Skew Dyck paths with air pockets

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Abstract

We yield bivariate generating function for the number of n-length partial skew Dyck paths with air pockets (DAPs) ending at a given ordinate. We also give an asymptotic approximation for the average ordinate of the endpoint in all partial skew DAPs of a given length. Similar studies are made for two subclasses of skew DAPs, namely valley-avoiding and zigzagging, valley-avoiding skew DAPs. We express these results as Riordan arrays. Finally, we present two one-to-one correspondences with binary words avoiding the patterns 00 and 0110, and palindromic compositions with parts in $\{2, 1, 3, 5, 7, \ldots\}$.

1 Introduction

Dyck paths with air pockets (DAPs) are introduced in a recent paper [5]. These paths consist in lattice paths in $\mathbb{N}^2$ starting at the origin, ending on the x-axis, and made of up-steps $U = (1, 1)$ and down-steps $D_k = (1, -k)$ where

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$k \geqslant 1$, and so that no two down-steps can be consecutive. The *length* of a path P is the number of steps in P. DAPs can be viewed as ordinary Dyck paths where maximal runs of down-steps are replaced by one large down-step. In sorting theory, DAPs also correspond to a stack evolution with (partial) reset operations that cannot be consecutive, see [14]. In [5], the authors enumerate these paths and their prefixes with respect to the length, the type (up or down) of the last step, and the ordinate of the endpoint. Moreover, they establish a one-to-one correspondence between DAPs of length n and peak-less Motzkin paths of length $n-1$. In a second paper, the authors [6] generalized DAPs by allowing them to go below the x-axis, calling them *grand Dyck paths with air pockets* (GDAPs). They yield enumerative results for these paths with respect to their length and various restrictions on their minimum and maximum ordinates. More recently, the definition of DAPs was extended to include horizontal steps under certain conditions, as described in [4, 7].

Let $\mathcal{D}$ be the set of all DAPs and $\mathcal{D}_n$ be the set of DAPs of length n. In this paper, we generalize DAPs by allowing back-down steps as follows.

Definition 1. *A* skew Dyck path with air pockets *(abbreviated as "skew DAP") is a lattice path in* $\mathbb{N}^2$*, consisting of steps in the set* $\{U, L, D_1, D_2, \ldots\}$*, where* $U = (1,1)$*,* $L = (-1,-1)$*, and* $D_k = (1,-k)$ *for every positive integer* k *(we abbreviate* D_1 *to* D *for convenience), starting at the origin* $(0,0)$*, ending somewhere on the* x*-axis, and such that any occurrence of the following consecutive patterns is forbidden:* UL*,* LU*, and* D_iD_j *for any* $i, j > 0$*.*

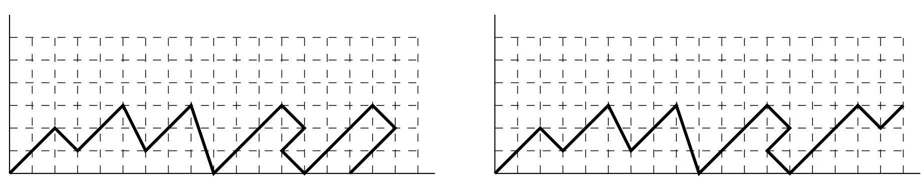

Figure 1: From left to right, a skew Dyck path with air pockets and a partial skew Dyck path with air pockets ending at ordinate 3, respectively.

We say a skew DAP has *length* n (where n is a nonnegative integer) if it consists of n steps (the empty path ε counts as a 0-length skew DAP). For all $n \geqslant 0$, we let $\mathcal{S}_n$ denote the set of n-length skew DAPs, and we set $\mathcal{S} = \bigcup_{n\geqslant 0} \mathcal{S}_n$.

Definition 2. *A skew DAP is* valley-avoiding *(v.a. skew DAP for short) if it contains no occurrence of the consecutive pattern* D_kU*, for all* $k > 0$*.*

For all $n \geqslant 0$, we let $\mathcal{V}_n$ denote the set of n-length v.a. skew DAPs, and we set $\mathcal{V} = \bigcup_{n\geqslant 0} \mathcal{V}_n$.

Definition 3. *A valley-avoiding skew DAP is* zigzagging *(z.v.a. skew DAP for short) if it contains no occurrence of the consecutive pattern LL.*

For all $n \geqslant 0$, we let $\mathcal{Z}_n$ denote the set of n-length z.v.a skew DAPs, and we set $\mathcal{Z} = \bigcup_{n\geqslant 0} \mathcal{Z}_n$.

Finally, for each type of skew Dyck paths defined above, we will use the terminology of *partial skew Dyck paths* to refer to a prefix of a skew DAP (i.e. the path does not necessarily end on the x-axis). For all $n \geqslant 0$, we let $\mathcal{PD}_n$ (resp. $\mathcal{PV}_n$, resp. $\mathcal{PZ}_n$) denote the set of n-length partial skew DAPs (resp. v.a. skew DAPs, resp. z.v.a. skew DAPs), and we use the notation $\mathcal{PD}$, $\mathcal{PV}$, $\mathcal{PZ}$ for the sets of all partial skew DAPs of each type. We refer to Figure 1 for an illustration of a skew DAP and a partial skew DAP.

For $k \geqslant 0$, we consider the generating function $f_k = f_k(z)$ (resp. $g_k = g_k(z)$, resp. $h_k = h_k(z)$), where the coefficient $[z^n]f_k$ (resp. $[z^n]g_k$, resp. $[z^n]h_k$) of z^n in its series expansion is the number of partial skew Dyck paths of length n, ending at ordinate k with an up-step U (resp., with a down-step D_k, $k \geqslant 1$, resp., with a back-down step L). Also, we introduce the bivariate generating functions

$$F(u, z) = \sum_{k\geqslant 0} u^k f_k(z), \quad G(u, z) = \sum_{k\geqslant 0} u^k g_k(z), \text{ and } \quad H(u, z) = \sum_{k\geqslant 0} u^k h_k(z).$$

For short, we use the notation $F(u)$, $G(u)$, and $H(u)$ for these functions. In Sections 3 and 4 respectively, the same notations are preserved for partial v.a. skew DAPs and partial z.v.a. skew DAPs, respectively.

Motivation and outline of the paper. The main goal of this work is to present enumerative results for three classes of partial skew Dyck paths, and to exhibit some links between these paths and restricted classes of binary words and compositions (incidentally, we express these results as Riordan arrays). More precisely, in Section 2, we give bivariate generating functions for the number of n-length partial skew DAPs ending at ordinate $k \geqslant 0$ with a given type of step (up, down or back-down step). We deduce the generating function for the total number of skew DAPs (resp. of partial skew DAPs). We also give asymptotic approximations for the corresponding cardinalities, and for the average ordinate of the endpoint in all partial skew DAPs of a given length. In Section 3, we make a similar study for partial v.a. skew DAPs. In addition, we present a bijection between these paths and binary words avoiding the patterns 00 and 0110. Section 4 presents the counterpart for z.v.a skew DAPs, and we provide a bijection between these paths and palindromic compositions with parts in $\{2, 1, 3, 5, 7, \ldots\}$. Some of the obtained results are summarized in the following table.

Type of paths	First terms	OEIS
Skew DAPs	$1, 0, 1, 1, 3, 5, 13, 26, 64, 143, \ldots$	New
Partial skew DAPs	$1, 1, 2, 4, 9, 19, 44, 100, 236, 558 \ldots$	New
v.a. skew DAPs	$1, 0, 1, 1, 2, 2, 4, 5, 9, 12, \ldots$	A124280
Partial v.a. skew DAPs	$1, 1, 2, 3, 5, 7, 11, 16, 25, 37, \ldots$	A130137
z.v.a skew DAPs	$1, 0, 1, 1, 2, 2, 3, 4, 6, 8, \ldots$	A103632
Partial z.v.a skew DAPs	$1, 1, 2, 3, 5, 7, 10, 14, 20, 28, \ldots$	New

2 Enumerating skew DAPs

In this part, we count partial skew DAPs of a given length, i.e., ending at a given abscissa, according to the type of the last step, and the ordinate of the endpoint.

Theorem 1. *We have*

$$F(u) = \frac{s_1}{s_1 - u}, \qquad G(u) = \frac{1 - s_1 z}{z(s_1 - u)}, \qquad H(u) = \frac{1 - s_1 z}{(s_1 - u)(s_1 - z)},$$

and thus,

$$F(u) + G(u) + H(u) = \frac{s_1(1 - z^2)}{z(s_1 - z)(s_1 - u)},$$

where $s_1 = \frac{A}{6z} + \frac{4z^4 - 2z^3 - \frac{4}{3}z^2 - \frac{2}{3}z + \frac{2}{3}}{zA} + \frac{z+1}{3z}$, *with*

$$A = \left(72z^5 - 72z^4 + 44z^3 + 12B\, z - 48z^2 - 12z + 8\right)^{\frac{1}{3}},$$

and

$$B = \sqrt{-96z^{10} + 144z^9 + 60z^8 - 108z^7 - 24z^6 - 48z^5 + 81z^4 - 18z^2 + 12z - 3}.$$

Proof. By convention, we fix $f_0 = 1$ to take into account the empty path consisting of the origin $(0, 0)$ only. A nonempty skew DAP of length n ending at ordinate $k \geqslant 1$ with an up-step U is uniquely obtained from a skew DAP of length $n-1$ ending at ordinate $k-1$ with either an up-step or a down-step, which implies the recurrence relation $f_k(z) = z(f_{k-1}(z) + g_{k-1}(z))$ for $k \geqslant 1$. A skew DAP of length n ending at ordinate $k \geqslant 0$ with a down-step D_i, $i \geqslant 1$, is uniquely obtained from a skew DAP of length $n-1$ ending at ordinate $k+i$, $i \geqslant 1$, with either an up-step U or a back-down-step L, which implies the recurrence relation $g_k(z) = z \sum_{i \geqslant 1} \left(f_{k+i}(z) + h_{k+i}(z)\right)$, for $k \geqslant 0$. A skew DAP of length n ending at ordinate $k \geqslant 0$ with an L-step is uniquely obtained from a skew DAP of length $n-1$ ending at ordinate $k+1$ with either a down-step or an L-step, which implies the recurrence relation $h_k(z) = z(g_{k+1}(z) + h_{k+1}(z))$ for $k \geqslant 0$. Hence, we obtain the following equations:

$$\begin{cases} & f_0(z) &=& 1, \\ \forall k > 0, & f_k(z) &=& z\left(f_{k-1}(z) + g_{k-1}(z)\right), \\ \forall k \geqslant 0, & g_k(z) &=& z \sum_{i \geqslant 1} \left(f_{k+i}(z) + h_{k+i}(z)\right), \\ \forall k \geqslant 0, & h_k(z) &=& z\left(g_{k+1}(z) + h_{k+1}(z)\right). \end{cases}$$

From the previous system, we multiply both sides of all four equations by u^k, then we sum over k, and using basic algebraic methods on generating functions. For instance, let us examine the case of $G(u)$, which is the least straightforward, we get:

$$G(u) = \sum_{k \geqslant 0} g_k u^k = \sum_{k \geqslant 0} z \sum_{i \geqslant 1} \left(f_{k+i}(z) + h_{k+i}(z)\right) u^k.$$

Interchanging the order of the double summation, the formula becomes:

$$G(u) = z \sum_{i \geqslant 1} (f_i + h_i) \sum_{k=0}^{i-1} u^k = z \sum_{i \geqslant 1} (f_i + h_i) \frac{1-u^i}{1-u}.$$

Then, bringing the factor $\frac{1}{1-u}$ to the front of the expression, and expanding the product inside of the summation, we deduce:

$$G(u) = \frac{z}{1-u}\left(\sum_{i \geqslant 1} f_i - \sum_{i \geqslant 1} f_i u^i + \sum_{i \geqslant 1} h_i - \sum_{i \geqslant 1} h_i u^i\right),$$

which can be rewritten as:

$$G(u) = \frac{z}{1-u}\left(F(1) - F(0) - F(u) + F(0) + H(1) - H(0) - H(u) + H(0)\right).$$

