

Catalan and Schröder permutations sortable by two restricted stacks*

Jean-Luc Baril¹, Giulio Cerbai², Carine Khalil¹, Vincent Vajnovszki¹

¹LIB, Université de Bourgogne Franche-Comté, B.P. 47 870, 21078 Dijon Cedex France
{barjl, carine.khalil, vvajnov}@u-bourgogne.fr

²Dipartimento di Matematica e Informatica U. Dini, University of Firenze, Firenze, Italy,
giulio.cerbai@unifi.it

Abstract

Pattern avoiding machines were introduced recently by Claesson, Cerbai and Ferrari as a particular case of the two-stacks in series sorting device. They consist of two restricted stacks in series, ruled by a right-greedy procedure and the stacks avoid some specified patterns. Some of the obtained results have been further generalized to Cayley permutations by Cerbai, specialized to particular patterns by Defant and Zheng, or considered in the context of functions over the symmetric group by Berlow. In this work we study pattern avoiding machines where the first stack avoids a pair of patterns of length 3 and investigate those pairs for which sortable permutations are counted by the (binomial transform of the) Catalan numbers and the Schröder numbers.

1 Introduction

Pattern avoiding machines were recently introduced in [8] in attempt to gain a better understanding of sortable permutations using stacks in series. They consist of two restricted stacks in series, equipped with a right-greedy procedure, where the first stack avoids a fixed pattern, reading the elements from top to bottom; and the second stack avoids the pattern 21 (which is a necessary condition for the machine to sort permutations). The authors of [8] provide a characterization of the avoided patterns for which sortable permutations do not form a class, and they show that those patterns are enumerated by the Catalan numbers. For specific patterns, such as 123 and the decreasing pattern of any length, a geometrical description of sortable permutations is also obtained. The pattern 132 has been solved later in [9]. Some of these results have been further generalized to Cayley permutations in [10]. More recently, Berlow [5] explores a single stack version of pattern avoiding machines, where the stack avoids a set of patterns and the sorting process is regarded as a function. Analogous machines, but based on the notion of consecutive patterns, have been introduced and discussed in [11].

In this work we study a variant of pattern-avoiding machines where the first stack avoids (σ, τ) , a pair of patterns of length three. Following [8], we call it (σ, τ) -machine. More specifically, we restrict ourselves to those pairs of patterns for which sortable permutations are counted by either the Catalan numbers or two of their close relatives: the binomial transform of Catalan numbers and the Schröder numbers. For the pair $(132, 231)$ we show that sortable permutations are those avoiding 1324 and 2314, a set whose enumeration is given by the large Schröder numbers. Under certain conditions on the avoided patterns, the output of the first stack is bijectively related to its input (see [5, 10]): it follows that for three pairs of patterns, namely $(123, 213)$, $(132, 312)$ and $(231, 321)$, sortable permutations are counted by the Catalan numbers. This result was conjectured in [3] and settled independently in [4, 5]. For the pair $(123, 132)$, we prove that sortable permutations are those avoiding the patterns 2314, 3214, 4213 and the generalized pattern $[24\bar{1}3]$. We prove that sortable permutations are enumerated by the Catalan numbers by showing that the distribution of the first element is given by the well-known

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Catalan triangle. Finally, we show that for the pair (123, 312) the corresponding counting sequence is the binomial transform of Catalan numbers.

This paper is the extended version of the conference presentation [4] and some of the presented results previously appear in [3].

2 Notations and some preliminary results

We start by recalling some classical definitions about pattern avoidance on permutations (see [13] for a more detailed introduction). Denote by \mathfrak{S}_n the set of permutations of length n and let $\mathfrak{S} = \cup_{n \geq 0} \mathfrak{S}_n$. Given two permutations σ of length k and $\pi = \pi_1 \cdots \pi_n$, we say that π *contains* the pattern σ if π contains a subsequence $\pi_{i_1} \cdots \pi_{i_k}$, with $i_1 < i_2 < \cdots < i_k$, which is order isomorphic to σ . In this case, we say that $\pi_{i_1} \cdots \pi_{i_k}$ is an *occurrence* of the pattern σ in π . Otherwise, we say that π *avoids* σ .

We say that π contains an occurrence of the (generalized) pattern $[\sigma]$ if π contains an occurrence of σ that involves the first element π_1 of π . For instance, an occurrence of [12 in π corresponds to a pair of elements $\pi_1 \pi_i$, with $i > 1$ and $\pi_i > \pi_1$. A *barred pattern* $\tilde{\sigma}$ is a pattern where some entries are barred. Let σ be the classical pattern obtained by removing all the bars from $\tilde{\sigma}$. Let τ be the pattern which is order isomorphic to the non-barred entries of $\tilde{\sigma}$ (i.e. obtained from $\tilde{\sigma}$ by removing all the barred entries and suitably rescaling the remaining elements). A permutation π avoids $\tilde{\sigma}$ if each occurrence of τ in π can be extended to an occurrence of σ . For instance, a permutation π avoids the pattern [2413 if for any subsequence $\pi_1 \pi_i \pi_j$, with $1 < i < j$ and $\pi_1 < \pi_j < \pi_i$, there is an index t , $i < t < j$, such that $\pi_1 \pi_i \pi_t \pi_j$ is an occurrence of 2413.

Given a set of (generalized) patterns T , denote by $\text{Av}_n(T)$ the set of permutations in \mathfrak{S}_n avoiding each pattern in T . Similarly, let $\text{Av}(T) = \cup_{n \geq 0} \text{Av}_n(T)$. If $T = \{\sigma\}$ is a singleton, we write $\text{Av}_n(\sigma)$ and $\text{Av}(\sigma)$. In his celebrated book [14], Knuth gave the following characterization of stack sortable permutations, which is often considered the starting point of stack sorting and permutation patterns disciplines.

Proposition 1 ([14]). *A permutation π is sortable using a classical stack (that is, a 21-avoiding stack) if and only if π avoids the pattern 231.*

Let T be a set of patterns. A T -stack is a stack that is not allowed to contain an occurrence of any pattern in T , reading its elements from top to bottom. Given a permutation π , denote by $\text{out}^T(\pi)$ the permutation obtained after passing π through the T -avoiding stack by applying a greedy procedure, i.e. by always pushing the next element of the input, unless it creates an occurrence of a forbidden pattern inside the stack. Denote by $\text{Sort}_n(T)$ the set of length n permutations that are sortable by the T -machine, that is, by passing π through the T -avoiding stack and then through the 21-avoiding stack. Permutations in $\text{Sort}_n(T)$ are called T -sortable, and $\text{Sort}(T)$ is the set of T -sortable permutations of any length. As a consequence of Proposition 1, $\text{Sort}(T)$ consists precisely of those permutations π for which $\text{out}^T(\pi)$ avoids 231. To ease notations, if T is either a singleton $T = \{\sigma\}$ or a pair of patterns $T = \{\sigma, \tau\}$, we will omit the curly brackets from the above notations. For instance, we will write $\text{Sort}(\sigma, \tau)$ instead of $\text{Sort}(\{\sigma, \tau\})$.

The authors of [8] showed that if π is a 12-sortable permutation of length n , then $\text{out}^{12}(\pi) = n(n-1) \cdots 1$. Moreover, by Proposition 1 and applying the complement operation on the processed permutation, we have that $\text{Sort}(12) = \mathfrak{S}(213)$. In order to refer to this result later, we state it below in a slightly more general form. A *partial permutation* of n is an injection $\pi : \{1, 2, \dots, k\} \rightarrow \{1, 2, \dots, n\}$, for some $0 \leq k \leq n$, and the integer k is said to be the length of π . We let a 12-stack act on a partial permutation π of n in the natural way by identifying π with the list of its images.

Proposition 2. *If π is a partial permutation of n which is 12-sortable, then $\text{out}^{12}(\pi)$ is the decreasing rearrangement of the symbols of π . Moreover, π is 12-sortable if and only if it avoids 213.*

An entry π_i of a permutation π is a *left-to-right minimum* if $\pi_i < \pi_j$, for each $j < i$. The *left-to-right minima decomposition* (briefly *ltr-min decomposition*) of π is $\pi = m_1 B_1 m_2 B_2 \cdots m_t B_t$, where $m_1 > m_2 > \cdots > m_t$ are the ltr-minima of π and the *block* B_i contains the elements of π between m_i and m_{i+1} , for $i = 1, \dots, t-1$. The last block B_t contains the elements that follow m_t in π . Note that $m_t = 1$. The notion of *left-to-right maximum* of a permutation π is defined similarly. The *ltr-max decomposition* of π is $\pi = M_1 B_1 M_2 B_2 \cdots M_t B_t$, where $M_1 < M_2 < \cdots < M_t$ are the ltr-maxima of π . In this case $M_t = n$, where n is the length of π .

Finally, the sequence $(c_n)_{n \geq 0}$, pervasive in this paper, is the sequence of Catalan numbers $c_n = \frac{1}{n+1} \binom{2n}{n}$ (A000108 in [16]).

3 Pair (132, 231)

This section is devoted to the analysis of the (123, 231)-machine.

Theorem 1. *Consider the $(132, \sigma)$ -machine, where $\sigma = \sigma_1 \cdots \sigma_{k-1} \sigma_k \in \mathfrak{S}_k$, with $k \geq 3$ and $\sigma_{k-1} > \sigma_k$. Given a permutation π of length n , let $m_1 B_1 \cdots m_t B_t = \pi$ be its ltr-min decomposition. Then:*

1. *Everytime a ltr-minimum m_i is pushed into the $(132, \sigma)$ -stack, the $(132, \sigma)$ -stack contains the elements m_{i-1}, \dots, m_2, m_1 , reading from top to bottom. Moreover, we have*

$$\text{out}^{132, \sigma}(\pi) = \tilde{B}_1 \cdots \tilde{B}_t m_t \cdots m_1,$$

where \tilde{B}_i is a rearrangement of B_i .

2. *If π is $(132, \sigma)$ -sortable, then \tilde{B}_i is decreasing for each i . Moreover, for each $i \leq t-1$, we have $B_i > B_{i+1}$ (i.e. $x > y$ for each $x \in B_i, y \in B_{i+1}$).*

Proof. 1. Let us consider the evolution of the $(132, \sigma)$ -stack on input π . Note that, since $k \geq 3$, the element m_1 remains at the bottom of the $(132, \sigma)$ -stack until the end of the process. Now, if B_1 is not empty then for each $x \in B_1$, the elements $m_2 x m_1$ form an occurrence of 132. Therefore the block B_1 is extracted before m_2 enters the $(132, \sigma)$ -stack. After m_2 is pushed, the $(132, \sigma)$ -stack contains $m_2 m_1$, reading from top to bottom. Since $m_2 < m_1$, but $\sigma_{k-1} > \sigma_k$ by hypothesis, m_2 cannot play the role of either σ_{k-1} in an occurrence of σ or of 3 in an occurrence of 132. Thus m_2 remains at the bottom of the $(132, \sigma)$ -stack until the end of the sorting procedure. The thesis follows by iterating the same argument on each block B_i , for $i \geq 2$.

2. Suppose that π is $(132, \sigma)$ -sortable. Assume, for a contradiction, that \tilde{B}_i is not decreasing, for some i . Then there are two consecutive elements $x < y$ in \tilde{B}_i . Therefore, by what proved above, $\text{out}^{132, \sigma}(\pi)$ contains an occurrence $x y m_t$ of 231, which is impossible due to Proposition 1. Finally, suppose that $x < y$, for $x \in B_i$ and $y \in B_{i+1}$. Then $x y m_t$ is an occurrence of 231 in $\text{out}^{132, \sigma}(\pi)$, a contradiction. □

Theorem 1 and Proposition 2 guarantee that if $\pi = m_1 B_1 \cdots m_t B_t$ is the the ltr-min decomposition of a $(132, 231)$ -sortable permutation π , then (with the notation above) $\tilde{B}_i = \text{out}^{12}(B_i)$, for each i . However, this is true even when the sortability requirement is relaxed.

Lemma 1. *Let $\pi = m_1 B_1 \cdots m_t B_t$ be the ltr-min decomposition of a permutation π . Write $\text{out}^{132, 231}(\pi) = \tilde{B}_1 \cdots \tilde{B}_t m_t \cdots m_1$ as in Theorem 1. Then $\tilde{B}_i = \text{out}^{12}(B_i)$, for each i .*

Proof. Consider the instant immediately after m_i is pushed into the $(132, 231)$ -stack and the non-empty block B_i has to be processed, for some i . By Theorem 1, at this point the $(132, 231)$ -stack contains m_i, m_{i-1}, \dots, m_1 , reading from top to bottom. We want to show that the behavior of the $(132, 231)$ -stack on B_i is equivalent to the behavior of an empty 12-stack on input B_i . We prove that the $(132, 231)$ -stack performs the pop operation of some $x \in B_i$ if and only if the 12-stack does the same. If either the next element of the input is m_{i+1} or x is the last element of π to be processed, then both the $(132, 231)$ -stack and the 12-stack perform a pop operation, as desired. Otherwise, suppose the next element of the input is y , for some y in the same block B_i , and the $(132, 231)$ -stack pops the element $x \in B_i$. This means that the $(132, 231)$ -stack contains two elements z, w , with z above w , such that yzw is an occurrence of either 132 or 231. Note that, since $z > w$, z is not a ltr-minimum. Therefore yz is an occurrence of 12 and the 12-stack performs a pop operation, as desired. Conversely, suppose that the 12-stack pops the element x , with $y \in B_i$ the next element of the input. This implies that the 12-stack contains an element z such that $z > y$. Therefore $yz m_i$ is an occurrence of 231 and the $(132, 231)$ -stack performs a pop operation, as desired. □

Corollary 1. *Let $\pi = m_1 B_1 \cdots m_t B_t$ be the ltr-min decomposition of a permutation π . Then the following are equivalent.*

1. B_i avoids 213 and $B_i > B_{i+1}$, for each i .
2. π is (132, 231)-sortable.
3. $\pi \in \text{Av}(1324, 2314)$.

Proof. Combining the first point in Theorem 1 and Lemma 1 we have:

$$\text{out}^{132,231}(\pi) = \text{out}^{12}(B_1) \cdots \text{out}^{12}(B_t) m_t \cdots m_1.$$

We will use this decomposition of $\text{out}^{132,231}(\pi)$ throughout the rest of the proof.

[1 \Rightarrow 2] Suppose, for a contradiction, that $\text{out}^{132,231}(\pi)$ contains an occurrence bca of 231. Note that, since $c > a$, while $m_t < \cdots < m_1$, c is not a ltr-minimum of π (and thus neither b is). Now, if b and c are in the same block B_j , then $\text{out}^{12}(B_j)$ is not decreasing. Thus, by Proposition 2, B_j contains 213, which is a contradiction. Otherwise, if $b \in B_j$ and $c \in B_k$, with $j < k$, then we have a contradiction with the hypothesis $B_i > B_{i+1}$ for each i .

[2 \Rightarrow 3] Suppose, for a contradiction, that $\pi \notin \text{Av}(1324, 2314)$. First, suppose that π contains an occurrence $acbd$ of 1324. Observe that b, c, d are not ltr-minima of π . Now, if b and d are in the same block B_j of π , for some j , then B_j contains an occurrence cbd of 213. Therefore $\text{out}^{12}(B_j)$ contains an occurrence of 231 due to Proposition 2, which contradicts the hypothesis. Otherwise, if $b \in B_j$ and $d \in B_k$, for some $j < k$, then $\text{out}^{132,231}(\pi)$ contains an occurrence bdm_k of 231, again a contradiction. The pattern 2314 can be addressed analogously, so we leave it to the reader.

[3 \Rightarrow 1] Let $\pi \in \text{Av}(1324, 2314)$. If B_i contains an occurrence bac of 213, then π contains an occurrence $m_i bac$ of 1324, which is impossible. Otherwise, if π contains two elements $x \in B_j$, $y \in B_k$, with $x < y$ and $j < k$, then $m_j x m_k y$ is an occurrence of 2314, contradicting the hypothesis. \square

The enumeration of $\text{Av}(1324, 2314)$ (or a symmetry of these patterns) can be found for instance in [2, 17]. Note that in [1], the authors provide a constructive bijection between these permutations and Schröder paths.

Corollary 2. *Permutations of length n in $\text{Sort}(132, 231)$ are enumerated by the large Schröder numbers (sequence A006318 in [16]).*

4 The $(\sigma, \hat{\sigma})$ -machine

For a permutation σ of length two or more, denote by $\hat{\sigma}$ the permutation obtained from σ by interchanging its first two entries. Let us regard a (σ, τ) -stack as an operator $\text{out}^{\sigma, \tau} : \mathfrak{S} \rightarrow \mathfrak{S}$. By conveniently modifying the proof of Corollary 4.5 in [10] (stated in the context of Cayley permutations), we have that $\text{out}^{\sigma, \tau}$ is a length preserving bijection on \mathfrak{S} if and only if $\tau = \hat{\sigma}$. More generally, Berlow [5] showed that for a set T of patterns, out^T is a length preserving bijection on \mathfrak{S} if and only if T is closed under the $\hat{}$ operator. In order for the paper to be self-contained, we shall give the following result, which is easier to prove (although weaker): $\text{out}^{\sigma, \hat{\sigma}}$ is a bijection for any pattern σ . An immediate consequence will be Theorem 2 below.

Let N^* be the set of finite length integer sequences. The action of the (σ, τ) -stack on input π can be naturally represented as a sequence of triples $(r; s; t) \in (N^*)^3$, where r is the current content of the output, s is the current content of the (σ, τ) -stack (read from top to bottom) and t is the current content of the input. The triple $(r; s; t)$ is said to be a *state of passing* of π through the (σ, τ) -stack. Clearly, r is a prefix of $\text{out}^{\sigma, \tau}(\pi)$, t is a suffix of π , the initial state is $(\lambda; \lambda; \pi)$ and the final one is $(\text{out}^{\sigma, \tau}(\pi); \lambda; \lambda)$, where λ is the empty sequence. Moreover a non-final state $(p_1 p_2 \cdots p_a; s_1 s_2 \cdots s_b; t_1 t_2 \cdots t_c)$ is followed by either the state

$$(p_1 p_2 \cdots p_a s_1; s_2 \cdots s_b; t_1 t_2 \cdots t_c),$$

if a pop operation is performed next, or

$$(p_1 p_2 \cdots p_a; t_1 s_1 s_2 \cdots s_b; t_2 \cdots t_c),$$

if a push operation is performed next.

For $p = p_1 \cdots p_n \in \mathbb{N}^n$, we denote by p^r the *reverse* of p , that is $p^r = p_n \cdots p_1$. We wish to show that the behavior of the $(\sigma, \hat{\sigma})$ -stack on π is strictly related to its behavior on $(\text{out}^{\sigma, \hat{\sigma}}(\pi))^r$. More precisely,

if $o_1 \cdots o_{2n}$ is the sequence of push/pop operations performed when π is passed through a $(\sigma, \hat{\sigma})$ -stack, then $o'_{2n} \cdots o'_1$ is the sequence of push/pop operations performed when $(\text{out}^{\sigma, \hat{\sigma}}(\pi))^r$ is passed through the $(\sigma, \hat{\sigma})$ -stack, where o'_i is a push (resp. pop) operation if o_i is a pop (resp. push) operation. This can be equivalently expressed by saying that the state $(p; s; t)$ is followed by $(u; v; w)$ if and only if the state $(w^r; v; u^r)$ is followed by $(t^r; s; p^r)$.

Lemma 2. *Consider the action of the $(\sigma, \hat{\sigma})$ -stack. Let $p, s, t \in \mathbb{N}^*$ and $x \in \mathbb{N}$.*

1. *If the state (p, xs, t) is followed by the state (px, s, t) (and thus a pop operation is performed) then the state (t^r, s, xp^r) is followed by the state (t^r, xs, p^r) (and thus a push operation is performed).*
2. *If the state (p, s, xt) is followed by the state (p, xs, t) (and thus a push operation is performed), then the state (t^r, xs, p^r) is followed by the state $(t^r x, s, p^r)$ (and thus a pop operation is performed).*

Proof. 1. Since xs is the content of the $(\sigma, \hat{\sigma})$ -stack in the state (p, xs, t) , we have that xs avoids σ and $\hat{\sigma}$. Thus a push operation is performed if s is the content of the $(\sigma, \hat{\sigma})$ -stack and x is the next element of the input.

2. If p is empty, the statement holds. Otherwise, let $p = p_1 \cdots p_a$ and $s = s_1 \cdots s_b$. Observe that p_a is the last element that has been extracted from the $(\sigma, \hat{\sigma})$ -stack before x enters. Therefore, when p_a is extracted, p_a plays the role of either σ_2 in an occurrence of σ or of $\hat{\sigma}_2$ in an occurrence of $\hat{\sigma}$. More precisely, one of the following four cases hold. We show the details for the first case only, the others being similar. Let z be the length of σ .

- $s_\ell p_a s_{i_3} \cdots s_{i_z}$ is an occurrence of σ , for some $\ell \geq 1$ and $\ell < i_3 < \cdots < i_z$. Then $p_a s_\ell s_{i_3} \cdots s_{i_z}$ is an occurrence of $\hat{\sigma}$ and therefore a pop operation is performed when p_a is the next element of the input and xs is the content of the $(\sigma, \hat{\sigma})$ -stack, as desired.
- $s_\ell p_a s_{i_3} \cdots s_{i_z}$ is an occurrence of $\hat{\sigma}$, for some $\ell \geq 1$ and $\ell < i_3 < \cdots < i_z$.
- $x p_a s_{i_3} \cdots s_{i_z}$ is an occurrence of σ , for some $i_3 < \cdots < i_z$.
- $x p_a s_{i_3} \cdots s_{i_z}$ is an occurrence of $\hat{\sigma}$, for some $i_3 < \cdots < i_z$.

□

A straightforward consequence of the previous lemma is that the map $(\text{out}^{\sigma, \hat{\sigma}})^r : \mathfrak{S} \rightarrow \mathfrak{S}$ is its own inverse, and thus a bijection. More specifically, for any permutation π , we have $(\text{out}^{\sigma, \hat{\sigma}})^{-1}(\pi) = (\text{out}^{\sigma, \hat{\sigma}}(\pi^r))^r$. Since π is $(\sigma, \hat{\sigma})$ -sortable if and only if $\text{out}^{\sigma, \hat{\sigma}}(\pi)$ avoids 231 (and the reverse map is bijective), we have that $\text{Sort}(\sigma, \hat{\sigma})$ is in bijection with $\text{Av}(231)$. The next theorem follows.

Theorem 2. *For any pattern σ , $\text{out}_n^{\sigma, \hat{\sigma}}$ is a bijection on \mathfrak{S}_n . Moreover, we have*

$$|\text{Sort}_n(123, 213)| = |\text{Sort}_n(132, 312)| = |\text{Sort}_n(231, 321)| = c_n,$$

the n th Catalan number.

5 Pair (123, 132)

We characterize $\text{Sort}(123, 132)$ in terms of pattern avoidance. Then we show that $(123, 132)$ -sortable permutations are enumerated by the Catalan numbers by exhibiting a link with the very well studied Catalan triangle.

Theorem 3. *A permutation π is $(123, 132)$ -sortable if and only if π avoids 2314, 3214, 4213 and $[24\bar{1}3]$.*

Proof. Suppose that π is $(123, 132)$ -sortable. For a contradiction, suppose that π contains $\tau \in \{2314, 3214, 4213\}$. Pick an occurrence $\pi_i \pi_j \pi_k \pi_\ell$ of τ , with $i < j < k < \ell$, where ℓ is chosen minimal, and k, j , and i are chosen maximal, in this order.

If $\tau = 2314$, due to our choice of i, j, k, ℓ , we have $\pi_i < \pi_u < \pi_j$, for $k < u < \ell$. Now, when π_k is pushed in the $(123, 132)$ -stack, at least one of π_i and π_j has already been extracted: otherwise the

(123, 132)-stack would contain an occurrence of 132, which is forbidden. For each u , $k + 1 \leq u \leq \ell$, we have $\pi_k < \pi_u$ and when π_u is pushed in the (123, 132)-stack, π_k is still in the (123, 132)-stack. Indeed, $\pi_u \pi_x \pi_y$ cannot be an occurrence of 123 nor of 132 with π_x above π_y , both in the tail of the (123, 132)-stack beginning by π_k . If π_i (resp. π_j) is extracted before π_k enters in the (123, 132)-stack, then $\pi_i \pi_{k+1} \pi_k$ (resp. $\pi_j \pi_\ell \pi_k$) creates a pattern 231 in $\text{out}^{123,132}(\pi)$, a contradiction.

