Quelques ballades combinatoires

Jean-Luc Baril et al.

April 18, 2025

- Mémoire de thèse
- Mémoire d'habilitation à diriger des recherches
- Ballades publiées dans des revues internationales à comités de lecture

N° d'ordre : 1455

THESE

Présentée à

L'UNIVERSITE BORDEAUX I

ECOLE DOCTORALE DE MATHEMATIQUES ET INFORMATIQUE

PAR **Jean-Luc BARIL**

POUR OBTENIR LE GRADE DE

DOCTEUR

SPECIALITE : MATHEMATIQUES PURES

AUTOUR DE L'ALGORITHME DE VORONOI :
CONSTRUCTION DE RESEAUX EUCLIDIENS

Soutenue le 25 janvier 1996

Après avis de Mme Vallée et M. Sigrist *Rapporteurs*

Devant la commission d'examen composée de :

M	F. SIGRIST	Professeur (Univ. de Neuchâtel)	**Président**
M	J. MARTINET	Professeur (Univ. Bordeaux I)	**Rapporteur**
M	Ch. BATUT	Maître de Conférences (Univ. Bordeaux I)	**Examinateurs**
Mme	A-M. BERGÉ	Professeur (Univ. Bordeaux I)	
M	A. JOUX	Ingénieur Armement (Rennes)	
Mme	G. NEBE	Maître de Conférences (Univ. d'Aix-La-Chapelle)	
Mme	B. VALLÉE	Professeur (Univ. de Caen)	

- 1996 -

Remerciements

Je tiens tout d'abord à exprimer mes remerciements à mon directeur de thèse Anne-Marie BERGÉ. Sa disponibilité, son aide, ses encouragements, son souci de rigueur et d'efficacité ont été déterminants pour l'aboutissement de ce travail. Tout au long de notre collaboration, j'ai pu apprécier ces grandes qualités humaines et scientifiques.

Je remercie également Jacques MARTINET. Il a su, lors de "conversations de couloir", me faire bénéficier d'une infime partie de sa culture mathématique. Une très grande part de ma reconnaissance lui revient.

Je voudrais aussi remercier l'ensemble des membres du laboratoire d'algorithmique, les auteurs du système de calculs PARI, Ch. BATUT, D. BERNARDI, H. COHEN et M. OLIVIER, et en particulier Ch. BATUT dont nous avons aussi utilisé certains programmes non intégrés au système PARI. Je n'oublie pas les deux ingénieurs Ch. LABESSE et P. CRUBILLÉ, qui, grâce à leurs compétences, m'ont initié aux joies du clavier.

Brigitte VALLÉE et François SIGRIST ont accepté d'être rapporteurs de cette thèse, et me font l'honneur de figurer dans le jury. Je tiens à leur exprimer toute ma gratitude pour avoir répondu favorablement à notre sollicitation.

Je remercie également G. NEBE qui m'a apporté plusieurs commentaires très profitables concernant mes travaux. Sa présence dans le jury, ainsi que celle de A. JOUX sont pour moi un grand honneur.

Je salue mes amis et collègues thésards des salles 200 et 300 ; ils ont créé une ambiance agréable dans le laboratoire. Plus particulièrement, je remercie H. NAPIAS pour nos échanges mathématiques et pour le travail qu'elle a fourni lors de la relecture de cette thèse.

Merci à Mauricette JAUBERT et Daniel YNBOURG, qui ont assuré l'impression de cette thèse. Je n'oublierai certainement pas les nombreuses discussions avec eux...

Enfin, je ne pourrais terminer sans une pensée affectueuse à ma famille et mes amis, qui ont su me soutenir en toutes occasions.

TABLE DES MATIÈRES

- INTRODUCTION ET NOTATIONS.
  Introduction.
  Notations.
    1 Quelques généralités.
    2 Réseaux-Formes quadratiques.
    3 Réseaux parfaits et eutactiques.
    4 Algorithme de Voronoï.

- Chapitre I. RÉSEAUX À SECTION HYPERPLANE PARFAITE.
  §1 Introduction et notations.
  §2 Description de l'algorithme de Laihem.
  §3 Amélioration de l'algorithme.
  §4 Résultats.
    §4.1 Dimension 8.
    §4.2 Dimension 9.
    §4.2 Dimension 10.
    §4.2 Dimension 11.

- Chapitre II. DENSITÉ DES RÉSEAUX EN PATCHWORK
  §1 Introduction et notations.
  §2 Étude de la constante d'Hermite sur $\mathcal{F}$.
  §3 Expression matricielle de la $\mathcal{F}$-perfection.
  §4 Algorithme de Voronoï pour la famille $\mathcal{F}$.
  §5 Améliorations de l'algorithme.
  §6 Résultats.

- Chapitre III. ALGORITHME DE VORONOI ET PROBLÈMES DE DUALITÉ
  §1 Introduction et notations.
  §2 Cas des dimensions 6 et 7.
    §2.1 Étude du chemin de Voronoï reliant $\mathbb{E}_6$ à $\mathbb{E}_6^*$.
    §2.2 Étude du chemin de Voronoï reliant $\mathbb{E}_7$ à $\mathbb{E}_7^*$.
  §3 Étude de la dimension 5.
  §4 Étude de la dimension 3.
  §5 Étude de la dimension 9.

- ANNEXES
- RÉFÉRENCES

Introduction

Cette thèse traite de problèmes de géométrie des nombres liés aux empilements de sphères associés à un réseau euclidien. En particulier, on construit algorithmiquement des réseaux pour lesquels la densité de cet empilement est un maximum local (réseaux extrêmes) : leur densité ne peut que diminuer sous l'effet d'une déformation assez petite. Pour étudier ces réseaux, Korkine et Zolotareff introduisent en 1875, en termes de formes quadratiques, la notion de perfection, plus faible que celle d'extrémalité mais qui a l'avantage de pouvoir s'exprimer dans le cadre de l'algèbre linéaire. Traduite géométriquement, cette notion fait intervenir les vecteurs minimaux du réseau (c'est-à-dire les plus courts vecteurs non nuls, dont le carré de la norme euclidienne est par définition la norme du réseau) :

Un réseau est parfait s'il perd des vecteurs minimaux dès qu'on le déforme. (La définition formelle est donnée à la fin de ce chapitre.)

En 1908, Voronoi reprend l'etude des réseaux parfaits. Il prouve que, pour une dimension n donnée, ils sont en nombre fini à similitude près. Il introduit, dans l'espace des réseaux, une relation de contiguïté entre réseaux parfaits pour laquelle le graphe est connexe. Il donne ainsi un algorithme permettant d'obtenir, pour une dimension donnée, toutes les classes de similitude des réseaux parfaits à partir de l'un d'entre eux (cf. 0.4). C'est autour de cet algorithme que s'articulent les trois chapitres de cette thèse.

Cet algorithme permet à Voronoï de retrouver les 8 réseaux parfaits de dimensions inférieures ou égales à 5 (liste établie par Korkine et Zolotareff) et d'amorcer l'étude de la dimension 6. [Incidemment, Voronoï découvre un réseau parfait non extrême, ce qui le conduit à introduire la notion d'eutaxie.] En 1957, cette étude est reprise avec succès (et sans ordinateur) par Barnes ([Bar1],[Bar2]).

Implanté sur ordinateur, cet algorithme énumère en quelques heures les réseaux parfaits jusqu'à la dimension 6. Toutefois, le nombre des réseaux à traiter croît très rapidement avec n, et jusqu'à la fin des années 80, la dimension 7 paraissait hors d'atteinte par cette méthode (bien que plusieurs résultats partiels aient été obtenus par Stacey, Conway et Sloane, etc.). Pourtant, F. Sigrist puis D.O. Jaquet reprennent cette méthode, qui permet à Jaquet de confirmer en 1990 que la liste des 33 réseaux parfaits de rang 7 établie par Stacey (par d'autres méthodes) est complète. Lors de cette recherche, Jaquet a dû faire tourner son programme plus de 100 jours pour obtenir tous les contigus du réseau de racines $\mathbb{E}_7$.

On peut donc penser que l'énumération complète des réseaux parfaits de rang 8 par cette méthode n'aboutirait pas au bout d'un temps raisonnable. Une première étape a toutefois été achevée par Jaquet, qui a déterminé les 48 contigus du réseau de racines $\mathbb{D}_8$ (ainsi que les 226 contigus du réseau $\mathbb{D}_9$ ([Jaq1], 1992)).

Afin de pouvoir prétendre traiter de façon analogue des dimensions supérieures, on est donc amené à restreindre l'étude à certaines *familles affines* de réseaux *a priori* intéressants

du point de vue de la densité, ou des symétries, ou de la dualité... et pour lesquelles on puisse utiliser un algorithme se déroulant dans un sous-espace de dimension raisonnable de l'espace des réseaux.

Un première idée consiste à définir une méthode de récurrence à partir de réseaux de dimensions inférieures connus. Systématisant la méthode d'*extension* introduite par Barnes ([Bar3]), on peut par exemple imposer aux réseaux une section hyperplane donnée de même norme. Cette méthode, mise au point par M. Laihem ([Lai], 1992) lui a permis d'obtenir, par un algorithme se développant dans un espace $\mathcal{T}$ de dimension 8, la liste des 1171 réseaux parfaits de rang 8 construits à partir des 30 réseaux parfaits de rang 7 différents de $\mathbb{A}_7$, $\mathbb{D}_7$, $\mathbb{E}_7$; il a aussi proposé une liste vraisemblable de 4 extensions parfaites pour ces trois réseaux. Pour ce dernier cas, la difficulté résidait dans l'énumération des contigus du réseau de racines $\mathbb{E}_8$.

Dans le premier chapitre de cette thèse, nous reprenons cette étude en perfectionnant l'algorithme pour réduire le temps d'exécution des programmes. Cela nous permet de confirmer qu'il n'y a que 4 réseaux parfaits de dimension 8 au-dessus des trois réseaux de racines de dimension 7. On construit aussi les extensions de réseaux classiques denses dans des dimensions supérieures (9, 10, 11). On obtient ainsi des réseaux (pour la plupart nouveaux) intéressants du point de vue de l'empilement de sphères associé.

Cette sélection de réseaux à section hyperplane parfaite élimine sans doute beaucoup de réseaux : alors que jusqu'à la dimension 6 presque tous les réseaux sont susceptibles d'une telle définition, seulement 15 des 33 réseaux parfaits de dimension 7 ont une section hyperplane parfaite ; et d'ailleurs H. Napias vient de construire, en dimension 8, par contiguïté à partir des réseaux les moins denses de la liste de Laihem, plus de cinq mille réseaux parfaits sans section hyperplane parfaite!

C'est pourquoi, dans le second chapitre de cette thèse, nous utilisons un autre procédé de recurrence. On restreint l'étude de la densité à une famille $\mathcal{F}$ de réseaux en *patchwork*, c'est-à-dire sommes directes de réseaux de même norme isométriques à des réseaux donnés. Comme précédemment, on recherche les réseaux de $\mathcal{F}$ les plus denses parmi ceux qui réalisent un maximum local de densité sur $\mathcal{F}$ (réseaux $\mathcal{F}$-extrêmes). On obtient pour ces réseaux des résultats analogues à ceux de Voronoï . Comme dans le cas classique, on les caractérise par des propriétés liées à leurs vecteurs minimaux ; notamment, on définit la notion de $\mathcal{F}$-perfection, moins restrictive que la notion classique de perfection, et qui, cependant, coïncide avec elle lorsque les sections imposées sont parfaites. Cette étude débouche donc bien sur un nouveau procédé de construction de réseaux parfaits par récurrence. Par exemple, lorsque chaque section imposée est de dimension 1, on obtient une méthode de construction des réseaux parfaits ayant une base de vecteurs minimaux. L'autre situation extrême consiste à imposer une section hyperplane et une section de rang 1. Il s'agit alors d'une étude proche de celle des extensions développée au premier chapitre, mais plus restrictive (il existe une extension de $\mathbb{D}_9$ qui n'est pas un tel patchwork).

D'une façon générale (que les réseaux de départ soient parfaits ou non) nous montrons que les réseaux $\mathcal{F}$-parfaits sont à similitude près en nombre fini et nous développons un algorithme à la Voronoï dans cette famille affine. Remarquons qu'une telle étude algorithmique de famille affine de réseaux, généralisant celle de Laihem, a été proposée par

Bergé, Martinet et Sigrist ([B-M-S], 1992), mais sous des hypothèses trop restrictives pour pouvoir être directement appliquée ici (voir chapitre II, Remarque suivant 3.3).

On applique cet algorithme à la construction de nouveaux réseaux parfaits de dimensions moyennes et, en particulier, on détermine tous les réseaux parfaits de rang 8, sommes directes d'un réseau parfait de rang 6 et du réseau parfait de rang 2 parmi lesquels on trouve exactement 53 réseaux parfaits sans section hyperplane parfaite.

On sait que les réseaux parfaits sont proportionnels à des réseaux entiers (sur lesquels le produit scalaire ne prend que des valeurs entières), et on constate que les plus denses d'entre eux sont généralement pairs (le produit scalaire ne prend que des valeurs paires). Notre étude de patchwork de réseaux impairs confirme ce fait expérimental.

Dans les deux premiers chapitres, on a utilisé l'algorithme de Voronoï (ou l'un de ses dérivés) pour construire des réseaux parfaits. Dans le troisième chapitre, nous l'utilisons comme moyen d'exploration de l'espace des réseaux. On s'y intéresse plus particulièrement aux propriétés de dualité. On construit notamment des familles à un paramètre stables par dualité, par exemple des chemins de Voronoï dans lesquels se situent un réseau et son dual, sur lesquels on recherche des réseaux *isoduaux* (semblables à leurs duals) : c'est ainsi que l'on a retrouvé les réseaux isoduaux les plus denses connus à ce jour pour les dimensions 5, 6 et 7, réseaux que Conway et Sloane ont construits par d'autres méthodes ([C-S1], 1994), et nous améliorons leur record pour la dimension 9. On étudie aussi, dans ce chapitre, la densité moyenne d'un réseau et de son dual. Nous construisons notamment un réseau extrême de dimension 9 qui réalise la valeur record de cette densité en dimension 9.

Nous remercions les auteurs du système de calcul PARI (Ch. Batut, D. Bernardi, H. Cohen et M. Olivier), et en particulier Ch. Batut dont nous avons aussi utilisé certains programmes non intégrés au système PARI.

Notations

Dans cette thèse, E désigne un espace euclidien de dimension n dont on note $x.y$ le produit scalaire. Par abus de langage, on appelle *norme* d'un vecteur x de E *le carré* $N(x) = x.x$ de sa norme euclidienne $\|x\| = \sqrt{x.x}$.

1. Quelques généralités.

Un *réseau* L de E est un sous-groupe discret de rang maximum, c'est à dire de rang n. C'est donc aussi un $\mathbb{Z}$-module libre engendré par une base β de E.

Soit L un réseau de E, on appelle *norme* (ou *minimum*) de L le nombre réel

$$N(L) = \inf_{x \in L \setminus \{0\}} N(x);$$

la *sphère* de L,

$$S(L) = \{x \in L,\ N(x) = N(L)\}\ ,$$

est l'ensemble des *vecteurs minimaux* de L, c'est-à-dire ceux dont la norme réalise le minimum de L. On note $s(L) = \frac{1}{2}|S(L)|$ le nombre de couples de vecteurs minimaux $(\pm x)$ de L. Classiquement, on associe au réseau L *l'empilement des sphères* de rayon $\frac{1}{2}\sqrt{N(L)}$ ayant pour centres les points du réseau ; l'entier $s(L)$ s'interprète alors comme la moitié du *kissing number*, nombre de contacts des sphères de l'empilement avec la sphère centrée à l'origine.

Soient $\epsilon = \{\epsilon_1, \dots, \epsilon_n\}$ une base orthonormée de E et $\beta = \{\beta_1, \dots, \beta_n\}$ une base de L, le *discriminant* de L est le réel positif

$$\Delta(L) = |\det_\epsilon(\beta_1, \dots, \beta_n)|.$$

C'est le volume du parallélotope $P = \{\sum_{i=1}^{n} a_i\beta_i, \quad 0 \le a_i \le 1\}$ engendré par la base β. On note $\det(L)$ le carré du discriminant ; c'est aussi le déterminant de la *matrice de Gram* de la base β,

$$\mathrm{Gram}(L, \beta) = (\beta_i.\beta_j).$$

On note

$$\gamma_n(L) = N(L)\det(L)^{-1/n}$$

l'invariant d'Hermite de L. Ce nombre, invariant par similitude, a d'ailleurs une interprétation géométrique : élevé à la puissance $\frac{n}{2}$, il est proportionnel à la densité de l'empilement de sphères associé au réseau. Un des problèmes de la géométrie des nombres est de trouver la valeur maximale $\gamma_n = \max_L \gamma_n(L)$ de cet invariant pour une dimension donnée. On ne connait cette constante (*constante d'Hermite*) que jusqu'à la dimension 8. Cependant, on en connait de bonnes majorations en toutes dimensions. Nous donnons une borne grossière de γ_n ci-dessous.

Rappelons les inégalités de Hadamard et d'Hermite. Elles comparent le déterminant d'un réseau L avec les normes des vecteurs d'une base, et donnent donc un encadrement pour son invariant d'Hermite.

Inégalité de Hadamard (1893) : Soient $e_1, \ldots, e_n$ n vecteurs indépendants d'un réseau L de E, alors

$$\Delta(L) \leq \|e_1\| \cdot \ldots \cdot \|e_n\|.$$

Inégalité d'Hermite : Tout réseau L de E possède une base $\beta = \{\beta_1, \ldots, \beta_n\}$ satisfaisant

$$\|\beta_1\| \cdot \ldots \cdot \|\beta_n\| \leq (4/3)^{\frac{n(n-1)}{4}} \Delta(L).$$

On déduit de cette dernière inégalité la majoration suivante de la constante d'Hermite d'un réseau L de E :

$$\gamma_n(L) \leq (4/3)^{\frac{(n-1)}{2}}.$$

[En fait, Minkowski a obtenu des majorations asymptotiques linéaires.]

Rappelons quelques propriétés liées à la dualité.
Soit L^* le *dual* de L, défini par

$$L^* = \{x \in E, \quad x.y \in \mathbb{Z} \quad \forall y \in L\}.$$

À chaque couple de réseaux (L, L^*), Berge-Martinet (cf. [B-M]) introduisent en 1989 une variante de γ_n : la *constante d'Hermite duale* $\gamma'_n(L)$ définie par

$$\gamma'^2_n(L) = \gamma_n(L).\gamma_n(L^*) = N(L).N(L^*).$$

La méthode classique (qui, en ce qui concerne γ_n, remonte à Korkine et Zolotareff) consiste à rechercher les valeurs maximales γ_n et γ'_n parmi les maxima locaux des invariants $\gamma_n(L)$ et $\gamma'_n(L)$. On montre qu'ils sont atteints sur des réseaux (*extrêmes, dual-extrêmes*) dont les sphères engendrent E : en adoptant la terminologie de A. Ash ("well rounded lattices"), nous qualifierons de tels réseaux de *bien entourés.* Remarquons que dans le cas de l'invariant $\gamma'_n(L)$ qui fait jouer un role symétrique à L et à son dual, la sphère du dual d'un réseau dual-extrême engendre également E. On doit donc avoir à la fois $s(L) \geq n$ et $s(L^*) \geq n$.

Voronoï a aussi montré que les réseaux extrêmes sont proportionnels à des réseaux *entiers*, c'est-à-dire sur lesquels le produit scalaire ne prend que des valeurs entieres (ce qui est équivalent à dire qu'ils sont contenus dans leurs duals). On verra au chapitre 3 qu'il n'en est pas de même pour la constante γ'.

2. Réseaux-formes quadratiques.

Soient L un réseau de E et β une base de L ; on note A la matrice de Gram de la base β. On définit sur $\mathbb{R}^n$ la *forme quadratique* q associée à la matrice A,

$$q(X) = {}^t XAX.$$

Elle est définie positive car, pour tout vecteur x de E, représenté dans la base β par le vecteur $X \in \mathbb{R}^n$, on vérifie facilement que

$$q(X) = N(x).$$

Nous dirons que A est *définie positive*. Réciproquement, toute matrice symétrique définie positive peut être interprétée comme associée à un réseau L dans une base convenable de E.

Les notions introduites pour le réseau L peuvent être traduites en terme de formes quadratiques ou de matrices. Par exemple, si x est un élément de L représenté dans la base β par le vecteur X de $\mathbb{Z}^n$, alors

$$N(L) = \min_{X \in \mathbb{Z}^n \setminus \{0\}} q(X)$$

est appelé *minimum* de q (ou de A) et noté $\min(q)$ (ou $\min(A)$). On note également $S(A) = S(q) = \{X \in \mathbb{Z}^n, q(X) = \min(q)\}$ l'ensemble des vecteurs *minimaux* de A ou de q, soit encore des représentations, dans la base β, des vecteurs minimaux de L.

D'une façon générale, par abus de langage, on emploiera pour les matrices symétriques le vocabulaire des formes quadratiques, et, si elles sont définies positives, celui des réseaux. En fait, algorithmiquement, on préfèrera le point de vue des formes quadratiques (identifiées parfois à leurs matrices).

Précisons la correspondance réseau-matrice.

Soit $GL_n(\mathbb{Z})$ le groupe des matrices entières d'ordre n de déterminant ± 1 et soit L un réseau de E de base β. La *matrice de passage* P de la base β à une autre base β' de L appartient au groupe linéaire, et l'on a :

$$\mathrm{Gram}(L, \beta') = {}^tP\mathrm{Gram}(L, \beta)P,$$

où l'on note ${}^tP$ la matrice *transposée* de P. De plus, deux réseaux isométriques ont dans des bases convenables la même matrice de Gram.

On fait ainsi correspondre à une classe d'isométrie de réseaux une *classe d'équivalence modulo* $GL_n(\mathbb{Z})$ de matrices symétriques définies positives, où l'équivalence de deux matrices est définie de la façon suivante :

$$A \sim B \Longleftrightarrow \exists P \in GL_n(\mathbb{Z}), \quad B = {}^tPAP.$$

3. Réseaux parfaits et eutactiques.

Afin d'étudier les formes quadratiques *extrêmes*, c'est-à-dire celles qui réalisent un maximum local de la constante d'Hermite, Korkine et Zolotareff introduisent en 1873 la notion moins restrictive de formes parfaites.

Adoptons, en vue des généralisations qui font l'objet des chapitres suivants, le point de vue des réseaux.

Plaçons-nous pour cela dans l'espace des *endomorphismes symétriques* de E

$$\mathrm{End}_s(E) = \{u \in \mathrm{End}(E), \quad {}^t u = u\},$$

où ${}^t u$ est l'endomorphisme transposé de u, défini par

$${}^t u(x).y = x.u(y) \quad \forall x, y \in E.$$

Cet espace, de dimension $\frac{n(n+1)}{2}$, est muni d'une structure euclidienne naturelle par le produit scalaire $\langle u, v\rangle = \mathrm{tr}(u.v)$, où tr est la forme linéaire trace.

3.1 Définition : Soient $x \in E$ et φ_x la forme linéaire $u \longrightarrow u(x).x$ définie sur $\mathrm{End}_s(E)$. Un réseau L de E est *parfait*, s'il vérifie la propriété suivante :

$$\varphi_x(u) = 0 \quad \forall x \in S(L) \Longrightarrow \quad u = 0.$$

Afin d'obtenir une condition nécessaire et suffisante pour l'extrémalité, Voronoï introduit la notion d'*eutaxie* dont on donne une définition équivalente ci-dessous.

3.2 Définition : Un réseau L est *eutactique* s'il existe des coefficients ρ_x strictement positifs ($x \in S(L)$) tels que l'endomorphisme trace s'écrive :

$$\mathrm{tr} = \sum_{x \in S(L)} \rho_x \varphi_x.$$

Voronoï montre alors qu'un réseau est extrême si et seulement s'il est parfait et eutactique. Il montre aussi que les réseaux parfaits sont à similitude près et pour une dimension donnée, en nombre fini, et il donne un algorithme qui permet de les trouver tous en un nombre fini d'étapes et que l'on décrit brièvement ci-dessous.

4. Algorithme de Voronoï .

Cet algorithme se déroule dans l'espace des matrices symétriques réelles $n \times n$, $\mathcal{M}_s(n \times n, \mathbb{R})$, que l'on munit d'une structure d'espace euclidien grâce au produit scalaire $\langle A, B\rangle =$ trace (AB). On appelera cet espace, version matricielle de l'espace des endomorphismes symétriques muni du produit scalaire trace, *espace de Voronoï* .

Soient alors A une matrice symétrique définie positive de minimum N et $S(A) = \{v_1, \ldots, v_s\} \subset \mathbb{Z}^n$ l'ensemble de ses vecteurs minimaux, définis par les relations :

$${}^t v_i A v_i = \min(A) = N \qquad \forall i \in \{1, \ldots, s\}.$$

On remarque que ${}^t v_i A v_i = \langle A, v_i {}^t v_i\rangle$, de sorte que, à chaque vecteur minimal v_i, on fait correspondre dans l'espace de Voronoï la matrice

$$V_i = v_i {}^t v_i.$$

Aux formes linéaires φ_x associées dans 3.1 à un réseau L, correspondent en langage matriciel les formes linéaires $M \mapsto \langle M, V_i \rangle$ sur $\mathcal{M}_s(n \times n, \mathbb{R})$, et l'on retrouve donc à partir de 3.1 la définition historique (et algorithmique) de perfection :

La matrice A est parfaite si elle est déterminée de façon unique par la donnée de son minimum N et de ses vecteurs minimaux $v_i \in \mathbb{Z}^n$, c'est-à-dire que le système suivant

$$\langle M, v_i {}^t v_i \rangle = N \quad \forall i \in \{1, \dots, s\},$$

a pour unique solution $M = A$.

Il revient au même de dire que le système des vecteurs V_i est de rang maximum.

Remarquons de même que, lorsqu'on représente un endomorphisme $u \in \mathrm{End}_s(E)$ par sa matrice (symétrique) M dans les bases duales β et β^*, on a $tr(u) = \langle A^{-1}, M \rangle$, où A désigne la matrice de Gram de β.

La définition 3.2 se traduit donc de la façon suivante pour la matrice A : il existe des coefficients ρ_x strictement positifs ($x \in S(L)$) tels que :

$$A^{-1} = \sum_{x \in S(L)} \rho_x v_i {}^t v_i.$$

On appelle *domaine de Voronoï de* A l'enveloppe convexe D_A des demi-droites portées par les V_i ($i \in \{1, \dots, s\}$). On voit donc que A est parfaite si et seulement si son domaine de Voronoï D_A est de rang maximum (c'est-à-dire $\frac{n(n+1)}{2}$). Supposons qu'il en soit ainsi. Une *face* de codimension 1 du domaine D_A de A parfaite est l'intersection de D_A avec un *hyperplan d'appui* F, que l'on peut représenter par un *vecteur de face* B orthogonal à l'hyperplan F et dirigé vers l'intérieur du domaine :

$$\langle B, V_i \rangle = 0 \quad \forall V_i \in F \quad \text{et} \quad \langle B, V_i \rangle \geq 0 \quad \forall V_i.$$

On définit de façon analogue une face de dimension quelconque. En particulier, on appelle *arête* du domaine une face de dimension 1.

Soit B un vecteur de face du domaine de A ; pour tout réel strictement positif ϵ suffisamment petit, la matrice $A + \epsilon B$ a encore pour minimum N mais a pour sphère le sous-ensemble des vecteurs minimaux v_i de A tels que $V_i \in F$. On pose

$$\rho = \sup\{\epsilon \mid \min(A + \epsilon B) = N\}.$$

On montre que cette valeur existe, est atteinte et que la matrice $A' = A + \rho B$ est de nouveau parfaite. On l'appelle *contiguë de* A *par la face* F. Voronoï montre que si l'on part d'une forme parfaite arbitraire et si l'on itère la recherche des formes contiguës en supprimant à chaque étape les matrices équivalentes à une matrice déjà trouvée, on obtient la liste de toutes les formes quadratiques parfaites (à équivalence près). Il donne alors l'algorithme suivant :

•0. Trouver une matrice A parfaite.

•1. Déterminer la liste des faces de son domaine de Voronoï .

•2. Déterminer la contiguë de A pour chaque face.

•3. Supprimer de la liste des contiguës toutes les formes équivalentes à une forme déjà rencontrée.

•4. Si l'on a trouvé une matrice nouvelle, aller en 1 ; sinon terminer.

Dans la pratique, cet algorithme a permis de trouver ou de retrouver les 48 formes parfaites de rang inférieur où égal à 7.

Chapitre I

Réseaux à section hyperplane parfaite

Dans sa thèse (1990), Laihem met au point un algorithme qui lui permet d'énumérer les réseaux parfaits de dimension 8 ayant une section hyperplane parfaite de même norme autre qu'un réseau de racines. Il donne aussi 4 extensions des réseaux de racines $\mathbb{E}_7$, $\mathbb{D}_7$ et $\mathbb{A}_7$ sans pouvoir prouver que ce sont les seules. Dans cette première partie, nous achevons la preuve grâce à une amélioration de l'algorithme de Laihem :

il y a exactement 1175 réseaux parfaits de dimension 8 ayant une section hyperplane parfaite de même norme.

Cet algorithme nous permet également de construire des réseaux assez denses dans des dimensions supérieures parmi les extensions de certains réseaux classiques de dimensions 8, 9, 10.

I.1. Introduction et notations

On se place dans un espace euclidien E de dimension n et l'on considère un réseau L_0 de dimension $(n-1)$ inclus dans E ; soit H l'hyperplan de E engendré par L_0. On note $\mathcal{R}$ la famille des réseaux L de E de même norme N que L_0, et tels que $L \cap H = L_0$:

$$\mathcal{R} = \{L \text{ réseau de } E, \quad L \cap H = L_0 \text{ et } N(L) = N(L_0) = N\}.$$

Par abus de langage, nous adopterons pour de tels réseaux L le terme d'*extensions de* L_0, introduit par Barnes pour les formes quadratiques. Dans ce chapitre, on détermine algorithmiquement les réseaux $\mathcal{R}$-extrêmes d'une telle famille $\mathcal{R}$, c'est-à-dire ceux qui réalisent un maximum local dans $\mathcal{R}$ de la constante d'Hermite $\gamma_n(L) = \frac{N(L)}{\Delta(L)^{2/n}}$.

Rappelons la méthode mise au point par Laihem. Comme dans le cas classique, il introduit une propriété plus faible, la $\mathcal{R}$-perfection, facile à tester :

un réseau L de $\mathcal{R}$ est $\mathcal{R}$-parfait si et seulement si les vecteurs minimaux de L hors de $H = \mathbb{R}.L_0$ engendrent E.

De plus,

dans le cas où L_0 est parfait au sens classique, la notion de $\mathcal{R}$-perfection est équivalente à celle de perfection.

Il s'agit donc d'une méthode de construction par récurrence de réseaux parfaits en dimension n à partir d'un réseau parfait de rang $n-1$, méthode implicitement contenue dans

l'article de Barnes ([Bar3], 1959). Laihem obtient aussi, pour les réseaux $\mathcal{R}$-parfaits, un résultat de *finitude* modulo les isométries de E qui conservent globalement L_0. Toutes ces propriétés justifient que l'on entreprenne la recherche des extensions de L_0 les plus denses par la détermination de celles qui sont $\mathcal{R}$-parfaites. D'autant plus qu'il adapte à cette situation l'algorithme de Voronoï en définissant une notion de $\mathcal{R}$-contiguïté entre réseaux $\mathcal{R}$-parfaits, relation pour laquelle le graphe est connexe. On donne dans le paragraphe I.2 les grandes lignes de cet algorithme. Disons seulement ici que les calculs se déroulent dans un espace de dimension n au lieu de $\frac{n(n+1)}{2}$.

En appliquant cet algorithme en dimension 8, Laihem trouve les 1171 réseaux parfaits de rang 8 à partir des 30 réseaux parfaits de rang 7 autres que les réseaux de racines $\mathbb{A}_7$, $\mathbb{D}_7$ et $\mathbb{E}_7$. Le traitement de ces trois derniers cas aurait exigé un temps de calcul prohibitif et l'on a arrêté le programme après qu'il eût fourni quatre extensions : les réseaux de racines $\mathbb{E}_8$, $\mathbb{D}_8$, $\mathbb{A}_8$ et le réseau $\mathbb{A}_8^2$ construit par Barnes comme extension de $\mathbb{A}_8$. L'ordinateur avait à explorer un nombre considérable de combinaisons de génératrices du domaine de $\mathbb{E}_8$ avant de trouver une face de ce domaine. (D'une façon générale, les temps de calcul croissent très vite avec la différence entre le nombre de vecteurs minimaux hors de la section imposée et la dimension de l'espace dans lequel l'algorithme se déroule.)

On donne dans ce premier chapitre, une méthode pour accélérer les programmes de recherche des contigus d'un réseau $\mathcal{R}$-parfait, en éliminant par une remarque de convexité certaines génératrices du domaine. Revenant à la dimension 8, cette méthode nous permet d'obtenir tous les contigus (au sens de Laihem) de $\mathbb{E}_8$ et de prouver que les quatre réseaux $\mathbb{E}_8$, $\mathbb{A}_8^2$, $\mathbb{D}_8$, $\mathbb{A}_8$ sont bien les seuls au-dessus des trois réseaux de racines de dimension 7.

Dans le dernier paragraphe, on applique l'algorithme en dimensions 9, 10, 11, notamment à partir de réseaux parfaits classiques (réseaux de racines, réseaux laminés ...), et l'on obtient dans chaque cas des listes complètes d'extensions parfaites dont nous étudions les nouvelles venues.

I.2. Description de l'algorithme de Laihem.

Dans ce paragraphe, on rappelle brièvement les différentes étapes de l'algorithme de Voronoï adapté à l'étude de la famille $\mathcal{R}$. Pour une étude plus approfondie, on réfère le lecteur aux travaux de thèse de M. Laihem [Laih].

Soit L appartenant à $\mathcal{R}$ où $\mathcal{R}$ est la famille des réseaux définie ci-dessus (L possède donc une section hyperplane isométrique au réseau L_0 ; de plus on a $N(L) = N(L_0) = N$ et donc $S(L_0) \subset S(L)$). Fixons une matrice de Gram A_0 de L_0 ; tout réseau L de la famille $\mathcal{R}$ admet alors une base ayant une matrice de Gram A de la forme :

$$\begin{pmatrix} & & & * \\ & A_0 & & \vdots \\ & & & * \\ * & \dots & * & * \end{pmatrix}.$$

Soit $\mathcal{M}_{\mathcal{R}}$ l'ensemble des matrices symétriques définies positives de la forme ci-dessus et de minimum N ($N = \min(A) = \min(A_0)$) ; c'est un cône convexe de la variété affine des

matrices qui ont un coin supérieur gauche égal à la matrice de Gram A_0. On note

$$\mathcal{T} = \left\{ \begin{pmatrix} & & & * \\ & 0 & & \vdots \\ & & & * \\ * & \dots & * & * \end{pmatrix} \right\}$$

le sous-espace vectoriel associé, qui est de dimension n et que l'on munit du produit scalaire $\langle\, , \rangle$. C'est relativement à cet espace que se déroulent l'algorithme de Voronoï adapté à la famille $\mathcal{R}$.

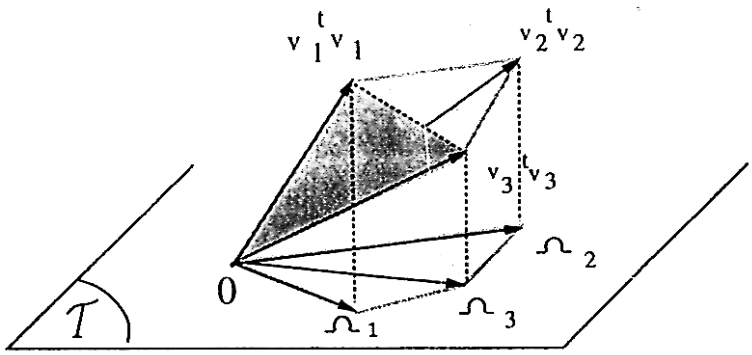
Dans la suite, on emploiera le vocabulaire des formes quadratiques (ou de matrices symétriques) plutot que celui des réseaux. Cela permet une approche plus algorithmique du problème.

Soit A une matrice de $\mathcal{M}_{\mathcal{R}}$ et $S(A) = \{v_1, \dots, v_s\} \subset \mathbb{Z}^n$ l'ensemble de ses vecteurs minimaux, définis par les relations :

$$^t v_i A v_i \ (= \langle A, v_i {}^t v_i \rangle) \ = N \qquad \forall i \in \{1, \dots, s\}.$$

Lorsque ces équations linéaires à n inconnues déterminent la matrice A de façon unique dans $\mathcal{M}_{\mathcal{R}}$, on dit que la matrice A est *$\mathcal{R}$-parfaite*. Remarquons que les vecteurs minimaux de la matrice A_0 n'interviennent pas dans la résolution du système : pour que A soit $\mathcal{R}$-parfaite, il faut et il suffit que ses vecteurs minimaux hors de la section A_0 engendrent $\mathbb{R}^n$.

Comme dans le cas classique, on associe à chaque forme $\mathcal{R}$-parfaite A un domaine convexe, contenu cette fois dans $\mathcal{T}$: *il s'agit de la projection orthogonale $\mathrm{proj}_{\mathcal{T}}(D_A)$ sur le sous-espace $\mathcal{T}$ du domaine de Voronoï D_A de A.* On montre facilement que A est $\mathcal{R}$-parfaite si et seulement si le domaine $\mathrm{proj}_{\mathcal{T}}(D_A)$ est de rang maximum dans $\mathcal{T}$. Ce domaine est engendré par les projections orthogonales Ω_i sur $\mathcal{T}$ des matrices $V_i = v_i {}^t v_i$ ($v_i \in S(A) \setminus S(A_0)$) (les vecteurs $v_i {}^t v_i$ correspondant à la matrice A_0 n'interviennent pas dans la construction du domaine $\mathrm{proj}_{\mathcal{T}}(D_A)$ puisqu'ils ont une projection orthogonale sur $\mathcal{T}$ nulle). On note s' le nombre de génératrices Ω_i du domaine $\mathrm{proj}_{\mathcal{T}}(D_A)$.



Ensuite, on obtient un résultat de finitude pour les formes $\mathcal{R}$-parfaites dans $\mathcal{M}_{\mathcal{R}}$ modulo l'équivalence définie par :

$$A \sim B \iff \exists P \in \mathrm{Gl}_n(\mathbb{Z}), \quad \text{tel que } {}^t PAP = B \quad \text{et} \quad {}^t P\mathcal{T}P \subset \mathcal{T}.$$

Comme pour le cas classique de l'algorithme de Voronoï, lorsque la matrice A est $\mathcal{R}$-parfaite, on considère une face d'appui hyperplane F du domaine $\mathrm{proj}_{\mathcal{T}}(D_A)$, représentée par un vecteur de face $B \in \mathcal{T}$ vérifiant :

$$\langle B, \Omega_i \rangle = 0 \quad \forall \Omega_i \in F \quad \text{et} \quad \langle B, \Omega_i \rangle \geq 0 \quad \forall \Omega_i.$$

On associe à cette face la nouvelle forme parfaite de $\mathcal{M}_{\mathcal{R}}$: $A' = A + \rho B$ où

$$\rho = \sup\{\epsilon > 0, \quad \min(A + \epsilon B) = \min(A) = N\}.$$

On obtient ainsi une relation de contiguïté dont le graphe est connexe, et donc un algorithme analogue à celui de Voronoï pour les déterminer, algorithme qui se déroule maintenant dans l'espace $\mathcal{T}$ de dimension n. La principale difficulté dans la pratique de cet algorithme se situe dans la recherche des faces du domaine (la détermination de ρ, dont on sait *a priori* qu'il est rationnel, s'avérant au contraire très rapide dans tous les cas que nous avons traités). En effet, le problème est de trouver tous les hyperplans d'appuis du domaine, et donc d'abord de chercher $(n-1)$ génératrices indépendantes parmi celles du domaine.

On a donc *a priori* $\frac{s'!}{(n-1)!(s'-n+1)!}$ tests d'indépendance à effectuer.

A titre d'exemple, lorsqu'on cherche les faces du domaine de $\mathbb{E}_8$ ($s(\mathbb{E}_8) = 120$) en imposant une section hyperplane semblable à $\mathbb{A}_7$ ($s(\mathbb{A}_7) = 28$), le nombre de génératrices du domaine $s' = 120 - 28 = 92$ est algorithmiquement très important. Expérimentalement, l'algorithme, tel qu'il est décrit ci-dessus, ne permet pas de résoudre dans un temps raisonnable le cas des réseaux pour lesquels la différence $(s' - n)$ est supérieure à 25. C'est à la résolution de cette difficulté qu'est consacré le paragraphe suivant.

I.3. Amélioration de l'algorithme.

Un phénomène nouveau apparaît, qui va permettre d'éliminer, lors de la détermination des faces, un grand nombre de génératrices Ω_i : il s'agit des génératrices qui sont combinaisons linéaires à coefficients strictement positifs d'autres génératrices. Contrairement au cas classique où cette situation ne se présente pas puisque chaque génératrice est une arête (ce résultat a été démontré par Jaquet [Jaq2]), on verra que l'on peut éliminer un grand nombre de génératrices.

Soit

$$\Omega = \sum_{i=1}^{k} \lambda_i \Omega_i \quad \lambda_i > 0$$

une génératrice du domaine $\mathrm{proj}_{\mathcal{T}}(D_A)$ combinaison linéaire à coefficients strictement positifs d'autres génératrices Ω_i, et soit F une face du domaine contenant Ω et B un vecteur de face correspondant. On a donc :

$$\langle B, \Omega \rangle = \sum_{i=1}^{k} \langle B, \Omega_i \rangle,$$

avec

$$\langle B, \Omega_i \rangle \geq 0 \quad \forall i \leq s'.$$

Comme les coefficients λ_i sont strictement positifs l'équation $\langle B, \Omega \rangle = 0$ entraîne la nullité de tous les termes $\langle B, \Omega_i \rangle$ $(1 \leq i \leq k)$; ce qui veut dire que toutes les génératrices Ω_i appartiennent à la face F.

Lors de la recherche de la face F, le vecteur Ω est inutile puisque l'on pourra construire la face en se servant des vecteurs Ω_i $(1 \leq i \leq k)$.

Remarquons d'abord que deux génératrices du domaine ne peuvent pas être proportionnelles.

En effet, si $v = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{Z}^n$ est un vecteur minimal de A, la génératrice $\Omega = \mathrm{proj}_{\mathcal{T}}(v^t v)$ correspond à la matrice : $x_n \begin{pmatrix} & & & x_1 \\ & 0 & & \vdots \\ & & & x_{n-1} \\ x_1 & \dots & x_{n-1} & x_n \end{pmatrix}$.

On détermine alors toutes les relations de la forme :

$$\Omega = \lambda_1 \Omega_1 + \lambda_2 \Omega_2$$

avec λ_1 et λ_2 positifs. [Dans la pratique, on applique la fonction **kerint()** de PARI pour chaque triplet de génératrices.] L'examen de ces relations permet d'éliminer un nombre suffisant de génératrices inutiles pour la détermination effective des faces du domaine.

On peut maintenant appliquer l'algorithme de Laihem aux formes ayant un grand nombre de vecteurs minimaux, et en particulier traiter le cas des réseaux de racines.

I.4. Résultats.

Dans les exemples ci-dessous, on construit principalement les extensions de réseaux de racines ou de réseaux qui leur sont liés. Nous obtenons dans chaque cas la liste complète des extensions $\mathcal{R}$-parfaites à isométrie près, ce qui nous permet de calculer la constante d'Hermite pour la famille $\mathcal{R}$. Dans certains cas, nous précisons le graphe de contiguïté des réseaux obtenus et leurs propriétés (perfection, eutaxie, extrémalité, etc). On les énumère par ordre de densité décroissante dans des tableaux dans lesquels figurent leurs caractéristiques : dans la première colonne figure s'il y a lieu le nom du réseau dans la terminologie de Conway et Sloane [C-S2] ou de Barnes [Bar2] . Tous les réseaux trouvés sont parfaits donc proportionnels à des réseaux entiers ; on donne dans les deuxième et troisième colonnes le déterminant (det) et la norme μ du plus petit homothétique entier. On précise ensuite les nombres s, s^* de couples $(\pm x)$ de vecteurs minimaux du réseau et de son dual. Enfin, on donne la constante d'Hermite, γ et le carré de la constante d'Hermite duale $\gamma'^2 = \gamma\gamma^*$ (où γ^* est l'invariant d'Hermite du dual), arrondies à trois décimales. Enfin, dans la dernière colonne, on étudie l'extrémalité du réseau (E, extrême ; N-E, non-extrême ; s-e, semi-eutactique).

I.4.1 Dimension 8 :

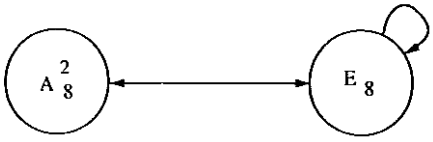
Dans ce paragraphe, on ne s'intéresse qu'aux extensions des trois réseaux de racines $\mathbb{E}_7$, $\mathbb{D}_7$ et $\mathbb{A}_7$. Pour les extensions des trente autres réseaux parfaits de rang 7, on se réfèrera à la liste informatisée sous forme PARI construite par Laihem ([Lai]). On donne dans chaque cas le graphe de contiguïté où l'on représente la relation de contiguïté entre deux réseaux $\mathcal{R}$-parfaits par une arête du graphe. A la fin de ce paragraphe, on rappelle, dans un tableau, les caractéristiques des réseaux trouvés (apparus depuis bien longtemps dans la littérature).

Les extensions de $\mathbb{E}_7$:

Les seuls réseaux parfaits ayant une section hyperplane parfaite de même norme semblable au réseau $\mathbb{E}_7$ sont le réseau de racines $\mathbb{E}_8$ et le réseau $\mathbb{A}_8^2$ de Barnes.

La méthode de réduction du nombre des vecteurs minimaux hors de la section imposée ne permet pas d'abréger les temps de calculs puisque seulement une génératrice a pu être supprimée. Il reste donc 56 génératrices du domaine. Sur une machine du type SPARC 10, on trouve les $\mathcal{R}$-voisins du réseau $\mathbb{E}_8$ en 345812 sCPU. Ce temps, lié à la machine, correspond à trois jours de calculs environ.

Graphe de contiguïté



Les extensions de $\mathbb{D}_7$:

Il n'y a que les deux réseaux de racines $\mathbb{D}_8$ et $\mathbb{E}_8$.

Après l'élimination des génératrices intérieures du nouveau domaine de $\mathbb{E}_8$ (on en supprime 46 sur 78), il a fallu 21477 sCPU pour déterminer les faces du $\mathcal{R}$-domaine de $\mathbb{E}_8$, ce qui correspond à une dizaine d'heures de calcul.

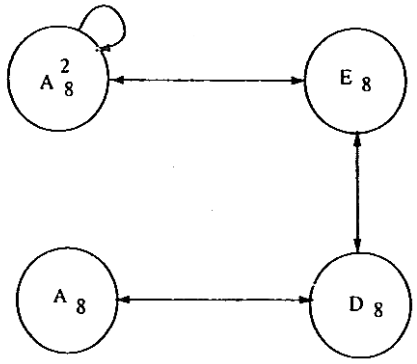
Graphe de contiguïté

Les extensions de $\mathbb{A}_7$ *:*

Dans ce cas, on obtient les trois réseaux de racines $\mathbb{A}_3$, $\mathbb{E}_8$, $\mathbb{D}_8$ *et le réseau* $\mathbb{A}_8^2$.

Ici, nous avons pu éliminer 64 génératrices intérieures au domaine de $\mathbb{E}_8$ (il en reste seulement 28) ; le temps de calcul est encore une dizaine d'heures.

Graphe de contiguïté



On vient donc de montrer le résultat suivant :

I.4.1.1. Théorème : *Il y a exactement* 4 *réseaux parfaits de rang* 8, *à savoir* $\mathbb{A}_8$, $\mathbb{D}_8$, $\mathbb{E}_8$ *et* $\mathbb{A}_8^2$, *ayant une section hyperplane de même norme isométrique à l'un des trois réseaux de racines de rang* 7.

En ajoutant à cette liste les 1171 réseaux trouvés par Laihem au-dessus des trente autres réseaux parfaits de rang 7, on montre :

I.4.1.2. Corollaire : *Il y a exactement 1175 réseaux parfaits de rang 8 ayant une section hyperplane parfaite de même norme.*

L'étendue de cette liste (et le peu d'intérêt de la "queue de liste") suffirait à dissuader d'entreprendre un travail d'énumération analogue pour des dimensions supérieures. Nous nous sommes donc bornés, pour les dimensions 9, 10 et 11, à énumérer les extensions de quelques réseaux intéressants soit du point de vue de la densité soit de celui de la dualité.

On rappelle ci-dessous les caractéristiques des 4 réseaux $\mathbb{E}_8$, $\mathbb{A}_8^2$, $\mathbb{D}_8$ et $\mathbb{A}_8$, puisque, au moins pour les trois premiers, nous allons en chercher les extensions.

	det	μ	(s, s^*)	$\lvert Aut\rvert$	γ	γ'^2	Extr
$\mathbb{E}_8$	1	2	(120,120)	$2^{14}3^5 5^2 7$	2	4	E
$\mathbb{A}_8^2$	576	4	(71,1)	$2^8 3^2 5.7$	1.807	1.333	E
$\mathbb{D}_8$	4	2	(56,8)	$2^{15}3^2 5.7$	1.681	2	E
$\mathbb{A}_8$	9	2	(36,1)	$2^8 3^4 5.7$	1.519	1.777	E

I.4.2 Dimension 9 :

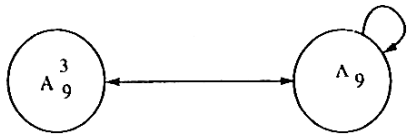
Les extensions de $\mathbb{E}_8$ *:*

Ici encore, on ne trouve que des réseaux connus, le réseau laminé Λ_9 conjecturalement le plus dense en dimension 9 et le réseau $\mathbb{A}_9^3$ construit par Barnes par raffinement de $\mathbb{A}_9$ (c'est aussi une extension de $\mathbb{A}_8^3 \sim \mathbb{E}_8$).

I.4.2.1. Proposition : *Les seules extensions parfaites de* $\mathbb{E}_8$ *sont les réseaux* Λ_9 *et* $\mathbb{A}_9^3$.

Remarquons dans ce cas, que la différence entre le nombre de génératrices du domaine de Λ_9 et la dimension ne dépasse pas 7. Les temps de calculs, pour obtenir le graphe de contiguïté suivant, sont inférieurs à une seconde.

Graphe de contiguïté



	det	μ	(s, s^*)	$\lvert Aut\rvert$	γ	γ'^2	Extr
Λ_9	512	4	(136,1)	10321920	2	2	E
$\mathbb{A}_9^3$	21870	6	(129,1)	725760	1.976	1.8	E

Les extensions de $\mathbb{D}_8$ *:*

On obtient les réseaux Λ_9, $\mathbb{D}_9$ et un réseau J_9, lui aussi extrême au sens classique, le plus dense des 226 contigus de $\mathbb{D}_9$ obtenus par D. O. Jaquet ([Jaq1]). Ses seules sections hyperplanes parfaites sont $\mathbb{D}_8$ et $\mathbb{A}_8$.

	det	μ	(s, s^*)	$\lvert Aut\rvert$	γ	γ'^2	Extr
Λ_9	512	4	(136,1)	10321920	2	2	E
J_9	17915904	12	(65,8)	80640	1.876	2.781	E
$\mathbb{D}_9$	4	2	(72,9)	185794560	1.714	2	E

Les extensions de $\mathbb{A}_8^2$ *:*

Parmi les 5 extensions de $\mathbb{A}_8^2$, on reconnait seulement la plus dense (le réseau Λ_9), et la moins dense (le réseau $\mathbb{A}_9^{2,4,2}$ de Barnes, extension de $\mathbb{A}_8$ et raffinement de $\mathbb{A}_9$). On donne les caractéristiques des cinq extensions dans le tableau suivant :

	det	μ	(s,s^*)	$\lvert Aut\rvert$	γ	γ'^2	Extr
Λ_9	512	4	(136,1)	10321920	2	2	E
	896	4	(91,1)	4608	1.879	2.285	E
	19595520	12	(80,2)	1440	1.857	2.314	N-E
	1152	4	(81,2)	4608	1.827	2	E
$\mathbb{A}_9^{2,4,2}$	1280	4	(80,2)	161280	1.806	1.8	E

On observe ici l'un des rares cas où l'ordre des constantes d'Hermite duales diffère de celui des constantes d'Hermite.

Les extensions de L_8^4 :

Le réseau de Barnes L_8^4, obtenu par raffinement du réseau $\mathbb{A}_2 \perp \mathbb{A}_2 \perp \mathbb{A}_2 \perp \mathbb{A}_2$ se retrouve également sous un autre pseudonyme, K'_8 : c'est une section du réseau K_{12} de Coxeter-Todd, qui appartient à la suite K'_n $(0 \leq n \leq 12)$ définie par Martinet ([Mar]) à partir de K_{12} par une méthode "d'antilamination". Notons que les réseaux K'_n sont extrêmes pour $n \geq 5$. Parmi les trois extensions de ce réseau $L_8^4 \sim K'_8$, on trouve donc le réseau K'_9 (obtenu aussi par Barnes par extension de L_8^4), intéressant du point de vue de la dualité. Nous avons en effet montré qu'il réalise un maximum local de la constante γ' : comme il est extrême, il suffit pour cela, d'après [B-M1], de vérifier qu'il existe des constantes $\rho_i > 0$ et $\rho'_j > 0$ telles que, si A désigne une matrice de Gram du réseau, V_i les matrices symétriques représentant ses vecteurs minimaux, V'_j celles de son inverse :

$$A \sum_{i=1}^{s} \rho_i V_i = \sum_{j=1}^{s^*} \rho_j V'_j A^{-1} \quad \text{(dual-eutaxie.)}$$

La valeur $\gamma'(K'_9)$ n'est toutefois pas la meilleure connue à ce jour ; le record est notamment détenu par le réseau $\mathbb{A}_9^5$ de Coxeter (voir chapitre III).

On reconnait dans la moins dense des trois extensions le réseau L_9^4 de Barnes.

	det	μ	(s,s^*)	$\lvert Aut\rvert$	γ	γ'^2	Extr
K'_9	972	4	(81,13)	2592	1.862	3	E
	1296	4	(68,1)	768	1.803	2.25	E
L_9^4	1458	4	(66,1)	62208	1.780	2	E

Les extensions de K_8 :

Jusqu'ici, nous avons étudié des familles $\mathcal{R}$ de réseaux pour lesquels on impose une section hyperplane parfaite, ce qui conduisait inévitablement à une sélection de réseaux parfaits.

Dans le cas suivant, on impose une section hyperplane isométrique au réseau K_8 de la suite K_n construite par Leech (par antilamination de K_{12}), réseau qui n'est pas parfait

(le défaut de perfection est 1), et l'on obtient 5 réseaux $\mathcal{R}$-parfaits, qui sont également parfaits au sens classique. Ce sont les premiers exemples de réseaux parfaits construits à partir d'une section hyperplane non-parfaite.

À notre connaissance, le seul de ces réseaux déjà apparu dans la littérature est le réseau K_9 (semblable au réseau L_9^2 de Barnes).

Le plus dense de la liste, qui est extrême, est une section (la deuxième en densité après Λ_9) du réseau laminé Λ_{10}, qui réalise la meilleure constante d'Hermite connue en dimension 10. Sa section la plus dense en dimension 8 (non parfaite) se trouve sur un chemin de contiguïté de $\mathbb{E}_8$ à lui-même sur lequel on a $s = 75$, ce qui est, d'après Watson, le record pour les réseaux distincts de $\mathbb{E}_8$ (le réseau K_8 n'en possède que 66).

	det	μ	(s, s^*)	$\lvert Aut\rvert$	γ	γ'^2	Extr
	768	4	(99,3)	18432	1.911	2.666	E
K_9	864	4	(90,3)	2304	1.887	2.666	E
	960	4	(84,1)	384	1.865	2.400	
	528384	8	(76,1)	96	1.850	2.232	
	1056	4	(80,1)	192	1.845	2.181	

I.4.3. Dimension 10 :

Les extensions de Λ_9 :

On trouve naturellement le réseau Λ_{10} obtenu par lamination de Λ_9. Il est intéressant de voir que les cinq autres réseaux sont presque aussi denses que lui. L'un d'eux, qui se distingue par un groupe d'automorphismes énorme ($g = 2^{17}.3^2.5.7$), est le réseau $B_{10}^{2,4,2}$ de Barnes, raffinement du réseau $\mathbb{D}_{10}$ qu'il contient avec l'indice 2. Ce réseau est en effet une extension de $B_9^{2,4,2} \sim \Lambda_9$.

	det	μ	(s, s^*)	$\lvert Aut\rvert$	γ	γ'^2	Extr
Λ_{10}	768	4	(168,3)	84736	2.058	2.666	E
	960	4	(154,2)	322560	2.012	2.133	E
$B_{10}^{2,4,2}$	1024	4	(154,2)	41287680	2	2	E
	60466176	12	(146,1)	5760	2	2	N-E
	65505024	12	(146,1)	80640	1.984	1.846	E

Les extensions de $\mathbb{A}_9^3$:

Dans ce cas, la famille $\mathcal{R}$ contient 16 réseaux parfaits qui ont au moins 139 (=10+129) vecteurs minimaux ; les constantes d'Hermite sont assez proches du record. Nous donnons dans le tableau suivant les caractéristiques des 8 réseaux les plus denses. On y observe une coïncidence étonnante : les deux réseaux les plus denses de cette liste ont les mêmes caractéristiques excepté l'ordre de leur groupe d'automorphismes. Ils diffèrent d'ailleurs aussi par la structure du groupe déterminant (L^*/L).

det	μ	(s,s^*)	$\lvert Aut\rvert$	γ	γ'^2	Extr
54675	6	(147,3)	4354560	2.015	2.4	E
54675	6	(147,3)	7776	2.015	2.4	E
3400690959	18	(141,1)	1440	2.005	2.227	N-E
561273046875	30	(139,2)	5760	2.005	2.283	N-E
59533056	12	(141,2)	288	2.003	2.257	
3419822835	18	(139,1)	2304	2.003	2.265	N-E
61344055296	24	(139,1)	1440	2.001	2.158	N-E
60466176	12	(146,1)	5760	2	2	N-E

Réseaux au-dessus de $\mathbb{D}_9$:

Parmi ces réseaux, on obtient bien sûr le réseau $\mathbb{D}_{10}$ et son raffinement unimodulaire $\mathbb{D}_{10}^+$. Le réseau le plus dense est le réseau $B_{10}^{2,4,2}$ déjà signalé comme extension de Λ_9. L'énumération des 9 extensions de $\mathbb{D}_9$ a été beaucoup plus longue que les précédentes. La difficulté ne provenait pas de l'absence d'une base de vecteurs minimaux pour le réseau $\mathbb{D}_{10}^+$ (situation à laquelle j'ai adapté l'algorithme standard), mais de la présence du réseau $B_{10}^{2,4,2}$ dont le $\mathcal{R}$-domaine de Voronoï a un grand nombre de génératrices (82, contre 18 pour son domaine au-dessus de Λ_9).

Les réseaux trouvés sont relativement peu denses, mais sont toutefois tous extrêmes et beaucoup plus intéressants du point de vue de la dualité que les précédents, bien que seuls $\mathbb{D}_{10}^+$ et $\mathbb{D}_{10}$ soient dual-extrêmes. Notons que la valeur record (à ce jour) de γ' est détenue notamment par le réseau $\mathbb{D}_{10}^+$.

	det	μ	(s,s^*)	$\lvert Aut\rvert$	γ	γ'^2	Extr
$B_{10}^{2,4,2}$	1024	4	(154,2)	41287680	2	2	E
$\mathbb{D}_{10}^+$	1024	4	(90,90)	1857945600	2	4	E
	65505024	12	(90,9)	725760	1.984	3.282	E
	1191182336	16	(82,9)	725760	1.979	3.380	E
	11600000000	20	(82,6)	34560	1.970	3.103	E
	1275068416	16	(85,5)	46080	1.965	2.947	E
	73903104	12	(92,4)	92160	1.960	2.727	E
	1310720	8	(108,3)	276480	1.955	2.4	E
$\mathbb{D}_{10}$	4	2	(90,10)	$2^{18}3^4 5^2 7$	1.741	2	E

I.4.4. Dimension 11 :

Les extensions de Λ_{10} *:*

Parmi les 6 extensions, on reconnait les deux réseaux laminés pour la dimension 11, $\Lambda_{11}^{\min}$ et $\Lambda_{11}^{\max}$, dont les caractéristiques ne diffèrent que par le nombre de vecteurs minimaux et leurs groupes d'automorphismes.

	det	μ	(s, s^*)	$\lvert Aut \rvert$	γ	γ'^2	Extr
$\Lambda_{11}^{\max}$	1024	4	(219,4)	3538944	2.130	3	E
$\Lambda_{11}^{\min}$	1024	4	(216,4)	98304	2.130	3	E
	1152	4	(204,2)	36864	2.107	2.666	s-e
	1408	4	(188,1)	9216	2.069	2.181	N-E
	262020096	12	(179,1)	192	2.060	2.076	N-E
	272097792	12	(179,1)	1152	2.053	2	N-E

I.4.5. Commentaires

Remarquons que les têtes de listes sont toujours de norme minimale $\mu = 4$, sauf bien entendu les extensions de $\mathbb{A}_9^3$ de norme 6. On observe également dans les tableaux ci-dessus un phénomène assez fréquent quoiqu'inexpliqué : lorsqu'on parcourt les listes, le nombre s varie dans le même sens que la densité sauf lorsque la norme μ diminue, à la notable exception du réseau $\mathbb{D}_{10}$ dans la suite des extensions du réseau $\mathbb{D}_9$. Notons que la constante d'Hermite des extensions de $\mathbb{A}_9^3$ de même que celle des extensions de Λ_{10}, est atteinte sur deux réseaux non semblables.

La construction de réseaux parfaits par cette méthode, sorte de généralisation de la méthode de lamination, permet, à partir d'un réseau de rang $n-1$ assez dense, d'obtenir des réseaux de rang n également intéressants du point de vue de la densité et du *kissing number*. On retrouve bien entendu la suite Λ_n, $8 \leq n \leq 11$ des réseaux laminés, qui réalisent les meilleures constantes d'Hermite connues, sauf pour $n = 11$.

Par contre, si l'on veut construire des réseaux intéressants du point de vue de la dualité, cette méthode est inefficace (la dualité ne se propage pas par extension), et on constate en effet, dans les tableaux précédents, que les nombres de vecteurs minimaux des duals valent pour la plupart 1, 2, 3, 4. Seuls le réseau de racines $\mathbb{E}_8$ et l'extension K'_9 de L_8^4 sont intéressants du point de vue de la dualité puisque $\mathbb{E}_8$ réalise naturellement la meilleure constante d'Hermite duale de dimension 8 ($\gamma'^2(\mathbb{E}_8) = 4$) et K'_9 donne la deuxième connue en dimension 9 ($\gamma'^2(K'_9) = 4$)

Chapitre II

Densité des réseaux en patchwork

Le but de ce chapitre est de construire des réseaux denses qui sont somme directe de réseaux de même norme. On adopte les méthodes locales d'extrémalité pour lesquelles on développe une théorie à la Voronoï. On obtient aussi un algorithme qui permet notamment de déterminer toutes les classes d'isométrie de réseaux parfaits sommes directes de réseaux parfaits donnés, algorithme que l'on applique dans le dernier paragraphe en dimensions moyennes.

II.1. Introduction et notations

Soit E un espace euclidien de dimension n. Dans ce chapitre, on restreint l'étude de la densité à une famille $\mathcal{F}$ de réseaux en *patchwork* de E, c'est-à-dire de réseaux de norme N donnée, sommes directes de r ($r \geq 2$) réseaux $(L_i)_{i \leq r}$ de norme N isométriques à des réseaux (L_i^0) donnés. En particulier, on recherche les réseaux les plus denses de $\mathcal{F}$ parmi ceux dont la constante d'Hermite réalise un maximum local sur cet ensemble (réseaux *$\mathcal{F}$-extrêmes*). L'existence de tels réseaux n'est pas assurée *a priori*, mais résulte de la compacité modulo isométrie de la famille $\mathcal{F}$ (cf §.2).

Dans l'espace des réseaux de E, paramatré par le cône des endomorphismes symétriques positifs de E, la famille $\mathcal{F}$ est associée à un sous-espace strict V de l'espace $\mathrm{End}_s(E)$ (elle s'identifie à un convexe de cet espace). Nous introduisons des notions de perfection et d'eutaxie relatives à ce sous-espace qui nous permettent une caractérisation à la Voronoï des réseaux $\mathcal{F}$-extrêmes. On s'intéresse par la suite davantage à la notion de *$\mathcal{F}$-perfection* que l'on caractérise par une propriété des vecteurs minimaux : *un réseau $\mathcal{F}$-parfait perd de ses vecteurs minimaux hors sections dès lors qu'il subit une perturbation (tout en restant dans $\mathcal{F}$)* (cf. II.3). De cette propriété d'unicité, nous déduisons les théorèmes attendus de finitude et de rationalité. Nous montrons de plus que ces réseaux sont parfaits (au sens classique) lorsque les sections imposées le sont. Tout ceci justifie notre entreprise d'énumération des réseaux $\mathcal{F}$-parfaits, à laquelle est consacré le paragraphe 4.

Notamment, lorsque chaque section L_i est de dimension 1, on obtient une méthode de construction des réseaux parfaits ayant une base de vecteurs minimaux.

On y adapte l'algorithme de Voronoï pour la détermination de toutes les formes quadratiques $\mathcal{F}$-parfaites (nous adoptons en effet maintenant le point de vue plus algorithmique des formes quadratiques). Cet algorithme se déroule dans le sous-espace de matrices symétriques $n \times n$ associé à V ce qui abaisse ainsi la taille des calculs et permet d'aborder des valeurs moyennes de n hors d'atteinte par l'algorithme de Voronoï usuel.

On associe à toute forme quadratique de la famille un domaine convexe de cet espace (les formes $\mathcal{F}$-parfaites correspondent aux domaines de dimension maximale $\dim(V)$). On définit ici encore une relation de contiguïté entre les formes $\mathcal{F}$-parfaites, pour laquelle nous obtenons un théorème de finitude et de connexité. Des phénomènes nouveaux apparaissent dans la pratique de cet algorithme, ne serait-ce que parce qu'il se déroule sur un compact et qu'il s'arrête donc au bout d'un nombre fini d'étapes sans qu'il soit nécessaire de tester les équivalences. Toutefois, on introduit une notion d'équivalence compatible avec la contiguïté pour améliorer l'efficacité de l'algorithme. D'autre part, on observe aussi le même phénomène que dans le chapitre I : certaines génératrices du $\mathcal{F}$-domaine de Voronoï de la forme $\mathcal{F}$-parfaite peuvent ne pas être des arêtes, et on utilisera la méthode du chapitre I pour les éliminer.

Enfin, dans le dernier paragraphe, on applique l'algorithme en dimensions $n \leq 10$. Tout d'abord, on teste la rapidité de notre algorithme en déterminant les réseaux parfaits en dimension 5 et 6 auxquels on a imposé une base de vecteurs minimaux (patchwork de n réseaux de rang 1). Nous construisons aussi en ces dimensions des réseaux "presque parfaits" assez denses que nous étudions du point de vue de la dualité, de l'eutaxie et de la norme.

En dimension 8, on ajoute à la liste des 1175 réseaux parfaits à section hyperplane parfaite (cf chap. I) 53 réseaux parfaits patchworks de réseaux parfaits de rang 2 et 6, dont on donne les caractéristiques principales dans l'annexe I.

C'est ainsi qu'en dimension 9 et 10, on ajoute en somme directe des réseaux parfaits de rang 2 et de rang 7 ou 8. Au-dessus du réseau critique $\mathbb{E}_7$, on a seulement retrouvé les réseaux $\mathbb{A}_9^3$ et $\mathbb{B}_9^2$ de Coxeter et Barnes. Par contre, on trouve 8 formes parfaites de rang 9 au-dessus du dual $\mathbb{E}_7^*$ (parmi lesquelles aucune n'est entière de norme 3). Le nombre d'extensions parfaites trouvées semble varier en raison inverse du nombre d'automorphismes du réseau de départ. Par exemple, on possède une liste de plus de 600 réseaux parfaits au-dessus de $\mathbb{P}_7^6$ (notation de Conway et Sloane) qui n'a que 192 automorphismes.

Au-dessus de $\mathbb{E}_8$, on trouve une douzaine de réseaux parfaits de rang 10. Les plus denses d'entre eux (parmi lesquels bien sûr le réseau Λ_{10}) ont déjà été rencontrés au chapitre I.

Enfin, situons l'étude du chapitre II par rapport aux études analogues de la littérature. Tout d'abord, la méthode d'extension exposée au chapitre I ne conduit pas *a priori* à des réseaux patchworks de réseaux de dimensions $n-1$ et 1 (voir l'exemple du réseau $\mathbb{D}_{10}^+$ extension de $\mathbb{D}_9$). Par ailleurs, on trouve dans [B-M1] une étude de densité dans des familles de réseaux paramétrées par des sous-groupes fermés du groupe linéaire. Il est facile de voir que les réseaux en patchwork ne sont pas susceptibles d'une telle définition.

Par contre, la notion de $\mathcal{F}$-perfection est un cas particulier de celle qui a été attachée dans [B-M-S] à un sous-espace vectoriel de matrices symétriques. Toutefois, on donne

un exemple (après II.3.3) qui prouve que les hypothèses faites dans [B-M-S] ne sont pas nécessairement vérifiées ici.

II.2. Etude de la constante d'Hermite sur $\mathcal{F}$

On se donne r réseaux L_i^0 de même norme N et de rangs n_i avec $\sum_{i=1}^{r} n_i = n$, et l'on se propose de paramétrer la famille patchwork $\mathcal{F}$ correspondante à partir de l'un de ses réseaux $L = \bigoplus_{i=1}^{r} L_i$, avec pour tout i $L_i \sim L_i^0$ et $N(L) = N$.

Posons $F_i = \mathbb{R}L_i \quad \forall i$ (on a donc $E = \bigoplus_{i=1}^{r} F_i$), et soit

$$V = \{v \in \mathrm{End}_s(E) \text{ tels que, } \forall\, i \in \{1, \cdots, r\}, \quad v(F_i) \subset F_i^{\perp}\}.$$

Alors on a :

$$\mathcal{F} = \{u(L)\ ,\ u \in\ GL(E),\ \textit{avec}\ {}^t u.u - id\ \in\ V\ \textit{et}\ N(u(L)) = N(L)\}.$$

La démonstration découle immédiatement du lemme suivant.

II.2.1 Lemme. *Soient F un sous-espace de E, M un réseau de F et $u \in GL(E)$. Alors la restriction de u à M est une isométrie si et seulement si l'on a :*

$$({}^t uu - id)(F) \subset F^{\perp},$$

où ${}^t u$ désigne l'endomorphisme transposé de u.

Démonstration :

Soit $u \in GL(E)$. Alors on a les équivalences suivantes.

$$\begin{aligned} u_{/M} \text{ est une isométrie} &\Leftrightarrow \forall\, x, y\ \in M \quad u(x).u(y) = x.y \\ &\Leftrightarrow\ \forall\, x, y\ \in M \quad ({}^t uu - id)(x).y = 0 \\ &\Leftrightarrow\ ({}^t uu - id)(F)\ \subset\ F^{\perp}. \end{aligned}$$

On a obtenu une nouvelle définition de $\mathcal{F}$ à partir de l'un de ses réseaux $L = \bigoplus_{i=1}^{r} L_i$:

$$\mathcal{F} = \{u(L)\ ,\ u \in\ GL(E),\ \text{avec}\ {}^t u.u - id\ \in\ V\ \text{et}\ N(u(L)) = N(L)\},$$

si V est le sous-espace de l'espace $\mathrm{End}_s(E)$ des endomorphismes symétriques de E :

$$V = \{v \in \mathrm{End}_s(E) \text{ tels que, } \forall\, i \in \{1, \cdots, r\}, \quad v(F_i) \subset F_i^{\perp}\},$$

où F_i est le sous-espace engendré par L_i.

Remarque sur la condition $N(u(L)) = N(L)$: Soit $S = S(L)$ l'ensemble des vecteurs minimaux de L ; l'application $u \longmapsto N(u(x))$ étant continue, pour u voisin de l'identité, les vecteurs minimaux du réseau $u(L)$ proviennent de ceux de L. En posant $v = {}^t u.u - id$,

$$N(u(L)) = \min_{x \in S} N(u(x)) = N(L) + \min_{x \in S} \varphi_x(v) \text{ où } \varphi_x(v) = v(x).x,$$

et donc $N(u(L)) = N(L)$ équivaut à $\min_{x \in S} \varphi_x(v) = 0$. On se place donc dans le cône $\mathcal{C}$ de V défini par :

$$\mathcal{C} = \{v \in V, \min_{x \in S} \varphi_x(v) = 0\} .$$

Notons que pour tout v suffisamment proche de 0 appartenant à ce cône (plus précisément à valeurs propres supérieures à -1), il existe $u \in \mathrm{End}_s(E)$ défini modulo $\mathcal{O}(E)$ (groupe orthogonal de E) tel que ${}^t u.u - id = v$. En particulier, si $v = 0$ alors u est orthogonal et le réseau $u(L)$ est isométrique à L.

Munissons pour toute la suite $\mathcal{F}$ de la topologie induite par la topologie naturelle sur l'ensemble des réseaux de E. Les voisinages dans $\mathcal{F}$ d'un réseau $L \in \mathcal{F}$ correspondent donc à des voisinages de 0 dans le cône $\mathcal{C}$ de V.
On introduit alors une notion analogue à l'extrémalité classique introduite par Korkine et Zolotareff [K-Z].

II.2.2 Définition Un réseau L de $\mathcal{F}$ est *$\mathcal{F}$-extrême* si sa constante d'Hermite réalise un maximum local dans $\mathcal{F}$.

L'existence de tels réseaux dans la famille $\mathcal{F}$ s'obtient grâce au théorème de compacité de Mahler [Cas].

II.2.3 Théorème (MAHLER). *Soit $\mathcal{L}$ une famille de réseaux vérifiant les deux conditions suivantes :*

(1) *Il existe $m > 0$ tel que l'on ait $N(L) \geq m$ pour tout $L \in \mathcal{L}$;*
(2) *Il existe $M > 0$ tel que l'on ait $\det(L) \leq M$ pour tout $L \in \mathcal{L}$.*
Alors, $\mathcal{L}$ est d'adhérence compacte.

Les réseaux de la famille $\mathcal{F}$ sont tous de même norme N ; de plus, d'après une généralisation évidente de l'inégalité d'Hadamard,

$$\begin{aligned} \det(L) &= \det(L_1 \oplus \ldots \oplus L_r) \\ &\leq \det(L_1). \ldots .\det(L_r) = \det(L_1^0). \ldots .\det(L_r^0) \quad \text{pour tout } L \in \mathcal{L}. \end{aligned}$$

Les deux hypothèses du théorème de Mahler étant vérifiées, l'ensemble fermé $\mathcal{F}$ est compact. La fonction $L \longmapsto \gamma(L) = \frac{N}{\det(L)^{1/n}}$ continue sur l'ensemble $\mathcal{F}$ et à valeurs dans $\mathbf{R}$ admet donc un maximum sur $\mathcal{F}$. Il existe donc au moins un réseau $\mathcal{F}$-extrême dans $\mathcal{F}$.

II.2.4 Caractérisation. *La $\mathcal{F}$-extrémalité équivaut à la condition suivante, où tr désigne la trace :*

$$\text{Si } v \in V \text{ vérifie } \min_{x \in S} \varphi_x(v) = 0 \quad \text{et} \quad \mathrm{tr}(v) \leq 0, \text{ alors il est nul.} \qquad (1)$$

Démonstration :

Soient L un réseau de $\mathcal{F}$ et S l'ensemble de ses vecteurs minimaux. Alors L est $\mathcal{F}$-*extrême* si et seulement si pour u assez voisin de l'identité on a : $\gamma(u(L)) \leq \gamma(L)$. Puisque $N(u(L)) = N(L)$, on a l'équivalence :

$$\gamma(u(L)) \leq \gamma(L) \Longleftrightarrow \det(u) \geq 1.$$

1) Supposons L $\mathcal{F}$-extrême, et montrons la condition (1). Soit v non nul appartenant à $\mathcal{C}$ tel que $\mathrm{tr}(v) \leq 0$. Prenons v sur la sphère unité Σ de V, et u tel que ${}^t u.u = id + \varepsilon v$ avec $\varepsilon > 0$ suffisamment petit pour que l'on ait l'implication :

$$v \in \mathcal{C} \Longrightarrow N(u(L)) = N(L).$$

Soit $P_v(X) = \det(X.\mathrm{id} - v) = X^n - T_1(v).X^{n-1} + ... + (-1)^n.T_n(v)$ le polynôme caractéristique de v. On a $T_1(v) = \mathrm{tr}(v) = \sum_{i=1}^{n} \lambda_i$ et $T_2(v) = \sum_{i<j} \lambda_i.\lambda_j$ où $\lambda_1, \lambda_2, ..., \lambda_n$ désignent les valeurs propres de v. (Elles sont toutes réelles car v est symétrique.)

On remarque que : $(T_1(v))^2 = 2.T_2(v) + \sum_{i=1}^{n} \lambda_i^2 > 2.T_2(v)$ puisque v est non nul, donc que $T_2(v)$ est strictement négatif si $T_1(v) = 0$.
Grâce au développement limité à l'ordre 2 suivant :

$$\det(id + \varepsilon v) = (-\varepsilon)^n P_v(-1/\varepsilon) = 1 + \varepsilon T_1(v) + \varepsilon^2 T_2(v) + o(\varepsilon^2), \qquad (2)$$

on obtient donc les implications (pour $\varepsilon > 0$ assez petit) :

$$\gamma(u(L)) \leq \gamma(L) \Longrightarrow \det(u) \geq 1 \Longrightarrow T_1(v) > 0.$$

Ceci est contraire à l'hypothèse $\mathrm{tr}(v) \leq 0$. v est donc nul.

2) Montrons la réciproque. Soit L vérifiant la condition :

$$\begin{cases} v \in \mathcal{C} \\ \mathrm{tr}(v) \leq 0 \end{cases} \Longrightarrow v = 0$$

(c'est en fait une condition sur $S = S(L)$ *via* le cône $\mathcal{C}$). Soit K l'intersection de Σ avec le cône $\mathcal{C} = \{v \in \mathrm{End}_s E,\ \min_{x \in S} \varphi_x(v) = 0\}$ alors K est compact. Supposons $\mathrm{tr}(v) > 0$ (v est donc non nul) ; d'après (2), il existe $\beta_v > 0$ tel que $\forall t, 0 < t \leq \beta_v$ entraîne $\det(tv + id) > 1$.

Par ailleurs, pour tout t, la fonction $v \longmapsto \det(tv + id)$ est continue donc en particulier, pour tout v dans K, il existe un voisinage $B(v)$ de K tel que $\forall w \in B(v)$ $\det(\beta_v w + id) > 1$.

Or la fonction $t \longmapsto \log(\det(tv+id))$ est strictement concave (il suffit de calculer la dérivée seconde), donc si $\log(\det(tv+id))$ est positif pour $t=\beta_v$ alors il en est de même sur tout l'intervalle $]0,\beta_v[$. On obtient donc :

$$\forall w \in B(v),\ \forall t \in]0,\beta_v[, \quad \det(tw+id) > 1.$$

Les voisinages $B(v)$ forment donc un recouvrement du compact K. En extrayant un sous-recouvrement fini $(\bigcup_{i=1}^{p} B(v_i))$ et en prenant $\beta = \min_{i\leq p} \beta_{v_i}$, on a :

$$\forall w \in K, \forall t \in]0,\beta[, \quad \det(tw+id) > 1.$$

Soit $L' = u(L)$ un réseau de $\mathcal{F}$ non isométrique à L, alors ${}^t u.u - id$ est non nul et appartient au cône C. D'après ce qui précède, si $0 < \|{}^t u.u - id\| < \beta$, alors le déterminant de u est strictement supérieur à 1, et donc $\gamma(u(L)) < \gamma(L)$.

Remarquons que le maximum local de la constante d'Hermite est strict, ce qui signifie qu'*à isométrie près, les réseaux extrêmes sont isolés.*

Pour exploiter la condition (1), qui ne fait intervenir que les restrictions à V des φ_x et de la trace, on introduit des notions de perfection et d'eutaxie relatives au sous-espace V.

II.2.5 Définition: Un réseau $L \in \mathcal{F}$ est *$\mathcal{F}$-parfait* si l'on a la propriété suivante :

$$(\varphi_x(v) = 0 \quad \forall\, x \in\ S(L) \text{ et } v \in\ V \quad \Rightarrow \quad v = 0).$$

II.2.6 Définition: $L \in \mathcal{F}$ est *$\mathcal{F}$-eutactique* s'il existe des coefficients $\rho_x > 0$ pour tout x dans $S(L)$ tels que :

$$\mathrm{tr}(v) = \sum_{x \in S(L)} \rho_x.\varphi_x(v) \qquad \forall v \in V.$$

Remarquons que ces notions sont plus faibles que les notions classiques correspondantes.

Rappelons un théorème de programmation linéaire que l'on utilisera pour la caractérisation des réseaux *$\mathcal{F}$-extrêmes.*

II.2.7 Théorème. *(Stiemke) Soient W un espace vectoriel réel de dimension finie et $\varphi_1,..,\varphi_p$ des formes linéaires sur W, alors les conditions suivantes sont équivalentes.*

(i) $\exists\ \rho_1,....,\rho_p$ tous strictement positifs tels que $\rho_1.\varphi_1 + ... + \rho_p.\varphi_p = 0$

(ii) On a l'implication :

$$\left\{\begin{array}{l} \varphi_1(v) \geq\ 0 \\ \quad\vdots \\ \varphi_p(v) \geq\ 0 \\ v \in\ W \end{array}\right. \Rightarrow \left\{\begin{array}{l} \varphi_1(v) = 0 \\ \quad\vdots \\ \varphi_p(v) = 0 \end{array}\right. .$$

On obtient alors l'équivalent du théorème de Voronoï [V] pour les réseaux de $\mathcal{F}$.

II.2.8 Théorème. *Soit $L \in \mathcal{F}$, il y a équivalence entre les deux propriétés suivantes :*

(i) L est $\mathcal{F}$-extrême.

(ii) L est $\mathcal{F}$-parfait et $\mathcal{F}$-eutactique.

Démonstration :

• ((i) $\Rightarrow$ (ii)) Supposons la $\mathcal{F}$-extrémalité, c'est-à-dire (d'après II.2.4)
$(\min_{x \in S(L)} \varphi_x(v) = 0$ et $\mathrm{tr}(v) \leq 0$ pour $v \in V \Rightarrow v = 0)$,
et montrons la $\mathcal{F}$-eutaxie.
La $\mathcal{F}$-extrémalité nous donne la condition (ii) du théorème de Stiemke pour l'espace V et les formes $(-\mathrm{tr})$ et $(\varphi_x)_{x \in S(L)}$, l'équivalence du théorème donne directement la $\mathcal{F}$-eutaxie.

Montrons la $\mathcal{F}$-perfection. Soit $v \in V$ tel que $\varphi_x(v) = 0 \quad \forall x \in S(L)$. On a donc $\min_{x \in S(L)} \varphi_x(v) = 0$, mais on a aussi $\varphi_x(-v) = 0 \ \forall x \in S(L)$, donc, quitte à changer v en $-v$, on peut supposer $\mathrm{tr}(v) \leq 0$, et par la caractérisation de la $\mathcal{F}$-extrémalité, v est nul.

•((ii) $\Rightarrow$ (i)) Soit maintenant un réseau *$\mathcal{F}$-eutactique* et *$\mathcal{F}$-parfait*, et montrons qu'il est *$\mathcal{F}$-extrême.*

Soit $v \in V$ tel que $\min_{x \in S(L)} \varphi_x(v) = 0$ et $\mathrm{tr}(v) \leq 0$; la relation de $\mathcal{F}$-eutaxie donne la condition (i) du théorème de Stiemke pour les formes $(-\mathrm{tr})$ et $(\varphi_x)_{x \in S(L)}$, et en appliquant la condition (ii) à v, on obtient $\varphi_x(v) = 0, \forall x \in S(L)$ (et $\mathrm{tr}(v)$=0) ; la $\mathcal{F}$-perfection permet de conclure : $v = 0$.

Remarque : Dans la pratique, pour trouver tous les réseaux *$\mathcal{F}$-extrêmes*, il est beaucoup plus facile de chercher dans un premier temps les réseaux *$\mathcal{F}$-parfaits*, puis d'examiner la $\mathcal{F}$-eutaxie (qui n'est pas toujours facile à détecter).

Enumérons certaines propriétés des vecteurs minimaux des réseaux *$\mathcal{F}$-parfaits* ; celles-ci nous permettront une simplification de l'étude algorithmique.

II.2.9 Caractérisation. *Soient $L = \bigoplus_{i=1..r} L_i$ un réseau de $\mathcal{F}$ et $S' = S(L) \setminus \bigcup_{i=1}^{r} S(L_i)$ l'ensemble des vecteurs minimaux hors des sections $(L_i)_{i \leq r}$. Alors L est $\mathcal{F}$-parfait si et seulement si les restrictions à V des formes φ_x pour $x \in S'$ engendrent V^*, l'espace dual de V.*

Démonstration :

Soient W le sous-espace de V^* engendré par les restrictions à V des $(\varphi_x)_{x \in S(L)}$ et $W^\perp = \{v \in V,\ \varphi_x(v) = 0 \quad \forall\ \varphi_x \in W\}$ son orthogonal dans V. Par définition de la $\mathcal{F}$-perfection, on a les équivalences :

$$L\ \mathcal{F}\text{-}parfait \Longleftrightarrow W^\perp = \{0\} \Longleftrightarrow W = V^*.$$

Par ailleurs, remarquons que les formes φ_x pour $x \in \bigcup_{i=1}^{r} S(L_i)$ n'interviennent pas dans la construction de $W^\perp$: en effet, si x est dans une section L_i alors $\forall\, v \in V \ \varphi_x(v) = v(x).x = 0$

par définition de V. Donc W est en fait engendré par les restrictions à V des φ_x pour $x \in S'$.

On en déduit qu'un réseau $\mathcal{F}$-parfait doit avoir suffisamment de vecteurs minimaux hors des sections L_i :

II.2.10 Corollaire. *Le nombre s' de couples $(\pm x)$ de vecteurs minimaux hors des sections $F_i = \mathbb{R}.L_i$ $(i \in \{1 \cdots r\})$ d'un réseau $\mathcal{F}$-parfait est supérieur ou égal à : dim $V = \sum_{1 \leq i < j \leq r} n_i.n_j$ (où $n_i = \dim F_i$).*

Démonstration :

Il est clair que, si L est $\mathcal{F}$-parfait, on doit avoir $s' \geq \dim(V)$ d'après II.2.9. Calculons donc la dimension de V. (dans des bases convenables) ayant ses blocs diagonaux nuls

Soit $\mathcal{M}_0$ l'ensemble des matrices symétriques $n \times n$ ayant des blocs diagonaux de dimensions $n_1, \ldots, n_r$ nuls, c'est-à-dire de la forme

$$\begin{pmatrix} \boxed{0} & A_1^2 & \cdots & \cdots A_1^r \\ {}^tA_1^2 & \boxed{0} & \cdots & \cdots \\ \cdots & \cdots & \boxed{0} & A_{r-1}^r \\ {}^tA_1^r & \cdots & {}^tA_{r-1}^n & \boxed{0} \end{pmatrix}.$$

Tout endomorphisme v de V peut être représenté par une matrice symétrique de $\mathcal{M}_0$. Fixons en effet une base β de E telle que $\beta = \bigcup_{j=1}^{r} \beta_j$ où $\beta_j = (e_i^j)_{i \in \{1, \cdots, n_j\}}$ est une base de F_j et soit β^* sa base duale. Soit $v \in V$. Sa matrice dans les bases (β, β^*) est symétrique ; de plus, pour tout $j \in \{1, \cdots, r\}$, la condition $v(F_j) \subset F_j^{\perp}$ équivaut à : $v(e_i^j).e_k^j = 0 \quad \forall\, i, k \in \{1, \cdots, n_j\}$. Donc nécéssairement $v(e_i^j)$ est nul sur l'espace engendré par $\{(e_1^j)^*, \cdots, (e_{n_j}^j)^*\}$. On aboutit alors à la représentation annoncée dans les bases (β, β^*). La dimension de V est donc : $\sum_{1 \leq i < j \leq r} n_i.n_j$, d'où le résultat.

Le résultat suivant va nous permettre de construire par récurrence des réseaux parfaits en "patchwork" de dimension donnée.

II.2.11 Théorème. *Soit $L = \bigoplus_{i=1}^{r} L_i$ avec chaque L_i parfait (au sens classique). Alors L est parfait si et seulement s'il est $\mathcal{F}$-parfait.*

Démonstration :

Rappelons qu'un réseau L est ***parfait*** si :

$$(\varphi_x(v) = 0 \quad \forall\, x \in S(L) \text{ et } v \in \mathrm{End}_s(E) \quad \Rightarrow \quad v = 0),$$

et que si L est ***parfait*** alors il est nécessairement $\mathcal{F}$-parfait. Montrons la réciproque sous les hypothèses du théorème.

Soit $v \in \mathrm{End}_s E$ tel que $\forall x \in S \quad v(x).x = 0$. Montrons que v est nul. Il se décompose de manière unique sous la forme $v = v_1 + w$ avec $v_1 \in V^\perp$ et $w \in V$.

Soient $1 \leq j \leq r$ et $(\pm x_k^j)_{k=1..s_j}$ les vecteurs minimaux de L_j ; ils appartiennent aussi à S car L et L_i ont même norme. On obtient donc :

$$v(x_k^j).x_k^j = 0 = v_1(x_k^j).x_k^j \ + \ w(x_k^j).x_k^j \qquad \forall k = 1 \ldots s_j.$$

Or puisque $x_k^j \in F_j$ et $w \in V$, $w(x_k^j) \in \ {F_j}^\perp$ donc le dernier terme est nul. Par conséquent $v_1(x_k^j).x_k^j = 0 \ \forall k$. La perfection des L_j nous conduit à $v_{1/F_j} = 0 \ \forall j \in \{1,..,r\}$ ce qui prouve que $v_1 = 0$. D'où $v = w$ appartient à V ; la $\mathcal{F}$-perfection de L permet de conclure : $v = 0$.

II.3. Expression matricielle

Dans la suite, on identifiera l'espace vectoriel réel des matrices symétriques sur $\mathbb{R}^n$ avec celui des formes quadratiques sur $\mathbb{R}^n$, et on appliquera aux matrices symétriques le vocabulaire des formes quadratiques (matrice positive, définie positive, $\mathcal{F}$-parfaite, minimum sur $\mathbb{Z}^n$, ensemble des vecteurs minimaux).

Pour tout j, $(1 \leq j \leq r)$, on se donne une représentation de la classe d'isométrie de L_j^0 par une matrice de Gram A_j.

Soit $L = \bigoplus_{i=1}^{r} L_i$ un réseau de $\mathcal{F}$, et soit $\beta = \bigcup_{j=1}^{r} \beta_j$ une base de L où β_j est une base de L_j ayant pour matrice de Gram A_j ; alors la matrice de Gram de cette base est de la forme :

$$\begin{pmatrix} \boxed{A_1} & \cdots & \cdots & \cdots & \cdots \\ \cdots & \boxed{A_2} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \ddots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \ddots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \boxed{A_r} \end{pmatrix}.$$

On note $\mathcal{M}_{\mathcal{F}}$ l'ensemble des matrices de ce type représentant un réseau de $\mathcal{F}$ (leurs blocs diagonaux A_j sont donc fixés, et elles ont pour minimum N). Toutes les bases utilisées dans la suite seront de la forme ci-dessus.

II.3.1 Proposition. *Soit $L = \bigoplus_{i=1}^{r} L_i$ un réseau de $\mathcal{F}$, alors il n'existe qu'un nombre fini de représentations de L dans $\mathcal{M}_{\mathcal{F}}$.*

Démonstration :

Soient $\beta = (e_i)_{i=1\ldots n}$ une base de L et A sa matrice de Gram dans $\mathcal{M}_{\mathcal{F}}$. Les blocs diagonaux de A étant fixés, le maximum D des termes diagonaux de la matrice A ne dépend pas de la base choisie. On a alors :

$$\forall i \leq n, \ e_i.e_i \leq D.$$

Or il n'y a qu'un nombre fini de vecteurs dans L vérifiant cette inégalité, donc il n'y a qu'un nombre fini de bases de L dont la matrice de Gram appartient à $\mathcal{M}_{\mathcal{F}}$.

Donnons une interprétation matricielle de la proposition II.2.9.

Soient $L = \bigoplus_{i=1}^{r} L_i$ un réseau de $\mathcal{F}$ et $\beta = \bigcup_{j=1}^{r} \beta_j$ une base de L de la forme habituelle. On note A sa matrice de Gram dans $\mathcal{M}_{\mathcal{F}}$. Tout vecteur minimal x de L peut être représenté dans cette base par $X = {}^t(x_1, \dots, x_n)$. On associe à x la projection orthogonale p_x sur la droite $\mathbb{R}x$ définie par $p_x(y) = \frac{x.y}{x.x}x$ pour $y \in E$ et la matrice $P_x = P_X = X\,{}^tX$ de l'endomorphisme $N(x)p_x$ exprimée dans les bases β^* et β (β^* est la base dual de β). Rappelons que tout endomorphisme symétrique φ_x dans V peut être représenté par une matrice de $\mathcal{M}_0$ dans les bases β^* et β, où $\mathcal{M}_0$ est défini au paragraphe II.2.

On munit $\mathrm{End}_s E$ (resp. $M_s(n \times n, \mathbb{R})$) du produit scalaire $\langle u, v\rangle = \mathrm{tr}(uv)$ (resp. $\langle U, V\rangle = \mathrm{tr}(U\,{}^tV)$).

II.3.2 Corollaire. *Soient $A \in \mathcal{M}_{\mathcal{F}}$ et S' l'ensemble des vecteurs minimaux hors des sections imposées. Alors A est $\mathcal{F}$-parfaite si et seulement si les projections orthogonales Ω_X des $P_X = X\,{}^tX$ sur $\mathcal{M}_0$ engendrent $\mathcal{M}_0$ lorsque X décrit S'.*

Remarque : On pose :

$$\mathcal{M}_0^{\perp} = \mathcal{M}_1 = \left\{ \begin{pmatrix} \boxed{M_1} & 0 & \cdots & \cdots & 0 \\ 0 & \boxed{M_2} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & \boxed{M_r} \end{pmatrix}, \quad M_i \in M_s(n_i \times n_i, \mathbb{R}) \right\}.$$

La projection orthogonale sur $\mathcal{M}_0$ d'une matrice de $\mathrm{End}_s E$ s'obtient en remplaçant les blocs diagonaux par des blocs de zéros. Soit $p_{\mathcal{M}_0}$ cette projection.

Démonstration :

D'après la proposition II.2.8, la $\mathcal{F}$-perfection d'un réseau L équivaut au fait que le système d'équations $\varphi_x(v) = 0$ pour x minimal hors sections n'a pas de solution non nulle dans V.

Soient $\beta = (e_i) = \bigcup_{j=1}^{r} \beta_j$ une base de L et v un endomorphisme symétrique de V ; alors v admet une représentation matricielle $B = (v_{ij}) \in \mathcal{M}_0$ dans les bases β et β^*. On a donc :

$$\begin{aligned} \varphi_x(v) = v(x).x = v(\sum_{i=1}^{n} x_i e_i).\sum_{i=1}^{n} x_i e_i &= \sum_{ij} x_j v_{ij} e_i^* \sum_{k=1}^{n} x_k e_k \\ &= \sum_{ij} x_j v_{ij} x_j v_{ij} x_i \\ &= \mathrm{tr}(BX\,{}^tX) = \langle B, X\,{}^tX\rangle. \end{aligned}$$

Il suffit donc de remarquer que toute matrice de $\mathcal{M}_0$ n'agit que sur la projection orthogonale Ω_X de P_X sur $\mathcal{M}_0$ (Ω_X est à blocs diagonaux nuls). Donc $\langle B, X\,{}^tX\rangle = \langle B, \Omega_x\rangle$. Il y a donc équivalence entre la $\mathcal{F}$-perfection d'un réseau L et le fait que les projections orthogonales Ω_X des $P_X = X\ {}^tX$ sur $\mathcal{M}_0$ engendrent $\mathcal{M}_0$ lorsque X décrit S'.

II.3.3 Proposition. *Une matrice A dans $\mathcal{M}_{\mathcal{F}}$ est $\mathcal{F}$-parfaite si et seulement si elle est déterminée de façon unique par les projections orthogonales Ω_X des P_X sur $\mathcal{M}_0$ lorsque X décrit $S(A)$ (ensemble des vecteurs minimaux de A).*

Démonstration :

Soit A et A' deux matrices de $\mathcal{M}_{\mathcal{F}}$ ayant le même ensemble des Ω_X ; on a donc :

$$\Omega_X = p_{\mathcal{M}_0}(P_X) = p_{\mathcal{M}_0}(P_{X'}) \quad \forall X \in S(A), \forall X' \in S(A')$$

On obtient alors :

$$\begin{aligned}\langle (A - A'), \Omega_X\rangle &= \langle (A - A'), P_X\rangle = \min(A) - \langle A', P_X\rangle \leq 0 \quad \text{car } \min(A) = \min(A') \\ &= \langle (A - A'), P_{X'}\rangle = \langle A, P_{X'}\rangle - \min(A') \geq 0.\end{aligned}$$

Donc :

$$\langle A, \Omega_X\rangle = \langle A', \Omega_X\rangle \qquad \forall X \in S(A)$$

Donc $A' - A$ appartient à l'orthogonal dans $\mathcal{M}_0$ du sous-espace engendré par les Ω_X. Puisque A est $\mathcal{F}$-parfaite, le sous-espace $< \Omega_X,\ X \in S(A) >$ est égal à $\mathcal{M}_0$, et on obtient $A = A'$.

Réciproquement, supposons L non $\mathcal{F}$-parfait ; d'après le corollaire II.3.2 on a :

$$\mathcal{M}_0 \neq < \Omega_X,\ X \in S(A) > .$$

Soit $B \neq 0$ dans $\mathcal{M}_0$ orthogonal à tous les Ω_X. Pour ε assez petit, le minimum de la matrice $A_\varepsilon = A + \varepsilon B$ est atteint sur certains vecteurs minimaux de A, c'est à dire : $S(A_\varepsilon) \subset S(A)$.

Il existe donc $X_0 \in S(A)$ tel que :

$$\begin{aligned}\min(A_\varepsilon) = \langle A_\varepsilon, P_{X_0}\rangle &= \langle (A + \varepsilon B), P_{X_0}\rangle \\ &= \langle A, P_{X_0}\rangle + \varepsilon\langle B, P_{X_0}\rangle \\ &= \min(A) + \varepsilon\langle B, \Omega_{X_0}\rangle \\ &= \min(A)\end{aligned}$$

Donc A_ε est bien une matrice de $\mathcal{M}_{\mathcal{F}}$ dont les vecteurs minimaux sont contenus dans $S(A)$. Le calcul précédent étant valable pour tout vecteur minimal X de A, on a aussi :

$$S(A) \subset S(A_\varepsilon).$$

Pour ε assez petit, l'ensemble des vecteurs minimaux de A_ε est égal à celui de A. A et A_ε sont donc deux matrices de $\mathcal{M}_\mathcal{F}$ possédant le même ensemble des Ω_X. Il n'y a donc pas unicité de la matrice A.

Remarque : La notion de $\mathcal{F}$-perfection est celle de [B-M-S 2.1.] ; toutefois l'hypothèse (II.2.2) sous laquelle les auteurs définissent et étudient leur algorithme peut ne pas être vérifiée. Voici un exemple de réseau $\mathcal{F}$-parfait dont les *vecteurs minimaux n'engendrent pas l'espace E.*

On prend en dimension 4

$$A_1 = A_2 = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix},$$

la matrice

$$A = \begin{pmatrix} 2 & 0 & -1 & -1 \\ 0 & 4 & 0 & -3 \\ -1 & 0 & 2 & 0 \\ -1 & -3 & 0 & 4 \end{pmatrix}$$

possède les 6 vecteurs minimaux

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

de rang 3 dans $\mathbb{R}^4$ mais les images Ω engendrent $\mathcal{T}$.

II.3.4 Corollaire. *Si les matrices A_i sont rationnelles, alors les matrices $\mathcal{F}$-parfaites dans $\mathcal{M}_\mathcal{F}$ le sont aussi.*

Démonstration :

La proposition précédente montre qu'une matrice $\mathcal{F}$-parfaite est solution unique d'un système d'équations affines de la forme suivante :

$$\langle A, X^t X \rangle = min(A) \quad \forall X \in S(A).$$

Or dans ces équations les coefficients des matrices $X^t X$ sont entiers. On a donc un système de Cramer à coefficients entiers. Les solutions sont donc rationnelles.

II.3.5 Théorème. *L'ensemble des classes d'isométrie des réseaux $\mathcal{F}$-parfaits de $\mathcal{F}$ est fini.*

Démonstration :

Soient $L \in \mathcal{F}$ un réseau *$\mathcal{F}$-parfait* et $\beta = \bigcup_{j=1}^{r} \beta_j$ une base de L telle que :

$$\forall j \in \{1 \dots r\} \quad \mathrm{Gram}(\beta_j) = A_j.$$

Soient D le maximum des termes diagonaux de la matrice de Gram de la base β et $x = \sum_{i=1}^{n} a_i e_i$, $a_i \in \mathbb{Z}$ un vecteur minimal de L exprimé dans la base $\beta = (e_i)$.

S'il n'est pas nul, $|a_i|$ est l'indice dans L du sous-réseau

$$L' =< e_1, \dots, e_{i-1}, x, e_{i+1}, \dots, e_n >$$

dont on majore le déterminant par l'inégalité d'Hadamard :

$$\det(L') \leq N(e_1) \dots N(e_{i-1}) N(x) N(e_{i+1}) \dots N(e_n) \leq D^{n-1} N.$$

Par ailleurs, le déterminant d'un réseau de rang n, de norme N, est minoré par $\frac{{\gamma_n}^n}{N^n}$ où γ_n est la constante d'Hermite en dimension n. On a donc :

$${a_i}^2 \leq \frac{{\gamma_n}^n}{N^n} D^{n-1} N = {\gamma_n}^n \frac{D^{n-1}}{N^{n-1}}.$$

Les a_i étant entiers, et la majoration étant indépendante du réseau étudié dans $\mathcal{F}$, on en déduit qu'il n'existe qu'un nombre fini de n-uplets d'entiers pouvant représenter un vecteur minimal de L. Or la matrice de Gram dans la base β d'un réseau *$\mathcal{F}$-parfait* est déterminée de façon unique par les composantes entières de ses vecteurs minimaux dans la base β (prop. II.3.3). On obtient donc un nombre fini de matrices dans $\mathcal{M}_{\mathcal{F}}$ représentant un réseau *$\mathcal{F}$-parfait* de $\mathcal{F}$.

Dans le paragraphe suivant, on adapte l'algorithme de Voronoï à la détermination de tous les réseaux *$\mathcal{F}$-parfaits* de $\mathcal{F}$. En fait, on adoptera le point de vue plus algorithmique des formes quadratiques sur $\mathbb{R}^n$.

II.4. Algorithme de Voronoï pour la famille $\mathcal{F}$.

On note $Q_{\mathcal{F}}$ l'ensemble des formes quadratiques définies par une matrice A de $\mathcal{M}_{\mathcal{F}}$. Il s'agit donc des formes quadratiques définies positives sur $\mathbb{R}^n$ de minimum N et telles que :

$$\begin{cases} Q(x_1, \dots, x_{n_1}, 0, \dots, 0) = Q_1(x_1, \dots, x_{n_1}) \\ Q(0, \dots, 0, x_{n_1+1}, \dots, x_{n_1+n_2}, 0, \dots, 0) = Q_2(x_{n_1+1}, \dots, x_{n_1+n_2}) \\ \vdots \\ Q(0, \dots, 0, x_{n-n_r+1}, \dots, x_n) = Q_r(x_{n-n_r+1}, \dots, x_n) \end{cases},$$

où les Q_i sont des formes quadratiques données sur $\mathbb{R}^{n_i}$ de même minimum N. Par abus de langage, on appelle sections les sous-espaces de $\mathbb{R}^n$ formés des vecteurs dont toutes les composantes sont nulles sauf peut-être celles d'indices $l \in [n_1 + \dots + n_{i-1} + 1, n_1 + \dots + n_{i-1} + n_i]$.

II.4.1 Définitions: Soient Q une forme quadratique de $\mathcal{Q}_{\mathcal{F}}$ et $S'(Q)$ l'ensemble des vecteurs minimaux hors des sections imposées, le *$\mathcal{F}$-domaine* de Voronoï $\mathcal{D}_Q$ de Q est l'enveloppe convexe dans $\mathcal{M}_0$ des demi-droites fermées portant les Ω_x pour $x \in S'(Q)$.

Rappelons que Ω_x est la projection orthogonale sur $\mathcal{M}_0$ de $P_x = x^t x$ et que la $\mathcal{F}$-perfection de Q équivaut au fait que l'intérieur du $\mathcal{F}$-domaine est non-vide dans $\mathcal{M}_0$. Si Q est $\mathcal{F}$-parfaite, on appelle *face* $\mathcal{R}$ du $\mathcal{F}$-domaine $\mathcal{D}_Q$ son intersection avec un hyperplan d'appui (dans M_0). On lui associe une matrice $B_\mathcal{R}$ de $\mathcal{M}_0$, unique à homothétie positive près, telle que :

$$\begin{cases} (i) & \langle B_\mathcal{R}, \Omega_x \rangle \geq 0 \qquad \forall x \in S(Q) \\ (ii) & \langle B_\mathcal{R}, \Omega_x \rangle = 0 \Longleftrightarrow \Omega_x \in \mathcal{R} \end{cases}.$$

Si $Q_\mathcal{R}$ est la forme quadratique définie par la matrice $B_\mathcal{R}$, on associe à chaque face $\mathcal{R}$ une nouvelle forme $Q_\theta = Q + \theta Q_\mathcal{R}$ qui est encore dans $\mathcal{Q}_\mathcal{F}$ puisque $B_\mathcal{R}$ est dans $\mathcal{M}_0$. Lorsque $\theta > 0$ est proche de 0, les vecteurs minimaux de la forme quadratique Q_θ proviennent de ceux de Q. Il existe donc x vecteur minimal à la fois de Q et de Q_θ de sorte que :

$$\min(Q_\theta) = Q_\theta(x) = Q(x) + \theta Q_\mathcal{R}(x) \geq \min(Q),$$

et il y a égalité sur l'ensemble des vecteurs minimaux x de Q dont l'image $P_x = x^t x$ dans l'espace de Voronoï se projette orthogonalement sur $\mathcal{M}_0$ dans la face $\mathcal{R}$. Pour $\theta > 0$ assez petit, les formes Q et Q_θ ont donc même minimum.

Soit $\rho > 0$ la borne supérieure des $\theta \geq 0$ tels que Q_θ soit positive de même minimum que Q. Notons que ρ est fini car la forme $Q_\mathcal{R}$ n'est pas positive (puisqu'elle appartient à $\mathcal{M}_0$). Il esiste alors un vecteur $x \in \mathbb{Z}^n$ tel que $Q_\mathcal{R}(x)$ soit négatif, et donc tel que la fonction $\theta \mapsto Q_\theta(x)$ soit décroissante. Le phénomène *d'impasse* rencontré dans d'autres variantes de l'algorithme ne se produira pas ici.

Alors, comme dans la cas classique, par définition de ρ, il existe $y \in \mathbb{Z}^n$ tel que :

$$\begin{cases} Q_\rho(y) = \min(Q) \\ Q_\theta(y) < \min(Q) \qquad \forall \theta > \rho \end{cases}.$$

Le vecteur y est donc minimal de Q_ρ et comme :

$$Q_\theta(y) = Q(y) + \theta Q_\mathcal{R}(y) < \min(Q) = Q(y) \quad \forall \theta > \rho,$$

on obtient :

$$Q_\mathcal{R}(y) = \langle B_\mathcal{R}, P_y \rangle = \langle B_\mathcal{R}, \Omega_y \rangle < 0.$$

La projection orthogonale Ω_y sur $\mathcal{M}_0$ de $P_y = y^t y$ n'est donc pas dans la face $\mathcal{R}$. L'ensemble des vecteurs Ω_x pour $x \in S(Q_\rho)$ contient, en plus des vecteurs qui appartiennent à l'hyperplan $\mathcal{R}$, $\Omega_y \notin \mathcal{R}$; il engendre donc $\mathcal{M}_0$ et la forme Q_ρ est $\mathcal{F}$-parfaite.

Cette nouvelle forme appartenant à $Q_\mathcal{F}$ sera appelée *contiguë* de Q par la face $\mathcal{R}$.

On construit ainsi à partir d'une forme $\mathcal{F}$-parfaite un graphe de contiguïté. Montrons, comme Voronoï l'a fait dans le cas classique et A.M. Bergé, J. Martinet, F. Sigrist [B-M-S] sous d'autres hypothèses, que toute forme $\mathcal{F}$-parfaite de $Q_\mathcal{F}$ appartient à ce graphe. Cela est basé sur le résultat suivant dont on trouve une démonstration dans [Mar].

II.4.2 Lemme (*Voronoï*). Soient Q une forme quadratique définie positive, K et m deux constantes strictement positives, alors l'ensemble des formes parfaites Q' de minimum m vérifiant $\langle Q, Q' \rangle \leq K$, est fini.

II.4.3 Proposition. *Soient Q et Q' deux formes quadratiques définies positives dans $\mathcal{Q}_{\mathcal{F}}$ de matrices A et A' dans $\mathcal{M}_{\mathcal{F}}$. Supposons Q $\mathcal{F}$-parfaite, alors il existe un chemin $Q_0 = Q, Q_1, \ldots, Q_k$ de formes $\mathcal{F}$-parfaites telles que Q_i est contiguë à Q_{i+1} et la projection orthogonale sur $\mathcal{M}_0$ de A' appartienne au $\mathcal{F}$-domaine de Q_k.*

Démonstration :

Supposons que $p_{\mathcal{M}_0}(A') \notin \mathcal{D}_Q$, alors il existe une face de $\mathcal{D}_Q$ dont l'orthogonal $\mathbb{R}.B$ dans $\mathcal{M}_0$ vérifie :

$$\langle B, p_{\mathcal{M}_0}(A')\rangle = \langle B, A'\rangle < 0.$$

Soit Q_1 la *contiguë* de Q à travers cette face. On a donc : $\langle Q_1, Q'\rangle < \langle Q, Q'\rangle$. On construit ainsi une suite strictement décroissante

$$\ldots < \langle Q_2, Q'\rangle < \langle Q_1, Q'\rangle < \langle Q, Q'\rangle.$$

Les formes Q_i étant *$\mathcal{F}$-parfaites* et de même minimum, le lemme II.4.2 permet de conclure que la suite s'arrête à partir d'un certain rang et donc qu'il existe $k > 0$ tel que $p_{\mathcal{M}_0}(A') \in \mathcal{D}_{Q_k}$.

II.4.4 Lemme. *(cf. [B-M-S]) Soient Q et Q' deux formes quadratiques $\mathcal{F}$-parfaites de même minimum et soit $\overset{\circ}{\mathcal{D}}_{Q'}$ l'intérieur du domaine $\mathcal{D}_{Q'}$; alors, si $\mathcal{D}_Q \cap \overset{\circ}{\mathcal{D}}_{Q'}$ est non vide, $Q = Q'$.*

Démonstration :

Soit $G \in \mathcal{D}_Q \cap \overset{\circ}{\mathcal{D}}_{Q'}$.

Alors $\forall x' \in S'(Q')$ il existe $\beta_{x'} > 0$ tel que $G = \sum_{x' \in S'(Q')} \beta_{x'}\Omega_{x'}$ (rappelons que $\Omega_{x'}$ est la projection orthogonale sur $\mathcal{M}_0$ des $P_{x'}$ pour $x' \in S'(Q')$).

En notant A et A' les matrices de Q et Q' dans $\mathcal{M}_{\mathcal{F}}$, on a :

$$\begin{aligned}
\langle G, A - A'\rangle &= \sum_{x' \in S'(Q')} \beta_{x'}\langle \Omega_{x'}, A - A'\rangle \\
&= \sum_{x' \in S'(Q')} \beta_{x'}\langle P_{x'}, A - A'\rangle \\
&= \sum_{x' \in S'(Q')} \beta_{x'}\langle P_{x'}, A\rangle - \sum_{x' \in S'(Q')} \beta_{x'}\langle P_{x'}, A'\rangle \\
&\geq \sum_{x' \in S'(Q')} \beta_{x'} \min(A) - \sum_{x' \in S'(Q')} \beta_{x'} \min(A') = 0
\end{aligned}$$

En faisant le même raisonnement avec les Ω_x pour $x \in S(Q)$, on obtient l'inégalité inverse ce qui nous prouve que $\langle G, A'\rangle = \langle G, A\rangle$.

D'où :

$$\begin{cases} \displaystyle\sum_{x' \in S'(Q')} \beta_{x'}\langle \Omega_{x'}, A - A'\rangle = 0 \\ \langle \Omega_{x'}, A - A'\rangle = \langle P_{x'}, A - A'\rangle \geq 0 \quad \forall x' \in S'(Q'). \end{cases}$$

Comme les $\beta_{x'}$ sont strictement positifs, on a les égalités :

$$\forall x' \in S'(Q'),\ \langle \Omega_{x'},\ A - A' \rangle = 0\ ;$$

or les $\Omega_{x'}$ engendrent $\mathcal{M}_0$, donc $A - A'$ (qui est dans $\mathcal{M}_0$) est nul.

II.4.5 Théorème de connexité. *Soient deux formes quadratiques Q et Q' $\mathcal{F}$-parfaites, alors il existe un chemin $Q_0 = Q, Q_1, \ldots, Q_k = Q'$ de formes $\mathcal{F}$-parfaites contiguës.*

Démonstration :

Prenons le cas non trivial où Q' n'est pas *contiguë* à Q. La $\mathcal{F}$-perfection de Q' nous assure que l'intérieur dans $\mathcal{M}_0$ de son *$\mathcal{F}$-domaine* $\mathcal{D}_{Q'}$ est non vide. Soit $G \in \overset{\circ}{\mathcal{D}}_{Q'}$, il existe alors un chemin $Q_0 = Q, Q_1, \ldots, Q_k$ de formes *$\mathcal{F}$-parfaites contiguës* tel que $G \in \mathcal{D}_{Q_k}$, donc d'après le lemme II.4.4 on a $Q_k = Q'$.

Remarque : On peut donc maintenant élaborer un programme permettant de trouver toutes les matrices $\mathcal{F}$-parfaites de $\mathcal{M}_{\mathcal{F}}$ à partir de l'une d'entre elles.

II.5. Améliorations de l'algorithme

Pour améliorer l'algorithme, et en particulier le temps d'exécution des programmes, on introduit une nouvelle notion d'équivalence de deux formes quadratiques.

II.5.1 Définition: Deux formes quadratiques Q et Q' appartenant à $Q_{\mathcal{F}}$, de matrices A et A' dans $\mathcal{M}_{\mathcal{F}}$, sont dites *$\mathcal{F}$-équivalentes* s'il existe une matrice $P \in \mathrm{GL}_n(\mathbb{Z})$ de la forme

$$\begin{pmatrix} \boxed{P_1^1} & 0 & \cdots & \cdots & 0 \\ 0 & \boxed{P_2^2} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & \boxed{P_r^r} \end{pmatrix}, \quad P_i^i \in \mathrm{GL}_{n_i}(\mathbb{Z})$$

telle que : ${}^tPAP = A'$.

II.5.2 Proposition. *La $\mathcal{F}$-équivalence entre deux formes $\mathcal{F}$-parfaites Q et Q', de matrices dans $\mathcal{M}_{\mathcal{F}}$ A et A', met en bijection les ensembles de leurs formes contiguës, et plus particulièrement, deux faces $\mathcal{F}$-équivalentes d'une forme stable par cette équivalence donnent deux contiguës $\mathcal{F}$-équivalentes.*

Démonstration

Soit $P \in \mathrm{GL}_n(\mathbb{Z})$ de la forme de la définition II.4.1 telle que ${}^tPAP = A'$. Soient $\mathcal{R}$ une face du $\mathcal{F}$-domaine de Q, et $\mathbb{R}.B$ son orthogonal dans $\mathcal{M}_0$. On note $A_1 = A + \rho B$ la contiguë de A à travers cette face.

Remarquons que la matrice P stabilise le sous-espace $\mathcal{M}_0$ et qu'elle définit une bijection entre les vecteurs minimaux de Q et ceux de Q'. En effet, si $x \in S(Q)$ alors $x' = P^{-1}x$ appartient à $S(Q')$.

On a alors pour $x' \in S(Q')$:

$$\begin{aligned}\langle {}^tPBP, \Omega_{x'}\rangle = \langle {}^tPBP, x'{}^tx'\rangle &= \mathrm{tr}({}^tPBPP^{-1}(x{}^tx){}^tP^{-1}) \\ &= \mathrm{tr}(Bx{}^tx) = \langle B, x{}^tx\rangle = \langle B, \Omega_x\rangle.\end{aligned}$$

La matrice ${}^tPBP$ est donc un représentant de l'orthogonal d'une face du $\mathcal{F}$-domaine de A'. Or on a :

$$A_1 = A + \rho B \quad \text{donc} \quad {}^tPA_1P = A' + \rho{}^tPBP.$$

On conclut alors que $A_2 = {}^tPA_1P$ est la contiguë de A' par la face orthogonale à ${}^tPBP$.

Remarque : Notons qu'on trouve dans [B-M-S] une notion d'équivalence moins restrictive : la condition imposée aux matrices P de changements de bases est simplement ${}^tP\mathcal{M}_0P \subset \mathcal{M}_0$, ce qui suffit à assurer la démonstration précédente. Toutefois, cette condition ${}^tP\mathcal{M}_0P \subset \mathcal{M}_0$, n'est *a priori* pas très facile à détecter d'un point de vue algorithmique.

D'un point de vue géométrique, notre $\mathcal{F}$-équivalence consiste à considérer seulement les changements de bases qui stabilisent les sections imposées et interdit donc "l'échange" de deux sections isométriques.

Cependant, si l'on considère une famille de patchwork $\mathcal{F}$ dont les sections imposées sont deux à deux non-isométriques, alors on vérifie facilement que les deux notions d'équivalence coïncident dans la pratique.

Le groupe $G_{\mathcal{F}}(A)$ des $\mathcal{F}$-équivalences stabilisant la forme A agit sur les faces de cette même forme. Pour éviter une recherche trop longue des faces d'une forme, on supprime, comme dans le premier chapitre, les génératrices "intérieures" (remarquons que dans ce cas, deux génératrices du domaine peuvent être proportionnelles), et on détermine seulement un représentant de chaque classe de face. On peut donc élaborer un algorithme semblable à celui de Voronoï pour les formes $\mathcal{F}$-parfaites.

•**0.** Trouver une forme $\mathcal{F}$-parfaite $A \in \mathcal{M}_{\mathcal{F}}$ (on utilise un programme de balayage).

•**1.** Trouver les vecteurs minimaux hors des sections imposées et stocker les Ω_x sous forme de vecteurs à $t = \sum_{1\leq i<j\leq r} n_i.n_j$ composantes.

•**2.** Déterminer le groupe $G_{\mathcal{F}}(A)$ des $\mathcal{F}$-équivalences conservant la forme ($i \leftarrow 0; f \leftarrow 0$).

•**3.** Supprimer des génératrices "intérieures".

•**4.** (Back-tracking) Choisir un Ω_x linéairement indépendant de ceux déjà stockés ($i \leftarrow i+1$).

•**5.** Si la classe modulo $G_{\mathcal{F}}(A)$ du sous-espace formé par les Ω_x a déjà été trouvée, ($i \leftarrow i-1$) et aller en 4.

•**6.** Sinon

+ Si $i < t-1$ aller en 4.
+ Sinon tester si c'est une face.
  * Si ce n'est pas une face, ($i \leftarrow i-1$) et aller en 4.
  * Sinon stocker la face ($f \leftarrow f+1; i \leftarrow i-1$) et aller en 4.

•7. Etablir la liste des contiguës $A_1, \ldots, A_f$.

•8. Supprimer de la liste toute forme $\mathcal{F}$-équivalente à une forme déjà rencontrée.

•9. Terminer si aucune forme nouvelle n'a été conservée, sinon aller en 2 (en prenant une nouvelle forme).

D.O. Jaquet constate dans [Jaq] que la complexité de l'algorithme de Voronoï classique dépend de la différence D entre le nombre s de paires de vecteurs minimaux et la dimension de l'espace des matrices symétriques, $\frac{n(n+1)}{2}$. Dans notre cas, on travaille dans un espace $\mathcal{M}_0$ de dimension inférieure ($\dim(\mathcal{M}_0)=\sum_{i<j} n_i.nj$) et seuls les vecteurs minimaux hors des sections interviennent dans l'algorithme (soit s' le nombre de ces paires de vecteurs). La complexité de notre algorithme dépend donc de la différence $D' = s'$-$\dim(\mathcal{M}_0)$.

Concrètement, on remarque que si D' est supérieur à 12, une semaine est nécessaire pour trouver toutes les contiguës d'une forme. Par ailleurs, si cette différence dépasse 25, on ne peut trouver toutes les contiguës en un temps raisonnable. Ces résultats sont tout à fait comparables à ceux trouvés dans le cas classique par D.O Jaquet.

Considérons plus particulièrement le cas où les sections imposées L_i sont de rang 1. La famille $\mathcal{F}$ est donc l'ensemble des réseaux de E de norme N et ayant une base de vecteurs minimaux, de sorte que les réseaux $\mathcal{F}$-parfaits sont les réseaux parfaits (au sens usuel) qui ont une base de vecteurs minimaux. Cette restriction ne change pas la différence D'. En effet, si s est le nombre de paires de vecteurs minimaux de la forme étudiée, on a : $s' = s - n$ et on vérifie facilement que :

$$D' = s' - \dim(\mathcal{M}_0) = s' - \frac{n(n-1)}{2} = s - \frac{n(n+1)}{2} = D.$$

Ceci suggère que le fait d'imposer une base de vecteurs minimaux ne réduit pas la complexité de l'algorithme, fait que j'ai constaté expérimentalement en dimensions 5 et 6.

L'utilisation du groupe des $\mathcal{F}$-équivalences qui stabilisent la forme permet (lorsqu'il n'est pas trivial, c'est-à-dire dans plus de la moitié des cas traités) d'accélérer la recherche des faces (on gagne environ 50% de temps d'exécution du programme). On donne dans l'annexe II un échantillonnage des temps de calculs en fonction du nombre de génératrices du domaine et de l'ordre du groupe des $\mathcal{F}$-équivalents.

II.6. Résultats

Nous nous sommes bornés dans les exemples ci-dessous à des patchwork de deux réseaux entiers L_1 et L_2, représentés par leurs matrices de Gram A_1 et A_2 normalisées à la même norme. Nous avons obtenu dans chaque cas la liste complète des réseaux $\mathcal{F}$-parfaits (qui sont rationnels (II.3.4)) à équivalence près, et donc la constante d'Hermite $\gamma_{\mathcal{F}} = \sup_{L\in\mathcal{F}} \gamma(L)$. Pour le réseau réalisant cette constante (il est en effet unique à isométrie près dans chacun des cas traités), nous précisons une matrice de Gram entière primitive (dont les coefficients sont premiers entre eux dans leur ensemble), son déterminant, son minimum μ, le nombre de couples de vecteurs minimaux s, sa constante d'Hermite γ,

nombre de couples de vecteurs minimaux dans le dual s* et enfin le carré de la constante d'Hermite duale γ'.

Afin de construire des réseaux denses de norme $\mu = 6$, nous avons souvent choisi l'un des réseaux de départ dans la suite W_i des réseaux de Watson [Wat] de dimension i, $2 \leq i \leq 7$, (les W_i sont des sections de norme 3 de $\sqrt{2}\mathbb{E}_7^*$; on a : $W_6 \sim \mathbb{D}_6$, $W_5 \sim \mathbb{A}_5^2$, $W_3 \sim \mathbb{A}_3^*$) renormalisés au besoin.

Nous utilisons aussi les réseaux de racines, et plus généralement les réseaux parfaits, pour lesquels nous adoptons les notations de Conway et Sloane [C-S].

Nous donnons d'abord des exemples de dimensions 5 et 6 pour lesquelles on ne cherche pas à construire des réseaux parfaits (puisque la liste en est déjà connue). On choisit donc l'une des sections non parfaite. Il s'agit en fait d'un réseau de Watson. L'autre est semblable à un réseau de racines $\mathbb{A}_i$ donc de norme paire. On obtient donc des réseaux entiers de normes multiples de 6 qui ne peuvent donc pas être parfaits au sens classique.

II.6.1 *Dimension 5 :* On choisit d'abord L_1 semblable au réseau critique en dimension 3 $\mathbb{A}_3$, et L_2 au réseau de Watson W_2 :

$$A_1 = \begin{pmatrix} 6 & 3 & 3 \\ 3 & 6 & 3 \\ 3 & 3 & 6 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 6 & 2 \\ 2 & 6 \end{pmatrix}$$

On obtient quatre matrices $\mathcal{F}$-parfaites. La plus dense,

$$M_1 = \begin{pmatrix} 6 & 3 & 3 & 3 & 3 \\ 3 & 6 & 3 & 3 & 3 \\ 3 & 3 & 6 & 0 & 3 \\ 3 & 3 & 0 & 6 & 2 \\ 3 & 3 & 3 & 2 & 6 \end{pmatrix},$$

dont les caractéristiques sont

$$(\det = 1188 \quad s = 16 \quad \gamma = 1.4560 \quad s^* = 1 \quad \mu = 6 \quad \gamma'^2 = 1.2792),$$

appartient au chemin de Voronoï reliant deux matrices de Gram de $\mathbb{D}_5$ le long duquel toutes les matrices ont 16 vecteurs minimaux. Elles ont bien entendu un défaut 1 de perfection, et ce sont les seules à avoir plus de $14 = n(n+1)/2 - 1$ vecteurs minimaux. M_1 est la seule matrice (à équivalence près) de norme 6 sur chemin.

Remarquons que le réseau le moins dense de cette famille est un réseau eutactique de norme 4, qui constitue un contre-exemple de dimension minimale à la question posée par Coxeter [Cox] "*les réseaux eutactiques ayant au moins* $\frac{n(n+1)}{2}$ *vecteurs minimaux sont-ils extrêmes?* "

La densité de M_1 le situe entre les réseaux parfaits $\mathbb{D}_5$ et $\mathbb{A}_5^3$:$\gamma(\mathbb{A}_5^3) = 1.4459 < \gamma_{\mathcal{F}} = 1.4560 < \gamma_5 = 1.5157$. Par contre, ce réseau (comme les trois autres) est médiocre du point de vue de la dualité ou des automorphismes : $\gamma'^2 = 1.2792$, $|\mathrm{Aut}(M_1)| = 48$, à comparer à $\gamma'^2(\mathbb{D}_5) = 2$, $|\mathrm{Aut}(\mathbb{D}_5)| = 3840$.

II.6.2 C'est maintenant L_1 qui est semblable à $\mathbb{W}_3$ (donc à $\mathbb{D}_3^*$), et L_2 au réseau $\mathbb{A}_2$ critique en dimension 2.

$$A_1 = \begin{pmatrix} 6 & -2 & -2 \\ -2 & 6 & -2 \\ -2 & -2 & 6 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 6 & 3 \\ 3 & 6 \end{pmatrix}$$

Les 5 réseaux $\mathcal{F}$-parfaits M'_i sont moins denses que les précédents (ils se situent entre les réseaux parfaits $\mathbb{A}_5^3$ et $\mathbb{A}_5$ et l'on a : $\gamma_{\mathcal{F}} = 1.4345$. Par contre, ils sont plus intéressants du point de vue de la dualité. En particulier, le réseau le plus dense, de matrice de Gram

$$M'_1 = \begin{pmatrix} 6 & -2 & -2 & 2 & 3 \\ -2 & 6 & -2 & 2 & -1 \\ -2 & -2 & 6 & -2 & -3 \\ 2 & 2 & -2 & 6 & 3 \\ 3 & -1 & -3 & 3 & 6 \end{pmatrix},$$

de caractéristiques

$$(\det = 1280 \quad s = 14 \quad \gamma = 1.4345 \quad s^* = 4 \quad \mu = 6 \quad \gamma'^2 = 1.8750),$$

fournit la troisième valeur connue pour γ'^2 : $\gamma'^2(M'_1) < \gamma'^2(L(1)) = 1.9599$, où $L(1)$ est le réseau isodual construit par Conway et Sloane dans [C-S1] (cf. chap. III). Remarquons que ce réseau n'a pas d'autre section parfaite que L_2, et que son dual n'en a aucune de dimension supérieure à 1.

Il appartient au chemin de Voronoï reliant les réseaux parfaits $\mathbb{D}_5$ et $\mathbb{A}_5^3$, et c'est ici encore le réseau de norme $\mu = 6$.

Dans le but de trouver de nouveaux de norme 3, nous avons aussi étudié la famille $\mathcal{F}$ patchwork du réseau W_4 et du réseau $\sqrt{3}\mathbb{Z}$. On obtient trois réseaux $\mathcal{F}$-parfaits dont un seul est de norme 3 : il s'agit naturellement du réseau W_5, mais le plus dense est un réseau de norme 6, à savoir le réseau M'_1 précédent. Le réseau le plus intéressant du point de vue de la dualité est W_5 : rappelons que son dual est semblable au réseau parfait $\mathbb{A}_5^3$.

II.6.3 *Dimension 6 :* on prend L_1 et L_2 semblables respectivement au réseau de racines $\mathbb{A}_4$ et au réseau de Watson $\mathbb{W}_2$, de matrices de Gram

$$A_1 = \begin{pmatrix} 6 & 3 & 3 & 3 \\ 3 & 6 & 3 & 3 \\ 3 & 3 & 6 & 3 \\ 3 & 3 & 3 & 6 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 6 & 2 \\ 2 & 6 \end{pmatrix}.$$

Il y a exactement 11 réseaux $\mathcal{F}$-parfaits (de norme 6 sauf un, de norme 12). Le plus dense, de matrice

$$M_2 = \begin{pmatrix} 6 & 3 & 3 & 3 & 3 & 3 \\ 3 & 6 & 3 & 3 & 3 & 2 \\ 3 & 3 & 6 & 3 & 0 & 0 \\ 3 & 3 & 3 & 6 & 0 & 3 \\ 3 & 3 & 0 & 0 & 6 & 2 \\ 3 & 2 & 0 & 3 & 2 & 6 \end{pmatrix},$$

est à peine moins dense que $\mathbb{P}_6^2 \sim \mathbb{E}_6^*$: $\gamma_{\mathcal{F}} = \gamma(M_2) = 1.5949 < \gamma(\mathbb{P}_6^2) = 1.6012$. Il appartient encore à un chemin de Voronoï à 28 couples de vecteurs minimaux au-dessus du réseau $\mathbb{D}_5$ joignant le réseau critique $\mathbb{E}_6$ à lui même, et c'est le seul de norme 6.

Le réseau le moins dense du chemin est encore un réseau eutactique de norme 4, autre contre-exemple à la question posée par Coxeter.

Ici encore, on étudie les extensions du réseau de Watson W_5. On obtient des résultats tout à fait analogues à ceux de II.6.2 : il y a trois extensions $\mathcal{F}$-parfaites, dont une seule, W_6, est de norme 3 ($\gamma(W_6) = 1.5$, $s(W_6) = 16$) ; mais la plus dense est un réseau de norme 6 presque parfait ($\gamma_{\mathcal{F}} = 1.516$, $s = 20$). Une étude analogue en dimension 7 donne 4 extensions $\mathcal{F}$-parfaites de W_6, mais dont la plus dense est évidemment le réseau de norme 3 $W_7 \sim \mathbb{E}_7^*$ (le second en densité de la liste des réseaux parfaits de rang 7). L'un des autres réseaux est également parfait (de norme 6) : il s'agit du réseau P_7^{12}.

Dans ce qui suit, on cherche plutôt à étendre les listes de réseaux parfaits connus, à partir de réseaux parfaits de rangs inférieurs.

II.6.4 *Dimension 8 :* Ici, on ajoute en somme directe un réseau parfait de rang 6 avec le réseau plan hexagonal de même norme. On obtient une liste exhaustive de réseaux parfaits de rang 8, tous plus denses que $\mathbb{A}_8$, ce qui est conforme à la conjecture de Coxeter : "$\mathbb{A}_n$ *is the least dense perfect lattice in n dimensions*". Parmi ces réseaux, 53 n'ont pas de section hyperplane parfaite. Compte tenu des résultats de Laihem, on possède donc maintenant une liste de 1228 réseaux parfaits en dimension 8 qui ont une section parfaite de codimension 1 ou 2.

Le tableau suivant fournit, pour $i = 1, ..., 7$ le nombre (à isométrie près) de réseaux parfaits dans $\mathbb{R}^8$ sommes directes du réseau hexagonal (semblable à $\mathbb{A}_2$) et d'un réseau parfait semblable à P_6^i où les P_6^i sont les 7 réseaux parfaits de rang 6 classés par ordre décroissant de densité. On donne aussi le nombre (shp) de formes sans section hyperplane parfaite, la meilleure constante d'Hermite $\gamma_{\mathcal{F}}$ et le nombre maximum s_{max} de couples $(\pm x)$ de vecteurs minimaux, atteints sur $\mathbb{E}_8$ et deux réseaux construits par Barnes, une extension $g(8,1)$ du réseau P_7^{26} et le réseau L_8^4 déjà rencontré au chapitre I.

$\mathcal{F}$	$P_6^1 = \mathbb{E}_6$	$P_6^2 = \mathbb{E}_6^*$	$P_6^3 = \mathbb{D}_6$	P_6^4	P_6^5	P_6^6	$P_6^7 = \mathbb{A}_6$
Nombre	5	5	5	71	33	84	≥ 9
shp	2	0	0	19	9	23	≥ 3
$\gamma_{\mathcal{F}}$	2 ($\mathbb{E}_8$)	1.754 (L_8^4)	2 ($\mathbb{E}_8$)	1.754 (L_8^4)	1.746 ($g(8,1)$)	1.754 (L_8^4)	2 ($\mathbb{E}_8$)
s_{max}	120	54	120	54	54	54	120

Donnons quelques statistiques sur les 53 réseaux sans section hyperplane parfaite. On trouvera dans *l'annexe I* les principales caractéristiques de ces réseaux.

Tous sont nouveaux sauf le seul réseau de norme 10 figurant dans la liste des contigus de $\mathbb{D}_8$ obtenue par Jaquet. Ils ne peuvent pas être dans la liste des réseaux de Barnes, qui possèdent tous une section hyperplane parfaite de même norme. Ils sont tous de norme 8 excepté deux de norme 6, cinq de norme 12 et un de norme 10. Seulement 15 sont eutactiques, donc extrêmes. Le plus dense de ces 53 réseaux ($\gamma = 1.7354$), de norme 6, de la forme $\mathbb{E}_6 \oplus \mathbb{A}_2$, est aussi celui qui a le plus de vecteurs minimaux ($s = 51$) et le plus d'automorphismes ($g = 144$). Du point de vue de la densité, il se place en

douzième position de la liste de tous les réseaux parfaits connus de rang 8. L'ordre de son groupe d'automorphismes n'est pas très important comparé par exemple au réseau de Barnes L_8^4 qui possède 62208 automorphismes. Plus généralement, les ordres des groupes d'automorphismes des 53 réseaux parfaits valent pour les 5/6-ièmes d'entre eux 2, 4 ou 8.

Du point de vue de la dualité, la meilleure constante d'Hermite duale atteinte sur un réseau au-dessus de $\mathbb{A}_6$, ($\gamma'^2_{max} = 2.6203$), est plus élevée que celle du réseau $\mathbb{D}_8$ ($\gamma'^2(\mathbb{D}_8) = 2$), mais ne dépasse pas celle du réseau L_8^4 ($\gamma'^2(L_8^4) = 2.6666$) qui est la meilleure constante connue, mise à part celle de $\mathbb{E}_8$ qui vaut 4. Le nombre de paires de vecteurs minimaux des duals est inférieur ou égal à 4, ce qui laissait présager de faibles constantes γ'.

Intéressons-nous plus particulièrement aux réseaux au-dessus des trois réseaux de racines $\mathbb{E}_6$, $\mathbb{D}_6$, $\mathbb{A}_6$.

Dans ces trois cas, il y a peu d'extensions ; l'expérience montre que le nombre d'extensions des réseaux P_6^i varie en sens inverse de l'ordre des groupes d'automorphismes des P_6^i. L'étape la plus longue de notre algorithme a ici encore consisté à déterminer les contigus du réseau $\mathbb{E}_8$ à cause du grand nombre de ses vecteurs minimaux (120). Grâce à la remarque de convexité déjà utilisée au chapitre I, on peut éliminer certains vecteurs minimaux pour l'énumération des faces du domaine et donc augmenter la rapidité des calculs ; il a toutefois fallu 20 jours de calculs sur une station SPARC 10 pour déterminer les voisins de $\mathbb{E}_8$ au-dessus de $\mathbb{A}_6$.

Les réseaux de racines ou liés à ces réseaux que l'on trouve au-dessus de $\mathbb{A}_6$ (resp. $\mathbb{D}_6$, resp. $\mathbb{E}_6$) sont $\mathbb{A}_8$, $\mathbb{D}_8$, $\mathbb{A}_8^2$ et $\mathbb{E}_8$ (resp. $\mathbb{D}_8$, $\mathbb{A}_8^2$ et $\mathbb{E}_8$, resp. $\mathbb{A}_8^2$ et $\mathbb{E}_8$).

Autre résultat : On a également construit des réseaux $\mathcal{F}$-parfaits sommes directes de réseaux semblables à $\mathbb{D}_4$ et au réseau de Watson W_4 respectivement. On a notamment trouvé un réseau parfait de norme 6 avec 47 couples de vecteurs minimaux qui contient un réseau semblable à $\mathbb{D}_5$ mais n'a pas de section parfaite de codimension 1 ou 2 et ne figure donc pas dans les listes précédentes. C'est donc un nouveau réseau à ajouter à la liste des 1228 réseaux parfaits de rang 8.

II.6.5 *Dimension 9 et 10 :* On ajoute dans ce paragraphe le réseau plan hexagonal à un réseau parfait de rang 7 ou 8.

On obtient alors une liste de réseaux parfaits en dimensions 9 et 10. En particulier, il y a 19 réseaux (resp. 8 et 14) au-dessus de $\mathbb{E}_7$ (resp. $\mathbb{E}_7^*$ et $\mathbb{E}_8$).

En dimension 9, les réseaux obtenus par extension de $\mathbb{E}_7$ sont assez denses (cf. tableau I). Cependant, les deux réseaux les plus intéressants du point de vue de la densité, du groupe d'automorphismes et du kissing number sont déjà bien connus : il s'agit des réseaux $B_9^2 \sim \Lambda_9$ et $\mathbb{A}_9^3$ de Barnes. Par contre, les extensions de $\mathbb{E}_7^*$ (dont aucune n'est de norme 3) ne sont pas très denses et possèdent au plus 62 vecteurs minimaux.

On a également obtenu une liste d'environ 600 réseaux parfaits de rang 9 de la forme $P_7^6 \oplus \mathbb{A}_2$, où P_7^6 est le sixième réseau parfait de rang 7. Leurs caractéristiques n'étant en général pas très intéressantes, nous avons arrêté la recherche. Nous étudions l'un d'eux au chapitre III.

En dimension 10, on ajoute en somme directe les réseaux $\mathbb{A}_2$ et $\mathbb{E}_8$. On trouve exactement 14 réseaux parfaits dont le réseau laminé Λ_{10} (cf. tableau II).

Dans les tableaux ci-dessous, dans lesquels figurent les 6 réseaux les plus denses au-dessus de $\mathbb{E}_7$ et $\mathbb{E}_8$, on précise dans la première colonne ceux d'entre eux qui ont déjà été rencontrés comme extensions.

Tableau I

Extensions de $\mathbb{E}_7$

	det	N	(s, s^*)	$\lvert Aut \rvert$	γ	γ'^2	Extr
$\mathbb{B}_9^2$	512	4	(136,1)	10321920	2	2	E
$\mathbb{A}_9^3$	21870	6	(129,1)	725760	1.976	1.8	E
$ext(K_8)$	768	4	(99,3)	18432	1.911	2.666	E
	32076	6	(90,2)	2880	1.894	2.318	E
	33534	6	(88,1)	288	1.885	2.217	N-E
$ext(\mathbb{A}_8^2)$	896	4	(91,1)	4608	1.879	2.285	E

Tableau II

Extensions de $\mathbb{E}_8$

	det	N	(s, s^*)	$\lvert Aut \rvert$	γ	γ'^2	Extr
Λ_{10}	768	4	(168,3)	884736	2.058	2.666	E
$ext(\mathbb{A}_9^3)$	54675	6	(147,3)	7776	2.015	2.4	E
$ext(\Lambda_9)$	960	4	(154,2)	322560	2.012	2.133	E
$ext(\mathbb{A}_9^3)$	3400690959	18	(141,1)	1440	2.005	2.227	N-E
$ext(\mathbb{A}_9^3)$	59533056	12	(141,2)	288	2.003	2.257	N-E
	61236	6	(141,2)	17280	1.992	2.142	E

Chapitre III

Algorithme de Voronoï et problèmes de dualité

En parcourant des chemins (stables par dualité ou à automorphisme imposé) joignant deux réseaux parfaits contigus, on construit des réseaux isoduaux (ou à constante duale élevée) en dimensions 3,5,6,7 et 9.

III.1. Introduction et notations

Dans la première partie de ce chapitre, nous utilisons l'algorithme de Voronoï pour la recherche de réseaux "presque" parfaits isoduaux, c'est-à-dire isométriques à leurs duals. Motivés par des questions de géométrie Riemannienne, Conway et Sloane ont introduit dans [C-S1] ce type de réseaux, et les ont étudiés du point de vue de la densité. L'algorithme de Voronoï ou une de ses variantes, introduit naturellement des chemins linéaires joignant, dans l'espace des réseaux, deux réseaux parfaits, éventuellement au sens large. L'un des intérêts de ces chemins, c'est que l'on y rencontre des réseaux à grand kissing number ($s \geq n(n+1)/2 - 1$), constant tout le long du chemin. Il est naturel pour la recherche de réseaux isoduaux, où plus géneralement semblables à leurs duals, de se limiter à des chemins de Voronoï stables par dualité. Plus correctement, il s'agit de chemins dans l'espace des matrices symétriques joignant deux matrices parfaites contiguës, et qui sont stables par inversion modulo normalisation et équivalence. Pour assurer cette condition dans la pratique, on leur impose de contenir (à similitude près) un couple de réseaux duals. Par exemple, les réseaux parfaits $\mathbb{A}_2$ et $\mathbb{A}_2^*$ (qui sont semblables) sont contigus, de même que les réseaux $\mathbb{D}_4$ et $\mathbb{D}_4^*$ (eux aussi semblables), que les réseaux $\mathbb{E}_6$ et $\mathbb{E}_6^*$, et que $\mathbb{E}_7$ et $\mathbb{E}_7^*$.

Dans les cas des dimensions 2 et 4 évoqués ci-dessus, tous les réseaux du chemin sont (après normalisation) isoduaux. Par contre, sur les chemins reliant $\mathbb{E}_6$ et $\mathbb{E}_6^*$ (resp. $\mathbb{E}_7$ et $\mathbb{E}_7^*$), nous avons trouvé un seul réseau isodual, le plus dense connu en dimension 6 (resp. 7), réseau construit par Conway et Sloane (par "collage" de réseaux de racines, cf. [C-S1])

Conway et Sloane ont également construit par "collage" les réseaux isoduaux de densité record en dimensions 3 et 5, mais ces réseaux ne possèdent pas assez de vecteurs minimaux pour pouvoir appartenir à un chemin de Voronoï au sens classique. Nous les trouvons ci-dessous sur un "raccourci" joignant encore deux réseaux parfaits. Enfin, nous montrons que le réseau de dimension 9 construit par Conway et Sloane n'est pas extrême parmi les isoduaux, et nous construisons un chemin d'isoduaux tous plus denses.

Précisons la notion d'isodualité (cf. [B-M1]). Un réseau L est dit σ-isodual s'il existe une transformation orthogonale σ de E telle que $\sigma(L) = L^*$ (ou $\sigma(L^*) = L$), en particulier,

il est dit orthogonal (resp. symplectique) s'il existe une telle isométrie σ avec $\sigma^2 = id$ (resp. $\sigma^2 = -id$, ce qui exige n pair). Un réseau σ-isodual est dit σ-extrême s'il réalise un maximum local de densité sur l'ensemble des réseaux σ-isoduaux. Nous avons vérifié que les réseaux isoduaux que nous construisons sont orthogonaux (et symplectiques dans le cas pair), et σ-extrêmes. (Dans le cas $n = 3$, Conway et Sloane ont montré que le réseau *ccc* est le plus dense des réseaux isoduaux de dimension 3.) Par contre, dans chaque cas, nous prouvons qu'ils réalisent le minimum de la constante γ' sur leur chemin.

Nous adoptons dans la pratique le point de vue matriciel. Dans chaque cas, on construit une famille de matrices symétriques définies positives de même norme, à un paramètre t :

$$A_t = A + tB, \quad t \in I =]a, b[,$$

pour laquelle il existe une matrice $P \in \mathrm{GL}_n(\mathbb{Z})$ telle quel que soit $t \in I$, alors ${}^tPA_t{}^{-1}P$ soit proportionnelle à une matrice A_{t^*} de la famille. La recherche du réseau isodual consiste alors à chercher une valeur de t telle que $t = t^*$.

III.2. Cas des dimensions 6 et 7.

III.2.1. Étude du chemin de Voronoï reliant $\mathbb{E}_6$ à $\mathbb{E}_6^*$.

Dans tout ce qui suit, on utilise l'algorithme de Voronoï à base de vecteurs minimaux imposée (cf. second chapitre).

On lance l'algorithme à partir d'une matrice de Gram A (normalisée à 4) du réseau de racines $\mathbb{E}_6$, et on l'arrête dès que l'on trouve l'unique orbite de faces (modulo l'équivalence classique) conduisant à une matrice de Gram d'un réseau semblable à E_6^*.

Soit B un vecteur de cette face, c'est-à-dire un représentant de l'orthogonal de la face. On étudie ici le *chemin de Voronoï* à un paramètre défini par $A + tB$ où t décrit la droite réelle. On obtient alors la matrice A_t dépendant du paramètre t définie par :

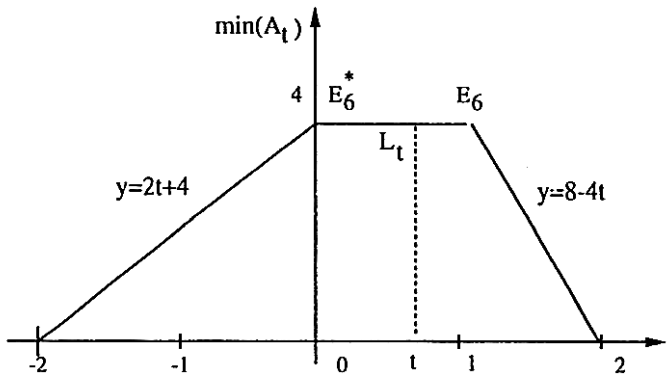
$$A_t = A + tB = \begin{pmatrix} 4 & 0 & 0 & 2 & 2 & 2 \\ 0 & 4 & 2 & 2 & 2 & 2 \\ 0 & 2 & 4 & 2 & 2 & 2 \\ 2 & 2 & 2 & 4 & 2 & 2 \\ 2 & 2 & 2 & 2 & 4 & 2 \\ 2 & 2 & 2 & 2 & 2 & 4 \end{pmatrix} + t \begin{pmatrix} 0 & -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & -1 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & -1 \\ 0 & -1 & 0 & -1 & 0 & -1 \\ 0 & -1 & 0 & -1 & -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & -t & t & 2 & 2 & 2 \\ -t & 4 & 2 & -t+2 & -t+2 & -t+2 \\ t & 2 & 4 & 2 & 2 & 2 \\ 2 & -t+2 & 2 & 4 & -t+2 & -t+2 \\ 2 & -t+2 & 2 & -t+2 & 4 & -t+2 \\ 2 & -t+2 & 2 & -t+2 & -t+2 & 4 \end{pmatrix}.$$

Soit L_t, le réseau associé à la matrice de Gram A_t (lorsqu'il existe, il est unique à isométries près).

Par construction, le minimum de la matrice $A_t = A + tB$ reste constant ($\min(A_t) = 4$) lorsque $t > 0$ est "assez petit", et pour les valeurs $t = 0$ et $t = 1$, on obtient les deux matrices parfaites A_0 et A_1 correspondant aux réseaux de racines $L_0 \sim \mathbb{E}_6$ et à son dual $L_1 \sim \mathbb{E}_6^*$.

Le déterminant du réseau L_t associé à la matrice de Gram A_t vaut $3(t-2)^2(t+2)^4$, ce qui prouve que la matrice n'est plus définie positive lorsque $|t| \geq 2$. On montre alors que si t appartient à $]-2,0[$ (resp. $]1,2[$), le minimum de la matrice A_t décroit linéairement suivant t selon le graphe ci-dessous.



Lemme 2.1.1 : Soit L_t un réseau associé à la matrice de Gram A_t, alors le dual L_t^* est semblable à L_{t^*} avec $t^* = \frac{4t-4}{t-4}$.

Démonstration : Il suffit de vérifier matriciellement la relation ${}^tP({A_t}^{-1})P = \frac{t-4}{6(t^2-4)} A_{\frac{4t-4}{t-4}}$, où P est la matrice d'équivalence suivante :

$$P = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 \end{pmatrix} \in \mathrm{GL}_6(\mathbb{Z}).$$

On vient donc de montrer que la dualié $t \mapsto t^*$ conserve le segment $[0,1]$ et échange les intervalles $]-2,0[$ et $]0,2[$. L_t^* est isométrique à $k(t)L_{t^*}$, avec $k(t) = \sqrt{\frac{t-4}{6(t^2-4)}}$.

Or, il n'existe qu'une valeur $t_0 \in]-2,2[$ telle que $t = \frac{4t-4}{t-4}$ ($t_0 = 2(2-\sqrt{3})$) pour laquelle L_{t_0} est semblable à $L_{t_0}^*$. En renormalisant L_{t_0} de façon qu'il ait même norme que son dual, on obtient un réseau isodual de rang 6, de norme $4k(t_0) = 1 + \frac{1}{\sqrt{3}}$, défini sur $\mathbb{Q}(\sqrt{3})$. Comme tous les réseaux représentés par les matrices A_t, $0 < t < 1$, il possède 24 couples de vecteurs minimaux. Ce réseau figure dans l'article de Conway et Sloane sous le nom de $M(\mathbb{E}_6)$, et il a une interprétation en géométrie riemannienne : J.R. Quine a montré

qu'il s'agit de la jacobienne d'une courbe de Picard de genre 3. Il a également prouvé, avec P.L. Zhang, que ce réseau est symplectique-extrême.

Soit β une base de L_{t_0} telle que $\mathrm{Gram}(\beta) = A_{t_0}$. La transformation σ représentée par P dans les bases β et β^* est du type symplectique. En effet, il suffit de vérifier que $\sigma^2 = -id$, c'est-à-dire, matriciellement :

$$({A_{t_0}}^{-1}P)^2 = -I_6,$$

où ${A_{t_0}}^{-1}P$ est la matrice de σ dans les bases (β, β).

Étudions la constante d'Hermite duale γ' sur un tel chemin.

Proposition III.2.1.2. *Sur un chemin de Voronoï stable par dualité, il existe au plus un réseau σ-isodual σ-extrême (avec $\sigma^2 = \pm Id$), et ce réseau réalise le minimum de la constante γ' sur le chemin.*

Démonstration :

Soit L un réseau σ-extrême, c'est-à-dire qu'il réalise un maximum local de la constante d'Hermite sur l'ensemble des réseaux σ-isoduaux. Un tel réseau doit vérifier une condition d'eutaxie relative à σ (cf. [B-M1]) : il existe des coefficients $\rho_x > 0$ tels que

$$\sum_{x \in S(L)} \rho_x (p_x - p_{\sigma(x)}) = 0,$$

où les p_x sont les projections orthogonales sur les droites $\mathbb{R}x$. Une telle relation s'écrit aussi :

$$\sum_{x \in S(L)} \rho_x p_x = \sum_{x \in S(L)} \rho_x p_{\sigma(x)},$$

où le vecteur $\sigma(x)$ décrit $S(L^*)$ lorsque x décrit $S(L)$. La relation précédente est donc un cas particulier de la relation de dual-eutaxie (cf. [B-M2]).

Par ailleurs, tout le long du chemin de Voronoï, les matrices de Gram A_t ont même ensemble de vecteurs minimaux S ; la stabilité de ce chemin par dualité fait que l'ensemble S^* des vecteurs minimaux de leurs matrices inverses ne dépend pas non plus de t. Ce chemin est donc contenu dans une "classe au sens dual" telle qu'elle est définie dans [Ber], classe qui contient, à similitude près, au plus un réseau dual-eutactique (car S engendre $\mathbb{R}^n$), sur lequel la constante γ' atteint son minimum.

Revenant à l'exemple précédent, nous notons que le réseau isodual n'est pas eutactique. L'unique réseau eutactique (à isométrie près) sur le chemin $\{L_t,\ 0 < t < 1\}$ correspond à la valeur $t = 2/3$, et réalise naturellement (cf. [B-M1]) un minimum de densité sur ce chemin : $\gamma(L_{2/3}) = 1.573 < \gamma(L_{t_0}) = 1.577 < \gamma(\mathbb{E}_6^*) = 1.601 < \gamma(\mathbb{E}_6) = 1.665$ (les valeurs de γ' sont $\gamma'(L_{t_0}) = 1.577 < \gamma'(L_{2/3}) = 1.581 < \gamma'(\mathbb{E}_6^*) = \gamma'(\mathbb{E}_6) = 1.632$).

III.2.2. Étude du chemin de Voronoï reliant $\mathbb{E}_7$ à $\mathbb{E}_7^*$.

On effectue en dimension 7 un travail similaire à celui du paragraphe précédent pour déterminer un réseau semblable à son dual.

Pour obtenir une paramétrisation du chemin, on applique l'algorithme de Voronoï dans le cas où une base de vecteurs minimaux est imposée en partant d'une matrice de Gram de $\mathbb{E}_7^*$ (dont le domaine de Voronoï est très facile à déterminer) et l'on obtient une famille de matrices A_t dépendante du paramètre t :

$$A_t = \begin{pmatrix} 2 & 2t & -2t & 2t & -t+1 & -t+1 & -t+1 \\ 2t & 2 & 2t & -t+1 & 2t & 2t & 2t \\ -2t & 2t & 2 & t-1 & -t+1 & -t+1 & -t+1 \\ 2t & -t+1 & t-1 & 2 & 2t & 2t & 2t \\ -t+1 & 2t & -t+1 & 2t & 2 & -t+1 & -t+1 \\ -t+1 & 2t & -t+1 & 2t & -t+1 & 2 & -t+1 \\ -t+1 & 2t & -t+1 & 2t & -t+1 & -t+1 & 2 \end{pmatrix}, \quad t \in]-1, 1/2[,$$

(le déterminant de cette matrice est $-2(2t-1)(t+1)^6$). Par construction les matrices A_0 et $A_{\frac{1}{3}}$ correspondent aux réseaux $\mathbb{E}_7$ et $\mathbb{E}_7^*$. On vérifie aisément que l'on a :

$${}^tP({A_t}^{-1})P = \frac{1}{2}\frac{-5t+3}{-4t^2-2t+2}A_{\frac{1-3t}{3-5t}}$$

où P est définie par :

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

On vient donc de prouver qu'un réseau L_t correspondant à la matrice A_t possède un dual semblable au réseau $L_{\frac{1-3t}{3-5t}}$. La valeur pour laquelle le réseau L_t est semblable à son dual est $t_0 = 1/5$.

$$5/2A_{t_0} = \begin{pmatrix} 5 & 1 & -1 & 1 & 2 & 2 & 2 \\ 1 & 5 & 1 & 2 & 1 & 1 & 1 \\ -1 & 1 & 5 & -2 & 2 & 2 & 2 \\ 1 & 2 & -2 & 5 & 1 & 1 & 1 \\ 2 & 1 & 2 & 1 & 5 & 2 & 2 \\ 2 & 1 & 2 & 1 & 2 & 5 & 2 \\ 2 & 1 & 2 & 1 & 2 & 2 & 5 \end{pmatrix}.$$

Ce réseau rationnel possède 27 couples de vecteurs minimaux et réalise la meilleure constante d'Hermite connue parmi les réseaux isoduaux. Il est de type orthogonal, bien que la transformation σ définie ci-dessus par la matrice P ne soit pas une symétrie. Or, le sous-groupe des transformations changeant $L_{\frac{1}{5}}$ en son dual est $\sigma.G = \{\sigma g,\, g \in G\}$ où

G est le groupe d'automorphismes du réseau $L_{\frac{1}{5}}$. La transformation $\sigma' \in \sigma.G$ représentée dans les bases β et β^* par la matrice

$$P' = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 \\ 1 & 1 & -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 \end{pmatrix},$$

vérifie la relation d'orthogonalité : ${\sigma'}^2 = Id$.

$$(A_{\frac{1}{5}}{}^{-1} * P')^2 = I = Mat(id, \beta, \beta).$$

On vérifie alors la σ'-extrémalité et notamment la σ'-eutaxie : on trouve des coefficients égaux (tout le long du chemin le groupe d'automorphismes est d'ordre $g = 2^8 3^4 5$, d'indice 28 dans celui du réseau $\mathbb{E}_7$). D'après la proposition III.2.1.2, ce réseau réalise un minimum de la constante γ' sur le chemin ($\gamma' = \gamma = 5/3$). Il n'est donc pas dual-parfait (la dual-perfection, notion introduite dans [B-M2], siginfie que les projections sur les vecteurs minimaux du réseau et ceux de son dual engendrent l'espace $\mathrm{End}_s(E)$) : tout le long du chemin, les projections sur les vecteurs minimaux du réseau L_t engendrent un hyperplan de l'espace, de même que celles relatives à son dual, et ces deux hyperplans viennent se confondre lorsque L_t est isodual. C'est le seul réseau non dual-parfait du chemin. On retrouve le même phénomène pour le réseau isodual de rang 6.

III.3. Étude de la dimension 5.

Dans ce paragraphe, le problème est différent de celui des dimensions 6 et 7. En effet, on vérifie qu'il n'existe pas de chemin de Voronoï reliant un réseau parfait à son dual. Cependant, en considérant un "raccourci" de deux chemins Voronoï consécutifs, on obtient un chemin à un paramètre stable par dualité.

Donnons ci-dessous la construction d'un tel chemin.

Soit

$$A = \begin{pmatrix} 4 & 2 & 0 & 2 & 0 \\ 2 & 4 & -2 & 2 & 0 \\ 0 & -2 & 4 & -2 & -2 \\ 2 & 2 & -2 & 4 & 2 \\ 0 & 0 & -2 & 2 & 4 \end{pmatrix}$$

une matrice de Gram du réseau $\mathbb{D}_5$, renormalisé à la norme 4. Le domaine de Voronoï de A possède exactement 4 classes d'équivalences de faces. (Deux classes différentes (l'une à 14 vecteurs minimaux, l'autre à 16) conduisent à une matrice de Gram de $\mathbb{D}_5$, et les deux autres classes à des matrices de Gram de $\mathbb{A}_5^3$ et $\mathbb{A}_5$.)

Comme dans les exemples précédents, on obtient une paramétrisation du chemin reliant $\mathbb{D}_5$ à $\mathbb{A}_5^3$ en appliquant l'algorithme de Voronoï à la matrice de Gram A du réseau $\mathbb{D}_5$, et

on stoppe la recherche des faces lorsque l'on a trouvé un représentant de l'unique orbite de face conduisant à $\mathbb{A}_5^3$ (soit B la matrice de Gram obtenue).

On considère maintenant le chemin de Voronoï reliant la matrice de Gram A avec une autre matrice de Gram C du réseau $\mathbb{D}_5$ par une face à 16 vecteurs minimaux (cf. II.6.1).

Ces deux chemins de Voronoï ne sont pas stables par dualité. Par contre, si l'on considère le "raccourci" reliant les deux matrices B et C, on obtient un chemin à un paramètre qui n'est pas un chemin de Voronoï, mais qui est stable par dualité.

Donnons ci-dessous une paramétrisation de ce chemin :

$$A_t = C + t * (B - C) = \begin{pmatrix} 4 & 2t & -t+2 & 2t & t-2 \\ 2t & 4 & t-2 & -t+2 & -2t \\ -t+2 & t-2 & 4 & t-2 & t-2 \\ 2t & -t+2 & t-2 & 4 & -t+2 \\ t-2 & -2t & t-2 & -t+2 & 4 \end{pmatrix}.$$

On a alors $A_0 = C$, $A_1 = B$. Le déterminant de cette matrice est $-2(3t-4)(t+2)^4$; la matrice A_t est définie positive lorsque $t \in]-2, 4/3[$, et on vérifie que l'on a :

$${}^tP(A_t)^{-1}P = \frac{7t-10}{8(3t^2+2t-8)} A_{\frac{10t-12}{7t-10}},$$

où P est la matrice de $\mathrm{GL}_5(\mathbb{Z})$ définie par

$$P = \begin{pmatrix} 0 & 0 & -1 & 1 & 1 \\ 0 & -1 & 0 & 0 & 1 \\ -1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & -1 \end{pmatrix}.$$

Un réseau L_t défini par la matrice A_t ($t \in]-2, 4/3[$) possède donc un dual semblable au réseau L_{t^*}, avec $t^* = \frac{10t-12}{7t-10}$. On obtient alors un réseau semblable à son dual pour la racine de l'équation $t = t^*$ appartenant à $]-2, 4/3[$: $t_0 = \frac{6}{7}$. La matrice de Gram rendue entière est

$$\frac{7}{4} A_{t_0} = \begin{pmatrix} 7 & 3 & 2 & 3 & -2 \\ 3 & 7 & -2 & 2 & -3 \\ 2 & -2 & 7 & -2 & -2 \\ 3 & 2 & -2 & 7 & 2 \\ -2 & -3 & -2 & 2 & 7 \end{pmatrix}.$$

Le réseau isodual correspondant est de norme 4/5. Il est de type σ-orthogonal pour l'endomorphisme σ associé à P, et réalise un maximum local de densité parmi les réseaux σ-isoduaux. Ici encore, les coefficients de σ-eutaxie sont tous égaux.

La dualité $t \leftrightarrow t^*$ échange les intervalles $\{-2, 6/7\}$ et $\{4/3, 6/7\}$. Le dual de $\mathbb{D}_5$ correspond à $t = 6/5$, ce qui place sa matrice à l'exterieur du chemin de Voronoï (sur lequel $s = 5$). Par contre le dual $\mathbb{A}_5^2$ de $\mathbb{A}_5^3$ est semblable à la matrice $A_{2/3}$ qui appartient au chemin proprement dit.

Les matrices A_t pour $t \in]0,1[$ ont le même ensemble S de vecteurs minimaux à 10 éléments. Elles ont aussi le même groupe d'automorphismes d'ordre 240. En particulier, elles sont invariantes sous l'action d'une matrice Q d'ordre 5. L'ensemble de matrices M telles que ${}^tQMQ = M$ dépend linéairement de trois paramètres. A.M. Bergé, J. Martinet et F. Sigrist ont mis au point un algorithme à groupe d'automorphismes G donné (cf. [B-M-S]). Dans le cas d'un groupe d'ordre 5, ils ont montré que la matrice C de $\mathbb{D}_5$ a 3 G-contiguës dont deux sont des matrices de Gram de $\mathbb{A}_5$, et la troisième représente le réseau $\mathbb{A}_5^3$. Le chemin de G-contiguïté reliant les matrices de $\mathbb{D}_5$ et $\mathbb{A}_5^3$ est précisément le segment $\{A_t, t \in]0,1[\}$ que nous avons pris comme "raccourci"entre B et C. C'est cette méthode à groupe d'automorphismes imposé que nous allons utiliser dans les exemples suivants.

III.4. Étude de la dimension 3.

Soient G un groupe cyclique d'ordre 4, σ un de ses générateurs et soit la représentation ρ de rang 3 définie par

$$\rho(\sigma) = Q = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}.$$

On considère l'ensemble des matrices symétriques définies positives dórdre 3 $\mathcal{T} = \{M, {}^tQMQ = M\}$. C'est l'espace vectoriel des matrices

$$\begin{pmatrix} x & y & -x-2y \\ y & x & y \\ -x-2y & y & x \end{pmatrix}, \quad x \in \mathbb{R}, y \in \mathbb{R}.$$

Lorsqu'elle sont définies positives, elles représentent les G-réseaux, c'est-à-dire les réseaux de E stables sous l'action du groupe $G \subset O(E)$, et qui sont des $\mathbb{Z}[G]$-modules libres.

Lorsque l'on impose un minimum 2 à ces matrices, on obtient la famille

$$A_t = \begin{pmatrix} 2 & t & -2t-2 \\ t & 2 & t \\ -2t-2 & t & 2 \end{pmatrix}, \quad t \in [-1, -1/2],$$

où A_{-1} représente le réseau parfait $\mathbb{A}_3$ et $A_{-1/2}$ le réseau le moins dense de la classe fragile à 5 vecteurs minimaux cité dans [C-S1]. Lorsque l'on applique l'algorithme de G-Voronoï à partir de la matrice A_{-1}, on obtient dans l'espace $\mathcal{T}$ un domaine à deux faces dont l'une est une impasse (le vecteur de face est une matrice positive), et l'autre face conduit au chemin ci-dessus.

Dans l'intervalle ouvert $]-1,-1/2[$, les matrices ont le même ensemble de vecteurs minimaux constituant une orbite à 4 éléments sous l'action de Q. Les matrices correspondant à $t = -1$ ou $t = -1/2$ ont une deuxième orbite de vecteurs minimaux. Il s'agit donc d'un chemin de G-Voronoï au sens de [B-M-S]. En fait, les matrices A_t sont définies positives pour $t \in]-2,0[$, et on constate que l'on a la relation

$${}^tPA_t{}^{-1}P = k(t)A_{t^*},$$

avec $k(t) = 2\frac{4t^2+8t}{t-2}$ et $t^* = \frac{2t+4}{t-2}$, et $P = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$.

Cette dualité échange la matrice parfaite A_{-1} avec la matrice $A_{-2/3}$ et conserve globalement le segment correspondant à l'intervalle $[-1, -2/3]$ (elle échange les intervalles $]-2/3, 0[$ et $]-2, -1[$). Sur l'intervalle $[-1, -2/3]$, l'équation $t = t^*$ a une unique solution $t_0 = 2 - 2\sqrt{2}$ qui correspond (après normalisation) á un réseau isodual de norme $2(\sqrt{k(t_0)} = 1/2 + \frac{1}{\sqrt{2}}$. On reconnait le réseau *ccc* de Conway et Sloane qui réalise le maximum absolu de la constante d'Hermite des réseaux isoduaux de dimension 3.

III.5. Étude de la dimension 9.

III.5.1. Un chemin intéressant du point de vue de la dualité.

Le réseau extrême $\mathbb{A}_9^5$ de Coxeter est intéressant du point de vue de la dualité. Il a 45 vecteurs minimaux ainsi que son dual $\sim \mathbb{A}_9^2$, et il réalise le maximum de la constante d'Hermite duale $\gamma' = \sqrt{16/5}$. À isométrie près, il a un seul contigu dont les caractéristiques sont : $\mu = 10$, $s = 45$, $s^* = 16$, $\gamma = 1.779$, $\gamma' = 1.755$. Son groupe d'automorphismes est d'indice 45 dans celui de $\mathbb{A}_9^5$. Le chemin les joignant ne peut pas être stable par dualité puisqu'il ne contient pas le réseau $\mathbb{A}_9^2$ (en effet, ce réseau est parfait et il serait donc le contigu de $\mathbb{A}_9^5$). D'aillleurs, tout le long du chemin ouvert, on a $s = 44 \neq s^* = 16$, ce qui exclut la présence d'un réseau isodual. Par contre, nous allons trouver un chemin joignant $\mathbb{A}_9^5$ à un autre réseau parfait, et tout le long duquel on aura $s = s^* = 40$. Pour cela, on utilise l'algorithme de Voronoï pour le groupe cyclique d'ordre 10 et sa représentation d'augmentation de degré 9.

Soit

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \in \mathrm{GL}_9(\mathbb{Z})$$

l'image d'un générateur par la représentation d'augmentation, et soit

$$T = \left\{ \begin{pmatrix} a & b & c & d & e & x & e & d & c \\ b & a & b & c & d & e & x & e & d \\ c & b & a & b & c & d & e & x & e \\ d & c & b & a & b & c & d & e & x \\ e & d & c & b & a & b & c & d & e \\ x & e & d & c & b & a & b & c & d \\ e & x & e & d & c & b & a & b & c \\ d & e & x & e & d & c & b & a & b \\ c & d & e & x & e & d & c & b & a \end{pmatrix} \right\}, \text{ avec } x = -a - 2b - 2c - 2d - 2e$$

l'espace des matrices invariantes par P.

Si l'on applique l'algorithme de G-Voronoï à partir d'une représentation dans $\mathcal{T}$ du réseau $\mathbb{A}_9^5$ ($a = 8$, $b = d = e = x = -2$, $c = 3$) on obtient, à équivalence près, trois contiguës parfaites. Les deux premières, à 70 couples de vecteurs minimaux, sont peu intéressantes du point de vue de la dualité et sont comparables à celles que nous avons construites dans les deux premiers chapitres. Le long du troisième chemin de contiguïté,

$$A_t = \begin{pmatrix} 8 & -2t-2 & -t+3 & 2t-2 & -2t-2 & 6t-2 & -2t-2 & 2t-2 & -t+3 \\ -2t-2 & 8 & -2t-2 & -t+3 & 2t-2 & -2t-2 & 6t-2 & -2t-2 & 2t-2 \\ -t+3 & -2t-2 & 8 & -2t-2 & -t+3 & 2t-2 & -2t-2 & 6t-2 & -2t-2 \\ 2t-2 & -t+3 & -2t-2 & 8 & -2t-2 & -t+3 & 2t-2 & -2t-2 & 6t-2 \\ -2t-2 & 2t-2 & -t+3 & -2t-2 & 8 & -2t-2 & -t+3 & 2t-2 & -2t-2 \\ 6t-2 & -2t-2 & 2t-2 & -t+3 & -2t-2 & 8 & -2t-2 & -t+3 & 2t-2 \\ -2t-2 & 6t-2 & -2t-2 & 2t-2 & -t+3 & -2t-2 & 8 & -2t-2 & -t+3 \\ 2t-2 & -2t-2 & 6t-2 & -2t-2 & 2t-2 & -t+3 & -2t-2 & 8 & -2t-2 \\ -t+3 & 2t-2 & -2t-2 & 6t-2 & -2t-2 & 2t-2 & -t+3 & -2t-2 & 8 \end{pmatrix} \qquad t \in]0, 1/3[,$$

on a constamment $s = s^* = 40$ (plus précisément, l'ensemble S des vecteurs minimaux des matrices est constant, et constitué de 4 orbites à 10 éléments, de même que l'ensemble S^* des vecteurs minimaux des matrices inverses). Malgré les apparences, ce chemin n'est pas stable par dualité. De plus, nous avons montré qu'il n'existe pas, sur ce chemin, de matrice ayant la même densité que son inverse.

Cependant, la contiguë G-parfaite $A_{1/3}$ correspondante présente une propriété remarquable : *elle réalise, ex-aequo avec l'autre extrémité du chemin, la meilleure valeur connue à ce jour ($\sqrt{16/5}$) de l'invariant d'Hermite dual.* Plus généralement, on a, le long du chemin,

$$\gamma'^2 = 16/5 \frac{t+5}{(t+1)(5-3t)},$$

qui atteint son minimum (3.149) pour $t = \frac{-15+4\sqrt{15}}{3}$, valeur supérieure à toutes celles que nous avons trouvées dans les chapitres précédents. La contiguë $A_{1/3}$, de norme 6, est extrême au sens classique ; son inverse qui a 40 vecteurs minimaux ne peut pas être parfaite ($n(n+1)/2 = 45$), elle n'est pas non plus eutactique. Nous avons toutefois vérifié la dual-eutaxie de ce couple de matrices (comme nous l'avons fait pour K'_9 au premier chapitre), qui sont donc dual-extrêmes. Les G-contiguës de la matrice $A_{1/3}$ sont presque parfaites (le défaut de perfection est égal à 1, bien qu'elles aient 45 couples de vecteurs minimaux) et ne présentent pas d'intérêt particulier vis-à-vis de la dualité.

III.5.2. Réseaux isoduaux.

Conway et Sloane, (dans [C-S1]), contruisent par "collage" un réseau isodual de rang 9. Il possède 21 couples de vecteurs minimaux et sa constante d'Hermite leur semble faible ("probably weak") ($\gamma = 1.7142$). Nous construisons, dans ce paragraphe, une matrice de Gram de ce réseau, et nous montrons qu'il ne réalise pas un maximum local de la constante d'Hermite parmi les isoduaux orthogonaux. Enfin, à partir de ce réseau isodual, on construit un chemin (non linéaire) de réseaux isoduaux tous plus denses.

Conway et Sloane considèrent le sous-réseau du réseau cubique de rang 4 orthogonal au vecteur $(2, 1, 1, \ldots, 1)$, dont une matrice de Gram est :

$$\begin{pmatrix} 5 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix}.$$

En "collant" ce réseau au réseau $\mathbb{A}_6$, on obtient un réseau isodual ayant la matrice de Gram suivante :

$$A = \begin{pmatrix} 12 & -5 & -5 & -5 & 4 & 5 & -5 & -5 & -5 \\ -5 & 12 & -2 & -2 & -4 & 2 & -2 & -2 & -2 \\ -5 & -2 & 12 & 5 & 3 & -5 & 5 & 5 & 5 \\ -5 & -2 & 5 & 12 & -4 & -5 & 5 & 5 & 5 \\ 4 & -4 & 3 & -4 & 13 & -3 & -4 & -4 & -4 \\ 5 & 2 & -5 & -5 & -3 & 12 & -5 & -5 & -5 \\ -5 & -2 & 5 & 5 & -4 & -5 & 12 & 5 & 5 \\ -5 & -2 & 5 & 5 & -4 & -5 & 5 & 12 & 5 \\ -5 & -2 & 5 & 5 & -4 & -5 & 5 & 5 & 12 \end{pmatrix}.$$

Soit une base β de E telle que $Gram(\beta) = A$, alors on vérifie facilement la σ-isodualité où σ a pour matrice dans les bases (β, β^*) :

$$P = \begin{pmatrix} 0 & 1 & -1 & -1 & 0 & 1 & -1 & -1 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & -1 & 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & 0 & 0 & -1 & 1 & 1 & 1 \\ 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 1 & 1 & -1 & -1 & -1 \\ -1 & 0 & 1 & 1 & 0 & -1 & 0 & 1 & 1 \\ -1 & 0 & 1 & 1 & 0 & -1 & 1 & 0 & 1 \\ -1 & 0 & 1 & 1 & 0 & -1 & 1 & 1 & 0 \end{pmatrix}.$$

Dans [B-M1], on paramètre les réseaux σ-isoduaux grâce au sous-espace

$$\mathcal{A} = \{v \in \mathrm{End}_s(E),\ v \circ \sigma = -\sigma \circ v\}$$

des endomorphismes anticommutant avec σ.

La composante connexe des réseaux σ-isoduaux correspondant à $CS9$ est alors :

$$\{e^v(CS9),\ v \in \mathcal{A}\}.$$

Ici, les valeurs propres $+1$ et -1 de σ étant d'ordres respectifs 4 et 5, l'espace $\mathcal{A}$ est de dimension 20 ; sa traduction matricielle est :

$$\mathcal{T} = \{M,\ {}^tPA^{-1}MA^{-1}P = -M\}.$$

La matrice A n'est pas $\mathcal{T}$-parfaite (son défaut de perfection est de 4). Ceci confirme la remarque dans Conway et Sloane ("probably weak") : le réseau $CS9$ n'est donc pas extrême parmi les réseaux σ-isoduaux.

Il est alors aisé de construire des familles (non linéaires) de réseaux σ-isoduaux à un paramètre dans lesquelles $CS9$ soit le moins dense : par exemple, prenons v tel que :

$$v^3 = v \ ;$$

on obtient alors :

$$\begin{cases} e^{tv} = Id + sh(t).v + (ch(t) - 1).v^2 \\ \quad\;\; = Id + \dfrac{\theta^2 - 1}{2\theta} v + \dfrac{\theta^2 - 2\theta + 1}{2\theta} v^2 \end{cases} \qquad \text{avec } \theta = e^t.$$

Si, de plus, on choisit v tel que

$$\begin{cases} Tr(v \circ p_x) = 0 \ \forall x \in S(CS9) \\ Tr(v^2 \circ p_x) > 0 \ \forall x \in S(CS9) \end{cases},$$

alors la constante d'Hermite du réseau isodual $e^{tv}(CS9)$ croît lorsque t est assez petit.

Prenons par exemple v ayant pour matrice V dans les bases (β, β^*) :

$$V = \frac{1}{16} \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & -10 & 0 & 0 & 3 & 3 & 3 \\ 0 & 0 & -10 & 0 & 0 & -10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & -10 & 0 & 0 & 3 & 3 & 3 \\ 0 & 0 & 3 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 3 & 0 & 0 & 0 \end{pmatrix}.$$

On obtient alors un chemin de réseaux isoduaux L_t ($t \in [-t_0, t_0]$) tous plus denses que le réseau $CS9$. Le réseau le plus dense, L_{t_0}, est défini sur $\mathbb{Q}(\sqrt{3823})$, ne possède que 4 couples de vecteurs minimaux mais sa constante d'Hermite vaut $1.714343 > \gamma(CS9) = 1.714285$.

Annexe I

Les 52 réseaux de rang 8 sans section hyperplane parfaite de même norme.

det	N	(s, s^*)	$\|Aut\|$	γ	γ'^2	Extr
20412	6	(51,2)	144	1.735	2.142	E
20736	6	(51,3)	96	1.732	2.250	E
223488	8	(42,1)	8	1.715	2.597	E
228352	8	(41,2)	48	1.711	2.345	E
232128	8	(41,2)	32	1.707	2.511	E
5996025	12	(39,2)	32	1.705	2.617	E
6200388	12	(37,1)	8	1.698	2.556	N-E
243984	8	(41,2)	4	1.696	2.433	N-E
244608	8	(39,1)	4	1.696	2.467	N-E
250752	8	(40,1)	4	1.691	2.367	N-E
250896	8	(39,1)	4	1.691	2.323	E
251136	8	(40,2)	8	1.690	2.363	N-E
251472	8	(39,4)	16	1.690	2.620	N-E
254736	8	(40,1)	2	1.687	2.330	N-E
255248	8	(39,2)	2	1.687	2.438	E
255744	8	(38,1)	8	1.687	2.270	N-E
257472	8	(39,1)	4	1.685	2.305	N-E
257808	8	(39,1)	2	1.685	2.302	N-E
6648480	12	(37,4)	16	1.684	2.451	N-E
259776	8	(39,2)	8	1.683	2.484	N-E
259776	8	(39,2)	4	1.683	2.444	E
262848	8	(39,1)	4	1.681	2.258	N-E
262976	8	(38,2)	4	1.681	2.536	E
6810804	12	(36,1)	4	1.678	2.562	N-E
267792	8	(38,1)	2	1.677	2.435	N-E
267648	8	(38,1)	2	1.677	2.257	N-E

det	N	(s, s^*)	$\|Aut\|$	γ	γ'^2	Extr
267024	8	(38,1)	2	1.677	2.395	E
267648	8	(38,2)	4	1.677	2.462	E
6858432	12	(37,3)	12	1.677	2.507	E
269136	8	(37,1)	2	1.676	2.443	N-E
268944	8	(37,1)	2	1.676	2.467	E
269568	8	(38,1)	2	1.675	2.241	N-E
271872	8	(39,1)	32	1.674	2.135	E
272016	8	(38,1)	2	1.674	2.182	N-E
272784	8	(38,1)	2	1.673	2.176	N-E
272384	8	(37,2)	4	1.673	2.486	N-E
273408	8	(37,1)	2	1.672	2.209	N-E
275520	8	(37,1)	2	1.671	2.342	N-E
275520	8	(37,1)	2	1.671	2.342	N-E
275968	8	(37,2)	4	1.671	2.500	N-E
275072	8	(36,1)	4	1.671	2.462	N-E
276240	8	(38,1)	2	1.670	2.149	N-E
281424	8	(37,2)	4	1.666	2.293	N-E
281424	8	(37,1)	2	1.666	2.293	N-E
282384	8	(36,1)	2	1.666	2.351	N-E
284480	8	(36,1)	4	1.664	2.381	N-E
285696	8	(36,1)	2	1.663	2.328	N-E
289536	8	(36,2)	2	1.661	2.316	N-E
289296	8	(36,1)	2	1.661	2.231	N-E
289296	8	(36,1)	2	1.661	2.231	N-E
299136	8	(36,1)	2	1.654	2.157	N-E
299136	8	(36,1)	2	1.654	2.157	N-E

Annexe II

Échantillon des temps de calculs.

Nbre vm	Nbre auto	Nbre face	CPU
12	2	10	1
12	4	5	0.9
12	8	3	0.6
12	96	1	0.6
13	1	24	6
13	2	12	3
14	1	32	48
14	2	13	6
15	1	15	33
15	2	19	91
16	1	17	76
16	2	19	262
16	4	15	127
16	32	9	33
17	1	16	461
17	2	13	168
17	4	16	188

Nbre vm	Nbre auto	Nbre face	CPU
17	8	10	128
18	1	17	397
18	2	27	1072
19	1	43	3378
19	2	46	790
21	1	140	6610
21	2	16	4090
22	1	44	23986
22	2	64	17043
22	4	24	2603
24	1	145	115674
24	4	24	8717
24	8	46	23182
25	1	86	171616
27	1	64	176753
27	2	65	262879
30	4	70	290500
30	2	217	1134635

BIBLIOGRAPHIE

[Bar1] E.S. Barnes, *The perfect and extreme senary forms*, Canad. J. Math. **9** (1957), 235–242.

[Bar2] E.S. Barnes, *The complete enumeration of extreme senary forms*, Phil. Trans. Roy. Soc. London (A), 249 (1957), 461–506.

[Bar3] E.S. Barnes, *The construction of perfect and extreme forms I, II*, Acta Arith. **5** (1959), 57–79, 461–506.

[Ber] A-M. Bergé, *Minimal vectors of pairs of dual lattices*, J. Number Theory **52** (1995), 284–298.

[B-M1] A-M. Bergé, J. Martinet, *Densité dans des familles de réseaux. Application aux réseaux isoduaux*, à paraître.

[B-M2] A-M. Bergé, J. Martinet, *Sur un problème de dualité lié aux sphères en géométrie des nombres*, J. Number Theory **32** (1989), 14–42.

[B-M-S] A-M. Bergé, J. Martinet, F. Sigrist, *Une généralisation de l'algorithme de Voronoï*, Astérisque **209** (1992), 137–158.

[Cas] J.W.S. Cassels, *An introduction to the geometry of numbers*, Springer-Verlag, **99** (1959).

[C-S] J.H. Conway, N.J.A. Sloane, *Low-dimensional lattices. III. Perfect forms*, Proc. Royal Soc. London **A 418** (1988), 43–80.

[C-S1] J.H. Conway, N.J.A. Sloane, *On lattices equivalent to their duals.*, J. Number Theory (1994), 373–382.

[C-S2] J.H. Conway, N.J.A. Sloane, *Sphere packings, Lattices and groups*, Springer-Verlag (1988).

[Cox] H.S.M. Coxeter, *Extreme forms*, Canad. J. Math. **3** (1951), 391–441.

[Jaq] D-O. Jaquet, *Thèse, Neuchâtel*, 1991.

[Ja1] D-O. Jaquet, *Description des voisines de E_7, D_7, D_8, D_9*, Sém. Th. Nombres de Bordeaux **4** (1992), 275–283.

[Ja2] D-O. Jaquet, *Domaines de Voronoï et algorithme de réduction des formes quadratiques définies positives*, Sém. Th. Nombres de Bordeaux **2** (1990), 163–215.

[Ja-S] D-O. Jaquet, F. Sigrist, *Formes quadratiques contiguës à D_7*, C. R. Acad. Sc. Paris **A 309** (1989), 641–644.

[K-Z] A. Korkine, G. Zolotareff, *Sur les formes quadratiques positives*, Math. Ann. **11** (1877), 242–292.

[Lai] M. Laïhem, *Thèse, Bordeaux*, 1992.

[Mar] J. Martinet, *Les réseaux parfaits des espaces euclidiens (à paraître).*

[Q-Z] J. R. Quine, P. L. Zhang, Preprint (1995).

[Vor] G. Voronoï, *Nouvelles applications des paramètres continus à la théorie des formes quadratiques : 1 Sur quelques propriétés des formes quadratiques positives parfaites*, J. reine angew. Math **133** (1908), 97–178.

[Wat] G.L. Watson, *The number of minimum points of a positive quadratic form having no perfect binary section with the same minimum*, Proc. London Math. Soc. **24** (1972), 625–646.

RESUME

Dans l'étude de la densité des empilements réguliers de sphères se dégage la notion de réseaux extrêmes, pour lesquels Voronoï a donné un algorithme d'énumération, inabordable dans la pratique au-delà de la dimension 7. Dans les premiers chapitres de cette thèse, nous élaborons de nouveaux algorithmes sélectionnant les réseaux extrêmes (laminés, en patchwork,...), et qui, implantés sur ordinateur, fournissent de nombreux réseaux nouveaux en dimensions moyennes. Au dernier chapitre, où l'on privilégie les questions de dualité, les algorithmes précédents sont plutôt utilisés à l'exploration de l'espace des réseaux. On y découvre par exemple un réseau qui réalise, en moyenne avec son dual, la densité record pour la dimension 9. On construit également en dimension 9 un chemin de réseaux isoduaux sur lequel il existe un réseau qui réalise actuellement le record de densité parmi les isoduaux.

ABSTRACT

In study of the density of regular sphere packing raises the notion of extremal lattices, for which Voronoï gave an enumeration algorithm unatainable in practice for dimension over 7. In the first chapters of this thesis, we create new algorithms selecting the extremal lattices (laminated, in patchwork, ...), and that, once settled on the computer, provide numerous new lattices in averages dimensions. In the last chapter, where question of duality are considered, we previous algorithms are rather used to explorate the space of lattices. We discover for exemple a lattice which achieve, in average with its dual, the record density for the dimension 9. As well, we build a path of isodual lattices on wich there exist a lattice achieving the record of density among the isodual.

MOTS-CLEFS

Réseaux, perfection, extrêmalité, algorithme de Voronoï, isodualité, réseau symplectique, réseau en patchwork, lamination.

Mémoire présenté par

Jean-Luc Baril

en vue de l'obtention de l'habilitation à diriger des recherches à

l'Université de Bourgogne, Dijon

Spécialité : Combinatoire

Génération exhaustive et étude de structures de classes combinatoires

Soutenu le 15 juin 2010 devant la commission d'examen composée de

Rapporteurs : Elena Barcucci (Florence, Italie)
Toufik Mansour (Haïfa, Israël)
Maurice Margenstern (Metz, France)

Examinateurs : Abderrafia Koukam (Belfort, France)
Jean-Marcel Pallo (Dijon, France)
Renzo Pinzani (Florence, Italie)
Vincent Vajnovszki (Dijon, France)

Laboratoire d'Électronique, Informatique et Image de Dijon - LE2I, UMR CNRS 5158
Faculté des Sciences et Techniques - B.P. 47870 - 21078 Dijon Cedex

Mis en page avec la classe Phdlasl

Remerciements

Une aide très importante m'a été apportée par mes trois collègues combinatoriciens Jean Pallo, Vincent Vajnovszki et Olivier Togni ; le fait que nous travaillons ensemble depuis plusieurs années sur des questions d'algorithmique combinatoire m'a conduit à leur faire de nombreux emprunts (parfois inconsciemment). Je les remercie tous très fortement.

Je remercie également les autres membres du laboratoire LE2I de l'université de Bourgogne (Dijon) pour leur soutien et plus particulièrement ceux avec qui j'ai collaboré pour mes tâches et responsabilités d'enseignements.

Je remercie très vivement les rapporteurs, Elena Barcucci, Toufik Mansour et Maurice Margenstern pour le temps qu'ils ont passé à la lecture et l'analyse de ce mémoire. Tous ont accepté spontanément cette tâche ; je leur en suis très reconnaissant. Je remercie également Abder Koukam et Renzo Pinzani d'avoir accepté de participer au jury.

Je remercie le Professeur Jacques Martinet de l'université de Bordeaux et j'ai également une pensée particulière pour Anne-Marie Bergé.

Je tiens à remercier très affectueusement Maïté et Eloïse.

Table des matières

Introduction générale **3**

Notations **5**

I Génération exhaustive de classes d'objets combinatoires **7**

I.1 Introduction . . . 7

I.2 Génération CAT . . . 8

I.2.1 Permutations avec un nombre donné d'excédences . . . 8

I.2.2 Permutations et mots de Fibonacci ou de Lucas . . . 11

I.2.2.1 Permutations et mots de Fibonacci généralisés . . . 11

I.2.2.2 Permutations de Lucas généralisées . . . 13

I.2.2.3 Algorithme et complexité . . . 14

I.3 Génération en code de Gray . . . 15

I.3.1 Mots de Lucas p-généralisés . . . 15

I.3.2 Dérangements . . . 18

I.3.3 Permutations ayant un nombre donné de cycles . . . 20

I.3.4 Permutations ayant un nombre donné de minima gauche-droite . . . 23

I.3.5 Permutations évitant un ensemble de motifs . . . 27

I.4 Conclusion - perspectives . . . 30

II Etude de structures énumérées par les nombres de Catalan **33**

II.1 Introduction . . . 33

II.2 Treillis de Catalan . . . 34

II.2.1 Le treillis 'Phagocyte' . . . 34

II.2.2 La transformation de taille-greffe . . . 38

II.3 Distance de rotation . . . 44

II.3.1 Sommet de type (2 :0) . . . 47

II.3.2 Sommet de type (1 :1) . . . 47

II.3.3 Sommet de type (3 :0) . . . 47

II.3.4 Sommet de type (2 :1) . . . 48

II.3.5 Algorithme pour le calcul de bornes inférieure et supérieure . . . 48

II.3.6 Résultats expérimentaux . . . 49

II.4 Conclusion . . . 50

III Motifs dans les permutations **51**
III.1 Introduction 51
III.2 Duplication en miroir du génome 51
III.2.1 Permutations à motifs exclus et le modèle de WM-duplication 54
III.2.2 Considérations algorithmiques 56
III.2.2.1 Un chemin de $12\ldots n$ vers $\sigma \in S_n$ obtenu en reculant à partir de σ 56
III.2.2.2 Un chemin de WM-duplications de $12\ldots n$ vers $\sigma \in S_n$ . . 57
III.2.3 Autres modèles de duplication 58
III.2.3.1 Une WM-duplication suivie par plusieurs W-duplications . 58
III.2.3.2 Une W- ou WM-duplication suivie par plusieurs W-duplications 58
III.2.4 Future direction de recherche 59
III.3 Cloture cyclique de $S(k(k-1)\ldots 21)$ 59
III.3.1 La cloture cylindrique 64
III.3.2 Discussion et Conclusion 65

IV Perspectives **67**

Annexes **69**

Liste des publications **75**

Bibliographie **81**

Introduction générale

Ce mémoire a pour objectif de fournir une synthèse d'une partie de mes activités de recherche réalisées post-thèse.
Commençons par retracer le parcours effectué depuis l'obtention de mon doctorat. En 1996, j'ai donc soutenu une thèse dans le domaine de la théorie des nombres à l'université de Bordeaux. J'ai obtenu ensuite (1998) un poste de professeur certifié (PRCE) à l'université de Dijon au sein de l'équipe d'informatique. Obtenant l'agrégation de mathématiques en 1999, j'interviens alors en qualité de professeur agrégé (PRAG). Une fois la charge d'enseignements digérée (384H/an), ma curiosité me guide naturellement vers l'équipe d'algorithmique combinatoire de Dijon. C'est alors que j'amorçe un parcours initiatique pour découvrir les bases de la combinatoire. Il est toujours difficile de percer les secrets d'un nouveau domaine de recherche, mais l'aide des membres de l'équipe de combinatoire a été sans aucun doute un facteur essentiel dans mon apprentissage. Je suis alors recruté Maître de Conférences au laboratoire LE2I de Dijon en 2004. C'est à ce moment que je decouvre la forte intéraction entre l'informatique théorique et les mathématiques.

En effet, l'informatique théorique apparaît essentiellement sous la forme de deux courants : l'algorithmique, et la logique. L'algorithmique provient de l'automatisation de la notion de calcul, alors que la logique formalise la notion de démonstration mathématique.

Dans l'antiquité, ces deux notions apparaissaient déjà. On peut citer par exemple, Archimède et Diophante qui obtiennent une règle de calcul de l'aire située sous une parabole ; Euclide pour la notion de système axiomatique ; et Aristote pour la logique propositionnelle. Plus tard, Newton, Leibniz, Euler ou Gauss, proposent des méthodes de calcul numérique permettant d'automatiser certains problèmes issus des mathématiques. Puis la théorie de la calculabilité est développée par Turing (1936) [89], von Neumann, Kleene, Church...

Durant ces dernières années (et sans auncune comparaison avec les noms cités précédemment), mes travaux se sont orientés dans les domaines de l'algorithmique dédiée à des problèmes issus de la combinatoire, la théorie des graphes et l'étude de classes combinatoires ayant une structure de treillis. La référence actuelle du domaine de l'algorithmique combinatoire est la collection de volumes *The art of Computer Programming* de Donald E. Knuth [98]. Knuth étudie de nombreux algorithmes ainsi que leur comportement en terme du nombre d'opérations élémentaires. Il dédie ainsi un volume entier à la génération de classes d'objets combinatoires. On peut aussi citer les travaux de Flajolet, Johnson, Korsh, Ruskey, Sedgwick, Trotter, Vajnovszki, Walsh,...[77, 91, 166, 149, 155, 100, 174]. Pour les références concernant les structures de treillis, on peut aussi considérer Davey et Grätzer [83, 60]. Tous mes travaux concernant les structures en treillis sont isssus de classes combinatoires énumérées par les nombres de Catalan A000108, [162]. Enfin, mes travaux concernant la théorie des graphes abordent essentiellement les colorations de graphes. No-

tons toutefois qu'il existe évidemment des liens très forts entre ces trois domaines. On peut citer par exemple le lien entre les cycles Hamiltoniens en théorie des graphes et les codes de Gray en algorithmique combinatoire.

Je n'exposerai pas dans ce mémoire mes travaux concernant les colorations de graphes dans le but d'obtenir un mémoire axé sur la combinatoire issue de structures liées aux permutations, mots binaires et arbres binaires. Si toutefois le lecteur souhaite obtenir mes travaux en théorie des graphes, je l'invite à consulter les articles [27, 28, 29, 30].

Le mémoire est organisé de la façon suivante.

Dans le premier chapitre, je présente une grande partie (non exhaustive) de mes travaux de recherche concernant la génération exhaustive. Ce chapitre contient une partie importante de mes résultats obtenus ces dernières années. Je propose ainsi plusieurs nouveaux algorithmes de génération dont la complexité est CAT (Constant Amortized Time), *i.e. le nombre d'opérations est proportionnel au nombre d'objets générés et ceci indépendamment de la taille des objets.* On étudie également plusieurs classes dont on a obtenu la génération en code de Gray.

Dans le second chapitre, j'expose les travaux concernant l'étude de structures comptées par les nombres de Catalan. En particulier, je donne deux nouvelles structures en treillis pour les parenthèsages bien formés ce qui induit deux nouvelles structures sur les arbres binaires. On s'intéressse aussi dans cette partie à la construction de distance sur ces treillis, et notamment on donne un algorithme polynomial d'approximation pour le calcul de la distance de rotation entre deux parenthésages dans le treillis de Tamari. Notons que l'on ne sait toujours pas aujourd'hui s'il existe un algorithme calculant en temps polynomial la distance de rotation entre deux parenthésages dans ce treillis.

Dans le dernier chapitre, j'expose deux contributions récentes dans le domaine de l'étude des motifs dans les permutations. La première est issue d'un problème de bioinformatique puisqu'elle consiste à caractériser les suites du génome obtenues après un nombre donné de duplications en miroir. La seconde repond à une question posée par Atkinson : est-ce que la cloture cyclique de $S(k(k-1)\dots 321)$ possède une base finie ?

Enfin, une conclusion donnera une approche globale de mes travaux de recherches en précisant les objectifs souhaités ainsi que les futures directions de recherches envisagées en proposant des questions encore à explorer.

Notations et définitions

Je présente dans cette partie, les principales notations et définitions utilisées dans ce mémoire.

On note $[n]$ l'ensemble $\{1, 2, \ldots, n\}$. L'ensemble des permutations sur l'ensemble $[n]$ sera noté S_n. L'entier n est appelé la taille de la permutation. On représentera les permutations en notation linéaire, *i.e.,* $\sigma \in S_n$ sera notée $\sigma_1\sigma_2\ldots\sigma_n$ si $\sigma(i) = \sigma_i$ pour tout i, $1 \leqslant i \leqslant n$. La permutation $\sigma \in S_n$ sera appelée un cycle lorsqu'il existe une suite d'indices $i_1, i_2, \ldots, i_n$ différents deux à deux dans $[n]$ tels que $\sigma(i_j) = i_{j+1}$ pour $1 \leqslant j \leqslant n-1$ et $\sigma(i_n) = i_1$. Une telle permutation pourra aussi être notée $\langle i_1, i_2, \ldots, i_n\rangle$. Toute permutation $\sigma \in S_n$ peut être uniquement décomposée en un produit de cycles à supports disjoints. On dira que σ a k cycles lorsque cette décomposition possède k cycles. L'ensemble des permutations de taille n ayant k cycles sera noté $S_{n,k}$.

Une valeur σ_i est un minimum gauche-droite (left-to-right minimum en anglais) si tous les éléments à sa gauche dans σ sont plus grands que σ_i. Etant donné que l'ensemble des permutations de taille n ayant k cycles est en bijection avec l'ensemble des permutations de taille n ayant k minima gauche-droite, nous utiliserons toujours la notation $S_{n,k}$ pour ce dernier ensemble.

Un dérangement σ est une permutation sans point fixe, c'est à dire qu'il n'existe pas d'indice i tel que $\sigma(i) = i$. L'ensemble des dérangements de taille n sera noté D_n. Dans le second chapitre, D_n designera l'ensemble des mots de Dyck avec n parenthèses ouvrantes et n fermantes.

Une excédence de σ est une valeur σ_i telle que $\sigma_i > i$. L'ensemble des permutations de taille n ayant k excédences est noté $E_{n,k}$.

Une permutation $\sigma \in S_n$ contient le motif $\pi \in S_k$ si et seulement s'il existe une suite d'indices $1 \leqslant i_1 < i_2 < \ldots < i_k \leqslant n$ tels que $\sigma(i_1)\sigma(i_2)\ldots\sigma(i_k)$ est ordonné comme π. Par exemple, la permutation **4132** contient (en gras) le motif 321 mais ne contient pas (ou évite) le motif 231. Si T est un ensemble de motifs, on notera $S_n(T)$ l'ensemble des permutations dans S_n qui évitent tous les motifs appartenant à T. Un motif barré $\bar{\pi} \in S_k$ est une permutation ayant une barre sur certaines valeurs. Posons $1 \leqslant r \leqslant k-1$. Soit π la permutation sur $[k]$ obtenue en enlevant les barres de $\bar{\pi}$ et $\hat{\pi}$ la permutation obtenue de $\bar{\pi}$ en enlevant toutes les $(k-r)$ valeurs barrées et renormalisée comme une permutation de $[k-r]$. Alors $\sigma \in S_n$ ne contient pas le motif $\bar{\pi}$ si tout motif $\hat{\pi}$ dans σ peut être étendu en un motif π dans σ. Par exemple, si $\bar{\pi} = 4\bar{1}32$, alors $\sigma = 58132674 \in S_8(4\bar{1}32)$ (voir, [71, 113, 160]). Il est aussi possible d'introduire des ' :' entre les valeurs d'un motif (exemple 13 : 24). Si deux valeurs consécutives du motif π sont séparées par un ' :' alors la permutation σ contient le motif π si les entrées correspondantes à ces éléments sont aussi adjacentes dans σ (et bien sur dans le même ordre que dans π). Par exemple, $\sigma = 13524$

ne contient pas le motif $\pi = 13 : 24$ mais contient le motif 1324. Notons qu'il s'agit d'un cas particulier de la définition des motifs définis dans [76].

Une liste $\mathcal{L}$ est un ensemble d'éléments totalement ordonnés. La liste $\overline{\mathcal{L}}$ est la liste $\mathcal{L}$ considérée (ou lue) du dernier élément au premier. Les éléments $prem(\mathcal{L})$ et $der(\mathcal{L})$ désignent respectivement les premiers et derniers éléments de la liste. Si $\mathcal{L}$ et $\mathcal{L}'$ sont deux listes alors on définit la liste $\mathcal{L} \circ \mathcal{L}'$ comme étant la concaténation des deux listes $\mathcal{L}$ et $\mathcal{L}'$ telle que $prem(\mathcal{L}) = prem(\mathcal{L} \circ \mathcal{L}')$.

Soit $(L_n)_{n \geqslant 1}$ une famille de listes L_n d'éléments de taille n. Si pour tout $n \geqslant 1$ deux objets consécutifs de la liste L_n diffèrent sur un nombre de positions indépendant de n, on dit que la liste est en ordre de *code de Gray*.

Chapitre I

Génération exhaustive de classes d'objets combinatoires

I.1 Introduction

Dans ce chapitre, on s'intéresse à des algorithmes permettant de générer exhaustivement (et sans répétition) les objets d'une classe combinatoire. L'intérêt de tels algorithmes est multiple : l'obtention d'une liste complète d'objets (en un temps acceptable) peut être utile pour la vérification où la découverte de propriétés ou conjectures, pour la résolution pratique de certains problèmes NP-complets,... La complexité des algorithmes est donc cruciale. Elle peut être constante en moyenne ou dans le pire des cas. Une complexité constante en moyenne signifie que le nombre d'opérations effectuées est proportionnel au nombre d'objets à générer (CAT - constant amortized time) et ceci indépendamment de la taille des objets ; une complexité constante dans le pire des cas signifie que le passage d'un objet à son successeur se fait toujours en un temps constant indépendant de la taille des objets (loopless). Lorsque deux objets consécutifs diffèrent sur un nombre de positions indépendant de la taille, on dit que la liste est en ordre de code de Gray. Un algorithme loopless liste nécessairement les objets en ordre de code de Gray ; la réciproque n'est pas forcément vraie.

Il existe de nombreux travaux concernant la génération exhaustive efficace de classes combinatoires. Par exemple, il existe des algorithmes efficaces pour la génération des permutations [91, 166, 155, 148], involutions [174], up-down permutations [100, 152], permutations avec un nombre donné d'inversions [151]... Il est toujours intéressant de relire le survol de Carla Savage [154] pour les travaux réalisés avant 1997 concernant les codes de Gray. On trouvera dans ce mémoire de nombreuses références plus récentes.

Plusieurs méthodes différentes ont été utilisées pour l'obtention de tels algorithmes. On peut bien sur citer la méthode de Johnson-Trotter [91, 166] pour la génération des permutations en ordre de code de Gray. Il y a également des méthodes ad-hoc liées à la structure particulière des objets à générer, mais principalement, deux méthodes sont très souvent utilisées : la méthode récursive et la méthode ECO.

La méthode récursive consiste à définir la liste des objets de taille n en fonction des listes

des objets de tailles inférieures à n. Par exemple, la classe B_n des mots binaires de taille n peut s'écrire récursivement sous la forme $B_n = B_{n-1}.\mathbf{0} \bigcirc \overline{B_{n-1}}.\mathbf{1}$, avec $B_1 = \{0,1\}$, où $\bigcirc$ est l'opérateur de concaténation des listes, et $\overline{B_{n-1}}$ est la liste B_{n-1} considérée dans l'ordre inverse (de la fin au début). On obtient ainsi, $B_2 = \{0\mathbf{0}, 1\mathbf{0}, 1\mathbf{1}, 0\mathbf{1}\}$ et $B_3 = \{00\mathbf{0}, 10\mathbf{0}, 11\mathbf{0}, 01\mathbf{0}, 01\mathbf{1}, 11\mathbf{1}, 10\mathbf{1}, 00\mathbf{1}\}$. Voir par exemple [84]. Dans cet exemple, la liste fournie est aussi en ordre de code de Gray.

La méthode ECO (Enumeration of Combinatorial Objects), introduite par Pinzani et al. [15] pour l'énumération de classes combinatoires, consiste à donner des règles de succession pour étendre un objet de taille n à plusieurs objets de tailles supérieures. Plus formellement, on part de (b), $b \in \mathbb{N}^+$, et on fournit des règles de succession Ω :

$$\{(k) \rightsquigarrow (e_1(k))(e_2(k)) \dots (e_k(k)), k \in \mathbb{N}\},$$

où $e_i : \mathbb{N}^+ \longrightarrow \mathbb{N}^+$; il s'agit d'expliquer l'évolution des successeurs $(e_1(k))$, $(e_2(k))$,..., $(e_k(k))$ en fonction de (k), $k \in \mathbb{N}^+$. Les entiers positifs (b), (k), $(e_i(k))$ sont appelés *étiquettes*. Cela induit un arbre de génération où (b) est l'étiquette de la racine et chaque noeud étiqueté (k) possède k successeurs étiquetés $(e_1(k))$, $(e_2(k))$,..., $(e_k(k))$. Par conséquent, (Ω) induit une suite d'entiers positifs $(a_n)_{n \geqslant 0}$ ou a_n est le nombre de noeuds sur le niveau n dans l'arbre de génération. Par exemple, la règle de succession Ω définie par $(2) \rightsquigarrow (2)(2)$ initialisée par l'étiquette (2) induit la suite $a_n = 2^n$, *i.e.*, chaque niveau n de l'arbre de génération associé contient 2^n noeuds. Les nombres de Catalan peuvent être obtenus par cette méthode à l'aide de la règle de succession $(k) \rightsquigarrow (2)(3) \dots (k+1)$ initialisée par l'étiquette (2).

La première partie de ce chapitre traîte de la génération CAT alors que la seconde traîte de la génération en code de Gray.

I.2 Génération CAT

Dans cette section, nous donnons les différents résultats obtenus concernant la génération CAT (constant amortized time) de certaines classes d'objets combinatoires. Rappelons que la génération CAT consiste à générer une et une seule fois chaque élément d'une classe en un temps proportionnel au nombre d'objets mais indépendant de la taille des objets. [Dans ce paragraphe, nous ne considérerons pas les générations en code de Gray ; en effet, ces générations sont aussi CAT mais possèdent la propriété supplémentaire que deux objets consécutifs diffèrent d'un nombre de positions indépendant de la taille.]
De nombreux algorithmes de générations efficaces existent dans la littérature. Par exemple, on peut se réfèrer aux deux survols suivants [98, 155] pour la génération de plusieurs classes de permutations.

On présente ci-dessous deux algorithmes de génération CAT permettant de générer efficacement les permutations ayant un nombre donné d'excédences, les permutations et les mots de Fibonacci et de Lucas.

I.2.1 Permutations avec un nombre donné d'excédences

Ce paragraphe est issu de l'article [20]. Soit une permutation $\sigma \in S_n$. On représente $\sigma = \sigma_1\sigma_2 \dots \sigma_n$ en notation linéaire. La valeur σ_i est une excédence de σ si $\sigma(i) > i$.

Les permutations ayant un nombre donné d'excédences ont été très étudiées dans le domaine de la combinatoire énumérative [39, 72, 116, 140]. Foata et Schützenberger [78] donnent plusieurs propriétés fondamentales pour ces objets. De nombreuses applications à l'analyse des algorithmes sont obtenues par Knuth [98] et à la combinatoire des mots par Lothaire [108]. La statistique relative aux excédences sur les permutations est appelée statistique Eulerienne. Je donne dans ce paragraphe un algorithme efficace (CAT) de génération des permutations avec k excédences ($k \geqslant 0$). On note $E_{n,k}$ l'ensemble des permutations de taille n ayant k excédences. On procède par la méthode récursive en remarquant que $E_{n,k}$ peut être construit à partir de $E_{n-1,k}$ et $E_{n-1,k-1}$.

Soit $\gamma \in E_{n-1,k}$ une permutation de taille $(n-1)$ ayant k excédences, $n \geqslant 2$, $0 \leqslant k \leqslant n-2$; soit i un entier, $1 \leqslant i \leqslant n-1$. Si on note par σ la permutation dans S_n obtenue à partir de γ en remplaçant $\gamma(i)$ par n et en ajoutant $\gamma(i)$ sur la droite de γ, alors on a :

(a) si $\gamma(i)$ est une excédence dans γ, alors $\sigma \in E_{n,k}$;

(b) sinon, $\sigma \in E_{n,k+1}$.

De plus, (c) si σ est obtenue de γ en ajoutant n sur sa droite, alors $\sigma \in E_{n,k}$. Réciproquement, chaque permutation de $E_{n,k}$, $n \geqslant 2$, peut être uniquement obtenue par l'une de ces trois constructions (a), (b) et (c).

Ainsi, nous définissons deux fonctions ϕ_n et ψ_n comme suit :

Définition I.1. *Pour $0 \leqslant k \leqslant n-2$, un entier $i \in [n-1]$ et une permutation $\gamma \in E_{n-1,k}$, on définit une permutation de taille n : $\sigma = \phi_{n-1}(i,\gamma)$ par*

$$\sigma(j) = \begin{cases} n & \text{si } j = i \\ \gamma(i) & \text{si } j = n \\ \gamma(j) & \text{sinon.} \end{cases}$$

On définit aussi ψ_{n-1} de $E_{n-1,k}$ vers S_n qui transforme $\gamma \in E_{n-1,k}$ en une permutation $\sigma \in E_{n,k}$ obtenue de γ en ajoutant n sur sa droite.

Par exemple, si $\gamma = 3142 \in E_{4,2}$, on a $\phi_4(3,\gamma) = 31524 \in E_{5,2}$; $\phi_4(2,\gamma) = 35421 \in E_{5,3}$ et $\psi_4(\gamma) = 31425 \in E_{5,2}$. Plus généralement, on peut remarquer :

$$\phi_{n-1}(i,\gamma) = \psi_{n-1}(\gamma) \cdot \langle i, n\rangle = \langle \gamma(i), n\rangle \cdot \psi_{n-1}(\gamma).$$

Pour des raisons de clarté, on omettra l'indice n dans ϕ_n et ψ_n.

Lemme I.2. *Soient n, m, ℓ, k quatre entiers naturels tels que $1 \leqslant m < n$, $0 \leqslant \ell \leqslant m-1$, et $0 \leqslant k \leqslant n-1$. Soit aussi $\gamma \in E_{m,\ell}$.*

- *S'il existe i, $1 \leqslant i \leqslant m-1$, tel que $\gamma(i)$ est une excédence de γ, alors il y a une permutation $\sigma \in E_{n,k}$ telle que σ est obtenue de $\phi(i,\gamma)$ (respectivement $\psi(\gamma)$) en appliquant plusieurs fois les fonctions ϕ et ψ si et seulement si : (i) $\ell \leqslant k$ et $m+1-\ell \leqslant n-k$.*
- *Maintenant nous considérons i, $1 \leqslant i \leqslant m$, tel que $\gamma(i)$ n'est pas une excédence de γ, alors il y a une permutation $\sigma \in E_{n,k}$ telle que σ est obtenue de $\phi(i,\gamma)$ en appliquant plusieurs fois les fonctions ϕ et ψ si et seulement si : (ii) $\ell+1 \leqslant k$ et $m-\ell \leqslant n-k$.*
- *Si $\sigma \in E_{n,k}$ est obtenue de $\gamma \in E_{m,\ell}$ par la première construction et $\tau \in E_{n,k}$ par la seconde, alors σ et τ sont différentes.*

Maintenant nous expliquons les principales difficultés pour l'implémentation de la procédure $\mathbf{gen}(m, \ell)$ donnée en figure I.1. La procédure $\mathbf{gen}(1, 0)$ génère récursivement toutes les permutations $\sigma \in E_{n,k}$. En effet, supposons que lorsque l'on exécute l'appel récursif $\mathbf{gen}(m, \ell)$, la permutation courante est $\sigma \in E_{m,\ell}$ et ses excédences sont en positions $i_1, i_2, \ldots, i_\ell$. Le tableau t_1 contient ces ℓ positions $i_1, i_2, \ldots, i_\ell$. D'un autre coté les indices dans $[m] \backslash \{i_1, i_2, \ldots, i_\ell\}$ sont stockés dans t_2 de taille $r = m - \ell$. Grâce au lemme I.2, la procédure $\mathbf{gen}(m, \ell)$ génère les permutations de $E_{m+1,\ell}$ ou de $E_{m+1,\ell+1}$ en appliquant sur σ les fonctions ϕ et/ou ψ. Pour obtenir :

- $\psi(\sigma) \in E_{m+1,\ell}$; on ajoute l'indice $(m+1)$ sur la droite de t_2 ($t_2[r+1] = m+1$) et on exécute $\mathbf{gen}(m+1, \ell)$.
- $\phi(i_j, \sigma)$ avec $j \in [\ell]$; on ajoute $(m+1)$ sur la droite de t_2 ($t_2[r+1] = m+1$). On actualise $\sigma = \sigma \cdot \langle t_1[j], m+1 \rangle$ et on exécute $\mathbf{gen}(m+1, \ell)$.
- $\phi(i, \sigma)$ avec $i \notin \{i_1, i_2, \ldots, i_\ell\}$; *i.e.*, pour chaque $j \leqslant r$, on pose $temp = t_2[j]$. On ajoute $temp$ sur la droite de t_1 ($t_1[\ell+1] = temp$) et on remplace $t_2[j]$ par $m+1$; alors on actualise $\sigma = \sigma \cdot \langle temp, m+1 \rangle$ et on exécute $\mathbf{gen}(m+1, \ell+1)$.

Après chaque appel récursif, on actualise t_1, t_2 et σ pour retrouver leur valeur avant l'appel. Ces opérations requièrent une complexité $\mathcal{O}(1)$. Voir l'algorithme en figure I.1.

```
procedure gen(m, ℓ)
r := m − ℓ
if m = n then output σ ;
else
    if ℓ ⩽ k and m + 1 − ℓ ⩽ n − k then
      t2[r + 1] := m + 1
      gen(m + 1, ℓ)
      for each v ∈ [ℓ]
        σ := σ · ⟨t1[v], m + 1⟩
        gen(m + 1, ℓ)
        σ := σ · ⟨t1[v], m + 1⟩
    if ℓ + 1 ⩽ k and  m − ℓ ⩽ n − k then
      for each v ∈ [r]
        t1[ℓ + 1] := t2[v]
        temp := t2[v]; t2[v] := m + 1
        σ := σ · ⟨temp, m + 1⟩
        gen(m + 1, ℓ + 1)
        σ := σ · ⟨temp, m + 1⟩
        t2[v] := temp;
end procedure
```

$E_{4,1}$		$E_{4,2}$		$E_{5,1}$			
1	**4231**	1	3**241**	1	**52**341	14	**52134**
2	1**432**	2	**3412**	2	1**5342**	15	1324**5**
3	1**243**	3	4**321**	3	12**543**	16	1**5243**
4	3214	4	**1342**	4	1**2354**	17	1423**5**
5	**4213**	5	2**431**	5	4231**5**	18	**15234**
6	1324	6	**2143**	6	**52314**	19	**2**1345
7	**1423**	7	3**421**	7	1432**5**	20	5134**2**
8	**2**134	8	**3142**	8	1**5324**	21	4132**5**
9	4**132**	9	**2**314	9	1243**5**	22	**51324**
10	**3**124	10	4**312**	10	**12534**	23	**3**1245
11	4123	11	2413	11	**3**2145	24	5124**3**
				12	5214**3**	25	4123**5**
				13	4213**5**	26	51234

FIGURE I.1 : Algorithme de génération pour les permutations avec un nombre fixé d'excédences et les listes $E_{4,1}$, $E_{4,2}$ et $E_{5,1}$.

L'algorithme produit les permutations de $E_{n,k}$ en temps amorti constant (CAT) ; en effet, dans la procédure le nombre de calculs est proportionnel au nombre d'appels récursifs. De plus, chaque appel récursif produit au moins un objet ou produit deux autres appels récursifs. Une implementation Java de cet algorithme peut être vue à l'adresse http ://www.u-bourgogne.fr/jl.baril/applet.html.

I.2.2 Permutations et mots de Fibonacci ou de Lucas

Le contenu de cette partie est issu de l'article [17]. Nous utiliserons ici la méthode ECO (Enumeration of Combinatorial Objects) définie en 1999 par Pinzani et al. [15]. Il s'agit de fournir des règles de successions permettant de construire un arbre de génération dont les noeuds d'un niveau n sont les objets de la classe combinatoire considérée. On présente ci-dessous des algorithmes de génération basés sur ECO permettant de générer efficacement (CAT) les mots de Fibonacci et de Lucas p-généralisés. L'arbre de génération obtenu nous permet d'obtenir plusieurs bijections entre ces classes et des classes de permutations à motif exclus. Nous obtenons ainsi la génération CAT de nouvelles classes de permutations.

I.2.2.1 Permutations et mots de Fibonacci généralisés

Un mot de Fibonacci p-généralisé de taille n est un mot binaire de taille n n'ayant pas p uns consécutifs. Par exemple, le mot 110001010111 est un mot de Fibonacci 4 généralisé de taille 12. Soit $F_{n,p}$ l'ensemble des mots de Fibonacci p-généralisés de taille n. La cardinalité de cette ensemble est donnée par la suite de Fibonacci généralisée $f_{n,p}$ définie par $f_{n,p} = f_{n-1,p} + \ldots + f_{n-p,p}$ initialisée par $f_{n,p} = 0$ si $n = 0$ et $f_{1,p} = 1$. En fait, on a plus précisément : $|F_{n,p}| = f_{n+2,p}$. On définira par la suite ce qu'est une permutation de Fibonacci généralisée.

Le théorème suivant donne les règles de successions pour la suite $f_{n,p}$ de Fibonacci p-généralisée.

Théorème I.3. *Pour $p \geqslant 2$, un système de règles de succession (Ω_p) pour la suite $f_{n,p}$ de Fibonacci p-généralisée est donné par :*

$$(\Omega_p) \begin{cases} (2_{p-1}) \\ (2_{p-1}) \rightsquigarrow (2_{p-1})(2_{p-2}) \\ (2_{p-2}) \rightsquigarrow (2_{p-1})(2_{p-3}) \\ \ldots \\ (2_1) \rightsquigarrow (2_{p-1})(1) \\ (1) \rightsquigarrow (2_{p-1}). \end{cases}$$

Par un simple calcul utilisant les matrices de production [64], on déduit la fonction génératrice de la suite engendrée par Ω_p [71] :

$$f_p(z) = \frac{1 + z + z^2 + \ldots + z^{p-1}}{1 - z - z^2 - \ldots - z^p} = \frac{1}{z} \cdot \left(\frac{1}{1 - \sum_{i=1}^{p} z^i} - 1 \right).$$

Puisque $f_p(z) \underset{p \to \infty}{\longrightarrow} \frac{1}{1-2z}$, Ω_p produit une sorte de continuité discrète entre les suites de Fibonacci p-généralisées ($p \geqslant 2$) et la suite 2^n.

Proposition I.4. *L'arbre de génération de (Ω_p) peut être codé par les mots de Fibonacci p-généralisés, i.e., les mots binaires n'ayant pas p uns consécutifs. Un mot de taille n est obtenu à partir d'un mot de taille $(n-1)$ en insérant la valeur 0 ou 1 en dernière position (Voir figure I.2).*

Dans toute la suite une position (ou site) dans un mot (ou une permutation) est un emplacement dans un mot avant la première lettre ou après la dernière lettre ou entre deux lettres du mot. Lorsque l'on permet l'insertion d'une lettre sur une position d'un mot, alors on dira que la position est un site actif.

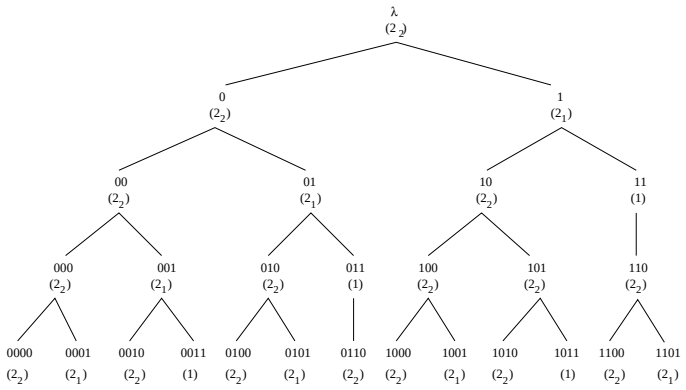


FIGURE I.2 : Les cinq premiers niveaux de l'arbre de génération codé par les mots de Fibonacci 3-généralisés.

L'arbre de génération précédent induit une bijection entre $F_{n,p}$ et deux ensembles de permutations à motifs exclus (voir la définition d'un motif dans une permutation dans le paragraphe Notations et Définitions). Ceci explique la raison pour laquelle on peut appeler permutation de Fibonacci p-généralisée une permutation appartenant à l'un de ces deux ensembles de permutations.

Théorème I.5. *L'arbre de génération de (Ω_p) peut être codé par les permutations π dans $S_n(321, 312, 234\ldots(p+1)1)$. Les sites actifs de π sont les deux dernières positions si l'étiquette de π est (2_i) ; sinon seulement la dernière position est un site actif. (Voir figure I.3).*

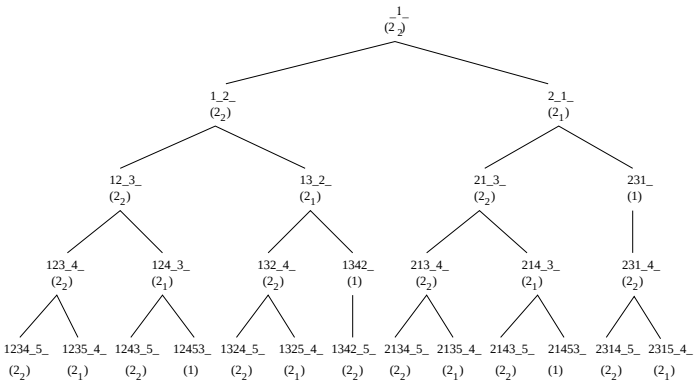


FIGURE I.3 : Les cinq permiers niveaux de l'arbre de génération de (Ω_3) codé par les permutations de Fibonacci 3-généralisées $S_n(321, 312, 2341)$. Les sites actifs sont représentés en soulignés.

Théorème I.6. *L'arbre de génération de (Ω_p) peut être codé par les permutations de Fibonacci généralisées π dans $S_n(231, 312, (p+1)p\ldots 321)$. Une permutation π a toujours un*

site actif sur sa dernière position ; de plus, dans le cas où le label est (2_i), un autre site actif se situe juste avant l'entrée maximale de la permutation π.

I.2.2.2 Permutations de Lucas généralisées

Un mot de Lucas p-généralisé est un mot de Fibonacci p-généralisé n'ayant pas p uns consécutifs lorsqu'on considère le mot circulairement. Par exemple, le mot 1010011 n'est pas un mot de Lucas 3-généralisé puisque 111 est contenu dans le mot (considéré circulairement). Soit $L_{n,p}$ l'ensemble des mots de Lucas p-généralisés de taille n. Pour $n < p$ on impose par convention que le mot 1^n est dans $L_{n,p}$. La cardinalité de $L_{n,p}$ est donnée par la suite $\ell_{n,p}$ définie par : $\ell_{n,p} = \ell_{n-1,p} + \ldots + \ell_{n-p,p}$ initialisée par $\ell_{n,p} = 2^n - 1$ si $0 < n \leqslant p$. En fait on a précisément $|L_{n,p}| = \ell_{n,p} + 1$ si $n < p$, et $|L_{n,p}| = \ell_{n,p}$ sinon. Le théorème suivant donne les règles de succession pour la suite de Lucas p-généralisée.

Théorème I.7. *Pour $p \geqslant 2$, un système de règles de succession (Φ_p) pour la suite de Lucas p-généralisée est :*

$$(\Phi_p)\begin{cases} (2'_{p-1}) \\ (2'_{p-1}) \rightsquigarrow (2_{p-1})(2'_{p-2}) \\ (2'_{p-2}) \rightsquigarrow (2_{p-2})(2'_{p-3}) \\ \ldots \\ (2'_1) \rightsquigarrow (2_1)(1') \\ (1') \rightsquigarrow (1) \\ (2_{p-1}) \rightsquigarrow (2_{p-1})(2_{p-2}) \\ (2_{p-2}) \rightsquigarrow (2_{p-1})(2_{p-3}) \\ \ldots \\ (2_1) \rightsquigarrow (2_{p-1})(1) \\ (1) \rightsquigarrow (2_{p-1}), \end{cases}$$

ce qui signifie que (Φ_p) produit la suite $\ell_{p,n}$ si $n > p$ et $\ell_{p,n} + 1 = 2^n$ sinon.

Par un simple calcul utilisant les matrices de production [64], la fonction génératrice de la suite associée à (Φ_p) est :

$$\ell_p(z) = \frac{\sum_{i=0}^{p-1}(z^i - i \cdot z^{2p-i})}{1 - z - z^2 - \ldots - z^p}.$$

On obtient une continuité discrète entre la suite de Lucas p-généralisée est la suite 2^n.

L'arbre de génération obtenu ne peut pas être codé directement par les mots de Lucas de $L_{n,p}$. Cependant, on peut coder cet arbre en utilisant un ensemble $K_{n,p} \subseteq F_{n,p}$; $K_{n,p}$ est l'ensemble les mots de $F_{n,p}$ qui contiennent au moins deux 0s dans leur préfixe de longueur $p + 1$.

Proposition I.8. *L'arbre de génération de (Φ_p) peut être codé par les mots de l'ensemble $K_{p,n}$. Un mot de taille n est obtenu d'un mot de taille $n - 1$ en insérant 0 ou 1 sur la dernière position.*

L'arbre de génération induit aussi une bijection entre $L_{n,p}$ et des ensembles de permutations à motifs exclus dont les éléments seront appelés permutations de Lucas p-généralisées.

Théorème I.9. *Soit*

$$T_p = \begin{cases} \overline{1}34\ldots(p+1)2:(p+3)(p+2) \\ \overline{1}34\ldots p2:(p+2)(p+3)(p+1) \\ \ldots \\ \overline{1}32:56\ldots(p+3)4. \end{cases}$$

Le système de règles de succession (Φ_p) *produit un arbre de génération codé par les permutations de Lucas p-généralisées* $S_n(321, 312, 234\ldots(p+1)1, T_p)$. *Les sites actifs sont les deux dernières positions si l'étiquette est* (2_i) *ou* $(2'_i)$ *; sinon, seulement la dernère position est active (voir figure I.4).*

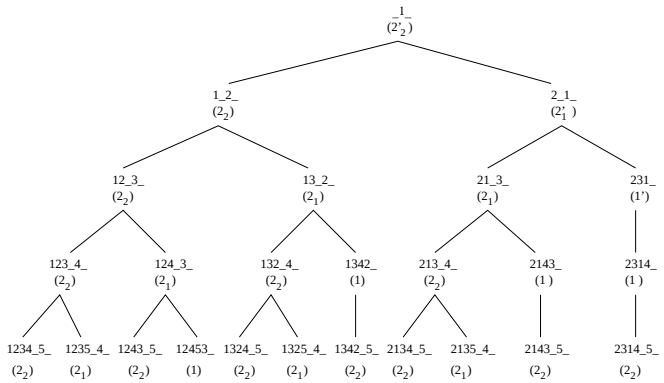


FIGURE I.4 : Les cinq premiers niveaux de l'arbre de génération de (Φ_3). Chaque noeud est codé par une permutation de Lucas dans $S_n(321, 312, 2341, \overline{1}342:65, \overline{1}32:564)$.

Théorème I.10. *Soit*

$$T'_p = \begin{cases} \overline{1}(p+1)p\ldots32:(p+3)(p+2) \\ \overline{1}p(p-1)\ldots32:(p+3)(p+2)(p+1) \\ \ldots \\ \overline{1}32:(p+3)(p+2)\ldots54. \end{cases}$$

L'arbre de génération de (Φ_p) *peut être codé par les permutations de Lucas* π *dans* $S_n(231, 312, (p+1)p\ldots321, T'_p)$. *Une permutation* π *a toujours un site actif en dernière position ; de plus, dans le cas où l'étiquette est* (2_i) *ou* $(2'_i)$, *elle en possède un autre qui est juste avant l'entrée maximale de* π.

I.2.2.3 Algorithme et complexité

Tous les ensembles étudiés précédemment peuvent être générés de façon plus ou moins efficace. Je donne ici seulement les algorithmes de génération des ensembles de Lucas $S_n(321, 312, 234\ldots(p+1)1, T_p)$ et $S_n(231, 312, (p+1)p\ldots321, T'_p)$. Un résumé de la compléxité des algorithmes trouvés est présenté à la fin de ce paragraphe dans la table I.1. Notons que le fait de pouvoir générer $K_{n,p}$ en $\mathcal{O}(1)$ nous permet de générer facilement $L_{n,p}$ en $\mathcal{O}(p)$.

```
l1gen(i, k)

if i = n then output σ ;
else
    if k ≠ 0 then
        f1gen(i + 1, k) ;
        σ = σ · ⟨i, i + 1⟩ ;
        l1gen(i + 1, k − 1) ;
        σ = σ · ⟨i, i + 1⟩ ;
    else f1gen(i + 1, 0) ;
end ;
```

```
l2gen(i, k, t)

if i = n then output σ ;
else
    if k ≠ 0 then
        f2gen(i + 1, k, i + 1) ;
        for j = i downto t
            σ = σ · ⟨j, j + 1⟩ ;
        l2gen(i + 1, k − 1, t) ;
        for j = t to i
            σ = σ · ⟨j, j + 1⟩ ;
    else f2gen(i + 1, 0, i + 1) ;
end ;
```

Ensembles généré	Complexité en moyenne
$F_{p,n}$	$\mathcal{O}(1)$
$S_n(321, 312, 234 \ldots (p+1)1)$	$\mathcal{O}(1)$
$S_n(231, 312, (p+1)p \ldots 321)$	$\mathcal{O}(p)$
$K_{p,n}$	$\mathcal{O}(1)$
$L_{p,n}$	$\mathcal{O}(p)$
$S_n(321, 312, 234 \ldots (p+1)1, T_p)$	$\mathcal{O}(1)$
$S_n(231, 312, (p+1)p \ldots 321, T'_p)$	$\mathcal{O}(p)$

Table I.1 : Complexité moyenne pour la génération de chaque classe.

I.3 Génération en code de Gray

Le contenu de cette partie est issu des cinq articles [19, 26, 25, 24, 31]. Elle présente des codes de Gray pour l'ensemble $L_{n,p}$ des mots de Lucas p-généralisés définis précédemment, pour l'ensemble des dérangements, *i.e*, l'ensemble des permutations n'ayant pas de point fixe, pour l'ensemble des permutations ayant un nombre donné de cycles, ou de minima gauche-droite, et enfin pour une famille large de classes de permutations évitant un ensemble de motifs.

I.3.1 Mots de Lucas p-généralisés

Rappelons que l'ensemble $L_{n,p}$ des mots de Lucas p-généralisés de taille n est l'ensemble des mots binaires de taille n ne contenant pas p uns consécutifs (les mots étant considé-

rés circulairement). Par exemple, le mot 100011 est dans $L_{6,4}$ mais n'est pas dans $L_{6,3}$. L'ensemble des mots de Fibonacci p-généralisés est encore noté $F_{n,p}$. Remarquons qu'un code de Gray est déjà connu pour $F_{n,p}$ [167]. Nous avons (dans le paragraphe précédent) obtenu une génération en $\mathcal{O}(p)$ de $L_{n,p}$. Nous donnons ici un code de Gray pour $L_{n,p}$ et un algorithme de génération en $\mathcal{O}(1)$.

Dans un premier temps on obtient une condition pour qu'il existe un code de Gray pour $L_{n,p}$ tel que deux éléments successifs diffèrent d'une et une seule position. Pour cela, il est nécessaire que la différence du nombre de mots de Lucas ayant un nombre pair de uns avec le nombre de mots de Lucas ayant un nombre impair de uns, soit comprise entre -1 et 1.

Soient $\{\phi_n\}_{n\geqslant 0}$ et $\{\lambda_n\}_{n\geqslant 0}$ les suites de différences de parité correspondantes aux suites généralisées de Fibonacci et Lucas :

- $\phi_n = card(F''_{n,p}) - card(F'_{n,p})$, et
- $\lambda_n = card(L''_{n,p}) - card(L'_{n,p})$.

ou $F''_{n,p}$ (resp. $L''_{n,p}$) est l'ensemble des mots de Fibonacci (resp. Lucas) ayant un nombre pair de uns, et $F'_{n,p}$ (resp. $L'_{n,p}$) est l'ensemble des mots de Fibonacci (resp. Lucas) ayant un nombre impair de uns.

Lemme I.11. *1. ϕ_n vérifie*

$$\phi_n = \phi_{n-1} - \phi_{n-2} + \cdots + (-1)^{p+1}\phi_{n-p}, \text{ pour } n \geqslant p+1, \tag{I.1}$$

2. λ_n est lié à ϕ_n par

$$\lambda_n = \phi_{n-2} - 2\cdot\phi_{n-3} + \cdots + (-1)^{p+1}p\cdot\phi_{n-p-1}, \text{ pour } n \geqslant p+2. \tag{I.2}$$

Proposition I.12. *Si $\phi(z)$ et $\lambda(z)$ sont les fonctions génératrices pour les suites $\{\phi_n\}_{n\geqslant 0}$ et $\{\lambda_n\}_{n\geqslant 0}$ respectivement alors*

1.

$$\phi(z) = \phi_0 + z\cdot(-z)^{p-1}\cdot\frac{1+z}{1-(-z)^{p+1}}, \tag{I.3}$$

2.

$$\lambda(z) = \sum_{j=0}^{p+1}\lambda_j z^j + z^{p+2}\cdot(-1)^{p+1}\cdot\frac{1-(p+1)(-z)^p + p(-z)^{p+1}}{(1-(-z)^{p+1})\cdot(1+z)}. \tag{I.4}$$

Corollaire I.13. *La suite $\{\lambda_n\}_{n\geq p+2}$ est périodique de période $2(p+1)$. De plus, si on définit $\lambda_n = \lambda_{n+2(p+1)}$ pour tout $n = 0, 1, \ldots, p+1$ alors sa fonction génératrice est*

$$\lambda(z) = \frac{(-z)^{p+1} + (p+1)z + p}{(1-(-z)^{p+1})\cdot(1+z)}. \tag{I.5}$$

Corollaire I.14. *La suite des différences de parités correspondant aux mots de Lucas vérifie*

$$\lambda_{n,p} = \begin{cases} (-1)^{n+1} & si \quad (p+1) \not| n, \\ (-1)^n\cdot p & si \quad (p+1)|n. \end{cases} \tag{I.6}$$

Théorème I.15. *Si l'ensemble $L_{n,p}$ admet un code de Gray alors $(p+1) \not| n$.*

Considérons la relation d'ordre suivante utilisée pour obtenir un code de Gray pour les mots de Fibonacci généralisés [167].

Définition I.16. *On dit que $\boldsymbol{x}$ est plus petit que $\boldsymbol{y}$ en* ordre local réfléchi*, noté par $\boldsymbol{x} \prec \boldsymbol{y}$, si $\sum_{j=1}^{i}(1-x_j)$ est impair et $\sum_{j=1}^{i}(1-y_j)$ est pair, où i est la position la plus à gauche telle que $x_i \neq y_i$.*

Soit alors $\mathcal{L}_{n,p}$ la liste ordonnée obtenue avec cette relation d'ordre. On peut montrer que :

Théorème I.17. *$\mathcal{L}_{n,p}$ est est un code de Gray optimal. Plus précisément,*

1. *Si $(p+1) \not| n$ alors $\mathcal{L}_{n,p}$ est un 1-code de Gray pour $L_{n,p}$.*
2. *Si $(p+1)|n$ alors $\mathcal{L}_{n,p}$ est un 2-code de Gray pour $L_{n,p}$ et il y a exactement $p-1$ mots vérifiant $d(\boldsymbol{x}, succ(\boldsymbol{x})) = 2$.*

La table suivante présente les codes de Gray obtenus pour l'ensemble $L_{n,p}$ lorsque $(n,p) = (4,2)$ et $(n,p) = (4,3)$.

$\mathcal{L}_{4,2}$	$\mathcal{L}_{4,3}$
0 1 0 **0**	0 1 **1** 0
0 **1** 0 1	0 1 0 **0**
0 0 0 **1**	0 **1** 0 1
0 0 **0** 0	0 0 0 **1**
0 0 1 0	0 0 **0** 0
1 0 **1** 0	0 0 1 **0**
1 0 0 0	**0** 0 1 **1**
	1 0 **1** 0
	1 0 0 **0**
	1 **0** 0 **1**
	1 1 0 0

Table I.2 : Les listes $\mathcal{L}_{4,2}$ et $\mathcal{L}_{4,3}$. Les modifications de bits sont en gras.

Un algorithme de génération CAT pour cette liste est naturellement obtenu par une simple adaptation de l'algorithme présenté dans [167].

I.3.2 Dérangements

Un *dérangement* de taille n est une permutation $\pi \in S_n$ sans point fixe, c'est à dire telle que $\pi(i) \neq i$ pour tout $i \in [n] = \{1, 2, \ldots, n\}$. Si D_n est l'ensemble de tous les dérangements de taille n, une relation de récurrence pour $d_n = card(D_n)$ est donnée par :

$$d_n = (n-1)(d_{n-1} + d_{n-2}) \tag{I.7}$$

pour $n \geqslant 2$, avec $d_1 = 0$ et $d_2 = 1$, voir par exemple [58, p. 180] ou [165, p. 67].

La construction du code de Gray ci-dessous pour l'ensemble D_n est basée sur une preuve combinatoire de la relation (I.7) ci-dessus.

Définition I.18. *1. Pour $n \geqslant 3$, un entier $i \in [n-1]$, et un dérangement $\tau \in D_{n-1}$, on définit une permutation $\sigma = \phi_n(i, \tau)$ de taille n par*

$$\sigma(j) = \begin{cases} n & si\ \tau(j) = i \\ i & si\ j = n \\ \tau(j) & sinon. \end{cases}$$

2. Pour $n \geqslant 4$, un entier $i \in [n-1]$ et un dérangement $\tau \in D_{n-2}$ on définit une permutation $\sigma = \psi_n(i, \tau)$ de taille n par

$$\sigma(j) = \begin{cases} i & si \quad j = n \\ n & si \quad j = i \\ \tau(j) & si \quad j < i \ et\ \tau(j) < i \\ \tau(j) + 1 & si \quad j < i \ et\ \tau(j) \geqslant i \\ \tau(j-1) & si \quad j > i \ et\ \tau(j-1) < i \\ \tau(j-1) + 1 & si \quad j > i \ et\ \tau(j-1) \geqslant i. \end{cases}$$

Remarque : Si $\sigma \in D_n$, $n \geqslant 4$, et $i = \sigma(n)$, alors

1. si $\sigma(i) \neq n$ (n n'est pas dans une transposition dans σ) alors $\sigma = \phi_n(i, \tau)$ avec τ la permutation représentée par les $n-1$ premières entrées de $\langle i, n \rangle \cdot \sigma$;
2. si $\sigma(i) = n$ (n est dans une transposition dans σ) alors $\sigma = \psi_n(i, \tau)$ avec τ la permutation représentée par la forme normale de la suite $(\sigma(1), \sigma(2), \ldots, \sigma(i-1), \sigma(i+1), \ldots, \sigma(n-1))$.

◇

Soit $\mathcal{D}_n$ la liste pour l'ensemble D_n définie par :

$$\begin{aligned} \mathcal{D}_n &= \phi(1, \mathcal{D}_{n-1}) \circ \psi(1, \overline{\mathcal{D}}_{n-2}) \\ &\quad \circ \psi(2, \mathcal{D}_{n-2}) \circ \phi(2, \overline{\mathcal{D}}_{n-1}) \\ &\quad \circ \phi(3, \mathcal{D}_{n-1}) \circ \psi(3, \overline{\mathcal{D}}_{n-2}) \\ &\quad \vdots \\ &= \bigcirc_{i=1}^{n-1} \left(\phi(i, \mathcal{D}_{n-1}) \circ \psi(i, \overline{\mathcal{D}}_{n-2})\right)^{(i+1)} \end{aligned} \tag{I.8}$$

pour $n \geqslant 3$, et $\mathcal{D}_1 = \psi(1, \emptyset) = \psi(2, \emptyset) = \emptyset$ et $\mathcal{D}_2 = (2, 1)$. La liste $\overline{\mathcal{D}_n}$ est la liste $\mathcal{D}_n$ lue du dernier élément au premier. On définit aussi $(\mathcal{D}_n)^{(i)} = \mathcal{D}_n$ si i est pair et $(\mathcal{D}_n)^{(i)} = \overline{\mathcal{D}_n}$ si i est impair.

La figure I.5 ci-dessous illustre la définition du code de Gray en représentant $\mathcal{D}_n$ par un parcours de sous-listes lues dans l'ordre habituel ou dans son sens inverse. Des exemples de listes sont donnés dans la table I.3 ci-dessous.

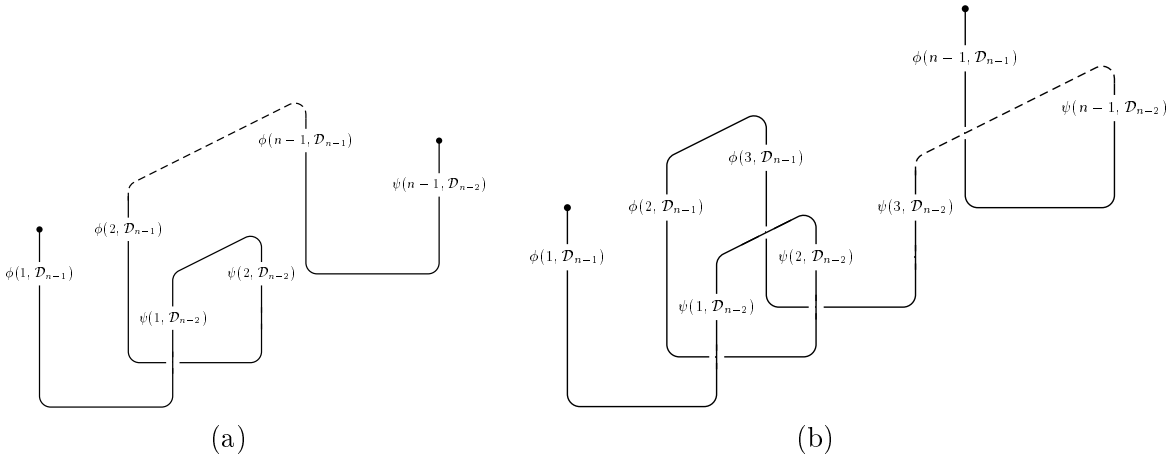


Figure I.5 : La liste $\mathcal{D}_n$: (a) n est pair, (b) n est impair.

Table I.3 : Les listes $\mathcal{D}_4$ et $\mathcal{D}_5$. Dans $\mathcal{D}_5$ les sous-listes $\phi(i, \mathcal{D}_4)$ et $\psi(i, \mathcal{D}_3)$, $1 \leqslant i \leqslant 4$, sont respectivement en gras et en italiques.

$\mathcal{D}_4$		$\mathcal{D}_5$							
1	2341	1	**23451**	12	*35412*	23	**25413**	34	*23154*
2	3421	2	**34251**	13	*45132*	24	**54213**	35	*31254*
3	4321	3	**43251**	14	**51432**	25	**45213**	36	**21534**
4	3412	4	**34521**	15	**41532**	26	**54123**	37	**51234**
5	3142	5	**35421**	16	**54132**	27	**51423**	38	**25134**
6	4312	6	**43521**	17	**43152**	28	**45123**	39	**53124**
7	2413	7	**24531**	18	**31452**	29	**24153**	40	**31524**
8	4123	8	**45231**	19	**34152**	30	**41253**	41	**35124**
9	2143	9	**25431**	20	**43512**	31	**21453**	42	**53214**
		10	*54231*	21	**34512**	32	*41523*	43	**35214**
		11	*53421*	22	**53412**	33	*24513*	44	**23514**

Théorème I.19. *Deux dérangements successifs dans la liste $\mathcal{D}_n$ diffèrent en au plus quatre positions.*

La démonstration de ce résultat consiste à étudier les différentes transitions entre les sous-listes de la définition récursive de $\mathcal{D}_n$.

Une génération CAT de la liste $\mathcal{D}_n$ peut être obtenue après quelques considérations assez techniques que je préfère occulter dans ce mémoire. J'invite le lecteur intéressé à se référer à l'article correspondant [31]. On trouvera également dans ce papier une généralisation aux permutations avec un nombre de points fixes situés entre deux constantes.

I.3.3 Permutations ayant un nombre donné de cycles

Pour $1 \leqslant k \leqslant n$, on désigne par $S_{n,k}$ l'ensemble des permutations de taille n admettant une décomposition en produit de k cycles disjoints. Naturellement, $\{S_{n,k}\}_{1\leqslant k\leqslant n}$ est une partition de S_n. La cardinalité de $S_{n,k}$ est donnée par les nombres de Stirling signés de première espèce $s(n,k)$ définis par :

$$s(n,k) = (n-1)\cdot s(n-1,k) + s(n-1,k-1) \tag{I.9}$$

avec la condition initiale $s(n,k) = 0$ si $n \leqslant 0$ ou $k \leqslant 0$, $(n,k) \neq (0,0)$, et $s(0,0) = 1$. Voir par exemple [165, 179].

Pour $n, k \geqslant 1$ on définit par $S'_{n,k}$ l'ensemble des permutations de taille n avec k cycles et telles que n est un point fixe (i.e. $\sigma(n) = n$) et $S''_{n,k} = S_{n,k} \backslash S'_{n,k}$ son complément dans $S_{n,k}$.

Les fonctions ϕ_n et ψ_n définies ci-dessous induisent deux bijections entre $S_{n-1,k-1}$ et $S'_{n,k}$, puis entre $[n-1] \times S_{n-1,k}$ et $S''_{n,k}$.

Définition I.20. *1. Pour $1 \leqslant k < n$, un entier $i \in [n-1]$ et une permutation $\gamma \in S_{n-1,k}$, on définit une permutation de taille n, $\sigma = \psi_n(i,\gamma)$ par*

$$\sigma(j) = \begin{cases} \gamma(i) & \text{si } j = n \\ n & \text{si } j = i \\ \gamma(j) & \text{sinon.} \end{cases}$$

2. Pour $n \geqslant k \geqslant 2$ et une permutation $\gamma \in S_{n-1,k-1}$, on définit une permutation de taille n, $\sigma = \phi_n(\gamma)$ par

$$\sigma(j) = \begin{cases} n & \text{si } j = n \\ \gamma(j) & \text{sinon.} \end{cases}$$

Remarquons qu'avec i et γ comme ci-dessus, il est facile de voir que :
- si $\gamma \in S_{n-1,k}$, $\psi_n(i,\gamma) \in S''_{n,k}$ et $\psi_n : [n-1] \times S_{n-1,k} \to S''_{n,k}$ est une bijection ; et
- si $\gamma \in S_{n-1,k-1}$, $\phi_n(\gamma) \in S'_{n,k}$ et $\phi_n : S_{n-1,k-1} \to S'_{n,k}$ est une bijection.

Lemme I.21. *Soit γ une permutation de taille $n-1$, si $n \geqslant 3$ et $1 \leqslant i, j \leqslant n-1$, $i \neq j$ alors $\psi(i,\gamma) = \psi(j,\gamma)\cdot\langle i,j,n\rangle$.*

Lemme I.22. *Soit γ et σ deux permutations de taille $n-1$ vérifiant $\sigma = \gamma\cdot\langle i,j,k\rangle$ avec i,j,k deux à deux différents, $1 \leqslant i,j,k \leqslant n-1$, $n \geqslant 3$ alors $\psi(j,\sigma) = \psi(k,\gamma)\cdot\langle i,j,k\rangle$.*

Lemme I.23. *Soit γ une permutation de taille $n-1$ et $\sigma = \gamma\cdot\langle i,j\rangle$ avec $i \neq j$, $1 \leqslant i,j \leqslant n-1$, $n \geqslant 3$ alors $\psi(i,\sigma) = \phi(\gamma)\cdot\langle i,n,j\rangle$.*

L'ensemble $S_{n,k}$ peut alors s'écrire :

$$S_{n,k} = \bigcup_{i=1}^{n-1} \psi(i, S_{n-1,k}) \cup \phi(S_{n-1,k-1}) \tag{I.10}$$

avec $\phi(S_{n,0})$ et $\psi(i, S_{n,n+1})$ vides.

On définit alors la liste $\mathcal{S}_{n,k}$ en distinguant les cinq cas ci-dessous. Comme dans le paragraphe précédent, la liste $\mathcal{S}^{(i)}_{n,k}$ correspond à la liste $\mathcal{S}_{n,k}$ si i est pair et à $\overline{\mathcal{S}_{n,k}}$ si i est impair.

* $\boxed{k=1, n \geqslant 1}$ Pour $n \geqslant 1$, on définit :

$$\mathcal{S}_{n,1} = \begin{cases} (1) & \text{si } n=1 \\ (2,1) & \text{si } n=2 \\ (3,1,2),(2,3,1) & \text{si } n=3 \\ \psi(1,\mathcal{S}_{3,1}) \circ \psi(2,\overline{\mathcal{S}_{3,1}}) \circ \psi(3,\overline{\mathcal{S}_{3,1}}) & \text{si } n=4 \\ \psi(1,\mathcal{S}_{n-1,1}) \circ \bigcirc_{i=n-1}^{4} \psi(i,\mathcal{S}_{n-1,1})^{(i+1)} \circ \psi(2,\mathcal{S}_{n-1,1}) \circ \psi(3,\overline{\mathcal{S}_{n-1,1}}) & \text{si } n \geqslant 5. \end{cases} \quad (a)$$

* $\boxed{k=n, n \geqslant 2}$ $\mathcal{S}_{n,n}$ contient un seul élément $(1,2,\ldots,n-1,n)$ pour $n \geqslant 2$, (b)

* $\boxed{k=n-1, n \geqslant 3}$ Pour $n \geqslant 3$, on définit $\mathcal{S}_{n,n-1} =$

$$\begin{cases} \psi(1,\mathcal{S}_{2,2}) \circ \psi(2,\mathcal{S}_{2,2}) \circ \phi(\mathcal{S}_{2,1}) & \text{si } n=3 \\ \psi(1,\mathcal{S}_{n-1,n-1}) \circ \bigcirc_{\substack{i=n-1 \\ i \neq n-2}}^{2} \psi(i,\mathcal{S}_{n-1,n-1}) \circ \psi(n-2,\mathcal{S}_{n-1,n-1}) \circ \phi(\overline{\mathcal{S}_{n-1,n-2}}) & \text{sinon.} \end{cases} \quad (c)$$

* $\boxed{k=n-2, n \geqslant 4}$ Pour $n \geqslant 4$, on définit

$$\mathcal{S}_{n,n-2} = \psi(1,\mathcal{S}_{n-1,n-2}) \circ \bigcirc_{i=3}^{n-1} \psi(i,\mathcal{S}_{n-1,n-2})^{(i)} \circ \phi(\mathcal{S}_{n-1,n-3}) \circ \psi(2,\overline{\mathcal{S}_{n-1,n-2}}). \quad (d)$$

* $\boxed{2 \leqslant k \leqslant n-3}$ Pour $2 \leqslant k \leqslant n-3$, on définit

$$\mathcal{S}_{n,k} = \psi(1,\mathcal{S}_{n-1,k}) \circ \bigcirc_{i=n-1}^{k+1} \psi(i,\mathcal{S}_{n-1,k})^{(i)} \circ \phi(\mathcal{S}_{n-1,k-1})^{(k)} \circ \bigcirc_{i=k}^{2} \psi(i,\mathcal{S}_{n-1,k})^{(i-1)}. \quad (e)$$

Théorème I.24. *Les listes $\mathcal{S}_{n,k}$ et $\mathcal{S}_{n,k}^{-1}$ sont des codes de Gray optimaux et cycliques. Une permutation est obtenue à partir de sa précédente en effectuant un produit par un cycle de taille 3.*

Table I.4 : Les listes $\mathcal{S}_{3,k}$, $1 \leqslant k \leqslant 3$, et $\mathcal{S}_{4,k}$ pour $1 \leqslant k \leqslant 4$. Par exemple, dans $\mathcal{S}_{4,2}$ les sous-listes de la relation (d), $\psi(1,\mathcal{S}_{3,2})$, $\overline{\psi(3,\mathcal{S}_{3,2})}$, $\phi(\mathcal{S}_{3,1})$ et $\overline{\psi(2,\mathcal{S}_{3,2})}$, sont alternativement en gras et italique.

$\mathcal{S}_{3,1}$		$\mathcal{S}_{3,2}$		$\mathcal{S}_{3,3}$		$\mathcal{S}_{4,1}$		$\mathcal{S}_{4,2}$		$\mathcal{S}_{4,3}$		$\mathcal{S}_{4,4}$	
1	**312**	1	**321**	1	**123**	1	**4123**	1	**4213**	1	**4231**	1	**1234**
2	*231*	2	*132*			2	**4312**	2	**4321**	2	*1243*		
		3	**213**			3	*2413*	3	**4132**	3	**1432**		
						4	*3421*	4	*2143*	4	*2134*		
						5	**2341**	5	*1342*	5	*1324*		
						6	**3142**	6	*3241*	6	*3214*		
								7	**3124**				
								8	**2314**				
								9	*2431*				
								10	*1423*				
								11	*3412*				

Toute permutation $\pi \in S_n$ a une unique décomposition comme un produit de transpo-

sitions de la forme :

$$\pi = \langle p_1, 1\rangle \cdot \langle p_2, 2\rangle \cdot \langle p_3, 3\rangle \cdot \ldots \cdot \langle p_n, n\rangle = \prod_{i=1}^{n} \langle p_i, i\rangle \tag{I.11}$$

avec $1 \leqslant p_i \leqslant i$. Donc (I.11) fournit une bijection $S_n \longrightarrow T_n$ avec $T_n = [1] \times [2] \times \ldots \times [n]$; de plus un mot $p_1p_2p_3 \ldots p_n \in T_n$ est une autre façon de représenter une permutation : on appelera cette repésentation *tableau des transpositions*. D'un autre coté, un mot $p_1p_2p_3 \ldots p_n \in T_n$ peut être vu comme la *table d'inversions* de la permutation $\sigma \in S_n$: p_i-1 est le nombre d'entrées $\sigma_j > \sigma_i$, $j < i$.

La relation entre ces deux repésentations n'a jamais été étudiée et est peut être une direction de recherche intéressante.

Soit maintenant l'ensemble $T_{n,k}$ des mots dans T_n avec exactement k points fixes, c'est-à-dire k entrées p_i telles que $p_i = i$. Puisque le nombre de 'transpositions' $\langle i, i\rangle$ dans (I.11) est aussi le nombre de cycles dans π, $T_{n,k}$ est l'ensemble des tableaux de transpositions des permutations dans $S_{n,k}$; la relation (I.11) induit une bijection $S_{n,k} \longrightarrow T_{n,k}$.

D'un autre coté, $T_{n,k}$ est l'ensemble des tables d'inversions des permutations dans S_n ayant k *minima gauche-droite* (en anglais left-to-right minima) ; un minimum gauche-droite dans une permutation est une entrée plus petite que toutes les entrées à sa gauche. Par conséquent, on obtient le résultat bien connu d'énumération : le nombre de permutations dans S_n avec k minima gauche-droite est aussi le nombre de Stirling signé $s(n, k)$. Voir également [165].

En général, une bijection entre deux ensembles ne conserve pas les petites modifications entre les objets ; C'est le cas de la bijection $S_n \longrightarrow T_n$ définie précédemment. Par exemple, si π et π' sont deux permutations telles que $\pi' = \pi \cdot \langle a, b, c\rangle$ alors la décomposition de π' peut différer de celle de π par beaucoup de transpositions : Prendre $\pi = (7, 1, 2, 3, 4, 5, 6) = \langle 1, 1\rangle \cdot \langle 1, 2\rangle \cdot \langle 1, 3\rangle \cdot \langle 1, 4\rangle \cdot \langle 1, 5\rangle \cdot \langle 1, 6\rangle \cdot \langle 1, 7\rangle$ et $\pi' = (2, 1, 4, 3, 7, 5, 6) = \pi \cdot \langle 1, 3, 5\rangle = \langle 1, 1\rangle \cdot \langle 1, 2\rangle \cdot \langle 3, 3\rangle \cdot \langle 3, 4\rangle \cdot \langle 5, 5\rangle \cdot \langle 5, 6\rangle \cdot \langle 5, 7\rangle$. Réciproquement, si deux décompositions diffèrent par au plus deux transpositions, alors les permutations correspondantes peuvent différer par beaucoup d'entrées : prendre $\pi = (2, 3, 4, 5, 6, 1) = \langle 1, 1\rangle \cdot \langle 1, 2\rangle \cdot \langle 2, 3\rangle \cdot \langle 3, 4\rangle \cdot \langle 4, 5\rangle \cdot \langle 5, 6\rangle$ et $\pi' = (2, 4, 1, 6, 3, 5) = \langle 1, 1\rangle \cdot \langle 1, 2\rangle \cdot \langle 2, 3\rangle \cdot \langle 2, 4\rangle \cdot \langle 4, 5\rangle \cdot \langle 4, 6\rangle$.

Soit maintenant la liste $\mathcal{T}_{n,k}$ pour l'ensemble $T_{n,k}$, obtenue en remplaçant chaque permutation de la liste $\mathcal{S}_{n,k}$ par son tableau de transpositions. De façon surprenante, deux suites consécutives dans $\mathcal{T}_{n,k}$ diffèrent en au plus deux positions et donc c'est un code de Gray code (voir la table 2). Ce résultat est présenté dans le prochain théorème. Malgrè la ressemblance avec le théorème I.24, ces deux théorèmes ne sont pas conséquences l'un de l'autre.

Théorème I.25. – *$\mathcal{T}_{n,k}$ est à la fois un code de Gray pour l'ensemble des permutations de taille n avec k cycles dans leur représentation en tableau de transposition, et pour l'ensemble des permutations de taille n avec k minimum gauche-droite dans leur représentation en table d'inversion.*

– *$\mathcal{T}_{n,k}$ est partitionnée selon les suffixes et satisfait les desiderata de Walsh : la liste peut donc être obtenue en une complexité loopless. [175] .*

Il est important de noter que si on considère la liste des permutations ayant pour tableau d'inversions $\mathcal{T}_{n,k}$, on n'obtient pas un code de Gray pour les permutations ayant un nombre

Table I.5 : Les listes $\mathcal{S}_{4,2}$ et $\mathcal{T}_{4,2}$. Chaque suite dans $\mathcal{T}_{4,2}$ est le tableau de transpositions de la permutation correspondante dans $\mathcal{S}_{4,2}$.

$\mathcal{S}_{4,2}$	$\mathcal{T}_{4,2}$
4213	1211
4321	1221
4132	1131
2143	1133
1342	1223
3241	1213
3124	1114
2314	1124
2431	1132
1423	1222
3412	1212

donné de minima gauche-droite. De façon récursive, et en utilisant les définitions (a), (b), (c), (d) et (e) on obtient naturellement un algorithme CAT pour la génération pour $\mathcal{S}_{n,k}$.

Le paragraphe suivant présente alors une construction ad-hoc permettant d'obtenir un code de Gray optimal pour cet ensemble.

I.3.4 Permutations ayant un nombre donné de minima gauche-droite

Soit $\sigma \in S_n$; alors σ_i est un minimum gauche-droite si $\sigma_i < \sigma_j$ pour tout $j < i$. Pour $1 \leqslant k \leqslant n$, on note $S_{n,k}$ l'ensemble des permutations de taille n avec exactement k minima gauche-droite. Par exemple, on a $S_{3,1} = \{123, 132\}$, $S_{3,2} = \{213, 231, 312\}$ et $S_{3,3} = \{321\}$. Naturellement, $\{S_{n,k}\}_{1 \leqslant k \leqslant n}$ forme une partition pour S_n et il est bien connu (voir par exemple [165], p. 20) que la cardinalité de $S_{n,k}$ est donnée par les nombres de Stirling signés de première espèce $s(n,k)$ définis dans le paragraphe précédent puisqu'ils énumèrent aussi les permutations de taille n ayant k cycles [165, 179]. C'est la raison pour laquelle on conserve la notation $S_{n,k}$ dans ce paragraphe.

Les fonctions ϕ_n et ψ_n définies dans le paragraphe précédent induisent une bijection entre $S_{n-1,k-1} \cup [n-1] \times S_{n-1,k}$ et $S_{n,k}$ telle que sa restriction à $S_{n-1,k-1}$ est ϕ_n et sa restriction à $[n-1] \times S_{n-1,k}$ est ψ_n.

Définition I.26. *1. Pour $1 \leqslant k < n$, un entier $i \in [n-1]$ et une permutation $\gamma \in S_{n-1,k}$, on définit une permutation de taille n, $\sigma = \psi_n(i, \gamma)$ par*

$$\sigma(j) = \begin{cases} \gamma(j) & \text{si } j \leqslant i \\ n & \text{si } j = i+1 \\ \gamma(j-1) & \text{sinon.} \end{cases}$$

2. Pour $n \geqslant k \geqslant 2$ et une permutation $\gamma \in S_{n-1,k-1}$, on définit une permutation de taille n, $\sigma = \phi_n(\gamma)$ par

$$\sigma(j) = \begin{cases} n & \text{si } j = 1 \\ \gamma(j-1) & \text{sinon.} \end{cases}$$

D'un autre coté, on a une autre bijection entre $S_{n-1,k-1} \cup [n]\backslash\{1\} \times S_{n-1,k}$ et $S_{n,k}$ en remplaçant ϕ_n et ψ_n avec les deux fonctions ϕ'_n et ψ'_n données ci-dessous.

Définition I.27. *1. Pour $1 \leqslant k < n$, un entier $i \in [n]\backslash\{1\}$ et une permutation $\gamma \in S_{n-1,k}$, on définit une permutation de taille n, $\sigma = \psi'_n(i,\gamma)$ par*

$$\sigma(j) = \begin{cases} \gamma(j) & si\ \gamma(j) \leqslant i-1 \\ i & si\ j = n \\ \gamma(j)+1 & sinon. \end{cases}$$

2. Pour $n \geqslant k \geqslant 2$ et une permutation $\gamma \in S_{n-1,k-1}$, on définit une permutation de taille n, $\sigma = \phi'_n(\gamma)$ par

$$\sigma(j) = \begin{cases} 1 & si\ j = n \\ \gamma(j)+1 & sinon. \end{cases}$$

Lemme I.28. *Soit γ une permutation de taille $n-1$, si $n \geqslant 3$ et $1 \leqslant i \leqslant n-2$, alors $\psi(i,\gamma) = \psi(i+1,\gamma) \cdot \langle i+1, i+2\rangle$.*

Lemme I.29. *Soit γ une permutation de taille $n-1$, si $n \geqslant 3$ et $1 \leqslant i \leqslant n-3$, alors $\psi(i,\gamma) = \psi(i+2,\gamma) \cdot \langle i+1, i+3, i+2\rangle$.*

Ces deux lemmes peuvent aussi s'écrire avec ψ' :

Lemme I.30. *Soit γ une permutation de taille $n-1$, si $n \geqslant 3$ et $2 \leqslant i \leqslant n-1$, alors $\psi'(i,\gamma) = \langle i, i+1\rangle \cdot \psi'(i+1,\gamma)$.*

Lemme I.31. *Soit γ une permutation de taille $n-1$, si $n \geqslant 3$ et $2 \leqslant i \leqslant n-2$, alors $\psi'(i,\gamma) = \langle i, i+1, i+2\rangle \cdot \psi'(i+2,\gamma)$.*

L'ensemble $S_{n,k}$ peut s'écrire :

$$S_{n,k} = \bigcup_{i=1}^{n-1} \psi(i, S_{n-1,k}) \cup \phi(S_{n-1,k-1}) \tag{I.12}$$

avec $\phi(S_{n,0})$ et $\psi(i, S_{n,n+1})$ vides. Si on considère maintenant ϕ' et ψ', la relation devient

$$S_{n,k} = \bigcup_{i=2}^{n} \psi'(i, S_{n-1,k}) \cup \phi'(S_{n-1,k-1}). \tag{I.13}$$

On définit alors la liste $\mathcal{S}_{n,k}$ en considérant les cinq cas suivants. On donne également une représentation de la liste en montrant l'enchaînement des différentes sous-listes qui la composent.

* $\boxed{k = n, n \geqslant 1}$ $\mathcal{S}_{n,n}$, $n \geqslant 1$, contient un seul élément $(n, n-1, \ldots, 2, 1)$, (a)

* $\boxed{k=1, n \geqslant 2}$ Pour $n \geqslant 2$, on définit les relations (b) :

• Pour $n = 2$, $\mathcal{S}_{n,1} = (1, 2)$.

• Pour $n = 2m \geqslant 2$,

$$\mathcal{S}_{n,1} = \psi(n-1, \mathcal{S}_{n-1,1}) \circ \bigcirc_{i=m-1}^{1} \psi(2i-1, \mathcal{S}_{n-1,1})^{(m+i-1)} \circ \bigcirc_{i=1}^{m-1} \psi(2i, \mathcal{S}_{n-1,1})^{(m+i)}.$$

• Pour $n = 2m+1 \geqslant 3$,

$$\mathcal{S}_{n,1} = \psi(n-1, \mathcal{S}_{n-1,1}) \circ \bigcirc_{i=m-1}^{1} \psi(2i, \mathcal{S}_{n-1,1})^{(m+i)} \circ \bigcirc_{i=1}^{m} \psi(2i-1, \mathcal{S}_{n-1,1})^{(m-1+i)}.$$

Figure I.6 : Les relations (b) pour $n = 8$ et $n = 9$. Chaque point i correspond à la liste $\psi(i, \mathcal{S}_{n-1,1})$ et chaque point encerclé i représente la liste $\overline{\psi(i, \mathcal{S}_{n-1,1})}$, *i.e.*, la liste $\psi(i, \mathcal{S}_{n-1,1})$ dans l'ordre inverse. Par exemple, si $n = 8$ alors $\mathcal{S}_{8,1} = \psi(7, \mathcal{S}_{7,1}) \circ \psi(5, \mathcal{S}_{7,1}) \circ \overline{\psi(3, \mathcal{S}_{7,1})} \circ \psi(1, \mathcal{S}_{7,1}) \circ \overline{\psi(2, \mathcal{S}_{7,1})} \circ \psi(4, \mathcal{S}_{7,1}) \circ \overline{\psi(6, \mathcal{S}_{7,1})}$.

* $\boxed{k=n-1, n \geqslant 3}$ On définit les relations (c) (voir figure I.7),

• Pour $n = 2m \geqslant 4$,

$$\mathcal{S}_{n,n-1} = \psi'(n, \mathcal{S}_{n-1,n-1}) \circ \phi'(\mathcal{S}_{n-1,n-2}) \circ \bigcirc_{i=m-1}^{1} \psi'(2i, \mathcal{S}_{n-1,n-1}) \circ \bigcirc_{i=2}^{m} \psi'(2i-1, \mathcal{S}_{n-1,n-1}).$$

• Pour $n = 2m+1 \geqslant 3$,

$$\mathcal{S}_{n,n-1} = \psi'(n, \mathcal{S}_{n-1,n-1}) \circ \phi'(\mathcal{S}_{n-1,n-2}) \circ \bigcirc_{i=m}^{2} \psi'(2i-1, \mathcal{S}_{n-1,n-1}) \circ \bigcirc_{i=1}^{m} \psi'(2i, \mathcal{S}_{n-1,n-1}).$$

Notons que ces relations utilisent ψ' et ϕ' au lieu de ψ et ϕ.

Figure I.7 : Les relations (c) pour $n = 8$ et $n = 9$. Chaque point $i \neq 0$ correspond à la liste $\psi'(i, \mathcal{S}_{n-1,n-1})$ et le point 0 représente la liste $\phi'(\mathcal{S}_{n-1,n-2})$. Par exemple, si $n = 8$ alors $\mathcal{S}_{8,7} = \psi'(8, \mathcal{S}_{7,7}) \circ \phi'(\mathcal{S}_{7,6}) \circ \psi'(6, \mathcal{S}_{7,7}) \circ \psi'(4, \mathcal{S}_{7,7}) \circ \psi'(2, \mathcal{S}_{7,7}) \circ \psi'(3, \mathcal{S}_{7,7}) \circ \psi'(5, \mathcal{S}_{7,7}) \circ \psi'(7, \mathcal{S}_{7,7})$.

* $\boxed{k=n-2, n \geqslant 4}$ On définit les relations (d) (voir figure I.8),

• Pour $n = 2m \geqslant 4$,

$$\mathcal{S}_{n,n-2} = \psi(n-1, \mathcal{S}_{n-1,n-2}) \circ \overline{\phi(\mathcal{S}_{n-1,n-3})} \circ \bigcirc_{i=m-1}^{1} \psi(2i-1, \mathcal{S}_{n-1,n-2})^{(m+i-1)}$$
$$\circ \bigcirc_{i=1}^{m-1} \psi(2i, \mathcal{S}_{n-1,n-2})^{(m+i)}.$$

• Pour $n = 2m+1 \geqslant 5$,

$$\mathcal{S}_{n,n-2} = \psi(n-1, \mathcal{S}_{n-1,n-2}) \circ \phi(\mathcal{S}_{n-1,n-3}) \circ \bigcirc_{i=m-1}^{1} \psi(2i, \mathcal{S}_{n-1,n-2})^{(m+i)} \circ \bigcirc_{i=1}^{m} \psi(2i-1, \mathcal{S}_{n-1,n-2})^{(m+1+i)}.$$

Figure I.8 : Les relations (d) pour $n = 8$ et $n = 9$. Chaque point $i \neq 0$ correspond à la liste $\psi(i, \mathcal{S}_{n-1,n-2})$ et le point 0 est la liste $\phi(\mathcal{S}_{n-1,n-3})$. Quand le point est encerclé, on considere la liste en ordre inverse. Par exemple, si $n = 8$ alors $\mathcal{S}_{8,6} = \psi(7, \mathcal{S}_{7,6}) \circ \overline{\phi(\mathcal{S}_{7,5})} \circ \psi(5, \mathcal{S}_{7,6}) \circ \overline{\psi(3, \mathcal{S}_{7,6})} \circ \psi(1, \mathcal{S}_{7,6}) \circ \overline{\psi(2, \mathcal{S}_{7,6})} \circ \psi(4, \mathcal{S}_{7,6}) \circ \overline{\psi(6, \mathcal{S}_{7,6})}$.

* $\boxed{2 \leqslant k \leqslant n-3}$ Si $2 \leqslant k \leqslant n-3$, on définit les relations (e) (voir figure I.9)

• Pour $n = 2m$, k impair,

$$\mathcal{S}_{n,k} = \psi(n-1, \mathcal{S}_{n-1,k}) \circ \bigcirc_{i=m-1}^{1} \psi(2i-1, \mathcal{S}_{n-1,k})^{(m+i)} \circ \bigcirc_{i=1}^{(k-1)/2} \psi(2i, \mathcal{S}_{n-1,k})^{(m+i+1)} \circ \phi(\mathcal{S}_{n-1,k-1})^{m+(k-1)/2} \circ \bigcirc_{i=(k+1)/2}^{m-1} \psi(2i, \mathcal{S}_{n-1,k-1})^{(m+i)}.$$

• Pour $n = 2m$, k pair,

$$\mathcal{S}_{n,k} = \psi(n-1, \mathcal{S}_{n-1,k}) \circ \bigcirc_{i=m-1}^{(k+2)/2} \psi(2i-1, \mathcal{S}_{n-1,k})^{(m+i)} \circ \phi(\mathcal{S}_{n-1,k-1})^{m+\frac{k}{2}}$$
$$\circ \bigcirc_{i=k/2}^{1} \psi(2i-1, \mathcal{S}_{n-1,k})^{(m+i-1)} \circ \bigcirc_{i=1}^{m-1} \psi(2i, \mathcal{S}_{n-1,k-1})^{(m+i)}.$$

• Pour $n = 2m+1$, k impair,

$$\mathcal{S}_{n,k} = \psi(n-1, \mathcal{S}_{n-1,k}) \circ \bigcirc_{i=m-1}^{(k+1)/2} \psi(2i, \mathcal{S}_{n-1,k})^{(m+i-1)} \circ \phi(\mathcal{S}_{n-1,k-1})^{m+\frac{k+1}{2}}$$
$$\circ \bigcirc_{i=(k-1)/2}^{1} \psi(2i, \mathcal{S}_{n-1,k})^{(m+i)} \circ \bigcirc_{i=1}^{m} \psi(2i-1, \mathcal{S}_{n-1,k-1})^{(m+i+1)}.$$

• Pour $n = 2m+1$, k pair,

$$\mathcal{S}_{n,k} = \psi(n-1, \mathcal{S}_{n-1,k}) \circ \bigcirc_{i=m-1}^{1} \psi(2i, \mathcal{S}_{n-1,k})^{(m+i-1)} \circ \bigcirc_{i=1}^{(k-1)/2} \psi(2i-1, \mathcal{S}_{n-1,k})^{(m+i)} \circ \phi(\mathcal{S}_{n-1,k-1})^{m+(k+1)/2} \circ \bigcirc_{i=(k+1)/2}^{m} \psi(2i-1, \mathcal{S}_{n-1,k-1})^{(m+i+1)}.$$

Théorème I.32. *La famille de listes $\{\mathcal{S}_{n,k}\}_{n \geqslant k \geqslant 1}$ est un code de Gray optimal.*

La démonstration se fait par récurrence et consiste à examiner toutes les transitions entre les sous-listes dans les relations (a)-(e).

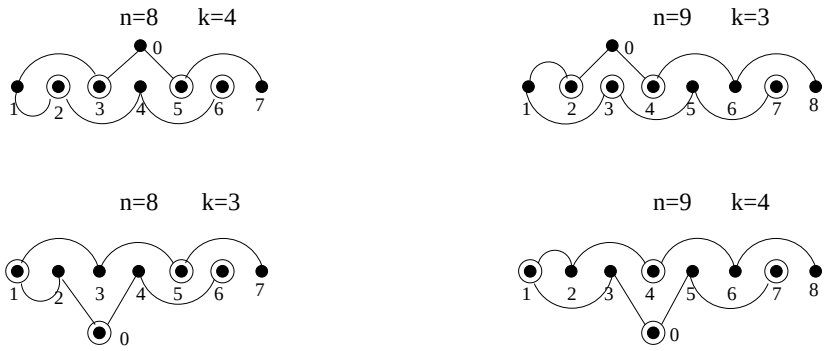


FIGURE I.9 : Les relations (e) pour $n = 8, 9$ et $k = 3, 4$. Chaque point $i \neq 0$ correspond à la liste $\psi(i, \mathcal{S}_{n-1,k})$ et le point 0 est la liste $\phi(\mathcal{S}_{n-1,k-1})$. Quand le point est encerclé, on considère la liste en ordre inverse. Par exemple, si $n = 8$ et $k = 4$ alors $\mathcal{S}_{8,4} = \psi(7, \mathcal{S}_{7,4}) \circ \overline{\psi(5, \mathcal{S}_{7,4})} \circ \phi(\mathcal{S}_{7,3}) \circ \overline{\psi(3, \mathcal{S}_{7,4})} \circ \psi(1, \mathcal{S}_{7,4}) \circ \overline{\psi(2, \mathcal{S}_{7,4})} \circ \psi(4, \mathcal{S}_{7,4}) \circ \overline{\psi(6, \mathcal{S}_{7,4})}$.

TABLE I.6 : Les listes $\mathcal{S}_{3,k}$, $1 \leqslant k \leqslant 3$, et $\mathcal{S}_{4,k}$ pour $1 \leqslant k \leqslant 4$. Par exemple, dans $\mathcal{S}_{4,2}$ les sous-listes de la relation (d), $\psi(3, \mathcal{S}_{3,2})$, $\overline{\phi(\mathcal{S}_{3,1})}$, $\psi(1, \mathcal{S}_{3,2})$, et $\overline{\psi(2, \mathcal{S}_{3,2})}$, sont alternativement en gras et en italique.

$\mathcal{S}_{3,1}$		$\mathcal{S}_{3,2}$		$\mathcal{S}_{3,3}$		$\mathcal{S}_{4,1}$		$\mathcal{S}_{4,2}$		$\mathcal{S}_{4,3}$		$\mathcal{S}_{4,4}$	
1	**123**	1	**213**	1	**321**	1	**1234**	1	**2134**	1	**3214**	1	**4321**
2	*132*	2	*231*			2	**1324**	2	**2314**	2	*3241*		
		3	**312**			3	*1423*	3	**3124**	3	*3421*		
						4	*1432*	4	*4132*	4	*4231*		
						5	**1342**	5	*4123*	5	**4312**		
						6	**1243**	6	**2413**	6	*4213*		
								7	**2431**				
								8	**3412**				
								9	*3142*				
								10	*2341*				
								11	*2143*				

Etant donné que l'on a utilisé une définition récursive du code de Gray, l'implantation d'un algorithme CAT pour l'obtention de cette liste s'obtient naturellement. Une applet java est disponible sur le site http ://www.u-bourgogne.fr/jl.baril.

I.3.5 Permutations évitant un ensemble de motifs

Dans ce paragraphe on donne une construction de codes de Gray unifiée pour un grand nombre de classes de permutations évitant un ensemble de motifs.

Soit $\sigma \in S_n$; les *sites* de σ sont les positions entre deux entrées consécutives, avant la première entrée et après la dernière. On suppose que les sites sont numérotés de la droite vers la gauche de 1 à $n + 1$. Par exemple, le troisième site de la permutation $\sigma = 463512$ est entre 5 et 1. De plus, soit $\sigma \in S_n(T)$, où T est un ensemble de motifs à exclure, on dira que le i-ième site de σ est *actif* si la permutation γ obtenue à partir de σ en insérant $(n+1)$ sur ce site appartient à $S_{n+1}(T)$. On notera cette permutation γ par $\phi(i, \sigma)$ et nous dirons que γ est un *successeur* (*ou fils*) de σ. Les sites actifs d'une permutation $\sigma \in S_n(T)$ sont *ressérés à droite* (voir [67]) si tous les sites à droite du site actif le plus à gauche sont aussi

actifs. On note par $\chi_T(i,\sigma)$ le nombre de sites actifs de $\phi(i,\sigma)$. Un ensemble de motifs T est appelé *régulier* [67] si pour tout $n \geqslant 1$ et $\gamma \in S_n(T)$,

- γ a au moins deux sites actifs, et les sites actifs de γ sont ressérés à droite,
- il existe $\sigma \in S_{n-1}(T)$ et un entier i, $1 \leqslant i \leqslant n$, tel que $\gamma = \phi(i,\sigma)$,
- $\chi_T(i,\sigma)$ ne dépend pas de σ mais seulement du nombre k de sites actifs de σ ; dans ce cas, on note $\chi_T(i,\sigma)$ par $\chi_T(i,k)$ si σ a k sites actifs.

Pour la suite de ce paragraphe, on ne considère que des ensembles de motifs T réguliers.

Maintenant, pour chaque $n \geqslant 1$ et pour chaque permutation $\sigma \in S_n(T)$, on associe :

- une *direction* : *haut* ou *bas* ; une permutation σ avec la direction *haut* (*bas* respectivement) sera notée par σ^1 (σ^0 respectivement). Une telle permutation sera appelée permutation *orientée*.
- Une liste de ses successeurs considérés avec leur directions, *i.e.* une liste contenant des $\phi(i,\sigma)^j$ où $j \in \{0,1\}$ et les indices i sont les sites actifs de σ dans un ordre que l'on va expliciter ci-dessous.

La liste des successeurs de σ^0 sera obtenue en renversant l'ordre de la liste des successeurs de σ^1 et en inversant la direction de chaque élément de la liste.

Soit σ^1 une permutation orientée dans $S_n(T)$ avec k sites actifs et soit L la suite unimodale d'entiers où $L(i)$ est le i-ième ($1 \leqslant i \leqslant k$) entier de la suite L suivante :

$$L = \begin{cases} 3,5,\ldots,(k-3),(k-1),k,(k-2),(k-4),\ldots,4,2,1 & \text{si } k \text{ est pair,} \\ 3,5,\ldots,(k-4),(k-2),k,(k-1),(k-3),\ldots,4,2,1 & \text{si } k \text{ est impair.} \end{cases}$$

On considère aussi la suite L' définie par :

$$L' = \begin{cases} 2,4,\ldots,(k-4),(k-2),k,(k-1),(k-3),\ldots,5,3,1 & \text{si } k \text{ est pair,} \\ 2,4,\ldots,(k-3),(k-1),k,(k-2),(k-4),\ldots,5,3,1 & \text{si } k \text{ est impair.} \end{cases}$$

Soit $\Pi(\sigma^1)$ la liste des successeurs de σ^1 contenant les k permutations orientées dans $S_{n+1}(T)$ telle que : le j-ième élément est $\phi(L(j),\sigma)^{(i-1) \bmod 2}$ pour $1 \leqslant j \leqslant k-1$ et le k-ième élément est $\phi(L(k),\sigma)^1$. De plus, si on remplace la suite L par L' dans cette définition, on obtient une liste notée $\Pi'(\sigma^1)$. Les figures I.10 et I.11 illustrent les différents cas pour les listes $\Pi(\sigma)$ et $\Pi'(\sigma)$ lorsque σ possède $k=7$ ou $k=8$ sites actifs.

Figure I.10 : Si $k = 7$ alors $\Pi(\sigma^1)$ est la liste $\phi(3,\sigma)^0, \phi(5,\sigma)^1, \phi(7,\sigma)^0, \phi(6,\sigma)^1, \phi(4,\sigma)^0, \phi(2,\sigma)^1, \phi(1,\sigma)^1$; si $k = 8$, $\Pi(\sigma^1)$ est $\phi(3,\sigma)^0, \phi(5,\sigma)^1, \phi(7,\sigma)^0, \phi(8,\sigma)^1, \phi(6,\sigma)^0, \phi(4,\sigma)^1, \phi(2,\sigma)^0, \phi(1,\sigma)^1$. Chaque point (resp. point encerclé) étiqueté par $\ell \in [1..k]$ représente $\phi(\ell,\sigma)^1$ (resp. $\phi(\ell,\sigma)^0$).

FIGURE I.11 : Si $k = 7$ alors $\Pi'(\sigma^1)$ est la liste $\phi(2,\sigma)^0, \phi(4,\sigma)^1, \phi(6,\sigma)^0, \phi(7,\sigma)^1, \phi(5,\sigma)^0, \phi(3,\sigma)^1, \phi(1,\sigma)^1$; si $k = 8$, $\Pi'(\sigma^1)$ est $\phi(2,\sigma)^0, \phi(4,\sigma)^1, \phi(6,\sigma)^0, \phi(8,\sigma)^1, \phi(7,\sigma)^0, \phi(5,\sigma)^1, \phi(3,\sigma)^0, \phi(1,\sigma)^1$. Chaque point (resp. point encerclé) étiqueté par $\ell \in [1..k]$ represente $\phi(\ell,\sigma)^1$ (resp. $\phi(\ell,\sigma)^0$).

Posons $d_n = \text{card}(S_n(T))$. On définit alors récursivement le code de Gray $\mathcal{S}_n(T)$ pour l'ensemble $S_n(T)$:

$$\mathcal{S}_n(T) = \begin{cases} 1^0 & \text{si } n = 1 \\ \bigcirc_{i=1}^{d_{n-1}} \tilde{\Pi}(\sigma[i]) & \text{sinon,} \end{cases}$$

où

- $\sigma[i]$ est la i-ième permutation orientée de la liste $\mathcal{S}_{n-1}(T)$,
- $\tilde{\Pi}(\sigma[i]) = \Pi'(\sigma[i])$ si : la direction de $\sigma[i]$ est 0 et la direction de $\sigma[i+1]$ est 1 ; $\sigma[i]$ et $\sigma[i+1]$ diffèrent en trois positions et ont respectivement au moins trois fils et exactement deux fils,
- $\tilde{\Pi}(\sigma[i]) = \Pi'(\sigma[i])$ si : la direction de $\sigma[i]$ est 1 et la direction de $\sigma[i-1]$ est 0 ; $\sigma[i]$ et $\sigma[i-1]$ diffèrent de trois positions et ont respectivement au moins trois fils et exactement deux fils,
- $\tilde{\Pi}(\sigma[i]) = \Pi(\sigma[i])$ sinon.

La table I.7 illustre notre construction pour $\mathcal{S}_5(321)$ et $S_5(312)$. Notons que les listes Π et Π' apparaîssent dans la construction du code Gray $\mathcal{S}_5(321)$; mais seulement les listes de la forme Π sont utilisées pour $\mathcal{S}_5(312)$.
Maintenant nous étudions plusieurs propriétés qui vont nous permettre de prouver que cette définition induit presque toujours des codes de Gray tels que deux éléments consécutifs de la liste diffèrent en au plus trois positions. Si $\mathcal{L}$ est une liste, on note par prem($\mathcal{L}$) et der($\mathcal{L}$) le premier et dernier élément de $\mathcal{L}$.

Proposition I.33. *Soit T un ensemble régulier de motifs. La liste $\mathcal{S}_n(T)$ contient exactement une et une seule fois chaque permutations de $S_n(T)$. Le premier élément de la liste est $123\ldots(n-1)n$ et le dernier est $2134\ldots(n-1)n$.*

Lemme I.34. *Soit T un ensemble régulier de motifs. Soit σ une permutation dans $\mathcal{S}_n(T)$, et soit $r \in \{0,1\}$ sa direction, alors deux permutations successives dans $\Pi(\sigma^r)$ diffèrent en au plus trois positions.*

Lemme I.35. *Soit T un ensemble régulier de motifs. Soient σ^1 et γ^0 deux permutations orientées consécutives dans la liste $\mathcal{S}_n(T)$. Alors $d(\sigma,\gamma) = d(der(\Pi(\sigma^1)), prem(\Pi(\gamma^0)))$.*

Lemme I.36. *Soit T un ensemble régulier de motifs. Soient σ^0 et γ^1 deux permutations orientées consécutives dans la liste $\mathcal{S}_n(T)$. Si les deux permutations σ et γ ont au moins trois successeurs alors on a : $d(\sigma,\gamma) = d(der(\Pi(\sigma^0)), prem(\Pi(\gamma^1)))$.*

Lemme I.37. *Soit T un ensemble régulier de motifs. Soient σ^0 et γ^1 deux permutations orientées consécutives dans la liste $\mathcal{S}_n(T)$. Si les deux permutations σ et γ ont deux successeurs, on a : $d(\sigma,\gamma) = d(der(\Pi(\sigma^0)), prem(\Pi(\gamma^1)))$.*

Lemme I.38. *Soit T un ensemble régulier de motifs. Soient σ^0 et γ^1 deux permutations orientées consécutives dans la liste $\mathcal{S}_n(T)$ telles que $\sigma = \gamma \cdot \langle n-1, n\rangle$. Si σ a au moins trois fils et γ a deux fils alors on a : $d(der(\Pi(\sigma^0)), prem(\Pi(\gamma^1))) = 3$.*

Lemme I.39. *Soit T un ensemble régulier de motifs. Soient σ^0 et γ^1 deux permutations orientées consécutives dans la liste $\mathcal{S}_n(T)$ telles que σ et γ diffèrent en trois positions. Si σ a deux fils (resp. au moins trois fils) et γ a au moins trois fils (resp. exactement deux fils), alors on a : $d(der(\Pi(\sigma^0)), prem(\Pi(\gamma^1))) = 3$.*

Lemme I.40. *Soit T un ensemble régulier de motifs. Soient σ^1 et γ^1 deux permutations orientées consécutives dans la liste $\mathcal{S}_n(T)$ telles que $\sigma = \gamma \cdot \langle n-1, n\rangle$. Si γ a au moins trois fils, alors on a : $d(der(\Pi(\sigma^1)), prem(\Pi(\gamma^1))) = 2$. D'un autre coté, si γ a deux fils alors on a : $d(der(\Pi(\sigma^1)), prem(\Pi(\gamma^1))) = 3$.*

Remarquons alors que :
- si $\chi_T(i,k) = 2$ pour tout i et k, alors la liste $\mathcal{S}_n(T)$ définie ci-dessus est un code de Gray cyclique pour $S_n(T)$ tel que deux éléments consécutifs dans la liste diffèrent sur deux positions. En effet, il suffit d'appliquer les lemmes I.34, I.35, et I.37.
- si $\chi_T(i,k) \geqslant 3$ pour tout i et k, alors la liste $\mathcal{S}_n(T)$ définie ci-dessus est un code de Gray cyclique pour $S_n(T)$. Deux éléments consécutifs dans la liste diffèrent sur au plus trois positions. Il suffit d'appliquer les lemmes I.34, I.35, I.36 et I.40.

Dans la table I.8, on donne plusieurs ensembles T de motifs pour lesquels on peut appliquer notre construction dans le but d'obtenir des codes de Gray pour $S_n(T)$. Notons que cette construction nous permet d'obtenir des codes de Gray optimaux pour $S_{2n+1}(321)$ et $S_{2n+1}(312)$, $n \geqslant 0$ (voir la table I.7). En effet, la différence entre le nombre de permutations paires et impaires dans $S_{2n+1}(321)$ (resp. $S_{2n+1}(312)$) est au moins deux (voir [160]). Donc nous ne pouvons pas avoir un code de Gray tel que deux permutations consécutives diffèrent par une transposition. Puisque la différence entre le nombre de permutations paires et impaires dans $S_{2n}(321)$ (resp. $S_{2n}(312)$) est zéro, nous ne pouvons pas déduire l'optimalité de notre code de Gray pour $S_{2n}(321)$ (resp. $S_{2n}(312)$) ainsi que pour d'autres classes. Le code de Gray pour $S_n(321, 312)$ est optimal et il correspond à celui trouvé dans [94] par une autre méthode.

I.4 Conclusion - perspectives

Ce chapitre expose mes travaux concernant la génération exhaustive d'une classe d'objets combinatoires. La plupart des résultats obtenus étudient la génération de classes de permutations particulières. Cependant, il reste plusieurs problèmes ouvert dans ce domaine. On peut se demander si les classes $S_{2n}(321)$ et $S_{2n}(312)$ admettent des codes de Gray tels que deux permutations successives diffèrent en au plus deux positions. On peut également se poser la question de l'existence d'un code de Gray pour la classe de permutations ayant un nombre donné de descentes ou de la classe des permutations ayant un nombre donné d'excédances. Il reste donc encore de nombreuses classes de permutations à explorer. Enfin, on peut aussi citer le problème du 'middle level' qui résiste encore à l'ensemble de la communauté. Il s'agit de trouver un code de Gray de l'ensemble des mots binaires de taille impaire $2n+1$ ayant n ou $n+1$ uns...

Table I.7 : Les codes de Gray optimaux pour $\mathcal{S}_5(312)$ et $\mathcal{S}_5(321)$. Deux éléments consécutifs diffèrent en au plus trois positions.

$\mathcal{S}_5(312)$			$\mathcal{S}_5(321)$		
12345	**321**45	**2**4**3**15	12345	1**3**2**4**5	**231**45
123**54**	321**54**	243**51**	123**54**	132**54**	231**54**
12**543**	32**541**	2**543**1	1**523**4	13**52**4	23**51**4
12**45**3	32**45**1	2**45**31	**51**234	13**452**	23**451**
124**35**	324**15**	2**3**541	**125**34	134**25**	234**15**
1**432**5	**432**15	23**45**1	12**453**	**341**25	2**413**5
143**52**	432**51**	234**15**	124**35**	341**52**	241**53**
1**543**2	4**532**1	23**154**	**412**35	34**51**2	24**51**3
1**45**32	**54**321	231**45**	412**53**	3**145**2	2**145**3
1**3**542	**435**21	2**143**5	4**512**3	314**25**	214**35**
13**45**2	**34**521	214**53**	41**52**3	31**52**4	21**53**4
134**25**	3**54**21	21**54**3	**14**523	3**51**24	2**51**34
13**254**	3**425**1	21**354**	14**25**3	3**125**4	2**135**4
132**45**	342**15**	213**45**	142**35**	312**45**	213**45**

Table I.8 : Classes de permutations évitant les motifs de T avec leur fonctions de succession $\chi_T(i,k)$. On donne aussi les différents lemmes utilisés pour la preuve du code de Gray. On compare aussi les nombres maximum des différences de position entre deux permutations consécutives dans [67] et dans notre code de Gray **B**. Un point dans la dernière colonne signifie qu'il n'y a pas de code de Gray connu pour cette classe dans la littérature.

Classes	T	$\chi_T(i,k)$	Lemmes	[67], **B**
2^{n-1} [11]	$\{321, 312\}$	2	$I.34, I.35, I.37$	$3, 2$
Catalan [176]	$\{312\}$	$i+1$	$I.34, I.35, I.36, I.38, I.40$	$5, 3$
	$\{321\}$	$k+1$ si $i=1$ i sinon	$I.34, I.35, I.36, I.38, I.39, I.40$	$5, 3$
Indice pair Fibonacci [11]	$\{321, 3412\}$	$k+1$ si $i=1$ 2 sinon	$I.34, I.35, I.37, I.38, I.40$	$5, 3$
	$\{321, 4123\}$	3 si $i=1$ i sinon	$I.34, I.35, I.36, I.37, I.38, I.40$	$5, 3$
	$\{321, 4321\}$	3 si $i=3$ & $k=3$ $i+1$ sinon	$I.34, I.35, I.36, I.38, I.40$	$\cdot, 3$
Pell [11]	$\{321, 3412, 4123\}$	3 si $i=1$ 2 sinon	$I.34, I.35, I.37, I.38, I.40$	$5, 3$
	$\{312, 4321, 3421\}$	3 si $i=2$ 2 sinon	$I.34, I.35, I.37, I.38, I.40$	$\cdot, 3$
Schröder [85]	$\{4321, 4312\}$	$k+1$ si $i=1$ ou $i=2$ i sinon	$I.34, I.35, I.36, I.38, I.40$	$5, 3$
	$\{4231, 4132\}$	$k+1$ si $i=1$ ou $i=k$ $i+1$ sinon	$I.34, I.35, I.36, I.38, I.40$	$5, 3$
	$\{4123, 4213\}$	$k+1$ si $k-1 \leqslant i \leqslant k$ $i+2$ sinon	$I.34, I.35, I.36, I.38, I.40$	$5, 3$
Coefficient binomial central [85]	$\{4321, 4231, 4312, 4132\}$	$k+1$ si $i=1$ 3 si $i=2$ i sinon	$I.34, I.35, I.36, I.38, I.40$	$5, 3$
	$\{4231, 4132, 4213, 4123\}$	3 si $i=1$ $i+1$ sinon	$I.34, I.35, I.36, I.38, I.40$	$5, 3$
Motifs généralisés [11, 54]	$\{321, (p+1)12\ldots p\}$	$k+1$ si $i=1$ & $k<p$ p si $i=1$ & $k=p$ i sinon	$I.34, I.35, I.36, I.38, I.39, I.40$	$5, 3$
	$\{321, 3412, (p+1)12\ldots p\}$	$k+1$ si $i=1$ & $k<p$ p si $i=1$ & $k=p$ 2 sinon	$I.34, I.35, I.37, I.38, I.40$	$5, 3$
Motzkin généralisé [10]	pour $p \geqslant 2$, $\{321, (p+2)\bar{1}(p+3)2\ldots(p+1)\}$	$k+1$ si $i=1$ i si $2 \leqslant i \leqslant p$ $i-1$ sinon	$I.34, I.35, I.36, I.38, I.39, I.40$	$\cdot, 3$
Schröder généralisé [102]	$\cup_{\tau \in S_{n-1}}\{(p+1)\tau p\}$	$k+1$ si $k \leqslant p$ ou $p+i \geqslant k+1$ $i+p-1$ sinon	$I.34, I.35, I.36, I.38, I.40$	$5, 3$

Chapitre II

Etude de structures énumérées par les nombres de Catalan

II.1 Introduction

Un grand nombre de classes d'objets combinatoires sont énumérées par les nombres de Catalan. C'est par exemple le cas, parmi d'autres, des suites de ballot, des arbres binaires, mots ou chemins de Dyck, des tableaux de Young, des permutations triables avec une pile, des bons parenthésages, des permutations évitant un motif de longueur trois, triangulations,... Une liste de plus de soixante sortes de telles classes combinatoires sont répertoriées par Stanley dans [164].

D'un point de vue de la théorie des ordres, il existe des ordres partiels sur les ensembles de Catalan qui induisent une structure de treillis. Le plus connu est le treillis de Tamari qui peut être obtenu de façon équivalente de trois façons différentes. En effet, les recouvrements correspondent aux transformations élémentaires dans trois classes de Catalan : une 'reparenthèsisation' $(x(yz)) \to ((xy)z)$ [79], une rotation sur les arbres binaires [132, 161], et une bascule (flip en anglais) d'une diagonale dans les triangulations [88, 161]. Voir aussi la référence [127]. Ensuite il y a le treillis des partitions non-croisées [103] : le treillis de Kreweras. Deux partitions π et π' vérifient $\pi \leqslant \pi'$ si chaque bloc de π est un sous-ensemble d'un bloc de π'. Voir [68, 69, 137, 158] et de nombreuses autres références dans le survey [159]. Il y a aussi le treillis de Stanley qui opère sur les chemins de Dyck [164]. Plus récemment, un lien entre les mots de Dyck, les partitions non-croisées et les permutations évitant le motif 321 permet de définir une structure de treillis distributif sur ces ensembles de Catalan [8, 75].

Dans la deuxième partie de ce chapitre, on présente deux nouvelles structures en treillis sur des ensembles énumérés par les nombres de Catalan. La première consiste en un treillis sur les mots de Dyck, ou de façon équivalente sur les parenthésages bien formés ; le treillis "phagocyte" opère une mutation sur les parenthèsages de la manière suivante : un sous-parenthésage $w((\cdots(())\cdots))$ est transformé en $((\cdots((w))\cdots))$. La seconde exhibe le treillis 'taille-greffe' (en anglais, pruning-grafting) sur les arbres binaires dont la trans-

formation élémentaire consiste à *tailler une cerise pour ensuite la greffer sur la branche immédiatement à gauche.*

Dans la troisième partie de ce chapitre, on étudie la distance de rotation entre deux arbres binaires. Le problème de déterminer en temps polynomial la taille du plus petit chemin entre deux arbres binaires situés dans le treillis de Tamari reste encore ouvert. On fournit ici un algorithme d'approximation de cette distance en temps polynomial.

II.2 Treillis de Catalan

II.2.1 Le treillis 'Phagocyte'

Soit D l'ensemble des mots de Dyck sur l'alphabet $\{(,)\}$. Il s'agit du langage généré par la grammaire $S \to \lambda|(S)|SS$, i.e. l'ensemble des mots constitués des bons parenthèsages. Notons D_n l'ensemble des mots de Dyck avec n parenthèses ouvrantes et n fermantes. La cardinalité de D_n est le *n-ième* nombre de Catalan : $card(D_n) = (2n)!/(n!(n+1)!)$. Les parenthèses ouvrantes (resp. fermantes) de $w \in D_n$ sont numérotées de 1 à n en parcourant w de la gauche vers la droite.
Une partition $b_1/b_2/\cdots/b_k$ de $[1,n] = \{1,2,\cdots,n\}$ en blocs b_i est dite non-croisée s'il n'existe pas quatre nombres $p < q < r < s$ tels que $p, r \in b_i$ et $q, s \in b_j$ avec $i \neq j$. Donc $146/23/5$ est une partition non-croisée de $[1,6]$ (*ncp* en abrégé) alors que $135/2/46$ est croisée. Notons NC_n l'ensemble de tous les *ncp* de $[1,n]$. Nous avons $card(NC_n) = card(D_n)$. Une *ncp* sera écrite en listant les éléments dans chaque bloc en ordre croissant, et les blocs en ordre coissant de leur minima. Il y a différentes façons de représenter une *ncp*. Par exemple, dans une représentation linéaire, $[1..n]$ apparaît classiquement sur une droite réelle et des éléments successifs dans un même bloc sont joints par un arc dans le premier quadrant. Dans cette section, on utilisera la représentation canonique : soit une *ncp* $\pi \in NC_n$, on définit la relation R_π sur $[1..n]$ par $iR_\pi j$ si et seulement si i est le plus petit élément du bloc de π qui contient j ($i \leqslant j$). La représentation canonique de π est obtenue en traçant le graphe orienté de R_π sur le premier quadrant. Dans ces deux représentations, la propriété 'non-croisée' de la partition correspond au fait que les arcs ne s'intersectent jamais.
Puisque $card(D_n) = card(NC_n)$, il y a une bijection explicite entre D_n et NC_n [81, 82, 158, 159]. Soit une *ncp* $\pi \in NC_n$ et sa représentation canonique ; pour chaque élément visité $i \in [1,n]$ (pour i allant de 1 à n), on écrit r parenthèses ouvrantes ($r \geqslant 0$) si il y a r arcs commençant en i, puis on écrit une parenthèse fermante puisqu'il y a toujours exactement un seul arc finissant en i. La figure $II.1$ illustre cette bijection.

$$\begin{aligned} w &= (((()(()))(()())()) \\ \pi_w &= 149/23/57/6/8 \\ c_w &= (1,2,2,1,5,6,5,8,1) \end{aligned}$$

On note par π_w la *ncp* correspondante au mot de Dyck w. Remarquons alors que le nombre $nbl(\pi_w)$ de blocs de π_w est égal au nombre d'occurrences de "()" dans w. Soit $w \in D_n$, on définit aussi le vecteur c_w ou $c_w[i]$ est le plus petit élément du bloc de π_w qui contient i ($1 \leqslant i \leqslant n$).

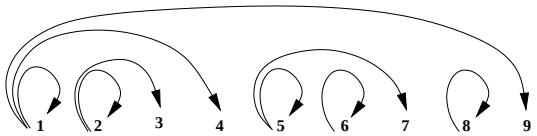


Figure II.1 : La représentation canonique du mot de Dyck $w \in D_9$

Dans la suite, la relation d'inclusion stricte (respectivement large) sera notée par le symbole $\subset$ (respectivement $\subseteq$).

Définition II.1. *La transformation 'phagocyte'* $\longrightarrow$ *sur* D_n *est définie par* $w \longrightarrow w'$ *si on a :* $w = u'v(^n)^n u''$ *et* $w' = u'(^n v)^n u''$*, où* v *est un mot de Dyck non-vide,* $n \geqslant 1$ *et* $u', u'' \in \{(,)\}^*$*. On note* $\stackrel{*}{\longrightarrow}$ *la cloture réflexive et transitive de* $\longrightarrow$*.*

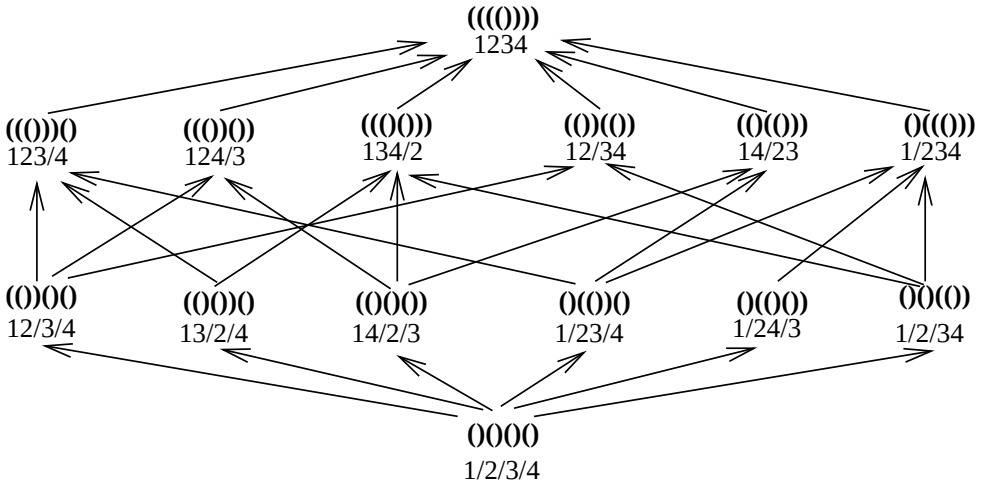


Figure II.2 : Le treillis phagocyte D_4.

Lemme II.2. *Pour tout* $w, w' \in D_n$*, on a* $w \longrightarrow w'$ *si et seulement si* π_w *et* $\pi_{w'}$ *vérifient :*
- *un bloc de* $\pi_{w'}$ *est l'union de deux blocs* b_1 *et* b_2 *de* π_w *où* $\min(b_1) < \min(b_2)$ *et* $b_2 = [\min(b_2), max(b_2)]$ *est un intervalle,*
- *chaque bloc de* π_w *différent de* b_1 *et* b_2 *est aussi un bloc de* $\pi_{w'}$*.*

Proposition II.3. $(D_n, \stackrel{*}{\longrightarrow})$ *est un ensemble partiellement ordonné avec un zéro* $\mathbf{0} = ()()\ldots()$ *et un* $\mathbf{1} = ((\ldots()\ldots))$ *; il est gradué par la fonction de rang* $r(w) = n - nbl(\pi_w)$*.*

Voir le diagramme de D_4 dans la figure $II.2$.

Théorème II.4. *Pour tout* $w, w' \in D_n$*, on a* $w \stackrel{*}{\longrightarrow} w'$ *si et seulement si les deux conditions suivantes sont vérifiées :*
- *(C1) chaque bloc de* π_w *est inclu dans un bloc de* $\pi_{w'}$*,*

– *(C2) pour tous blocs* $b_1, b_2 \in \pi_w$, $b' \in \pi_{w'}$ *tels que* $b_1 \cup b_2 \subset b'$ *et* $\min(b_1) < \min(b_2)$ *alors* $[\min(b_2), \max(b_2)] \subset b'$.

Remarque : La condition (C1) est l'ordre bien connu sur les partitions [103, 159]. La condition (C2) signifie que tout bloc $b' \in \pi_{w'}$ est fait d'un bloc $b_i \in \pi_w$ et d'un ou plusieurs intervalles disjoints $[k, l]$ pour lesquels un bloc $b_j \in \pi_w$ $(j > i)$ existe tel que $k = \min(b_j)$ et $l = \max(b_j)$.

◇

$$w = (()(())(())((()))))$$

$$w_1 = (()((((())(()))))))$$

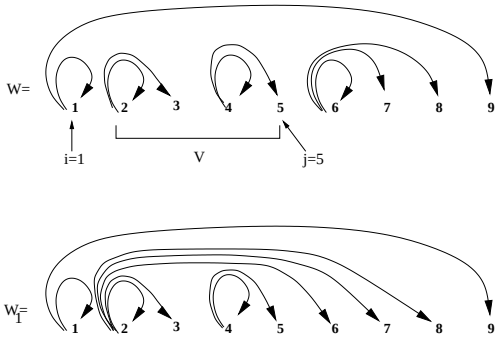


FIGURE II.3 : Une transformation 'phagocyte' $w \longrightarrow w_1$; $\pi_w = 19/23/45/678$ et $\pi_{w_1} = 19/23678/45$.

Théorème II.5. *Pour tout* n*, l'ensemble partiellement ordonné* $(D_n, \xrightarrow{*})$ *est un treillis.*

Dans la proposition suivante, on énumère les éléments max-irréductibles et min-irréductibles de D_n. Rappelons que $x \in D_n$ est max-(resp. min)-irréductible si $x = a \vee b$ (resp. $x = a \wedge b$) implique $x = a$ ou $x = b$. En d'autres termes, les éléments max-(resp. min)-irréductibles sont ceux qui ont un unique prédécesseur (resp. successeur).

Proposition II.6. *Le nombre d'éléments max-irréductibles dans le treillis 'phagocyte'* D_n *est* $n(n-1)/2$.

Proposition II.7. *Le nombre d'éléments min-irréductibles dans le treillis 'phagocyte'* D_n *est* $2^{n-1} - 1$.

En utilisant le théorème $II.4$, on donne ici deux algorithmes pour calculer le min $w \wedge w'$ et le max $w \vee w'$ de deux mots de Dyck $w, w' \in D_n$.

Algorithm min :
for $i := 1$ **to** n **do** $c_{w \wedge w'}[i] := 0$ **enddo**
for $i := 1$ **to** n **do**

```
    if c_{w∧w'}[i] = 0 then
      if c_w[i] < i or c_{w'}[i] < i then
        j_0 := i ;
        for j := i to n do
          if c_{w∧w'}[j] = 0 and c_w[j] = c_w[i] and c_{w'}[j] = c_{w'}[i] then
            c_{w∧w'}[j] := j_0 ;
          else j_0 := j ; endif
        enddo
      else
        for j := i to n do
          if c_w[j] = i and c_{w'}[j] = i then
            c_{w∧w'}[j] := i ; endif
        enddo
      endif
    endif
enddo
```

La complexité en espace est $O(n)$. La complexité en temps est $O(n^2)$.

On peut aussi définir un algorithme récursif pour calculer le max $\pi \sqcup \pi'$ de deux *ncp* π et π' dans le treillis classique des partitions non-croisées. Donc on suppose que $c_{w \sqcup w'}$ est déjà calculé pour tout $w, w' \in D_n$. Supposons que c_w et $c_{w'}$ sont déterminés. Soit d_w (respectivement $d_{w'}$) le vecteur défini par $d_w[i]$ (respectivement $d_{w'}[i]$) comme étant le plus grand élément du bloc de π_w (respectivement $\pi_{w'}$) qui contient i.

Algorithm max.

```
for i := 1 to n do
    if c_{w∨w'}[i] = 0 then
      for j := i to n do
        if c_{w⊔w'}[j] = c_{w⊔w'}[i] then
          c_{w∨w'}[j] := c_{w⊔w'}[i] ;
          if c_w[j] ≠ c_w[i] then
            for k := j + 1 to d_w[j] do
              c_{w∨w'}[k] := c_{w⊔w'}[i] ; enddo
          endif
          if c_{w'}[j] ≠ c_{w'}[i] then
            for k := j + 1 to d_{w'}[j] do
              c_{w∨w'}[k] := c_{w⊔w'}[i] ; enddo
          endif
        endif
      enddo
    endif
enddo
```

Par exemple, si $\pi_w = 149/23/57/6/8$ et $\pi_{w'} = 147/23/56/89$, les algorithmes donnent $\pi_{w \wedge w'} = 14/23/5/6/7/8/9$ et $\pi_{w \vee w'} = 1456789/23$.

En conclusion, nous avons défini une nouvelle structure sur les mots de Dyck (parenthésages).
La transformation $v() \to (v)$ définie dans [81] est un cas particulier de la transformation 'phagocyte'. Cependant, l'ensemble partiellement ordonné obtenu par cette transformation n'est pas un treillis. Il est toutefois semi-modulaire ce qui permet à Pallo et Germain de calculer la distance du plus court chemin [117].
Il reste à examiner plusieurs problèmes.
Est-ce qu'il existe un algorithme pour calculer la fonction de Möbius du treillis 'phagocyte' comme dans [131] ?
Des calculs expérimentaux montrent que le nombre de recouvrements $cov(n)$ de D_n est égal à $3(2n+2)!/((n+1)!(n+4)!)$ pour de petites valeurs de n ($n \leqslant 6$). Nous obtenons les premiers termes de la suite **A003517**$=[1, 6, 27, 110, \cdots]$ trouvée dans *Sloane Encyclopedia* [162]. Peut-on prouver que c'est bien cette suite ? Est-ce qu'il existe un algorithme en temps polynomial pour calculer la distance minimale entre deux mots de Dyck dans le treillis 'phagocyte' [117] ? Si oui, une nouvelle distance du plus court chemin pourrait être obtenue et ajoutée aux distances déjà existantes sur les ensembles de Catalan [40, 81, 130, 124, 132, 128, 161].

II.2.2 La transformation de taille-greffe

Dans ce paragraphe, on définit une nouvelle transformation sur les arbres binaires qui peut être caractérisée à l'aide des suites de distribution. Cela induit une nouvelle structure de treillis sur B_n.

Un *arbre binaire* avec une racine, ordonné, est un arbre non étiqueté dans lequel tous les noeuds sont soit une feuille $\square$ (i.e. un noeud sans fils) soit un noeud interne $\bigcirc$ ayant deux fils. On note par B_n l'ensemble des arbres binaires avec n noeuds internes (donc avec $n+1$ feuilles). Il est bien connu que $\mathrm{card}(B_n) = |B_n|$ est égal au nombre de Catalan $\binom{2n}{n}/(n+1)$. L'ensemble B des arbres binaires est récursivement défini par $B = \square + \bigcirc BB$ en notation polonaise. Il est bien connu qu'une suite de n cercles et $(n+1)$ carrés correspond à la notation polonaise d'un arbre de B_n si et seulement si dans tout préfixe propre de la suite, le nombre de cercles est plus grand ou égal au nombre de carrés, et le nombre total de carrés de la suite est exactement le nombre total de cercles plus un. Soit T_L et T_R les sous-arbres gauche et droit de $T \in B_n$. Alors on peut écrire $T = \bigcirc T_L T_R$ et on définit l'abre miroir $\bar{T} = \bigcirc \bar{T}_R \bar{T}_L$. Les feuilles de l'arbre T sont numérotées en préordre transversal (i.e. de la gauche vers la droite). La *suite des profondeurs* de T est la suite d'entiers $\ell_T = (\ell_T(1), \ell_T(2), \ldots, \ell_T(n+1))$ où $\ell_T(i)$ est le *niveau* de la i-ième feuille, i.e. la longueur de l'unique chemin entre la i-ième feuille et la racine [147]. Soit aussi $\bar{\ell}$ la suite miroir $\bar{\ell} = (\ell_T(n+1), \ell_T(n), \ldots, \ell_T(1))$ laquelle est aussi la suite des profondeurs de l'arbre miroir $\bar{T}$ récursivement défini par $\bar{T} = \bigcirc \bar{T}_R \bar{T}_L$ et $\bar{\square} = \square$. Par exemple, l'arbre défini en notation polonaise par $\bigcirc\square\bigcirc\bigcirc\square\bigcirc\square\bigcirc\square\square\bigcirc\bigcirc\square\square\square$ a la suite des profondeurs $(1, 3, 4, 5, 5, 4, 4, 3)$. Son miroir est $\bigcirc\bigcirc\bigcirc\square\bigcirc\square\square\bigcirc\bigcirc\bigcirc\square\square\square\square\square$ et a la suite des profondeurs $(3, 4, 4, 5, 5, 4, 3, 1)$. Maintenant considérons une suite $(a(1), a(2), \ldots, a(n+1))$ avec un entier $k \in [2, n+1]$ tel que $a(k-1) = a(k) = q$. Le processus qui consiste à remplacer la paire $(a(k-1), a(k)) = (q, q)$ par $q-1$ dans le but d'obtenir une nouvelle suite $(a(1), a(2), \ldots, a(k-2), q-1, a(k+1), \ldots, a(n+1))$ est appelé une *réduction*. Ruskey et Hu [147] donnent la condition nécessaire et suffisante pour qu'une suite de taille $(n+1)$

représente un arbre binaire de B_n. Une suite $(a(1), a(2), \ldots, a(n+1))$ est une suite des profondeurs si et seulement si une série de n réductions de la gauche vers la droite produit la suite 0.

De plus, la suite des profondeurs d'un arbre binaire obéit à ce qu'on appelle l'inégalité de Kraft ([80], p. 45). Une condition nécessaire [99, p. 404, ex. 3] pour une suite $(a(1), a(2), \ldots, a(n+1))$ pour être une suite des profondeurs est que :

$$\sum_{i=1}^{n+1} 2^{-a(i)} = 1.$$

Lemme II.8. *[99, p. 404, ex. 3] Soit $\ell_T = (\ell_T(1), \ell_T(2), \ldots, \ell_T(n+1))$ une suite des profondeurs d'un arbre binaire T et soit T' un sous arbre de T tel que la racine de T' est au niveau k dans T. Si $m_1 < m_2$ et $(\ell_T(m_1), \ell_T(m_1+1), \ldots, \ell_T(m_2-1), \ell_T(m_2))$ est la sous-suite de ℓ_T correspondant aux feuilles de T' alors*

$$\sum_{i=m_1}^{m_2} 2^{-\ell_T(i)} = 2^{-k}.$$

La suite $2^{-\ell_T} = (2^{-\ell_T(1)}, 2^{-\ell_T(2)}, \ldots, 2^{-\ell_T(n+1)})$ est appelée la *suite de densité* de l'arbre binaire (la somme de ses valeurs est un). Alors on lui associe sa *suite de distribution* comme la suite croissante :

$$L_T = \quad (2^{-\ell_T(1)}, \quad 2^{-\ell_T(1)} + 2^{-\ell_T(2)}, \quad \ldots, \quad \sum_{j=1}^{i} 2^{-\ell_T(j)}, \quad \ldots, \quad 1).$$

Dans la suite de ce paragraphe, les suites des profondeurs sont notées en minuscules (ℓ par exemple) et les suites de distribution en majuscules (L). On suppose que $\leqslant$ est l'ordre habituel pour deux suites de tailles n, i.e. si $\ell = (\ell(1), \ldots, \ell(n+1))$ et $\ell' = (\ell'(1), \ldots, \ell'(n+1))$, on dit que $\ell \leqslant \ell'$ si $\ell(i) \leqslant \ell'(i)$ pour tout $i \in [1, n+1]$.

Définition II.9. *La transformation de taille-greffe $\to$ sur B_n avec $n \geqslant 2$ est définie par le recouvrement $T \to T'$ s'il existe $k \geqslant 1$, $\tau_1 \in \bigcirc\{\bigcirc, \square\}^*$ et $\tau_2 \in \{\bigcirc, \square\}^*$ tel que $T = \tau_1 \square \bigcirc^k \square\square\tau_2$ et $T' = \tau_1 \bigcirc \square\square \bigcirc^{k-1} \square\tau_2$. Soit $\overset{*}{\to}$ la cloture réflexive et transitive de $\to$ sur B_n.*

La transformation de taille-greffe "taille" en remplaçant une 'petite branche' $T_1 = \bigcirc\square\square$ de T par une feuille et greffe cette petite branche $T_1 = \bigcirc\square\square$ à la place de la feuille $\square$ située juste avant T_1 dans la notation polonaise de T. Voir la figure II.4 par exemple.

Proposition II.10. *Soient $T, T' \in B_n$ et ℓ_T, $\ell_{T'}$ leurs suites des profondeurs. Alors on a $T \to T'$ si et seulement s'il existe $p \geqslant 1$, $q \geqslant 2$ et $2 \leqslant i \leqslant n$ tels que*

$$\begin{array}{llllll} \ell_T & = (\ell_T(1), \ell_T(2), \ldots, \ell_T(i-2) & ,p & ,q & ,q & ,\ell_T(i+2), \ldots, \ell_T(n+1)) \text{ et} \\ \ell_{T'} & = (\ell_T(1), \ell_T(2), \ldots, \ell_T(i-2) & ,p+1 & ,p+1 & ,q-1 & ,\ell_T(i+2), \ldots, \ell_T(n+1)). \end{array}$$

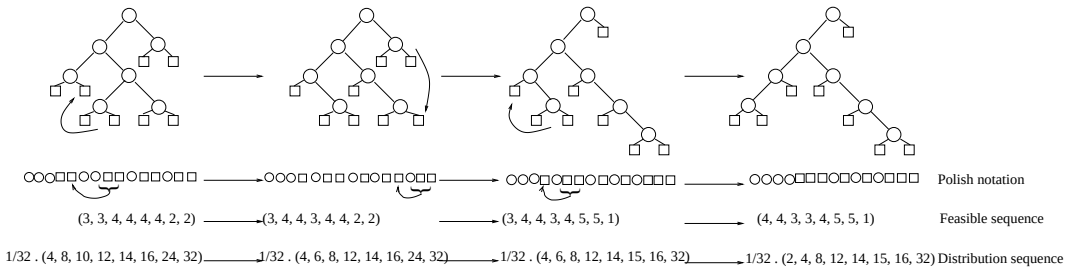


Figure II.4 : Trois transformations de taille-greffe dans B_7.

Corollaire II.11. *Soient $T, T' \in B_n$ et L_T, $L_{T'}$ leurs suites de distribution. Alors on a $T \to T'$ si et seulement s'il existe $p \geqslant 1$, $q \geqslant 2$ et $2 \leqslant i \leqslant n$ tels que $L_T = (L_T(1), \ldots, L_T(i-2), L_T(i-2) + 2^{-p}, L_T(i-2) + 2^{-p} + 2^{-q}, L_T(i+1), \ldots, 1)$ et $L_{T'} = (L_T(1), \ldots, L_T(i-2), L_T(i-2) + 2^{-(p+1)}, L_T(i-2) + 2^{-p}, L_T(i+1), \ldots, 1)$.*

Définition II.12. *Soient T et T' deux arbres différents dans B_n. Soit $\sigma_1 \in \bigcirc\{\bigcirc, \square\}^*$ le plus long préfixe commun de T et T' dans la notation polonaise. Supposons que*

$$T = \sigma_1 \sigma_2 \square \bigcirc^j \underbrace{\bigcirc \square \square}_{\tau} \sigma_3$$

ou

- *$j \geqslant 0$, et*
- *σ_2 est le mot vide ou $\sigma_2 \in \square\{\bigcirc, \square\}^*$ et $\sigma_2\square$ ne contient pas d'occurence de $\tau = \bigcirc\square\square$ (i.e. $\tau = \bigcirc\square\square$ est l'occurence la plus à gauche de $\bigcirc\square\square$ dans T après σ_1), et*
- *$\sigma_3 \in \{\bigcirc, \square\}^*$.*

Alors on a nécessairement $T' = \sigma_1\sigma_2'$ avec $\sigma_2' \in \bigcirc\{\bigcirc, \square\}^*$.
On définit l'arbre $U(T, T') \in B_n$ (U pour UP) tel que $T \to U(T, T')$ et

$$U(T, T') = \sigma_1 \sigma_2 \underbrace{\bigcirc \square \square}_{\tau} \bigcirc^j \square \sigma_3.$$

De plus, on pose $U(T, T) = T$.

Par exemple, si :

$$\begin{aligned} T &= \bigcirc\bigcirc\square\square\bigcirc\square\bigcirc\bigcirc\square\bigcirc\bigcirc\square\square\square\square \\ T' &= \bigcirc\bigcirc\square\square\bigcirc\bigcirc\square\square\bigcirc\square\bigcirc\bigcirc\square\square\square \end{aligned}$$

alors $\sigma_1 = \bigcirc\bigcirc\square\square\bigcirc$ $\sigma_2 = \square\bigcirc\bigcirc$, $\sigma_3 = \square\square$ et $j = 1$. On obtient $U(T, T') = \bigcirc\bigcirc\square\square\bigcirc\square\bigcirc\bigcirc\bigcirc\square\square\bigcirc\square\square\square$.

Lemme II.13. *Soient T et T' deux arbres dans B_n tels que leurs suites de distribution L_T et $L_{T'}$ vérifient $L_T > L_{T'}$. Alors $U(T, T')$ existe et sa suite de distribution $L_{U(T,T')}$ vérifie $L_T > L_{U(T,T')} \geqslant L_{T'}$.*

Définition II.14. *Soient T et T' deux arbres différents dans B_n avec $n \geqslant 2$. Soit $\sigma_1 \in \bigcirc\{\bigcirc, \square\}^*$ le plus long préfixe commun en notation polonaise. Supposons que $T = \sigma_1 \square \sigma_2$ avec $\sigma_2 \in \{\bigcirc, \square\}^*$. Alors on a nécessairement*

$$T' = \sigma_1 \sigma_2' \underbrace{\bigcirc\square\square}_{\tau} \bigcirc^j \square \sigma_3'$$

où

- *$j \geqslant 0$, et*
- *σ_2' est le mot vide ou $\sigma_2' \in \bigcirc\{\bigcirc, \square\}^*$ tel que σ_2' ne contient pas d'occurrence de $\tau = \bigcirc\square\square$ (i.e. τ est l'occurence de $\bigcirc\square\square$ la plus à gauche dans T' après σ_1), et*
- *$\sigma_3' \in \{\bigcirc, \square\}^*$.*

On définit l'arbre $D(T, T') \in B_n$ (D pour DOWN) tel que $D(T, T') \to T'$ et

$$D(T, T') = \sigma_1 \sigma_2' \square \bigcirc^j \underbrace{\bigcirc\square\square}_{\tau} \sigma_3'.$$

De plus, on pose $D(T, T) = T$.

Par exemple, si :

$$\begin{array}{ll} T & = \bigcirc\bigcirc\square\square\bigcirc\square\bigcirc\square\bigcirc\bigcirc\square\bigcirc\square\square\square \\ T' & = \bigcirc\bigcirc\square\square\bigcirc\bigcirc\square\bigcirc\square\square\bigcirc\square\bigcirc\square\square \end{array}$$

alors $\sigma_1 = \bigcirc\bigcirc\square\square\bigcirc$, $\sigma_2' = \bigcirc\square$, $\sigma_3' = \bigcirc\square\square$ et $j = 1$. On obtient $D(T, T') = \bigcirc\bigcirc\square\square\bigcirc\bigcirc\square\square\bigcirc\bigcirc\square\square\bigcirc\square\square$.

Lemme II.15. *Soient T et T' deux arbres dans B_n tels que leurs suites de distribution L_T et $L_{T'}$ vérifient $L_T > L_{T'}$. Alors $D(T, T')$ existe et sa suite de distribution $L_{D(T,T')}$ vérifie $L_T \geqslant L_{D(T,T')} > L_{T'}$.*

Théorème II.16. *Soient T et $T' \in B_n$, L_T et $L_{T'}$ leur suites de distribution. Alors on a $T \xrightarrow{*} T'$ si et seulement si $L_T \geqslant L_{T'}$.*

Théorème II.17. *Pour tout n, l'ensemble partiellement ordonné $(B_n, \xrightarrow{*})$ est un treillis.*

Notons qu'il existe des relations entre les treillis de Catalan déjà connus. En effet, rappelons que si T et T' sont deux éléments dans le treillis de Kreweras (partition non-croisées) tels que $T \leqslant T'$, alors on a $T \leqslant T'$ dans le treillis de Tamari. Si $T \leqslant T'$ dans le treillis de Tamari, alors $T \leqslant T'$ est aussi vrai dans le treillis de Stanley. Si $T \leqslant T'$ dans le treillis 'phagocyte' défini dans la section précédente [21], alors $T \leqslant T'$ est aussi vrai dans le treillis de Kreweras. L'ensemble des recouvrements du treillis 'phagocyte' est inclu dans l'ensemble des recouvrements du treillis de Kreweras. Cependant, le treillis 'taille-greffe' ne semble pas avoir des relations similaires avec d'autres treillis. Voir les figures II.5 et II.6.

Le treillis $(B_n, \xrightarrow{*})$ possède un plus grand élément $\mathbf{1}$ et un plus petit élément $\mathbf{0}$ et leur suites des profondeurs sont respectivement $\ell_{\mathbf{1}} = (n, n, n-1, \ldots, 2, 1)$ et $\ell_{\mathbf{0}} = (1, 2, \ldots, n-1, n, n)$. $(B_n, \xrightarrow{*})$ est symétrique car $T \xrightarrow{*} T'$ si et seulement si $\bar{T}' \xrightarrow{*} \bar{T}$. Le treillis $(B_n, \xrightarrow{*})$ n'est pas modulaire puisqu'il contient des pentagones (voir les figures II.5 et II.6).

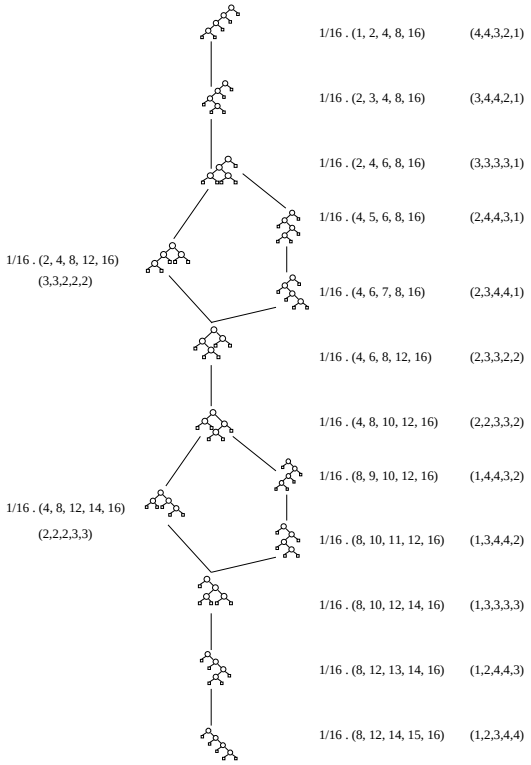


FIGURE II.5 : Le treillis 'taille-greffe' B_4. Chaque arbre est codé par ses suite de profondeurs et de distribution.

Proposition II.18. *Soient T et T' deux arbres dans B_n tels que $T \xrightarrow{*} T'$, alors*

(i) $\overline{U(\bar{T}, \bar{T'})} \to T'$,

(ii) $T \to \overline{D(\bar{T}, \bar{T}')}$,

(iii) le chemin $T' \leftarrow D(T,T') \leftarrow D(T, D(T,T')) \leftarrow D(T, D(T, D(T,T'))) \leftarrow \ldots \leftarrow T$ *est un plus long chemin entre T et T',*

(iv) le chemin $T \to U(T,T') \to U(U(T,T'),T') \to U(U(U(T,T'),T'),T') \to \ldots \to T'$ *est un plus court chemin entre T et T'.*

Corollaire II.19. *Soit $\mathcal{C}$ le chemin défini par :*

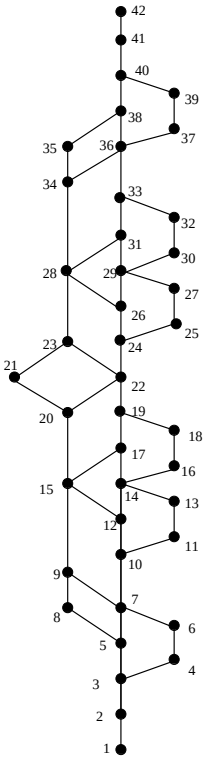
$$\mathbf{1} \leftarrow D(\mathbf{0},\mathbf{1}) \leftarrow D(\mathbf{0}, D(\mathbf{0},\mathbf{1})) \leftarrow D(\mathbf{0}, D(\mathbf{0}, D(\mathbf{0},\mathbf{1}))) \leftarrow \ldots \leftarrow \mathbf{0}.$$

T appartient à $\mathcal{C}$ si et seulement si T vérifie la condition (A) définie par : T a soit une unique occurence de $\bigcirc\square\square$, soit deux occurrences de $\bigcirc\square\square$ telles qu'il n'existe pas de feuilles entre elles en notation polonaise de T. De plus, si T et T' appartiennent au chemin $\mathcal{C}$ et vérifient $T \to T'$, alors T' est obtenu à partir de T par une transformation de taille-greffe de l'occurence $\bigcirc\square\square$ la plus à droite dans T.

Corollaire II.20. *Soit $\mathcal{C}'$ le chemin défini par :*

$$\mathbf{0} \to U(\mathbf{0},\mathbf{1}) \to U(U(\mathbf{0},\mathbf{1}),\mathbf{1}) \to U(U(U(\mathbf{0},\mathbf{1}),\mathbf{1}),\mathbf{1}) \to \ldots \to \mathbf{1}.$$

1.	(1, 2, 3, 4, 5, 5)	1/32.(16,24,28,31,31,32)
2.	(1, 2, 3, 5, 5, 4)	1/32.(16,24,28,29,30,32)
3.	(1, 2, 4, 4, 4, 4)	1/32.(16,24,26,28,30,32)
4.	(1, 2, 4, 5, 5, 3)	1/32.(16,24,26,27,28,32)
5.	(1, 3, 3, 3, 4, 4)	1/32.(16,20,24,28,30,32)
6.	(1, 2, 5, 5, 4, 3)	1/32.(16,24,25,26,28,32)
7.	(1, 3, 3, 4, 4, 3)	1/32.(16,20,24,26,28,32)
8.	(2, 2, 2, 3, 4, 4)	1/32.(8,16,24,28,30,32)
9.	(2, 2, 2, 4, 4, 3)	1/32.(8,16,24,26,28,32)
10.	(1, 3, 4, 4, 3, 3)	1/32.(16,20,22,24,28,32)
11.	(1, 3, 4, 5, 5, 2)	1/32.(16,20,22,23,24,32)
12.	(1, 4, 4, 3, 3, 3)	1/32.(16,18,20,24,28,32)
13.	(1, 3, 5, 5, 4, 2)	1/32.(16,20,21,22,24,32)
14.	(1, 4, 4, 4, 4, 2)	1/32.(16,18,20,22,24,32)
15.	(2, 2, 3, 3, 3, 3)	1/32.(8,16,20,24,28,32)
16.	(1, 4, 5, 5, 3, 2)	1/32.(16,18,19,20,24,32)
17.	(2, 2, 3, 4, 4, 2)	1/32.(8,16,20,22,24,32)
18.	(1, 5, 5, 4, 3, 2)	1/32.(16,17,18,20,24,32)
19.	(2, 2, 4, 4, 3, 2)	1/32.(8,16,18,20,24,32)
20.	(2, 3, 3, 2, 3, 3)	1/32.(8,12,16,24,28,32)
21.	(3, 3, 2, 2, 3, 3)	1/32.(4,8,16,24,28,32)



22.	(2, 3, 3, 3, 3, 2)	1/32.(8,12,16,20,24,32)
23.	(3, 3, 2, 3, 3, 2)	1/32.(4,8,16,20,24,32)
24.	(2, 3, 4, 4, 2, 2)	1/32.(8,12,14,16,24,32)
25.	(2, 3, 4, 5, 5, 1)	1/32.(8,12,14,15,16,32)
26.	(2, 4, 4, 3, 2, 2)	1/32.(8,10,12,16,24,32)
27.	(2, 3, 5, 5, 4, 1)	1/32.(8,12,13,14,16,32)
28.	(3, 3, 3, 3, 2, 2)	1/32.(4,8,12,16,24,32)
29.	(2, 4, 4, 4, 4, 1)	1/32.(8,10,12,14,16,32)
30.	(2, 4, 5, 5, 3, 1)	1/32.(8,10,11,12,16,32)
31.	(3, 3, 3, 4, 4, 1)	1/32.(4,8,12,14,16,32)
32.	(2, 5, 5, 4, 3, 1)	1/32.(8,9,10,12,16,32)
33.	(3, 3, 4, 4, 3, 1)	1/32.(4,8,10,12,16,32)
34.	(3, 4, 4, 2, 2, 2)	1/32.(4,6,8,16,24,32)
35.	(4, 4, 3, 2, 2, 2)	1/32.(2,4,8,16,24,32)
36.	(3, 4, 4, 3, 3, 1)	1/32.(4,6,8,12,16,32)
37.	(3, 4, 5, 5, 2, 1)	1/32.(4,6,7,8,16,32)
38.	(4, 4, 3, 3, 3, 1)	1/32.(2,4,8,12,16,32)
39.	(3, 5, 5, 4, 2, 1)	1/32.(4,5,6,8,16,32)
40.	(4, 4, 4, 4, 2, 1)	1/32.(2,4,6,8,16,32)
41.	(4, 5, 5, 3, 2, 1)	1/32.(2,3,4,8,16,32)
42.	(5, 5, 4, 3, 2, 1)	1/32.(1,2,4,8,16,32)

FIGURE II.6 : Le treillis 'taille-greffe' B_5. Chaque arbre est codé par ses suites des profondeurs et de distribution.

Si $T \to T'$ est sur le chemin $\mathcal{C}'$ alors T' est obtenu à partir de T par la transformation de taille-greffe la plus à gauche dans T.

Remarque 1 Par le Corollaire II.19, le plus long chemin entre **0** et **1** a une longueur de $2^n - (n+1)$. Ce sont les nombres Euleriens donnés par la suite **A000295** dans [162]. Le plus court chemin entre **0** et **1** a une longueur de $(n-1)^2$.

Maintenant nous montrons que les éléments max-(resp. min) irréductibles dans B_n sont énumérés par les nombres Euleriens. Rappelons que $x \in B_n$ est un élément max-(resp. min)-irréductible si $x = a \vee b$ (resp. $x = a \wedge b$) implique $x = a$ ou $x = b$.

Théorème II.21. *Le nombre d'éléments max-(resp. min)-irréductibles dans le treillis 'taille-greffe' $(B_n, \overset{*}{\to})$ est donné par les nombres Euleriens $2^n - (n+1)$.*

Théorème II.22. *Le nombre $cov(n)$ de recouvrements dans B_n est égal à $\binom{2n}{n-1}$.*

Une nouvelle structure a été définie sur les arbres binaires via une transformation naturelle. La simplicité de la définition de la transformation de 'taille-greffe' est en contraste avec la difficulté de la preuve du théorème qui caractérise cette transformation.
Des problèmes doivent encore être résolus.
Existe-t-il un algorithme efficace et non-récursif pour calculer la fonction de Möbius du treillis 'taille-greffe' des arbres binaires comme dans [131] ? On conjecture que la fonction

de Möbius prend ses valeurs dans $\{-1, 0, +1\}$.
Est-ce qu'il existe un algorithme en temps polynomial calculant la longueur minimale d'un chemin entre deux arbres binaires dans le treillis [117] ? Si un tel algorithme existe, une nouvelle distance de chemin le plus court serait obtenue et pourrait être ajoutée dans le catalogue des distances déjà existantes sur les ensembles de Catalan [81, 130, 132, 128, 161].
Définissons la valuation v sur $(B_n, \xrightarrow{*})$ par : $v(T)$ est la longueur du plus long chemin entre T et $\mathbf{1}$. On conjecture que v vérifie que pour tout T et T', $v(T)+v(T') \leqslant v(T \vee T')+v(T \wedge T')$ et $T < T' \Longrightarrow v(T) > v(T')$. Si tel était le cas, alors $d(T, T') = v(T) + v(T') - 2v(T \vee T')$ serait une distance sur $(B_n, \xrightarrow{*})$ [32, 41].

Guttmann, Krattenthaler et Viennot [86] énumèrent les k-chaînes dans les treillis de Stanley. Kreweras [103] énumère les k-chaînes dans les treillis des partitions non-croisées. Plus généralement, Chapoton prouve dans [47] que la suite **A000260** de [162] énumère le nombre d'intervalles dans le treillis de Tamari, i.e. le nombre de paires (T, T') telles que $T \leqslant T'$: voir aussi [82, p. 27]. Bernardi et Bonichon [34] construisent des bijections entre l'ensemble des intervalles de ces treillis et certaines triangulations (realizers). On a obtenu expérimentalement le nombre d'intervalles pour les treillis taille-greffe de petites tailles :

$$1, 3, 15, 101, 818, 7486, 74648, 793005, 8843056, 102464586, ...$$

Cette suite n'apparaît pas dans [162]. Est-il possible d'obtenir la fonction génératrice de cette suite ?

II.3 Distance de rotation

Culik et Wood ont défini en 1982 la distance de rotation entre une paire d'arbres binaires comme le nombre minimum de rotations nécessaires pour transformer l'un en l'autre [59] (voir aussi la figure II.7). Il existe une bijection explicite bien connue entre les arbres binaires et les triangulations. Par conséquent, un système qui est isomorphe aux arbres binaires munis de la rotation l'est aussi pour les triangulations des polygones convexes munis de l'opération de bascule d'une diagonale (diagonal-flip en anglais). Une transformation qui bascule une diagonale est une opération qui convertit une triangulation en une autre en supprimant une diagonale dans la triangulation et en ajoutant la diagonale qui subdivise le quadrilatère restant dans le sens opposé (voir la figure II.8). Alors la distance de rotation des arbres binaires et la distance de bascule de diagonale des triangulations sont équivalentes.
Un problème ouvert est la complexité du calcul de la distance de rotation entre deux arbres. Lucas a présenté un algorithme en temps quadratique pour calculer la distance de rotation entre les arbres binaires d'une forme donnée en [109]. Mais dans le cas général, il reste le problème ouvert de savoir si les distances peuvent être calculées en temps polynomial.

Des bornes supérieures de ces distances ont été exhibées. Quelques auteurs approchent le problème en limitant la transformation élémentaire à une version plus restrictive de la rotation [40, 55, 56, 53, 110, 128, 146]. Evidemment ces distances de rotation restreintes majorent la distance de rotation usuelle. Une autre approche pour calculer une borne supérieure utilise l'opération de 'flexion' sur les arbres binaires [132].
Dans [130], une borne inférieure assez large de la distance de rotation est obtenue.

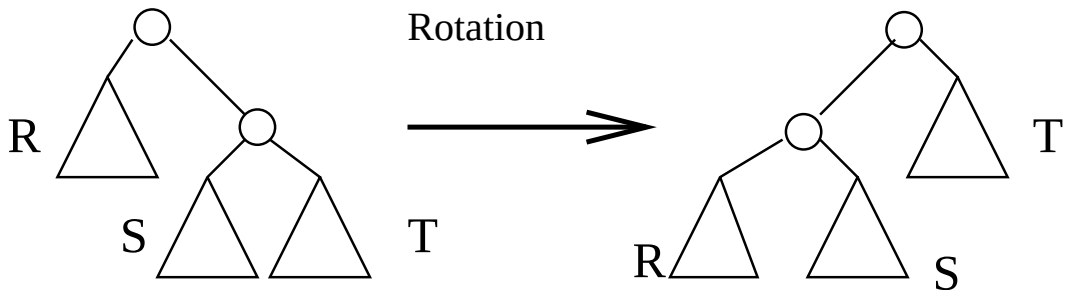


Figure II.7 : La rotation sur les arbres binaires.

Dans cette section, on présente un algorithme polynomial pour calculer des bornes inférieures et supérieures. Des calculs expérimentaux montrent que ces bornes sont assez efficaces. Dans la suite de cette section, on exposera nos résultats en terme de triangulation plutôt que d'arbres binaires.

Considérons un n-gon, i.e. un polygone convexe avec n cotés et avec un coté que l'on distingue en le mettant en haut horizontalement. On étiquète les sommets de 0 à $n-1$ dans le sens des aiguilles d'une montre tel que les deux sommets en haut ont 0 et $n-1$. Chaque triangulation d'un n-gone a $n-2$ triangles et $n-3$ diagonales qui ne s'intersectent pas. Soit $\mathcal{T}_n$ l'ensemble des triangulations d'un n-gon. Il y a une bijection explicite entre $\mathcal{T}_n$ et l'ensemble des arbres binaires avec $n-2$ noeuds internes (et donc $n-1$ feuilles) [161]. Voir la figure II.8. Si $T, S \in \mathcal{T}_n$, on note par $dist(T, S)$ le nombre minimum de bascules de diagonales nécessaires pour transformer T en S.

Le degré interne $d_i(T)$ d'un sommet i de $T \in \mathcal{T}_n$ est le nombre de diagonales incidentes à i ($0 \leq i \leq n-1$). Pour une paire (T, S) de triangulations de $\mathcal{T}_n$, le degré 'composé' $cd_i(T, S)$ est le nombre total de diagonales incidentes au sommet i pour les deux triangulations. Si $T, S \in \mathcal{T}_n$ ont une diagonale $\{i, j\}$ en commun, on dit que $\{i, j\}$ est une $(2, 2)$-diagonale si $cd_i(T, S) = cd_j(T, S) = 2$. Soient deux triangulations de $\mathcal{T}_n$, une diagonale basculable est une diagonale de la triangulation qui peut être basculée en une diagonale appartenant à l'autre triangulation [141]. Soient $T, S \in \mathcal{T}_n$, on définit le type d'un sommet i par $type_i(T, S) = (k : l)$ ou $k = d_i(T)$ et $l = d_i(S)$.

Définition II.23. *([161]) Soit $T \in \mathcal{T}_n$, on définit la triangulation normalisée $N_T(i, j)$ (respectivement $N'_T(i, j)$) en respectant la diagonale $\{i, j\}$ comme suit :*
(1) $N_T(i, j)$ et $N'_T(i, j)$ contiennent la diagonale $\{i, j\}$
(2) $N_T(i, j)$ et $N'_T(i, j)$ contiennent chaque diagonale de T qui ne croisent pas la diagonale $\{i, j\}$
(3) si T contient une diagonale $\{a, b\}$ qui croise la diagonale $\{i, j\}$, alors $N_T(i, j)$ (respectivement $N'_T(i, j)$) contient les diagonales $\{a, j\}$ et $\{b, j\}$ (respectivement $\{a, i\}$ et $\{b, i\}$).

Définition II.24. *Dans le cas où $j = i + 2 \mod n$, on dit que $N_T(i, i+2)$ (respectivement $N'_T(i, i+2)$) est la normalisation dans le sens contraire des aiguilles d'une montre (respectivement sens des aiguilles d'une montre) en respectant le sommet $i+1$ et on note $N_T(i, i+2) = N_T(i+1)$ (respectivement $N'_T(i, i+2) = N'_T(i+1)$).*

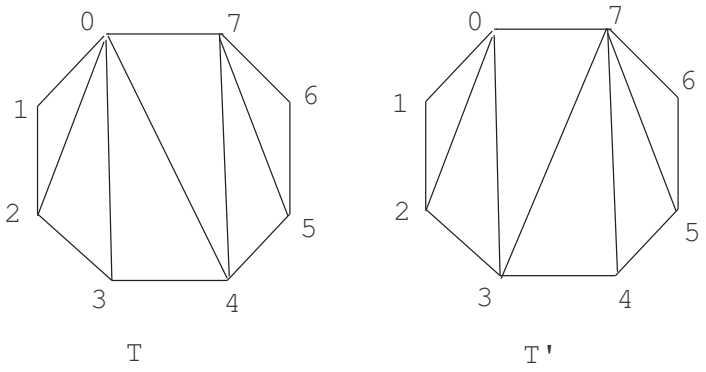


Figure II.8 : Une bascule de diagonale dans $\mathcal{T}_8$

Définition II.25. *Soit $\{i,j\}$ une diagonale de $T \in \mathcal{T}_n$ telle que $d_i(T) = d_j(T) = 2$. Alors T peut être doublement normalisée en respectant la diagonale $\{i,j\}$ pour créer une nouvelle triangulation $N''_T(i,j)$ comme suit :*
(1) on bascule dans T la diagonale $\{i,j\}$
(2) puis on bascule la diagonale adjacente au sommet i
(3) et on bascule la diagonale adjacente au sommet j.

Définition II.26. *Soit i un sommet de $T \in \mathcal{T}_n$ tel que $d_i(T) = 3$. On note $\{a,i\}$, $\{b,i\}$ et $\{c,i\}$ les trois diagonales incidentes au sommet i avec les sommets a,b,c dans le sens des aiguilles d'une montre. Alors T peut être pseudo-normalisée en respectant le sommet i pour créer une nouvelle triangulation $N'''_T(i)$ comme suit : on bascule les deux diagonales $\{a,i\}$ et $\{c,i\}$, et on bascule la diagonale du milieu $\{b,i\}$.*

Il est important de noter que chacune de ces normalisations crée au moins une diagonale de la forme $\{i-1, i+1\} \mod n$. Par la suite, les arêtes $\{i-1,i\}$ et $\{i,i+1\}$ seront 'grignotées' par l'algorithme exposé dans le paragraphe II.3.5.

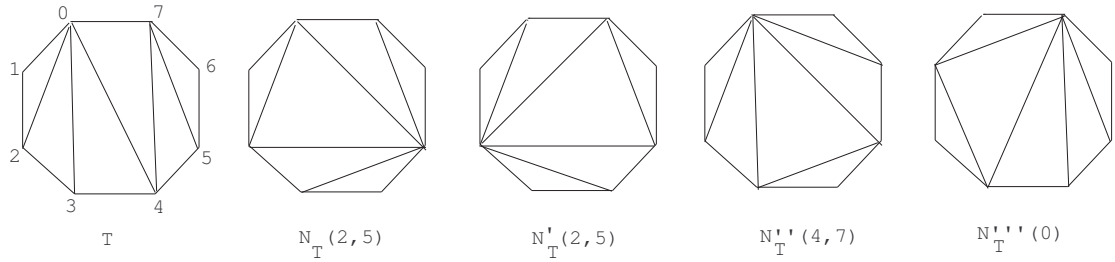


Figure II.9 : Quatre normalisations $T \in \mathcal{T}_6$

Lemme II.27. *([161]) Soient $T, S \in \mathcal{T}_n$; s'il est possible de basculer une diagonale de T pour créer T_1 tel que T_1 a une diagonale de plus en commun avec S qu'elle n'a avec T, alors il existe un plus court chemin de T à S dans lequel la première opération de bascule crée T_1.*

Lemme II.28. *([161]) Soit $T, S \in \mathcal{T}_n$, si T et S ont une diagonale en commun, alors un plus court chemin de T à S ne bascule jamais cette diagonale.*

Lemme II.29. *([161]) Si $T, S \in \mathcal{T}_n$ ont une diagonale commune, cette diagonale partage T (respectivement S) en deux sous-triangulations $T^{'}$ et $T^{''}$ (respectivement $S^{'}$ et $S^{''}$) de telles sortes que $T^{'}$ et $S^{'}$ (et donc $T^{''}$ et $S^{''}$) ont les mêmes sommets. Alors on a l'égalité : $dist(T, S) = dist(T^{'}, S^{'}) + dist(T^{''}, S^{''})$.*

Lemme II.30. *Si $T, S \in \mathcal{T}_n$, alors il existe un sommet i tel que $cd_i(T, S) \leq 3$.*

En vertu du lemme II.30, on étudie par la suite les différents types de sommets qui peuvent apparaître, i.e. $(2:0)$, $(1:1)$, $(3:0)$ et $(2:1)$. Le cas $(1:0)$ est un cas particulier du lemme II.27.
Maintenant on suppose que les hypothèses des lemmes II.27 et II.28 ne sont pas vérifiées.

II.3.1 Sommet de type (2 :0)

Lemme II.31. *Soient $T, S \in \mathcal{T}_n$ avec un sommet i tel que $type_i(T, S) = (2:0)$, il existe un plus court chemin de T à S dans lequel les deux premières bascules créent soit $N_T(i)$ ou soit $N_T^{'}(i)$.*

Théorème II.32. *Soient $T, S \in \mathcal{T}_n$ avec un sommet i tel que $type_i(T, S) = (2 : 0)$, alors on a :*
$dist(T, S) = min(dist(N_T(i), S), dist(N_T^{'}(i), S)) + 2$ *et*
$dist(T, S) \geq max(dist(N_T(i), S), dist(N_T^{'}(i), S)) + 1$.

Théorème II.33. *Soient $T, S \in \mathcal{T}_n$ avec une $(2, 2)$-diagonale $\{i, j\}$ telle que $d_T(i) = 2$, $d_S(i) = 0$, $d_T(j) = 2$ et $d_S(j) = 0$, on a :*
$dist(T, S) = dist(N_T^{''}(i, j), S) + 3$.

II.3.2 Sommet de type (1 :1)

Théorème II.34. *Soient $T, S \in \mathcal{T}_n$ avec un sommet i tel que $type_i(T, S) = (1:1)$ alors les inégalités suivantes sont vraies :*
$dist(N_T(i), N_S(i)) + 1 \leq dist(T, S) \leq dist(N_T(i), N_S(i)) + 2$.

II.3.3 Sommet de type (3 :0)

Théorème II.35. *Soient $T, S \in \mathcal{T}_n$ avec un sommet i tel que $type_i(T, S) = (3:0)$ alors les inégalités suivantes sont vraies :*
$dist(N_T^{'''}(i), S) + 1 \leq dist(T, S) \leq dist(N_T^{'''}(i), S) + 3$.

II.3.4 Sommet de type (2 :1)

Théorème II.36. *Soient $T, S \in \mathcal{T}_n$ avec un sommet i tel que $type_i(T,S) = (2:1)$ alors les inégalités suivantes sont vraies :*
$dist(T,S) \geq max(dist(N_T(i), N_S(i)), dist(N'_T(i), N'_S(i))) + 1$
$dist(T,S) \leq min(dist(N_T(i), N_S(i)), dist(N'_T(i), N'_S(i))) + 3.$

II.3.5 Algorithme pour le calcul de bornes inférieure et supérieure

L'objectif de l'algorithme suivant est de 'grignoter' les arêtes (et par conséquent les sommets) des deux triangulations en appliquant le lemme II.27 et les théorèmes II.32, II.33, II.34, II.35, II.36. Il est nécessaire de vérifier au départ s'il existe une diagonale pouvant être 'basculée' directement. Dans ce cas, on fait cette bascule immédiatement. Dans le cas contraire, l'existence des types (1 : 0), (2 : 0), (1 : 1), (3 : 0) et (2 : 1) (dans cet ordre) est alors vérifiée. En fait quelque soit l'ordre, l'algorithme est correct ; on choisit cependant cet ordre pour des raisons statistiques. Cela conduit statistiquement (et surtout expérimentalement) à un plus petit nombre de bascules et donc fournit de meilleures bornes de la distance de rotation.

Maintenant nous donnons un algorithme récursif (basé sur le lemme II.30) pour trouver des bornes inférieure et supérieure (dans notre algorithme *low* et *up*) pour la distance de rotation entre deux triangulations $T, S \in \mathcal{T}_n$. Rappelons que T, S et donc aussi $(i:j)$, $(j:i)$ jouent le même rôle dans notre algorithme.

Algorithme (Calcul de *low* et *up*)
Soient $T, S \in \mathcal{T}_n$

procedure low-up(T,S,n)
si $n \geqslant 3$ **alors**
si T et S ont une diagonale en commun **alors**
  $T := (T_1, T_2)$; $S := (S_1, S_2)$;
  $n_1 := size(T_1) = size(S_1)$; $n_2 := size(T_2) = size(S_2)$;
  low-up(T_1, S_1, n_1) ;
  low-up(T_2, S_2, n_2) ;
else
  si T et S vérifient le Lemma II.27 **alors**
    on bascule la diagonale dans T ou dans S
    $low := low + 1; up := up + 1$
    low-up(T, S, n) ;
  sinon
    si T et S ont un sommet i de type (2 :0) **alors**
      si T et S ont une (2,2)-diagonale $\{i, j\}$ **alors**
        $T := N''_T(i,j)$ ou $S := N''_S(i,j)$
        $low := low + 3; up := up + 3$
        low-up$(T, S, n-2)$;
      sinon

$T := N_T(i)$ ou $S := N_S(i)$
$low := low + 1; up := up + 2$
low-up$(T, S, n-1)$;
sinon
si T et S ont un sommet i de type (1 :1) **alors**
$T := N_T(i)$ et $S := N_S(i)$
$low := low + 1; up := up + 2$
low-up$(T, S, n-1)$;
sinon
si T et S ont un sommet i de type (3 :0) **alors**
$T := N'''_T(i)$ ou $S := N'''_S(i)$
$low := low + 1; up := up + 3$
low-up$(T, S, n-1)$;
sinon T et S ont un sommet i de type (2 :1)
$T := N_T(i)$ et $S := N_S(i)$
$low := low + 1; up := up + 3$
low-up$(T, S, n-1)$;
fin low-up

Exemples :
Si (T, S) sont deux triangulations $\in \mathcal{T}_{23}$ de Rogers ([141], p. 88), alors l'algorithme précédent fournit $20 \leq dist(T, S) \leq 29$. La distance exacte de rotation est 21.
Si (T, S) sont les deux triangulations $\in \mathcal{T}_{20}$ de Lucas ([110], p. 261), alors l'algorithme fournit $17 \leq dist(T, S) \leq 25$. La valeur exacte de la distance de rotation est 20.

Une applet Java pour calculer les bornes inférieures et supérieures est disponible sur le site web : http ://www.u-bourgogne.fr/jl.baril/titi.html. Cette applet choisit la meilleure borne entre les résultats obtenus ci-dessus et la borne donnée dans [132]. La compléxité en temps est polynomiale : $\mathcal{O}(n^3)$.

II.3.6 Résultats expérimentaux

On présente ici quelques résultats exhaustifs et statistiques. Dans le premier tableau on obtient des résultats exhaustifs pour $n \leqslant 13$ concernant le ratio des pairs (T, S) vérifiant $(up - low) \leqslant k$ avec $k = 0, 1, 2, n/2$. Dans le second tableau, on donne des résultats probabilistes pour $n \in [14, 100]$ en générant aléatoirement un grand nombre de paires (T, S). On remarque que notre algorithme donne des bornes efficaces pour des grandes valeurs de n : 99% des paires (T, S) ont des bornes inférieures et supérieures vérifiant $(up - low) \leqslant n/3$.

Résultats exhaustifs

$n - gons$	5	6	7	8	9	10	11	12	13
$Up = Low$	100(%)	96	88	78	67	57	47	38	33
$Up - Low \leqslant 1$	100	100	99	99	95	90	82	73	54
$Up - Low \leqslant 2$	100	100	100	100	100	99	96	91	64
$Up - Low \leqslant n/2$	100	100	100	100	100	100	99,9	99,9	99,9

Résultats statistiques

$n-gons$	14	15	16	17	18	20	30	50	100
$Up-Low \leqslant n/10$	$81,5(\%)$	77	75	71	69	65	59	45	39
$Up-Low \leqslant n/5$	98	97	97	96	95	95	95	94	93
$Up-Low \leqslant n/3$	$99,9$	$99,9$	$99,9$	$99,9$	$99,8$	$99,8$	$99,8$	$99,7$	$99,7$
$Up-Low \leqslant n/2$	$99,9$	$99,9$	$99,9$	$99,9$	$99,9$	$99,9$	$99,9$	$99,9$	$99,9$

II.4 Conclusion

Dans cette partie, nous avons exhibé deux nouvelles structures naturelles sur des ensembles de Catalan (parenthésages et arbres binaires). Il reste toutefois de nombreuses questions concernant ces structures. Par exemple, peut-on trouver la longueur d'un plus court chemin entre deux éléments de la structure en un temps polynomial ? Cette question reste encore un problème ouvert pour le treillis de Tamari. Toutefois, nous avons réussi dans la dernière section à obtenir un algorithme d'approximation pour la distance de rotation.

Il reste donc de nombreuses directions de recherche prometteuses concernant le treillis de Tamari et ces deux nouvelles structures (phagocyte, taille-greffe)...

Chapitre III

Motifs dans les permutations

III.1 Introduction

Dans ce dernier chapitre, je présente des travaux de recherche concernant l'étude des motifs dans les permutations. Il s'agit d'un domaine très abordé depuis ces dix dernières années. En effet, de nombreux travaux existent pour l'énumération de classes de permutations à motifs exclus ainsi que pour la génération exhaustive des objets de telles classes [3, 178, 163, 38, 48, 49, 45, 96, 113, 112].
La première partie traite d'un problème issu de la bioinformatique présenté en terme de permutations. Une suite du génôme est alors modélisée par une permutation où chaque valeur correspond à un gène. On étudie alors l'évolution de la permutation identité lorsqu'on lui applique plusieurs duplications en miroir avec une perte aléatoire de certains gènes. On caractérise alors les permutations obtenues après un nombre donné de duplications et on obtient des algorithmes efficaces de construction d'un chemin de duplications entre l'identité et une permutation donnée quelconque.

Dans la seconde partie, on répond à une question proposée par Atkinson et al. : *la cloture cyclique des permutations évitant le motif* $k(k-1)\dots 321$ *a-t-elle une base finie ?*. L'étude de ce problème nous donne une nouvelle classe d'objets combinatoires énumérée par les nombres de Catalan. On donne enfin des résultats généraux concernant les clotures cylindrique et torique.

III.2 Duplication en miroir du génome

La duplication de génome consiste à copier une partie du génome et de l'insérer dans le génome, suivi par une perte d'une copie de chaque gène dupliqué (voir [36, 52, 87, 105, 153] pour une explication des différentes méthodes de duplication). D'un point de vue formel, un génome de n gènes est représenté par une permutation de taille n. Dans un article précédent, Chaudhuri et al. [50] étudient une variante appelée 'tandem duplication-random loss model' : La partie dupliquée (de taille K) du génome est insérée immédiatement après la partie originale, suivie par la procédure de perte. Ce modèle est issu de la biologie évolutionnaire où il est appliqué au génome mitochondrial des vertébrés. On donne un

exemple du processus de duplication de la permutation 123456 suivi par la procédure de perte aléatoire.

$$123456 \rightsquigarrow 1\ 2\ 3\ 4\ 5\ 6\ \underbrace{1\ 2\ 3\ 4\ 5\ 6}_{\text{duplication}} \rightsquigarrow \underbrace{1\ 2\ \not{3}\ 4\ \not{5}\ 6\ \not{1}\ \not{2}\ 3\ \not{4}\ 5\ \not{6}}_{\text{procedure de perte}} \rightsquigarrow 124635.$$

Ce modèle de duplication sera appélée la W-duplication (W pour *Whole*). Chaudhuri et al. introduisent une notion de distance entre deux génomes et ils fournissent un algorithme pour la calculer efficacement dans certains cas. Bouvel et Rossin [43] ont aussi étudié ce modèle. Ils prouvent que la classe des permutations obtenues à partir de l'identité après p étapes (la partie dupliquée est ici de longueur K) est aussi une classe de permutations à motifs exclus. Plus particulièrement, dans le cas restreint d'une duplication totale (*whole duplication* ou *W-duplication* pour abréger), ils obtiennent après p W-duplications, la classe des permutations évitant toutes les permutations minimales ayant 2^p descentes, minimale dans le sens de la relation d'inclusion des motifs dans une permutation. De plus, ils calculent le nombre nécessaire et suffisant d'étapes de duplications de largeur K pour obtenir n'importe quelle permutation à partir de l'identité. Plus récemment, Bouvel et Pergola [42] ont montré une caractérisation locale et simple et plusieurs propriétés sur l'ensemble des permutations minimales avec 2^p descentes. Récemment Mansour et Yan [115] ont aussi donné propriétés sur les permutations minimales.

Dans cette section, on étudie la duplication totale en miroir (*whole mirror duplication-random loss model* ou (*WM-duplication* pour abréger) : C'est le même processus que pour la *W-duplication* sauf que l'on copie le miroir de la portion originale. Ce modèle apparaît couramment dans les génomes eubactériens dans la plupart des chromosomes [51, 121]. La figure III.1 illustre une *WM-duplication* pour la permutation $\sigma = 5421673$:

$$5421673 \rightsquigarrow 5\ 4\ 2\ 1\ 6\ 7\ 3\ 3\ 7\ 6\ 1\ 2\ 4\ 5 \rightsquigarrow 5\ 4\ \not{2}\ \not{1}\ 6\ \not{7}\ \not{3}\ 3\ 7\ \not{6}\ 1\ 2\ \not{4}\ \not{5} \rightsquigarrow 5463712.$$

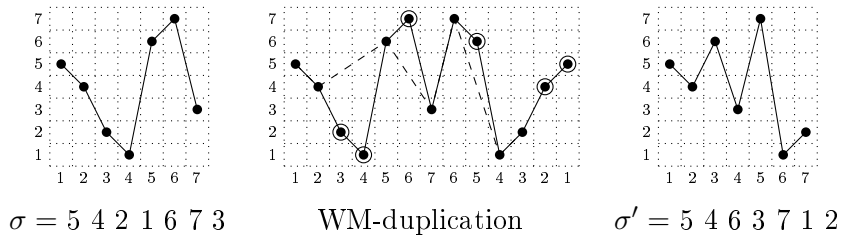


Figure III.1 : Une WM-duplication de la permutation 5421673. Les points encerclés sont supprimés lors de la procédure de perte.

Soit $\mathcal{S}_n$ l'ensemble des suites s de taille n , $s = s_1 s_2 \dots s_n$, formées d'entiers positifs. Le *miroir* de s est $\bar{s} = s_n s_{n-1} \dots s_1$. Une *sous-suite* de s est $s_{i_1} s_{i_2} \dots s_{i_m}$ pour $1 \leqslant i_1 < i_2 < \dots < i_m \leqslant n$. Une sous-suite est appélée une *sous-chaîne* lorsque l'ensemble $\{i_j,\ 1 \leqslant j \leqslant m\}$ est un intervalle, *i.e.* lorsque la sous-suite est formée d'éléments consécutifs de s. Une *montée* (resp. une *descente*) de s est une position i $(1 \leqslant i \leqslant n-1)$ avec $s_i < s_{i+1}$ (resp. $s_i > s_{i+1}$). Un *parcours montant* (resp. *un parcours descendant*) de s est une sous-chaîne (de longueur au moins un) dans laquelle les éléments sont en ordre

croissant (resp. ordre décroissant). Plus généralement, un parcours montant (ou parcours descendant) est dit *maximal* lorsqu'il ne peut pas être étendu en un plus long parcours montant (resp. parcours descendant) dans la suite. Voir Rodney et Wilf [46] pour des résultats concernant l'énumération des permutations qui ont un nombre donné de parcours montant et descendant. Par exemple, si $s = 5467312$ alors la position 4 est une descente, la sous-chaîne 46 est un parcours montant, et 467 est un parcours montant maximal. Notons que 5 est aussi un parcours montant maximal. Une *vallée* de s est une sous-chaîne qui est un parcours descendant suivi d'un parcours montant l'un des deux ayant une longueur au moins deux, *i.e.* une sous-chaîne $s_k s_{k+1} \dots s_\ell$, $1 \leqslant k < \ell \leqslant n$ telle qu'il existe j $(k < j < \ell)$ vérifiant $s_k > s_{k+1} > \dots > s_j$ et $s_j < s_{j+1} < \dots < s_\ell$. Une vallée est *maximale* si la sous-chaîne est maximale pour cette propriété. Dans l'exemple ci-dessus, les sous-chaînes 5467 et 7312 sont les seules vallées maximales de s. On note par $val(s)$ le nombre de vallées maximales dans s, *i.e.* la cardinalité de l'ensemble $\{j, s_j < \min\{s_{j-1}, s_{j+1}\}\}$. Une formule récursive énumérant les permutations avec un nombre donné de vallées peut être trouvée dans [139].

D'un autre coté, une suite s de taille n est une *permutation* lorsque chaque s_i est un élément différent dans l'ensemble $[n] = \{1, 2, \dots, n\}$. Dans la suite, les permutations seront notées par des lettres grecques : σ, π, $\tau, \dots$ Soit S_n l'ensemble de toutes les permutations de taille n $(n \geqslant 1)$. En relation avec les définitions précédentes, toute permutation σ contient au plus $\lfloor \frac{n-1}{2} \rfloor$ vallées. Une permutation $\sigma \in S_n$ sera dite *alternantes* si $\sigma_1 > \sigma_2 < \sigma_3 > \sigma_4 < \sigma_5 > \dots$ Dans la littérature [33], les permutations alternantes sont aussi appelées *down-up* et sont énumérées par les nombres d'Euler (A000111 [162]). Par exemple, la permutation 324165 est alternante. Notons qu'une permutation alternante de taille n contient exactement $\lfloor \frac{n-1}{2} \rfloor$ vallées.

On rappelle ici la définition d'un motif dans une permutation. Une permutation π de taille k, $k \leqslant n$, est un *motif* (pattern en anglais) de la permutation $\sigma \in S_n$ s'il existe une sous-suite de σ qui est isomorphe à π en conservant l'ordre des éléments ; *i.e.*, s'il y a une sous-suite $\sigma_{i_1}\sigma_{i_2}\dots\sigma_{i_k}$ de σ (avec $1 \leqslant i_1 < i_2 < \dots < i_k \leqslant n$) telle que $\sigma_{i_\ell} < \sigma_{i_m}$ lorsque $\pi_\ell < \pi_m$. On écrit $\pi \prec \sigma$ pour noter que π est un motif de σ. Une permutation σ qui ne contient pas un motif π est dite *qui évite* π. par exemple, $\sigma = 1423$ contient les motifs 132, 312 et 123 ; mais σ évite le motif 321. La classe de toutes les permutations évitant les motifs $\pi_1, \pi_2, \dots, \pi_k$ est notée $S(\pi_1, \pi_2, \dots, \pi_k)$, et $S_n(\pi_1, \pi_2, \dots, \pi_k)$ est l'ensemble des permutations de taille n évitant $\pi_1, \pi_2, \dots$ et π_k. On dit aussi que $S(\pi_1, \pi_2, \dots, \pi_k)$ est une classe de permutations à motifs exclus de base $\{\pi_1, \pi_2, \dots, \pi_k\}$. Une classe $\mathcal{C}$ de permutations est *stable* pour $\prec$ si, pour $\sigma \in \mathcal{C}$, pour $\pi \prec \sigma$, alors on a aussi $\pi \in \mathcal{C}$. On donne maintenant une remarque cruciale pour la suite de notre étude.

Remarque : Si une classe $\mathcal{C}$ de permutations est stable pour $\prec$ alors $\mathcal{C}$ est aussi une classe de permutations à motifs exclus possédant la base $\mathcal{B} = \{\sigma \notin \mathcal{C}, \forall \pi \prec \sigma \text{ avec } \pi \neq \sigma, \pi \in \mathcal{C}\}$.

$\diamond$

Dans le prochain paragraphe, on prouve que la classe des permutations obtenues à partir de l'identité après un nombre donné p de WM-duplications est la classe des permutations avec au plus $2^{p-1} - 1$ vallées. C'est aussi la classe des permutations évitant les permutations alternantes de taille $2^p + 1$. De plus, on obtient la longueur du plus court chemin entre une permutation et l'identité. Ensuite dans le paragraphe suivant, on fournit deux algorithmes (d'une complexité polynomiale) permettant de reconstruire un plus court chemin entre

l'identité et une permutation quelconque. Le premier trouve le chemin en reculant, c'est à dire en partant de la permutation et en allant vers l'identité. Le deuxième utilise un algorithme efficace pour générer le code de Gray binaire réfléchi [37, 84] et reconstruit le chemin en partant de l'identité et en allant vers la permutation. Dans la dernière partie, on donne des résultats pour d'autres modèles de duplications mélangeant les W- et les WM-duplications et découlant de notre étude.

III.2.1 Permutations à motifs exclus et le modèle de WM-duplication

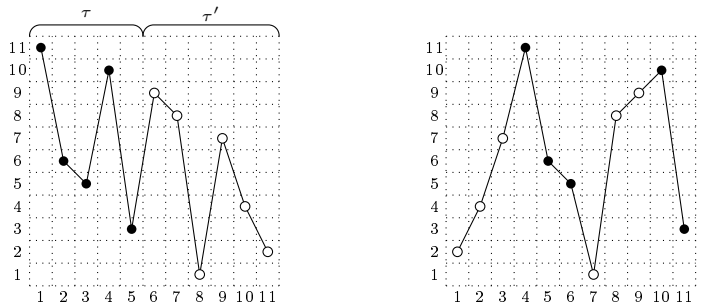
Nous étudions ici le modèle de WM-duplication en terme de permutations à motifs exclus. Nous établissons que la classe $\mathcal{C}(p)$ obtenue à partir de l'identité après p WM-duplications est exactement la classe des permutations évitant toutes les permutations alternantes de taille $2^p + 1$.

Lemme III.1. *Soit σ et π deux permutations différentes telles que $\pi \prec \sigma$. Alors σ contient au moins autant de vallées que π.*

Lemme III.2. *Une permutation obtenue à partir de l'identité après un nombre donné p de WM-duplications contient au plus $2^{p-1} - 1$ vallées.*

Théorème III.3. *La classe $\mathcal{C}(p)$ des permutations obtenues à partir de l'identité après un nombre donné p de WM-duplications est la classe des permutations avec au plus $2^{p-1} - 1$ vallées.*

Preuve : En considérant le lemme III.2, il suffit de prouver que chaque permutation σ avec au plus $2^{p-1} - 1$ vallées peut être obtenue à partir de l'identité après p WM-duplications. On procède par récurrence sur p. En effet, soit σ une permutation avec k vallées telle que $2^{p-2} - 1 < k \leqslant 2^{p-1} - 1$. Alors σ peut s'écrire $\sigma = \tau\tau'$ où τ correspond au plus long préfixe de σ contenant exactement $2^{p-2} - 1$ vallées et τ' le suffixe restant. On décompose la permutation $\tau = u_1 d_1 u_2 d_2 \dots u_\ell d_\ell$, où u_i et d_i sont respectivement des parcours montant et parcours descendant définis comme suit : u_1 est le premier parcours montant maximal sans prendre sa plus grande valeur ; d_1 le parcours descendant juste après u_1 ; u_2 le parcours montant juste après d_1 sans prendre sa plus grande valeur, ainsi de suite... Notons que u_1 peut être vide ce qui ne peut pas se produire pour d_ℓ. Par exemple, $\tau = 5421673$ a la décomposition : u_1 est vide, $d_1 = 5421$, $u_2 = 6$ et $d_2 = 73$. Soit aussi $\tau' = u_{\ell+1} d_{\ell+1} \dots u_k d_k$ la décomposition similaire pour τ'. Maintenant appliquons le processus suivant : on classe en ordre décroissant les valeurs apparaîssant dans d_ℓ ou u_1 afin de créer un parcours descendant D_ℓ ; on classe en ordre croissant les valeurs situées dans u_ℓ ou $d_{\ell+1}$ afin de créer un parcours montant U_ℓ ; on construit alors la suite $S = U_\ell D_\ell$. On classe dans une suite décroissante $D_{\ell-1}$ les valeurs situées dans $d_{\ell-1}$ ou $u_{\ell+2}$, et ainsi de suite. A chaque étape j, on insère les suite obtenues U_j et D_j au début de S. La permutation $S = \dots U_{\ell-1} D_{\ell-1} U_\ell D_\ell$ obtenue à la fin du processus contient au plus $2^{p-2} - 1$ vallées. Voir figure III.2 pour un exemple de construction de S. Donc, par récurrence S peut être obtenue à partir de l'identité après $(p-1)$ WM-duplications. Par construction, σ est atteint avec une WM-duplication de S ce qui signifie que σ peut être construite à partir de l'identité avec p WM-duplications.
Notons que la permutation σ a été décomposée en une partition de parcours montant



$\sigma = 11\ 6\ 5\ 10\ 3\ 9\ 8\ 1\ 7\ 4\ 2$ $\qquad$ $S = 2\ 4\ 7\ 11\ 6\ 5\ 1\ 8\ 9\ 10\ 3$

Figure III.2 : Décomposition $\sigma = \tau\tau'$ et la permutation S obtenue après le processus de la preuve du Tthéorème III.3. On a $\tau = 11\ 6\ 5\ 10\ 3$, $\tau' = 9\ 8\ 1\ 7\ 4\ 2$, u_1 est vide, $d_1 = 11\ 6\ 5$, u_2 est vide, $d_2 = 10\ 3$, u_3 est vide, $d_3 = 9\ 8\ 1$, u_4 est vide, $d_4 = 7\ 4\ 2$ et, $D_2 = 10\ 3$, $U_2 = 1\ 8\ 9$; $D_1 = 11\ 6\ 5$; $U_1 = 2\ 4\ 7$. Donc $S = U_1D_1U_2D_2 = 2\ 4\ 7\ 11\ 6\ 5\ 1\ 8\ 9\ 10\ 3$.

et parcours descendant, u_j et d_j. On utilisera cette décomposition pour l'algorithme de reconstruction d'un plus court chemin entre l'identité et σ.

□

Corollaire III.4. *Soit σ une permutation et $val(\sigma)$ le nombre de ses vallées. Dans le modèle de WM-duplication, $\lceil log_2(val(\sigma)+1)\rceil + 1$ étapes sont nécessaires et suffisantes pour obtenir σ à partir de l'identité.*

On fournira par la suite deux algorithmes permettant de reconstituer un chemin entre l'identité et une permutation.

Théorème III.5. *La classe $\mathcal{C}(p)$ des permutations obtenues après un nombre donné p de WM-duplications est la classe des permutations évitant les permutations alternantes de taille $2^p + 1$.*

Par exemple, $\mathcal{C}(1) = S(213, 312)$ et $\mathcal{C}(2) = S(21435, 31425, 41325, 32415, 42315, 21534, 31524, 51324, 32514, 52314, 41523, 51423, 42513, 52413, 43512, 53412)$.

On obtient aussi le résultat plus général (voir théorème III.6) pour la classe des permutations ayant au plus p vallées. Notons, avec [139, 95, 97], une fonction génératrice bivariée pour cette classe est :

$$\frac{1}{1-y}(1 - \frac{1}{y} + \frac{1}{y}\sqrt{y-1} \cdot tan(x\sqrt{y-1} + arctan(\frac{1}{\sqrt{y-1}}))),$$

où y est la variable dédiée au nombre maximal de vallées dans la permutation et x est dédiée à la longueur de la permutation.

Théorème III.6. *La classe des permutations ayant au plus p vallées est la classe des permutations évitant les permutations alternantes de taille $2p + 3$.*

III.2.2 Considérations algorithmiques

On présente ici deux algorithmes pour reconstituer un scenario possible de WM-duplications pour une permutation $\sigma \in S_n$: le premier construit un plus court chemin entre l'identité et σ en reculant de σ vers l'identité ; le deuxième algorithme donne une construction du chemin de l'identité vers σ en utilisant le code de Gray binaire réfléchi [84] et en parcourant le chemin de l'identité vers σ. On discute aussi de leur complexité. Notons que si l'on applique σ^{-1} à chaque élément du chemin on obtient naturellement un plus court chemin de WM-duplications entre σ^{-1} et l'identité.

III.2.2.1 Un chemin de $12\ldots n$ vers $\sigma \in S_n$ obtenu en reculant à partir de σ

On explique ici comment on peut établir un algorithme pour construire un scenario de WM-duplications de l'identité vers $\sigma \in S_n$. L'algorithme reconstituera ce chemin en reculant à partir de σ. On partitionne la permutation σ comme suit : $\sigma = u_1 d_1 u_2 d_2 \ldots u_k d_k$ où $u_1 = \sigma_1 \ldots \sigma_{i_1}$ est le parcours montant maximal contenant la première entrée σ_1 ; d_k est le parcours descendant maximal contenant σ_n ; d_1 est le parcours descendant maximal contenant σ_{i_1+1} ; u_k est le parcours montant maximal juste avant d_k, *i.e.* la valeur juste avant d_k dans σ appartient à u_k ; on continue le processus en alternant les parcours montant et les parcours descendant jusqu'à ce que la permutation σ soit entièrement partitionnée. Attention, cette décomposition n'est pas la même que celle utilisée dans la preuve du théorème III.3.
Pour i de 1 à $\ell = \left\lfloor \frac{k+1}{2} \right\rfloor$, on définit par U_i (resp. D_i) le classement en ordre croissant (resp. décroissant) de u_i et d_{k-i+1} (resp. d_i et u_{k-i+1}). Soit $\pi = U_1 D_1 U_2 D_2 \ldots U_\ell D_\ell$ la concaténation de toutes les sous-chaînes classées U_i et D_i où D_ℓ peut être vide. Alors π contient au plus $(\ell - 1) = \left\lfloor \frac{k+1}{2} \right\rfloor - 1 = \left\lfloor \frac{val(\sigma)+2}{2} \right\rfloor - 1 = \left\lfloor \frac{val(\sigma)}{2} \right\rfloor$ vallées. Cette étape demande $\mathcal{O}(n)$ calculs puisqu'on peut simultanément détecter et classer les parcours montant et les parcours descendant. En itérant ce processus avec cette nouvelle permutation π, le Corollaire III.4 nous garantit que la procédure BackPath (voir Algorithm 1) construit un chemin de l'identité vers σ en $(\lceil log_2(val(\sigma)+1) \rceil + 1)\mathcal{O}(n)$, *i.e.* $\mathcal{O}(n \cdot log_2(n))$ dans le pire des cas. Par exemple, ce processus appliqué à la permutation $\sigma = \mathbf{5}421\mathbf{6}73$ donne le chemin $\sigma \leftarrow \mathbf{357}6421 \leftarrow 1234567$ (les parcours montant sont en gras).

procedure BackPath
while $\sigma \neq 12\ldots n$ **do**
- $\pi \leftarrow$ *empty*
- Soit $\sigma = u_1 d_1 u_2 d_2 \ldots u_k d_k$ la partition en parcours montant et descendant
for $i \leftarrow 1$ **to** $\left\lfloor \frac{k+1}{2} \right\rfloor$ **do**
  - Classer en order croissant u_i et d_{k-i+1} et ajouter le résultat sur la droite de π
  - Classer en ordre décroissant d_i et u_{k-i+1} et ajouter le résultat sur la droite de π
end for
- $\sigma \leftarrow \pi$
end while

Algorithm 1: La procédure BackPath fournissant un scenario de WM-duplications de l'identité vers $\sigma \in S_n$.

III.2.2.2 Un chemin de WM-duplications de $12\ldots n$ vers $\sigma \in S_n$

Ici on construit un chemin de WM-duplications à partir de $12\ldots n$ vers la permutation $\sigma \in S_n$. Contrairement à l'algorithme précédent, ici le chemin est construit directement en commençant avec l'identité et en allant vers la permutation σ. On décompose la permutation $\sigma = u_1 d_1 u_2 d_2 \ldots u_k d_k$ en parcours montant et descendant de la même manière que pour la preuve du théorème III.3. On rappelle cette décomposition : u_i et d_i sont respectivement parcours montant descendant définis comme suit : u_1 est le premier parcours montant sauf sa plus grande valeur ; d_1 le parcours descendant juste après u_1 ; u_2 le parcours montant juste après d_1 sauf sa plus grande valeur, ainsi de suite... Notons que u_1 peut être vide : une telle décomposition est donnée en exemple dans la preuve du théorème III.3.

On étiquète maintenant chaque parcours (montant et descendant) de la décomposition avec le code de Gray binaire réfléchi [84] : on peut le faire en utilisant un algorithme loopless introduit par Bitner, Ehrlich, et Reingold [37]. Par exemple, la figure III.3 montre un tel étiquetage. La structure du code de Gray binaire réfléchi B_n est cruciale : $B_n = 0 \cdot B_{n-1} \bigcup 1 \cdot \overline{B_{n-1}}$ initialisé par $B_1 = \{0, 1\}$ où $\overline{B_n}$ est la liste B_n considérée en ordre inverse. Supposons que $\sigma = u_1 d_1 \ldots u_\ell d_\ell u_{\ell+1} d_{\ell+1} \ldots u_k d_k$ où tous les parcours de u_1 à d_ℓ ont une étiquette avec le dernier bit 0, et les autres parcours ont 1 comme dernier bit. En accord avec la preuve du théorème III.3, on reconstitue la permutation ci-dessus π de σ en concaténant (sur la gauche) le classement en ordre décroissant D_ℓ de d_ℓ et $u_{\ell+1}$, le classement en ordre croissant U_ℓ de u_ℓ et $d_{\ell+1}$, ainsi de suite...

A contrario, à l'étape j de notre algorithme on fait une WM-duplication de π en : (i) conservant dans la première copie de π les éléments étiquetés 0 sur le $j$ième bit significatif, et, (ii) en conservant dans le miroir les éléments étiquetés 1 sur le $j$ième bit significatif (voir la figure III.3 pour un exemple). Cette étape demande seulement $\mathcal{O}(n)$ calculs.

En itérant ce processus, le Corollaire III.4 nous garantit que la procédure Path (voir Algorithm 2) construit un chemin de l'identité à σ en $(\lceil log_2(val(\sigma) + 1) \rceil + 1)\mathcal{O}(n)$, *i.e.* $\mathcal{O}(n \cdot log_2(n))$ dans le pire des cas.

procedure Path
- $\pi = 12\ldots n$
- Partitionner $\sigma = u_1 d_1 u_2 d_2 \ldots u_k d_k$ en parcours montant et descendant
- Etiqueter les parcours montant et descendant avec le code de Gray réfléchi (loopless algorithm [37])

for $j = 1$ **to** $1 + \lfloor log_2(k-1) \rfloor$ **do**
  - Faire une WM-duplication sur π qui conserve dans la première copie de π exactement les éléments étiquetés 0 sur le $j$ième bit.

end for

Algorithm 2: La procédure Path donnant un scénario de WM-duplications de $12\ldots n$ vers $\sigma \in S_n$.

Une Applet Java pour ce dernier algorithme est donnée sur le site internet http ://www.u-bourgogne.fr/jl.baril

000 001 011 010 110 111

2 3 1 **6** 8 4 **5** 9 7

$$123456789 \to 245897631 \to 231679854 \to 231684597$$

Figure III.3 : Etiquetage des parcours montant et descendant de $\sigma = 231684597$ avec le code de Gray réfléchi. Exemple de chemin de WM-duplications à partir de l'identité vers σ.

III.2.3 Autres modèles de duplication

Ici, on explore deux variantes du modèle de WM-duplication : 1) on fait une WM-duplication de l'identité suivi par plusieurs W-duplications et, 2) on fait une WM- et/ou W-duplication de l'identité suivi par plusieurs W-duplications. Pour ces deux cas on fournit une caractérisation de la classe des permutations obtenues après p duplications.

III.2.3.1 Une WM-duplication suivie par plusieurs W-duplications

On a vu auparavant qu'une WM-duplication de l'identité donne la classe des permutations ne contenant pas de vallée. Donc, on obtient par récurence que la classe des permutations obtenues après une WM-duplication de l'identité suivie par $(p-1)$ W-duplications est la classe des permutations avec au plus $2^{p-1}-1$ vallées. On déduit le théorème suivant :

Théorème III.7. *La classe des permutations obtenues après une WM-duplication de l'identité suivie par $(p-1)$ W-duplications est la classe des permutations évitant les permutations alternantes de longueur 2^p+1.*

III.2.3.2 Une W- ou WM-duplication suivie par plusieurs W-duplications

Ici, on fait une W- ou WM-duplication de l'identité suivie par plusieurs W-duplications.

Théorème III.8. *La classe des permutations obtenues après une W- ou WM-duplication de l'identité suivie par p W-duplications est l'union de deux classes : (i) la classe des permutations ayant au plus 2^p-1 vallées et, (ii) la classe des permutations ayant au plus $2^{p+1}-1$ descentes.*

Corollaire III.9. *La classe $\mathcal{C}'(p)$ des permutations obtenues après une W- ou WM-duplication de l'identité suivie par p W-duplications est la classe des permutations évitant les permutations de taille $3 \cdot 2^p + 1$ ayant exactement 2^p vallées et 2^{p+1} descentes.*

Par exemple, $\mathcal{C}'(0) = S(4132, 3142, 4312, 3241, 3214, 4231, 4213, 2143)$; on a calculé expérimentalement la cardinalité de la base de $\mathcal{C}'(1)$, qui est 720. On déduit alors le résultat plus général :

Corollaire III.10. *La classe des permutations ayant au plus p vallées et au plus $2p+1$ descentes est la classe des permutations évitant les permutations de taille $3p+4$ et ayant exactement $p+1$ vallées et $2p+2$ descentes.*

Notons que l'ensemble des permutations de taille $3p+4$ ayant exactement $p+1$ vallées et $2p+2$ descentes est aussi l'ensemble des permutations de même taille, avec le même nombre de vallées et où chaque montée est immédiatement précédée par une descente (ce qui définit une vallée). Cet ensemble a été étudié par Shapiro et al. [157] (voir aussi la suite A101280 dans [162]). En effet, ils énumèrent les permutations de taille n ayant k 'peaks' et avec la propriété supplémentaire que chaque montée est immédiatement suivie par une descente. Donc, la cardinalité de notre ensemble est obtenue quand $n = 3p+4$ et $k = p+1$. Par exemple, les premières cardinalités pour $p = 0, 1, 2, 3, 4$ sont $8, 720, 230144, 179266560, 277662253056$.

III.2.4 Future direction de recherche

Beaucoup de problèmes relatifs restent ouverts. Par exemple, soit σ_1 et σ_2 deux permutations dans S_n. Peut-on exhiber un algorithme efficace pour calculer la permutation π qui minimise la somme $d(\pi, \sigma_1) + d(\pi, \sigma_2)$ où $d(\pi, \sigma)$ est le nombre minimum de WM-duplications nécessaires pour transformer π en σ ? Peut-on caractériser la classe de permutations à motifs exclus correspondante aux permutations obtenues à partir de l'identité après une W-duplication suivie par p WM-duplications ? Peut-on obtenir des résultats similaires pour la duplication miroir partielle (non totale) ? Peut-on faire un travail similaire lorsque l'on permet de faire à chaque étape une W ou WM duplication ?

III.3 Cloture cyclique de $S(k(k-1)\dots 21)$

Rappelons quelques notations utiles pour la suite de ce paragraphe. Soit S (respectivement S_n) la classe des permutations (respectivement permutations de taille n). On a $S = \bigcup_{n\geqslant 0} S_n$. Comme nous l'avons fait depuis le début de ce mémoire, on représente les permutations en notation linéaire, *i.e.* si $i_1, i_2, \dots, i_n$ sont n valeurs distinctes dans $[n] = \{1, 2, \dots, n\}$, on note la permutation $\sigma \in S_n$ par la suite $i_1\ i_2\ \dots\ i_n$ si $\sigma(k) = i_k$ pour $1 \leqslant k \leqslant n$. Par exemple, la permutation identité de taille n, Id_n, sera écrite $1\ 2\ \dots\ n$. Un *segment* d'une permutation σ est une sous-suite de la forme $\sigma_i\sigma_{i+1}\dots\sigma_{j-1}\sigma_j$ ou $\sigma_j\sigma_{j+1}\dots\sigma_n\sigma_1\dots\sigma_{i-1}\sigma_i$ avec $1 \leqslant i \leqslant j \leqslant n$. La *cardinalité* (ou *longueur*) d'un segment W sera notée $|W|$. Une permutation $\sigma = \sigma_1\sigma_2\dots\sigma_n \in S_n$ contient le *motif* $\pi \in S_k$, $k \geqslant 2$, si et seulement s'il existe une suite d'indices $1 \leqslant i_1 < i_2 < \dots < i_k \leqslant n$ telle que $\sigma(i_1)\sigma(i_2)\dots\sigma(i_k)$ est ordonnée comme π. On écrit $\pi \prec \sigma$ pour dire que π est un motif de σ. Une permutation σ qui ne contient pas π comme motif est dite *qui évite* π. Soit B un ensemble de permutations. On denote par $S_n(B)$ (respectivement $S(B)$) l'ensemble (respectivement la classe) des permutations dans S_n (respectivement S) évitant tous les motifs $\pi \in B$. Par exemple, $25314 \notin S_5(123)$ mais $43152 \in S_5(123)$. Voir par exemple [39, 160]. Dans la suite, une sous-suite d'une permutation σ ordonnée suivant le motif $D_k = k(k-1)\dots 21$ sera appelée un D_k-motif de σ. Une classe $\mathcal{C}$ de permutations est *fermée* (or *stable*) pour la relation $\prec$ si, pour tout $\sigma \in \mathcal{C}$, pour tout $\pi \prec \sigma$, alors on a aussi $\pi \in \mathcal{C}$. Une telle classe peut toujours être définie par une classe de permutations $S(B)$ évitant des motifs. Dans le cas où B est minimale (relativement à la relation $\prec$), B est dite une *base*.

Soit $\sigma = \sigma_1\sigma_2\dots\sigma_n$ une permutation dans S_n. On définit deux bijections r (*rotation cyclique*) et ℓ (*rotation cylindrique*) sur S_n de la manière suivante : $r(\sigma) = \sigma_n\sigma_1\sigma_2\dots\sigma_{n-1}$

et $\ell(\sigma) = (\sigma_1 \mod (n)+1)(\sigma_2 \mod (n)+1)\dots(\sigma_n \mod (n)+1)$. Par soucis de simplicité, on dira qu'une permutation $\sigma' \in S_n$ est une *rotation cyclique* (resp. *rotation cylindrique*) d'une permutation $\sigma \in S_n$ si $\sigma' = r^k(\sigma)$ (resp. $\sigma' = \ell^k(\sigma)$) pour un entier $k \in [1..n]$. Par exemple, 231 et 312 sont toutes les deux des rotations cycliques de $123 \in S_3$; 213 et 132 sont elles des rotations cylindrique de $321 \in S_3$. La *cloture cyclique* $cc(\mathcal{C})$ (resp. *cloture cylindrique* $c\ell(\mathcal{C})$) d'une classe $\mathcal{C}$ est la classe des permutations qui peuvent être obtenues par une rotation cyclique (resp. rotation cylindrique) d'une permutation de $\mathcal{C}$. Enfin, la *cloture torique* $tr(\mathcal{C})$ de $\mathcal{C}$ est définie par $cc(c\ell(\mathcal{C}))$ qui est aussi égal à $c\ell(cc(\mathcal{C}))$.

De nombreuses études ont été faites sur les clotures de classes de permutations : énumération des permutations d'une longueur donnée dans la classe ; caractérisation de la base associée à la classe ; et récemment leurs liens avec les classes fermées relativement à la rotation cyclique. Rappelons maintenant les principaux résultats obtenus par Albert et al. (Voir [2]).

Soit $\mathcal{C}$ la classe $S(B)$ ou B est un ensemble de motifs. Nous savons que $cc(\mathcal{C})$ est une classe à motifs exclus où sa base est l'ensemble des permutations qui sont minimales et qui n'appartiennent pas à $cc(\mathcal{C})$. Une permutation est minimale relativement à l'ensemble de motif B lorsque la permutation contient au moins un motif de B et que tout motif propre à la permutation exclut tous les motifs de B. Soit σ une permutation dans $\mathcal{C}$ et θ un motif de σ (relativement à B). Le *segment témoin* $W(\theta)$ de θ est le segment de σ (considéré circulairement) commençant par la valeur juste après la dernière de θ et finissant par la première valeur de θ dans σ. Albert et al. donne une condition nécessaire et suffisante pour que σ appartienne (ou pas) à la cloture cyclique de $\mathcal{C}$.

Lemme III.11. *[2] $\sigma \notin cc(\mathcal{C})$ si et seulement si les segments témoins couvrent σ.*

De plus, ils obtiennent une condition suffisante pour que la classe $cc(\mathcal{C})$ ait une base de cardinalité finie.

Proposition III.12. *[2] Soit $\mathcal{C} = S(B)$, où B est finie et supposons qu'il existe une borne Δ dépendant de B seulement telle que, pour tout $\sigma \notin cc(\mathcal{C})$, il y a une collection d'au plus Δ segments témoins qui couvrent σ. Alors $cc(\mathcal{C})$ possède une base finie.*

Albert et al. déduisent de la proposition III.12 que $cc(S(321))$, $cc(S(231))$ et $cc(S(4321))$ ont des bases finies. Ils montrent aussi que la cloture cyclique d'une classe peut avoir une base infinie (c'est le cas de la cloture de $S(265143)$). Ils donnent également plusieurs résultats d'énumération très intéressants.

Cette partie est organisée de la façon suivante. Dans un premier temps, on prouve que la cloture cyclique de $S(k(k-1)\dots 21)$ a une base finie B. De plus, on prouve que la plus petite taille d'une permutation minimale de B est $2k-1$, et que ces permutations sont énumérées par $(2k-1).c_k$ où c_k est le k-ième nombre de Catalan. On caractérise aussi ces permutations. Ensuite, on étudie les clotures cylindriques de classes fermées. On montre une dualité avec la cloture cyclique ce qui nous permet de prouver que la cloture cylindrique de $S(k(k-1)\dots 21)$ est aussi à base finie. Enfin, on étudie la cloture torique. On donne une condition suffisante pour que la classe $tr(\mathcal{C})$ ait une base finie. On propose ainsi plusieurs questions ouvertes concernant la cloture torique de certaines classes de permutations. Plus précisément, on a obtenu la preuve que $tr(S(321))$ est à base finie, mais nous ne la donnons pas ici. En effet, la preuve contient énormément de cas à vérifier

et ne contient pas de concepts particulièrement intéressants. On finit en présentant d'autres questions ouvertes concernant l'énumération de certaines clotures toriques de classes.

Soit $\sigma = \sigma_1 \dots \sigma_n$ une permutation de taille n telle que $\sigma \notin cc(S(D_k))$ pour $k \geqslant 3$. Pour $x \in \{1, \dots, n\}$, on définit $\mathcal{W}_\sigma(x)$ par l'ensemble des sous-suites $\theta = \theta_1\theta_2 \dots \theta_{k-1}\theta_k$ de σ (considérée circulairement), ordonnées comme $D_k = k(k-1)\dots 21$ et telles que $x \in W(\theta)$. Soit $\mathcal{W}_\sigma$ l'union de tous les $\mathcal{W}_\sigma(x)$ pour $1 \leqslant x \leqslant n$. Dans la suite, on omet l'indice σ pour $\mathcal{W}_\sigma(x)$ et $\mathcal{W}_\sigma$ puisque ce sera implicite en fonction du contexte. Pour $\theta, \theta' \in \mathcal{W}$, on définit une relation d'ordre $\trianglelefteq$ par : $\theta' \trianglelefteq \theta$ si et seulement si (a) $|W(\theta')| < |W(\theta)|$; ou (b) $|W(\theta')| = |W(\theta)|$ et le mot associé $\theta'_1\theta'_2 \dots \theta'_k$ est plus petit que $\theta_1\theta_2 \dots \theta_k$ en ordre lexicographique. Alors $\mathcal{W}$ est un ensemble fini non vide totalement ordonné. Soit θ sont élément maximum. Alors, σ a une structure spéciale illustrée par la figure III.4 et décrite ci-dessous.

En effet, σ peut s'écrire (modulo rotation) :

$$\sigma = \theta_1\alpha_1\theta_2\alpha_2 \dots \theta_{k-1}\alpha_{k-1}\theta_k\alpha_k$$

ou $W(\theta) = \alpha_k\theta_1$, $\theta_i \in \{1, 2, \dots, n\}$ et α_i sont des segments de σ pour $i \in [1, k]$. Par convenance, on pose $\theta_{k+1} = 0$ et $\theta_0 = n+1$. Pour $i \in [1, k-1]$, on définit la sous-suite α_i^- (resp. α_i^+) des éléments dans α_i qui sont plus petit que θ_{i+1} (resp. plus grand que θ_{i-1}). D'un autre coté, $\alpha_{i,j}^-$, $i \in [1, k-2]$, $j \in [i+2, k+1]$, (resp. $\alpha_{i,j}^+$, $i \in [2, k-1]$, $j \in [0, i-2]$,) est la sous-suite de α_i^- (resp. α_i^+) constituée par toutes les valeurs plus grandes que θ_j (resp. moins grandes que θ_j).

Avec ces hypothèses, σ vérifie les propriétés structurales suivantes.

Fait 1. Pour $i \in [1, k-1]$, α_i ne contient pas de valeur dans l'intervalle $[\theta_{i+1}, \theta_i]$.
S'il existe $y \in \alpha_i$ tel que $y \in [\theta_{i+1}, \theta_i]$, alors $\theta' = \theta_1 \dots \theta_i y \theta_{i+1} \dots \theta_{k-1}$ est une sous-suite de σ appartenant à $\mathcal{W}$ tel que $|W(\theta')| > |W(\theta)|$ ce qui donne une contradiction avec la maximalité de θ.

Fait 2. Pour $i \in [1, k-1]$, α_i ne contient pas de valeur dans $[\theta_i, \theta_{i-1}]$.
S'il existe $y \in \alpha_i$ tel que $y \in [\theta_i, \theta_{i-1}]$, alors $\theta' = \theta_1 \dots \theta_{i-1} y \theta_{i+1} \dots \theta_k$ est une sous-suite de σ appartenant à $\mathcal{W}$ telle que $\theta \trianglelefteq \theta'$ ce qui donne une contradiction.

Fait 3. Pour $i \in [1, k-2]$, $j \in [i+2, k]$, $\alpha_{i,j}^-$ appartient à $S(D_{j-i})$ et α_i^- appartient à $S(D_{k-i})$.
Si $\alpha_{i,j}^-$ contient une sous-suite $\rho_1\rho_2 \dots \rho_{j-i}$ ordonnée comme D_{j-i}, alors $\theta' = \theta_1 \dots \theta_i\rho_1 \dots \rho_{j-i}\theta_j \dots \theta_{k-1}$ appartient à $\mathcal{W}$ et vérifie $|W(\theta')| > |W(\theta)|$ ce qui donne une contradiction.

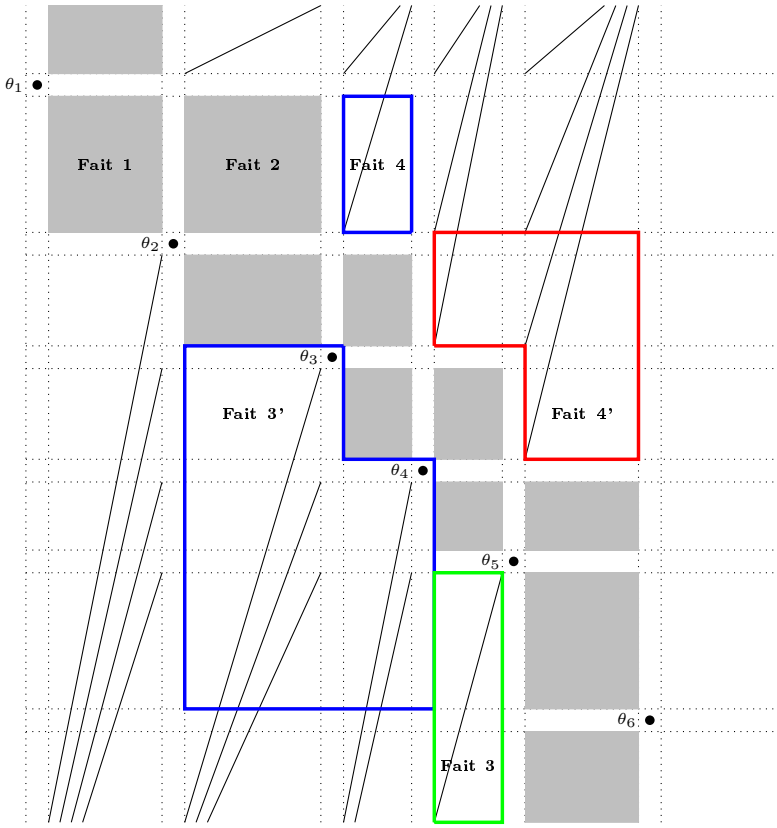
Fait 4. Pour $i \in [2, k-1]$, $j \in [0, i-2]$, $\alpha_{i,j}^+$ appartient à $S(D_{i-j})$.
Si $\alpha_{i,j}^+$ contient une sous-suite $\rho_1\rho_2 \dots \rho_{j-i}$ ordonnée comme D_{i-j}, alors $\theta' = \theta_1 \dots \theta_j\rho_1 \dots \rho_{i-j}\theta_{i+1} \dots \theta_{k-1}$ appartient à $\mathcal{W}$ et vérifie $\theta \trianglelefteq \theta'$ ce qui donne une contradiction.

Les faits suivants sont une simple généralisation des faits 3 et 4. Les preuves sont obtenues *mutatis mutandis*.

Fait 3'. Pour $i \in [1, k-2]$, $j \in [i+2, k]$, $\ell \in [i+1, k-1]$, $\alpha_{i,j}^-\theta_{i+1}\alpha_{i+1,j}^-\theta_{i+2} \dots \alpha_{\ell-1,j}^-\theta_\ell$ appartient à $S(D_{j-i})$ et $\alpha_i^-\theta_{i+1}\alpha_{i+1}^-\theta_{i+2} \dots \alpha_{\ell-1}^-\theta_\ell$ appartient à $S(D_{k-i})$.

Fait 4'. Pour $i \in [2, k-1]$, $\ell \in [i+1, k]$, $j \in [0, \ell-3]$, $\alpha_{i,j}^+\theta_{i+1}\alpha_{i+1,j}^+\theta_{i+2}\alpha_{i+2,j}^+ \dots \alpha_{\ell-1,j}^+\theta_\ell$ appartient à $S(D_{\ell-j-1})$.

Figure III.4 : La structure spéciale d'une permutation $\sigma \notin S(D_k)$. Une aire remplie ne contient pas de point de la forme (i, σ_i), $1 \leqslant i \leqslant n$. Une suite de j diagonales croissantes dans une aire signifie que l'aire ne contient pas de motif $(j+1)j \dots 21$.



Théorème III.13. *Soit $k \geqslant 3$ un entier. Alors $cc(S(D_k))$ possède une base finie.*

Preuve. Soit $\sigma = \sigma_1 \dots \sigma_n$ une permutation de taille n qui n'appartient pas à $cc(S(D_k))$. Le point crucial de la preuve est de trouver un nombre borné (indépendamment de n) de segments témoins qui couvrent σ. Finalement, on conclura grâce à la proposition III.12.

Soit $\theta = \theta_1 \dots \theta_k$ un élément maximum de $\mathcal{W}$. Modulo une rotation cyclique, σ peut se décomposer comme suit :

$$\sigma = \theta_1 \alpha_1 \theta_2 \alpha_2 \dots \alpha_{k-1} \theta_k \alpha_k.$$

Notons que α_k et $\theta_1 = \sigma_1$ sont évidemment couverts par $W(\theta)$ puisque $W(\theta) = \alpha_k \theta_1$. Donc il suffit de prouver que $\alpha_i \theta_{i+1}$ pour $1 \leqslant i \leqslant k-1$ sont couverts par un nombre borné de segments témoins.

Soit i tel que $1 \leqslant i \leqslant k-1$ et σ^i la rotation de σ définie par $\sigma^i = \alpha_i \theta_{i+1} \dots \theta_k \alpha_k \theta_1 \alpha_1 \dots \theta_{i-1} \alpha_{i-1} \theta_i$. Par hypothèse, il existe une sous-suite ρ de σ^i (considéré circulairement) telle que $\theta_{i+1} \in W(\rho)$. Si α_i est vide, il n'y a rien à faire puisque $\alpha_i \theta_{i+1}$ est couvert par $W(\rho)$. Maintenant, supposons que α_i est non vide, et considérons la première valeur x_1 de α_i. Alors, il existe une sous-suite $\rho = \rho_1 \rho_2 \dots \rho_k$ de σ^i ordonnée pareil que D_k telle que $x_1 \in W(\rho)$. On prend pour ρ l'élément maximal de $\mathcal{W}(x_1)$.

On discute sur la première valeur ρ_1 de ρ : (i) $\rho_1 \notin \alpha_i$; (ii) $\rho_1 \in \alpha_i^-$ et (iii) $\rho_1 \in \alpha_i^+$. Le cas (i) est immédiatement résolu puisque cela signifie que $\alpha_i\theta_{i+1}$ est contenu dans $W(\rho)$.

Regardons le cas (ii) : $\rho_1 \in \alpha_i^-$. Notons que ρ_k apparaît sur la droite de θ_k (dans σ^i), puisque sinon θ ne serait pas maximal dans $\mathcal{W}$ (car $\theta_1 \in W(\rho)$ et $|W(\rho)| > |W(\theta)|$). Soit j le plus petit entier, $1 \leqslant j \leqslant k-1$, tel que ρ_j appartient à $\alpha_i^-\alpha_{i+1}^-\dots\alpha_{k-1}^-$ mais pas à ρ_{j+1}. Alors, la structure particulière (Fait 3') de σ induit que la sous-suite $\rho' = \theta_{i+1}\theta_{i+2}\dots\theta_{i+j}\rho_{j+1}\rho_{j+2}\dots\rho_k$ est ordonnée comme D_k et $x \in W(\rho')$. Cela donne une contradiction avec le fait que ρ est un élément maximal de $\mathcal{W}(x_1)$. Donc, le cas (ii) ne se produit jamais.

Regardons le cas (iii) : $\rho_1 \in \alpha_i^+$. S'il existe j, $1 \leqslant j \leqslant k-1$ tel que $\rho_j \in \alpha_i^+$ et $\rho_{j+1} \in \alpha_i^-$, alors on considère ℓ comme étant le plus petit entier $j+1 \leqslant \ell \leqslant k-1$, tel que ρ_ℓ appartient à $\alpha_i^-\dots\alpha_{k-1}^-$ mais pas $\rho_{\ell+1}$ (ℓ existe car ρ_k est sur la droite de θ_k). Avec le même argument que pour le cas (ii), Fait 3' induit que $\rho' = \rho_1\dots\rho_j\theta_{i+1}\theta_{i+2}\dots\theta_{i+\ell-j}\rho_{\ell+1}\dots\rho_k$ est une sous-suite dans $\mathcal{W}(x_1)$ tel que $\rho \trianglelefteq \rho'$. C'est une contradiction. Donc la sous-suite $\rho^1 = \rho$ vérifie la propriété que toute valeur de ρ^1 est nécessairement dans $\alpha_i^+\theta_{i+1}\alpha_{i+1}^+\dots\alpha_{k-1}^+\theta_k\alpha_k\theta_1\alpha_1^-\theta_2\alpha_2^-\dots\alpha_{i-1}^-\theta_i$. Moins formellement, on dit que ρ_1 est situé au-dessus de θ dans σ^i. Puisque cette propriété est cruciale pour la suite de la preuve, on appelle cette propriété la *propriété de dominance* de ρ^1 sur θ.

Maintenant on considère le plus petit j_1, $1 \leqslant j_1 \leqslant k-1$, tel que $\rho^1_{j_1}$ appartient à α_i mais pas $\rho^1_{j_1+1}$. En considérant le Fait 4 (ou 4') avec la propriété de dominance de ρ^1 sur θ, on déduit $j_1 < i-1$.

Maintenant, on remplace $x_1 = x$ par la valeur x_2 de α_i juste après ρ^1_1 (si elle existe ; sinon on prend $x_2 = \theta_{i+1}$). Par hypothèse, x_2 est aussi couvert par le segment témoin $W(\rho^2)$ où ρ^2 est un autre motif ordonné comme D_k, et tel que ρ^2_k appartient à σ^i sur la doite de ρ^1_k ou sur la gauche de x_2. Avec le même argument que ci-dessus, ρ^2 a la propriété de dominance sur ρ^1, *i.e*, moins formellement, la suite ρ^2 est au-dessus ρ^1. Donc, ρ^2 contient au plus $j_2 \leqslant j_1 - 1$ valeurs dans α_i^+.

En itérant ce raisonnement pour la valeur juste après ρ^2, on construit une sous-suite ρ^3 telle que le nombre j_3 de ses valeurs dans α_i^+ vérifie $j_3 \leqslant j_2 - 1$.

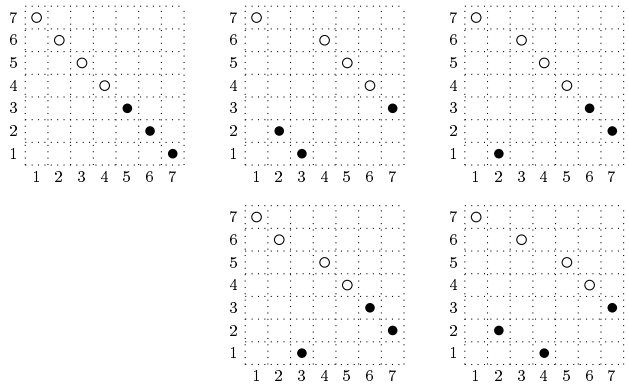
Le processus finit après au plus $i-1$ étapes. Cela signifie que α_i peut être couvert par les segments témoins $W(\rho^\ell)$ pour certains ℓ, tel que $1 \leqslant \ell \leqslant i-2 \leqslant k-2$, *i.e.* avec au plus $i-2$ segments témoins, ce qui est une borne dépendant de k seulement. Par conséquent $\alpha_i\theta_{i+1}$ peut être couvert par au plus $i-1$ segments témoins. Donc, σ est couvert par au plus $1 + \sum_{\ell=1}^{k-1}\ell = 1 + \frac{k(k-1)}{2}$ segments témoins. □

Théorème III.14. *$cc(S_{2k-\ell}(D_k)) = S_{2k-\ell}$ pour $2 \leqslant \ell \leqslant 2k-1$.*

Théorème III.15. *$cc(S_{2k-1}(D_k))$ est énuméré par $(2k-1)! - (2k-1)c_{k-1}$ ou $(c_k)_{k\geqslant 1}$ est la suite de Catalan.*

Notons qu'une permutation de taille $(2k-1)$ $\sigma \notin cc(S(D_k))$ commençant par $2k-1$ ne contient pas de motif de la forme 123 (cette remarque est déduite de la preuve du théorème précédent). De plus, la preuve du théorème précédent induit une bijection constructive entre l'ensemble des bons parenthésages de taille $2k-2$ et l'ensemble des permutations de taille $2k-1$, commençant par $2k-1$, et contenant un motif D_k dans chaque rotation cyclique. En effet, on considère la représentation binaire $b = b_1\dots b_{2k-2}$ des bons parenthèsages, *i.e.*

Figure III.5 : Les cinq permutations σ commençant par 7 telles que $\sigma \notin cc(S_7(D_4))$: **7654**321, **7**21**6543**, **7165432**, **7615432**, **7261543**. Le motif le plus à gauche θ est illustré en gras dans la notation linéaire de σ et illustré par des points vides dans la représentation ci-dessous. Les bons parenthèsages correspondant sont respectivement ((())), (())(), (()()), ()(()) et ()()().



$b_i = 0$ si la i-ième parenthèse est une parenthèse fermante, $b_i = 1$ sinon). Par convenance, on ajoute $b_{2k-1} = 0$ sur la droite de b. Maintenant, soit j, $1 \leqslant j \leqslant 2k-1$, le rang du zéro le plus à droite à gauche du un le plus à droite dans b (si j n'existe pas on pose $j = 2k-1$). Alors, on traverse b de la droite vers la gauche (de l'indice j à 1), et on étiquète chaque zéro en ordre croissant à partir de 1. Finalement, on continue d'étiqueter (en ordre croissant) les éléments restants de b de la droite vers la gauche. Par exemple, le parenthèsage ()((()())((()))) a une représentation binaire 1011101**0**11100000 où $j = 9$ (en gras). Sa permutation correspondante 17 4 16 15 14 3 13 21 12 11 10 9 8 7 6 5 contient le motif D_9 dans chaque rotation cyclique. Voir la figure III.5 dans le cas $k = 4$.

Le théorème III.13 montre que la base B de $cc(S(D_k))$ est finie. Les théorèmes III.14 et III.15 induisent que la plus petite taille d'une permutation minimale est $2k-1$, et que ces permutations sont énumérées par $(2k-1) \cdot c_{k-1}$. Il reste à caractériser les autres éléments de la base B, ce que je n'ai pas réussi à faire. Notons que Albert et al. ont obtenu expérimentalement les éléments de la base de $cc(S(321))$: 15432, 14325, 164253, 163254 et 1472536. Nous avons obtenus expérimentalement que la base de $cc(S(4321))$ contient 5 permutations minimales de taille 7, 32 de taille 8, 54 de taille 9, et 136 de taille 10.

Théorème III.16. *Les inversions de taille $2k-1$ qui n'appartiennent pas à $cc(S(D_k))$ sont énumérées par $\frac{k(k+1)}{2}$.*

Pour $k = 3$, il y a 6 involutions de taille 5 qui n'appartiennent pas à $cc(S(D_3))$: 54321, 15432, 21543, 32154, 43215 et 14325.

III.3.1 La cloture cylindrique

Dans ce paragraphe, on étudie la cloture cylindrique $c\ell(\mathcal{C})$ d'une classe fermée $\mathcal{C}$, *i.e.*, la classe des permutations pouvant être obtenues par une rotation cylindrique d'une per-

mutation dans $\mathcal{C}$. On fournit plusieurs résultats généraux faisant des liens entre les clotures cyclique et cylindrique.

Lemme III.17. *Si X est un ensemble de permutations alors $\pi \in cc(X) \iff \pi^{-1} \in c\ell(X^{-1})$. Par conséquent, si $X = S(B)$ alors $cc(S(B)) = c\ell(S(B^{-1}))^{-1}$. De plus, si la base B est stable par inversion, i.e. $B = B^{-1}$, alors $cc(S(B)) = c\ell(S(B))^{-1}$.*

Maintenant nous posons $\mathcal{C} = S(B)$ où B est une base. Comme pour le cas de la rotation cyclique, $c\ell(\mathcal{C})$ est aussi une classe de permutations à motifs exclus. Soit σ une permutation dans $\mathcal{C}$ et θ un motif (appartenant à B) dans σ. On définit par *intervalle témoin* de σ l'intervalle cyclique $W'(\theta) =]\max(\theta)..\min(\theta)] = [n]\backslash]\min(\theta)..\max(\theta)]$. Notons que $W'(\theta)$ est un intervalle dans $[n]$ considéré circulairement, mais ce n'est pas forcément un segment dans σ. On remarque facilement que $W'(\theta)$ est aussi un segment témoin de σ^{-1} (considéré circulairement) relativement au motif θ^{-1}. Deux résultats similaires aux lemme III.11 et proposition III.12 peuvent être déduit :

Lemme III.18. *$\sigma \notin c\ell(\mathcal{C})$ si est seulement si les intervalles témoins $W'(\theta)$ couvrent $[n]$ (ou de façon équivalente σ).*

Proposition III.19. *Soit $\mathcal{C} = S(B)$, où B est de cardinalité finie et supposons qu'il existe une borne Δ dépendante de B seulement telle que, pour tout $\sigma \notin c\ell(X)$, il y a une collection d'au plus Δ intervalles témoins qui couvrent σ (ou de façon équivalente $[n]$). Alors $c\ell(X)$ a une base finie.*

Proof. La preuve est obtenue de la même façon que celle de la proposition III.12. □

Théorème III.20. *Soit $k \geqslant 2$ un entier. Alors $c\ell(S(D_k))$ admet une base finie.*

Tous les résultats d'énumération obtenus dans [2] pour la cloture cyclique sont aussi valides pour la cloture cylindrique.

Théorème III.21. *$c\ell(S_{2k-1}(D_k))$ est énuméré par $(2k-1)! - (2k-1)c_{k-1}$ ou $(c_k)_{k\geqslant 1}$ est la suite de Catalan.*

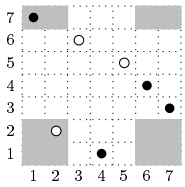
III.3.2 Discussion et Conclusion

Dans ce paragraphe, on donne des résultats généraux concernant la cloture torique $tr(\mathcal{C})$ pour une classe $\mathcal{C}$ de permutations, *i.e.* $tr(\mathcal{C}) = c\ell(cc(\mathcal{C})) = cc(c\ell(\mathcal{C}))$. Soit $f_{u,v}$, $1 \leqslant u, v \leqslant n$, les fonctions définies sur S_n par $f_{u,v}(\sigma) = \sigma'$ où $\sigma'_j = (\sigma_{(u+j-2) \mod (n)+1} - v) \mod (n) + 1$, pour $j \in [1..n]$. L'ensemble $\{f_{u,v}(\sigma),\ 1 \leqslant u, v \leqslant n\}$ contient exactement toutes les rotations toriques de σ puisque la permutation $f_{u,v}(\sigma)$ est égale à la permutation $r^{n-u+1}(\ell^{n-v+1}(\sigma))$.

Maintenant supposons que $\mathcal{C} = S(B)$ où B est un ensemble fini de permutations. Soit θ un motif (relativement à B) dans $\sigma \in \mathcal{C}$. L'*aire témoin* $W''(\theta)$ de θ dans σ est le produit direct de $\sigma^{-1}(W(\theta))$ de θ avec l'intervalle témoin $W'(\theta)$: $W''(\theta) = \sigma^{-1}(W(\theta)) \times W'(\theta)$. Par exemple, la permutation **7261**543 contient le motif 132 ($\theta = 265$), $W(\theta)$ est le segment 4372, alors $\sigma^{-1}(W(\theta)) = \{6, 7, 1, 2\}$, $W'(\theta)$ est l'intervalle 712, et $W''(\theta)$ est l'aire $\{6, 7, 1, 2\} \times \{7, 1, 2\}$. Voir la figure III.6 pour une illustration.

On obtient alors un résultat similaire aux résultats obtenus précédemment concernant l'appartenance d'une permutation à la cloture torique d'une classe.

Figure III.6 : L'aire témoin $W''(265)$ pour la permutation 7261543.



Lemme III.22. *$\sigma \notin tr(\mathcal{C})$ si et seulement si les aires témoins couvrent $[n] \times [n]$ où n est la longueur de σ.*

Proposition III.23. *Soit $\mathcal{C} = S(B)$, où B est de cardinalité finie ; supposons qu'il existe une borne Δ dépendante de B seulement telle que, pour tout $\sigma \notin tr(\mathcal{C})$, il y a une collection d'au plus Δ aires témoins qui couvrent $[n] \times [n]$. Alors $tr(\mathcal{C})$ a une base finie.*

Nous avons étudié la base pour la cloture de $tr(S(321))$. Des calculs expérimentaux suggèrent que $tr(S(321))$ a une base finie. De plus la base de $tr(S(321))$ contient probablement une permutation minimale de longueur 5, 2 de longueur 7, 2 de longueur 8, 39 de longueur 9 et 2 de longueur 10.

Cependant, nous nous sommes intéréssés à prouver ce résultat. Il semblerait que la preuve ait été forcée mais je préfère ne pas présenter celle ci pour les raisons suivantes. La preuve demande l'étude de beaucoup de cas et chacun d'eux induisent des sous-cas assez lourds à expliciter. En attendant l'obtention d'une preuve élégante de ce résultat, si elle existe, je considère que ce problème n'est pas encore résolu. Je propose maintenant trois questions ouvertes.

Problème 1 : Est-ce que $tr(S(321))$ a une base finie ?
Problème 2 : Est-ce que $tr(S(231))$ a une base finie ?
Problème 3 : Plus généralement, soit $k \geqslant 2$ un entier ; est-ce que $tr(S(D_k))$ a une base finie ?

Il reste cependant de nombreuses questions en suspens notamment dans le domaine de l'énumération des classes qui sont clotures toriques d'une classe de permutations à motifs exclus.

Chapitre IV

Perspectives

J'ai exposé dans ce mémoire les principaux travaux de recherche concernant : *la génération exhaustive de classes d'objets combinatoires*, *l'étude de structures en treillis*, et *l'étude des motifs dans les permutations.*

Je propose ci-dessous quelques exemples de problèmes encore ouverts concernant les domaines traités dans ces trois chapitres.

Génération exhaustive

Trouver un algorithme permettant de générer efficacement les permutations de taille n ayant un nombre donné de descentes (même question en imposant que la liste soit un code de Gray).

Trouver un algorithme permettant de générer efficacement en code de Gray les permutations de taille n ayant un nombre donné d'excédences.

Trouver un algorithme permettant de générer en code de Gray optimal les ensembles $S_{2n}(321)$ et $S_{2n}(312)$.

Construire un paquetage Maple (ou autre) permettant d'implanter tous les algorithmes obtenus.

Structure en treillis

Poursuivre les recherches concernant l'obtention d'un algorithme permettant de calculer en temps polynomial la distance de rotation entre deux arbres dans le treillis de Tamari.

Exhiber des algorithmes permettant de calculer en temps polynomial la distance entre deux arbres dans les treillis 'phagocyte' et le treillis 'taille-greffe'.

Résoudre la conjecture énoncée dans ce mémoire : Si $v(T)$ est la longueur du plus long chemin entre $\mathbf{1}$ et T alors on conjecture que $v(T) + v(T') \leqslant v(T \vee T') + v(T \wedge T')$ et si $T < T'$ alors $v(T) < v(T')$. Si tel est le cas, alors $d(T, T') = v(T) + v(T') - 2v(T \vee T')$ serait une distance sur ce treillis.

Enumérer le nombre d'intervalles dans les deux nouveaux treillis.

Trouver un algorithme efficace non-récursif pour calculer la fonction de Möbius du treillis 'taille-greffe' et prouver qu'elle prend ses valeurs dans $\{-1, 0, +1\}$.

Motifs dans les permutations

Trouver un algorithme efficace permettant de trouver la permutation π qui minimise la somme $d(\pi, \sigma_1) + d(\pi, \sigma_2)$ où $d(\pi, \sigma)$ est le nombre minimum de WM-duplications nécessaires pour transformer π en σ.

Caractériser la classe de permutations à motifs exclus correspondante aux permutations obtenues à partir de W ou WM-duplications.

Montrer que la cloture torique de $S(321)$ est à base finie. Même question pour $S(231)$.

Examiner la finitude de la base de la cloture torique de $S(k(k-1)\ldots 321)$.

Autres directions de recherche

Continuer à explorer en terme de permutation des problèmes issus de la bioinformatique (duplication de gène).

Etudier certains problèmes combinatoires issus du domaine de l'Image (tomographie).

Annexes

Liste des tableaux

I.1 Complexité moyenne pour la génération de chaque classe. 15

I.2 Les listes $\mathcal{L}_{4,2}$ et $\mathcal{L}_{4,3}$. Les modifications de bits sont en gras. 17

I.3 Les listes $\mathcal{D}_4$ et $\mathcal{D}_5$. Dans $\mathcal{D}_5$ les sous-listes $\phi(i, \mathcal{D}_4)$ et $\psi(i, \mathcal{D}_3)$, $1 \leqslant i \leqslant 4$, sont respectivement en gras et en italiques. 19

I.4 Les listes $\mathcal{S}_{3,k}$, $1 \leqslant k \leqslant 3$, et $\mathcal{S}_{4,k}$ pour $1 \leqslant k \leqslant 4$. Par exemple, dans $\mathcal{S}_{4,2}$ les sous-listes de la relation (d), $\psi(1, \mathcal{S}_{3,2})$, $\overline{\psi(3, \mathcal{S}_{3,2})}$, $\phi(\mathcal{S}_{3,1})$ et $\overline{\psi(2, \mathcal{S}_{3,2})}$, sont alternativement en gras et italique. 21

I.5 Les listes $\mathcal{S}_{4,2}$ et $\mathcal{T}_{4,2}$. Chaque suite dans $\mathcal{T}_{4,2}$ est le tableau de transpositions de la permutation correspondante dans $\mathcal{S}_{4,2}$. 23

I.6 Les listes $\mathcal{S}_{3,k}$, $1 \leqslant k \leqslant 3$, et $\mathcal{S}_{4,k}$ pour $1 \leqslant k \leqslant 4$. Par exemple, dans $\mathcal{S}_{4,2}$ les sous-listes de la relation (d), $\psi(3, \mathcal{S}_{3,2})$, $\overline{\phi(\mathcal{S}_{3,1})}$, $\psi(1, \mathcal{S}_{3,2})$, et $\overline{\psi(2, \mathcal{S}_{3,2})}$, sont alternativement en gras et en italique. 27

I.7 Les codes de Gray optimaux pour $\mathcal{S}_5(312)$ et $\mathcal{S}_5(321)$. Deux éléments consécutifs diffèrent en au plus trois positions. 31

I.8 Classes de permutations évitant les motifs de T avec leur fonctions de succession $\chi_T(i, k)$. On donne aussi les différents lemmes utilisés pour la preuve du code de Gray. On compare aussi les nombres maximum des différences de position entre deux permutations consécutives dans [67] et dans notre code de Gray **B**. Un point dans la dernière colonne signifie qu'il n'y a pas de code de Gray connu pour cette classe dans la littérature. 32

Table des figures

I.1 Algorithme de génération pour les permutations avec un nombre fixé d'excédences et les listes $E_{4,1}$, $E_{4,2}$ et $E_{5,1}$. 10

I.2 Les cinq premiers niveaux de l'arbre de génération codé par les mots de Fibonacci 3-généralisés. 12

I.3 Les cinq permiers niveaux de l'arbre de génération de (Ω_3) codé par les permutations de Fibonacci 3-généralisées $S_n(321, 312, 2341)$. Les sites actifs sont représentés en soulignés. 12

I.4 Les cinq premiers niveaux de l'arbre de génération de (Φ_3). Chaque noeud est codé par une permutation de Lucas dans $S_n(321, 312, 2341, \overline{1}342{:}65, \overline{1}32{:}564)$. . . 14

I.5 La liste $\mathcal{D}_n$: (a) n est pair, (b) n est impair. 19

I.6 Les relations (b) pour $n = 8$ et $n = 9$. Chaque point i correspond à la liste $\psi(i, \mathcal{S}_{n-1,1})$ et chaque point encerclé i représente la liste $\overline{\psi(i, \mathcal{S}_{n-1,1})}$, *i.e.*, la liste $\psi(i, \mathcal{S}_{n-1,1})$ dans l'ordre inverse. Par exemple, si $n = 8$ alors $\mathcal{S}_{8,1} = \psi(7, \mathcal{S}_{7,1}) \circ \psi(5, \mathcal{S}_{7,1}) \circ \overline{\psi(3, \mathcal{S}_{7,1})} \circ \psi(1, \mathcal{S}_{7,1}) \circ \overline{\psi(2, \mathcal{S}_{7,1})} \circ \psi(4, \mathcal{S}_{7,1}) \circ \overline{\psi(6, \mathcal{S}_{7,1})}$. 25

I.7 Les relations (c) pour $n = 8$ et $n = 9$. Chaque point $i \neq 0$ correspond à la liste $\psi'(i, \mathcal{S}_{n-1,n-1})$ et le point 0 représente la liste $\phi'(\mathcal{S}_{n-1,n-2})$. Par exemple, si $n = 8$ alors $\mathcal{S}_{8,7} = \psi'(8, \mathcal{S}_{7,7}) \circ \phi'(\mathcal{S}_{7,6}) \circ \psi'(6, \mathcal{S}_{7,7}) \circ \psi'(4, \mathcal{S}_{7,7}) \circ \psi'(2, \mathcal{S}_{7,7}) \circ \psi'(3, \mathcal{S}_{7,7}) \circ \psi'(5, \mathcal{S}_{7,7}) \circ \psi'(7, \mathcal{S}_{7,7})$. 25

I.8 Les relations (d) pour $n = 8$ et $n = 9$. Chaque point $i \neq 0$ correspond à la liste $\psi(i, \mathcal{S}_{n-1,n-2})$ et le point 0 est la liste $\phi(\mathcal{S}_{n-1,n-3})$. Quand le point est encerclé, on considere la liste en ordre inverse. Par exemple, si $n = 8$ alors $\mathcal{S}_{8,6} = \psi(7, \mathcal{S}_{7,6}) \circ \overline{\phi(\mathcal{S}_{7,5})} \circ \psi(5, \mathcal{S}_{7,6}) \circ \overline{\psi(3, \mathcal{S}_{7,6})} \circ \psi(1, \mathcal{S}_{7,6}) \circ \overline{\psi(2, \mathcal{S}_{7,6})} \circ \psi(4, \mathcal{S}_{7,6}) \circ \overline{\psi(6, \mathcal{S}_{7,6})}$. 26

I.9 Les relations (e) pour $n = 8, 9$ et $k = 3, 4$. Chaque point $i \neq 0$ correspond à la liste $\psi(i, \mathcal{S}_{n-1,k})$ et le point 0 est la liste $\phi(\mathcal{S}_{n-1,k-1})$. Quand le point est encerclé, on considère la liste en ordre inverse. Par exemple, si $n = 8$ et $k = 4$ alors $\mathcal{S}_{8,4} = \psi(7, \mathcal{S}_{7,4}) \circ \overline{\psi(5, \mathcal{S}_{7,4})} \circ \phi(\mathcal{S}_{7,3}) \circ \overline{\psi(3, \mathcal{S}_{7,4})} \circ \psi(1, \mathcal{S}_{7,4}) \circ \overline{\psi(2, \mathcal{S}_{7,4})} \circ \psi(4, \mathcal{S}_{7,4}) \circ \overline{\psi(6, \mathcal{S}_{7,4})}$. 27

I.10 Si $k = 7$ alors $\Pi(\sigma^1)$ est la liste $\phi(3, \sigma)^0, \phi(5, \sigma)^1, \phi(7, \sigma)^0, \phi(6, \sigma)^1, \phi(4, \sigma)^0, \phi(2, \sigma)^1, \phi(1, \sigma)^1$; si $k = 8$, $\Pi(\sigma^1)$ est $\phi(3, \sigma)^0, \phi(5, \sigma)^1, \phi(7, \sigma)^0, \phi(8, \sigma)^1, \phi(6, \sigma)^0, \phi(4, \sigma)^1, \phi(2, \sigma)^0, \phi(1, \sigma)^1$. Chaque point (resp. point encerclé) étiqueté par $\ell \in [1..k]$ représente $\phi(\ell, \sigma)^1$ (resp. $\phi(\ell, \sigma)^0$). 28

I.11 Si $k = 7$ alors $\Pi'(\sigma^1)$ est la liste $\phi(2, \sigma)^0, \phi(4, \sigma)^1, \phi(6, \sigma)^0, \phi(7, \sigma)^1, \phi(5, \sigma)^0, \phi(3, \sigma)^1, \phi(1, \sigma)^1$; si $k = 8$, $\Pi'(\sigma^1)$ est $\phi(2, \sigma)^0, \phi(4, \sigma)^1, \phi(6, \sigma)^0, \phi(8, \sigma)^1, \phi(7, \sigma)^0, \phi(5, \sigma)^1, \phi(3, \sigma)^0, \phi(1, \sigma)^1$. Chaque point (resp. point encerclé) étiqueté par $\ell \in [1..k]$ represente $\phi(\ell, \sigma)^1$ (resp. $\phi(\ell, \sigma)^0$). 29

II.1 La représentation canonique du mot de Dyck $w \in D_9$ 35
II.2 Le treillis phagocyte D_4. 35
II.3 Une transformation 'phagocyte' $w \longrightarrow w_1$; $\pi_w = 19/23/45/678$ et $\pi_{w_1} = 19/23678/45$. 36
II.4 Trois transformations de taille-greffe dans B_7. 40
II.5 Le treillis 'taille-greffe' B_4. Chaque arbre est codé par ses suite de profondeurs et de distribution. 42
II.6 Le treillis 'taille-greffe' B_5. Chaque arbre est codé par ses suites des profondeurs et de distribution. 43
II.7 La rotation sur les arbres binaires. 45
II.8 Une bascule de diagonale dans $\mathcal{T}_8$ 46
II.9 Quatre normalisations $T \in \mathcal{T}_6$ 46

III.1 Une WM-duplication de la permutation 5421673. Les points encerclés sont supprimés lors de la procédure de perte. 52
III.2 Décomposition $\sigma = \tau\tau'$ et la permutation S obtenue après le processus de la preuve du Tthéorème III.3. On a $\tau = 11\ 6\ 5\ 10\ 3$, $\tau' = 9\ 8\ 1\ 7\ 4\ 2$, u_1 est vide, $d_1 = 11\ 6\ 5$, u_2 est vide, $d_2 = 10\ 3$, u_3 est vide, $d_3 = 9\ 8\ 1$, u_4 est vide, $d_4 = 7\ 4\ 2$ et, $D_2 = 10\ 3$, $U_2 = 1\ 8\ 9$; $D_1 = 11\ 6\ 5$; $U_1 = 2\ 4\ 7$. Donc $S = U_1D_1U_2D_2 = 2\ 4\ 7\ 11\ 6\ 5\ 1\ 8\ 9\ 10\ 3$. 55
III.3 Etiquetage des parcours montant et descendant de $\sigma = 231684597$ avec le code de Gray réfléchi. Exemple de chemin de WM-duplications à partir de l'identité vers σ. 58
III.4 La structure spéciale d'une permutation $\sigma \notin S(D_k)$. Une aire remplie ne contient pas de point de la forme (i, σ_i), $1 \leqslant i \leqslant n$. Une suite de j diagonales croissantes dans une aire signifie que l'aire ne contient pas de motif $(j+1)j \ldots 21$. 62
III.5 Les cinq permutations σ commençant par 7 telles que $\sigma \notin cc(S_7(D_4))$: **7654**321, **7**2**16543**, **7**1**65**432, **7**6**15**432, **7**2**61543**. Le motif le plus à gauche θ est illustré en gras dans la notation linéaire de σ et illustré par des points vides dans la représentation ci-dessous. Les bons parenthèsages correspondant sont respectivement ((())), (())(), (()()), ()(()) et ()()(). 64
III.6 L'aire témoin $W''(265)$ pour la permutation 7261543. 66

Liste des publications

Revues internationales

1. J.-L. Baril, J.-M. Pallo, Matchings in three Catalan lattices, *Intern. J. Computer Math.*, Vol. 80(2), pp. 193-200, 2003.
2. J.-L. Baril, Gray code for Cayley permutations, *Computer Science Journal of M.*, Vol. 11 (2), pp. 124-136, 2003.
3. J.-L. Baril, V. Vajnovszki, Gray Code for Derangements, *Discrete Applied Mathematics*, Vol. 140 (1-3), pp. 207-221, 2004.
4. J.-L. Baril, V. Vajnovszki, Minimal change list for Lucas Strings and some graph theoritic consequences, *Theoretical Computer Science*, A dedicated to the memory of Alberto Del Lungo, Vol. 346 (2-3), pp. 189-199, 2005.
5. J.-L. Baril, H. Kheddouci, O. Togni, Irregularity strength of circulant graphs, *Discrete Math.*, Vol. 304 (1-3), pp. 1-10, 2005.
6. J.-L. Baril, H. Kheddouci, O. Togni, Adjacent vertex distinguishing edge-colorings of meshes and hypercubes, *Australasian Journal of Combinatorics*, Vol. 35, pp. 89-102, 2006.
7. J.-L. Baril, J.-M. Pallo, The Phagocyte Lattice of Dyck Words, *Order*, Vol 23(2-3), pp. 97-107, 2006.
8. J.-L. Baril, J.-M. Pallo, Efficient lower and upper bounds of the diagonal-flip distance between triangulations, *Information Processing Letters*, Vol. 100, pp. 131-136, 2006.
9. J.-L. Baril, Gray code for permutations with a fixed number of cycles, *Discrete Math.*, Vol. 307, pp. 1559-1571, 2007.
10. J.-L. Baril, Hamiltonian paths for involutions in the square of a Cayley graph, *Journal of Discrete Mathematical Sciences & Cryptography*, Vol. 10(2), 2008.
11. J.-L. Baril, P.-T. Do, ECO-generation for p-generalized Fibonacci and Lucas permutations, *Journal of Pure Mathematics and Applications*, Vol. 17(1-2), pp. 19-37, 2006.
12. J.-L. Baril, H. Kheddouci, O. Togni, Vertex distinguishing edge- and total-colorings of cartesian and other product graphs, A paraître dans *Ars Combinatoria*, 2010.
13. J.-L. Baril, J.-M. Pallo, The pruning-grafting lattice of binary trees, *Theoretical Computer Science*, Vol. 409(3), pp. 382-293, 2008.
14. J.-L. Baril, C. Moreira Dos Santos, Gray code for compositions of n with parts 1 and p, *Advances and Applications in Discrete Mathematics*, Vol. 3(1), pp. 67-84, 2009.
15. J.-L. Baril, More restrictive Gray codes for some classes of pattern avoiding permutations, *Information Processing Letters*, Vol. 109, pp. 799-804, 2009.
16. J.-L. Baril, O. Togni, Neighbor-distinguishing k-tuple edge-colorings of graphs, *Discrete Mathematics*, Vol. 309(16), pp. 5147-5157, 2009.
17. J.-L. Baril, Efficient generating algorithm for permutations with a fixed number of ex-

cedances, *Pure Mathematics and Applications/Algebra and Theoretical Computer Science*, Vol 19(3-4), pp. 61-70, 2009.
18. J.-L. Baril, Gray code for permutations with a fixed number of left-to-right minima, A paraître dans *Ars Combinatoria*, 2010.
19. J.-L. Baril, R. Vernay, Whole mirror duplication-random loss model and pattern avoiding permutations, *Information Process. Letters*, Vol. 110, pp. 474-480, 2010.
20. J.-L. Baril, Cyclic and lift closures of $S_n(k(k-1)\dots 321)$., Submitted 2010

Livres

J.-L. Baril, M. Poneti, *Pure Mathematics and Applications/ Algebra and Theoretical Computer Science*, Firenze, 2008.

Conferences internationales

1. J.-L. Baril, V. Vajnovszki, Gray codes for order p Lucas strings, *Proceedings of the 4th International Conference on Combinatorics on Words*, September 2003, Turku, Finland (ed. Tero Harju and Juhani Karhumaki), pp. 149-158. TUCS General Publication, No 27, August 2003.
2. J.-L. Baril, P.-T. Do, Generalized Fibonacci permutations generation by the ECO method, *Conference on Combinatorics, Automata and Number Theory*, Liège, 8-19 May 2006.
3. J.-L. Baril, P.-T. Do, ECO generation for Fibonacci and Lucas permutations, *Gascom'06*, Dijon, 12-15 September 2006.
4. J.-L. Baril, Gray code for permutations with exactly k-cycles, *Gascom'06*, Dijon, 12-15 September 2006.
5. J.-L. Baril, A CAT algorithm for generating permutations with a fixed number of excedances, *Gascom'08*, Dedicated to Jean-Guy Penaud on the occasion of his 65th birthday, Bibbiena, Italy, 16-20 june 2008.
6. J.-L. Baril, H. Kheddouci, O. Togni, VD-colorings of product graphs, *Gascom'08*, Dedicated to Jean-Guy Penaud on the occasion of his 65th birthday, Bibbiena, Italy, 16-20 june 2008.
7. J.-L. Baril, C. Moreira Dos Santos, More restrictive Gray code for (1,p)-compositions and relatives, *Journées Montoises d'Informatique Théorique*, UMH, Université de Mons-Hainaut, Belgique, 28 aout 2008.
8. J.-L. Baril, P.-T. Do, ECO generation for compositions and their restrictions, *Pattern permutations*, University of Otago, New Zealand, 16 - 20 June, 2008.
9. J.-L. Baril, P.-T. Do, Efficient generating algorithms for some combinatorial object classes. *Combinatorics and Statistical Mechanics, Erwin Schrödinger International Institute for Mathematical Physics*, Vienna, Austria, July 2008.
10. J.-L. Baril, R. Vernay, Whole mirror duplication-random loss model and pattern avoiding permutations, *Pattern permutations*, University of Firenze, Italy, 13 - 17 July, 2009.

Conférences nationales

1. J.-L. Baril, V. Vajnovszki, Hamiltonisme du cube de Lucas d'ordre p, *Journées Graphes 2003*, Dijon.
2. J.-L. Baril, H. Kheddouci, O. Togni, Force d'irrégularité des graphes circulants, *Journées Graphes 2003*, Dijon.
3. J.-L. Baril, R. Vernay, Autour de la duplication du génome, *EJCIM' 09*, Clermont, 2009.

Citations de mes travaux

Listes des auteurs citant mes travaux

Barcucci E. :[19],
Baskoro E.T. :[20],
Batut Ch. :[5,9],
Bergé A.M. :[11],
Bernini A. :[18,21,19],
Cleary S. :[22,38,37],
Cuckler B. :[30],
Diaconis P. :[28],
Domaratzki M. :[14],
Dukes W.M.B. :[29],
Dutour-Sikiric M. :[17],
Flanagan M.F. :[29],
Fulman J. :[28],
Grazzini E. :[21],
Greenhill C. :[16],
Guralnick R. :[28],
Kokosinski Z. :[25],
Huemer C. :[35],
Hurtado F. :[35],
Jezek J. :[3],
Juarna A. :[23],
Kim Y. :[42],
Korsh J. :[12],
Lafolette P.S. :[12],
Laihem M. :[11],
Lazenik F. :[30],
Li S. :[32],
Lin X. :[36,43],
Luccio F. :[22],
Mansour T. :[29,33,34],
Martinet J. :[41,6],
Nawawi M.S. :[20],
Nassar G. :[33],
Noy M. :[35],
Nurdin :[20],
Omana-Pulido E. :[35],
Pagli L. :[22],
Pergola E. :[21],
Pinzani R. :[21],
Poneti M. :[40,19],
Rucinski A. :[16],
Schürmann A. :[13,17],
Sloane N.J.A. :[2,4],
St-John K. :[38,37],
Stojmenovic I. :[15],
Tong C. :[36,43],
Vallentin F. :[13,17],
Vajnovszki V. :[23,24,29,40,26,27],
Wang D. :[32],
Wang L. :[43],
Wang X. :[32],
Yang Y. :[36,43],
Zhang S. :[32],
Zhihe L. :[36],

Liste des publications citant mes travaux

1. J. Martinet, *http ://www.math.u-bordeaux.fr/ martinet*
2. N.J.A. Sloane, *http ://www.research.att.com/ njas/lattices/perfect.html*

3. J. Jezek, *http ://www.karlin.mff.cuni.cz/ jezek.*
4. N.J.A. Sloane, *Eulerian Numbers, A000295, http ://www.research.att.com/ njas/sequences/*
5. Ch. Batut, J. Martinet, *Radiographie des réseaux parfaits, Experimental Mathematics, 3(1), 39-49*, 1994.
6. J. Martinet, *Les réseaux parfaits dans les espaces euclidiens. Editions Dunod*, 1996.
7. J. Martinet, *Une famille de réseaux dual-extrêmes, Journal de Théorie des Nombres de Bordeaux, 9, 169-181*, 1997.
8. J. Martinet, *Algebraic Constructions of Lattices ; Isodual Lattices, Acts of the 1996 Eger conference, Györy, Pethö, Sós, eds, Walter de Gruyter GmbH, Berlin-New York, pp. 349 - 360*, 1998.
9. Ch. Batut, *Classification of quintic eutactic forms, Math. Comp. 70, 395-417*, 2001.
10. J. Martinet, *Perfect lattices in euclidean spaces. Springer-Verlag*, 2002.
11. A.-M. Bergé, M. Laihem, *Lattices with a prescribed hyperplane perfect section, preprint*, 2002.
12. J. Korsh, P. S. Lafolette, *Constant time generation of derangements, Inf. Process. Lett., 90 (4), 181-186*, 2004.
13. A. Schürmann, F. Vallentin, *Methods in the local theory of packing and covering lattices, arXiv :math.MG/0412320v1*, 2004.
14. M. Domaratzki, *Trajectory based operations, Thesis, Queen's University, Kingston, Ontario, Canada*, 2004.
15. I. Stojmenovic, *Listing combinatorial objects in parallel, The international Journal of Parallel, Emergent and Distributed Systems, 21 (2), 127-146*, 2006.
16. C. Greenhill, A. Rucinski, *Neighbour-distinguishing edge colorings of random regular graphs, The electronic journal of combinatorics, 13, R77, 1-12*, 2006.
17. M. Dutour Sikiric, A. Schürmann, F. Vallentin, *Classification of eight dimensional perfect forms, Electron. Res. Announc. Amer. Math. Soc. 13, 21-32. math/0609388*, 2007.
18. A. Bernini, *Some properties of pattern avoiding permutations, Thèse, Florence, Italie*, 2006.
19. E. Barcucci, A. Bernini, M. Poneti, *From Fibonacci to Catalan permutations, Journal of Pure Mathematics and Applications, Vol. 17(1-2), pp. 1-17.*, 2006.
20. Nurdin, Baskoro E.T., Nawawi M.S.A., *The total edge irregular strengths of union graphs of $K_{2,n}$, International Conference on Mathematics and Natural Sciences*, 2006.
21. A. Bernini, E. GrazZini, E. Pergola, R. Pinzani, *A general exhaustive generation algorithm for Gray structures, Acta Informatica, vol. 44, 5, 361-376*, 2007.
22. S. Cleary, F. Luccio, L. Pagli, *Refined upper bounds for right-arm rotation distances, Theoretical Computer Science, 377, 277-281*, 2007.
23. A. Juarna, V. Vajnovszki, *Some Generalizations of a Simion-Schmidt Bijection, The Computer Journal, 50(5), 574-580*, 2007.

24. V. Vajnovszki, *Gray code order for Lyndon words, Discrete Mathematics and Theoretical Computer Science, 9 (2), 145-152*, 2007.
25. Z. Kokosinski, *On parallel generation of partial derangements, derangements and permutations, Proc. 7th Int. Conference "Parallel Processing and Applied Mathematics" PPAM'2007, Gdansk, Poland [in :] Lecture Notes in Computer Science*, 2008.
26. V. Vajnovszki, *Discrete Mathematics and Theoretical Computer Science, 9,2,145-152*, 2007.
27. V. Vajnovszki, *More restrictive Gray codes for necklaces and Lyndon words, IPL, 106, 3, 96-99*, 2007.
28. P. Diaconis, J. Fulman, R. Guralnick, *On fixed points of permutations, Journal of Algebraic Combinatorics*, 2008.
29. W.M.B. Dukes, M.F. Flanagan, T. Mansour, V. Vajnovszki, *Combinatorial Gray code for classes of pattern avoiding permutations, Theoretical computer science, vol. 396, 1-3, 35-49*, 2008.
30. B. Cuckler, F. Lazebnik, *Irregularity Strength of Denses Graphs, journal of graph theory, 58, 4, 299-313*, 2008.
31. Jacques Martinet, *Anne-Marie Bergé - In Memoriam, Journal de Théorie des Nombres de Bordeaux, Vol. 20, i-xi*, 2008.
32. D. Wang, X. Wang, S. Li, S. Zhang, Diagonal-flip distance algorithms of three type triangulations, *International conference on computer science and software engineering, Wuhan, China, IEEE*, 2008.
33. T. Mansour, G. Nassar, *Gray codes, Loopless algorithm and partitions, Journal of Mathematical Modelling and Algorithms 7 :3, 291-310*, 2008.
34. S. Heubach, T. Mansour, *Combinatorics of compositions and words, Discrete Mathematics and its Applications, Chapman and Hall Book, Taylor and Francis Group*, 2008.
35. C. Huemer, F. Hurtado, M. Noy, E. Omana-Pulido, *Gray codes for non-crossing partitions and dissections of a convex polygon, Discrete Applied Mathematics, 157, 7, 1509-1520*, 2009.
36. C. Tong, X. Lin, Y. Yang, L. Zhihe, *Equitable total coloring of $C_m \square C_n$, Discrete Applied Mathematics, 157, 596-601*, 2009.
37. S. Cleary, K. St. John, *A Linear-Time Approximation Algorithm for Rotation Distance, arXiv :math.CO/09030199v1*, 2009.
38. S. Cleary, K. St. John, *Rotation Distance is Fixed-Parameter Tractable, arXiv :math.CO/09030197v1*, 2009.
39. F. Luccio, L. Pagli, *The Fermat star of binary trees, Information Processing Letters, 109, 11, 568-571*, 2009.
40. M. Poneti, V. Vajnovszki, *Generating restricted classes of involutions, Bell and Stirling permutations, European Journal of Combinatorics*, 2009.
41. J. Martinet, *Compléments pour le livre : Les réseaux parfaits dans les espaces euclidiens. Editions Dunod*, 2009.

42. Y. Kim, *On semistability of root lattices and perfect lattices*, 2010.

43. C. Tong, X. Lin., Y. Yang, L. Wang., *Irregular total labellings of some families of graphs, Indian Journal of Pure and Applied Mathematics, 40 (3), 155-181*, 2009.

Bibliographie

[1] S.G. Akl, A new algorithm for generating derangements, *BIT*, 20 (1980), 2–7.

[2] M.H. Albert, R.E.L. Aldred, M.D. Atkinson, H.P. van Ditmarsch, C.C. Handley, D.A. Holton, D.J. McCaughan and C.W. Monteith, Cyclically closed pattern classes of permutations, *Australasian Journal of Combinatorics*, 38 (2007), 87–100.

[3] R. Arratia, On the stanley-Wilf conjecture for the number of permutations avoiding a given pattern, *Electronic J. Combin.*, 1 (1999), N1.

[4] M.D. Atkinson, Restricted permutations, *Discrete Math.*, 195 (1998), 27–38.

[5] M.D. Atkinson and T. Stitt, Restricted permutations and the wreath product, *Discrete Math.*, 259 (2002), 19–36.

[6] S. Bacchelli, E. Barcucci, E. Grazzini and E. Pergola, Exhaustive Generation of Combinatorial Objects by ECO, *Acta Informatica*, 40 (2004), 585–602.

[7] C. Banderier, P. Flajolet, D. Gardy, M. Bousquet-Melou, A. Denise, and D. Gouyou-Beauchamps. Generating functions for generating trees. *Discrete Mathematics*, 246 (2002), 29–55.

[8] E. Barcucci, A. Bernini, L. Ferrari, M. Poneti, A distributive lattice structure connecting Dyck paths, noncrossing partitions and 312-avoiding permutations, *Order*, 22(4) (2005), 311–328.

[9] E. Barcucci, A. Del Lungo, E. Pergola and R. Pinzani, A methodology for the enumeration of combinatorial objects, *Disc. Journal of Difference Equations and Applications*, 5 (1999), 435–490.

[10] E. Barcucci, A. Del Lungo, E. Pergola and R. Pinzani, From Motzkin to Catalan permutations, *Discrete Mathematics*, 217(1-3) (2000), 33–49.

[11] E. Barcucci, A. Bernini and M. Poneti, From Fibonacci to Catalan permutations, *PuMA*, 17(1-2) (2006), 1–17.

[12] E. Barcucci, A. Del Lungo, and E. Pergola. Random generation of trees and other combinatorial objects. *Theoretical Computer Science*, 218(2) (1999), 218–232.

[13] E. Barcucci, A. Del Lungo, E. Pergola, and R. Pinzani. A methodology for plane tree enumeration. *Discrete Mathematics*, 180 (1998), 45–64.

[14] E. Barcucci, A. Del Lungo, E. Pergola, and R. Pinzani. Directed animals, forests of trees and permutations. *Discrete Mathematics*, 204 (1999), 41–71.

[15] E. Barcucci, A. Del Lungo, E. Pergola, and R. Pinzani. ECO : a methodology for the enumeration of combinatorial objects. *Journal of Difference Equations and Applications*, 5 (1999), 435–490.

[16] J.L. Baril and P.T. Do. p-Generalized Fibonacci permutations generation by the ECO method. *CANT06*, (2006), May 8 - 19.

[17] J.L. Baril, P.T. Do. ECO-generation for p-generalized Fibonacci and Lucas permutations, *Journal of Pure Mathematics and Applications*, 17(1-2) (2006), 19–37.

[18] J.L. Baril, J.M. Pallo, Matchings in three Catalan lattices, *Intern. J. Computer Math.*, 8 (2003), 193–200.

[19] J.-L. Baril, Gray code for permutations with a fixed number of cycles, *Disc. Math.*, 30(13) (2007), 1559–1571.

[20] J.L. Baril, Efficient generating algorithm for permutations with a fixed number of excedances, *Pure Mathematics and Applications/Algebra and Theoretical Computer Science*, 19(3-4) (2009), 61–70.

[21] J.L. Baril, J.M. Pallo, The phagocyte lattice of Dyck words, *Order*, 23(2-3) (2006), 97–107.

[22] J.L. Baril, J.M. Pallo, Efficient lower and upper bounds of the diagonal-flip distance between triangulations, *Inform. Process. Lett.*, 100 (2006), 131–136.

[23] J.-L. Baril and V. Vajnovszki, Gray codes for order p Lucas strings, *Proceedings of the 4th International Conference on Combinatorics on Words*, Turku, Finland (ed. Tero Harju and Juhani Karhumäki), 149-158. TUCS General Publication, No 27, August 2003.

[24] J.L. Baril and V. Vajnovszki. Minimal change list for Lucas strings and some graph theoretic consequences. *Theoretical Computer Science.*, 346 (2005), 189–199.

[25] J.-L. Baril, Gray code for permutations with a fixed number of left-to-right minima, *A paraître dans Ars Combinatoria*, 2010.

[26] J.-L. Baril, More restrictive Gray codes for some classes of pattern avoiding permutations, *Information Processing Letters*, 109 (2009), 799–804.

[27] J.-L. Baril, H. Kheddouci and O. Togni, Irregularity strength of circulant graphs, *Discrete Math.*,304(1-3) (2005), 1–10.

[28] J.-L. Baril, H. Kheddouci and O. Togni, Adjacent vertex distinguishing edge-colorings of meshes and hypercubes, Australasian Journal of Combinatorics, 35 (2006), 89–102.

[29] J.-L. Baril, H. Kheddouci and O. Togni, Vertex distinguishing edge- and total-colorings of cartesian and other product graphs, To be published in Ars Combinatoria, 2010.

[30] J.-L. Baril, Olivier Togni, Neighbor-distinguishing k-tuple edge-colorings of graphs, *Discrete Mathematics*, 309(16) (2009), 5147–5157.

[31] J.L. Baril and V. Vajnovszki. Gray code for derangements. *Discrete Appl. Math.*, 140(1-3) (2004), 207–221.

[32] J.P. Barthélemy, B. Leclerc, B. Monjardet, On the use of ordered sets on problems of comparison and consensus of classifications, *Journal of Classification*, 3 (1986), 187–224.

[33] B. Bauslaugh and F. Ruskey, Generating alternating permutations lexicographically, *BIT*, 30 (1990), 17–26.

[34] O. Bernardi, N. Bonichon Catalan's intervals and realizers of triangulations, *FPSAC*, (2007), Tianjin, China.

[35] A. Bernini, E. Grazzini, E. Pergola and R. Pinzani, A general exhaustive generation algorithm for Gray structures, *Acta Informatica*, 44(5) (2007), 361–376.

[36] S. Bérard, A. Bergeron, C. Chauve, and C. Paul, Perfect sorting by reversals is not always difficult, *IEEE ACM Transactions on Computational Biology and Bioinformatics*, 4(1) (2007), 4–16.

[37] J. R. Bitner, G. Ehrlich, and E. M. Reingold, Efficient generation of the binary reflected Gray code and its applications, *Communications of the ACM*, 19(9) (1976), 517–521.

[38] M. Bóna, Permutations avoiding certains patterns ; the case of length 4 and generalizations. *Discrete Math.*, 175(1-3) (1997), 55–67.

[39] M. Bóna, Combinatorics of Permutations, *Chapman & Hall, 2004.*

[40] A. Bonnin and J.M. Pallo, A shortest path metric on unlabeled binary trees, *Pattern Recognition Lett.* 13 (1992), 411–415.

[41] S.A. Boorman, D.C. Olivier, Metrics on spaces of finite trees, *Journal of Mathematical Psychology*, 10 (1973), 26–59.

[42] M. Bouvel and E. Pergola, Posets and permutations in the duplication-loss model, *Pure Math. and Appl.*, 2009.

[43] M. Bouvel and D. Rossin, A variant of the tandem duplication-random loss model of genome rearrangement, *Theoretical Computer Science*, 410(8-10) (2009), 847–858.

[44] S. Brlek, E. Duchi, E. Pergola, and S. Rinaldi. On the equivalence problem for succession rules. *Discrete Math.*, 298 (2005), 142–154.

[45] A. Burstein, Enumeration of words with forbidden patterns, *Ph.D. thesis, University of Pennsylvania*, 1998.

[46] E. R. Canfield and H. S. Wilf, Counting permutations by their alternating runs, *Journal of Combinatorial Theory*, 115 (2008), 213–225.

[47] F. Chapoton,Sur le nombre d'intervalles dans les treillis de Tamari, *Séminaire Lotharingien de combinatoire*, 55 (2006).

[48] L. Carlitz, Enumeration of up-down sequences, Discrete Math., 4 (1973), 273–286.

[49] L. Carlitz, T. Vaughan, Enumeration of sequences of given specification according to rises, falls and maxima, Discrete Math., 8 (1974), 147–167.

[50] K. Chaudhuri, K. Chen, R. Mihaescu, and S. Rao, On the tandem duplication-random loss model of genome rearrangement, In *Proceedings of the seventeenth annual ACM-SIAM symposium on Discrete algorithm*, p. 564–570. ACM New York, NY, USA, 2006.

[51] H. D. Chen, W. L. Fan, S. G. Kong, H. Lee, B. Zheng, and N. Zhou, Inverse symmetry in genomes and whole-genome inverse duplication, In *International Bioinformatics Workshop (IBW2008).* Yunnan University, Kunming, Yunnan, China, 2008.

[52] M. C. Chen and R. C. T. Lee, Sorting by transpositions based on the first increasing substring concept, In *BIBE'04 Proceedings of the 4th IEEE Symposium on Bioinformatics and Bioengineering*, page 553, Washington, DC, USA, 2004. IEEE Computer Society.

[53] Y.J. Chen, J.M. Chang, Y.L. Wang, An efficient algorithm for estimating rotation distance between two binary trees, *Intern. J. Comput. Math.*, 82 (2005) 1095–1106.

[54] T. Chow and J. West, Forbidden sequences and Chebyshev polynomials, *Disc. Math.*, 204 (1999), 119–128.

[55] S. Cleary, Restricted rotation distance between binary trees, *Inform. Process. Lett.*, 84 (2002), 333–338.

[56] S. Cleary, J. Taback, Bounding restricted rotation distance, *Inform. Process. Lett.*, 88 (2003), 251–256.

[57] W. Chu, Ch. Colbourn and P. Dukes. Constructions for permutation codes in power-line communications. *Designs, Codes and Cryptography*, 32 (2004), 51–64.

[58] L. Comtet, Advanced Combinatorics : The Art of Finite and In0nite Expansions, Reidel, Dordrecht, Holland, 1974.

[59] K. Cukik and D. Wood, A note on some tree similarity measures, *Inform. Process. Lett.*, 15 (1982), 39–42.

[60] B.A. Davey, H.A. Priestley, Introduction to lattices and order, *Cambridge university press*, 2002.

[61] E. Dedó, D. Torri and N. Zagaglia Salvi, The observability of the Fibonacci and the Lucas cubes, *Combinatorics '98 (Palermo). Discrete Math.*, 255(1-3) (2002), 55–63.

[62] A. Del Lungo, A. Frosini, and S. Rinaldi, ECO method and the exhaustive generation of convex polyominoes. *DMTCS*, (2003), 103–116.

[63] E. Deutsch : Private communication, 2008.

[64] E. Deutsch, L. Ferrari, and S. Rinaldi, Production matrices. *Advances in Applied Mathematics*, 34 (2005),101–122.

[65] E. Duchi, A. Frosini, R. Pinzani, and S. Rinaldi, A note on rational succession rules. *Journal of Integer Sequences, Article 03.1.7*, 6 (2003).

[66] D. DuffusMatching in modular lattice, *J. Combin. Theory*, 32 (1982), 303–314.

[67] W.M.B. Dukes, M. F. Flanagan, T. Mansour, V. Vajnovszki, Combinatorial Gray codes for classes of pattern avoiding permutations, *Theoretical Computer Sciences*, 396 (2008), 35–49.

[68] P.H. Edelman, Chain enumeration and noncrossing partitions, *Discrete Math.*, 31 (1980), 171–180.

[69] P.H. Edelman and R. Simion, Chains on the lattice of noncrossing partitions, *Discrete Math.*, 126 (1994), 107–119.

[70] S. Effler and F. Ruskey, A CAT algorithm for generating permutations with a fixed number of inversions, *Information Process. Letters*, 86 (2003), 107–112.

[71] E. Egge and T. Mansour, Restricted permutations, Fibonacci numbers, and k-generalized Fibonacci numbers, *Integers Electronic Journal of Combinatorial Number Theory, Article A01*, 5 (2005).

[72] R. Ehrenborg and E. Steingrímsson, The excedance set of a permutation, *Advances in Applied Mathematics*, 24 (2000), 284–299.

[73] L. Ferrari, E. Pergola, R. Pinzani, and S. Rinaldi, An algebraic characterization of the set of succession rules, *Theoretical Computer Science*, 281 (2002), 351–367.

[74] L. Ferrari, E. Pergola, R. Pinzani, and S. Rinaldi, Jumping succession rules and their generating functions, *Discrete Math.*, 271 (2003), 29–50.

[75] L. Ferrari and R. Pinzani, Lattices of lattice paths, *J. Stat. Plan. Inference*, 135 (2005), 77–92.

[76] G. Firro and T. Mansour, Restricted permutations and polygons, *The Third International Conference on Permutations Patterns, Gainsville, Florida.*, March 7-11, 2005.

[77] P. Flajolet and R. Sedgewick, *Introduction à l'analyse des algorithmes*, Thomson Publishing, 1996.

[78] D. Foata and M.P. Schützenberger, Théeorie géométrique des polynômes Eulériens, *Lectures Notes in Mathematics*, 138, Springer-Verlag, Berlin, 1970.

[79] H. Friedman, D. Tamari, Problèmes d'associativité : une structure de treillis fini induite par une loi de demi-associative, *J. Comb. Theory*, 2 (1967), 215–242.

[80] R.G. Gallager, Information Theory and Reliable Communications, *J. Wiley*, New York, (1968).

[81] C. Germain and J.M. Pallo, Two shortest path metrics on well-formed parentheses strings, *Inform. Process. Lett.*, 60 (1996), 283–287.

[82] C. Germain and J.M. Pallo, The number of coverings in four Catalan lattices, *Intern. J. Computer Math.*, 61 (1996), 19–28.

[83] G. Grätzer, *General Lattice Theory*, Birkäuser Verlag, Basel, 1998.

[84] F. Gray, Pulse code communication, *U.S. Patent 2632058*, (1953).

[85] O. Guibert, Combinatoire des permutations à motifs exclus en liaison avec mots, cartes planaires et tableaux de Young, *PhD thesis, Universitte Bordeaux 1, 1995.*

[86] A.J. Guttmann, C. Krattenthaler and X.G. Viennot, Vicious walkers, friendly walkers and Young tableaux II :between two walls, *Journal of Statistical Physics*, 110 (2003), 1069–1086.

[87] M. T. Hallett and J. Lagergren, New algorithms for the duplication-loss model, In *Proceedings of the fourth annual international conference on Computational molecular biology*, pages 138–146. ACM New York, NY, USA, 2000.

[88] S. Hanke, T. Ottmann, S.J. Schuierer, The edge-flipping distance of triangulations, *J. Univers. Comp. Sci.*, 2 (1996), 570–579.

[89] A. Hodges, A. Turing,The Enigma, Simon & Schuster Vintage edition, 1992.

[90] W-J. Hsu, Fibonacci cubes – a new interconnection topology, *IEEE Transactions on Parallel and Distributed Systems*, 4(1) (1993), 3–12.

[91] S.M. Johnson, Generation of permutations by adjacent transpositions, *Mathematics of computation*, 17 (1963), 282–285.

[92] A. Juarna and V. Vajnovszki, Some generalizations of a Simion-Schmidt bijection, *The Computer Journal*, 50 (2007), 574–580.

[93] A. Juarna and V. Vajnovszki, More about a Simion-Schmidt's bijection, *Preprint.*

[94] A. