Discrete Applied Mathematics 224 (2017) 9–15

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

A permutation code preserving a double Eulerian bistatistic

Jean-Luc Baril, Vincent Vajnovszki *

LE2I, Université de Bourgogne Franche-Comté, BP 47870, 21078 Dijon Cedex, France

ARTICLE INFO

Article history:
Received 19 October 2016
Received in revised form 10 February 2017
Accepted 16 February 2017
Available online 22 March 2017

Keywords:
Set-valued/Eulerian statistic on permutation
Permutation/Lehmer code
Subexcedant sequence
Constructive bijection

ABSTRACT

In 1977 Foata proved bijectively, among other things, that the joint distribution of ascent and distinct nonzero value numbers on the set of subexcedant sequences is the same as that of descent and inverse descent numbers on the set of permutations, and the generating function of the corresponding bistatistics is the double Eulerian polynomial. In 2013 Foata's result was rediscovered by Visontai as a conjecture, and then reproved by Aas in 2014.

In this paper, we define a permutation code (that is, a bijection between permutations and subexcedant sequences) and show the more general result that two 5-tuples of set-valued statistics on the set of permutations and on the set of subexcedant sequences, respectively, are equidistributed. In particular, these results give another bijective proof of Foata's result.

1. Introduction

In enumerative combinatorics, it is a classical result that the descent number des and the inverse descent number ides (defined as $\mathsf{ides}\,\pi = \mathsf{des}\,\pi^{-1}$) on permutations are Eulerian statistics, and their distributions on the set $\mathfrak{S}_n$ of length-n permutations are given by the nth Eulerian polynomial A_n, that is

$$A_n(u) = \sum_{\pi\in\mathfrak{S}_n} u^{\mathsf{des}\,\pi+1} = \sum_{\pi\in\mathfrak{S}_n} u^{\mathsf{ides}\,\pi+1},$$

and the joint distribution of des and ides is given by the nth double Eulerian polynomial,

$$A_n(u,v) = \sum_{\pi\in\mathfrak{S}_n} u^{\mathsf{des}\,\pi+1} v^{\mathsf{ides}\,\pi+1},$$

see for instance [2,8].

An alternative way to represent a permutation is its Lehmer code [5], which is a subexcedant sequence. The ascent number asc on the set S_n of subexcedant sequences is still an Eulerian statistic (see for example [9]), and in [7] the statistic that counts the number of distinct nonzero symbols in a subexcedant sequence (that following [1] we denote by row) is proved to be still Eulerian; this result is credited to Dumont by the authors of [7]. In terms of generating functions, we have

$$A_n(u) = \sum_{s\in S_n} u^{\mathsf{asc}\,s+1} = \sum_{s\in S_n} u^{\mathsf{row}\,s+1}.$$

* Corresponding author.
E-mail addresses: barjl@u-bourgogne.fr (J.-L. Baril), vvajnov@u-bourgogne.fr (V. Vajnovszki).

http://dx.doi.org/10.1016/j.dam.2017.02.014
0166-218X/

In 1977 Foata [4] proved, among other things, that the joint distribution of des and ides on the set of permutations is the same as that of asc and row on the set of subexcedant sequences, that is

$$A_n(u,v) = \sum_{\pi\in\mathfrak{S}_n} u^{\mathrm{des}\,\pi+1} v^{\mathrm{ides}\,\pi+1} = \sum_{s\in S_n} u^{\mathrm{asc}\,s+1} v^{\mathrm{row}\,s+1}.$$

In the present paper, we define a bijection between permutations and subexcedant sequences (i.e., a permutation code) and show that the tuple of set-valued statistics

(Des, Ides, Lrmax, Lrmin, Rlmax) on the set of permutations, and
(Asc, Row, Posz, Max, Rlmax) on the set of subexcedant sequences

have the same distribution (each of the occurring statistics is defined below). In particular, our bijection provides set-valued partners for Asc (answering to a question in [1]) and gives a alternative proof of Foata's result.

2. Notation and definitions

A length-n *word* w over the alphabet A is a sequence $w_1w_2\dots w_n$ of symbols in A, and we will consider only finite alphabets $A\subset\mathbb{N}$.

Statistics
A *statistic* on a set X of words is simply a function from X to $\mathbb{N}$; a *set-valued statistic* is a function from X to $2^{\mathbb{N}}$; and a *multistatistic* is a tuple of statistics.

Let $w = w_1w_2\dots w_n$ be a length-n word. A *descent* in w is a position i in w, $1\le i<n$, with $w_i > w_{i+1}$, and the *descent set* of w is

$$\mathrm{Des}\,w = \{i : 1\le i<n \text{ with } w_i > w_{i+1}\}.$$

A *left-to-right maximum* in w is a position i in w, $1\le i\le n$, with $w_j < w_i$ for all $j<i$, and the *set of left-to-right maxima* is

$$\mathrm{Lrmax}\,w = \{i : 1\le i\le n \text{ with } w_j < w_i \text{ for all } j<i\}.$$

Clearly, $1\in\mathrm{Lrmax}\,w$, and Des and Lrmax are classical examples of set-valued statistics on words. We define similarly the sets Asc w of *ascents*, Lrmin w of *left-to-right minima*, Rlmax w of *right-to-left maxima* and Rlmin w of *right-to-left minima* in w.