If $\tau = 3214$, due to our choice of i, j, k, ℓ , we have $\pi_i > \pi_u > \pi_j$, for $k < u < \ell$; and $\pi_u < \pi_{u+1}$, for $k \leq u < \ell$. As above, when π_k is pushed in the (123, 132)-stack, at least one of π_i and π_j has already been extracted: otherwise the (123, 132)-stack would contain an occurrence of 123. Since $\pi_{k+1} > \pi_k$, the next step pushes π_{k+1} in the (123, 132)-stack. (i) Assume that π_i and π_j both had left the (123, 132)-stack. Then π_k is just below π_{k+1} in the (123, 132)-stack, and $\pi_k < \pi_j < \pi_{k+1}$. This implies that $\pi_j \pi_{k+1} \pi_k$ is an occurrence of 231 in $\text{out}^{123,132}(\pi)$, a contradiction. (ii) Assume that π_i is still in the (123, 132)-stack and π_j had left this stack. Again, $\pi_j \pi_{k+1} \pi_k$ is an occurrence of 231 in $\text{out}^{123,132}(\pi)$, a contradiction. (iii) Assume that π_j is still in the (123, 132)-stack and π_i had left this stack. Since $\pi_u < \pi_{u+1}$ for $k \leq u \leq \ell - 1$, the next steps of the process push successively all entries $\pi_{k+1}, \dots, \pi_\ell$ in the (123, 132)-stack. As above, $\pi_i \pi_\ell \pi_k$ is an occurrence of 231 in $\text{out}^{123,132}(\pi)$, again a contradiction.

The case $\tau = 4213$ can be treated similarly.

Finally, suppose that π contains $[24\bar{1}3]$. Equivalently, there are two indices $i < j$ such that $\pi_1 \pi_i \pi_j$ is an occurrence of 132 and $\pi_k > \pi_1$ for each $i < k < j$. Observe that, by choosing j minimal and i maximal (in this order), we can assume $j = i + 1$. Now, if π_i is still in the (123, 132)-stack when π_{i+1} enters, then $\text{out}^{123,132}(\pi)$ contains an occurrence $\pi_{i+1} \pi_i \pi_1$ of 231, which is impossible due to the sortability of π . Therefore π_i is extracted before π_{i+1} enters. This means that there are two elements π_u, π_v in the (123, 132)-stack, with $u < v$ (and thus π_v above π_u), such that $\pi_{i+1} \pi_v \pi_u$ is an occurrence of either 123 or 132. Choose u, v minimal amongst those indices¹, so that π_u is still in the (123, 132)-stack when π_{i+1} enters. Notice that $\pi_{i+1} < \pi_u$ (and thus $u \neq 1$). Moreover, it must be $\pi_u < \pi_i$, otherwise $\pi_i \pi_u \pi_1$ would be an occurrence of 231 in the (123, 132)-stack, which is forbidden. But then $\pi_{i+1} \pi_u \pi_1$ is an occurrence of 231 in $\text{out}^{123,132}(\pi)$, a contradiction.

Conversely, suppose that π is not (123, 132)-sortable. We shall prove that π contains at least one of the patterns 3214, 2314, 4213 or $[24\bar{1}3]$. By hypothesis $\text{out}^{123,132}(\pi)$ contains an occurrence bca of 231. Let $b = \pi_j$ and $c = \pi_k$, for some indices j, k . We distinguish two cases, according whether $j < k$ or $j > k$.

- Suppose that $j < k$ and thus π_j is extracted from the (123, 132)-stack before π_k enters. Then there are two elements π_u, π_v in the (123, 132)-stack, with $u < v$ (and thus π_v above π_u), such that $\pi_z \pi_v \pi_u$ is an occurrence of either 123 or 132, where π_z is the next element of the input. Notice that $\pi_j \geq \min\{\pi_u, \pi_v\}$, since otherwise $\pi_j \pi_v \pi_u$ would be an occurrence of either 123 or 132 in the (123, 132)-stack, which is impossible. Thus $\pi_k > \pi_j > \pi_z$. If $\pi_z \pi_v \pi_u$ is an occurrence of 123, then $\pi_u \pi_v \pi_z \pi_k$ is an occurrence of either 4213, if $\pi_u > \pi_k$, or 3214, if $\pi_u < \pi_k$. Finally, if $\pi_z \pi_v \pi_u$ is an occurrence of 132, then $\pi_u \pi_j \pi_z \pi_k$ is an occurrence of 2314.
- Suppose instead that $j > k$ and thus π_k is still in the (123, 132)-stack when π_j enters. Observe that $k \neq 1$, since π_1 is the last element of $\text{out}^{123,132}(\pi)$. Therefore, when π_j enters the (123, 132)-stack, the (123, 132)-stack contains the elements $\pi_j \pi_k \pi_1$, reading from top to bottom. Notice that $\pi_j > \pi_1$, otherwise $\pi_j \pi_k \pi_1$ would realize an occurrence of the forbidden 132 inside the (123, 132)-stack. Moreover, for each entry π_t , with $k < t < j$, we have $\pi_t > \pi_1$. Otherwise $\pi_t \pi_k \pi_1$ would be an occurrence of 132 and π_k would be extracted before π_j , which is impossible due to our assumptions. Thus $\pi_1 \pi_k \pi_j$ is an occurrence of $[24\bar{1}3]$. This completes the proof. □

Proposition 3. *The distribution of the first element in $\text{Sort}(123, 132)$ is given by the Catalan triangle (sequence A009766 in [16]).*

Proof. Let $A_n(k)$ be the set of (123, 132)-sortable permutations of length n and starting with k . Let $A_n^1(k)$ be the subset of $A_n(k)$ consisting of those permutations $\pi = \pi_1 \pi_2 \dots \pi_n$ where any occurrence

¹In other words, pick the deepest such elements π_u, π_v in the (123, 132)-stack

$\pi_1\pi_i\pi_\ell$ of [231 with $\pi_\ell = \pi_1 - 1$ can be extended into an occurrence $\pi_1\pi_i\pi_j\pi_\ell$ of [3412. Set $A_n^2(k) = A_n(k) \setminus A_n^1(k)$ and let $k \geq 2$. We shall provide bijections $\alpha : A_n^1(k) \rightarrow A_n(k-1)$ and $\beta : A_n^2(k) \rightarrow A_{n-1}(k)$.

