

Dyck paths with a first return decomposition constrained by height

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June 6, 2017

Abstract

We study the enumeration of Dyck paths having a first return decomposition with special properties based on a height constraint. We exhibit new restricted sets of Dyck paths counted by the Motzkin numbers, and we give a constructive bijection between these objects and Motzkin paths. As a byproduct, we provide a generating function for the number of Motzkin paths of height k with a flat (resp. with no flats) at the maximal height.

Keywords: Enumeration, Dyck and Motzkin paths, first return decomposition, statistics, height, peak.

1 Introduction and notations

A *Dyck path* of semilength $n \geq 0$ is a lattice path starting at $(0, 0)$, ending at $(2n, 0)$, and never going below the x -axis, consisting of up steps $U = (1, 1)$ and down steps $D = (1, -1)$. Let \mathcal{D}_n , $n \geq 0$, be the set of all Dyck paths of semilength n , and let $\mathcal{D} = \cup_{n \geq 0} \mathcal{D}_n$. The cardinality of \mathcal{D}_n is given by the n th Catalan number, which is the general term $\frac{1}{n+1} \binom{2n}{n}$ of the sequence [A000108](#) in the on-line encyclopedia of integer sequences of N.J.A. Sloane [17]. A large number of various classes of combinatorial objects are enumerated by the Catalan numbers such as planar trees, Young tableaux, stack sortable permutations, Dyck paths, and so on. A list of over 60 types of such combinatorial classes has been compiled by Stanley [19]. In combinatorics,

many papers deal with Dyck paths. Most of them consist in enumerating Dyck paths according to several parameters, such as length, number of peaks or valleys, number of double rises, number of returns to the x -axis (see for instance [2, 8, 9, 10, 11, 12, 13, 14, 15, 16, 18]). Other studies investigate restricted classes of Dyck paths avoiding some patterns or having a specific structure. For instance, Barucci et al. [1] consider non-decreasing Dyck paths which are those having a non-decreasing sequence of heights of valleys (see also [6, 7]), and it is well known [5] that Dyck paths avoiding the triple rise UUU are enumerated by the Motzkin numbers (see A001006 in [17]).

Any non-empty Dyck path $P \in \mathcal{D}$ has a unique first return decomposition [8] of the form $P = U\alpha D\beta$ where α and β are two Dyck paths in \mathcal{D} . See Figure 1 for an illustration of this decomposition.

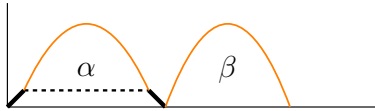


Figure 1: First return decomposition $U\alpha D\beta$ of a Dyck path $P \in \mathcal{D}$.

Based on this decomposition, we construct a new collection of subsets of \mathcal{D} as follows. Given a function $s : \mathcal{D} \rightarrow \mathbb{N}$, called *statistic*, and a comparison operator \diamond on \mathbb{N} (for instance \geq or $>$), the set $\mathcal{D}^{s,\diamond}$ is the union of the empty Dyck path with all Dyck paths P having a first return decomposition $P = U\alpha D\beta$ satisfying the three conditions:

$$\begin{cases} \alpha \in \mathcal{D}^{s,\diamond}, \\ \beta \in \mathcal{D}^{s,\diamond}, \\ s(U\alpha D) \diamond s(\beta). \end{cases} \quad (1)$$

For $n \geq 0$, we denote by $\mathcal{D}_n^{s,\diamond}$ the set of Dyck paths of semilength n in $\mathcal{D}^{s,\diamond}$. Thus, we have $\mathcal{D}^{s,\diamond} = \bigcup_{n \geq 0} \mathcal{D}_n^{s,\diamond}$.

For instance, if the operator \diamond is $=$ and s is a constant statistic (*i.e.*, $s(P) = 0$ for any $P \in \mathcal{D}$), then we obviously have $\mathcal{D}_n^{s,\diamond} = \mathcal{D}_n$ for $n \geq 0$.

If s is the number of returns (*i.e.*, $s(P)$ is the number of down steps D that returns the path P to the x -axis) and $s(U\alpha D) \diamond s(\beta)$ is $s(U\alpha D) \geq s(\beta)$, then it is straightforward to see that $\mathcal{D}_n^{s,\geq}$ consists of Dyck paths built over the grammar $S \rightarrow \epsilon \mid USD \mid USUDUSD$. So, the generating function $S(x)$ for the cardinalities of $\mathcal{D}_n^{s,\geq}$, $n \geq 0$, satisfies the functional equation

$S(x) = 1 + xS(x) + x^2S(x)^2$. The solution of this equation is the well-known generating function for the Motzkin numbers ([A001006](#) in [17]).

In this paper, we focus on the sets $\mathcal{D}^{h,\diamond}$ where the statistic h is the maximal height of a Dyck path, *i.e.*, $h(P)$ is the maximal ordinate reached by the path P .

