A permutation code preserving a double Eulerian bistatistic

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Abstract

In 1977 Foata proved bijectively, among other things, that the joint distribution of ascent and distinct nonzero values on the set of subexcedant sequences is the same as that of descent and inverse descent numbers on the set of permutations, and the generating function of the corresponding bistatistics is the double Eulerian polynomial. In 2013 Foata’s result was rediscovered by Visontai as a conjecture, and then reproved by Aas in 2014.

In this paper, we define a permutation code (that is, a bijection between permutations and subexcedant sequences) and show the more general result that two 5-tuples of set-valued statistics on the set of permutations and on the set of subexcedant sequences, respectively, are equidistributed. In particular, these results give another bijective proof of Foata’s result.

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1. Introduction

In enumerative combinatorics, it is a classical result that the descent number des and the inverse descent number ides (defined as ides π = des π⁻¹) on permutations are Eulerian statistics, and their distributions on the set Sn of length-n permutations are given by the nth Eulerian polynomial Aₙ, that is

\[ A_n(u) = \sum_{\pi \in S_n} u^{\text{des} \pi + 1} = \sum_{\pi \in S_n} u^{\text{ides} \pi + 1}, \]

and the joint distribution of des and ides is given by the nth double Eulerian polynomial,

\[ A_n(u, v) = \sum_{\pi \in S_n} u^{\text{des} \pi + 1} v^{\text{ides} \pi + 1}, \]

see for instance [2,8].

An alternative way to represent a permutation is its Lehmer code [5], which is a subexcedant sequence. The ascent number asc on the set Sn of subexcedant sequences is still an Eulerian statistic (see for example [9]), and in [7] the statistic that counts the number of distinct nonzero symbols in a subexcedant sequence (that following [1] we denote by row) is proved to be still Eulerian; this result is credited to Dumont by the authors of [7]. In terms of generating functions, we have

\[ A_n(u) = \sum_{s \in S_n} u^{\text{asc} s + 1} = \sum_{s \in S_n} u^{\text{row} s + 1}. \]
In 1977 Foata [4] proved, among other things, that the joint distribution of des and ides on the set of permutations is the same as that of asc and row on the set of subexcedant sequences, that is

\[ A_n(u, v) = \sum_{\pi \in S_n} u^{\text{des}\pi + 1} v^{\text{ides}\pi + 1} = \sum_{s \in S_n} u^{\text{asc} s + 1} v^{\text{row} s + 1}. \]

In the present paper, we define a bijection between permutations and subexcedant sequences (i.e., a permutation code) and show that the tuple of set-valued statistics

(Des, Ides, Lrmax, Lrmin, Rlmax) on the set of permutations, and

(Asc, Row, Posz, Max, Rlmax) on the set of subexcedant sequences

have the same distribution (each of the occurring statistics is defined below). In particular, our bijection provides set-valued partners for Asc (answering to a question in [1]) and gives a alternative proof of Foata’s result.

2. Notation and definitions

A length-\( n \) word \( w \) over the alphabet \( A \) is a sequence \( w_1 w_2 \ldots w_n \) of symbols in \( A \), and we will consider only finite alphabets \( A \subset \mathbb{N} \).

Statistics

A statistic on a set \( X \) of words is simply a function from \( X \) to \( \mathbb{N} \); a set-valued statistic is a function from \( X \) to \( 2^{\mathbb{N}} \); and a multistatistic is a tuple of statistics.

Let \( w = w_1 w_2 \ldots w_n \) be a length-\( n \) word. A descent in \( w \) is a position \( i \) in \( w \), \( 1 \leq i < n \), with \( w_i > w_{i+1} \), and the descent set of \( w \) is

\[ \text{Des} w = \{ i : 1 \leq i < n \text{ with } w_i > w_{i+1} \}. \]

A left-to-right maximum in \( w \) is a position \( i \) in \( w \), \( 1 \leq i \leq n \), with \( w_i < w_j \) for all \( j < i \), and the set of left-to-right maxima is

\[ \text{Lrmax} w = \{ i : 1 \leq i \leq n \text{ with } w_j < w_i \text{ for all } j < i \}. \]

Clearly, \( 1 \in \text{Lrmax} w \), and Des and Lrmax are classical examples of set-valued statistics on words. We define similarly the sets \( \text{Asc} w \) of ascents, \( \text{Lrmin} w \) of left-to-right minima, \( \text{Rlmax} w \) of right-to-left maxima and \( \text{Rlmin} w \) of right-to-left minima in \( w \).