To each set-valued statistic St corresponds an (integer-valued) statistic st defined as $\mathrm{st}\,w = \mathrm{card}\,\mathrm{St}\,w$, for example des w and lrmax w count, respectively, the number of descents and the number of left-to-right maxima in w.

Let X and X' be two sets of words, and st and st$'$ be two statistics defined on X and X', respectively. We say that st on X has the *same distribution* as st$'$ on X' (or equivalently, st and st$'$ are *equidistributed*) if, for any integer u,

$$\mathrm{card}\{w\in X : \mathrm{st}\,w = u\} = \mathrm{card}\{w\in X' : \mathrm{st}'\,w = u\},$$

and the multistatistic $(\mathrm{st}_1,\mathrm{st}_2,\dots,\mathrm{st}_p)$ defined on X has the same distribution as the multistatistic $(\mathrm{st}'_1,\mathrm{st}'_2,\dots,\mathrm{st}'_p)$ defined on X' (or the two multistatistics are equidistributed) if, for any integer p-tuple $u=(u_1,u_2,\dots,u_p)$,

$$\mathrm{card}\{w\in X : (\mathrm{st}_1,\mathrm{st}_2,\dots,\mathrm{st}_p)\,w = u\} = \mathrm{card}\{w\in X' : (\mathrm{st}'_1,\mathrm{st}'_2,\dots,\mathrm{st}'_p)\,w = u\}.$$

The notion of equidistribution of (multi)statistics can naturally be extended to set-valued (multi)statistics.

Permutations, subexcedant sequences and codes
This paper deals with two particular classes of words: permutations and subexcedant sequences. A permutation is a length-n word over $\{1,2,\dots,n\}$ with distinct symbols. Alternatively, a permutation is an element of the symmetric group on $\{1,2,\dots,n\}$ written in one line notation, and $\mathfrak{S}_n$ denotes the set of length-n permutations. If two permutations $\pi = \pi_1\pi_2\dots\pi_n$ and $\sigma = \sigma_1\sigma_2\dots\sigma_n$ are such that $\sigma_{\pi_1}\sigma_{\pi_2}\cdots\sigma_{\pi_n} = 12\cdots n$ (i.e., the identity in $\mathfrak{S}_n$), then σ is the *inverse* of π, which is denoted by π^{-1}.

A length-n *subexcedant sequence*[1] is a word $s = s_1s_2\dots s_n$ over $\{0,1,\dots,n-1\}$ with $0\le s_i\le i-1$ for $1\le i\le n$, and S_n denotes the set of length-n subexcedant sequences; and we have $S_n = \{0\}\times\{0,1\}\times\cdots\times\{0,1,\dots,n-1\}$.

Some statistics are consistently defined only on particular classes of words, *e.g.* permutations or subexcedant sequences.

For a permutation $\pi\in\mathfrak{S}_n$, an *inverse descent* (*ides* for short) in π is a position i for which π_i+1 appears to the left of π_i in π. Equivalently, i is an ides in π if π_i is a descent in π^{-1}. The *ides set* is defined as

$$\mathrm{Ides}\,\pi = \{i : 1<i\le n \text{ with } \pi_i+1 \text{ appears in } \pi \text{ to the left of } \pi_i\},$$

and $\mathrm{ides}\,\pi = \mathrm{des}\,\pi^{-1}$, but in general $\mathrm{Ides}\,\pi$ is not equal to $\mathrm{Des}\,\pi^{-1}$.

Let $s = s_1s_2\dots s_n$ be a subexcedant sequence in S_n. The Posz statistic gives the positions of 0s in s,

$$\mathrm{Posz}\,s = \{i : 1\le i\le n, s_i = 0\},$$

1 Known in the literature also as *inversion sequence*, *inversion table* or *subexceedant function*.

and obviously $1 \in \mathsf{Posz}\, s$. The Max statistic is defined as

$$\mathsf{Max}\, s = \{i : 1 \le i \le n, s_i = i - 1\},$$

and as above, $1 \in \mathsf{Max}\, s$.

A *last-value position* in s is a position i in s such that $s_i \neq 0$ and s_i does not occur in the suffix $s_{i+1}s_{i+2} \dots s_n$ of s. The *last-value position set*, denoted by Row, is defined as

$$\mathsf{Row}\, s = \{i : s_i \neq 0 \text{ and } s_i \text{ does not occur in the suffix } s_{i+1}s_{i+2} \dots s_n\}.$$

Clearly, $1 \notin \mathsf{Row}\, s$, and $\mathsf{row}\, s = \mathrm{card}\, \mathsf{Row}\, s$ counts the number of distinct nonzero symbols in s.

