Two kinds of partial Motzkin paths with air pockets[*]

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Abstract

Motzkin paths with air pockets (MAP) are defined as a generalization of Dyck paths with air pockets by allowing some horizontal steps with certain conditions. In this paper, we introduce two generalizations. The first one consists of lattice paths in $\mathbb{N}^2$ starting at the origin, made of steps $U = (1, 1)$, $D_k = (1, -k)$, $k \geqslant 1$ and $H = (1, 0)$, where two down steps cannot be consecutive, while the second one are lattice paths in $\mathbb{N}^2$ starting at the origin, made of steps U, D_k and H, where each step D_k and H is necessarily followed by an up step, except for the last step of the path. We provide enumerative results for these paths according to the length, the type of the last step, and the height of its end-point. A similar study is made for these paths read from right to left. As a byproduct, we obtain new classes of paths counted by the Motzkin numbers. Finally, we express our results using Riordan arrays.

Keywords: Enumeration, Motzkin paths, kernel method, Riordan array.

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1 Introduction

In a recent paper [2], the authors introduce, study and enumerate special classes of lattice paths, called *Dyck paths with air pockets* (DAP for short). Such paths are non empty lattice paths in the first quadrant of $\mathbb{Z}^2$ starting at the origin, and consisting of up-steps $U = (1, 1)$ and down-steps $D_k = (1, -k)$, $k \geqslant 1$, where two down steps cannot be consecutive.

The *length* of a path is the number of its steps. These paths can be viewed as ordinary Dyck paths where each maximal run of down-steps is condensed into one large down step. As mentioned in [2], they also correspond to a stack evolution with (partial) reset operations that cannot be consecutive (see for instance [7]). Whenever the last point is on the x-axis, they prove that DAP of length n are in one-to-one correspondence with the peakless Motzkin paths of length $n - 1$. They also investigate the popularity of many patterns in these paths and they give asymptotic approximations. We also refer to [10] for the enumeration of DAP with respect to the length, the type (up or down) of the last step and the height of the end-point (i.e., the ordinate of the end-point). In a second work [3], the authors make a study for a generalization of these paths by allowing them to go below the x-axis. They call these paths Grand Dyck paths with air pockets (GDAP), and they also yield enumerative results for these paths according to the length and several restrictions on the height.

In this paper, we introduce two generalizations of partial Dyck paths of air pockets by allowing some possible horizontal steps $H = (1, 0)$ with some conditions. These two kinds of paths can be viewed as special partial Motzkin paths (lattice paths in $\mathbb{N}^2$ starting at the origin and made of $U = (1, 1)$, $D = (1, -1)$, and $H = (1, 0)$), where each maximal run of down-steps is condensed into one large down step.

Definition 1.1. A partial Motzkin path with air pocket of the first kind is a lattice path in $\mathbb{N}^2$ starting at the origin, consisting of steps U, D_k and H, where two down steps cannot be consecutive. Let $\mathcal{M}_1$ be the set of these paths.

Definition 1.2. A partial Motzkin path with air pocket of second kind is a lattice path in $\mathbb{N}^2$ starting at the origin, consisting of steps U, D_k and H, where any step H and D_k (except the last step of the path) is immediately followed by an up step U. Let $\mathcal{M}_2$ be the set of these paths.

Whenever these paths end on the x-axis, we call them *Motzkin paths with air pockets* of the first and second kinds, respectively. For short, we denote by PMAP all paths in $\mathcal{M}_i$, $i \in \{1, 2\}$, and we denote by MAP all paths ending on the x-axis.

For instance, $UUUDHUD_2 \in \mathcal{M}_1$ and $UUUDUHUD_2 \in \mathcal{M}_2$ are two PMAP of the first and second kinds, respectively. The paths $UUUDHUD_3UD$ and $UUUDUHUD_4$ UD are MAP of the first and second kinds, respectively. See also Figure 1 and Figure 5 for other examples of PMAP and MAP.

We also consider lattice paths obtained from MAP in $\mathcal{M}_1$ and $\mathcal{M}_2$ by reading them from right to left and by replacing down-steps with up-steps and *vice versa*. More precisely, we define the two sets $\mathcal{M}_1'$ and $\mathcal{M}_2'$ as follows.

Definition 1.3. Let $\mathcal{M}_1'$ be the set of paths in $\mathbb{N}^2$ starting at the origin, consisting of up steps $U_k = (1, k)$, $k \geqslant 1$, horizontal step H and down step D, where two up steps cannot be consecutive.

Definition 1.4. Let $\mathcal{M}_2'$ be the set of paths in $\mathbb{N}^2$ starting at the origin, consisting of up steps $U_k = (1, k)$, $k \geqslant 1$, horizontal step H and down step D, where any H and U_k, $k \geqslant 1$, (except the first step of the path) is preceded by a down step D.

All paths in $\mathcal{M}_1'$ and $\mathcal{M}_2'$ will be be called *PMAP from right to left*, and in the case where they end on the x-axis, we call them *MAP from right to left*. For instance, we have $UDU_3DHUD \in \mathcal{M}_1'$, $UDU_4DHDU \in \mathcal{M}_2'$, and the path $UDU_3DHUDDD$ (resp.

$UDU_4DHDUDDD$) is a MAP of the first (resp. second) kind from right to left. See also Figure 3 and Figure 7 for other examples of PMAP and MAP from right to left. Notice that $D_1 = D$ and $U_1 = U$, and we will use both notations.

Throughout the paper, and for each class of paths $\mathcal{M}_i$ and $\mathcal{M}'_i$, $i \in \{1, 2\}$, described above, we will use the following notations. For $k \geqslant 0$, we consider the generating function $f_k = f_k(z)$ (resp. $g_k = g_k(z)$, resp. $h_k = h_k(z)$), where the coefficient of z^n in the series expansion is the number of partial Motzkin paths with air pockets of length n ending at height k with an up-step, (resp. with a down-step, resp. with a horizontal step H).

We introduce the bivariate generating functions

$$F(z, u) = \sum_{k \geqslant 0} u^k f_k(z), \quad G(z, u) = \sum_{k \geqslant 0} u^k g_k(z), \text{ and } H(z, u) = \sum_{k \geqslant 0} u^k h_k(z).$$

For short, we also use the notation $F(u), G(u)$ and $H(u)$ for these functions, and we introduce the generating function

$$\texttt{Total}(z, u) = F(u) + G(u) + H(u).$$

Finally, for any bivariate generating function $H(z, u) = H(u)$, we will use the notation $[u^k]H(u)$ for the coefficient of u^k in $H(u)$.

The outline of this paper is the following. In Section 2, we present enumerative results for partial Motzkin paths with air pockets of the first kind, and for these paths when we read them from right to left. We provide bivariate generating functions that count these paths with respect to the length, the type of the last step (up, down or horizontal step) and the height of the end-point. In Section 3, we make a similar study for PMAP of second kind, and we present new classes of lattice paths counted by the well known Motzkin numbers. All these results are obtained algebraically by using the famous kernel method for solving several systems of functional equations. More precisely, Sections 2.1, 2.2, 3.1 and 3.2 have the same structure: we show how to obtain a system of equations involving f_k, g_k and h_k, and we apply the kernel method in order to provide some expressions of $F(u)$, $G(u)$ and $H(u)$. Finally, in Section 4 we express our results using Riordan arrays and we deduce closed forms for PMAP of length n ending at height k.

2 PMAP of the first kind

In this section, we focus on PMAP of the first kind, i.e. lattices paths in $\mathbb{N}^2$ starting at the origin, made of steps U, D_k and H, such that two down steps cannot be consecutive. The first subsection considers the paths in $\mathcal{M}_1$, while the second subsection handles the paths in $\mathcal{M}'_1$ (see Introduction for the definition of these two sets). We yield enumerative results for these paths according to the length, the type of the last step, and the height of its end-point.

2.1 PMAP in $\mathcal{M}_1$ - From left to right

In this part, we consider PMAP in $\mathcal{M}_1$. Figure 1 shows two examples of such paths.

Let P be a length n PMAP in $\mathcal{M}_1$ ending at height $k \geqslant 0$. If the last step of P is U, then $k \geqslant 1$ and P can be written $P = QU$ where Q is a length $(n-1)$ PMAP ending at height $k - 1$. So, we obtain the first relation $f_k = zf_{k-1} + zg_{k-1} + zh_{k-1}$ for $k \geqslant 1$, anchored with $f_0 = 1$ by considering the empty path. If the last step of P is a down step D_a, $a \geqslant 1$, then we have $P = QD_a$ where Q is a length $(n-1)$ PMAP ending at height
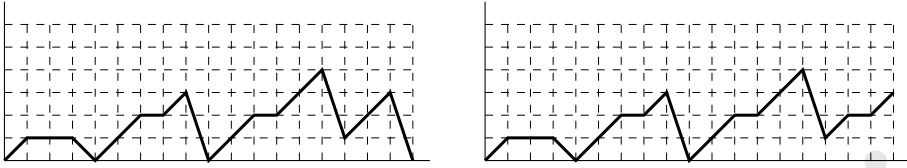
Figure 1: The left drawing shows a Motzkin path with air pockets of length 18 in $\mathcal{M}_1$. The right drawing shows a partial Motzkin path with air pockets of length 18 ending at height 3 in $\mathcal{M}_1$.

$\ell = a + k \geqslant k + 1$ with no down step at its end. So, we obtain the second relation $g_k = z \sum_{\ell \geqslant k+1} f_\ell + z \sum_{\ell \geqslant k+1} h_\ell$. If the last step of P is a horizontal step H, then we have $P = QH$ where Q is a length $(n-1)$ PMAP ending at height k, which implies $h_k = z f_k + z g_k + z h_k$.

Therefore, we have to solve the following system of equations.

$$
\begin{cases}
f_0 = 1, \text{ and } f_k = z f_{k-1} + z g_{k-1} + z h_{k-1}, & k \geqslant 1, \\
g_k = z \sum_{\ell \geqslant k+1} f_\ell + z \sum_{\ell \geqslant k+1} h_\ell, & k \geqslant 0, \\
h_k = z f_k + z g_k + z h_k, & k \geqslant 0.
\end{cases}
\tag{2.1}
$$

Summing the recursions in (2.1), we have:

$$
\begin{aligned}
F(u) &= 1 + z \sum_{k \geqslant 1} u^k f_{k-1} + z \sum_{k \geqslant 1} u^k g_{k-1} + z \sum_{k \geqslant 1} u^k h_{k-1} \\
&= 1 + zu F(u) + zu G(u) + zu H(u), \\
G(u) &= z \sum_{k \geqslant 0} u^k \Big(\sum_{\ell \geqslant k+1} f_\ell \Big) + z \sum_{k \geqslant 0} u^k \Big(\sum_{\ell \geqslant k+1} h_\ell \Big) \\
&= z \sum_{k \geqslant 1} f_k (1 + u + \ldots + u^{k-1}) + z \sum_{k \geqslant 1} h_k (1 + u + \ldots + u^{k-1}) \\
&= z \sum_{k \geqslant 1} \frac{u^k - 1}{u - 1} f_k + z \sum_{k \geqslant 1} \frac{u^k - 1}{u - 1} h_k \\
&= \frac{z}{u - 1} (F(u) - F(1) + H(u) - H(1)), \\
H(u) &= z F(u) + z G(u) + z H(u).
\end{aligned}
$$

Notice that we have $F(1) - H(1) = 1$ by considering the difference of the first and third equations. Now, setting $a := F(1)$ and solving these functional equations, we obtain

$$
\begin{aligned}
F(u) &= \frac{2au\,z^2 - u\,z^2 + uz + z^2 - u - z + 1}{u^2 z + u\,z^2 + z^2 - u - z + 1}, \\
G(u) &= -\frac{z\,(2auz + 2az - uz - 2a - z + 2)}{u^2 z + u\,z^2 + z^2 - u - z + 1}, \\
H(u) &= \frac{z\,(2az - u - 2z + 1)}{u^2 z + u\,z^2 + z^2 - u - z + 1}.
\end{aligned}
$$

In order to compute $a = F(1)$, we use the kernel method (see [1, 9]) on $F(u)$. This method consists in cancelling the denominator by finding u as an algebraic function of z, $s(z)$. So, if we substitute u with $s(z)$ in the numerator, then it necessarily equals zero (in order to counterbalance the cancellation of the denominator), which allows to find $a = F(1)$. Finally, we can deduce the generating function $F(u)$.

We can write the denominator (which is a polynomial in u of degree 2), as $z(u - r)$ $(u - s)$ with

$$r = \frac{1 - z^2 + \sqrt{z^4 - 4z^3 + 2z^2 - 4z + 1}}{2z},$$

$$s = \frac{1 - z^2 - \sqrt{z^4 - 4z^3 + 2z^2 - 4z + 1}}{2z}.$$

Replacing u with s (which have a Taylor expansion at $z = 0$) in order to cancel the numerator of $F(u)$, we obtain the equation

$$2as\, z^2 - s\, z^2 + sz + z^2 - s - z + 1 = 0.$$

Using $zrs = z^2 - z + 1$, a straightforward calculation provides

$$a = F(1) = H(1) + 1 = \frac{r(s - 1)}{2z}.$$

Finally, after simplifying by the factor $(u - s)$ in the numerators and denominators, we obtain

$$F(u) = \frac{r}{r - u}, \quad G(u) = \frac{r(s - 1) - z}{r - u}, \quad \text{and} \quad H(u) = \frac{1}{r - u}.$$

Extracting the coefficient of u^k in the series expansion of each generating function, we obtain

$$[u^k]F(u) := f_k = \frac{1}{r^k},$$

$$[u^k]G(u) := g_k = \frac{s - 1}{r^k} - \frac{z}{r^{k+1}},$$

$$[u^k]H(u) := h_k = \frac{1}{r^{k+1}}.$$

Theorem 2.1. *The bivariate generating function for the total number of PMAP in $\mathcal{M}_1$ with respect to the length and the height of the end-point is given by*

$$\text{Total}(z, u) = \frac{1}{z(r - u)},$$

and we have

$$[u^k]\text{Total}(z, u) = \frac{1}{zr^{k+1}}.$$

Finally, setting $t(n, k) := [z^n][u^k]\text{Total}(z, u)$, we have for $n \geqslant 2$, $k \geqslant 1$,

$$t(n, k) = t(n, k - 1) + t(n - 1, k) - t(n - 1, k - 2) - t(n - 2, k) - t(n - 2, k - 1),$$

and setting $t_n := t(n, 0)$, then we have $t_0 = t_1 = 1$, and for $n \geqslant 2$,

$$t_n = t_{n-1} + t_{n-2} + \sum_{k=0}^{n-3} t_k t_{n-k-3} + \sum_{k=2}^{n-1} (t_k - t_{k-1})\, t_{n-k-1}.$$

Proof. Since $\texttt{Total}(z, u) = F(u) + G(u) + H(u)$, the first two equalities are immediately deduced from the previous results. The third equality is obtained using the Mathematica package $\texttt{Guess.m}$ (written by Manuel Kauers [6]) for guessing recurrence relations. After this, it suffices to check algebraically:

$$\texttt{Total}(z, u) = (u + z - u^2 z - z^2 - uz^2)\texttt{Total}(z, u) - u + \frac{(z^2 - z + 1)(1 + rs - z)}{r}.$$

Now, let us prove the last equality. Any length n MAP of the first kind is of the form (i) HP, or (ii) UDP, or (iii) $UPHDQ$ where P, Q are some MAP so that the length k of P lies into $[0, n - 3]$, and the length of Q is $n - k - 3$, or (iv) $P^\sharp Q$ where $P^\sharp = UP'D_\ell$, $\ell \geqslant 1$, and P' is a PMAP such that $P'D_{\ell-1}$ is a MAP of length k lying into $[2, n - 1]$, and the length of Q is $n - k - 1$. Taking into account all these cases, we obtain the result. $\square$

Let $\mathcal{T}$ be the infinite matrix $\mathcal{T} := [t(n, k)]_{n \geqslant 0, k \geqslant 0}$. The first few rows of the matrix $\mathcal{T}$ are

$$\mathcal{T} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
2 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
5 & 5 & 3 & 1 & 0 & 0 & 0 & 0 & \cdots \\
13 & 14 & 9 & 4 & 1 & 0 & 0 & 0 & \cdots \\
36 & 40 & 28 & 14 & 5 & 1 & 0 & 0 & \cdots \\
105 & 118 & 87 & 48 & 20 & 6 & 1 & 0 & \cdots \\
317 & 359 & 273 & 161 & 75 & 27 & 7 & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.$$

Corollary 2.2. *The generating function that counts the PMAP in $\mathcal{M}_1$ with respect to the length is given by*

$$\texttt{Total}(z, 1) = \frac{1}{z(r - 1)}.$$

The first few terms of the series expansion of $\texttt{Total}(z, 1)$ are $1 + 2z + 5z^2 + 14z^3 + 41z^4 + 124z^5 + 385z^6 + 1220z^7 + 3929z^8 + 12822z^9 + O(z^{10})$, which correspond to the sequence A159771 in [13] counting the n-leaf binary trees that do not contain $(()((()())(()())()))$ as a subtree (see [11]). See Figure 2 for an illustration of the 14 PMAP of length 3.

Corollary 2.3. *The generating function that counts the MAP in $\mathcal{M}_1$ with respect to the length is given by*

$$\texttt{Total}(z, 0) = \frac{1}{zr}.$$

The first few terms of the series expansion of $\texttt{Total}(z, 0)$ are $1 + z + 2z^2 + 5z^3 + 13z^4 + 36z^5 + 105z^6 + 317z^7 + 982z^8 + 3105z^9 + O(z^{10})$ which correspond to the sequence A114465 in [13] counting Dyck paths of length $2n$ having no ascents of length 2 that start at an odd level. We leave open the question of finding a constructive bijection between these sets.
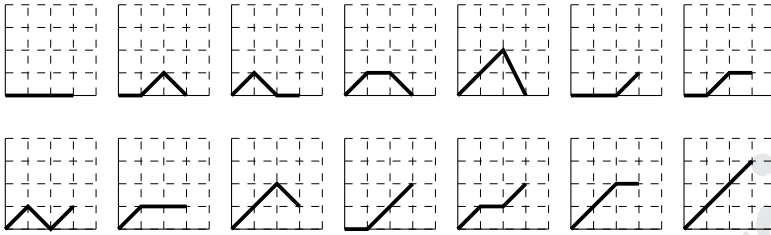
Figure 2: The 14 PMAP of length three in $\mathcal{M}_1$. Notice that five paths end on the x-axis, five paths end at height 1, three paths end at height 2, and one path ends at height 3, which correspond to the fourth row of $\mathcal{T}$.

