EQUIVALENCE CLASSES OF SKEW DYCK PATHS MODULO SOME PATTERNS

Jean-Luc Baril[1]
LIB, Univ. Franche-Comt, France
barjl@u-bourgogne.fr

José Luis Ramírez
Departamento de Matemticas, Universidad Nacional de Colombia, Bogot, Colombia
jlramirezr@unal.edu.co

Lina Maria Simbaqueba
Departamento de Matemticas, Universidad Nacional de Colombia, Bogot, Colombia
lmsimbaquebam@unal.edu.co

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Abstract

For any pattern $p \in \{U, L, UU, UD, DU, DD\}$, we enumerate the equivalence classes of skew Dyck paths, where two skew Dyck paths of the same semilength are p-equivalent whenever the positions of the occurrences of the pattern p are the same. In this paper we use generating functions, bijective arguments, and recurrence relations to obtain the main results.

1. Introduction and Notation

A *skew Dyck path* is a lattice path in the first quadrant of the xy-plane that starts at the origin, ends on the x-axis, and is made of up-steps $U = (1, 1)$, down-steps $D = (1, -1)$, and left steps $L = (-1, -1)$ so that up and left steps do not overlap. Whenever we do not permit the step L, we retrieve the well known definition of Dyck paths (see [5]). We let $\mathcal{SD}$ denote the set of all skew Dyck paths, $\mathcal{D}$ the set of Dyck paths, and $|P|$ the *length* of the path P, i.e., the number of its steps, which is an even non-negative integer. Let λ be the skew Dyck path of length zero. For example, Figure 1 shows all skew Dyck paths of length 6, or equivalently of semilength 3.

[1] Corresponding author

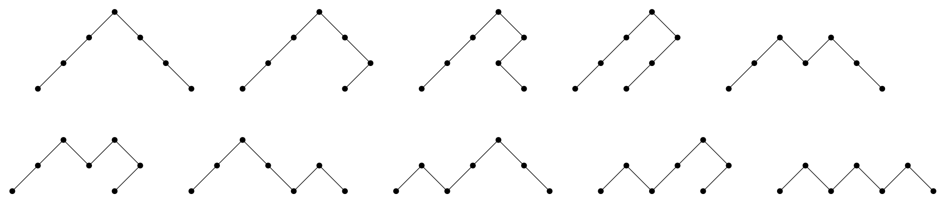

Figure 1: Skew Dyck paths of semilength 3.

The concept of skew Dyck path was introduced by Deutsch, Munarini, and Rinaldi [6]. Some additional studies can be found in [4, 7, 11], where the authors present enumerative results according to different parameters and some bijections with other combinatorial objects, as hex trees, tree-like polyhexes, and 3-Motzkin paths.

In the following, a *pattern* consists of consecutive steps in a path. We will say that a pattern is at *position* $i \geq 1$ in a path whenever the first step of the pattern appears at the i-th step of the path, the second step at the $(i+1)$-th step, and so on. The *height* of an occurrence of a pattern is the minimal ordinate reached by its points. For instance, the skew Dyck path $P = UDUUDL$ contains two occurrences of the pattern UD at positions 1 and 4, and the heights of these occurrences are respectively 0 and 1.

Recently in [1, 2, 3, 12], the authors investigate equivalence relations on the sets of Dyck paths, Motzkin paths, Łukasiewicz paths, and Ballot paths where two paths of the same length are equivalent whenever they coincide on all occurrences of a given pattern. In this paper, we extend these studies for skew Dyck paths by considering the analogous equivalence relation on $\mathcal{SD}$:

Two skew Dyck paths of the same semilength are p-equivalent whenever they have the same positions of the occurrences of the pattern p.

For instance, the skew Dyck path $UDUUDL$ is L-equivalent with $UUDUDL$ since the occurrences of L appear at the same positions in the two paths.

For some patterns p of length one or two, we provide generating functions for the number of p-equivalence classes in $\mathcal{SD}$ with respect to the semilength. The general method used consists in providing bijections between equivalence classes and some subsets of skew Dyck paths, and then, evaluating algebraically the generating functions for these subsets. We handle the cases $p \in \{U, L, UU, UD, DU, DD\}$, and we leave the other cases as open problems. As a byproduct, we characterize skew Dyck paths entirely fixed by the positions of its left steps L, and we count them using generating functions and recurrence relations. We also provide and conjecture asymptotic approximations for the number of p-equivalence classes of skew Dyck paths of semilength n.

2. Equivalence Classes Modulo Patterns of Length One

In this part, we focus on the patterns p of length one, that is $p \in \{U, D, L\}$. Table 1 presents the first few values for the number of p-equivalence classes. We do not succeed to solve the case D which is left as an open question (for this case, values in Table 1 are experimentally obtained).

Pattern	Sequence	OEIS([13])	a_n, $1 \le n \le 9$
U	Catalan	A000108	1, 2, 5, 14, 42, 132, 429, 1430, 4862
L		New	1, 2, 4, 9, 21, 50, 123, 308, 781
D	Open problem	New	1, 3, 10, 35, 129, 488, 1881, 7341, 28876

Table 1: Number of p-equivalence classes for skew Dyck paths.

2.1. The Pattern U

The number of U-equivalence classes is given by the Catalan numbers $c_n = \frac{1}{n+1}\binom{2n}{n}$ since we can establish a bijection between Dyck paths and the set of U-equivalence classes of skew Dyck paths. Indeed, each equivalence class of skew Dyck paths of length $2n$ can be represented by a word of length $2n$ using the symbols `1` and `0`. The symbol `1` represents a diagonal up step U, and the symbol `0` represents an absence of the step U, or a down step D in a Dyck path. For example, Figure 2 shows the case for the paths of semilength 2.

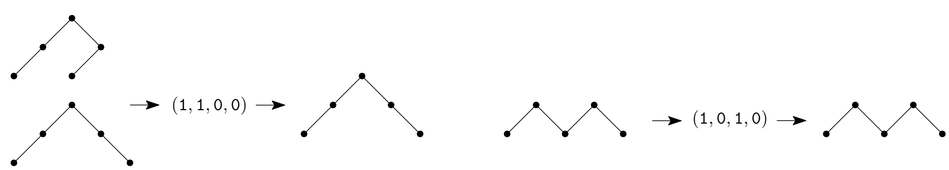

Figure 2: Bijection between Dyck paths and U-equivalence classes of $\mathcal{SD}$.