To each set-valued statistic \( \text{St} \) corresponds an (integer-valued) statistic \( s \) defined as \( s w = \text{card} \text{St} w \), for example des \( w \) and \( \text{Lrmax} w \) count, respectively, the number of descents and the number of left-to-right maxima in \( w \).

Let \( X \) and \( X' \) be two sets of words, and \( s \) and \( s' \) be two statistics defined on \( X \) and \( X' \), respectively. We say that \( s \) on \( X \) has the same distribution as \( s' \) on \( X' \) (or equivalently, \( s \) and \( s' \) are equidistributed) if, for any integer \( u \),

\[ \text{card}\{ w \in X : s w = u \} = \text{card}\{ w \in X' : s' w = u \}, \]

and the multistatistic \( (s_1, s_2, \ldots , s_p) \) defined on \( X \) has the same distribution as the multistatistic \( (s'_1, s'_2, \ldots , s'_p) \) defined on \( X' \) (or the multistatistics are equidistributed) if, for any integer \( p \)-tuple \( u = (u_1, u_2, \ldots , u_p) \),

\[ \text{card}\{ w \in X : (s_1, s_2, \ldots , s_p) w = u \} = \text{card}\{ w \in X' : (s'_1, s'_2, \ldots , s'_p) w = u \}. \]

The notion of equidistribution of (multi)statistics can naturally be extended to set-valued (multi)statistics.

Permutations, subexcedant sequences and codes

This paper deals with two particular classes of words: permutations and subexcedant sequences. A permutation is a length-\( n \) word over \( \{1, 2, \ldots , n\} \) with distinct symbols. Alternatively, a permutation is an element of the symmetric group on \( \{1, 2, \ldots , n\} \) written in one line notation, and \( S_n \) denotes the set of length-\( n \) permutations. If two permutations \( \pi = \pi_1 \pi_2 \ldots \pi_n \) and \( \sigma = \sigma_1 \sigma_2 \ldots \sigma_n \) are such that \( \sigma_1 \sigma_2 \cdots \sigma_n = 1 \cdot 2 \cdots n \) (i.e., the identity in \( S_n \)), then \( \sigma \) is the inverse of \( \pi \), which is denoted by \( \pi^{-1} \).

A length-\( n \) subexcedant sequence\(^1\) is a word \( s = s_1 s_2 \ldots s_n \) over \( \{0, 1, \ldots , n - 1\} \) with \( 0 \leq s_i \leq i - 1 \) for \( 1 \leq i \leq n \), and \( S_n \) denotes the set of length-\( n \) subexcedant sequences; and we have \( S_n = \{0\} \times \{0, 1\} \times \cdots \times \{0, 1, \ldots , n - 1\} \).

Some statistics are consistently defined only on particular classes of words, e.g. permutations or subexcedant sequences.

For a permutation \( \pi \in S_n \), an inverse descent (ides for short) in \( \pi \) is a position \( i \) for which \( \pi_i + 1 \) appears to the left of \( \pi_i \) in \( \pi \). Equivalently, \( i \) is an ides in \( \pi \) if \( \pi_i + 1 \) is a descent in \( \pi^{-1} \). The ides set is defined as

\[ \text{Ides} \pi = \{ i : 1 \leq i \leq n \text{ with } \pi_i + 1 \text{ appears in } \pi \text{ to the left of } \pi_i \}, \]

and Ides \( \pi = \text{des} \pi^{-1} \), but in general Ides \( \pi \) is not equal to Des \( \pi^{-1} \).

Let \( s = s_1 s_2 \ldots s_n \) be a subexcedant sequence in \( S_n \). The Posz statistic gives the positions of 0s in \( s \),

\[ \text{Posz} s = \{ i : 1 \leq i \leq n, s_i = 0 \}, \]

\(^1\) Known in the literature also as inversion sequence, inversion table or subexcedant function.
and obviously 1 ∈ Pos s. The Max statistic is defined as
\[ \text{Max } s = \{i : 1 \leq i \leq n, s_i = i - 1\}, \]
and as above, 1 ∈ Max s.

A last-value position in s is a position i in s such that s_i ≠ 0 and s_i does not occur in the suffix s_{i+1}s_{i+2} \ldots s_n of s. The last-value position set, denoted by Row, is defined as
\[ \text{Row } s = \{i : s_i ≠ 0 \text{ and } s_i \text{ does not occur in the suffix } s_{i+1}s_{i+2} \ldots s_n\} \]
Clearly, 1 ∉ Row s, and row s = card Row s counts the number of distinct nonzero symbols in s.