2.2 PMAP in $\mathcal{M}_1'$ - From right to left

Here, we consider the paths of the previous section, but we read them from right to left. This means that down steps become up steps and *vice versa*, and horizontal steps are unchanged, which implies that two up steps cannot be consecutive now. See Definition 1.3 and Figure 3 for two examples of such paths.
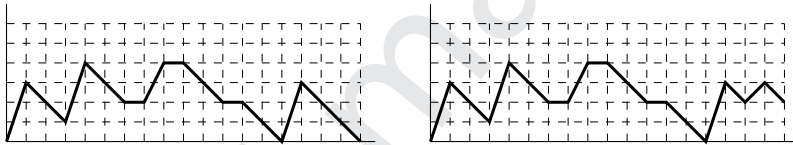

Figure 3: The left drawing shows a Motzkin path with air pockets of length 18 in $\mathcal{M}_1'$. The right drawing shows a partial Motzkin path with air pockets of length 18 ending at height 2 in $\mathcal{M}_1'$.

Let P be a length n PMAP in $\mathcal{M}_1'$ ending at height $k \geqslant 0$. If the last step of P is U_a, $a \geqslant 1$, then $k \geqslant a$ and we have $P = QU_a$ where Q is a length $(n-1)$ PMAP ending at height $\ell = k - a$ with a horizontal step or a down step. So, we obtain the first relation $f_k = z + z(g_0 + g_1 + \ldots + g_{k-1}) + z(h_0 + h_1 + \ldots + h_{k-1})$ for $k \geqslant 1$, anchored with $f_0 = 1$ by considering the empty path. If the last step of P is a down step D, then we have $P = QD$ where Q is a length $(n-1)$ PMAP ending at height $k + 1$. So, we obtain the second relation $g_k = zf_{k+1} + zg_{k+1} + zh_{k+1}$. If the last step of P is a horizontal step H, then we have $P = QH$ where Q is a length $(n-1)$ PMAP ending at height k, which implies $h_k = zf_k + zg_k + zh_k$.

Therefore, we have to solve the following system of equations.

$$\begin{cases} f_0 = 1, \\ f_k = z + z(g_0 + g_1 + \ldots + g_{k-1}) + z(h_0 + h_1 + \ldots + h_{k-1}), \quad k \geqslant 1, \\ g_k = zf_{k+1} + zg_{k+1} + zh_{k+1}, \quad k \geqslant 0, \\ h_k = zf_k + zg_k + zh_k, \quad k \geqslant 0. \end{cases} \tag{2.2}$$

Summing the recursions in (2.2), we have:

$$
\begin{aligned}
F(u) &= 1 + \frac{zu}{1-u} + z\sum_{k\geqslant 1}(g_0 + \ldots + g_{k-1})u^k + z\sum_{k\geqslant 1}(h_0 + \ldots + h_{k-1})u^k \\
&= 1 + \frac{zu}{1-u} + z\sum_{k\geqslant 0} g_k \frac{u^{k+1}}{1-u} + z\sum_{k\geqslant 0} h_k \frac{u^{k+1}}{1-u} \\
&= 1 + \frac{zu}{1-u}(1 + G(u) + H(u)), \\
G(u) &= z\sum_{k\geqslant 0} f_{k+1}u^k + z\sum_{k\geqslant 0} g_{k+1}u^k + z\sum_{k\geqslant 0} h_{k+1}u^k \\
&= \frac{z}{u}(F(u) - F(0) + G(u) - G(0) + H(u) - H(0)), \\
H(u) &= zF(u) + zG(u) + zH(u).
\end{aligned}
$$

Notice that we have $H(0) = \frac{z(1+G(0))}{1-z}$ by considering the third equation and $F(0) = 1$. Solving these functional equations using $a := G(0)$, we obtain

$$
\begin{aligned}
F(u) &= \frac{-u^2z^3 + a\,uz^2 + 3\,u^2z^2 - uz^3 - 3\,u^2z + 2\,uz^2 + u^2 + uz - z^2 - u + z}{(1-z)\,(u^2z^2 - u^2z + uz^2 + u^2 - u + z)}, \\
G(u) &= \frac{z\,(a\,uz^2 - a\,uz + a\,u + a\,z + uz - a)}{(-1+z)\,(u^2z^2 - u^2z + uz^2 + u^2 - u + z)}, \\
H(u) &= \frac{\left(a\,uz^2 + u^2z^2 - a\,uz - 2\,u^2z + uz^2 + a\,z + u^2 - u + z\right)z}{(1-z)\,(u^2z^2 - u^2z + uz^2 + u^2 - u + z)}.
\end{aligned}
$$

In order to compute $a = G(0)$, we use the kernel method on $G(u)$. We can write the denominator (which is a polynomial in u of degree 2), as $(z-1)(z^2 - z + 1)(u-r)(u-s)$ with

$$
r = \frac{1 - z^2 + \sqrt{z^4 - 4z^3 + 2z^2 - 4z + 1}}{2(z^2 - z + 1)},
$$

$$
s = \frac{1 - z^2 - \sqrt{z^4 - 4z^3 + 2z^2 - 4z + 1}}{2(z^2 - z + 1)}.
$$

Replacing u with s (which have a Taylor expansion at $z = 0$) in order to cancel the numerator of $G(u)$, we obtain the equation

$$
a(sz^2 - sz + s + z - 1) + sz = 0.
$$

Using $sr(z^2 - z + 1) = z$, we deduce

$$
a = G(0) = \frac{1-r}{r} - sz.
$$

Finally, after simplifying by the factor $(u - s)$ in the numerators and denominators, we obtain

$$
F(u) = 1 + \frac{sru}{r-u}, \quad G(u) = \frac{s(1 - r + rz)}{r-u}, \quad \text{and} \quad H(u) = \frac{sr - sru(1-z)}{r-u},
$$

which implies that

$$[u^k]F(u) := f_k = [k = 0] + [k \neq 0] \cdot \frac{s}{r^{k-1}},$$

$$[u^k]G(u) := g_k = \frac{s(1 - r + rz)}{r^{k+1}},$$

$$[u^k]H(u) := h_k = \frac{s}{r^k} - [k \neq 0] \cdot \frac{(1 - z)s}{r^{k-1}},$$

where $[k = 0]$ (resp. $[k \neq 0]$) equals 1 whenever $k = 0$ (resp. $k \neq 0$), and 0 otherwise.

Theorem 2.4. *The bivariate generating function for the total number of PMAP (read from right to left) with respect to the length and the height of the end-point is given by*

$$\texttt{Total}(z, u) = 1 + \frac{s(1 + rz + ruz)}{r - u},$$

and we have

$$[u^k]\texttt{Total}(z, u) = [k = 0] + \frac{s(rz + 1)}{r^{k+1}} + [k \neq 0] \cdot \frac{sz}{r^{k-1}}.$$

Finally, setting $t(n, k) := [z^n][u^k]\texttt{Total}(z, u)$, we have for $n \geqslant 2$, $k \geqslant 1$,

$$t(n, k) = t(n - 2, k - 1) + t(n - 2, k) - t(n - 1, k - 1) + t(n - 1, k + 1) + t(n, k - 1),$$

and setting $t_n := t(n, 0)$, we have $t_0 = t_1 = 1$, and for $n \geqslant 2$,

$$t_n = t_{n-1} + t_{n-2} + \sum_{k=0}^{n-3} t_k t_{n-k-3} + \sum_{k=2}^{n-1} (t_k - t_{k-1}) \, t_{n-k-1}.$$

Proof. The first two equalities are directly deduced from the previous results. Since we have

$$(u - u^2 z^2 - uz^2 + zu^2 - z - u^2)\texttt{Total}(z, u) + u^2(1 - z) - (1 + s - sz)u + s = 0,$$

we deduce the third relation. The last equality is already given in Theorem 2.1, since the number of MAP in $\mathcal{M}_1'$ is obviously equal to the number of MAP in $\mathcal{M}_1$. □

Let $\mathcal{T}$ be the infinite matrix $\mathcal{T} := [t(n, k)]_{n \geqslant 0, k \geqslant 0}$. The first few rows of the matrix $\mathcal{T}$ are

$$\mathcal{T} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 2 & 3 & 3 & 3 & 3 & 3 & \cdots \\ 5 & 8 & 10 & 12 & 14 & 16 & \cdots \\ 13 & 23 & 33 & 43 & 53 & 63 & \cdots \\ 36 & 69 & 107 & 149 & 195 & 245 & \cdots \\ 105 & 212 & 348 & 512 & 704 & 924 & \cdots \\ 317 & 665 & 1141 & 1753 & 2509 & 3417 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Since there is an infinite number of PMAP of length n, we do not provide an ordinary generating function (with respect to the length) for these paths. So, we get around this by counting PMAP ending on a point $(x, n - x)$ for a given $n \geqslant 0$.

Corollary 2.5. *The generating function that counts the partial PMAP ending on the line $y = n - x$ is given by*

$$\texttt{Total}(z, z) = 1 + \frac{s(1 + rz + rz^2)}{r - z}.$$

The first few terms of the series expansion of $\texttt{Total}(z, z)$ are $1 + z + 3z^2 + 9z^3 + 25z^4 + 73z^5 + 223z^6 + 697z^7 + 2217z^8 + 7161z^9 + O(z^{10})$, which correspond to the sequence A101499 in [13], which is a Chebyshev transform of the Catalan number that counts peakless Motzkin paths of length n where horizontal steps at level at least one come in 2 colors. See Figure 4 for an illustration of the 9 PMAP in $\mathcal{M}'_1$ ending on the line $y = 3 - x$.

Notice that, from Theorem 2.4 we have

$$\texttt{Total}(z, 0) = 1 + \frac{s(1 + rz)}{r},$$

which is obviously equal to the expression derived in Corollary 2.3 that counts MAP of the first kind with respect to the length.
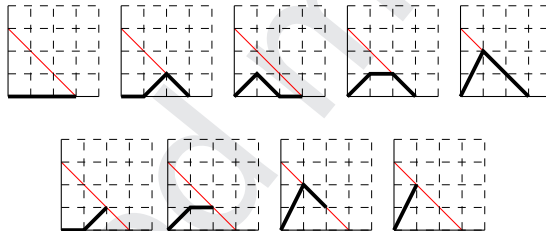
Figure 4: The 9 PMAP ending on the line $y = 3 - x$ in $\mathcal{M}'_1$. Notice that five paths end on the x-axis, three paths end at height 1, and one path ends at height 2, which correspond to the fourth diagonal of $\mathcal{T}$.

3 PMAP of the second kind

In this section, we focus on PMAP of the second kind. See Figure 5 for an illustration of such paths. The first subsection considers paths in $\mathcal{M}_2$, while the second handles paths in $\mathcal{M}'_2$. We yield enumerative results for these paths according to the length, the type of the last step, and the height of the end-point.

3.1 PMAP in $\mathcal{M}_2$ - From left to right

In this part, we consider PMAP in $\mathcal{M}_2$, i.e. lattice paths in $\mathbb{N}^2$ starting at the origin, consisting of steps U, D_k and H, and where any down step or horizontal step (except for the last step of the path) is immediately followed by an up step. Figure 5 shows two examples of such paths.

Figure 5: The left drawing shows a MAP of length 18 in $\mathcal{M}_2$. The right drawing shows a PMAP of length 18 ending at height 3 in $\mathcal{M}_2$.

Let P be a length n PMAP in $\mathcal{M}_2$ ending at height $k \geqslant 0$. If the last step of P is U, then $k \geqslant 1$ and we have $P = QU$ where Q is a length $(n-1)$ PMAP ending at height $k-1$. So, we obtain the first relation $f_k = z f_{k-1} + z g_{k-1} + z h_{k-1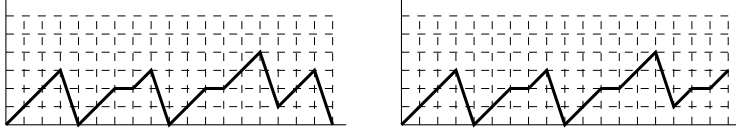}$ for $k \geqslant 1$, anchored with $f_0 = 1$ by considering the empty path. If the last step of P is a down step D_a, $a \geqslant 1$, then we have $P = Q D_a$ where Q is a length $(n-1)$ PMAP ending at height $\ell = a + k \geqslant k+1$ with an up step. So, we obtain the second relation $g_k = z \sum_{\ell \geqslant k+1} f_\ell$. If the last step of P is a horizontal step H, then we have $P = QH$ where Q is a length $(n-1)$ PMAP ending at height k with an up step, which implies $h_k = z f_k$.

So, we have to solve the following system of equations.

$$
\begin{cases}
f_0 = 1, \text{ and } f_k = z f_{k-1} + z g_{k-1} + z h_{k-1}, & k \geqslant 1, \\
g_k = z \sum_{\ell \geqslant k+1} f_\ell, & k \geqslant 0, \\
h_k = z f_k, & k \geqslant 0.
\end{cases}
\tag{3.1}
$$

Summing the recursions in (3.1), we have:

$$
\begin{aligned}
F(u) &= 1 + z \sum_{k \geqslant 1} u^k f_{k-1} + z \sum_{k \geqslant 1} u^k g_{k-1} + z \sum_{k \geqslant 1} u^k h_{k-1} \\
&= 1 + zu F(u) + zu G(u) + zu H(u), \\
G(u) &= z \sum_{k \geqslant 0} u^k \Big(\sum_{\ell \geqslant k+1} f_\ell \Big) = z \sum_{k \geqslant 1} \frac{u^k - 1}{u - 1} f_k \\
&= \frac{z}{u-1} (F(u) - F(1)), \\
H(u) &= z F(u).
\end{aligned}
$$

Now, setting $a := F(1)$ and solving these functional equations, we deduce

$$
F(u) = \frac{au\, z^2 - u + 1}{u^2 z^2 + u^2 z - uz - u + 1},
$$

$$
G(u) = -\frac{z \left(au\, z^2 + auz - a + 1 \right)}{u^2 z^2 + u^2 z - uz - u + 1}, \quad H(u) = \frac{z \left(au\, z^2 - u + 1 \right)}{u^2 z^2 + u^2 z - uz - u + 1}.
$$

In order to compute $a = F(1)$, we use the kernel method on $F(u)$. We can write the denominator (which is a polynomial in u of degree 2), as $(z^2 + z)(u - r)(u - s)$ with

$$
r = \frac{1 + z + \sqrt{-3z^2 - 2z + 1}}{2z\,(z+1)}, \text{ and } s = \frac{1 + z - \sqrt{-3z^2 - 2z + 1}}{2z\,(z+1)}.
$$

Replacing u with s (which have a Taylor expansion at $z = 0$) in order to cancel the numerator of $F(u)$, we obtain the equation

$$as\,z^2 - s + 1 = 0,$$

and thus

$$a = F(1) = \frac{s-1}{sz^2}.$$

Finally using $z(1+z)rs = 1$ and simplifying by the factor $(u - s)$ in the numerators and denominators, we obtain

$$F(u) = \frac{r}{r-u}, \quad G(u) = \frac{s-1}{sz(r-u)}, \quad \text{and} \quad H(u) = \frac{zr}{r-u},$$

which implies that

$$[u^k]F(u) := f_k = \frac{1}{r^k},$$

$$[u^k]G(u) := g_k = (1+z) \cdot \frac{s-1}{r^k},$$

$$[u^k]H(u) := h_k = \frac{z}{r^k}.$$

Theorem 3.1. *The bivariate generating function for the total number of PMAP with respect to the length and the height of the end-point is given by*

$$\texttt{Total}(z, u) = \frac{1}{z(r-u)},$$

and we have

$$[u^k]\texttt{Total}(z, u) = \frac{1}{zr^{k+1}}.$$

Finally, setting $t(n, k) = [z^n][u^k]\texttt{Total}(z, u)$, we have for $n \geqslant 2$, $k \geqslant 1$,

$$t(n, k) = t(n, k-1) + t(n-1, k-1) - t(n-1, k-2) - t(n-2, k-2),$$

and setting $t_n := t(n, 0)$, we have $t_0 = 1$, and for $n \geqslant 1$,

$$t_n = t_{n-1} + \sum_{k=1}^{n-2} t_k t_{n-1-k}.$$

Proof. The first two equalities are immediately deduced from the previous results. The third equality is obtained using the Mathematica package Guess.m ([6]) for guessing recurrence relations. After this, it suffices to check algebraically:

$$\texttt{Total}(z, u) = (u + uz - u^2 z - u^2 z^2)\texttt{Total}(z, u) - u(1 + z) + \frac{1}{zr}.$$

For the last equality, it suffices to remark that the generating function of the sequence $(t_n)_{n \geqslant 0}$, that is $1/(zr)$, generates a shift of the well known Motzkin sequence A001006 in in [13]. $\qquad\square$

Let $\mathcal{T}$ be the infinite matrix $\mathcal{T} := [t(n,k)]_{n \geqslant 0, k \geqslant 0}$. The first few rows of the matrix $\mathcal{T}$ are

$$\mathcal{T} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
2 & 3 & 3 & 1 & 0 & 0 & 0 & 0 & \cdots \\
4 & 6 & 6 & 4 & 1 & 0 & 0 & 0 & \cdots \\
9 & 13 & 13 & 10 & 5 & 1 & 0 & 0 & \cdots \\
21 & 30 & 30 & 24 & 15 & 6 & 1 & 0 & \cdots \\
51 & 72 & 72 & 59 & 40 & 21 & 7 & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.$$

Corollary 3.2. *The generating function that counts the PMAP with respect to the length is given by*

$$\texttt{Total}(z, 1) = \frac{1}{z(r-1)}.$$

The first few terms of the series expansion of $\texttt{Total}(z, 1)$ are $1 + 2z + 4z^2 + 9z^3 + 21z^4 + 51z^5 + 127z^6 + 323z^7 + 835z^8 + 2188z^9 + O(z^{10})$, which correspond to a shift of the sequence A001006 in [13] that counts the Motzkin paths of a given length. See Figure 6 for an illustration of the 9 paths of length 3.

Corollary 3.3. *The generating function that counts the MAP with respect to the length is given by*

$$\texttt{Total}(z, 0) = \frac{1}{zr}.$$

The first few terms of the series expansion of $\texttt{Total}(z, 0)$ are $1 + z + z^2 + 2z^3 + 4z^4 + 9z^5 + 21z^6 + 51z^7 + 127z^8 + 323z^9 + O(z^{10})$ which correspond to a shift of the sequence A001006 in [13] that counts Motzkin paths of a given length.

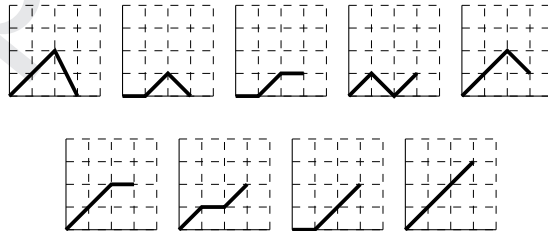Figure 6: The 9 PMAP of length three in $\mathcal{M}_2$. Notice that two paths end on the x-axis, three paths end at height 1, three paths end at height 2, and one path ends at height 3, which correspond to the fourth row of $\mathcal{T}$.

3.2 PMAP in $\mathcal{M}_2'$ - From right to left

Here, we consider the paths of the previous section, but we read them from right to left. This means that down steps become up steps and *vice versa*, and horizontal steps are unchanged, which implies that any up step or horizontal step (except the first step of the path) is preceded by a down step. See Definition 1.4 and Figure 7 for two examples of such paths.
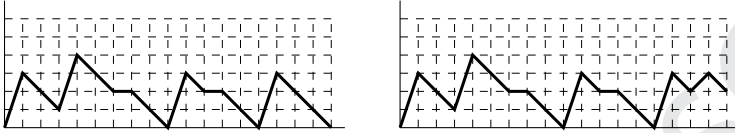
Figure 7: The left drawing shows a Motzkin path with air pockets of length 18 in $\mathcal{M}_2'$ (read from right to left). The right drawing shows a partial Motzkin path with air pockets of length 18 ending at height 2 in $\mathcal{M}_2'$.

Let P be a length n PMAP in $\mathcal{M}_2'$ ending at height $k \geqslant 0$. If the last step of P is $U_a, a \geqslant 1$, then $k \geqslant a$ and we have $P = QU_a$ where Q is a length $(n-1)$ PMAP ending at height $\ell = k - a$ with a down step. So, we obtain the first relation $f_k = z + z(g_0 + g_1 + \ldots + g_{k-1})$ for $k \geqslant 1$, anchored with $f_0 = 1$ by considering the empty path. If the last step of P is a down step D, then we have $P = QD$ where Q is a length $(n-1)$ PMAP ending at height $k+1$. So, we obtain the second relation $g_k = zf_{k+1} + zg_{k+1} + zh_{k+1}$. If the last step of P is a horizontal step H, then we have $P = QH$ where Q is a length $(n-1)$ PMAP ending at height k with a down step whenever it is nonempty. If $k \geqslant 1$, then we have $h_k = zg_k$; if $k = 0$ then we have $h_0 = z + zg_0$ where the monomial z corresponds to the path $P = H$.

So, we have to solve the following system of equations.

$$\begin{cases} f_0 = 1, \text{ and } f_k = z(1 + g_0 + g_1 + \ldots + g_{k-1}), \quad k \geqslant 1, \\ g_k = zf_{k+1} + zg_{k+1} + zh_{k+1}, \quad k \geqslant 0, \\ h_0 = z + zg_0, \text{ and } h_k = zg_k, \quad k \geqslant 1. \end{cases} \tag{3.2}$$

Using the same notations as in the previous sections, and summing the recursions in (3.2), we have:

$$\begin{aligned} F(u) &= 1 + \sum_{k \geqslant 1} u^k f_k = 1 + z \sum_{k \geqslant 1} (1 + g_0 + \ldots + g_{k-1}) u^k \\ &= 1 + \frac{zu}{1-u}(1 + G(u)), \\ G(u) &= z \sum_{k \geqslant 0} (f_{k+1} + g_{k+1} + h_{k+1}) u^k \\ &= \frac{z}{u}(F(u) - F(0) + G(u) - G(0) + H(u) - H(0)), \\ H(u) &= z + zG(u). \end{aligned}$$

Notice that $F(0) = 1$ and $H(0) = z + zG(0)$ by the third relation. Now, setting

$a := G(0)$ and solving these functional equations, we deduce

$$F(u) = \frac{a\,uz^3 + a\,uz^2 + uz^3 - u^2z + uz^2 + u^2 - uz + z^2 - u + z}{u^2 - uz + z^2 - u + z},$$

$$G(u) = -\frac{z\,(a\,uz + a\,u - a\,z + uz - a)}{u^2 - uz + z^2 - u + z},$$

$$H(u) = -\frac{z\,(a\,uz^2 + a\,uz - a\,z^2 + uz^2 - a\,z - u^2 + uz - z^2 + u - z)}{u^2 - uz + z^2 - u + z}.$$

In order to compute $a = G(0)$, we use the kernel method on $F(u)$. We can write the denominator (which is a polynomial in u of degree 2), as $(u - r)(u - s)$ with

$$r = \frac{1 + z + \sqrt{-3\,z^2 - 2\,z + 1}}{2}, \text{ and } s = \frac{1 + z - \sqrt{-3\,z^2 - 2\,z + 1}}{2}.$$

Replacing u with s (which have a Taylor expansion at $z = 0$) in oder to cancel the numerator of $F(u)$, we obtain the equation

$$a\,sz^3 + a\,sz^2 + sz^3 - s^2z + sz^2 + s^2 - sz + z^2 - s + z = 0.$$

Using $rs = z(1 + z)$, we deduce

$$a = G(0) = \frac{1 - r}{r}.$$

Finally, after simplifying by the factor $(u - s)$ in the numerators and denominators, we obtain

$$F(u) = \frac{u(z - 1) + r}{r - u}, \quad G(u) = \frac{1 - r}{r - u}, \quad \text{and} \quad H(u) = \frac{z(1 - u)}{r - u},$$

which implies that

$$[u^k]F(u) := f_k = \frac{1}{r^k} + [k \neq 0] \cdot \frac{z - 1}{r^k},$$

$$[u^k]G(u) := g_k = \frac{1 - r}{r^{k+1}},$$

$$[u^k]H(u) := h_k = \frac{z}{r^{k+1}} - [k \neq 0] \cdot \frac{z}{r^k}.$$

Theorem 3.4. *The bivariate generating function for the total number of PMAP with respect to the length and the height of the end-point is given by*

$$\texttt{Total}(z, u) = \frac{1 - u + z}{r - u}.$$

$$[u^k]\texttt{Total}(z, u) = \frac{z + 1}{r^{k+1}} - [k \neq 0] \cdot \frac{1}{r^k}.$$

Finally, setting $t(n, k) = [z^n][u^k]\texttt{Total}(z, u)$, we have for $n \geqslant 1$, $k \geqslant 1$,

$$t(n, k) = t(n, k - 1) - t(n - 1, k) + t(n - 2, k + 1) + t(n - 1, k + 1),$$

and setting $t_n := t(n, 0)$, we have $t_0 = 1$, and for $n \geqslant 2$,

$$t_n = t_{n-1} + \sum_{k=1}^{n-2} t_k t_{n-1-k}.$$

Proof. The proof are obtained *mutatis mutandis* as for the previous theorems.   □

Let $\mathcal{T}$ be the infinite matrix $\mathcal{T} := [t(n,k)]_{n \geqslant 0, k \geqslant 0}$. The first few rows of the matrix $\mathcal{T}$ are

$$\mathcal{T} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 4 & 6 & 8 & 10 & 12 & 14 & 16 & \cdots \\ 9 & 15 & 22 & 30 & 39 & 49 & 60 & \cdots \\ 21 & 36 & 54 & 75 & 99 & 126 & 156 & \cdots \\ 51 & 91 & 142 & 205 & 281 & 371 & 476 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Since there is an infinite number of PMAP of length n, we do not provide an ordinary generating function (with respect to the length) for these paths. So, we get around this by counting PMAP ending on a point $(x, n - x)$ for a given $n \geqslant 0$.

Corollary 3.5. *The generating function that counts the partial PMAP ending on the line $y = n - x$ is given by*

$$\mathtt{Total}(z, z) = \frac{1}{r - z}.$$

The first few terms of the series expansion of $\mathtt{Total}(z, z)$ are $1 + z + 2z^2 + 4z^3 + 9z^4 + 21z^5 + 51z^6 + 127z^7 + 323z^8 + 835z^9 + O(z^{10})$, which correspond to the sequence A001006 in [13] that counts the Motzkin paths with respect to the length. See Figure 8 for the illustration of the 9 PMAP in $\mathcal{M}'_2$ ending on the line $y = 4 - x$.

Notice that we obviously retrieve the results of Corollary 3.3, i.e., the generating function $\mathtt{Total}(z, 0)$ that counts the MAP with respect to the length is also a shift of the Motzkin sequence A001006 in [13].
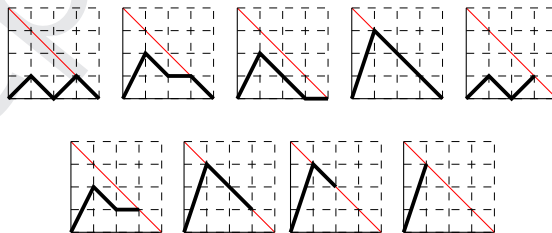

Figure 8:   The 9 PMAP ending on the line $y = 4 - x$ in $\mathcal{M}'_2$. Notice that four paths end on the x-axis, three paths end at height 1, one path ends at height 2, and one path ends at height 3, which correspond to the fifth diagonal of $\mathcal{T}$.

4   A Riordan array point of view

In this section, we make links between the previous matrices $\mathcal{T} = [t_{n,k}]_{n \geqslant 0, k \geqslant 0}$ and some Riordan arrays or almost Riordan arrays. We first give a short background on Riordan arrays [4, 5, 12].

An infinite column vector $(a_0, a_1, \dots)^T$ has generating function $f(z)$ if $f(z) = \sum_{n \geqslant 0} a_n z^n$. A *Riordan array* is an infinite lower triangular matrix whose k-th column has generating function $g(z) f(z)^k$ for all $k \geqslant 0$, for some formal power series $g(z)$ and $f(z)$, with $g(0) \neq 0$, $f(0) = 0$, and $f'(0) \neq 0$. Such a Riordan array is denoted by $(g(z), f(z))$. If we multiply this matrix by a column vector $(b_0, b_1, \dots)^T$ having generating function $h(z) = \sum_{n \geqslant 0} b_n z^n$, then the resulting column vector has generating function $g(z) h(f(z))$. This property is known as the *fundamental theorem of Riordan arrays*.

The product of two Riordan arrays $(g(z), f(z))$ and $(h(z), l(z))$ is defined by

$$(g(z), f(z)) * (h(z), l(z)) = (g(z) h(f(z)), l(f(z))) .$$

Under the operation "$*$", the set of all Riordan arrays is a group [12]. The identity element is $I = (1, z)$, and the inverse of $(g(z), f(z))$ is

$$(g(z), f(z))^{-1} = \left(1 / \left(g \circ \overline{f} \right) (z), \overline{f}(z) \right),$$

where $\overline{f}(z)$ denotes the compositional inverse of $f(z)$.

Moreover, if a matrix $\mathcal{T} = [t_{n,k}]_{n \geqslant 0, k \geqslant 0}$ is a *Riordan array* $(g(z), f(z))$ then $t_{n,k}$ equals the coefficient of $z^n u^k$ in the series expansion of the bivariate generating function

$$\frac{g(z)}{1 - u f(z)},$$

and we say that this is the bivariate generating function of the matrix $\mathcal{T}$.

An *almost Riordan array* $\mathcal{T}'$ is a matrix that consists of an initial column vector $(d_0, d_1, \dots)^T$ with generating function $g_0(z) = \sum_{n \geqslant 0} d_n z^n$, followed by a vertically shifted Riordan array $(g(z), f(z))$ as illustrated below.

$$\mathcal{T}' = \left(\begin{array}{c|ccccc} d_0 & 0 & \cdots & 0 & 0 & \cdots \\ d_1 & & & & & \\ d_2 & & & & & \\ d_3 & & & (g(z), f(z)) & & \\ d_4 & & & & & \\ d_5 & & & & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array} \right)$$

Therefore, the bivariate generating function for this matrix is given by

$$g_0(z) + zu \frac{g(z)}{1 - u f(z)}.$$

Finally, we will say that a matrix $\mathcal{M} = [m_{n,k}]_{n \geqslant 0, k \geqslant 0}$ is the *rectification of the Riordan array* $(g(z), f(z))$ whenever $m_{n,k}$ equals the coefficient of $z^n u^k$ in the series expansion of

the bivariate generating function

$$\frac{g(z)}{1 - u\frac{f(z)}{z}}.$$

In this section, $C(z) = \frac{1-\sqrt{1-4z}}{2z}$ will be the generating function where the coefficient of z^n in its series expansion is the Catalan number $c_n = \frac{1}{n+1}\binom{2n}{n}$.

4.1   Comment on Section 2.1

Let $\mathcal{T} = [t(n,k)]_{n \geqslant 0, k \geqslant 0}$ be the matrix given in Section 2.1 where $t(n,k)$ is the number of length n PMAP in $\mathcal{M}_1$ ending at height k.

Proposition 4.1. *The matrix $\mathcal{T} = [t(n,k)]_{n \geqslant 0, k \geqslant 0}$ is a Riordan array defined by*

$$\left(\frac{1}{1-z^2} C\left(\frac{z(1-z+z^2)}{(1-z^2)^2} \right), \frac{z}{1-z^2} C\left(\frac{z(1-z+z^2)}{(1-z^2)^2} \right) \right).$$

Proof. Considering r defined in Section 2.1, we have

$$[u^k]\texttt{Total}(z,u) = \frac{1}{zr^{k+1}} = \left(\frac{1}{zr} \right) \left(\frac{1}{r} \right)^k$$

$$= \frac{2}{1 - z^2 + \sqrt{1 - 4z + 2z^2 - 4z^3 + z^4}} \left(\frac{2z}{1 - z^2 + \sqrt{1 - 4z + 2z^2 - 4z^3 + z^4}} \right)^k.$$

Therefore, the array $\mathcal{T}$ satisfies

$$\mathcal{T} = \left(\frac{2}{1 - z^2 + \sqrt{1 - 4z + 2z^2 - 4z^3 + z^4}}, \frac{2z}{1 - z^2 + \sqrt{1 - 4z + 2z^2 - 4z^3 + z^4}} \right)$$

$$= \left(\frac{1 - z^2 - \sqrt{1 - 4z + 2z^2 - 4z^3 + z^4}}{2z(1 - z + z^2)}, \frac{1 - z^2 - \sqrt{1 - 4z + 2z^2 - 4z^3 + z^4}}{2(1 - z + z^2)} \right)$$

$$= \left(\frac{1}{1-z^2} C\left(\frac{z(1-z+z^2)}{(1-z^2)^2} \right), \frac{z}{1-z^2} C\left(\frac{z(1-z+z^2)}{(1-z^2)^2} \right) \right).$$

$\square$

Proposition 4.2. *The general term $t(n,k)$ equals*

$$\sum_{i=0}^{n-k} \binom{k + \frac{n-k-i}{2}}{\frac{n-k-i}{2}} \frac{1 + (-1)^{n-k-i}}{2} \sum_{j=0}^{i} C_{k+j,k} \sum_{m=0}^{j} \binom{j}{m}(-1)^m \sum_{p=0}^{m} \binom{m}{p}(-1)^p \binom{2j - 1 + \frac{i-j-m-p}{2}}{\frac{i-j-m-p}{2}} \frac{1 + (-1)^{i-j-m-p}}{2},$$

where $C_{n,k} = \frac{k+1}{n+1}\binom{2n-k}{n-k}$ is the general term of the Catalan matrix (A033184 in [13]).

Proof. Setting $Z = \frac{z(1-z+z^2)}{(1-z^2)^2}$, we have

$$
\begin{aligned}
t(n,k) &= [z^n]z^k \frac{1}{(1-z^2)^{k+1}} C(Z)^{k+1} \\
&= [z^{n-k}] \frac{1}{(1-z^2)^{k+1}} C(Z)^{k+1} \\
&= \sum_{i=0}^{n-k} [z^{n-k-i}] \frac{1}{(1-z^2)^{k+1}} [z^i]C(Z)^{k+1} \qquad \text{(Product rule [8])} \\
&= \sum_{i=0}^{n-k} [z^{n-k-i}] \frac{1}{(1-z^2)^{k+1}} \sum_{j=0}^{i} [z^j]C(z)^{k+1}[z^i]Z^j \quad \text{(Composition rule [8])} \\
&= \sum_{i=0}^{n-k} [z^{n-k-i}] \frac{1}{(1-z^2)^{k+1}} \sum_{j=0}^{i} [z^j] \frac{1}{z^k} C(z)(zC(z))^k [z^i]Z^j \\
&= \sum_{i=0}^{n-k} \binom{k + \frac{n-k-i}{2}}{\frac{n-k-i}{2}} \frac{1+(-1)^{n-k-i}}{2} \sum_{j=0}^{i} C_{k+j,k}[z^i]Z^j \qquad \text{(See [8]).}
\end{aligned}
$$

Since we have

$$
\begin{aligned}
[z^i]Z^j &= [z^i]\left(\frac{z(1-z+z^2)}{(1-z^2)^2} \right)^j \\
&= \sum_{m=0}^{j} \binom{j}{m}(-1)^m \sum_{p=0}^{m} \binom{m}{p}(-1)^p \binom{2j-1+\frac{i-j-m-p}{2}}{\frac{i-j-m-p}{2}} \frac{1+(-1)^{i-j-m-p}}{2},
\end{aligned}
$$

the result follows. $\qquad \square$

4.2 Comment on Section 2.2

Let $\mathcal{T} = [t(n,k)]_{n \geqslant 0, k \geqslant 0}$ be the matrix given in Section 2.2 where $t(n,k)$ is the number of length n PMAP in $\mathcal{M}'_1$ ending at height k.

Proposition 4.3. *The matrix* $\mathcal{T} = [t(n,k)]_{n \geqslant 0, k \geqslant 0}$ *can be written*

$$
\mathcal{T} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
1 & 1 & 1 & 1 & 1 & 1 & \cdots \\
2 & 3 & 3 & 3 & 3 & 3 & \cdots \\
5 & 8 & 10 & 12 & 14 & 16 & \cdots \\
13 & 23 & 33 & 43 & 53 & 63 & \cdots \\
36 & 69 & 107 & 149 & 195 & 245 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix} = A \cdot B
$$

where

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 3 & 0 & 0 & 0 & 0 & \cdots \\ 5 & 8 & 2 & 0 & 0 & 0 & \cdots \\ 13 & 23 & 10 & 0 & 0 & 0 & \cdots \\ 36 & 69 & 38 & 4 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 0 & 0 & 1 & 2 & 3 & 4 & \cdots \\ 0 & 0 & 0 & 1 & 3 & 6 & \cdots \\ 0 & 0 & 0 & 0 & 1 & 4 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

are defined as follows:

- *The matrix $B = [b_{n,k}]_{n,k \geqslant 0}$ is defined by $b_{0,0} = 1$, and $b_{n,0} = b_{0,n} = 0$ if $n \geqslant 1$, and $b_{n,k} = \binom{k-1}{n-1}$ otherwise, which is a kind of Pascal matrix.*

- *The matrix $A = [a_{n,k}]_{n,k \geqslant 0}$ is the almost Riordan array with initial column of generating function*

$$g_0(z) = \frac{1 - z^2 - \sqrt{1 - 4z + 2z^2 - 4z^3 + z^4}}{2z(1 - z + z^2)},$$

followed by the shifted Riordan array

$$\left(\frac{1 - 3z + z^2 - z^3 - (1-z)\sqrt{1 - 4z + 2z^2 - 4z^3 + z^4}}{2z^3(1 - z + z^2)}, \frac{1 - 2z - z^2 - \sqrt{1 - 4z + 2z^2 - 4z^3 + z^4}}{2z} \right).$$

The second column of $\mathcal{T}$ $1, 3, 8, 23, 69, \ldots$ with generating function

$$\frac{1 - 3z + z^2 - z^3 - (1-z)\sqrt{1 - 4z + 2z^2 - 4z^3 + z^4}}{2z^3(1 - z + z^2)}$$

is the convolution of the first column of $\mathcal{T}$ $1, 1, 2, 5, 13, 36, \ldots$ (A114465 in [13]) and the sequence $1, 2, 4, 10, 28, \ldots$ (A187256 in [13]).

Proof. An almost Riordan array is represented by an initial column vector with generating function $g_0(z)$, followed by a vertically shifted Riordan array $(g(z), f(z))$. The bivariate generating function of this matrix is then given by $g_0(z) + zu\frac{g(z)}{1 - uf(z)}$. In our case, for the almost Riordan array A, we have

$$g_0(z) = \frac{1 - z^2 - \sqrt{1 - 4z + 2z^2 - 4z^3 + z^4}}{2z(1 - z + z^2)},$$

$$g(z) = \frac{1 - 3z + z^2 - z^3 - (1-z)\sqrt{1 - 4z + 2z^2 - 4z^3 + z^4}}{2z^3(1 - z + z^2)},$$

$$f(z) = \frac{1 - 2z - z^2 - \sqrt{1 - 4z + 2z^2 - 4z^3 + z^4}}{2z}.$$

We let $G(z, u) = g_0(z) + zu\frac{g(z)}{1-uf(z)}$, the bivariate generating function of the almost Riordan array. Now, it suffices to check that the generating function corresponding to the matrix $A \cdot B$, that is

$$G(z, \frac{u}{1-u}) = \frac{(1-u+uz)(1-2z+2uz-(1+2u)z^2 - \sqrt{1-4z+2z^2-4z^3+z^4}}{2(1-z+z^2)(u(u-1)+(1-u^2)z+u(u+1)z^2)},$$

coincides with the generating function $\texttt{Total}(z, u)$ of the matrix $\mathcal{T}$. □

Proposition 4.4. *The matrix*

$$[t(n,k)]_{n\geqslant 1, k\geqslant 1} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & \cdots \\ 3 & 3 & 3 & 3 & 3 & \cdots \\ 8 & 10 & 12 & 14 & 16 & \cdots \\ 23 & 33 & 43 & 53 & 63 & \cdots \\ 69 & 107 & 149 & 195 & 245 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is the rectification of the Riordan array $(g(z), h(z))$ with

$$h(z) = \frac{1 - z^2 - \sqrt{1 - 4z + 2z^2 - 4z^3 + z^4}}{2}, \quad and$$

$$g(z) = \frac{1 - 3z + z^2 - z^3 - (1-z)\sqrt{1 - 4z + 2z^2 - 4z^3 + z^4}}{2z^3(1 - z + z^2)}.$$

Proof. It suffices to check that the generating function of $\mathcal{T}$, i.e. $\texttt{Total}(z, u)$, given in Theorem 2.4, equals to

$$\texttt{Total}(z, 0) + zu\frac{g(z)}{1 - u\frac{h(z)}{z}}.$$

□

We can express $h(z)$ and $g(z)$, respectively, in the following form

$$h(z) = \frac{z(1 - z + z^2)}{1 - z^2}C\left(\frac{z(1 - z + z^2)}{(1 - z^2)^2}\right),$$

$$g(z) = \frac{1}{1 - 3z + z^2 - z^3}C\left(\frac{z^3(1 - z + z^2)}{(1 - 3z + z^2 - z^3)^2}\right).$$

Then $g(z)$ expands to give the first column $1, 3, 8, 23, \ldots$, whose n-th term v_n can be expressed

$$v_n = \sum_{k=0}^{n}\sum_{j=0}^{k}\binom{k}{j}(-1)^j\sum_{i=0}^{j}\binom{j}{i}(-1)^i\sum_{\ell=0}^{n-3k-j-i}\binom{2k+\ell}{\ell}\sum_{m=0}^{\ell}\binom{\ell}{m}3^{\ell-m}(-1)^m\binom{m}{n - 3k - j - i - \ell - m}(-1)^{n-3k-j-i-\ell-m}c_k.$$

Using v_n, we can deduce the following.

Proposition 4.5. *The general term* $t(n,k)$ *equals*

$$\sum_{i=0}^{n+k} v_{n+k-i} \sum_{j=0}^{i} \sum_{m=0}^{j} M_{m,k} \sum_{p=0}^{m} \binom{m}{p}(-1)^p \sum_{q=0}^{p} \binom{p}{q}(-1)^q \binom{2m-1+\frac{i-m-p-q}{2}}{\frac{j-m-p-q}{2}} \frac{1+(-1)^{j-m-p-q}}{2} \binom{k}{\frac{i-j}{2}}(-1)^{\frac{i-j}{2}} \frac{1+(-1)^{i-j}}{2},$$

where

$$M_{n,k} = \begin{cases} [k=0] & \text{if } n=0, \\ \frac{n}{k}\binom{2n-k-1}{n-k} & \text{otherwise,} \end{cases}$$

is the general term of Riordan array $(1, zC(z))$ *(see* A106566 *in* [13]*).*

4.3 Comment on Section 3.1

Let $\mathcal{T} = [t(n,k)]_{n\geqslant 0, k\geqslant 0}$ be the matrix given in Section 3.1 where $t(n,k)$ is the number of length n PMAP in $\mathcal{M}_2$ ending at height k.

Proposition 4.6. *The matrix* $\mathcal{T} = [t(n,k)]_{n\geqslant 0, k\geqslant 0}$ *is the Riordan array*

$$\mathcal{T} = (1 + zM(z), z(1 + zM(z)))$$
$$= \left(C\left(\frac{z}{1+z}\right), zC\left(\frac{z}{1+z}\right) \right),$$

where $M(z) = \frac{1-z-\sqrt{1-2z-3z^2}}{2z^2}$ *is the generating function of the Motzkin numbers (see* A001006 *in* [13]*).*

Proof. It suffices to check that $\text{Total}(z,u)$ given in Theorem 3.1 satisfies

$$\text{Total}(z,u) = \frac{C\left(\frac{z}{1+z}\right)}{1 - uzC\left(\frac{z}{1+z}\right)},$$

and that $1 + zM(z) = C\left(\frac{z}{1+z}\right)$. $\square$

This triangle $\mathcal{T}$ corresponds to A091836 in [13] where the coefficient of row $n-1$ and column k is the number of Motzkin paths of length n having k points on the horizontal axis (besides the first and last point). As mentioned in [13], we obtain

$$t(n,k) = \begin{cases} 1, & \text{if } n=k, \\ \frac{k+1}{n+1}\sum_{j=1}^{n-k} j(-1)^{n-k-j}\binom{n+j}{j}\sum_{i=0}^{n-k}\frac{1}{n-k}\binom{i}{n-k-i+j}\binom{n-k}{i}, & \text{otherwise.} \end{cases}$$

A second expression for $t(n,k)$ is given by the following proposition.