Example 1. If $\pi = 6\,2\,5\,8\,7\,3\,1\,4 \in \mathfrak{S}_8$ and $s = 0\,1\,1\,0\,2\,3\,6\,3 \in S_8$, then

$$\begin{aligned}
\mathsf{Des}\, \pi &= \mathsf{Asc}\, s = \{1, 4, 5, 6\},\\
\mathsf{Ides}\, \pi &= \mathsf{Row}\, s = \{3, 5, 7, 8\},\\
\mathsf{Lrmax}\, \pi &= \mathsf{Posz}\, s = \{1, 4\},\\
\mathsf{Lrmin}\, \pi &= \mathsf{Max}\, s = \{1, 2, 7\},\\
\mathsf{Rlmax}\, \pi &= \mathsf{Rlmin}\, s = \{4, 5, 8\}.
\end{aligned}$$

An *inversion* in a permutation $\pi = \pi_1\pi_2 \dots \pi_n \in \mathfrak{S}_n$ is a pair (i, j) with $i < j$ and $\pi_i > \pi_j$. The set $\mathfrak{S}_n$ is in bijection with S_n, and any such bijection is called *permutation code*. The *Lehmer code L* defined in [5] is a classical example of permutation code; it maps each permutation $\pi = \pi_1\pi_2 \dots \pi_n$ to a subexcedant sequence $s_1s_2 \dots s_n$ where, for all j, $1 \le j \le n$, s_j is the number of inversions (i, j) in π (or equivalently, the number of entries in π larger than π_j and on its left). For example $L(6\,2\,5\,8\,7\,3\,1\,4) = 0\,1\,1\,0\,1\,4\,6\,4$. See also [11] for a family of permutation codes in the context of Mahonian statistics on permutations.

In [3] is showed that dmc statistic which counts the number of distinct nonzero symbols in the Lehmer code of a permutation π (the statistic $\pi \mapsto \mathsf{row}\, L(\pi)$ with the above notations) is Eulerian, and so has the same distribution as des, asc or ides on $\mathfrak{S}_n$. See also [10] where Dumont's statistic dmc is extended to words.

Although the following properties are folklore, they are easy to check.

Property 1. *If $\pi \in \mathfrak{S}_n$ and $L(\pi) \in S_n$ is its Lehmer code, then* $\mathsf{Des}\, \pi = \mathsf{Asc}\, L(\pi)$, $\mathsf{Lrmax}\, \pi = \mathsf{Posz}\, L(\pi)$, $\mathsf{Lrmin}\, \pi = \mathsf{Max}\, L(\pi)$, *and* $\mathsf{Rlmax}\, \pi = \mathsf{Rlmin}\, L(\pi)$.

3. The permutation code *b*

We define a mapping $b: \mathfrak{S}_n \to S_n$ and Theorem 1 shows that b is a bijection, that is, a permutation code, and it is the main tool in proving that (Des, Ides, Lrmax, Lrmin, Rlmax) on $\mathfrak{S}_n$ has the same distribution as (Asc, Row, Posz, Max, Rlmax) on S_n (see Theorem 2).

A position i in $\pi = \pi_1\pi_2 \dots \pi_n \in \mathfrak{S}_n$, $1 \le i \le n$, can satisfy the following properties:

P1: $\pi_i + 1$ occurs in π at the right of π_i,
P2: $\pi_i - 1$ occurs in $\pi 0$ at the right of π_i,

where $\pi 0$ is the permutation of $\{0, 1, \dots, n\}$ obtained by adding a 0 at the end of π. And, to each position i in π, we associate an integer $\lambda_i(\pi) \in \{0, 1, 2, 3\}$ according to i satisfies both, one, or none of these properties:

$$\lambda_i(\pi) = \begin{cases} 0, & \text{if } i \text{ satisfies both P1 and P2},\\ 1, & \text{if } i \text{ satisfies P2 but not P1},\\ 2, & \text{if } i \text{ satisfies P1 but not P2},\\ 3, & \text{if } i \text{ satisfies neither P1 nor P2},\end{cases}$$

and we denote it simply by λ_i when there is no ambiguity.

Alternatively, using the Iverson bracket notation ($[P] = 1$ iff the statement P is true), we have the more concise expression: $\lambda_i = [\pi_i = n \text{ or } \pi_i + 1 \text{ occurs at the left of } \pi_i] + 2 \cdot [\pi_i - 1 \text{ occurs at the left of } \pi_i]$.

For example, for any $\pi \in \mathfrak{S}_n$ we have $\lambda_1 = 0$ except $\lambda_1 = 1$ if $\pi_1 = n$; and when $n > 1$, then $\lambda_n = 3$ except $\lambda_n = 2$ if $\pi_n = 1$. Each λ_i is uniquely determined by π, for instance if $\pi = 6\,2\,5\,8\,7\,3\,1\,4$, then $\lambda_1, \lambda_2, \dots, \lambda_8 = 0, 0, 1, 1, 3, 2, 1, 3$, see Fig. 2.

An *interval* $I = [a, b]$, $a \le b$, is the set of integers $\{x : a \le x \le b\}$; and a *labeled interval* is a pair (I, ℓ) where I is an interval and ℓ is an integer. In order to give the construction of the mapping b, we define below the slices of a permutation, and some of their properties are given in Remark 1.

Definition 1. For a permutation $\pi = \pi_1\pi_2 \dots \pi_n \in \mathfrak{S}_n$ and an i, $0 \le i < n$, the ith *slice* of π is the sequence of labeled intervals $U_i(\pi) = (I_1, \ell_1), (I_2, \ell_2), \dots, (I_k, \ell_k)$, defined by the following process (see Fig. 1).

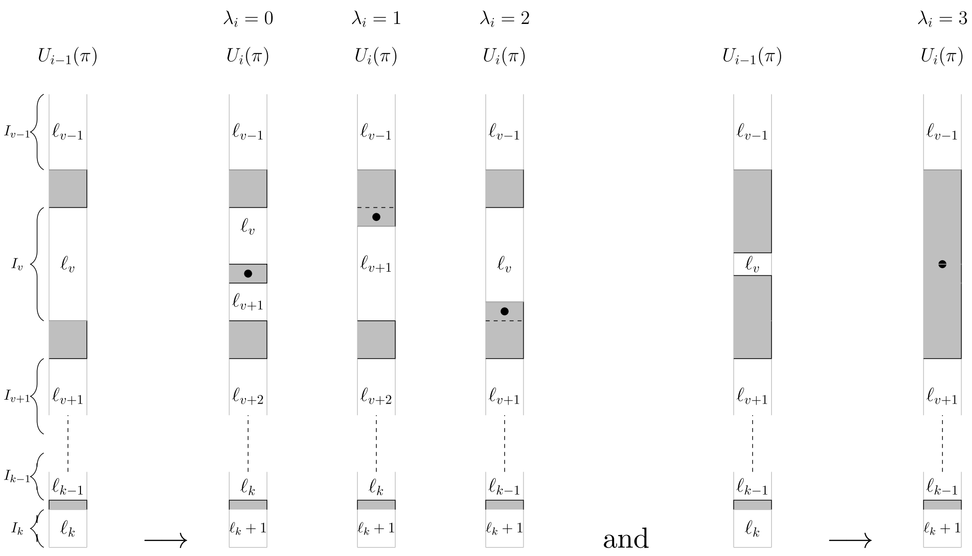
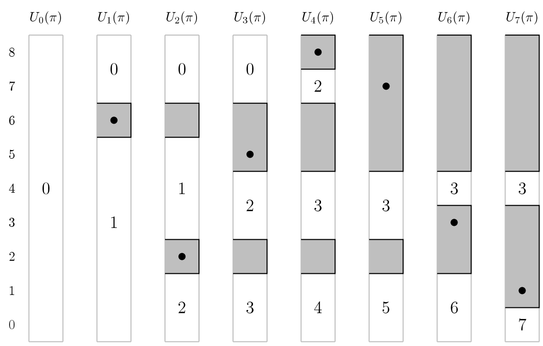


Fig. 1. The four cases in Definition 1.

- $U_0(\pi) = ([0, n], 0)$.
- For $i \geq 1$, let $U_{i-1}(\pi) = (I_1, \ell_1), (I_2, \ell_2), \ldots, (I_k, \ell_k)$ be the $(i-1)$th slice of π and v, $1 \leq v \leq k$, be the integer such that $\pi_i \in I_v$. The ith slice $U_i(\pi)$ of π is defined according to λ_i:
  - If $\lambda_i = 0$ (or equivalently, $\min I_v < \pi_i < \max I_v$), then
  $$U_i(\pi) = (I_1, \ell_1), \ldots, (I_{v-1}, \ell_{v-1}), (H, \ell_v), (J, \ell_{v+1}), (I_{v+1}, \ell_{v+2}), \ldots, (I_{k-1}, \ell_k), (I_k, \ell_k + 1),$$
  where $H = [\pi_i + 1, \max I_v]$ and $J = [\min I_v, \pi_i - 1]$;
  - If $\lambda_i = 1$ (or equivalently, $\min I_v < \max I_v = \pi_i$), then
  $$U_i(\pi) = (I_1, \ell_1), \ldots, (I_{v-1}, \ell_{v-1}), (J, \ell_{v+1}), (I_{v+1}, \ell_{v+2}), \ldots, (I_{k-1}, \ell_k), (I_k, \ell_k + 1),$$
  where $J = [\min I_v, \pi_i - 1]$;
  - If $\lambda_i = 2$ (or equivalently, $\min I_v = \pi_i < \max I_v$), then
  $$U_i(\pi) = (I_1, \ell_1), \ldots, (I_{v-1}, \ell_{v-1}), (J, \ell_v), (I_{v+1}, \ell_{v+1}), \ldots, (I_{k-1}, \ell_{k-1}), (I_k, \ell_k + 1),$$
  where $J = [\pi_i + 1, \max I_v]$;
  - If $\lambda_i = 3$ (or equivalently, $\min I_v = \max I_v = \pi_i$), then
  $$U_i(\pi) = (I_1, \ell_1), \ldots, (I_{v-1}, \ell_{v-1}), (I_{v+1}, \ell_{v+1}), \ldots, (I_{k-1}, \ell_{k-1}), (I_k, \ell_k + 1).$$

Example 2. For the permutation $\pi = 6\,2\,5\,8\,7\,3\,1\,4$ in Fig. 2, $\lambda_1(\pi), \lambda_2(\pi), \ldots, \lambda_8(\pi) = 0, 0, 1, 1, 3, 2, 1, 3$, and the process described in Definition 1 gives the slices below.

$$U_0(\pi) = ([0, 8], 0);$$
$$U_1(\pi) = ([7, 8], 0), ([0, 5], 1);$$
$$U_2(\pi) = ([7, 8], 0), ([3, 5], 1), ([0, 1], 2);$$
$$U_3(\pi) = ([7, 8], 0), ([3, 4], 2), ([0, 1], 3);$$
$$U_4(\pi) = ([7, 7], 2), ([3, 4], 3), ([0, 1], 4);$$
$$U_5(\pi) = ([3, 4], 3), ([0, 1], 5);$$
$$U_6(\pi) = ([4, 4], 3), ([0, 1], 6);$$
$$U_7(\pi) = ([4, 4], 3), ([0, 0], 7).$$



Remark 1. Let $U_i(\pi) = (I_1, \ell_1), (I_2, \ell_2), \ldots, (I_k, \ell_k)$ be the ith slice of π, $0 \leq i < n$. Then, the following properties can be easily checked:

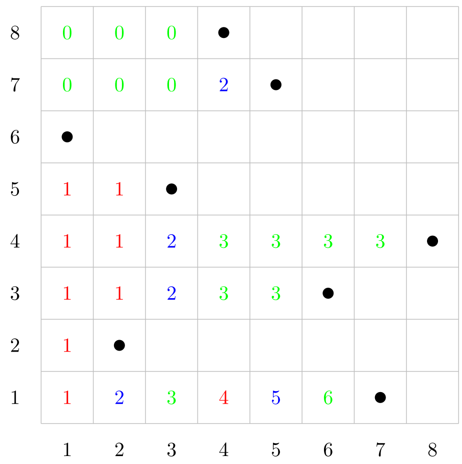


Fig. 2. The permutation $\pi = 6\,2\,5\,8\,7\,3\,1\,4$ with $b(\pi) = 0\,1\,1\,0\,2\,3\,6\,3$ and $\lambda_1(\pi), \lambda_2(\pi), \ldots, \lambda_8(\pi) = 0, 0, 1, 1, 3, 2, 1, 3$.

- the intervals $I_1, I_2, \ldots, I_k$ are in decreasing order, that is $\max I_{j+1} < \min I_j$ for any j, $1 \le j < k$;
- the sequence $\ell_1, \ell_2, \ldots, \ell_k$ is increasing, that is $\ell_j < \ell_{j+1}$ for any j, $1 \le j < k$;
- $\{\ell_1, \ell_2, \ldots, \ell_k\} \subseteq [0, i]$, and $\ell_k = i$;
- $0 \in I_k$;
- $\cup_{j=1}^{k} I_j = \{\pi_{i+1}, \pi_{i+2}, \ldots, \pi_n\} \cup \{0\}$;
- the $(i+1)$th entry of the Lehmer code of π is given by the number of entries $\pi_j > \pi_{i+1}$, with $j < i+1$, that is the cardinality of $[\pi_{i+1}, n] \setminus \cup_{j=1}^{k} I_j$.

A byproduct of Definition 1 is the construction of $b\colon \mathfrak{S}_n \to S_n$ defined below.

Definition 2. Let $\pi = \pi_1\pi_2 \ldots \pi_n \in \mathfrak{S}_n$. For each i, $1 \le i \le n$, we define $b_i = \ell_v$, where v is such that (I_v, ℓ_v) is a labeled interval in the $(i-1)$th slice of π with $\pi_i \in I_v$, and we denote by $b(\pi)$ the sequence $b_1 b_2 \ldots b_n$.

From Remark 1 it follows that $b(\pi)$ is a subexcedant sequence, see for instance Example 2 and Fig. 2.

Proposition 1. *Let $\pi = \pi_1\pi_2 \ldots \pi_n \in \mathfrak{S}_n$, $b(\pi) = b_1 b_2 \ldots b_n$, and let an i, $1 \le i \le n$.*

1. *i is a descent in π iff i is an ascent in $b(\pi)$;*
2. *i is an ides in π iff b_i does not occur in $b_{i+1} b_{i+2} \ldots b_n$;*
3. *i is a left-to-right maximum in π iff $b_i = 0$;*
4. *i is a left-to-right minimum in π iff $b_i = i - 1$;*
5. *i is a right-to-left maximum in π iff i is right-to-left minimum in b.*

Proof. Points 1 and 2 obviously follow from the definition of b.

