clearly, the time complexity of $\text{gen}_\text{up}$ is proportional to the total number of recursive calls. Since each call produces at least one new derangement the time complexity of $\text{gen}_\text{up}(n,i,r)$ is in $\mathcal{O}(d_n)$. A $C$ implementation of our algorithm is available at http://www.u-bourgogne.fr/v.vincent/AA/.

4. Permutations with a given number of fixed points

Here, we generalize the Gray code in the previous sections to permutations with a given number of fixed points and permutations with a bounded number of fixed points.
Let \( c = (c(1), c(2), \ldots, c(n)) \) be an \( n \)-combination of \( m, n \leq m \), in integer sequence representation, so that \( 1 \leq c(i) < c(i+1) \leq m \) for \( i = 1, 2, \ldots, n - 1 \). Also, let \( t_c = t \) be the binary representation of \( c \), i.e., \( t = (t(1), t(2), \ldots, t(m)) \) with \( t(i) = 1 \) if there exists a \( j \) such that \( c(j) = i \), and \( t(i) = 0 \) elsewhere. With those notations, for a derangement \( d = (d(1), d(2), \ldots, d(n)) \in D_n \) we define the length-\( m \) sequence \( u = (u(1), u(2), \ldots, u(m)) \), denoted by \( \text{shuffle}(c; d) \), as

\[
u(i) = \begin{cases} 
i & \text{if } t(i) = 0, \\
c(d(j)) & \text{if } t(i) \text{ is the } j\text{th } 1 \text{ in } t \end{cases}
\]

and we call \( u = \text{shuffle}(c; d) \) the shuffle of \( c \) by \( d \) on the trajectory \( t \).

In other words, \( u \) acts on indices \( c(1), c(2), \ldots, c(n) \) as \( d \), and fixes the other indices. The shuffle operator over combinatorial objects was formally defined in a larger context in [15,16]. It is not hard to show that \( \text{shuffle}(c; d) \) is a permutation of \( [m] \) with exactly \( n \) “deranged” points (i.e. with exactly \( m - n \) fixed points), and in addition, each such permutation can be uniquely constructed by shuffle operation from an appropriate combination and a derangement. More formally, if \( u = (u(1), u(2), \ldots, u(m)) \) is a permutation of \( [m] \) with exactly \( m - n \) fixed points then \( u = \text{shuffle}(c; d) \), where

- \( c = (c(1), c(2), \ldots, c(n)) \) is the \( n \)-combination of \( m \) corresponding to the subset of \( [m] \) where \( u(i) \neq i \), and
- \( d \) is the normal form of the sequence \( (u(c(1)), u(c(2)), \ldots, u(c(n))) \).

**Example.** If \( n = 3, m = 6 \), \( c = (2, 5, 6) \), and \( d = (2, 3, 1) \) then \( t = (0, 1, 0, 1, 1) \) and \( \text{shuffle}(c; d) = (1, 5, 3, 4, 6, 2) \); or if \( n = 4, m = 6 \), \( c = (1, 2, 4, 5) \), and \( d = (2, 4, 1, 3) \) then \( t = (1, 1, 0, 1, 0) \) and \( \text{shuffle}(c; d) = (2, 5, 3, 1, 4, 6) \).

See also [18] for a similar approach. To summarize, we have:

**Lemma 10.** If \( C_{m,n} \) is the set of all \( n \)-combinations of \( [m] \) and \( S_{m,n} \) the set of all permutations of \( [m] \) with exactly \( m - n \) fixed points then

\[
\text{shuffle} : C_{m,n} \times D_n \to S_{m,n}
\]

is a bijection.

Also, we extend the shuffle operation in a natural way to lists of derangements: if \( \mathcal{D} = d_1, d_2, \ldots \) is a sublist of \( D_n \) and \( c \in C_{m,n} \) then \( \text{shuffle}(c; \mathcal{D}) \) is the list \( \text{shuffle}(c; d_1), \text{shuffle}(c; d_2), \ldots \), and \( \text{shuffle}(c; \mathcal{D}) = \text{shuffle}(c; D_n) \).

A strong Gray code for the set \( C_{m,n} \) of \( n \)-combinations of \( m \), in integer sequence representation, is a list for \( C_{m,n} \) where two successive sequences, say \( c = (c(1), c(2), \ldots, c(n)) \) and \( c' = (c'(1), c'(2), \ldots, c'(n)) \), are such that, for some \( 1 \leq j \leq m \), \( c(i) = c'(i) \) for all \( i \neq j \); see [3,5,12] for such a Gray code.

**Lemma 11.** If \( \mathcal{C}_{m,n} \) is a strong Gray code for the set \( C_{m,n} \) then the list \( \mathcal{J}_{m,n} \) defined by

\[
\mathcal{J}_{m,n} = \bigcup_{c \in \mathcal{C}_{m,n}} \text{shuffle}(c; \mathcal{D}_n)
\]

is a Gray code for the set \( C_{m,n} \).
is a Gray code for the set $S_{m,n}$, where $r$ is the rank of $c$ in $C_{m,n}$ (the first combination in $C_{m,n}$ has rank zero) and $\mathcal{D}_n^{(r)}$ is $D_n$ or $\overline{D}_n$ according as $r$ is even or odd.

**Proof.** The list $\mathcal{S}_{m,n}$ has no repetitions, and, disregarding the order, it equals the set $S_{m,n}$. Moreover, for a fixed $c$ in $C_{m,n}$, the Hamming distance between two derangements in $D_n$, say $d$ and $d'$, equals the Hamming distance between $\mathcal{U}(c; d)$ and $\mathcal{U}(c; d')$. So, any successive permutations in $\mathcal{U}(c; D_n)$—or equivalently in $\mathcal{U}(c; \overline{D}_n)$—differ in at most four positions. If $c'$ is the successor of $c$ in $C_{m,n}$ then $t' = t_{c'}$ and $t = t_c$, the binary representations of $c'$ and $c$, differ in exactly two positions, say $k$ and $\ell$, with $t(k) = t'(k) = 0$ and $t(k) = t(\ell) = 1$. Since $C_{m,n}$ is a strong Gray code, the permutations $\sigma = \mathcal{U}(c; d)$ and $\sigma = \mathcal{U}(c'; d)$ differ in exactly three positions, namely $k$, $\ell$ and $i$, where $i$ is such that $\sigma(i) = \ell$ and $\sigma'(i) = k$. Moreover, the index $i$ can be computed in constant time if $d$ is the first or last derangement in $D_n$. 

The next lemma extends the result of the previous one to permutations where the number of fixed points is bounded between two constants. In this case $C_{m,n}$ denotes the Eades–McKay Gray code for combinations, and it has (see [5,12])

- first($C_{m,n}$) = $(1, 2, \ldots, n)$, and
- last($C_{m,n}$) = $(m - n + 1, m - n + 2, \ldots, m)$.

**Lemma 12.** Let $1 \leq k \leq \ell \leq m$ and $S_{m,k,\ell}$ be the set of all permutations in $S_m$ with $i$ “deranged” points, $k \leq i \leq \ell$. Then the list

$$
\mathcal{S}_{m,k,\ell} = \bigcup_{i=k}^{\ell} \mathcal{S}_{m,i}^{(k-i)}
$$

(12)

is a Gray code for the set $S_{m,k,\ell}$.

**Proof.** It is sufficient to prove that the last permutation in $\mathcal{S}_{m,i}^{(k-i)}$ and the first one in $\mathcal{S}_{m,i+1}^{(k-i+1)}$ differ in at most four positions. But last($\mathcal{S}_{m,i}^{(k-i)}$) = $\mathcal{U}(c, e_i)$, and first($\mathcal{S}_{m,i+1}^{(k-i+1)}$) = $\mathcal{U}(c', e_{i+1})$, with

(i) $c =$ last($C_{m,i}$) and $c' =$ last($C_{m,i+1}$) if $k - i$ is even, or
(ii) $c =$ first($C_{m,i}$) and $c' =$ first($C_{m,i+1}$) if $k - i$ is odd,

and $e_j = f_j$ or $e_j = \ell_j$; see Lemma 3 and the remark that follows. If $u = \mathcal{U}(c, e_i)$ and $u' = \mathcal{U}(c', e_{i+1})$ then: in case (i), $u(j) = u'(j) = j$ for all $j = 1, 2, \ldots, m - i - 1$ and $u(j) = u'(j) = j + 1$ for all $j = m - i + 1, m - i + 2, \ldots, m - 3$; in case (ii), $u(j) = u'(j) = j + 1$ for all $j = 1, 2, \ldots, i - 3$ and $u(j) = u'(j) = j$ for all $j = i + 2, i + 3, \ldots, m$. In both cases $u$ differs from $u'$ in at most four positions. 

**Algorithmic considerations**

The lists $\mathcal{U}(c; \mathcal{D}_n^{(r)})$ in (11) is $c$-similar to $D_n$ (c is regarded as a subset of $[m]$) and the procedure $\text{gen}_\uparrow$ and $\text{gen}_\downarrow$ can easily be transformed to generate these.
lists. In addition, with an efficient algorithm to compute the successor of \( c \) in \( \mathcal{C}_{m,n} \) and with appropriate initial values for the variables and transition statements between lists, the iterative call of \texttt{gen.up} and \texttt{gen.down} produces \( \mathcal{S}_{m,n} \) in constant average time. See [16,18] for loopless generating algorithms for \( \mathcal{C}_{m,n} \). Similar considerations hold for the list \( \mathcal{S}_{m,k,l} \) defined in relation (12). The loopless generation of those lists remains an open problem.

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Note added in proof. We recently learned of Korsh’ and LaFollette’s [10] algorithm for generating derangements. Their algorithm has the remarkable properties (a) that successive permutations differ by only one transposition or one rotation of three elements and (b) it is loopless. Our algorithm is based on a recursive counting relation and so has the advantage of being simpler to describe.

Appendix.

```plaintext
procedure gen_down(n,t;r)
var i,run;
begin
tail := pred[tail];
if t = \psi then remove(r); endif
if n = 3 then d := d · \langle head,tail,succ[head]\rangle;
else run := pred[tail]
for i := n − 1 downto 1 do
if i is odd
then if n > 4 then
   gen_up(n − 2,\psi,run);
endif
   update d as in (9);
   gen_down(n − 1,\phi,run);
if i \neq 1
then update d as in (6); run := pred[run];
endif
else gen_up(n − 1,\phi,run);
   update d as in (7) or (8);
if n > 4 then
   gen_down(n − 2,\psi,run);
endif
   update d as in (4); run := pred[run];
```

endif
enddo
endif
if \( t = \psi \) then append\((r)\); endif
tail := succ\([\text{tail}]\);
end

procedure remove\((r)\)
begin
if \( r = \text{head} \) then head := succ\([r]\);
else if \( r = \text{tail} \) then tail := pred\([\text{tail}]\)
else succ\([\text{pred}[r]]\) := succ\([r]\);
pred\([\text{succ}[r]]\) := pred\([r]\);
endif
endif
end

procedure append\((r)\)
begin
if \( r < \text{head} \) then head := pred\([\text{head}]\);
else if \( r > \text{tail} \) then tail := succ\([\text{tail}]\)
else succ\([\text{pred}[r]]\) := \( r \);
pred\([\text{succ}[r]]\) := \( r \);
endif
endif
end

References