Ultimately, we derive the following system for $F(u)$, $G(u)$, and $H(u)$:

$$\begin{cases} F(u) &=& 1 + zu(F(u) + G(u)), \\ G(u) &=& \frac{z}{1-u}\left(F(1) - F(u) + H(1) - H(u)\right), \\ H(u) &=& \frac{z}{u}\left(H(u) - H(0) + G(u) - G(0)\right). \end{cases}$$

Solving the previous linear system for $F(u)$, $G(u)$, and $H(u)$, we get:

$$F(u) = \frac{u\,z^2\,(u-z)\,(F(1)+H(1)) + u\,z^3(G(0)+H(0)) - u^2 + (z+1)\,u + z^2 - z}{u^3 z - 2u\,z^3 - u^2 z + z^2 u - u^2 + zu + z^2 + u - z},$$

$$G(u) = -\frac{z\left((zu-1)\,(u-z)\,(F(1)+H(1)) + \left(z^2 u - z\right)(G(0)+H(0)) + u - z\right)}{u^3 z - 2u\,z^3 - u^2 z + z^2 u - u^2 + zu + z^2 + u - z},$$

$$H(u) = -\frac{\left(\left(z^2 u + \left(u^2 - u\right) z - u + 1\right)(G(0)+H(0)) + z\,(zu-1)\,(F(1)+H(1)) + z\right) z}{u^3 z - 2u\,z^3 - u^2 z + z^2 u - u^2 + zu + z^2 + u - z}.$$

All three fractions share the same denominator, which we can be rewritten as $z(u-s_1)(u-s_2)(u-s_3)$, where s_1, s_2, s_3 are the roots of $u^3 z - 2u\,z^3 - u^2 z + z^2 u - u^2 + zu + z^2 + u - z$. We observe that only two roots have a Taylor expansion around $z = 0$ (without loss of generality, we assume that these roots are s_2 and s_3). According to the kernel method [16], $u - s_2$ and $u - s_3$ are bad factors that can be cancelled both numerator and denominator. Thus, the numerator is simplified into the coefficient of u^2, and the denominator becomes $z(u - s_1)$. Finally, we obtain:

$$\begin{cases} F(u) &=& \frac{1-z^2(F(1)+H(1))}{z(s_1-u)}, \\ G(u) &=& \frac{z(F(1)+H(1))}{s_1-u}, \\ H(u) &=& \frac{z(G(0)+H(0))}{s_1-u}, \end{cases}$$

where the third root s_1 is equal to $s_1 = \frac{A}{6z} + \frac{4z^4-2z^3-\frac{4}{3}z^2-\frac{2}{3}z+\frac{2}{3}}{zA} + \frac{z+1}{3z}$, with

$$A = \left(72z^5 - 72z^4 + 44z^3 + 12B\,z - 48z^2 - 12z + 8\right)^{\frac{1}{3}},$$

and

$$B = \sqrt{-96z^{10} + 144z^9 + 60z^8 - 108z^7 - 24z^6 - 48z^5 + 81z^4 - 18z^2 + 12z - 3}.$$

Fixing $u = 0$ and $u = 1$ in the above equations, we obtain the values of $G(0)$, $H(0)$, $F(1)$ and $H(1)$:

$$\begin{cases} G(0) &=& \frac{1-zs_1}{zs_1}, \\ H(0) &=& \frac{zs_1-1}{(z-s_1)s_1}, \\ F(1) &=& \frac{(z-1)\left(-s_1^2 z+2s_1\,z^2+z^2+s_1-z\right)}{z(s_1-1)(z-s_1)}, \\ H(1) &=& \frac{(-2s_1-1)z^4+\left(s_1^2+2s_1+2\right)z^3+\left(-2s_1^2-1\right)z^2+\left(s_1^3-s_1^2+2s_1-1\right)z-s_1^2+s_1}{z^2(s_1-1)(z-s_1)}. \end{cases}$$

Plugging those expressions back into $F(u)$, $G(u)$, and $H(u)$, we obtain the expected result. $\square$

The first terms of the series expansion of $F(u) + G(u) + H(u)$ are

$$\begin{aligned}1 &+ uz + (u^2+1)z^2 + (u^3+2u+1)z^3 + (u^4+3u^2+2u+3)z^4 \\ &+ (u^5+4u^3+3u^2+6u+5)z^5 + (u^6+5u^4+4u^3+10u^2+11u+13)z^6 \\ &+ (u^7+6u^5+5u^4+15u^3+19u^2+28u+26)z^7 + O(z^8).\end{aligned}$$

Now, we deduce the coefficient $[u^k](F(u)+G(u)+H(u))$ of u^k in the series expansion of $F(u) + G(u) + H(u)$ by using the well known series expansion $\frac{1}{s_1-u} = \frac{1}{s_1}\sum_{k\geqslant 0} {s_1}^{-k}\cdot u^k$.

Corollary 1. *We have*

$$[u^k]\,(F(u)+G(u)+H(u)) = \frac{1-z^2}{z(s_1-z)}s_1^{-k}.$$

The next remark makes a link with Riordan arrays theory. We refer to [8, 9, 17] for a background on Riordan arrays.

Remark 1. *Let $\mathcal{P}$ be the matrix $[p_{n,k}]_{n,k\geqslant 0}$, where $p_{n,k}$ is the number of skew DAPs of length n ending at ordinate k, i.e. the coefficient of z^n in the series expansion of $[u^k]\,(F(u)+G(u)+H(u))$. The first values of $\mathcal{P}$ are*

$$\mathcal{P} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \\ 1 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & \\ 3 & 2 & 3 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 5 & 6 & 3 & 4 & 0 & 1 & 0 & 0 & \\ 13 & 11 & 10 & 4 & 5 & 0 & 1 & 0 & \\ 26 & 28 & 19 & 15 & 5 & 6 & 0 & 1 & \\ & & & & \vdots & & & & \ddots \end{pmatrix}.$$

Since $F(u) + G(u) + H(u) = \frac{g(z)}{1-uf(z)}$ with $f(z) = s_1^{-1}$ and $g(z) = \frac{1-z^2}{z(s_1-z)}$, the matrix $\mathcal{P}$ corresponds to the Riordan array

$$\left(\frac{1-z^2}{z(s_1-z)}, \frac{1}{s_1}\right).$$

Now we plug in $u = 0$ and $u = 1$ to get the generating functions for skew DAPs and partial skew DAPs, respectively, and using classical methods [12, 15], we provide an asymptotic approximation of the coefficient of z^n.

Theorem 2. *The generating function for the total number of skew DAPs with respect to the length is given by*

$$F(0,z)+G(0,z)+H(0,z)=\frac{z^2-1}{z\,(z-s_1)},$$

and an asymptotic approximation of the n-th term is

$$0.5292\cdot 2.7309^n\cdot n^{-3/2}.$$

The leading terms of the series expansion of $F(0,z)+G(0,z)+H(0,z)$ are

$$1+z^2+z^3+3z^4+5z^5+13z^6+26z^7+64z^8+143z^9+\mathrm{O}\big(z^{10}\big)\,.$$

Theorem 3. *The generating function for the total number of partial skew DAPs with respect to the length is given by*

$$F(1,z)+G(1,z)+H(1,z)=\frac{s_1(1-z^2)}{z(s_1-z)(s_1-1)},$$

and an asymptotic approximation of the n-th term is

$$2.4909\cdot 2.7309^n\cdot n^{-3/2}.$$

The leading terms of the series expansion of $F(1,z)+G(1,z)+H(1,z)$ are

$$1+z+2z^2+4z^3+9z^4+19z^5+44z^6+100z^7+236z^8+558z^9+\mathrm{O}\big(z^{10}\big)\,.$$

By calculating $\partial_u(F(u,z)+G(u,z)+H(u,z))|_{u=1}$, we obtain the following.

Corollary 2. *An asymptotic approximation for the average of the ordinate of the endpoint in all partial skew DAPs of a given length is* 2.4859*.*

3 Enumerating v.a. skew DAPs

In this part, we use the same methodology and notation as in Section 2 in order to enumerate (partial) valley-avoiding skew Dyck paths with air pockets (v.a. skew DAPs for short), i.e. skew DAPs that do not contain a down-step followed by an up-step.

Theorem 4. *We have*

$$F(u) = \frac{1}{1-zu}, \qquad G(u) = \frac{z^2 - z^4}{(1-zu)(1-z-z^2+z^3-z^4)},$$
$$H(u) = \frac{z^4}{(1-zu)(1-z-z^2+z^3-z^4)},$$

and thus,

$$F(u) + G(u) + H(u) = \frac{1-z+z^3-z^4}{(1-zu)(1-z-z^2+z^3-z^4)}.$$

Proof. By convention, we fix $f_0 = 1$ to take into account the empty path consisting of the origin $(0,0)$ only. A nonempty v.a. skew DAP of length n ending at ordinate $k \geqslant 1$ with an up-step U is uniquely obtained from a skew v.a. DAP of length $n-1$ ending at ordinate $k-1$ with an up-step, which implies the recurrence relation $f_k(z) = zf_{k-1}(z)$ for $k \geqslant 1$. The two recurrence relations for $g_k(z)$ and $h_k(z)$ are the same as in the previous section. So, we refer to Section 2 for an explanation on how they are derived, and we have the following system of equations:

$$\begin{cases} & f_0(z) &=& 1, \\ \forall k > 0, & f_k(z) &=& zf_{k-1}(z), \\ \forall k \geqslant 0, & g_k(z) &=& z\sum_{i\geqslant 1}(f_{k+i}(z) + h_{k+i}(z)), \\ \forall k \geqslant 0, & h_k(z) &=& z(g_{k+1}(z) + h_{k+1}(z)). \end{cases}$$

Using the same method as for the proof of Theorem 1, the previous system induces the following equations for $F(u)$, $G(u)$, and $H(u)$:

$$\begin{cases} F(u) &=& \frac{1}{1-zu}, \\ G(u) &=& \frac{z}{1-u}(F(1) - F(u) + H(1) - H(u)), \\ H(u) &=& \frac{z}{u}(G(u) - G(0) + H(u) - H(0)). \end{cases}$$

Solving the previous linear system for $F(u)$, $G(u)$, and $H(u)$, we get:

$$\begin{cases} F(u) &=& \frac{1}{1-zu}, \\ G(u) &=& \frac{C}{u^3z^2-u^2z^3-u\,z^4-u^3z+2u\,z^3+z^3+u^2-2z^2-u+z}, \\ H(u) &=& \frac{D}{u^3z^2-u^2z^3-u\,z^4-u^3z+2u\,z^3+z^3+u^2-2z^2-u+z}, \end{cases}$$

with

$$\begin{aligned} C = -\,z\Big(&u^2z^2H(1) - u^2z(1+H(1)) + uz^3(G(0)+H(0)-H(1)) \\ &+ uz^2(1-G(0)-H(0)+H(1)) + uz(1-H(1)) + uH(1) \\ &- z^2(1+G(0)+H(0)-H(1)) + z(G(0)+H(0)-H(1))\Big), \end{aligned}$$

and

$$
\begin{aligned}
D = & - z\Big(u^2z^2(G(0)+H(0)) - u^2z(G(0)+H(0)) + uz^3H(1) \\
& - uz^2(1+G(0)+H(0)+H(1)) + u(G(0)+H(0)) \\
& + z^2(1-H(1)) + z(G(0)+H(0)+H(1)) - G(0) - H(0)\Big).
\end{aligned}
$$

Here, $G(u)$ and $H(u)$ share the same denominator, which we write as $(z^2-z)(u-r_1)(u-r_2)(u-r_3)$, where r_1, r_2, and r_3 are the roots of $u^3z^2 - u^2z^3 - u\,z^4 - u^3z + 2u\,z^3 + z^3 + u^2 - 2z^2 - u + z$. Since two of them have a Taylor expansion around $z=0$ (without loss of generality, we assume that these roots are r_2 and r_3), the kernel method [16] once again tells us that $u-r_2$ and $u-r_3$ are both bad factors and can be thus cancelled both numerator and denominator in $G(u)$ and $H(u)$. This leaves us with the numerators being simplified into the coefficient of u^2, and the denominators both being simplified into $(z^2-z)(u-r_1)$. Moreover, r_1 happens to be equal to $\frac{1}{z}$, which conveniently leads to simplifications further on. The previous system is then simplified into:

$$
\begin{cases}
F(u) & = & \frac{1}{1-zu}, \\
G(u) & = & \frac{z((z-1)H(1)-1)}{(1-z)(u-1/z)}, \\
H(u) & = & \frac{-z(G(0)+H(0))}{u-1/z}.
\end{cases}
$$