2.2. The Pattern L

In order to study the equivalence classes modulo L, we define the subclass $\mathcal{L}$ of skew Dyck paths avoiding the patterns UDD and DDD, and where all occurrences of UDU and DDU are at height 0.

Theorem 1. *There is a bijection between $\mathcal{L}$ and the set of L-equivalence classes of $\mathcal{SD}$.*

Proof. First, we will prove that for every $P \in \mathcal{SD}$ there exists $P' \in \mathcal{L}$ such that P and P' belong to the same equivalence class modulo left steps. Let us consider the sequence of skew Dyck paths $P = P_0, P_1, \ldots, P_k = P'$ with $k \geq 1$, where for any i, $0 \leq i \leq k-1$, P_{i+1} is obtained from P_i by performing the first possible transformation among the four described below, until the path belongs to $\mathcal{L}$.

(1) Remove occurrences UDU at height greater than 0.

If P_i contains such a pattern, $P_i = \alpha UDU\beta$, then we define $P_{i+1} = \alpha DUU\beta$.

Notice that if P_{i+1} avoids UDU at height $h > 0$ and contains an occurrence of DD at height $k > 0$, then before it, there is necessarily an occurrence UUU at height $k-1$.

(2) Remove occurrences DDU at height greater than 0.

If P_i contains such a pattern, then $P_i = \alpha DDU\beta$ where α contains the pattern UUU. Considering the rightmost pattern UUU in α, we have $P_i = \alpha_1 UUU\alpha_2 DDU\beta$ and we define $P_{i+1} = \alpha_1 UDU\alpha_2 DUU\beta$.

(3) Remove occurrences UDD.

If P_i contains such a pattern, $P_i = \alpha UDD\beta$, then we define $P_{i+1} = \alpha DUD\beta$.

(4) Remove occurrences DDD.

If P_i contains such a pattern, then $P_i = \alpha DDD\beta$ where α has the pattern UUU. Considering the rightmost UUU in α, we have $P_i = \alpha_1 UUU\alpha_2 DDD\beta$ and we define $P_{i+1} = \alpha_1 UDU\alpha_2 DUD\beta$.

Since the process do not modify the positions of the left steps L, the paths P and P' belong to the same equivalence class. An example of this process is shown in Figure 3.

Now, let us prove that if P and P' with the same length ℓ both belong to $\mathcal{L}$ and are in the same equivalence class modulo L, then $P = P'$. Any $P \in \mathcal{L}$ can be decomposed

$$P = \left(\prod_{i=0}^{n-1} \alpha_i L^{k_i}\right) \alpha_n,$$

where L does not belong in α_i, $0 \leq i \leq n$, and $k_i \geq 1$ for $0 \leq i \leq n-1$.

First, if P does not contain L, then P is a Dyck path, and as $P \in \mathcal{L}$, $P = (UD)^{\ell/2}$. With a similar argument for P', we obtain directly $P = P'$.

The second case is when P and P' have at least one occurrence of L. Since P belongs to $\mathcal{L}$, we can determine the form of α_i.

- Case $i = 0$. We have $\alpha_0 = (UD)^{s_1} U^{s_2} D$ with $s_1 \geq 0$ and $s_2 \geq 2$.
- Case $1 \leq i < n$. The endpoint of $\alpha_{i-1} L^{k_{i-1}}$ must be at the height $h \geq 1$.
  - If $h = 1$, $\alpha_i = D(UD)^{t_1} U^{t_2} D$ with $t_1 \geq 0$ and $t_2 \geq 2$.

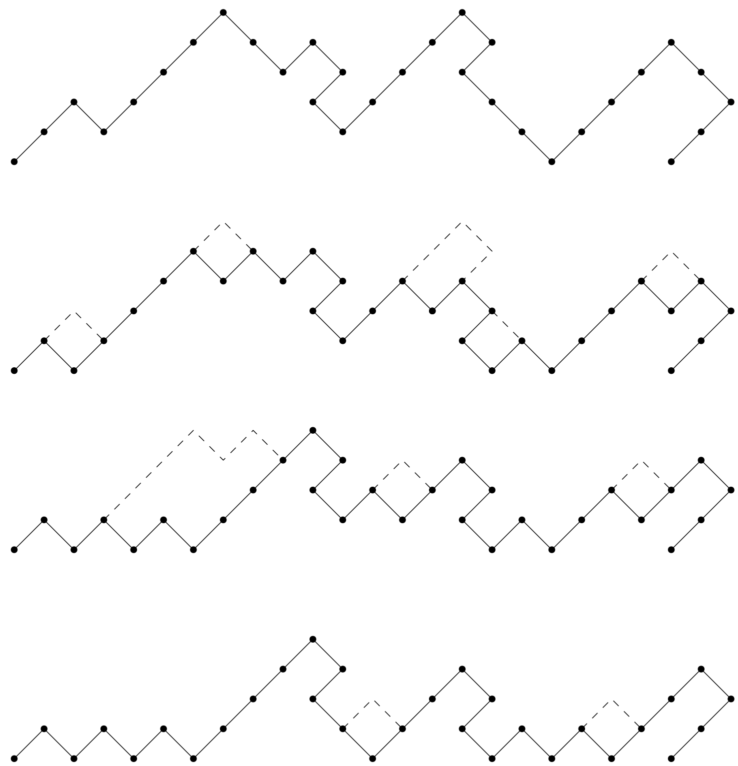

Figure 3: Construction of a path in $\mathcal{L}$ from a skew Dyck path.

  - If $h = 2$, α_i is either D, $D^2(UD)^{t_3}U^{t_4}D$, or $DU^{t_5}D$ with $t_3 \geq 0$, $t_4 \geq 2$, $t_5 \geq 1$.
  - If $h \geq 3$, α_i is either D, D^2, or DU^tD with $t \geq 1$.

- Case $i = n$. The endpoint of $\alpha_{n-1}L^{k_{n-1}}$ must be at height $h = 0, 1, 2$.
  - If $h = 0$, $\alpha_n = \lambda$.
  - If $h = 1$, $\alpha_n = D(UD)^{r_1}$ with $r_1 \geq 0$.
  - If $h = 2$, $\alpha_n = D^2(UD)^{r_2}$ with $r_2 \geq 0$.

Now, let us suppose that $P \neq P'$. Since P and P' are in the same class, we have $P' = \left(\prod_{i=0}^{n-1} \alpha'_i L^{k_i}\right) \alpha'_n$ with $|\alpha_j| = |\alpha'_j|$, and there exists j such that $\alpha_j \neq \alpha'_j$. We take the greatest j satisfying this condition.

Let us assume $j = n$. If $r = |\alpha_n| = |\alpha'_n|$ is even, then we have either $\alpha_n = \alpha'_n = \lambda$ or $\alpha_n = \alpha'_n = D^2(UD)^{\frac{r-2}{2}}$; if $r = |\alpha_n| = |\alpha'_n|$ is odd, then we have $\alpha_n = \alpha'_n = D(UD)^{\frac{r-1}{2}}$ which gives a contradiction with $P \neq P'$.

Let us assume $1 \leq j < n$. In this case, the endpoints of α_j and α'_j are at the same height h. Since $P \neq P'$ and using the form of α_j defined above, we necessarily have $|\alpha_j| \geq 3$. So, let us suppose that α_j and α'_j are of the form DU^tD or $D(UD)^{t_1}U^{t_2}D$.

Let analyze the following two cases:

- $\alpha_j = DU^{t_1}D$ and $\alpha'_j = DU^{t_2}D$. Since $|\alpha_j| = |\alpha'_j|$, we have $\alpha_j = \alpha'_j$ which is a contradiction.

- $\alpha_j = DU^tD$ and $\alpha'_j = D(UD)^{t_1}U^{t_2}D$ with $t_1, t_2 \geq 1$. Due to the fact that they have the same length, we conclude that $t = 2t_1 + t_2$, and due to the fact that they belong to $\mathcal{L}$, we conclude that $t_2 = h + 1$, as the occurrence UDU must appear at height 0. As a result, $t > h + 1$ and consequently, the path α_j is not well defined because it crosses the x-axis. So, this case throws a contradiction.

- $\alpha_j = D(UD)^{t_1}U^{t_2}D$ and $\alpha'_j = D(UD)^{s_1}U^{s_2}D$ with $t_1, s_1 \geq 1$ $t_2, s_2 \geq 2$. As they have the same length, $2t_1 + t_2 = 2s_1 + s_2$. If $\alpha_j \neq \alpha'_j$ then without loss of generality we can suppose $t_1 < s_1$ and conclude that $t_2 \geq 2 + s_2$. This establishes a contradiction because α_j would have a pattern UDU at height greater than 2.

A similar reasoning allows us to conclude that $\alpha_0 = \alpha'_0$ and therefore, $P = P'$. □

Theorem 2. *The generating function of equivalence classes modulo L is given by $L(x)$, where $L(x)$ is a root of*

$$(4x - 1)L^4(x) - 3L^3(x) - (7x - 10)L^2(x) + (5x - 8)L(x) - x + 2 = 0.$$

The series expansion of $L(x)$ is

$$1 + x + 2x^2 + 4x^3 + 9x^4 + 21x^5 + 50x^6 + 123x^7 + 308x^8 + 781x^9 + 2008x^{10} + O(x^{11}).$$

Proof. Let us define the following subsets of $\mathcal{SD}$:

- $\mathcal{A}$ is the subset of paths in $\mathcal{SD}$ avoiding UDU, DDU, UDD, and DDD, and not ending with an occurrence of UD or DD;

- $\mathcal{B}$ is the subset of paths in $\mathcal{SD}$ avoiding UDU, DDU, UDD, and DDD;

- $\mathcal{C}$ is the subset of paths in $\mathcal{SD}$ avoiding UDU, DDU, UDD, and DDD, and not ending with an occurrence of D.

In order to find the generating function of $\mathcal{L}$ we consider the first return decomposition of a path $P \in \mathcal{L}$: either P is empty, or $P = U\alpha D\beta$, or $P = U\gamma L$, with $\alpha \in \mathcal{A}$, $\beta \in \mathcal{L}$ and $\gamma \in \mathcal{B}\backslash\{\lambda\}$. Consequently, if $L(x), A(x), B(x)$, and $C(x)$ are respectively the generating functions for the sets $\mathcal{L}, \mathcal{A}, \mathcal{B}$, and $\mathcal{C}$, we obtain the functional equation (cf. [8])

$$L(x) = 1 + xA(x)L(x) + x(B(x) - 1).$$

A nonempty path $P \in \mathcal{A}$ is either $P = U\alpha D\beta$ or $P = U\gamma L$ with $\alpha \in \mathcal{C}\backslash\{\lambda\}$, $\beta \in \mathcal{A}$ and $\gamma \in \mathcal{B}\backslash\{\lambda\}$. Therefore, we have

$$A(x) = 1 + x(C(x) - 1)A(x) + x(B(x) - 1).$$

A nonempty path $P \in \mathcal{B}$ is either $P = UD$, or $P = U\alpha D\beta$ or $P = U\alpha' D$, or $P = U\gamma L$, with $\alpha \in \mathcal{C}\backslash\{\lambda\}$, β, $\gamma \in \mathcal{B}\backslash\{\lambda\}$, and $\alpha' \in \mathcal{A}\backslash\{\lambda\}$. Therefore we have

$$B(x) = 1 + x + x(C(x) - 1)(B(x) - 1) + x(A(x) - 1) + x(B(x) - 1).$$

A nonempty path $P \in \mathcal{C}$ is either $P = U\alpha D\beta$, or $P = U\gamma L$ with $\alpha, \beta \in \mathcal{C}\backslash\{\lambda\}$ and $\gamma \in \mathcal{B}\backslash\{\lambda\}$. Therefore we have

$$C(x) = 1 + x(C(x) - 1)^2 + x(B(x) - 1).$$

Using Gröbner basis on the polynomial equations for $L(x), A(x), B(x)$, and $C(x)$ we obtain the desired result. $\square$

Remark 1. Since the generating function of equivalence classes modulo L satisfies an algebraic equation of order four, the counting sequence $a_n := [x^n]L[x]$ satisfies a recurrence relation with polynomial coefficients. This can be automatically solved with Kauers's algorithm [9]. In particular we obtain that a_n satisfies the recurrence relation

$$p_0(n)a_n + p_1(n)a_{n+1} + p_2(n)a_{n+2} + p_3(n)a_{n+3} + p_4(n)a_{n+4} + p_5(n)a_{n+5} + p_6(n)a_{n+6} = 0,$$

for $n \geq 6$, where $p_i(n)$ $(i = 0, 1, \ldots, 6)$ are polynomials in n. From the package `Asymptotics` for Mathematica, see [10], we conjecture that

$$a_n \sim c \cdot \frac{\left(\frac{3}{2} + \sqrt{2}\right)^n}{n^{3/2}},$$

where $c \approx 2.111031048$.

2.3. Equivalence Classes of Size One

In the previous section, we proved that equivalence classes of size one are in one-to-one correspondences with the set $\mathcal{B}$, which means that every skew Dyck path in $\mathcal{B}$ is entirely fixed by the position of its L steps. Consequently, the number of skew Dyck paths in $\mathcal{B}$ with exactly k occurrences of L is finite. In this section, we study the number of skew Dyck paths in $\mathcal{B}$ having exactly n left steps L. We take two points of view: first, we provide an expression of the generating function $B(y) = \sum_{n\geq 0} b_n y^n$, where b_n is the number of skew Dyck paths in $\mathcal{B}$ with n left steps L, and next we provide a recursive formula for b_n.

First, it is worth noticing that as a skew path in $\mathcal{B}$ avoids UDU, DDU, UDD, and DDD, the last left step of a path must be at the last four positions. Consequently, as shown in Figure 4, there are four paths with one left step.

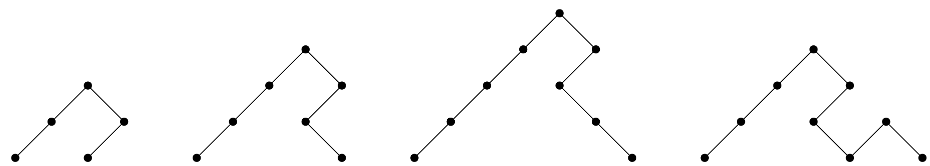

Figure 4: Skew Dyck paths in $\mathcal{B}$ with one occurrence of L.

2.3.1. Using Generating Functions

Theorem 3. *The generating function $B(y)$ for the number of skew Dyck paths in $\mathcal{B}$ with respect to the number of left steps L is a root of*

$$y^2B^4(y) + (y^3 + y^2)B^3(y) + (2y^2 + y)B^2(y) + (3y - 1)B(y) + 1 = 0.$$

The series expansion of $B(y)$ is

$$1 + 4y + 24y^2 + 181y^3 + 1549y^4 + 14312y^5 + 139142y^6 + 1402646y^7 + 14527909y^8 + O(x^9).$$

Proof. As mentioned above, a nonempty skew Dyck path in $\mathcal{B}\backslash\{UD\}$ ends with either L, LD, LDD, or $LDUD$. Let $B_1(y)$ (resp. $B_2(y)$, $B_3(y)$, $B_4(y)$) be the generating function for the number of skew Dyck paths in $\mathcal{B}$ ending with L (resp. LD, LDD, $LDUD$).

A skew Dyck path P ending with L can be written $P = \alpha U\beta L$ where $\alpha \in \mathcal{B}$ is either empty or ends with LD and $\beta \in \mathcal{B}$. So, we have the functional equation

$$B_1(y) = (1 + B_2(y))y(1 + B_1(y) + B_2(y) + B_3(y) + B_4(y)).$$

A skew Dyck path P ending with LD can be written $P = \alpha U\beta D$ where $\alpha \in \mathcal{B}$ is either empty or ends with LD and $\beta \in \mathcal{B}$ and ends with L. So, we have

$$B_2(y) = (1 + B_2(y))B_1(y).$$

A skew Dyck path P ending with LDD can be written $P = \alpha U\beta D$ where $\alpha \in \mathcal{B}$ is either empty or ends with LD and $\beta \in \mathcal{B}$ and ends with LD. So, we have

$$B_3(y) = (1 + B_2(y))B_2(y).$$

A skew Dyck path P ending with $LDUD$ can be written $P = \alpha UD$ where $\alpha \in \mathcal{B}$ ends with LD. So, we have

$$B_4(y) = B_2(y).$$

Using Gröbner basis on the polynomial equations for each generating function we obtain the desired result. □

Remark 2. Let b_n be the n-th coefficient of $B[y]$, that is, $b_n := [y^n]B[y]$. From a similar approach as in the Remark 1, we conjecture that

$$b_n \sim c \cdot \frac{\alpha^n}{n^{3/2}},$$

where $c \approx 0.6011640677$ and $\alpha = 0.650096$.

2.3.2. Using Recurrence Relation

Theorem 4. *Let $a_i(n)$ denote a family of sequences where $a_0(n) = n$, and for all $i \geq 1$*

$$a_i(n) = a_{i-1}(n) + a_{i-1}(n+1) + \sum_{k=0}^{n-i-3} a_{i-1}(n+2-k).$$

The number of skew Dyck paths in $\mathcal{B}$ with exactly $k \geq 2$ left steps, is given by $a_{k-1}(k+3)$.

Proof. Let us denote by $\mathcal{B}_l$ the set of all prefixes ending with L of skew Dyck paths in $\mathcal{B}$. Such a path will be called a *meander*.

Let us prove by induction on i that $a_i(n)$ is the number of meanders in $\mathcal{B}_l$ with $i+2$ occurrences of L and ending with an occurrence of L at height $n-i-5$.

First, let us assume that $i = 0$. A meander in $\mathcal{B}_l$ ending at height $n-5$, with two occurrences of L, ends with either LL, LDL, $LDDL$, or LDU^kDL, $1 \leq k \leq n-3$. Therefore, there are $n-3+3 = a_0(n)$ paths. Figure 5 shows the case when $n = 5$.

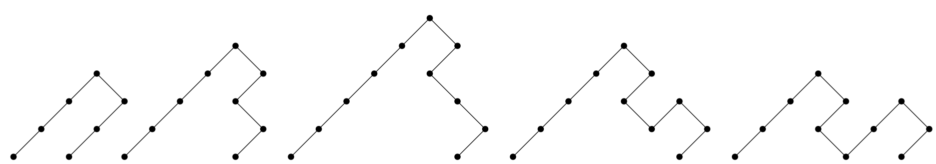

Figure 5: Skew Dyck paths in $\mathcal{B}$ with 2 occurrences of L and $s_1 = 1$

Now, assume that $a_j(n)$ satisfies the statement for $j \leq i$. Let us count the number $a_{i+1}(n)$ of meanders in $\mathcal{B}_l$ ending at height $n-i-6$, with $i+3$ occurrences of L. Taking into account all possible ways of a meander in $\mathcal{B}_l$ ends:

- there are $a_i(n)$ such paths ending with LL,
- there are $a_i(n+1)$ such paths ending with LDL,
- there are $a_i(n+2)$ such paths ending with $LDDL$,
- there are $a_i(n+2-k)$ paths ending with LDU^kDL, $1 \leq k \leq n-i-4$.

Consequently, we have:

$$a_{i+1}(n) = a_i(n) + a_i(n+1) + a_i(n+2) + \sum_{k=1}^{n-i-4} a_i(n+2-k)$$

which completes the induction.

Finally, as skew Dyck paths in $\mathcal{B}\backslash\{UD\}$ end with L, LD, LDD, or $LDUD$, this implies that the number of skew Dyck paths in $\mathcal{B}$ with exactly $k \geq 2$ left steps is given by

$$a_{k-1}(k+3) = a_{k-2}(k+3) + a_{k-2}(k+4) + a_{k-2}(k+5) + a_{k-2}(k+4). \quad \square$$

3. Equivalence Classes Modulo Patterns of Length Two

In this section, we focus on equivalence classes modulo patterns of length two. We start by giving a general result that allows us to solve the cases of patterns that do not contain occurrences of L and DD. Indeed, for these patterns the number of p-equivalence classes on the set $\mathcal{SD}$ of skew Dyck paths also is the same on the set $\mathcal{D}$ of Dyck paths, which is already given in [1]. We also deal with the pattern DD and leave as an open question the cases of patterns in $\{DL, LD, LL\}$. We refer to Table 2 for an overview of these numbers for small values of the length (the last three cases are obtained experimentally).

Pattern	Sequence	OEIS	a_n, $1 \leq n \leq 9$
UU	$\frac{1-x+\sqrt{1-2x-3x^2}}{1-3x+x^2+x^3+(1-x^2)\sqrt{1-2x-3x^2}}$	A244886	1, 2, 4, 9, 22, 56, 147, 393, 1065
UD	$\frac{(1-x)(1-5x+7x^2-x^3)}{(1-2x)^2(1-3x+x^2)}$	A244885	1, 2, 5, 14, 41, 121, 354, 1021, 2901
DU	$\frac{1-2x}{1-3x+x^2}$	A001519	1, 2, 5, 13, 34, 89, 233, 610, 1597
DD	$\frac{2(1+x)}{x+x^2+(2+x)\sqrt{1-2x-3x^2}}$	New	1, 2, 5, 12, 31, 81, 216, 583, 1590
DL	Open question	New	1, 2, 3, 6, 12, 23, 49, 102, 212
LD	Open question	New	1, 1, 2, 4, 7, 15, 31, 62, 136
LL	Open question	New	1, 1, 2, 4, 8, 15, 30, 63, 134

Table 2: Number of p-equivalence classes for skew Dyck paths.

Proposition 1. *If p is a pattern of length at least 2 avoiding L and DD, then the number of p-equivalence classes is the same in $\mathcal{SD}$ and $\mathcal{D}$.*

Proof. Let p be a pattern of length at least 2 avoiding L and DD, and let us consider $P \in \mathcal{SD}$. We will show that there exists a path $P' \in \mathcal{D}$ such that P and P' belong to the same equivalence class.

We can decompose $P = \alpha_0 \prod_{i=1}^{n} p^{d_i} \alpha_i$ or $P = \alpha_0$, where α_i is a sub-skew Dyck path avoiding p and $d_i \geq 1$. We can define $P' = \alpha'_0 \prod_{i=1}^{n} p^{d_i} \alpha'_i$ or $P' = \alpha'_0$, depending on the decomposition of P, where each α'_i is obtained from α_i by replacing all steps L with D. As p and α'_i avoid L, $P' \in \mathcal{D}$. Moreover, in a skew Dyck path, L cannot be contiguous with a step U, which implies that the operation of changing a step L by D does not create a pattern p whenever p avoids DD. Consequently, P and P' belong to the same p-equivalence class. □

3.1. Pattern DD

Let $\mathcal{E}$ denote the set of skew Dyck paths where all occurrences of UDU are at height 0 or 1, all occurrences of UD^kL, with $k \geq 1$, are at height 0, and the patterns $UUDU$ and $UUDL$ do not appear. For instance, Figure 6 shows two skew Dyck paths that do not belong to $\mathcal{E}$, whereas Figure 7 shows a skew Dyck path that belongs to $\mathcal{E}$.

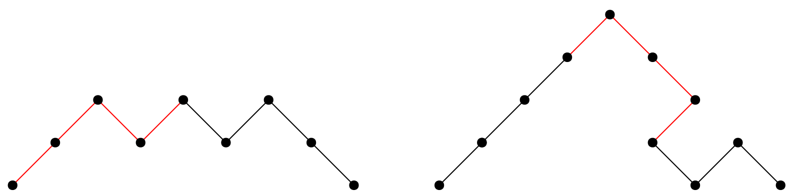

Figure 6: Skew Dyck paths that do not belong to $\mathcal{E}$.

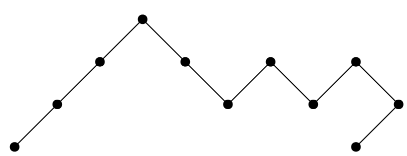

Figure 7: Skew Dyck path that belongs to $\mathcal{E}$.

Lemma 1. *For every skew Dyck path P, there exists a skew Dyck path $P' \in \mathcal{E}$ in the same equivalence class modulo DD.*

Proof. Let P be a skew Dyck path such that $P \notin \mathcal{E}$. Consider the sequence of skew Dyck paths $P = P_0, P_1, P_2, \ldots, P_{k-1}, P_k = P'$, $k \geq 1$, defined as follows:

For any i, $1 \leq i \leq k$, the skew Dyck path P_{i+1} is obtained from P_i by performing the first possible transformation among the four described below, until the path belongs to $\mathcal{E}$:

(1) Remove occurrences of $UUDU$.

If P_i has such a pattern, then $P_i = \alpha U^k DU\beta$ with $k \geq 2$ and α does not end with U. So, we define $P_{i+1} = \alpha UDU^k\beta$. Figure 8 shows a step of this process. Repeat this operation until the path does not contain $UUDU$.

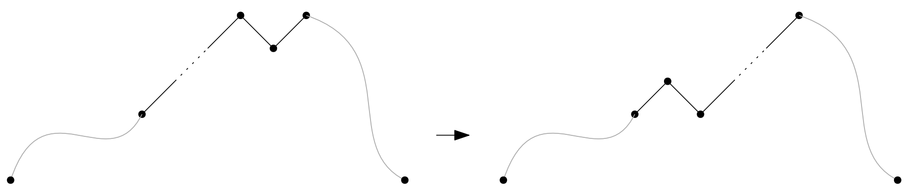

Figure 8: Removing $UUDU$.

(2) Remove occurrences $UUDL$ at height 0.

If P_i has such a pattern, $P_i = \alpha UUDL$, where α is a skew Dyck path, then we define $P_{i+1} = \alpha UDUD$. Figure 9 shows this process.

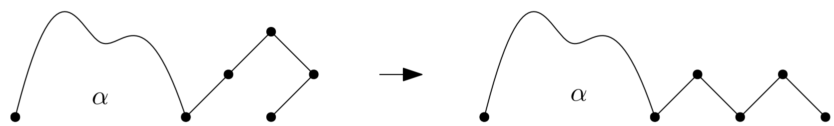

Figure 9: Removing $UUDL$ at height 0.

(3) Remove maximal occurrences of UD^kL^m, for $k, m \geq 1$, except when the occurrence is at height 0 with $m = 1$.

Since m is maximal, the chain L^m is followed by an occurrence of D^s for $s \geq 0$. We write $P_i = (\prod_{i=1}^{d-1} U^{s_i}\alpha_i)U^{s_d}D^kL^m\beta$ where α_i is nonempty and avoids U, and β is either empty or starts with D^s with $s \geq 1$. We consider three cases depending if $m \geq 3$, $m = 2$ and $m = 1$.

- Case $m \geq 3$. The maximum ordinate reached by the occurrence UD^kL^m is $p \geq m + k \geq 4$. Since the path reaches the ordinate $p \geq 4$, and since P_i avoids $UUDU$, there is at least one s_i such that $s_i \geq 3$; we choose the rightmost i.

  The path P_{i+1} is obtained by replacing U^{s_i} with $UDUU^{s_i-3}$, and by replacing UD^kL^m with UD^kUDL^{m-2}. This operation deletes the occurrence UD^kL^m and creates another occurrence UDL with maximum ordinate $p - 3$. See Figure 10.

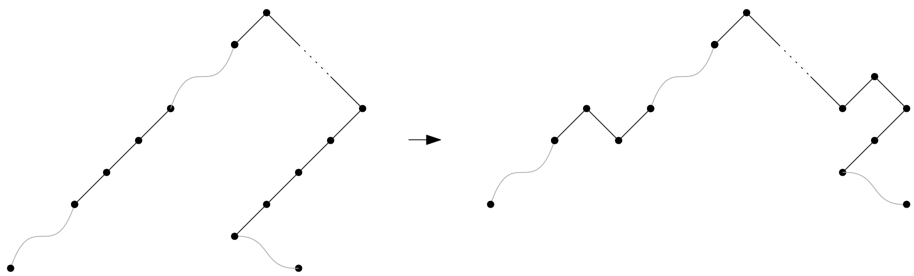

Figure 10: Removing UD^kL^m if $m \geq 3$.

- Case $m = 1$. The maximal ordinate reached by the occurrence $UD^kL^m = UD^kL$ is at least three, and (as above) there is some s_i such that $s_i \geq 3$; we choose the rightmost i.

  The path P_{i+1} is obtained by replacing U^{s_i} with $UDUU^{s_i-3}$, and by replacing $UD^kL^m = UD^kL$ with UD^kU. This operation deletes the occurrence UD^kL^m. See Figure 11.

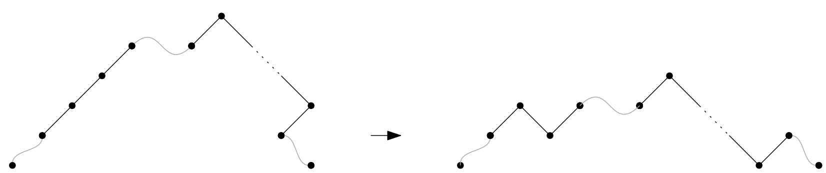

Figure 11: Removing UD^kL^m if $m = 1$.

- Case $m = 2$. The maximal ordinate p reached by the occurrence $UD^kL^m = UD^kL^2$ is at least three, and (as above) there is some s_i such that $s_i \geq 3$; we choose the rightmost i. Moreover, if $p \geq 5$ then either there is $s_i \geq 5$ or there are two indices i_0, i_1, such that $3 \leq i_0 \leq 4$ and $3 \leq i_1 \leq 4$; we choose the rightmost indices with these properties.

  If $s = 0$ then UD^kLL is at the end of P_i, and P_{i+1} is obtained by replacing UD^kLL with UD^kUD, and by replacing U^{s_i} with $UDUU^{s_i-3}$.

  Now, let us consider $s \geq 1$.

  - If UD^kLL is followed by D and P_i ends after D, then P_{i+1} is obtained by replacing UD^kLLD with UD^kUDL and by replacing the rightmost U^{s_i} for $s_i \geq 3$ with $UDUU^{s_i-3}$.
  - If UD^kLL is followed by DUU then we replace UD^kLLDU with UD^kUDUD and we replace the rightmost U^{s_i} for $s_i \geq 3$ with $UDUU^{s_i-3}$.

- If UD^kLL is followed by DUD then we replace UD^kLLDUD with UD^kUUDLD and we replace the rightmost U^{s_i} for $s_i \geq 3$ with $UDUU^{s_i-3}$.
- If UD^kLL is followed by DL then we replace UD^kLLDL with UD^kUUDL and either we replace the rightmost U^{s_i} for $s_i \geq 5$ with $UDUDUU^{s_i-5}$, or we replace $U^{s_{i_0}}$ with $UDUU^{s_{i_0}-3}$ and $U^{s_{i_1}}$ with $UDUU^{s_{i_1}-3}$ where i_0 and i_1 are defined above.
- If UD^kLL is followed by DD then we replace UD^kLLDD with UD^kUUDD and either we replace the rightmost U^{s_i} for $s_i \geq 5$ with $UDUDUU^{s_i-5}$, or we replace $U^{s_{i_0}}$ with $UDUU^{s_{i_0}-3}$ and $U^{s_{i_1}}$ with $UDUU^{s_{i_1}-3}$ where i_0 and i_1 are defined above.

All previous transformations either delete an occurrence UD^kL at height at least one, or decreases by at least one the maximal ordinate of one occurrence UD^kL, or decrease by one m in a subcase of $m = 2$. See Figure 12.

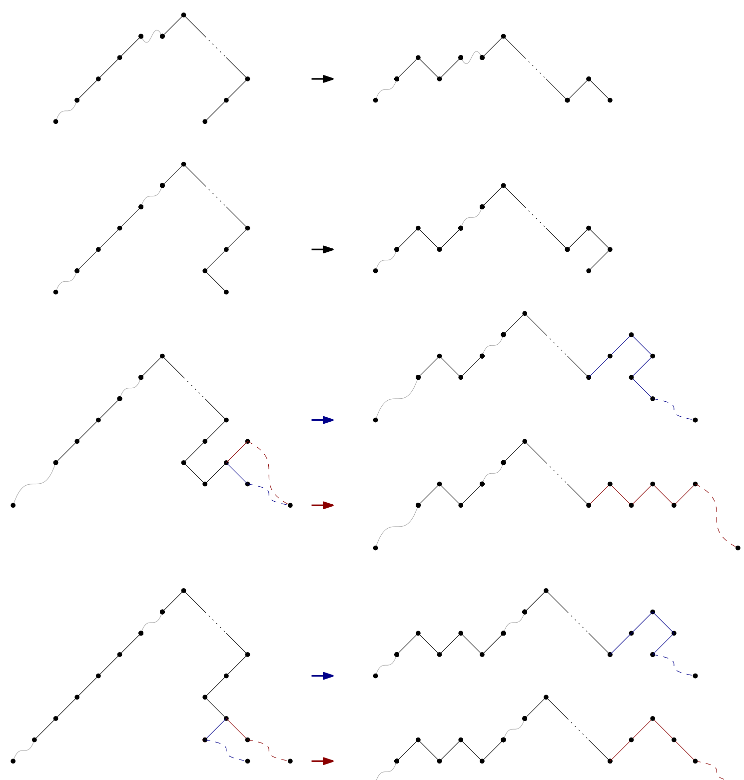

Figure 12: Removing UD^kL^m if $m = 2$.

(4) Remove UDU at height greater than 1. As the maximal ordinate p reached by the occurrence UDU satisfies $p \geq 3$, and as P_i avoids $UUDU$, there exists an occurrence U^3 at height $p-3$ at the left (we take the rightmost possible). The path P_{i+1} is obtained by exchanging the two occurrences U^3 and UDU, which decreases by at least one the height of the occurrence UDU.

After applying the previous process, we obtain a path $P' \in \mathcal{E}$. Since all transformations do not change the positions of occurrences DD, P and P' belong to the same equivalence class. An example of this process is shown in Figure 13.

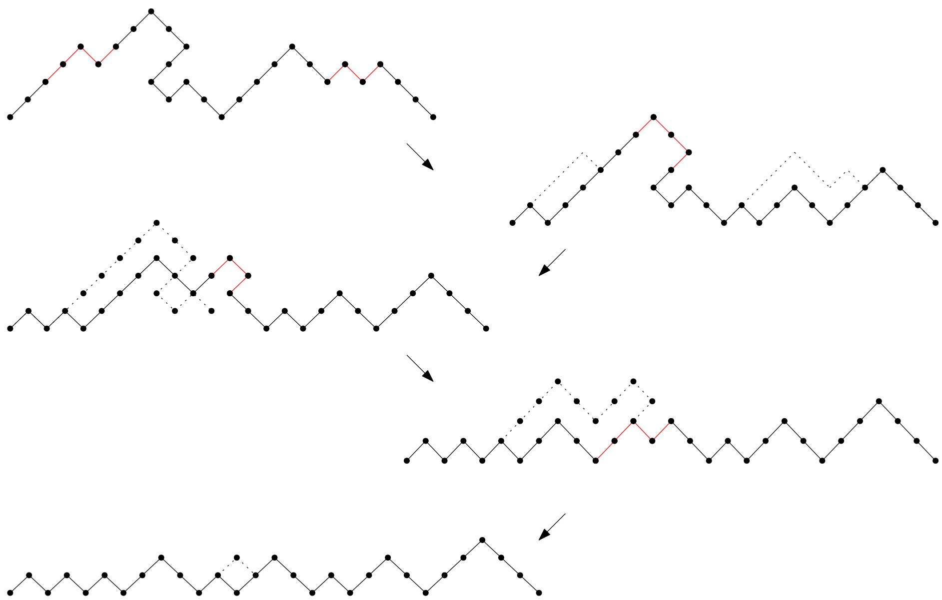

Figure 13: An example of the process described in the proof of Lemma 3.2

□

Theorem 5. *There is a bijection between $\mathcal{E}$ and the set of DD-equivalence classes of $\mathcal{SD}$.*

Proof. Considering Lemma 3.2, it suffices to prove that if P and P' have the same length in $\mathcal{E}$ and lie in the same equivalence class, then $P = P'$. We decompose $P = \alpha_0 \prod_{i=1}^{n} D^{k_{i-1}} \alpha_i$ (resp. $P' = \alpha'_0 \prod_{i=1}^{n} D^{k_{i-1}} \alpha'_i$) where α_i (resp. α'_i) do not contain the pattern DD and $k_i \geq 2$ (resp. $k'_i \geq 2$) are taken to be maximal.

First, if P and P' do not have the pattern DD, then $P = \alpha_0$ and $P' = \alpha'_0$. Moreover $P = \alpha_0$ avoids $UUDU$ and $UUDL$, which implies that $\alpha_0 = (UD)^m$ with

$m = |P|/2$. By a similar argument, we obtain $P = P'$.

Secondly, let us assume that P and P' have at least one occurrence of DD. Since P and P' belong to the same equivalence class, we have $|\alpha_i| = |\alpha'_i|$ for $0 \le i \le n$. Notice that we necessarily have $n \ge 1$. Now, we determine the form of α_i and α'_i.

- α_i and α'_i for $0 \le i \le n-1$.

  Since α_0 (resp. α'_0) avoids DD, $UUDU$, and $UUDL$, we can write $\alpha_0 = (UD)^{s_1}U^{s_2}$ with $s_2 \ge 2$ and $s_1 \ge 0$ (resp. $\alpha'_0 = (UD)^{s'_1}U^{s'_2}$ with $s'_2 \ge 2$ and $s'_1 \ge 0$). If $n > 1$, α_1 cannot starts with L (otherwise P contains UD^kL), thus it starts with U, and with the same argument as above it has the same form as α_0, i.e. $\alpha_1 = (UD)^{t_1}U^{t_2}$ with $t_2 \ge 2$ and $t_1 \ge 0$. Repeating this argument, α_i and α'_i are all of the same form for $1 \le i \le n-1$.

- α_n and α'_n.

  We have three cases depending on the final ordinate of

  $$Q = \alpha_0 \left(\prod_{i=1}^{n-1} D^{k_{i-1}} \alpha_i \right) D^{k_{n-1}}.$$

  Case 1. The path Q ends at height 0. Since P avoids $UUDU$ and $UUDL$, α_n does not contain an occurrence UU. Therefore α_n is either the empty path λ or of the form $(UD)^s$.

  Case 2. The path Q ends at height 1. The only one possibility is $\alpha_n = (UD)^sL$.

  Case 3. The path Q ends at height greater than 1. This case is not possible because α_n does not start with D, avoids DD and avoids UD^kL at height greater than 0.

Now let us prove that that $\alpha_i = \alpha'_i$ for every i. With the reasoning done above, $\alpha_n \in \{\lambda, (UD)^sL, (UD)^s\}$ and $\alpha'_n \in \{\lambda, (UD)^tL, (UD)^t\}$. Since $|\alpha_n| = |\alpha'_n|$, we have $\alpha_n = \alpha'_n$.

For a contradiction we suppose that there exists $i < n$, such that $\alpha_i \ne \alpha'_i$ (we take the greatest j satisfying this condition). With the reasoning above, we have $\alpha_i = (UD)^{s_1}U^{s_2}$ and $\alpha'_i = (UD)^{s'_1}U^{s'_2}$ with $2s_1 + s_2 = 2s'_1 + s'_2$ since $|\alpha_i| = |\alpha'_i|$. Without loss of generality we can assume $s_1 < s'_1$ because α_i and α'_i are different. This implies that $s_2 \ge 2 + s'_2$. Since α_i and α'_i end at the same height in P and P', this means that P' contains an occurrence UDU at height at least 2 which is a contradiction.

In summary, $\alpha_i = \alpha'_i$ for every i and consequently, $P = P'$. □

Before proving Theorem 1, we need the preliminary results shown in Lemmas 2 and 3. Let $\mathcal{F}$ be the set of all Dyck paths where all occurrences of UDU are at height 0 and not starting with UDU; let $\mathcal{G}$ be the set of Dyck paths that do not contain UDU and, let $\mathcal{H}$ be the set of Dyck paths where all occurrences of UDU are at height 0. It is well known (see [14]) that the generating function of $\mathcal{G}$, $G(x)$, is given by the expression $G(x) = 1+xM(x)$, where $M(x)$ is the generating function for the number of Motzkin paths, i.e., $G(x) = (1+x-\sqrt{1-2x-3x^2})/2x$.

Lemma 2. *The generating function of the set $\mathcal{H}$ is given by the expression*

$$H(x) = \frac{1}{1-xG(x)}.$$

Proof. A Dyck path in $\mathcal{H}$ is either empty or of the form $U\alpha D\beta$, with $\alpha \in \mathcal{G}$ and $\beta \in \mathcal{H}$. So, $H(x)$ satisfies the relation $H(x) = xG(x)H(x) + 1$ that induces the required result. □

Lemma 3. *The generating function of the set $\mathcal{F}$ is given by*

$$F(x) = \frac{x^2+x-2-x\sqrt{-3x^2-2x+1}}{x-1-\sqrt{-3x^2-2x+1}}.$$

Proof. A Dyck path in $\mathcal{F}$ is either empty, or UD, or $U\alpha D\beta$ where $\alpha \in \mathcal{G}\backslash\{\lambda\}$ and $\beta \in \mathcal{H}$. We conclude that $F(x)$ satisfies the relation $F(x) = 1+x+x(G(x)-1)H(x)$ which gives the required result. □

Theorem 6. *The generating function of the set $\mathcal{E}$ is given by*

$$E(x) = \frac{2(1+x)}{x+x^2+(2+x)\sqrt{1-2x-3x^2}}.$$

The series expansion of $E(x)$ is

$$1+x+2x^2+5x^3+12x^4+31x^5+81x^6+216x^7+583x^8+1590x^9+O\left(x^{10}\right).$$

Proof. A skew Dyck path in $\mathcal{E}$ is empty, is $U\alpha D\beta$, or is $U\gamma L$, where $\alpha \in \mathcal{F}$, $\beta \in \mathcal{E}$ and $\gamma \in \mathcal{F}\backslash\{\lambda, UD\}$. So, $E(x)$ satisfies the relation $E(x) = 1+xF(x)E(x)+x(F(x)-1-x)$, which is equivalent to

$$E(x) = \frac{1+x(F(x)-1-x)}{1-xF(x)}.$$

□

Let e_n be the number of DD-equivalence classes for skew Dyck paths. That is, $e_n = [x^n]E(x)$ for all $n \geq 0$. In Theorem 7 we give an asymptotic approximation for the sequence e_n. To accomplish this goal we use the singularity analysis method for finding an asymptotic expression of the coefficients of a generating function (see for example [8] for the details).

Theorem 7. *The sequence e_n has the asymptotic approximation*

$$e_n \sim 21\sqrt{\frac{3}{4\pi n^3}} \cdot 3^n.$$

Proof. The dominant singularity of the generating function $E(x)$ is $1/3$, that is, the smallest positive root of $1 - 2x - 3x^2$. Around the point $1/3$ the expansion of $E(x)$ is given by

$$E(x) = 6 - 21\sqrt{3(1-3x)} + O(1-3x).$$

The singularity analysis allows the transfer of the above equality to the asymptotic approximation of the coefficients. □

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