In Section 2, we deal with the case in which operator \diamond is a strict inequality $>$. We prove that the cardinalities of the sets $\mathcal{D}_n^{h,>}$, $n \geq 0$, are given by the sequence [A045761](#) in [17]. This sequence corresponds to the coefficients of the series $\lim_{k \rightarrow \infty} P_k(x)$ where $P_k(x)$ is a polynomial recursively defined by $P_0(x) = x$, $P_1(x) = x^2$, $P_k(x) = P_{k-1}(x) + P_{k-2}(x)$ if k is even, and $P_k(x) = P_{k-1}(x) \cdot P_{k-2}(x)$ if k is odd.

In Section 3, we focus on the set $\mathcal{D}^{h,\geq}$ where $h(U\alpha D) \geq s(\beta)$ (the operator \diamond is \geq). Using generating functions and continued fractions, we prove that the cardinalities of the $\mathcal{D}_n^{h,\geq}$, $n \geq 0$, are given by the Motzkin numbers ([A001006](#) in [17]). Moreover, we give a constructive one-to-one correspondence ϕ between Dyck paths in $\mathcal{D}_n^{h,\geq}$ and Motzkin paths of length n . Also, we show how ϕ transports peaks UD into peaks UD and flats F in Motzkin paths. Finally, we deduce generating function for the popularity of peaks in $\mathcal{D}_n^{h,\geq}$.

Table 1 presents the two main enumerative results of $\mathcal{D}_n^{h,\diamond}$ obtained in Sections 2 and 3.

\diamond -constraint	Sequence	Sloane	$a_n, 1 \leq n \leq 9$
$h(U\alpha D) > h(\beta)$		A045761	1, 1, 2, 3, 6, 12, 24, 50, 107
$h(U\alpha D) \geq h(\beta)$	Motzkin	A001006	1, 2, 4, 9, 21, 51, 127, 323, 835

Table 1: Cardinalities of $\mathcal{D}_n^{h,\diamond}$ according to the \diamond -constraint.

2 Enumeration of $\mathcal{D}_n^{h,>}$

In this section, we enumerate the set $\mathcal{D}_n^{h,>}$ of Dyck paths of semilength $n \geq 0$ with a first return decomposition satisfying $h(U\alpha D) > h(\beta)$ where h is the maximal height of a Dyck path. For instance, we have $\mathcal{D}_1^{h,>} = \{UD\}$, $\mathcal{D}_2^{h,>} = \{UUDD\}$, and $\mathcal{D}_3^{h,>} = \{UUDDUD, UUUDDD\}$.

Let $A_k(x) = \sum_{n \geq 0} a_{n,k} x^n$ (resp. $B_k(x) = \sum_{n \geq 0} b_{n,k} x^n$) be the generating function where the coefficient $a_{n,k}$ (resp. $b_{n,k}$) is the number of Dyck

paths in $\mathcal{D}_n^{h,>}$ with a maximal height equal to k (resp. of at most k). So, we have $B_k(x) = \sum_{i=0}^k A_i(x)$ and the generating function $B(x)$ for the set $\mathcal{D}_n^{h,>}$ is given by $B(x) = \lim_{k \rightarrow +\infty} B_k(x)$.

Any Dyck path of height k in $\mathcal{D}^{s,>}$ is either empty, or of the form $U\alpha D\beta$ where α is a Dyck path in $\mathcal{D}^{s,>}$ of height $k-1$ and $\beta \in \mathcal{D}^{s,>}$ such that $h(\beta) \leq k-1$. So, we deduce easily the following functional equations:

$$\begin{cases} A_0(x) = B_0(x) = 1, \\ A_k(x) = xA_{k-1}(x) \cdot B_{k-1}(x) \text{ for } k \geq 1. \end{cases} \quad (2)$$

Theorem 1 *We have for $k \geq 0$,*

$$B_k(x) = \frac{P_{2k}(x)}{x}$$

where P_k is the polynomial recursively defined by $P_0(x) = x$, $P_1(x) = x^2$, $P_{2k}(x) = P_{2k-1}(x) + P_{2k-2}(x)$ and $P_{2k+1}(x) = P_{2k}(x) \cdot P_{2k-1}(x)$. As consequence, we have for $k \geq 1$

$$A_k(x) = \frac{P_{2k}(x) - P_{2k-2}(x)}{x},$$

and $B(x)$ is generating function of the sequence [A045761](#) in [17].

Proof. We proceed by induction on k . For $k = 0$, the property holds since $P_0(x) = x = xB_0(x)$. Assuming the property for $0 \leq i \leq k-1$, we prove it for k . Taking into account that $A_k(x) = B_k(x) - B_{k-1}(x)$ in equation (2), we obtain

$$xB_k(x) = xB_{k-1}(x) + x^2B_{k-1}(x)(B_{k-1}(x) - B_{k-2}(x)).$$

With the recurrence hypothesis, we have

$$\begin{aligned} xB_k(x) &= P_{2k-2}(x) + P_{2k-2}(x)(P_{2k-2}(x) - P_{2k-4}(x)) \\ &= P_{2k-2}(x) + P_{2k-2}(x)P_{2k-3}(x) \\ &= P_{2k-2}(x) + P_{2k-1}(x) = P_{2k}(x). \end{aligned}$$

An induction on k completes the proof and the expression of $A_k(x)$ is deduced from $A_k(x) = B_k(x) - B_{k-1}(x)$. \square

For instance, we have $B_2(x) = 1 + x + x^2 + x^3$, $B_3(x) = 1 + x + x^2 + 2x^3 + 2x^4 + 2x^5 + 2x^6 + x^7$, and the first ten terms of $B(x)$ are $1 + x + x^2 + 2x^3 + 3x^4 + 6x^5 + 12x^6 + 24x^7 + 50x^8 + 107x^9$. We refer to Table 2 for an overview of the coefficients $a_{n,k}$ for $1 \leq n \leq 10$ and $1 \leq k \leq 9$.

Notice that the family of sets $\mathcal{D}^{s,>}$, $n \geq 1$, seems to be the first example of combinatorial objects enumerated by the sequence [A045761](#) in [17].

$k \setminus n$	1	2	3	4	5	6	7	8	9	10
1	1									
2		1	1							
3			1	2	2	2	1			
4				1	3	5	8	11	13	15
5					1	4	9	18	33	56
6						1	5	14	33	71
7							1	6	20	54
8								1	7	27
9									1	...
Σ	1	1	2	3	6	12	24	50	107	...

Table 2: Number $a_{n,k}$ of Dyck paths of height k in $\mathcal{D}_n^{h,>}$, $1 \leq n \leq 10$ and $1 \leq k \leq 9$.

3 Enumeration of $\mathcal{D}_n^{h,\geq}$

3.1 Using generating function

In this section, we enumerate the set $\mathcal{D}_n^{h,\geq}$ of Dyck paths of semilength $n \geq 0$ with a first return decomposition satisfying $h(U\alpha D) \geq h(\beta)$ where h is the maximal height of a Dyck path. For instance, we have $\mathcal{D}_1^{h,\geq} = \{UD\}$, $\mathcal{D}_2^{h,\geq} = \{UDUD, UUDD\}$, and $\mathcal{D}_3^{h,\geq}$ consists of the four Dyck paths $UDUDUD, UUDDUD, UUDUD$ and $UUUDDD$.

Let $C_k(x) = \sum_{n \geq 0} c_{n,k} x^n$ (resp. $D_k(x) = \sum_{n \geq 0} d_{n,k} x^n$) be the generating function where the coefficient $c_{n,k}$ (resp. $d_{n,k}$) is the number of Dyck paths in $\mathcal{D}_n^{h,\geq}$ with a maximal height equal to k (resp. of at most k). So, we

have $D_k(x) = \sum_{i=0}^k C_i(x)$ and the generating function $D(x)$ for the set $\mathcal{D}_n^{h,\geq}$ is given by $D(x) = \lim_{k \rightarrow +\infty} D_k(x)$.

Any Dyck path of height k in $\mathcal{D}^{s,\geq}$ is either empty, or of the form $U\alpha D\beta$ where α (resp. β) is a Dyck path in $\mathcal{D}^{s,\geq}$ of height $k-1$ (resp. of height at most k). So, we deduce the following functional equations:

$$\begin{cases} C_0(x) = 1, \\ C_k(x) = xC_{k-1}(x) \cdot D_k(x) \text{ for } k \geq 1. \end{cases} \quad (3)$$

Theorem 2 provides recursive expressions for the two generating functions $D_k(x)$ and $C_k(x)$. As a consequence, $D(x)$ can be expressed as an infinite product of terms $\frac{1}{1-xC_k(x)}$.

Theorem 2 *We have $D_0(x) = C_0(x) = 1$, $C_1(x) = \frac{x}{1-x}$ and*

$$D_k(x) = \prod_{i=0}^{k-1} (1 - xC_i(x))^{-1} \text{ for } k \geq 1,$$

$$C_k(x) = \frac{C_1(x)^k}{\prod_{i=1}^{k-1} (1 - xC_i(x))^{k-i}} \text{ for } k \geq 2,$$

and the generating function $D(x)$ where the coefficient of x^n is the cardinality of $\mathcal{D}_n^{h,\geq}$ satisfies

$$D(x) = \prod_{i=0}^{\infty} (1 - xC_i(x))^{-1}.$$

Proof. Since we have $C_k(x) = D_k(x) - D_{k-1}(x)$, equation (3) implies

$$D_k(x) = \frac{D_{k-1}(x)}{1 - xC_{k-1}(x)},$$

and starting from $D_0(x) = 1$, a straightforward induction on k provides $D_k(x) = \prod_{i=0}^{k-1} (1 - xC_i(x))^{-1}$. Moreover, from equation (3) and $C_1(x) = \frac{x}{1-x}$, we deduce

$$\frac{C_k(x)}{C_{k-1}(x)} = xD_k(x) = C_1(x) \prod_{i=1}^{k-1} (1 - xC_i(x))^{-1}.$$

An induction on k completes the proof. \square

Now, we will show how $D(x) = \lim_{k \rightarrow +\infty} D_k(x)$ can be expressed as a continued fraction. For this, let us define the function

$$f : u \mapsto \frac{1}{1 - \frac{x}{1 - \frac{x}{1 - x^2 u}}}.$$

For $n \geq 1$, we denote by f^n the function recursively defined by $f^n(u) = f(f^{n-1}(u))$ anchored with $f^0(u) = u$. A simple calculation (using Maple for instance) proves that the map f satisfies Remark 1.

Remark 1 *If $X = \frac{Y-1}{x(xY+1)}$ then the map f satisfies*

$$f\left(\frac{Y}{1-x(Y-X)}\right) = \frac{f(Y)}{1-x(f(Y)-f(X))}.$$

Lemma 1 *For $k \geq 1$, we have $D_{k-1}(x) = \frac{D_k(x)-1}{x(xD_k(x)+1)}$.*

Proof. We proceed by induction on k . Since $D_0(x) = 1$ and $D_1(x) = \frac{1}{1-x}$, it is easy to check that $D_0(x) = \frac{D_1(x)-1}{x(xD_1(x)+1)}$.

Assuming $D_{i-1}(x) = \frac{D_i(x)-1}{x(xD_i(x)+1)}$ for $1 \leq i \leq k-2$, we prove the result for $i = k-1$. From equation (3) and the recurrence hypothesis on $D_{k-2}(x)$ we obtain

$$D_k(x) = \frac{D_{k-1}(x)}{1-x\left(D_{k-1}(x) - \frac{D_{k-1}(x)-1}{x(xD_{k-1}(x)+1)}\right)}.$$

Isolating $D_{k-1}(x)$, we obtain $D_{k-1}(x) = \frac{D_k(x)-1}{x(xD_k(x)+1)}$ which completes the induction. \square

Theorem 3 *For $k \geq 0$, we have*

$$D_k(x) = f^{\lfloor \frac{k}{4} \rfloor}(D_{k \bmod 4}(x))$$

with the initial cases $D_0(x) = 1$, $D_1(x) = \frac{1}{1-x}$, $D_2(x) = \frac{1}{1-x-x^2}$ and $D_3(x) = \frac{1}{1-\frac{x}{1-x}}$. \square

$k \setminus n$	1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1	1
2		1	2	4	7	12	20	33	54	88
3			1	3	8	19	43	94	201	423
4				1	4	13	37	99	254	634
5					1	5	19	62	187	536
6						1	6	26	95	316
7							1	7	34	137
8								1	8	43
9									1	...
Σ	1	2	4	9	21	51	127	323	835	...

Table 3: Number $c_{n,k}$ of Dyck paths of height k in $\mathcal{D}_n^{h,\geq}$, $1 \leq n \leq 10$ and $1 \leq k \leq 9$.

Proof. We proceed by induction on k . Since we have $D_0(x) = f^0(D_0(x))$, $D_1(x) = f^0(D_1(x))$, $D_2(x) = f^0(D_2(x))$ and $D_3(x) = f^0(D_3(x))$, the basic case holds.

Assuming the result for $0 \leq i \leq k$, we prove it for $k+1$.

From equation (3) we have $D_{k+1}(x) = \frac{D_k(x)}{1-x(D_k(x)-D_{k-1}(x))}$. Using the recurrence hypothesis for $D_k(x)$ and $D_{k-1}(x)$, we obtain for $k \geq 4$:

$$\begin{aligned}
D_{k+1}(x) &= \frac{f^{\lfloor \frac{k}{4} \rfloor}(D_{k \bmod 4}(x))}{1-x\left(f^{\lfloor \frac{k}{4} \rfloor}(D_{k \bmod 4}(x))-f^{\lfloor \frac{k-1}{4} \rfloor}(D_{k-1 \bmod 4}(x))\right)} \\
&= \frac{f\left(f^{\lfloor \frac{k-4}{4} \rfloor}(D_{k-4 \bmod 4}(x))\right)}{1-x\left(f\left(f^{\lfloor \frac{k-4}{4} \rfloor}(D_{k-4 \bmod 4}(x))\right)-f\left(f^{\lfloor \frac{k-5}{4} \rfloor}(D_{k-5 \bmod 4}(x))\right)\right)}.
\end{aligned}$$

The recurrence hypothesis implies

$$D_{k+1}(x) = \frac{f(D_{k-4}(x))}{1-x(f(D_{k-4}(x))-f(D_{k-5}(x)))},$$

and using Remark 1, we deduce

$$\begin{aligned}
D_{k+1}(x) &= f\left(\frac{D_{k-4}(x)}{1-x(D_{k-4}(x)-D_{k-5}(x))}\right) = f(D_{k-3}(x)) \\
&= f^{1+\lfloor\frac{k-3}{4}\rfloor}(D_{k-3 \bmod 4}(x)) = f^{\lfloor\frac{k+1}{4}\rfloor}(D_{k+1 \bmod 4}(x)).
\end{aligned}$$

The induction is completed. \square

For instance, we have $D_{13}(x) = \frac{1-6x+10x^2-9x^4+2x^5+x^6}{1-7x+15x^2-5x^3-15x^4+9x^5+3x^6-x^7}$ and the first terms of its Taylor expansion are $1 + x + 2x^2 + 4x^3 + 9x^4 + 21x^5 + 51x^6 + 127x^7 + 323x^8 + 835x^9 + 2188x^{10} + 5798x^{11} + 15511x^{12} + 41835x^{13} + 113633x^{14} + 310557x^{15} + 853333x^{16}$.

Theorem 4 *The sets $\mathcal{D}_n^{h,\geq}$, $n \geq 0$, are enumerated by the Motzkin numbers.*

Proof. We have $D(x) = \lim_{k \rightarrow +\infty} D_k(x)$. Using Theorem 3, we directly obtain that

$$D(x) = \frac{1}{1 - \frac{x}{1 - \frac{x}{1 - \frac{x^2}{1 - \frac{x}{1 - \frac{x}{1 - \frac{x^2}{1 - \frac{x}{1 - \frac{x}{\dots}}}}}}}}}$$

which is the continued fraction for the Motzkin numbers (P. Barry [3]). \square

3.2 A constructive bijection

In this section, we exhibit a constructive bijection between $\mathcal{D}_n^{h,\geq}$ and the set \mathcal{M}_n of Motzkin paths of length n , *i.e.*, lattice paths starting at $(0, 0)$, ending at $(n, 0)$, never going below the x -axis, consisting of up-steps $U = (1, 1)$, down steps $D = (1, -1)$ and flat steps $F = (1, 0)$. We set $\mathcal{M} = \cup_{n \geq 0} \mathcal{M}_n$.

Let us define recursively the map ϕ from $\mathcal{D}^{h,\geq}$ to \mathcal{M} as follows. For $P \in \mathcal{D}^{h,\geq}$, we set

$$\phi(P) = \begin{cases} \epsilon & \text{if } P = \epsilon, \\ \phi(\alpha)F & \text{if } P = \alpha UD, \\ \phi(\alpha)\phi(\gamma)U\phi(\beta)D & \text{if } P = \alpha UU\beta D\gamma D. \end{cases}$$

Due to the recursive definition, the image by ϕ of a Dyck path of semilength n is a Motzkin path of length n . For instance, the images of $UDUDUD$, $UUDDUD$, $UUDUDD$, $UUUDDD$, $UUUUDDDDUUUDDUDD$ are respectively FFF , UDF , FUD , UFD and $UUDDFUFD$. We refer to Figure 2 for an illustration of this mapping.

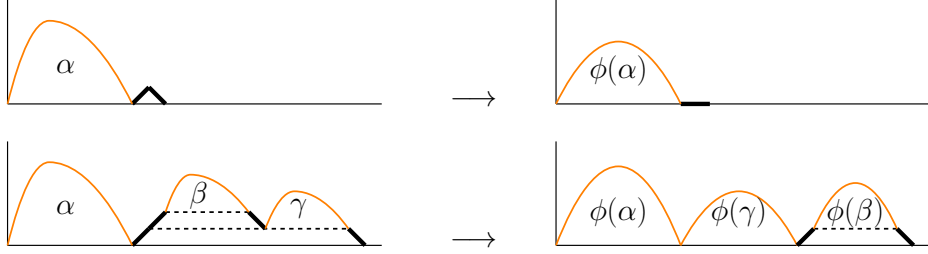


Figure 2: Illustration of the bijection between $\mathcal{D}_n^{h,\geq}$ and \mathcal{M}_n .

Remark 2 If $\alpha, \beta \in \mathcal{D}_n^{h,\geq}$ and $\alpha\beta \in \mathcal{D}_n^{h,\geq}$, then we have $\phi(\alpha\beta) = \phi(\alpha)\phi(\beta)$.

Theorem 5 The map $\phi : \mathcal{D}_n^{h,\geq} \rightarrow \mathcal{M}_n$ defined above is a bijection satisfying for any $P \in \mathcal{D}_n^{h,\geq}$,

$$h(\phi(P)) = \left\lfloor \frac{h(P)}{2} \right\rfloor.$$

Proof. We proceed by induction on n . Obviously, for $n = 1$, we have $\phi(UD) = F$ and $h(F) = 0 = \lfloor \frac{h(UD)}{2} \rfloor$. For $k \leq n$, we assume that ϕ is a bijection from $\mathcal{D}_k^{h,\geq}$ to \mathcal{M}_k such that $h(\phi(P)) = \lfloor \frac{h(P)}{2} \rfloor$ for any $P \in \mathcal{D}_k^{h,\geq}$, and we prove the result for $n + 1$.