Point 3. Let j be such that $\pi_j = n$; $b_i = 0$ iff $i \le j$ and π_i lies in the first interval of the $(i-1)$th slice of π, which in turn is equivalent to i is a left-to-right maximum in π.

Point 4. Similarly, let j be such that $\pi_j = 1$; $b_i = i - 1$ iff $i \le j$ and π_i lies in the last interval of the $(i-1)$th slice of π, which in turn is equivalent to i is a left-to-right minimum in π.

Point 5. By the construction of b, i is a right-to-left maximum in π iff π_i is the largest element of the first interval of the $(i-1)$th slice of π, which in turn is equivalent to b_i is smaller than any of $b_{i+1}, b_{i+2}, \ldots, b_n$. $\square$

See for instance Example 1, where $s = b(\pi)$.

For a length-n subexcedant sequence $b = b_1 b_2 \ldots b_n$, let us consider the following properties that a position i, $1 \le i \le n$, can satisfy:

R1: b_i occurs in the suffix $b_{i+1} b_{i+2} \ldots b_n$ of b,
R2: $i - 1$ occurs in b.

The next proposition shows that each $\lambda_i(\pi)$ can be obtained solely from $b(\pi)$.

Proposition 2. *Let $\pi \in \mathfrak{S}_n$ and $b(\pi) = b_1 b_2 \ldots b_n$. Then, for any i, $1 \le i \le n$, we have*

$$\lambda_i(\pi) = \begin{cases} 0, & \text{if } i \text{ satisfies both R1 and R2,} \\ 1, & \text{if } i \text{ satisfies R2 but not R1,} \\ 2, & \text{if } i \text{ satisfies R1 but not R2,} \\ 3, & \text{if } i \text{ satisfies neither R1 nor R2.} \end{cases}$$

Proof. By the construction given in Definition 2 for $b(\pi)$ from the slices of π, it follows that the position i in π satisfies property P1 (resp. P2) if and only if the position i in $b(\pi)$ satisfies property R1 (resp. R2), and the statement holds. □

Proposition 3. *Let $\pi, \sigma \in \mathfrak{S}_n$ with $b(\pi) = b(\sigma)$. Then*

1. *$\lambda_i(\pi) = \lambda_i(\sigma)$ for any i, $1 \le i \le n$.*
2. *If $(I_1, \ell_1), (I_2, \ell_2), \ldots, (I_k, \ell_k)$ is the ith slice of π, and $(J_1, m_1), (J_2, m_2), \ldots, (J_p, m_p)$ that of σ, for some i, $1 \le i < n$, then $k = p$ and $\ell_j = m_j$, for $1 \le j \le k$.*

Proof. The first point is a consequence of Proposition 2.

The second point follows by the next considerations. The ith slice of π, $i \le 1 < n$, has the same number of intervals as its $(i-1)$th slice, except in two cases: $\lambda_i(\pi) = 0$ (when an interval is split into two intervals); and when $\lambda_i(\pi) = 3$ (when a one-element interval is removed). The result follows by considering the first point and by induction on i. □

The sequence $b(\pi) = b_1 b_2 \ldots b_n \in S_n$ was defined by means of the slices of π, but in proving the bijectivity of b we need rather the complement of these slices. Let $\pi \in \mathfrak{S}_n$ and $U_i(\pi) = (I_1, \ell_1), (I_2, \ell_2), \ldots, (I_k, \ell_k)$ be the ith slice of π for an i, $1 \le i < n$. The ith *profile* of π is the sequence $X_1, X_2, \ldots, X_p$ of decreasing nonempty maximal intervals (that is, $\max X_{j+1} < \min X_j$, and none of them has the form $X_j \cup X_{j+1}$) with $\cup_{j=1}^{p} X_j = \{1, 2, \ldots, n\} \setminus \cup_{j=1}^{k} I_j$. And, clearly, $\cup_{j=1}^{p} X_j$ is the set of entries in π to the left of π_{i+1}, and $\sum_{j=1}^{p} \operatorname{card} X_j = i$.

Example 3. The vertical gray regions on the right side of Example 2 correspond to the profiles of $\pi = 6\,2\,5\,8\,7\,3\,1\,4$ in Fig. 2. These profiles are $[6, 6]$; $[6, 6], [2, 2]$; $[5, 6], [2, 2]$; $[8, 8], [5, 6], [2, 2]$; $[5, 8], [2, 2]$; $[5, 8], [2, 3]$; and $[5, 8], [1, 3]$.

In the proof of Theorem 1, we need the next result.