Define $\alpha : A_n^1(k) \rightarrow A_n(k-1)$ by $\alpha(\pi) = \pi'$, where π' is obtained from π by swapping the two entries π_1 and $\pi_1 - 1$ in π . Since $\pi \in A_n^1(k)$, it is easy to check that π' avoids [2413. In addition, swapping π_1 and $\pi_1 - 1$ does not affect the avoidance of the three patterns 3214, 2314, 4213, which implies (see Theorem 3) that $\alpha(\pi) \in A_n(k-1)$.

Next define $\beta : A_n^2(k) \rightarrow A_{n-1}(k)$ by $\beta(\pi) = \pi''$, where π'' is obtained from π by deleting the entry π_ℓ immediately before $k-1$ and by decreasing by one all entries of π greater than π_ℓ . Notice that $\beta(\pi) \in A_{n-1}(k)$. Let us now sketch the proof that β is bijective, leaving some technical details to the reader. We shall explicitly define the inverse map of β . Given $\pi \in A_{n-1}(k)$, choose an integer ℓ as follows:

- ℓ is the minimal entry $l = \pi_u > \pi_1$, with $1 < u < i$, such that there is an index v with $\pi_v < \pi_i$ and $u < v < i$, if such entry exists.
- Otherwise, set $\ell = n$.

The preimage π is obtained by inserting ℓ immediately before $\pi_i = k-1$ and then increasing by one all the entries π_j of π with $\pi_j \geq \ell$.

Finally, setting $a_n^k = |A_n(k)|$, we have that $a_n^k = a_n^{k-1} + a_{n-1}^k$, for $2 \leq k \leq n$. Since $A_n(1) = \{123 \cdots n\}$ and $A_n(n)$ is the set of length n permutations avoiding 213 and starting with n , the initial conditions are given by $a_n^1 = 1$ and $a_n^n = c_{n-1}$, where c_n is the n th Catalan number. Therefore, a_n^k generates the well-known Catalan triangle (see Table 1 and [7, 12, 15]). \square

$k \setminus n$	1	2	3	4	5	6	7	8
1	1	1	1	1	1	1	1	1
2		1	2	3	4	5	6	7
3			2	5	9	14	20	27
4				5	14	28	48	75
5					14	42	90	165
6						42	132	297
...						
\sum	1	2	5	14	42	132	429	1430

Table 1: The Catalan triangle $a_n^k = |A_n(k)|$, with $1 \leq n \leq 8$ and $1 \leq k \leq 6$.

Corollary 3. *Permutations of length n in $\text{Sort}(123, 132)$ are enumerated by the Catalan numbers.*

Proof. With the previous notations we have $|\text{Sort}_n(123, 132)| = \sum_{k=1}^n a_n^k = c_n$, the n th Catalan number (see again [7, 12, 15]). \square

6 Pair (123, 312)

We start by giving a ltr-max counterpart of Theorem 1.

Theorem 4. *Consider the $(312, \sigma)$ -machine, where $\sigma = \sigma_1 \cdots \sigma_{k-1} \sigma_k \in \mathfrak{S}_k$, with $k \geq 3$ and $\sigma_{k-1} < \sigma_k$. Given a permutation π of length n , let $\pi = M_1 B_1 \cdots M_t B_t$ be its ltr-max decomposition. Then:*

1. *Everytime a ltr-maximum M_i is pushed into the $(312, \sigma)$ -stack, the $(312, \sigma)$ -stack contains the elements $M_i, M_{i-1}, \dots, M_2, M_1$, reading from top to bottom. Moreover, we have*

$$\text{out}^{312, \sigma}(\pi) = \tilde{B}_1 \cdots \tilde{B}_t M_t \cdots M_1,$$

where \tilde{B}_i is a rearrangement of B_i .

2. *If π is $(312, \sigma)$ -sortable, then $M_1, M_2, \dots, M_t = n - t + 1, n - t + 2, \dots, n$.*

Proof. 1. The proof is identical to that of the first part of Theorem 1.

2. If π is $(312, \sigma)$ -sortable, then $\text{out}^{312, \sigma}(\pi)$ avoids 231 by Proposition 1. Suppose, for a contradiction, that there is an element $j \in \{n-t+1, \dots, n\}$ which is not a ltr-maximum. Note that $j \neq \pi_1 = M_1$ and $j \neq n = M_t$. Then, $\text{out}^{312, \sigma}(\pi)$ contains an occurrence jnM_1 of 231, which is a contradiction. \square

Instantiating σ by 123 in the previous theorem we have the next result.

Theorem 5. *Let π be a $(123, 312)$ -sortable permutation and let $\pi = M_1B_1 \cdots M_tB_t$ be its ltr-max decomposition. Then:*

1. B_i avoids 213 for each i .
2. $\tilde{B}_i = \text{out}^{12}(B_i)$, for each i .

Proof. Let $i \geq 2$. Notice that, as a consequence of Theorem 4, immediately after M_i has been pushed in the $(123, 312)$ -stack, this stack contains the elements $M_i \cdots M_2M_1$, reading from top to bottom. Moreover, these elements remain at the bottom of the $(123, 312)$ -stack until the end of the sorting procedure, since they are the last elements of $\text{out}^{123, 312}(\pi)$. This fact will be used for the rest of the proof.