Using the enumerative result of Theorem 4, it suffices to prove that ϕ is surjective from $\mathcal{D}_{n+1}^{h,\geq}$ to \mathcal{M}_{n+1} . So, let M be a Motzkin path in \mathcal{M}_{n+1} . We distinguish two cases: (i) $M = \sigma F$ with $\sigma \in \mathcal{M}_n$, and (ii) $M = \sigma U \pi D$ where σ and π are two Motzkin paths in \mathcal{M} .

(i) Using the recurrence hypothesis, there is $P \in \mathcal{D}_n^{h, \geq}$ such that $\sigma = \phi(P)$ and $h(\sigma) = \lfloor \frac{h(P)}{2} \rfloor$. So, the Dyck path PUD belongs to $\mathcal{D}_{n+1}^{h, \geq}$ and $\phi(PUD) = \sigma F$ which proves that M belongs to the image by ϕ of $\mathcal{D}_{n+1}^{h, \geq}$. Moreover we have $h(\phi(PUD)) = h(\sigma F) = h(\sigma) = \lfloor \frac{h(P)}{2} \rfloor = \lfloor \frac{h(PUD)}{2} \rfloor$.

(ii) We suppose $M = \sigma U \pi D$. Let us define the longest suffix σ_s of σ (possibly empty) such that $\sigma_s \in \mathcal{M}$ and $h(\phi^{-1}(\sigma_s)) \leq 1 + h(\phi^{-1}(\pi))$ (σ_s exists since the empty path ϵ satisfies the inequality, and the recurrence hypothesis ensures the existence and the unicity of $\phi^{-1}(\sigma_s)$ and $\phi^{-1}(\pi)$). Let σ_r be the Motzkin path obtained from σ by deleting the suffix σ_s , and let $S \in \mathcal{D}_n^{h, \geq}$ (resp. $R \in \mathcal{D}_n^{h, \geq}$) such that $\phi(S) = \sigma_s$ and $h(\sigma_s) = \lfloor \frac{h(S)}{2} \rfloor$ (resp. $\phi(R) = \sigma_r$ and $h(\sigma_r) = \lfloor \frac{h(R)}{2} \rfloor$). Also there is $T \in \mathcal{D}_n^{h, \geq}$ such that $\phi(T) = \pi$ with $h(\pi) = \lfloor \frac{h(T)}{2} \rfloor$.

Due to the maximality of σ_s , σ_r is either empty or $1 + h(T) < h(\phi^{-1}(\sigma_r \sigma_s))$. Using Remark 2 we obtain $h(\phi^{-1}(\sigma_r \sigma_s)) = h(RS) = h(R)$, and the last inequality can be written as $1 + h(T) < h(R)$.

- If $\sigma_r = \epsilon$ then the condition $h(S) \leq 1 + h(T)$ implies that $UUTDSD$ belongs to $\mathcal{D}_n^{h, \geq}$ and we have $\phi(UUTDSD) = \phi(S)U\phi(T)D = \sigma_s U \pi D = \sigma U \pi D$. Moreover, we have $h(\phi(UUTDSD)) = h(\phi(S)U\phi(T)D) = \max\{h(S), 1 + h(T)\} = \max\{\lfloor \frac{h(S)}{2} \rfloor, 1 + \lfloor \frac{h(T)}{2} \rfloor\}$. With $h(S) \leq 1 + h(T)$, we deduce $h(\phi(UUTDSD)) = 1 + \lfloor \frac{h(T)}{2} \rfloor = \lfloor \frac{h(T)+2}{2} \rfloor = \lfloor \frac{h(UUTDSD)}{2} \rfloor$ as desired.
- If $\sigma_r \neq \epsilon$ then we have $h(S) \leq 1 + h(T) < h(R)$ which implies that the Dyck path $RUUTDSD$ belongs to $\mathcal{D}_{n+1}^{h, \geq}$. Moreover, we have $h(\phi(RUUTDSD)) = h(\phi(R)\phi(S)U\phi(T)D) = \max\{h(R), h(S), 1 + h(T)\} = \max\{\lfloor \frac{h(R)}{2} \rfloor, \lfloor \frac{h(S)}{2} \rfloor, 1 + \lfloor \frac{h(T)}{2} \rfloor\}$. From $h(S) \leq 1 + h(T) < h(R)$, we obtain $\lfloor \frac{h(S)+1}{2} \rfloor \leq \lfloor \frac{h(T)+2}{2} \rfloor \leq \lfloor \frac{h(R)}{2} \rfloor$ which induces that $h(\phi(RUUTDSD)) = \lfloor \frac{h(R)}{2} \rfloor = \lfloor \frac{h(RUUTDSD)}{2} \rfloor$ as desired.