Proposition 4. *Let $\pi, \sigma \in \mathfrak{S}_n$ with $b(\pi) = b(\sigma)$, and let an i, $1 \le i < n$. If $X_1, X_2, \ldots, X_p$ and $Y_1, Y_2, \ldots, Y_m$ are the ith profiles of π and of σ, then*

- *$n \in X_1$ if and only if $n \in Y_1$,*
- *$p = m$, and*
- *$\operatorname{card} X_j = \operatorname{card} Y_j$ for any j, $1 \le j \le p$.*

Proof. It is easy to see that $n \in X_1$ iff 0 does not appear in $b_{i+1}(\pi) \ldots b_n(\pi) = b_{i+1}(\sigma) \ldots b_n(\sigma)$, that is, iff $n \in Y_1$. And, if $i = 1$, then the first profile of π and of σ are one-element intervals, and the statement holds.

From the first point of Proposition 3, we have $\lambda_i(\pi) = \lambda_i(\sigma)$. Suppose that the statement is true for $i - 1$, and we will prove it for i.

In passing from the $(i-1)$th profiles of π and of σ to their ith profiles, the following cases can occur (we refer the reader to Definition 1 and Fig. 1).

- If $\lambda_i(\pi) = \lambda_i(\sigma) = 0$, or $\lambda_i(\pi) = \lambda_i(\sigma) = 1$ and $b_i(\pi) = b_i(\sigma) = 0$, then a new one-element interval is added to the ith profile of π and of σ. Moreover, since $b(\pi) = b(\sigma)$, by the second point of Proposition 3, it follows that these intervals are both, for some k, the kth intervals in the ith profile of π and σ.
- If $\lambda_i(\pi) = \lambda_i(\sigma) = 1$ and $b_i(\pi) = b_i(\sigma) \neq 0$, then for some k, a new element is added to the kth interval of both ith profiles of π and σ; this element is the smallest one in the obtained intervals.
- If $\lambda_i(\pi) = \lambda_i(\sigma) = 2$, or $\lambda_i(\pi) = \lambda_i(\sigma) = 3$ and $b_i(\pi) = b_i(\sigma) = 0$, then for some k, a new element is added to the kth interval of both ith profiles of π and σ; this element is the largest one in the obtained intervals.
- If $\lambda_i(\pi) = \lambda_i(\sigma) = 3$ and $b_i(\pi) = b_i(\sigma) \neq 0$, then two consecutive intervals are merged in the ith profiles of π and of σ: the kth and $(k+1)$th ones, for some k. □

Now, we explain how the Lehmer code $c_1 c_2 \ldots c_n$ is linked to the profiles of a permutation. By definition, $c_1 = 0$ and c_i, $i > 1$, is the number of entries in π at the left of π_i and larger than π_i. If $X_1, X_2, \ldots, X_p$ is the $(i-1)$th profile of π, it follows that $c_i = \sum_{j=1}^{u} \operatorname{card} X_j$, where u is such that $\cup_{j=1}^{u} X_j$ is the set of entries in π at the left of π_i and larger than π_i, and so $c_i = \operatorname{card} \cup_{j=1}^{u} X_j$.

Theorem 1. *The mapping $b: \mathfrak{S}_n \to S_n$ is a bijection.*

Proof. Let $\pi, \sigma \in \mathfrak{S}_n$ with $b(\pi) = b(\sigma)$, and $c_1 c_2 \ldots c_n$ and $d_1 d_2 \ldots d_n$ be the Lehmer codes of π and σ. Let also i be an integer, $1 < i \le n$, and $(I_1, \ell_1), (I_2, \ell_2), \ldots, (I_k, \ell_k)$ be the $(i-1)$th slice of π, and v such that $\pi_i \in I_v$ (see Definition 2). If $X_1, X_2, \ldots, X_p$ is the $(i-1)$th profile of π, then

if $n \in X_1$, it follows that $c_i = \sum_{j=1}^{v} \operatorname{card} X_j$, and

if $n \notin X_1$, it follows that $c_i = \sum_{j=1}^{v-1} \operatorname{card} X_j$.

Since $b(\pi) = b(\sigma)$, combining Proposition 4 and the second point of Proposition 3, we have that $c_i = d_i$. It follows that the Lehmer code of π and of σ are equal, and so are π and σ, and thus b is injective. And, by cardinality reasons, it follows that b is bijective. □

It is straightforward to see that the 4-tuple of statistics (Des, Lrmax, Lrmin, Rlmax) on $\mathfrak{S}_n$ has the same distribution as (Asc, Posz, Max, Rlmin) on S_n. Indeed, for the Lehmer code $L(\pi)$ of a permutation π, we have (Des, Lrmax, Lrmin, Rlmax) π = (Asc, Posz, Max, Rlmin) $L(\pi)$, see Property 1. But, generally, Ides π is different from Row $L(\pi)$. For example, if $\pi = 6\,2\,5\,8\,7\,3\,1\,4$, then $L(\pi) = 0\,1\,1\,0\,1\,4\,6\,4$, Ides $\pi = \{3, 5, 7, 8\}$ and Row $L(\pi) = \{5, 7, 8\}$.