1. Suppose, for a contradiction, that B_i contains an occurrence of 213, for some i , and let bac be such an occurrence with a ‘minimal’, in the sense that there is no $a' < a$ where $ba'c$ is an occurrence of 213 in B_i . Therefore, since abM_i is an occurrence of 123, b is extracted from the $(123, 312)$ -stack before a enters. In addition, when c enters into the $(123, 312)$ -stack, a is still in this stack. Indeed, no entry in B_i between a and c together with a produces a forbidden pattern in the $(123, 312)$ -stack. It follows that $\text{out}^{123, 312}(\pi)$ contains bca which is an occurrence of 231, yielding a contradiction with the sortability of π .

2. Let us consider the action of the $(123, 312)$ -stack on the block B_i . We wish to show that the behavior of the $(123, 312)$ -stack when processing B_i is equivalent to the behavior of an empty 12-stack on input B_i . In other words, we prove that the restriction of the $(123, 312)$ -stack is triggered if and only if the next element of the input forms an occurrence of 12 together with some other element in the $(123, 312)$ -stack. Immediately after M_i has been pushed (i.e. before the first element of B_i is processed), the $(123, 312)$ -stack contains the elements $M_i \cdots M_2M_1$, reading from top to bottom. Observe that B_i avoids 213 by what proved above, therefore the $(123, 312)$ -stack cannot be triggered by an occurrence of 312 when processing B_i . Suppose that the next element of the input x forms an occurrence xy of 12 with some $y \in B_i$. Then xyM_i is an occurrence of 123 in the $(123, 312)$ -stack, and so this stack behaves as a 12-stack. Conversely, suppose that the $(123, 312)$ -stack is triggered by an occurrence of xyz of 123, where x is the next element of the input. Since $M_i > M_{i-1} > \cdots > M_1$, necessarily $y \in B_i$. Thus xy is an occurrence of 12 that triggers the 12-stack, as wanted. \square

As a consequence of what proved so far in this section, for any $(123, 312)$ -sortable permutation $\pi = M_1B_1 \cdots M_tB_t$ of length n , we have $B_i \in \text{Av}(213)$ and $M_1, \dots, M_t = n-t+1, \dots, n$. Moreover, by Proposition 2, each \tilde{B}_i in $\text{out}^{123, 312}(\pi) = \tilde{B}_1 \cdots \tilde{B}_tM_t \cdots M_1$ is decreasing. Therefore, for any three elements x, y, z , with $x \in B_i$, $y \in B_j$ and $z \in B_k$, with $i < j \leq k$, xyz is not an occurrence of 231. Otherwise xyz would still be an occurrence of 231 in $\text{out}^{123, 312}(\pi)$, contradicting the fact that π is $(123, 312)$ -sortable. From now on, we say that xyz is an occurrence of $2-3-1$ if $z < x < y$, with $x \in B_i$, $y \in B_j$ and $z \in B_k$, with $i < j < k$. Similarly, when $j = k$, we say that xyz is an occurrence of $2-31$.

Theorem 6. *Let $\pi = M_1B_1 \cdots M_tB_t$ be the ltr-max decomposition of a permutation π of length n . Write $\text{out}^{123, 312}(\pi) = \tilde{B}_1 \cdots \tilde{B}_tM_t \cdots M_1$ as in Theorem 4. Then π is $(123, 312)$ -sortable if and only if the following conditions are satisfied:*

1. $M_j = n-t+j$, for each $j = 1, \dots, t$.
2. B_i avoids 213 for each i (and thus \tilde{B}_i is decreasing for each i).
3. $\text{out}^{123, 312}(\pi)$ avoids $2-3-1$.
4. $\text{out}^{123, 312}(\pi)$ avoids $2-31$.

Proof. If π is (123, 312)-sortable, then π satisfies all the above conditions as a consequence of what proved before in this section. Conversely, it is easy to check that if π satisfies the above conditions, then $\text{out}^{123,312}(\pi)$ avoids 231. Thus π is (123, 312)-sortable. \square

Reformulating the third condition of Theorem 6 we obtain the following lemma, whose easy proof is omitted.

Lemma 3. *Let $\pi = M_1 B_1 \cdots M_t B_t$ be the ltr-max decomposition of the (123, 312)-sortable permutation π . Write $\text{out}^{123,312}(\pi) = \tilde{B}_1 \cdots \tilde{B}_t M_t \cdots M_1$ as in Theorem 4. Then $\text{out}^{123,312}(\pi)$ avoids 2-31 if and only if for each $x \in B_i, y \in B_j$, with $i < j$, we have:*

- if $y > x$, then $B_j > x$.
- If $y < x$, then $B_j < x$.

In other words, Lemma 3 says that each block B_j of a (123, 312)-sortable permutation π is bounded between two previous elements of π . The following result is obtained by restating this lemma and Theorem 6 in terms of pattern avoidance.

Theorem 7. *A permutation π is (123, 312)-sortable if and only if π avoids the three generalized patterns [132, [42531 and [42153.*

Next we prove that (123, 312)-sortable permutations are enumerated by the binomial transform of Catalan numbers. We shall exploit the above characterization in terms of patterns in order to provide a bijection with a certain set of partial permutations, whose enumeration is straightforward.

Recall from Section 2 that a partial permutation of n is an injection $\pi : \{1, 2, \dots, k\} \rightarrow \{1, 2, \dots, n\}$, for some $0 \leq k \leq n$. The partial permutation of length zero will be denoted by λ . Denote by \mathcal{A}_n the set of all partial permutations of n . For instance, we have $\mathcal{A}_3 = \{\lambda, 1, 2, 3, 12, 21, 13, 31, 23, 32, 123, 132, 213, 231, 312, 321\}$. There is a natural bijection between the set of permutations in \mathfrak{S}_n avoiding the pattern [132 and \mathcal{A}_{n-1} . Indeed, from a length n permutation π avoiding [132, we associate the unique partial permutation $\alpha(\pi) \in \mathcal{A}_{n-1}$ defined as follows:

$$\alpha(\pi)_{\pi_i} = i - 1, \text{ for } \pi_i < \pi_1.$$

In other words, $\alpha(\pi)$ is obtained by recording the indices (minus one) of the elements $\pi_i < \pi_1$, from the smallest to the largest one. For instance, if $\pi = 52461783$, then $\alpha(\pi) = 4172$. Notice that $\alpha(\pi) = \lambda$ if and only if $\pi_1 = 1$. Let us now define two pattern containments on \mathcal{A}_n . Let $a = a_1 a_2 \cdots a_m$ be a partial permutation of n , with $m \leq n$, and let $i < j < k$. Then $a_i a_j a_k$ is an occurrence of the pattern 31|2 if it is an occurrence of 312 such that at least one value of the interval $[a_j, a_k]$ does not appear in a . Moreover, we say that $a_i a_j a_k$ is an occurrence of the pattern $\bar{2}1\bar{3}$ if it is an occurrence of 213 such that $a_i = a_k - 1$.² By interpreting Theorem 7 in terms of partial permutations, we obtain easily:

Theorem 8. *A permutation π is (123, 312)-sortable if and only if $\alpha(\pi)$ avoids 31|2 and $\bar{2}1\bar{3}$.*

Let $\mathcal{A}_n(31|2, \bar{2}1\bar{3})$ be the set of partial permutations of n avoiding the two patterns 31|2 and $\bar{2}1\bar{3}$, and $\mathcal{A}_n(213)$ be the set of partial permutations of n avoiding the classical pattern 213.

Theorem 9. *For any $n \geq 1$, there is a bijection ϕ between $\mathcal{A}_n(31|2, \bar{2}1\bar{3})$ and $\mathcal{A}_n(213)$.*

Proof. Let us define recursively the map ϕ from $\mathcal{A}_n(31|2, \bar{2}1\bar{3})$ to $\mathcal{A}_n(213)$. If $\pi = \lambda$, then we set $\phi(\pi) = \lambda$. Otherwise, π has a unique decomposition of the form $\pi = A \min(\pi) B$ where A and B are disjoint partial permutations of n . We distinguish three cases:

- (i) If at least one of A or B is empty, then we set $\phi(\pi) = \phi(A) \min(\pi) \phi(B)$;
- (ii) If both A and B are not empty and $\min(A) > \max(B)$, then we set $\phi(\pi) = \phi(A) \min(\pi) \phi(B)$. It is worth noting that the hypothesis that π avoids 31|2 implies that any value $x \in [\min(\pi), \max(B)]$ occurs in B .

²This is analogous to the notion of bivincular pattern on classical permutations

(iii) Suppose that both A and B are not empty and $\min(A) < \max(B)$. Since π avoids $\bar{213}$, there exists $x \in [\min(A), \max(B)]$ such that x does not occur in π . We choose the smallest x with this property, so that any value of the interval $[\min(A), x]$ occurs in A . Moreover, since π avoids $31|2$, it must be $\max(A) = x - 1$. An illustration of this case is depicted in Figure 1. Let r be the maximum value of B that is lower than $\min(A)$ and consider the string B' obtained from B by decreasing by $x - r - 1$ all entries greater than x . Similarly, let A' be obtained from A by increasing by $\max(B) - x + 1$ all its entries. Obviously, A' and B' belong to $\mathcal{A}_n(31|2, \bar{213})$, whereas $\pi' = A' \min(\pi) B'$ contains $31|2$. Then we set $\phi(\pi) = \phi(A') \min(\pi) \phi(B')$ (see again Figure 1 for an illustration of this mapping). It is worth noting that the value $r + 1$ does not occur in both A' and B' , which implies that there exists $y \in [\min(\pi), r + 1]$ such that y does not occur in $\phi(B')$.

Next we prove that ϕ is an injective map. We proceed by induction on the length of partial permutations. Let π be a partial permutation. Due to the remarks at the end of (ii) and (iii), the image of π under ϕ satisfying (ii) is a partial permutation π' such that any value $x \in [\min(\pi), \max(\phi(B))]$ occurs in $\phi(B)$, which is not true for a permutation π satisfying (iii). Then, for two non-empty partial permutations π and σ in $\mathcal{A}_n(31|2, \bar{213})$, $\phi(\pi) = \phi(\sigma)$ implies that π and σ have the same length and they belong to the same case (i), (ii) or (iii). The recurrence hypothesis induces $\pi = \sigma$ which completes the induction.

Finally, observe that any partial permutation π avoiding 213 is of the form $A \min(\pi) B$ where $\min(A) > \max(B)$ and both A and B avoid 213 . According to the geometrical shape of π (as in the proof of injectivity), π fits exactly in one of the cases (i), (ii) and (iii) in the definition of ϕ . Therefore the surjectivity of ϕ can be showed by using its recursive definition and induction on A and B . We leave the details to the reader. \square

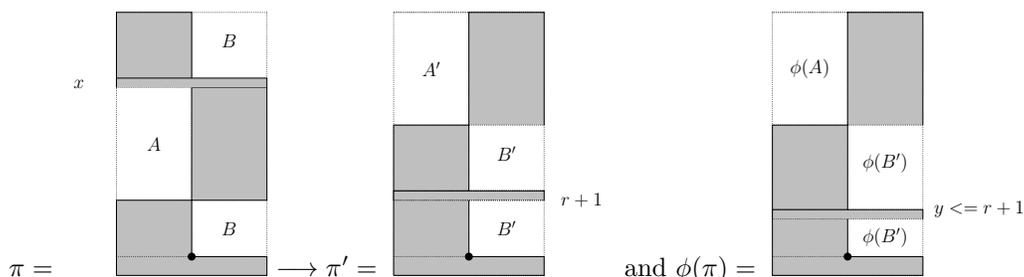


Figure 1: Illustration of ϕ in the case (iii) of the proof of Theorem 8.

Now, it is easy to enumerate the set $\mathcal{A}_n(213)$. Indeed any partial permutation $\pi \in \mathcal{A}_n(213)$ can be obtained by choosing k integers from $\{1, 2, \dots, n\}$ and then arranging them according to the partial order of a permutation in $\text{Av}_k(213)$ (there are c_k such permutations). Therefore, we have:

$$|\mathcal{A}_n(213)| = \sum_{k=0}^n \binom{n}{k} c_k,$$

the binomial transform of Catalan numbers (sequence A007317 in [16]). The enumeration of $\text{Sort}(123, 312)$ follows immediately.

Corollary 4. *For each $n \geq 1$, we have:*

$$|\text{Sort}_{n+1}(123, 312)| = \sum_{k=0}^n \binom{n}{k} c_k.$$

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