The induction is completed. \square

Corollary 1 *Let P be a Dyck path in $\mathcal{D}_n^{h, \geq}$, $n \geq 1$. The Motzkin path $\phi(P)$ contains a flat step F at height $h(\phi(P))$ if and only if $h(P)$ is odd.*

Proof. We proceed by induction on n . For $n = 1$, we have $P = UD$ and $\phi(P) = F$ and the result holds since $h(P) = 1$ is odd. Assuming the result for $i \leq n$, we prove it for $n + 1$. We distinguish two cases: (i) $P = \alpha UD$ and (ii) $P = \alpha UU\beta D\gamma D$, where α, β and γ belong to $\mathcal{D}^{h, \geq}$.

(i) For $\alpha \neq \epsilon$, the recurrence hypothesis means that $\phi(\alpha)$ contains a flat step F at height $h(\phi(\alpha))$ if and only if $h(\alpha)$ is odd. Since we have $\phi(P) = \phi(\alpha)F$ and $h(P) = h(\alpha)$, $\phi(P)$ contains a flat step F a height $h(\phi(P)) = h(\phi(\alpha))$ if and only if $h(P)$ is odd.

(ii) If $P = \alpha UU\beta D\gamma D$ then we have $h(\alpha) \geq 2 + h(\beta) \geq 1 + h(\gamma)$ and $\phi(P) = \phi(\alpha)\phi(\gamma)U\phi(\beta)D$.

- As we have done in (i), a flat step F appears at height $h(\phi(P)) = h(\phi(\alpha))$ in $\phi(\alpha)$ if and only if $h(\alpha) = h(P)$ is odd.
- $\phi(P)$ contains a flat step at height $h(\phi(P))$ in $\phi(\beta)$ if and only if a flat step appears at height $h(\phi(\beta))$ in $\phi(\beta)$ and $h(\phi(P)) = h(\phi(\beta)) + 1$, i.e., $\lfloor \frac{h(P)}{2} \rfloor = \lfloor \frac{h(\beta)}{2} \rfloor + 1$. Using the recurrence hypothesis, this is equivalent to $h(\beta)$ is odd, which means $2\lfloor \frac{h(\beta)}{2} \rfloor = h(\beta) + 1$. Since $h(P) \geq h(\beta) + 2$, this is equivalent to $h(P) = h(\beta) + 2$ and thus $h(P)$ is odd.
- Let us prove by contradiction that a flat step cannot occur at height $h(\phi(P))$ in $\phi(\gamma)$. Indeed, this should imply the following:

$$h(\phi(P) = h(\phi(\gamma)) \geq 1 + h(\phi(\beta)).$$

With Theorem 5 we obtain

$$\left\lfloor \frac{h(\gamma)}{2} \right\rfloor \geq 1 + \left\lfloor \frac{h(\beta)}{2} \right\rfloor.$$

Due to the recurrence hypothesis, $h(\gamma)$ is odd, so

$$\left\lfloor \frac{1 + h(\gamma)}{2} \right\rfloor - 1 \geq \left\lfloor \frac{2 + h(\beta)}{2} \right\rfloor, \text{ and } \left\lfloor \frac{h(U\gamma D)}{2} \right\rfloor > \left\lfloor \frac{h(UU\beta DD)}{2} \right\rfloor,$$

which implies

$$h(U\gamma D) > h(UU\beta DD).$$

This last inequality contradicts $P \in \mathcal{D}^{h, \geq}$.

The induction is completed. \square

As a byproduct, we derive in Corollary 2 three generating functions for the number of Motzkin paths of height k according to a constraint on the maximal height of a flat. The two first results do not seem to appear in the literature, while the third is presented in [4]. See Tables 4 and 5 for numerical data.

Corollary 2 *The generating function $\overline{M}_k(x)$ where the coefficient of x^n is the number Motzkin paths in \mathcal{M}_n of height exactly k and where there is a flat F at height k is*

$$\overline{M}_k(x) = D_{2k+1}(x) - D_{2k}(x).$$

The generating function $\widehat{M}_k(x)$ where the coefficient of x^n is the number Motzkin paths in \mathcal{M}_n of height exactly k and where there is no flat F at height k is

$$\widehat{M}_k(x) = D_{2k}(x) - D_{2k-1}(x).$$

The generating function $M_k(x)$ where the coefficient of x^n is the number Motzkin paths in \mathcal{M}_n of height exactly k is

$$M_k(x) = D_{2k+1}(x) - D_{2k-1}(x).$$

Proof. The proof is directly deduced from Theorem 5 and Corollary 1. \square

For instance, we have $\overline{M}_1(x) = \frac{x^3}{(1-2x)(1-x-x^2)}$ and the first terms of its Taylor expansion are $x^3 + 3x^4 + 8x^5 + 19x^6 + 43x^7 + 94x^8 + 201x^9 + 423x^{10} + 880x^{11}$ (see sequence [A008466](#) in [17]). Also, $\widehat{M}_1(x) = \frac{x^2}{(1-x-x^2)(1-x)}$ and the first terms of its Taylor expansion are $x^2 + 2x^3 + 4x^4 + 7x^5 + 12x^6 + 20x^7 + 33x^8 + 54x^9 + 88x^{10} + 143x^{11}$ (see sequence [A000071](#) in [17]). Notice that the two sequences defined by $\sum_{k \geq 0} \overline{M}_k(x)$ and $\sum_{k \geq 0} \widehat{M}_k(x)$ do not appear in [17]. We leave open the question of finding closed forms for their generating functions.

Corollary 3 *The bivariate generating function $Q(x, y)$ where the coefficient of $x^n y^k$ is the number of Dyck paths of $\mathcal{D}_n^{h, \geq}$ containing exactly k peaks UD is*

$$\frac{1-x^2y+x^2-xy-\sqrt{x^4y^2-2x^4y+2x^3y^2+x^4-2x^3y+x^2y^2-2x^2y-2x^2-2xy+1}}{2x^2}.$$

Proof. Due to the recursive definition of the bijection ϕ , the number of peaks UD in Dyck paths of $\mathcal{D}_n^{h, \geq}$ is equal to the number of peaks UD and flats F in Motzkin paths in \mathcal{M}_n . Since a non empty Motzkin path in \mathcal{M} is either αF or $\alpha U \beta D$ where $\alpha, \beta \in \mathcal{M}$, the generating function $Q(x, y)$ satisfies the functional equation $Q(x, y) = 1 + xyQ(x, y) + x^2yQ(x, y) + x^2Q(x, y)^2$. A simple calculation (with Maple for instance) completes the proof. \square

As a consequence, we deduce in Corollary 4 the popularity of peaks in $\mathcal{D}_n^{h, \geq}$.

$k \setminus n$	1	2	3	4	5	6	7	8	9	10
0	1	1	1	1	1	1	1	1	1	1
1			1	3	8	19	43	94	201	423
2					1	5	19	62	187	536
3							1	7	34	137
4									1	9
Σ	1	1	2	4	10	25	64	164	424	1106

Table 4: Number of Motzkin paths of length n and height k with a flat at height k , $1 \leq n \leq 10$ and $0 \leq k \leq 4$.

$k \setminus n$	1	2	3	4	5	6	7	8	9	10
1		1	2	4	7	12	20	33	54	88
2				1	4	13	37	99	254	634
3						1	6	26	95	316
4								1	8	43
5										1
Σ	0	1	2	5	11	26	63	159	411	1082

Table 5: Number of Motzkin paths of length n and height k with no flats at height k , $1 \leq n \leq 10$ and $1 \leq k \leq 5$.

Corollary 4 *The generating function where the coefficient of x^n is the total number of peaks in all Dyck paths of $\mathcal{D}_n^{h, \geq}$ is*

$$\frac{1 - x^2 - (1 + x)\sqrt{1 - 2x - 3x^2}}{2x\sqrt{1 - 2x - 3x^2}}.$$

Proof. Using Corollary 3, the result is given by $\frac{\partial Q(x,y)}{\partial y} \Big|_{y=1}$. □

The popularity of peaks in $\mathcal{D}_n^{h, \geq}$, $n \geq 1$, is given by the sequence [A025566](#) in [17]. The first terms are $x + 3x^2 + 8x^3 + 22x^4 + 61x^5 + 171x^6 + 483x^7 + 1373x^8 + 3923x^9 + 11257x^{10}$.

We say that a peak UD is odd whenever the maximal sub-path of the form $U^k D$ that contains it has an odd number of U -steps, *i.e.*, k is odd.

Corollary 5 *The bivariate generating function $R(x, y)$ where the coefficient of $x^n y^k$ is the number of Dyck paths of $\mathcal{D}_n^{h, \geq}$ containing exactly k odd peaks UD is*

$$\frac{2 - xy - \sqrt{1 - 2xy - 4x^2 + x^2y^2}}{2x^2}.$$

Proof. Using the recursive definition of ϕ , it is easy to check that the number of odd peaks on $\mathcal{D}_n^{h, \geq}$ is also the number of flats on \mathcal{M}_n . Since a non empty Motzkin path in \mathcal{M} is either αF or $\alpha U \beta D$ where $\alpha, \beta \in \mathcal{M}$, the generating function $R(x, y)$ satisfies the functional equation $R(x, y) = 1 + xyR(x, y) + x^2R(x, y)^2$. A simple calculation (with Maple for instance) completes the proof. \square

Corollary 6 *The generating function where the coefficient of x^n is the total number of odd peaks in all Dyck paths of $\mathcal{D}_n^{h, \geq}$ is*

$$\frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x\sqrt{1 - 2x - 3x^2}}.$$

Proof. Using Corollary 5, the result is given by $\left. \frac{\partial R(x, y)}{\partial y} \right|_{y=1}$. \square

The popularity of odd peaks is given by the sequence [A005717](#) in [17]. The first terms are $x + 2x^2 + 6x^3 + 16x^4 + 45x^5 + 126x^6 + 357x^7 + 1016x^8 + 2907x^9 + 8350x^{10}$.

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