Combining Theorem 1 and Proposition 1, it follows that b behaves not only as the Lehmer code for the above 4-tuples of statistics, but also it transforms Ides π into Row $b(\pi)$. Formally, we have the next theorem, which subsequently gives Row as a set-valued partner for Asc, thereby answering to an open question stated in [1].

Theorem 2. *For any* $\pi \in \mathfrak{S}_n$,

$$(\mathsf{Des}, \mathsf{Ides}, \mathsf{Lrmax}, \mathsf{Lrmin}, \mathsf{Rlmax})\,\pi = (\mathsf{Asc}, \mathsf{Row}, \mathsf{Posz}, \mathsf{Max}, \mathsf{Rlmin})\,b(\pi),$$

and so the multistatistic (Des, Ides, Lrmax, Lrmin, Rlmax) *on* $\mathfrak{S}_n$ *has the same distribution as* (Asc, Row, Posz, Max, Rlmin) *on* S_n.

The next corollaries are consequences of Theorem 2. The first of them is Foata's result in [4] saying that (asc, row) on subexcedant sequences is a double Eulerian bistatistic.

Corollary 1. *The bistatistics* (asc, row) *on the set of subexcedant sequences has the same distribution as* (des, ides) *on the set of permutations.*

Corollary 2. *The bistatistics* (asc, row) *and* (row, asc) *are equidistributed on the set of subexcedant sequences.*

Proof. Let $s \in S_n$ and let us define $t = b(\sigma)$ where $\sigma = \pi^{-1}$ with $\pi = b^{-1}(s)$. It is clear that (asc, row) s = (des, ides) π = (ides, des) σ = (row, asc) t. □

Final remarks: Recently, Lin and Kim [6] defined a bijection Ψ between two combinatorial classes counted by the large Schröder numbers, namely the set $S_n(021)$ of subexcedant sequences avoiding the pattern 021, and the set $\mathfrak{S}_n(2413, 4213)$ of permutations avoiding the patterns 2413 and 4213. In [6], it is shown that Ψ proves the equidistribution of two 6-tuples of set-valued statistics, that is

$$(\mathsf{Asc}, \mathsf{Row}, \mathsf{Posz}, \mathsf{Max}, \mathsf{Rlmin}, \mathsf{Expo})\,s = (\mathsf{Des}, \mathsf{Ides}, \mathsf{Lrmax}, \mathsf{Lrmin}, \mathsf{Rlmax}, \mathsf{Rlmin})\,\Psi(s),$$

for any $s \in S_n(021)$. Here, Expo is the *exposed positions statistic* define in [6], and if the last statistic of both 6-tuples is dropped out, then the 5-tuples in our Theorem 2 are obtained. Also in [6] is pointed out that the restriction of our mapping b yields a bijection between $\mathfrak{S}_n(2413, 4213)$ and $S_n(021)$, but unlike Ψ^{-1}, this restriction of b does not transform Rlmin to Expo.

References

[1] E. Aas, The double Eulerian polynomial and inversion tables, arXiv:1401.5653, 2014.
[2] M. Beck, S. Robins, Computing the Continuous Discretely: Integer-Point Enumeration in Polyhedra, in: Undergraduate Texts in Mathematics, Springer, New York, 2007.
[3] D. Dumont, Interprétations combinatoires des nombres de Genocchi, Duke Math. J. 41 (2) (1974) 305–318.
[4] D. Foata, Distributions Eulériennes et Mahoniennes sur le groupe des permutations, in: Higher Combinatorics, in: NATO Adv. Study Inst. Ser., Ser. C: Math. Phys. Sci., vol. 31, Reidel, Dordrecht, Boston, Mass, 1977, pp. 27–49. (in French). With a comment by Richard P. Stanley.
[5] D.H. Lehmer, Teaching combinatorial tricks to a computer, in: Proc. Sympos. Appl. Math., Vol. 10, Amer. Math. Soc., 1960, pp. 179–193.
[6] Z. Lin, D. Kim, A sextuple equidistribution arising in pattern avoidance, arXiv:1612.02964v2, 2016.
[7] R. Mantaci, F. Rakotondrajao, A permutation representation that knows what Eulerian means, Discrete Math. Theor. Comput. Sci. 4 (2) (2001) 101–108 (electronic).
[8] T.K. Petersen, Two-sided Eulerian numbers via balls in boxes, Math. Mag. 86 (2013) 159–176.
[9] C.D. Savage, M.J. Schuster, Ehrhart series of lecture hall polytopes and Eulerian polynomials for inversion sequences, J. Combin. Theory Ser. A 119 (4) (2012) 850–870.
[10] M. Skandera, Dumont's statistic on words, Electron. J. Combin. 8 (1) (2001) #R11.
[11] V. Vajnovszki, Lehmer code transforms and Mahonian statistics on permutations, Discrete Math. 313 (2013) 581–589.