**Example 1.** If π = 62587314 ∈ S_8 and s = 01102363 ∈ S_8, then
\[
\begin{align*}
\text{Des } π &= \text{Asc } s = \{1, 4, 5, 6\}, \\
\text{Ides } π &= \text{Row } s = \{3, 5, 7, 8\}, \\
\text{Lrmax } π &= \text{Posz } s = \{1, 4\}, \\
\text{Lrmin } π &= \text{Max } s = \{2, 7\}, \\
\text{Rlmax } π &= \text{Rlmin } s = \{4, 5, 8\}.
\end{align*}
\]

An inversion in a permutation π = π_1π_2 \ldots π_n ∈ S_n is a pair (i, j) with i < j and π_i > π_j. The set S_n is in bijection with P_n, and any such bijection is called a permutation code. The Lehmer code L defined in [5] is a classical example of permutation code; it maps each permutation π = π_1π_2 \ldots π_n to a subexcedant sequence s_1s_2 \ldots s_n where, for all j, 1 ≤ j ≤ n, s_j is the number of inversions (i, j) in π (or equivalently, the number of entries in π larger than π_j and on its left). For example, L(62587314) = 01101464. See also [11] for a family of permutation codes in the context of Mahonian statistics on permutations.

In [3] is showed that dmc statistic which counts the number of distinct nonzero symbols in the Lehmer code of a permutation π (the statistic π ↦ row L(π) with the above notations) is Eulerian, and so has the same distribution as des, asc or ides on S_n. See also [10] where Dumont’s statistic dmc is extended to words.

Although the following properties are folklore, they are easy to check.

**Property 1.** If π ∈ S_n and L(π) ∈ S_n is its Lehmer code, then Des π = Asc L(π), Lrmax π = Posz L(π), Lrmin π = Max L(π), and Rlmax π = Rlmin L(π).

3. The permutation code b

We define a mapping b: S_n → S_n and Theorem 1 shows that b is a bijection, that is, a permutation code, and it is the main tool in proving that (Des, Ides, Lrmax, Lrmin, Rlmax) on S_n has the same distribution as (Asc, Row, Posz, Max, Rlmax) on S_n (see Theorem 2).

A position i in π = π_1π_2 \ldots π_n ∈ S_n, 1 ≤ i ≤ n, can satisfy the following properties:

P1: π_i + 1 occurs in π at the right of π_i,

P2: π_i - 1 occurs in π0 at the right of π_i,

where π_0 is the permutation of \{0, 1, \ldots, n\} obtained by adding a 0 at the end of π. And, to each position i in π, we associate an integer \( \lambda_i(\pi) ∈ \{0, 1, 2, 3\} \) according to i satisfies both, one, or none of these properties:

\[
\lambda_i(\pi) = \begin{cases} 
0, & \text{if } i \text{ satisfies both P1 and P2,} \\
1, & \text{if } i \text{ satisfies P2 but not P1,} \\
2, & \text{if } i \text{ satisfies P1 but not P2,} \\
3, & \text{if } i \text{ satisfies neither P1 nor P2,}
\end{cases}
\]

and we denote it simply by \( \lambda_i \) when there is no ambiguity.

Alternatively, using the Iverson bracket notation \([P] = 1 \text{ if the statement } P \text{ is true}\), we have the more concise expression: \( \lambda_i = [\pi_i = n \text{ or } \pi_i + 1 \text{ occurs at the left of } \pi_i] + 2 \cdot [\pi_i - 1 \text{ occurs at the left of } \pi_i] \).

For example, for any π ∈ S_n we have \( \lambda_1 = 0 \) except \( \lambda_1 = 1 \) if π_1 = n; and when n > 1, then \( \lambda_n = 3 \) except \( \lambda_n = 2 \) if \( \pi_n = 1 \). Each \( \lambda_i \) is uniquely determined by π, for instance if π = 62587314, then \( \lambda_1, \lambda_2, \ldots, \lambda_8 = 0, 0, 1, 1, 3, 2, 1, 3 \), see Fig. 2.

An interval \( I = [a, b], a ≤ b, \) is the set of integers \( \{x : a ≤ x ≤ b\} \); and a labeled interval is a pair \( (I, \ell) \) where \( I \) is an interval and \( \ell \) is an integer. In order to give the construction of the mapping b, we define below the slices of a permutation, and some of their properties are given in Remark 1.

**Definition 1.** For a permutation \( π = π_1π_2 \ldots π_n ∈ S_n \) and an \( i, 0 ≤ i < n, \) the ith slice of π is the sequence of labeled intervals \( U_i(π) = (I_1, \ell_1), (I_2, \ell_2), \ldots, (I_k, \ell_k) \), defined by the following process (see Fig. 1).
\begin{itemize}
    \item $U_0(\pi) = ([0, n], 0)$.
    \item For $i \geq 1$, let $U_{i-1}(\pi) = (I_1, \ell_1), (I_2, \ell_2), \ldots, (I_k, \ell_k)$ be the $(i-1)$th slice of $\pi$ and $v, 1 \leq v \leq k$, be the integer such that $\pi_i \in I_v$. The $i$th slice $U_i(\pi)$ of $\pi$ is defined according to $\lambda_i$:
    \begin{enumerate}
        \item If $\lambda_i = 0$ (or equivalently, $\min l_v < \pi_i < \max l_v$), then
            \begin{align*}
                U_i(\pi) &= (I_1, \ell_1), \ldots, (I_{v-1}, \ell_{v-1}), (H, \ell_v), (J, \ell_{v+1}), (I_{v-1}, \ell_{v-2}), \ldots, (I_{k-1}, \ell_k), (I_k, \ell_k + 1),
            \end{align*}
            where $H = [\pi_i + 1, \max l_v]$ and $J = [\min l_v, \pi_i - 1]$.
        \item If $\lambda_i = 1$ (or equivalently, $\min l_v < \max l_v = \pi_i$), then
            \begin{align*}
                U_i(\pi) &= (I_1, \ell_1), \ldots, (I_{v-1}, \ell_{v-1}), (J, \ell_v), (I_{v+1}, \ell_{v+2}), \ldots, (I_{k-1}, \ell_k), (I_k, \ell_k + 1),
            \end{align*}
            where $J = [\min l_v, \pi_i - 1]$.
        \item If $\lambda_i = 2$ (or equivalently, $\min l_v = \pi_i < \max l_v$), then
            \begin{align*}
                U_i(\pi) &= (I_1, \ell_1), \ldots, (I_{v-1}, \ell_{v-1}), (I_v, \ell_v), (I_{v+1}, \ell_{v+2}), \ldots, (I_{k-1}, \ell_{k-1}), (I_k, \ell_k + 1),
            \end{align*}
            where $J = [\pi_i + 1, \max l_v]$.
        \item If $\lambda_i = 3$ (or equivalently, $\min l_v = \pi_i = \max l_v$), then
            \begin{align*}
                U_i(\pi) &= (I_1, \ell_1), \ldots, (I_{v-1}, \ell_{v-1}), (I_{v+1}, \ell_{v+1}), \ldots, (I_{k-1}, \ell_{k-1}), (I_k, \ell_k + 1).
            \end{align*}
    \end{enumerate}
\end{itemize}

**Example 2.** For the permutation $\pi = 6 \, 2 \, 5 \, 8 \, 7 \, 3 \, 1 \, 4$ in Fig. 2, $\lambda_1(\pi), \lambda_2(\pi), \ldots, \lambda_8(\pi) = 0, 0, 1, 3, 2, 1, 3$, and the process described in Definition 1 gives the slices below.

\begin{itemize}
    \item $U_0(\pi) = ([0, 8], 0)$;
    \item $U_1(\pi) = ([7, 8], 0), ([0, 5], 1)$;
    \item $U_2(\pi) = ([7, 8], 0), ([3, 5], 1), ([0, 1], 2)$;
    \item $U_3(\pi) = ([7, 8], 0), ([3, 4], 2), ([0, 1], 3)$;
    \item $U_4(\pi) = ([7, 7], 2), ([3, 4], 3), ([0, 1], 4)$;
    \item $U_5(\pi) = ([3, 4], 3), ([0, 1], 5)$;
    \item $U_6(\pi) = ([4, 4], 3), ([0, 1], 6)$;
    \item $U_7(\pi) = ([4, 4], 3), ([0, 0], 7)$.
\end{itemize}

**Remark 1.** Let $U_i(\pi) = (I_1, \ell_1), (I_2, \ell_2), \ldots, (I_k, \ell_k)$ be the $i$th slice of $\pi$, $0 \leq i < n$. Then, the following properties can be easily checked:

\begin{itemize}
    \item $U_0(\pi) = ([0, n], 0)$.
    \item For $i \geq 1$, let $U_{i-1}(\pi) = (I_1, \ell_1), (I_2, \ell_2), \ldots, (I_k, \ell_k)$ be the $(i-1)$th slice of $\pi$ and $v, 1 \leq v \leq k$, be the integer such that $\pi_i \in I_v$. The $i$th slice $U_i(\pi)$ of $\pi$ is defined according to $\lambda_i$:
    \begin{enumerate}
        \item If $\lambda_i = 0$ (or equivalently, $\min l_v < \pi_i < \max l_v$), then
            \begin{align*}
                U_i(\pi) &= (I_1, \ell_1), \ldots, (I_{v-1}, \ell_{v-1}), (H, \ell_v), (J, \ell_{v+1}), (I_{v-1}, \ell_{v-2}), \ldots, (I_{k-1}, \ell_k), (I_k, \ell_k + 1),
            \end{align*}
            where $H = [\pi_i + 1, \max l_v]$ and $J = [\min l_v, \pi_i - 1]$.
        \item If $\lambda_i = 1$ (or equivalently, $\min l_v < \max l_v = \pi_i$), then
            \begin{align*}
                U_i(\pi) &= (I_1, \ell_1), \ldots, (I_{v-1}, \ell_{v-1}), (J, \ell_v), (I_{v+1}, \ell_{v+2}), \ldots, (I_{k-1}, \ell_k), (I_k, \ell_k + 1),
            \end{align*}
            where $J = [\min l_v, \pi_i - 1]$.
        \item If $\lambda_i = 2$ (or equivalently, $\min l_v = \pi_i < \max l_v$), then
            \begin{align*}
                U_i(\pi) &= (I_1, \ell_1), \ldots, (I_{v-1}, \ell_{v-1}), (I_v, \ell_v), (I_{v+1}, \ell_{v+2}), \ldots, (I_{k-1}, \ell_{k-1}), (I_k, \ell_k + 1),
            \end{align*}
            where $J = [\pi_i + 1, \max l_v]$.
        \item If $\lambda_i = 3$ (or equivalently, $\min l_v = \pi_i = \max l_v$), then
            \begin{align*}
                U_i(\pi) &= (I_1, \ell_1), \ldots, (I_{v-1}, \ell_{v-1}), (I_{v+1}, \ell_{v+1}), \ldots, (I_{k-1}, \ell_{k-1}), (I_k, \ell_k + 1).
            \end{align*}
    \end{enumerate}
\end{itemize}
- the intervals $I_1, I_2, \ldots, I_k$ are in decreasing order, that is $\max I_{j+1} < \min I_j$ for any $j, 1 \leq j < k$;
- the sequence $\ell_1, \ell_2, \ldots, \ell_k$ is increasing, that is $\ell_j < \ell_{j+1}$ for any $j, 1 \leq j < k$;
- $\{\ell_1, \ell_2, \ldots, \ell_k\} \subseteq [0, i]$, and $\ell_k = i$;
- $0 \in I_1$;
- $\bigcup_{j=1}^k I_j = \{\pi_{i+1}, \pi_{i+2}, \ldots, \pi_n\} \cup \{0\}$;
- the $(i + 1)$th entry of the Lehmer code of $\pi$ is given by the number of entries $\pi_j > \pi_{i+1}$, with $j < i + 1$, that is the cardinality of $[\pi_{i+1}, n] \setminus \bigcup_{j=1}^k I_j$.

A byproduct of Definition 1 is the construction of $b : S_n \to S_n$ defined below.

Definition 2. Let $\pi = \pi_1 \pi_2 \ldots \pi_n \in S_n$. For each $i, 1 \leq i \leq n$, we define $b_i = \ell_v$, where $v$ is such that $(l_v, \ell_v)$ is a labeled interval in the $(i − 1)$th slice of $\pi$ with $\pi_1 \in I_v$, and we denote by $b(\pi)$ the sequence $b_1 b_2 \ldots b_n$.

From Remark 1 it follows that $b(\pi)$ is a subexcedant sequence, see for instance Example 2 and Fig. 2.

Proposition 1. Let $\pi = \pi_1 \pi_2 \ldots \pi_n \in S_n$, $b(\pi) = b_1 b_2 \ldots b_n$, and let an $i, 1 \leq i \leq n$.

1. $i$ is a descent in $\pi$ iff $i$ is an ascent in $b(\pi)$;
2. $i$ is an ides in $\pi$ iff $b_i$ does not occur in $b_{i+1} b_{i+2} \ldots b_n$;
3. $i$ is a left-to-right maximum in $\pi$ iff $b_i = 0$;
4. $i$ is a left-to-right minimum in $\pi$ iff $b_i = i - 1$;
5. $i$ is a right-to-left maximum in $\pi$ iff $i$ is right-to-left minimum in $b$.

Proof. Points 1 and 2 obviously follow from the definition of $b$.

Point 3. Let $j$ be such that $\pi_j = n$; $b_j = 0$ iff $i \leq j$ and $\pi_i$ lies in the first interval of the $(i − 1)$th slice of $\pi$, which in turn is equivalent to $i$ is a left-to-right maximum in $\pi$.

Point 4. Similarly, let $j$ be such that $\pi_j = 1$; $b_j = i - 1$ iff $i \leq j$ and $\pi_i$ lies in the last interval of the $(i − 1)$th slice of $\pi$, which in turn is equivalent to $i$ is a left-to-right minimum in $\pi$.

Point 5. By the construction of $b$, $i$ is a right-to-left maximum in $\pi$ iff $\pi_i$ is the largest element of the first interval of the $(i − 1)$th slice of $\pi$, which in turn is equivalent to $b_i$ is smaller than any of $b_{i+1}, b_{i+2}, \ldots, b_n$. □

See for instance Example 1, where $s = b(\pi)$.

For a length-$n$ subexcedant sequence $b = b_1 b_2 \ldots b_n$, let us consider the following properties that a position $i, 1 \leq i \leq n$, can satisfy:

R1: $b_i$ occurs in the suffix $b_{i+1} b_{i+2} \ldots b_n$ of $b$.
R2: $i - 1$ occurs in $b$.

The next proposition shows that each $\lambda_i(\pi)$ can be obtained solely from $b(\pi)$.

Proposition 2. Let $\pi \in S_n$ and $b(\pi) = b_1 b_2 \ldots b_n$. Then, for any $i, 1 \leq i \leq n$, we have

$$
\lambda_i(\pi) = \begin{cases} 
0, & \text{if } i \text{ satisfies both R1 and R2}, \\
1, & \text{if } i \text{ satisfies R2 but not R1}, \\
2, & \text{if } i \text{ satisfies R1 but not R2}, \\
3, & \text{if } i \text{ satisfies neither R1 nor R2}.
\end{cases}
$$
Proof. By the construction given in Definition 2 for $b(\pi)$ from the slices of $\pi$, it follows that the position $i$ in $\pi$ satisfies property P1 (resp. P2) if and only if the position $i$ in $b(\pi)$ satisfies property R1 (resp. R2), and the statement holds. □

Proposition 3. Let $\pi, \sigma \in \mathcal{S}_n$ with $b(\pi) = b(\sigma)$. Then

1. $\lambda_i(\pi) = \lambda_i(\sigma)$ for any $i$, $1 \leq i \leq n$.
2. If $(l_1, \xi_1), (l_2, \xi_2), \ldots, (l_k, \xi_k)$ is the $i$th slice of $\pi$, and $(f_1, m_1), (f_2, m_2), \ldots, (f_p, m_p)$ that of $\sigma$, for some $i$, $1 \leq i < n$, then $k = p$ and $\xi_j = m_j$, for $1 \leq j \leq k$.

Proof. The first point is a consequence of Proposition 2.

The second point follows by the next considerations. The $ith$ slice of $\pi$, $i \leq 1 < n$, has the same number of intervals as its $(i - 1)th$ slice, except in two cases: $\lambda_i(\pi) = 0$ (when an interval is split into two intervals); and when $\lambda_i(\pi) = 3$ (when a one-element interval is removed). The result follows by considering the first point and by induction on $i$. □

The sequence $b(\pi) = b_1 b_2 \ldots b_n \in S_n$ was defined by means of the slices of $\pi$, but in proving the bijectivity of $b$ we need rather the complement of these slices. Let $\pi \in \mathcal{S}_n$ and $U_i(\pi) = (l_i, \xi_i), (l_2, \xi_2), \ldots, (l_k, \xi_k)$ be the $ith$ slice of $\pi$ for an $i$, $1 \leq i < n$. The $ith$ profile of $\pi$ is the sequence $X_1, X_2, \ldots, X_p$ of decreasing nonempty maximal intervals (that is, $\max X_{i+1} < \min X_i$, and none of them has the form $X_i \cup X_{i+1}$ with $\bigcup_{j=1}^p X_j = \{1, 2, \ldots, n\} \setminus \bigcup_{j=1}^p I_j$. And, clearly, $\bigcup_{j=1}^p X_j$ is the set of entries in $\pi$ to the left of $\pi_{i+1}$, and $\sum_{j=1}^p \text{card } X_j = i$.

Example 3. The vertical gray regions on the right side of Example 2 correspond to the profiles of $\pi = 62587314$ in Fig. 2. These profiles are $[6, 6, 2, 2]; [5, 6, 2, 2]; [8, 8, 5, 6, 2, 2]; [5, 8, 2, 2]; [5, 8, 2, 3]; [5, 8, 1, 3].$

In the proof of Theorem 1, we need the next result.

Proposition 4. Let $\pi, \sigma \in \mathcal{S}_n$ with $b(\pi) = b(\sigma)$, and let an $i$, $1 \leq i < n$. If $X_1, X_2, \ldots, X_p$ and $Y_1, Y_2, \ldots, Y_m$ are the $ith$ profiles of $\pi$ and of $\sigma$, then

- $n \in X_i$ if and only if $n \in Y_i$,
- $p = m$, and
- $\text{card } X_j = \text{card } Y_j$ for any $j$, $1 \leq j \leq p$.

Proof. It is easy to see that $n \in X_i$ if $0$ does not appear in $b_{i+1}(\pi) \ldots b_n(\pi) = b_{i+1}(\sigma) \ldots b_n(\sigma)$, that is, iff $n \in Y_i$. And, if $i = 1$, then the first profile of $\pi$ and $\sigma$ are one-element intervals, and the statement holds.

From the first point of Proposition 3, we have $\lambda_i(\pi) = \lambda_i(\sigma)$. Suppose that the statement is true for $i - 1$, and we will prove it for $i$.

In passing from the $(i - 1)th$ profiles of $\pi$ and of $\sigma$ to their $ith$ profiles, the following cases can occur (we refer the reader to Definition 1 and Fig. 1).

- If $\lambda_i(\pi) = \lambda_i(\sigma) = 0$, or $\lambda_i(\pi) = \lambda_i(\sigma) = 1$ and $b_i(\pi) = b_i(\sigma) = 0$, then a new one-element interval is added to the $ith$ profile of $\pi$ and of $\sigma$. Moreover, since $b(\pi) = b(\sigma)$, by the second point of Proposition 3, it follows that these intervals are both, for some $k$, the $k$th intervals in the $ith$ profile of $\pi$ and $\sigma$.
- If $\lambda_i(\pi) = \lambda_i(\sigma) = 1$ and $b_i(\pi) = b_i(\sigma) \neq 0$, then for some $k$, a new element is added to the $k$th interval of both $ith$ profiles of $\pi$ and $\sigma$; this element is the smallest in the obtained intervals.
- If $\lambda_i(\pi) = \lambda_i(\sigma) = 3$ and $b_i(\pi) = b_i(\sigma) = 0$, then for some $k$, a new element is added to the $k$th interval of both $ith$ profiles of $\pi$ and $\sigma$; this element is the largest in the obtained intervals.
- If $\lambda_i(\pi) = \lambda_i(\sigma) = 3$ and $b_i(\pi) = b_i(\sigma) \neq 0$, then two consecutive intervals are merged in the $ith$ profiles of $\pi$ and of $\sigma$: the $k$th and $(k + 1)$th ones, for some $k$. □

Now, we explain how the Lehmer code $c_1 c_2 \ldots c_n$ is linked to the profiles of a permutation. By definition, $c_1 = 0$ and $c_i, i > 1$, is the number of entries in $\pi$ at the left of $\pi_i$ and larger than $\pi_i$. If $X_1, X_2, \ldots, X_p$ is the $(i - 1)th$ profile of $\pi$, it follows that $c_i = \sum_{j=1}^p \text{card } X_j$, where $u$ such that $\bigcup_{j=1}^p X_j$ is the set of entries in $\pi$ at the left of $\pi_i$ and larger than $\pi_i$, and so $c_i = \sum_{j=1}^p \text{card } X_j$.

Theorem 1. The mapping $b: \mathcal{S}_n \rightarrow S_n$ is a bijection.

Proof. Let $\pi, \sigma \in \mathcal{S}_n$ with $b(\pi) = b(\sigma)$, and $c_1 c_2 \ldots c_n$ and $d_1 d_2 \ldots d_n$ be the Lehmer codes of $\pi$ and $\sigma$. Let also $i$ be an integer, $1 < i \leq n$, and $(l_1, \xi_1), (l_2, \xi_2), \ldots, (l_k, \xi_k)$ be the $(i - 1)th$ slice of $\pi$, and $u$ such that $\pi_i \in l_r$ (see Definition 2). If $X_1, X_2, \ldots, X_p$ is the $(i - 1)th$ profile of $\pi$, then

- if $n \in X_i$, it follows that $c_i = \sum_{j=1}^p \text{card } X_j$, and
- if $n \notin X_i$, it follows that $c_i = \sum_{j=1}^{p-1} \text{card } X_j$.

Since $b(\pi) = b(\sigma)$, combining Proposition 4 and the second point of Proposition 3, we have that $c_i = d_i$. It follows that the Lehmer code of $\pi$ and of $\sigma$ are equal, and so are $\pi$ and $\sigma$, and thus $b$ is injective. And, by cardinality reasons, it follows that $b$ is bijective. □
It is straightforward to see that the 4-tuple of statistics \((\text{Des}, \text{Lrmax}, \text{Lrmin}, \text{Rlmax})\) on \(S_n\) has the same distribution as \((\text{Asc}, \text{Posz}, \text{Max}, \text{Rlmin})\) on \(S_n\). Indeed, for the Lehmer code \(L(\pi)\) of a permutation \(\pi\), we have \((\text{Des}, \text{Lrmax}, \text{Lrmin}, \text{Rlmax})\) \(\pi = (\text{Asc}, \text{Posz}, \text{Max}, \text{Rlmin}) \ L(\pi)\), see Property 1. But, generally, \(\text{Ides} \ \pi\) is different from \(\text{Row} \ L(\pi)\). For example, if \(\pi = 62587314\), then \(L(\pi) = 01101464\), \(\text{Ides} \ \pi = \{3, 5, 7, 8\}\) and \(\text{Row} \ L(\pi) = \{5, 7, 8\}\).

Combining Theorem 1 and Proposition 1, it follows that \(b\) behaves not only as the Lehmer code for the above 4-tuples of statistics, but also it transforms \(\text{Ides} \ \pi\) into Row \(b(\pi)\). Formally, we have the next theorem, which subsequently gives Row as a set-valued partner for Asc, thereby answering to an open question stated in [1].

**Theorem 2.** For any \(\pi \in S_n\),

\[
(\text{Des}, \text{Ides}, \text{Lrmax}, \text{Lrmin}, \text{Rlmax}) \ \pi = (\text{Asc}, \text{Row}, \text{Posz}, \text{Max}, \text{Rlmin}) \ b(\pi),
\]

and so the multistatistic \((\text{Des}, \text{Ides}, \text{Lrmax}, \text{Lrmin}, \text{Rlmax})\) on \(S_n\) has the same distribution as \((\text{Asc}, \text{Row}, \text{Posz}, \text{Max}, \text{Rlmin})\) on \(S_n\).

The next corollaries are consequences of Theorem 2. The first of them is Foata’s result in [4] saying that \((\text{asc}, \text{row})\) on subexcedant sequences is a double Eulerian bistatistic.

**Corollary 1.** The bistatistics \((\text{asc}, \text{row})\) on the set of subexcedant sequences has the same distribution as \((\text{des}, \text{ides})\) on the set of permutations.

**Corollary 2.** The bistatistics \((\text{asc}, \text{row})\) and \((\text{row}, \text{asc})\) are equidistributed on the set of subexcedant sequences.

**Proof.** Let \(s \in S_n\) and let us define \(t = b(\sigma)\) where \(\sigma = \pi^{-1}\) with \(\pi = b^{-1}(s)\). It is clear that \((\text{asc}, \text{row})\) \(s = (\text{des}, \text{ides})\) \(\pi = (\text{ides}, \text{des}) \ \sigma = (\text{row}, \text{asc}) \ t\). \(\Box\)

**Final remarks:** Recently, Lin and Kim [6] defined a bijection \(\Psi\) between two combinatorial classes counted by the large Schröder numbers, namely the set \(S_n(021)\) of subexcedant sequences avoiding the pattern 021, and the set \(S_n(2413, 4213)\) of permutations avoiding the patterns 2413 and 4213. In [6], it is shown that \(\Psi\) proves the equidistribution of two 6-tuples of set-valued statistics, that is

\[
(\text{Asc}, \text{Row}, \text{Posz}, \text{Max}, \text{Rlmin}, \text{Expo}) \ s = (\text{Des}, \text{Ides}, \text{Lrmax}, \text{Lrmin}, \text{Rlmin}) \ \Psi \ (s),
\]

for any \(s \in S_n(021)\). Here, \(\text{Expo}\) is the exposed positions statistic defined in [6], and if the last statistic of both 6-tuples is dropped out, then the 5-tuples in our Theorem 2 are obtained. Also in [6] is pointed out that the restriction of our mapping \(b\) yields a bijection between \(S_n(2413, 4213)\) and \(S_n(021)\), but unlike \(\Psi^{-1}\), this restriction of \(b\) does not transform \(\text{Rlmin}\) to \(\text{Expo}\).

**References**