Evaluating the second equation at $u=0$, and the third one at $u=0$ and $u=1$ respectively, we derive the values of $G(0)$, $H(0)$, and $H(1)$:

$$
\begin{cases}
G(0) & = & \frac{(z^2-1)z}{z^4-z^3+z^2+z-1}, \\
H(0) & = & \frac{-z^4}{z^4-z^3+z^2+z-1}, \\
H(1) & = & \frac{z^4}{z^5-2z^4+2z^3-2z+1}.
\end{cases}
$$

Plugging those expressions back into $F(u)$, $G(u)$, and $H(u)$, we obtain the expected result. $\square$

The first terms of the series expansion of $F(u)+G(u)+H(u)$ are

$$
\begin{aligned}
1 + uz + & \left(u^2+1\right)z^2 + \left(u^3+u+1\right)z^3 + \left(u^4+u^2+u+2\right)z^4 \\
& + \left(u^5+u^3+u^2+2u+2\right)z^5 + \left(u^6+u^4+u^3+2u^2+2u+4\right)z^6 \\
& + \left(u^7+u^5+u^4+2u^3+2u^2+4u+5\right)z^7 + \mathrm{O}\!\left(z^8\right).
\end{aligned}
$$

Now, we deduce the coefficient $[u^k](F(u)+G(u)+H(u))$ of u^k in the series expansion of $F(u)+G(u)+H(u)$.

Corollary 3. *We have*

$$[u^k](F(u)+G(u)+H(u)) = \frac{1-z+z^3-z^4}{1-z-z^2+z^3-z^4} z^k.$$

Remark 2. *Let $\mathcal{P}^{\mathcal{V}}$ be the matrix $\mathcal{P}^{\mathcal{V}} = [p^{\mathcal{V}}_{n,k}]_{n,k\geqslant 0}$, where $p^{\mathcal{V}}_{n,k}$ is the number of v.a. skew DAPs of length n ending at ordinate k, i.e. the coefficient of z^n in the series expansion of $[u^k](F(u)+G(u)+H(u))$. The first values of $\mathcal{P}^{\mathcal{V}}$ are*

$$\mathcal{P}^{\mathcal{V}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & \\ 2 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 2 & 2 & 1 & 1 & 0 & 1 & 0 & 0 & \\ 4 & 2 & 2 & 1 & 1 & 0 & 1 & 0 & \\ 5 & 4 & 2 & 2 & 1 & 1 & 0 & 1 & \\ & & & & \vdots & & & & \ddots \end{pmatrix}.$$

Since $F(u)+G(u)+H(u) = \frac{g(z)}{1-uf(z)}$, with $f(z)=z$ and $g(z) = \frac{1-z+z^3-z^4}{1-z-z^2+z^3-z^4}$, the matrix $\mathcal{P}^{\mathcal{V}}$ corresponds to the Riordan array

$$\left(\frac{1-z+z^3-z^4}{1-z-z^2+z^3-z^4}, z\right).$$

Now we plug in $u=0$ and $u=1$ to get the generating function for v.a. skew DAPs and partial v.a. skew DAPs, respectively.

Theorem 5. *The generating function for the total number of v.a. skew DAPs with respect to the length is given by:*

$$F(0,z)+G(0,z)+H(0,z) = \frac{1-z+z^3-z^4}{1-z-z^2+z^3-z^4},$$

and an asymptotic of the n-th term is

$$\frac{-a^4+a^3-a+1}{4a^4-3a^3+2a^2+a}\cdot\left(a^3-a^2+a+1\right)^n \approx 0.3051\cdot 1.5129^n,$$

where $a \approx 0.6609925319$ is the smallest root (modulus-wise) of $z^4-z^3+z^2+z-1$.

A simple calculation on generating functions allows to prove that the number a_n of n-length v.a. skew DAPs satisfies $a_0 = a_2 = a_3 = 1$, $a_1 = 0$, $a_4 = 2$ and $a_n = 2a_{n-2} + a_{n-5}$ for $n \geqslant 5$. The leading terms of the series expansion of $F(0, z) + G(0, z) + H(0, z)$ are:

$$1 + z^2 + z^3 + 2z^4 + 2z^5 + 4z^6 + 5z^7 + 9z^8 + 12z^9 + \mathrm{O}\big(z^{10}\big)\,.$$

The coefficients spell out sequence A124280 in [18], and the n-th term a_n satisfies

$$a_n = \sum_{k=0}^{\lfloor \frac{n-2}{2} \rfloor} \sum_{j=0}^{n-2k-2} \binom{j}{n-2k-j-2} \binom{k}{n-2k-j-2}.$$

Theorem 6. *The generating function for the total number of partial v.a. skew DAPs with respect to the length is given by:*

$$F(1, z) + G(1, z) + H(1, z) = \frac{1 + z^3}{1 - z - z^2 + z^3 - z^4},$$

and an asymptotic of the n-th term is

$$\frac{a^3 + 1}{4a^4 - 3a^3 + 2a^2 + a} \cdot \big(a^3 - a^2 + a + 1\big)^n \approx 0.9000 \cdot 1.5129^n,$$

where $a \approx 0.6609925319$ is the smallest root (modulus-wise) of $z^4 - z^3 + z^2 + z - 1$.

A simple calculation on generating functions allows to prove that the number b_n of n-length partial v.a. skew DAPs satisfies $b_0 = b_1 = 1$, $b_2 = 2$, $b_3 = 3$, $b_4 = 5$ and $b_n = 2b_{n-2} + b_{n-5}$ for $n \geqslant 5$. The leading terms of the series expansion of $F(1, z) + G(1, z) + H(1, z)$ are:

$$1 + z + 2z^2 + 3z^3 + 5z^4 + 7z^5 + 11z^6 + 16z^7 + 25z^8 + 37z^9 + \mathrm{O}\big(z^{10}\big)\,.$$

The coefficients spell out sequence A130137 in [18].

In the same way as we did at the end of Section 2, we obtain the following.

Corollary 4. *An asymptotic approximation for the average of the ordinate of the endpoint in all partial v.a. skew DAPs of a given length is* 1.9497.

We end this section by exhibiting a constructive bijection between the set $\mathcal{PV}_n$ of n-length partial v.a. skew v.a. DAPs, and the set $\mathcal{B}_{n-1}$ of $(n-1)$-length binary words avoiding the patterns 00 and 0110. Let $\mathcal{B} = \bigcup_{n \geqslant 0} \mathcal{B}_n$.

Definition 4. *We recursively define the map χ from the set $\mathcal{PV}\backslash\{\varepsilon\}$ to $\mathcal{B}$ as follows. For $\beta \in \mathcal{PV}\backslash\{\varepsilon\}$, we set:*

$$\chi(\beta) = \begin{cases} \epsilon & \text{if } \beta = U & (i) \\ 1^{k-1}0 & \text{if } \beta = U^k D_k,\ k \geqslant 1 & (ii) \\ \chi(\alpha)1 & \text{if } \beta = U\alpha,\ \alpha \in \mathcal{PV}\backslash\{\varepsilon\} & (iii) \\ \chi(\alpha)10 & \text{if } \beta = U\alpha L,\ \alpha \in \mathcal{V}\backslash\{\varepsilon\} & (iv) \\ \overline{\chi(\alpha)}1^k0 & \text{if } \beta = U^k\alpha D_k,\ k \geqslant 1,\ \alpha \in \mathcal{V}\backslash\{\varepsilon\} \text{ ending with } L & (v) \end{cases}$$

where the operator $\overline{\bullet}$ acts on a binary word ending with zero, by replacing this last zero with a one, i.e. $\overline{w_1 w_2 \dots w_{n-1}0} = w_1 w_2 \dots w_{n-1}1$.

Notice that χ is defined so that the image of any nonempty partial v.a. skew DAP is a binary word that avoids both patterns 00 and 0110. Furthermore, the image of any nonempty v.a. skew DAP ends with a 0. For instance, we have

$$\chi(U^4DLD_2) = \overline{\chi(U^2DL)}110 = \overline{\chi(UD)10}110 = \overline{010}110 = 011110.$$

We refer to Figure 2 for an illustration of the cases in the definition of the bijection χ.

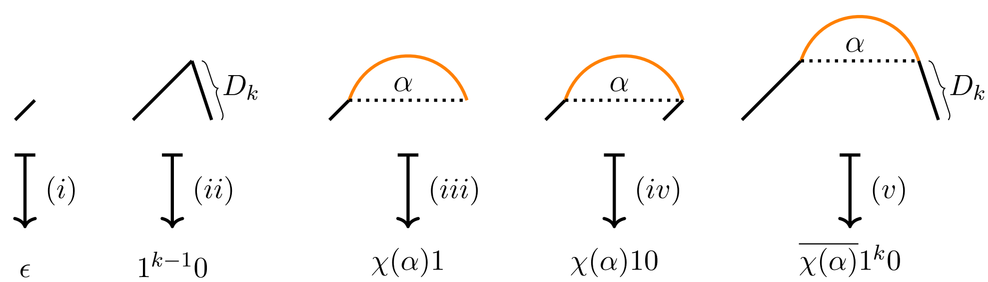

Figure 2: Illustration of the map χ.

Theorem 7. *The map χ induces a bijection between $\mathcal{PV}_n\backslash\{\varepsilon\}$ and $\mathcal{B}(n-1)$ for all $n \geqslant 1$.*

Proof. It follows from Theorem 6 and [18] (entry A130137) that $\mathcal{PV}_n\backslash\{\varepsilon\}$ and $\mathcal{B}(n-1)$ have the same cardinality for all $n \geqslant 1$. Thus, it is enough to prove the injectivity of χ. We proceed by induction on n. The statement is trivial for $n = 1, 2$. Now, let $n \geqslant 3$, and let $\alpha, \beta \in \mathcal{PV}_n\backslash\{\varepsilon\}$ such that $\chi(\alpha) = \chi(\beta)$.

If $\chi(\alpha) = 1^k0$, then both $\chi(\alpha)$ and $\chi(\beta)$ belong to case (ii) in the definition of χ. If not, say for example $\alpha = UAL$ with $A \in \mathcal{V}\backslash\{\varepsilon\}$, then

$1^k0 = \chi(\alpha) = \chi(UAL) = \chi(A)10$, which implies $\chi(A) = 1^{k-1}$, and in turn $A = U^k$, which contradicts the fact that A is an element of $\mathcal{V}\backslash\{\varepsilon\}$. The hypothesis that α is of the form $U^\ell AD_\ell$ with $\ell \geqslant 1$ and $A \in \mathcal{V}\backslash\{\varepsilon\}$ can be ruled out with a similar reasoning. It now follows from the definition of χ that $\alpha = U^{k+1}D_{k+1} = \beta$.

Otherwise, depending on their ending letters, $\chi(\alpha)$ and $\chi(\beta)$ both either belong to case (*iii*) ($\chi(\alpha)$ ends with 1), (*iv*) ($\chi(\alpha)$ ends with 010), or (*v*) ($\chi(\alpha)$ ends with 110) in the definition of χ. Say they both belong to case (*v*), for instance. Then, from the definition of χ, it follows that $\alpha = U^kAD_k$ and $\beta = U^\ell BD_\ell$ for some $k, \ell \geqslant 1$ and $A, B \in \mathcal{V}\backslash\{\varepsilon\}$ ending with L. Thus, we have $\chi(\alpha) = \overline{\chi(A)}1^k0$ and $\chi(\beta) = \overline{\chi(B)}1^\ell 0$. Suppose for a contradiction that $k \neq \ell$, and without loss of generality, let us assume $k > \ell$. Then, $\chi(\alpha) = \chi(\beta)$ implies $\overline{\chi(A)}1^{k-\ell} = \overline{\chi(B)}$. Since $\chi(A)$ avoids 00 and ends with 0 ($A \in \mathcal{V}\backslash\{\varepsilon\}$), it ends with 10, which implies $\overline{\chi(B)}$ ends with 111. Now, B cannot be of the form UXL with $X \in \mathcal{V}\backslash\{\varepsilon\}$, otherwise $\chi(B)$ would end with 010, and in turn $\overline{\chi(B)}$ would not end with 111 anymore; moreover, B cannot end with a down-step since $\beta = U^\ell BD_\ell$, which yields a contradiction. Hence, we have $k = \ell$, which implies $\overline{\chi(A)} = \overline{\chi(B)}$. Since $A, B \in \mathcal{V}\backslash\{\varepsilon\}$, we have $\chi(A) = \chi(B)$, and, by induction, $A = B$; thus, $\alpha = \beta$.

Cases (*iii*) and (*iv*) are handled *mutatis mutandis*, which completes the induction. The cardinality argument then proves the bijectivity. □

4 Enumerating z.v.a. skew DAP

Once again, we use the same methodology and notation as in Sections 2 and 3 in order to enumerate z.v.a. skew DAPs and partial z.v.a. skew DAPs. Then, we provide here all main results without the details of the proofs. The first system of equations is:

$$\begin{cases} & f_0(z) = 1, \\ \forall k > 0, & f_k(z) = zf_{k-1}(z), \\ \forall k \geqslant 0, & g_k(z) = z\sum_{i\geqslant 1}(f_{k+i}(z) + h_{k+i}(z)), \\ \forall k \geqslant 0, & h_k(z) = zg_{k+1}(z). \end{cases}$$

Redoing the same work as in Sections 2 and 3, we are led to the following results.

Theorem 8. *We have*

$$F(u) = \frac{1}{1-zu}, \qquad G(u) = \frac{z^2}{(1-zu)(1-z-z^4)},$$

$$H(u) = \frac{z^4}{(1 - zu)(1 - z - z^4)},$$

and thus,

$$F(u) + G(u) + H(u) = \frac{1 - z + z^2}{(1 - zu)(1 - z - z^4)}.$$

The first terms of the series expansion of $F(u) + G(u) + H(u)$ are

$$\begin{aligned}1 + uz &+ \left(u^2 + 1\right) z^2 + \left(u^3 + u + 1\right) z^3 + \left(u^4 + u^2 + u + 2\right) z^4 \\ &+ \left(u^5 + u^3 + u^2 + 2u + 2\right) z^5 + \left(u^6 + u^4 + u^3 + 2u^2 + 2u + 3\right) z^6 \\ &+ \left(u^7 + u^5 + u^4 + 2u^3 + 2u^2 + 3u + 4\right) z^7 + \mathrm{O}\left(z^8\right).\end{aligned}$$

Therefore, we can obtain the coefficient $[u^k](F(u) + G(u) + H(u))$ of u^k in the series expansion of $F(u) + G(u) + H(u)$.

Corollary 5. *We have*

$$[u^k](F(u) + G(u) + H(u)) = \frac{1 - z + z^2}{1 - z - z^4} z^k.$$

Remark 3. *Let $\mathcal{P}^{\mathcal{Z}}$ be the matrix $\mathcal{P}^{\mathcal{Z}} = [p^{\mathcal{Z}}_{n,k}]_{n,k\geqslant 0}$, where $p^{\mathcal{Z}}_{n,k}$ is the number of z.v.a. skew DAPs of length n ending at ordinate k, i.e. the coefficient of z^n in the series expansion of $[u^k](F(u) + G(u) + H(u))$. The first values of $\mathcal{P}^{\mathcal{Z}}$ are*

$$\mathcal{P}^{\mathcal{Z}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & \\ 2 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 2 & 2 & 1 & 1 & 0 & 1 & 0 & 0 & \\ 3 & 2 & 2 & 1 & 1 & 0 & 1 & 0 & \\ 4 & 3 & 2 & 2 & 1 & 1 & 0 & 1 & \\ & & & & \vdots & & & & \ddots \end{pmatrix}.$$

Since $F(u) + G(u) + H(u) = \frac{g(z)}{1-uf(z)}$, with $f(z) = z$ and $g(z) = \frac{1-z+z^2}{1-z-z^4}$, the matrix $\mathcal{P}^{\mathcal{Z}}$ corresponds to the Riordan array

$$\left(\frac{1 - z + z^2}{1 - z - z^4}, z\right).$$

Now we plug in $u = 0$ and $u = 1$ to get the generating function for v.a. skew DAPs and partial v.a. skew DAPs, respectively.

Theorem 9. *The generating function for the total number of z.v.a. skew DAPs with respect to the length is given by:*

$$F(0,z)+G(0,z)+H(0,z)=\frac{1-z+z^2}{1-z-z^4},$$

and an asymptotic of the n-th term is

$$\frac{a^2-a+1}{4a^4+a}\cdot\left((a+1)\left(a^2-a+1\right)\right)^n\approx 0.4382\cdot 1.3803^n,$$

where $a\approx 0.7244919590$ is the smallest root (modulus-wise) of z^4+z-1.

A simple calculation on generating functions allows to prove that the number c_n of n-length z.v.a. skew DAPs satisfies $c_0=c_2=c_3=1$, $c_1=0$, and $c_n=c_{n-1}+c_{n-4}$ for $n\geqslant 4$. The leading terms of the series expansion of $F(0,z)+G(0,z)+H(0,z)$ are:

$$1+z^2+z^3+2z^4+2z^5+3z^6+4z^7+6z^8+8z^9+\mathrm{O}\left(z^{10}\right).$$

The coefficients spell out sequence A103632 in [18], and the n-th term c_n satisfies

$$c_n=\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{\lfloor\frac{2n-3k-1}{2}\rfloor}{n-2k}.$$

Theorem 10. *The generating function for the total number of partial z.v.a. skew DAPs with respect to the length is given by:*

$$F(1,z)+G(1,z)+H(1,z)=\frac{z^2-z+1}{(-1+z)\left(z^4+z-1\right)},$$

and an asymptotic of the n-th term is

$$\frac{a^2-a+1}{a\left(4a^3+1\right)(1-a)}\cdot\left((a+1)\left(a^2-a+1\right)\right)^n\approx 1.5905\cdot 1.3803^n,$$

where $a\approx 0.7244919590$ is the smallest root (modulus-wise) of z^4+z-1.

A simple calculation on generating functions allows to prove that the number d_n of n-length partial z.v.a. skew DAPs satisfies $d_0=d_1=1$, $d_2=2$, $d_3=3$, $d_4=5$ and $d_n=2d_{n-1}-d_{n-2}+d_{n-4}-d_{n-5}$ for $n\geqslant 5$. The leading terms of the series expansion of $F(1,z)+G(1,z)+H(1,z)$ are:

$$1+z+2z^2+3z^3+5z^4+7z^5+10z^6+14z^7+20z^8+28z^9+\mathrm{O}\left(z^{10}\right).$$

Finally, we obtain the following.

Corollary 6. *An asymptotic approximation for the average of the ordinate of the endpoint in all partial z.v.a. skew DAPs of a given length is* 2.6296*.*

It follows from Theorem 9 and [18] (entry A103632) that n-length z.v.a. skew DAPs are in bijection with palindromic compositions of $n-2$ that have parts in $\{2,1,3,5,7,9,\ldots\}$, where a palindromic composition of n is a composition $(c_1, c_2, \ldots, c_k)$, $c_1 + c_2 + \ldots + c_k = n$, that reads the same backwards as forwards; for instance, $(3,1,5,2,5,1,3)$ is a palindromic composition of 20 (see [13]). Let $\mathcal{C}(n-2)$ be the set of such compositions, and let $\mathcal{C} = \bigcup_{n\geqslant 0} \mathcal{C}(n)$. We shall now provide an explicit bijection.

Definition 5. *Let us recursively define the map $\psi : \mathcal{Z} \longrightarrow \mathcal{C}$ as follows. For $\beta \in \mathcal{Z}$, we set:*

$$\psi(\beta) = \begin{cases} \epsilon & \text{if } \beta = UD & (i) \\ (1) & \text{if } \beta = U^2D_2 & (ii) \\ (2) & \text{if } \beta = U^2DL & (iii) \\ (3) & \text{if } \beta = U^3D_2L & (iv) \\ (1, \psi(U^aD_{k-2}B), 1) & \text{if } \beta = U^{a+2}D_kB,\ a \geqslant 1, k \geqslant 3 & (v) \\ (2, \psi(U^aB), 2) & \text{if } \beta = U^{a+2}DLB,\ a \geqslant 1, B \neq \varepsilon & (vi) \\ {}_{+2}\psi(U^aD_{k+1}B)_{+2} & \text{if } \beta = U^{a+2}D_2LD_kB,\ a \geqslant 2, k \geqslant 1 & (vii) \end{cases}$$

where B is a suffix of a z.v.a. skew DAP, and where ${}_{+2}(x_1, x_2, \ldots, x_{n-1}, x_n)_{+2} := ((x_1+2), x_2, \ldots, x_{n-1}, (x_n+2))$ for $n \geqslant 2$; for the case $n = 1$, we define ${}_{+2}(x_1)_{+2} := (x_1 + 4)$.

For instance, we have

$$\begin{aligned} \psi(U^7D_2LD_2LD) &= {}_{+2}\psi(U^5D_3LD)_{+2} = {}_{+2}(1, \psi(U^3DLD), 1)_{+2} \\ &= {}_{+2}(1, 2, \psi(UD), 2, 1)_{+2} = (3,2,2,3). \end{aligned}$$

We refer to Figure 3 for an illustration of the nontrivial cases in the definition of the bijection ψ.

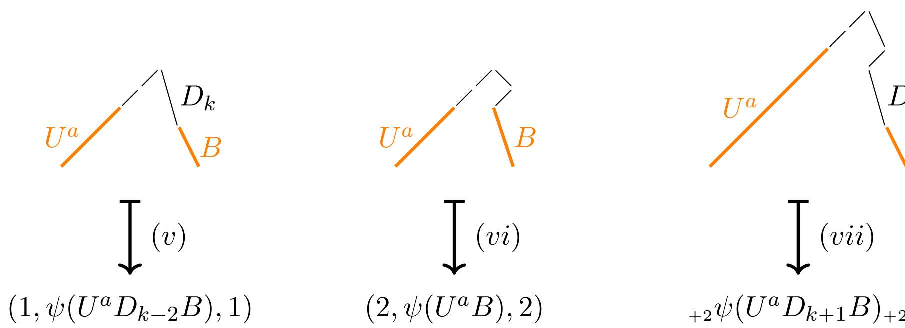

Figure 3: Illustration of the map ψ (the three nontrivial cases).

Theorem 11. *The map ψ induces a bijection between $\mathcal{Z}_n$ and $\mathcal{C}(n-2)$ for all $n \geqslant 2$.*

Proof. Since $\mathcal{Z}_n$ and $\mathcal{C}(n-2)$ have the same cardinality for all $n \geqslant 2$, it is enough to prove that ψ induces an injection from $\mathcal{Z}_n$ to $\mathcal{C}(n-2)$. We proceed by induction on n. The statement is trivial for $n = 2, 3$. Now, let $n \geqslant 4$, and let $\alpha, \beta \in \mathcal{Z}_n$ such that $\psi(\alpha) = \psi(\beta)$. If $\psi(\alpha) \in \{(2), (3)\}$, then we immediately get $\alpha = \beta$. Otherwise, depending on their starting letter, $\psi(\alpha)$ and $\psi(\beta)$ both either belong to case (v), (vi), or (vii) in the definition of ψ. Say they both belong to case (v), for instance. Then, from the definition of ψ, it follows that $\alpha = U^{a_1+2} D_{k_1} B_1$ and $\beta = U^{a_2+2} D_{k_2} B_2$ for some $a_1, a_2, k_1, k_2, B_1, B_2$. Thus, we have $\psi(\alpha) = (1, \psi(U^{a_1} D_{k_1-2} B_1), 1)$ and $\psi(\beta) = (1, \psi(U^{a_2} D_{k_2-2} B_2), 1)$, and in turn, $\psi(U^{a_1} D_{k_1-2} B_1) = \psi(U^{a_2} D_{k_2-2} B_2)$. Since $U^{a_1} D_{k_1-2} B_1$ and $U^{a_2} D_{k_2-2} B_2$ are both elements of $\mathcal{Z}_{n-2}$, and ψ is (by induction) injective from $\mathcal{Z}_{n-2}$ to $\mathcal{C}(n-4)$, we deduce $U^{a_1} D_{k_1-2} B_1 = U^{a_2} D_{k_2-2} B_2$, which implies $\alpha = \beta$. Cases (vi) and (vii) are handled *mutatis mutandis*, which completes the induction. The cardinality argument then proves the bijectivity. $\square$

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