Proposition 4.7. *The general term* $t(n,k)$ *of the Riordan array* $(1+zM(z), z(1+zM(z)))$ *is given by*

$$t(n,k) = \begin{cases} 1 & \text{if } n=k, \\ \frac{k+1}{n-k}\sum_{j=0}^{k}\binom{k}{j}\sum_{i=0}^{n-k}\binom{n-k}{i}\binom{i}{n-k-i-j-1} & \text{otherwise.} \end{cases}$$

Proof. We prove this using Lagrange inversion, using the fact that

$$\frac{z}{1 + z + z^2} M\left(\frac{z}{1 + z + z^2}\right) = z,$$

which means that the compositional inverse $(zM(z))^{-1}$ of $zM(z)$ is

$$(zM(z))^{-1} = \frac{z}{1 + z + z^2}.$$

Thus we have

$$
\begin{aligned}
t(n,k) &= [z^n](1 + zM(z))(z(1 + zM(z)))^k \\
&= [z^{n-k}](1 + zM(z))^{k+1} \\
&= [z^{n-k}]G(zM(z)), \quad \text{with} \quad G(z) = (1+z)^{k+1} \\
&= \frac{1}{n-k}[z^{n-k-1}]G'(z)\left(\frac{z}{(zM(z))^{-1}}\right)^{n-k} \quad \text{(Lagrange inversion)} \\
&= \frac{1}{n-k}[z^{n-k-1}](k+1)(1+z)^k(1+z+z^2)^{n-k} \\
&= \frac{k+1}{n-k}[z^{n-k}]\sum_{j=0}^{k}\binom{k}{j}z^j\sum_{i=0}^{n-k}\binom{n-k}{i}z^i(1+z)^i \\
&= \frac{k+1}{n-k}[z^{n-k}]\sum_{j=0}^{k}\binom{k}{j}z^j\sum_{i=0}^{n-k}\binom{n-k}{i}z^i\sum_{\ell=0}^{i}\binom{i}{\ell}z^\ell \\
&= \frac{k+1}{n-k}\sum_{j=0}^{k}\binom{k}{j}\sum_{i=0}^{n-k}\binom{n-k}{i}\binom{i}{n-k-i-j-1}.
\end{aligned}
$$

$\square$

Remark 4.8. The Riordan array $(1 + zM(z), z(1 + zM(z)))$ is a pseudo-involution in the Riordan group (see [5, Example 8]), that is, the matrix $[(-1)^k t_{n,k}]_{n,k \geqslant 0}$ is idempotent. Thus, this work yields a significant lattice path interpretation of this array.

4.4 Comment on Section 3.2

Let $\mathcal{T} = [t(n,k)]_{n \geqslant 0, k \geqslant 0}$ be the matrix given in Section 3.2 where $t(n,k)$ is the number of length n PMAP in $\mathcal{M}'_2$ ending at height k.

Proposition 4.9. *The matrix $\mathcal{T} = [t(n,k)]_{n \geqslant 0, k \geqslant 0}$ can be written*

$$
\mathcal{T} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
1 & 1 & 1 & 1 & 1 & 1 & \cdots \\
1 & 1 & 1 & 1 & 1 & 1 & \cdots \\
2 & 3 & 4 & 5 & 6 & 7 & \cdots \\
4 & 6 & 8 & 10 & 12 & 14 & \cdots \\
9 & 15 & 22 & 30 & 39 & 49 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix} = A \cdot B
$$

where

$$
A = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
1 & 1 & 0 & 0 & 0 & 0 & \cdots \\
1 & 1 & 0 & 0 & 0 & 0 & \cdots \\
2 & 3 & 1 & 0 & 0 & 0 & \cdots \\
4 & 6 & 2 & 0 & 0 & 0 & \cdots \\
9 & 15 & 7 & 1 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\quad and \quad
B = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 1 & 1 & 1 & 1 & 1 & \cdots \\
0 & 0 & 1 & 2 & 3 & 4 & \cdots \\
0 & 0 & 0 & 1 & 3 & 6 & \cdots \\
0 & 0 & 0 & 0 & 1 & 4 & \cdots \\
0 & 0 & 0 & 0 & 0 & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
$$

are defined as follows:

- *The matrix $B = [b_{n,k}]_{n,k \geqslant 0}$ is defined by $b_{0,0} = 1$, and $b_{n,0} = b_{0,n} = 0$ if $n \geqslant 1$, and $b_{n,k} = \binom{k-1}{n-1}$ otherwise, which is the same as in* Proposition 4.3.

- *The matrix $A = [a_{n,k}]_{n,k \geqslant 0}$ is the almost Riordan array with initial column whose generating function is*

$$
g_0(z) = 1 + zM(z) = \frac{1 + z - \sqrt{1 - 2z - 3z^2}}{2z},
$$

which is followed by the shifted Riordan array $(g(z), z^2 g(z))$ where

$$
g(z) = \frac{1 - z - 2z^2 - \sqrt{1 - 2z - 3z^2}}{2z^3(1 + z)}.
$$

Proof. The almost Riordan array A has generating function

$$
g_0(z) + zu \frac{g(z)}{1 - z^2 u g(z)}.
$$

Setting $G(z, u) = g_0(z) + zu \frac{g(z)}{1 - z^2 u g(z)}$, it suffices to check that the generating function corresponding to the matrix $A \cdot B$, that is $G(z, \frac{u}{1-u})$, coincides with the generating function $\mathtt{Total}(z, u)$ of the matrix $\mathcal{T}$ given in Theorem 3.4. $\qquad \square$

Proposition 4.10. *The matrix*

$$
[t(n, k)]_{n \geqslant 1, k \geqslant 1} = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & \cdots \\
1 & 1 & 1 & 1 & 1 & \cdots \\
3 & 4 & 5 & 6 & 7 & \cdots \\
6 & 8 & 10 & 12 & 14 & \cdots \\
15 & 22 & 30 & 39 & 49 & \cdots \\
36 & 54 & 75 & 99 & 126 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
$$

*is the rectification of the Riordan array $(M(z), zR(z))$ where $M(z) = \frac{1 - z - \sqrt{1 - 2z - 3z^2}}{2z^2}$ is the generating function of the Motzkin numbers, and $R(z) = \frac{1 + z - \sqrt{1 - 2z - 3z^2}}{2z(1+z)}$ is the generating function of the Riordan numbers (*A005043 *in* [13]*).*

Proof. It suffices to check that the generating function of $\mathcal{T}$, i.e. $\texttt{Total}(z, u)$, given in Theorem 3.4, equals to $g_0(z) + zu\frac{M(z)}{1-u\frac{zR(z)}{z}}$. $\square$

We let m_n denote the n-th Motzkin number $m_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k}c_k$ where c_k is the k-th Catalan defined above.

Corollary 4.11. *We have*

$$t(n,k) = \begin{cases} [k=0] & \text{if } n=0, \\ r(n-1,k) & \text{otherwise,} \end{cases}$$

where

$$r(n,k) = \sum_{i=0}^{n} \frac{m_i \cdot (k + [n=i])}{n+k-i+[n=i]} \sum_{j=0}^{n-i} (-1)^{n-i-j} \binom{n+k-i+j-1}{j} \sum_{\ell=0}^{n+k-i} \binom{n+k-i}{\ell} \binom{\ell}{n-i-j-\ell}.$$

Proof. We have $(M(z), zR(z))^{-1} = \left(\frac{(1-x)^2}{1-x+x^2}, \frac{x(1-x)}{1-x+x^2} \right)$. If we denote by $(v(z), u(z))$ this inverse Riordan array, then we obtain

$$(M(z), zR(z)) = \left(\frac{1}{v(\bar{u}(z))}, \bar{u}(z) \right),$$

where $\bar{u}$ is the compositional inverse [8] of u. Using the definition of a Riordan array, and Lagrange inversion, we find that the Riordan array $(M(z), zR(z))$ has general term $\tilde{r}(n,k)$ given by

$$\sum_{i=0}^{n} \frac{m_i \cdot (k + [n=k+i])}{n-i+[n=k+i]} \sum_{j=0}^{n-k-i} (-1)^{n-k-i-j} \binom{n-i+j-1}{j} \sum_{\ell=0}^{n-i} \binom{n-i}{\ell} \binom{\ell}{n-k-i-j-\ell}.$$

This array begins as follows:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 1 & 1 & 0 & 0 & 0 & \cdots \\ 4 & 3 & 1 & 1 & 0 & 0 & \cdots \\ 9 & 6 & 4 & 1 & 1 & 0 & \cdots \\ 21 & 15 & 8 & 5 & 1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

To rectify this array and thus to obtain the array $(M(z), R(z))$, we change n to $n+k$, and we obtain $r(n,k)$ above. To this array we must now prepend the row $(1,0,0,0,\ldots)$, and the result follows. $\square$

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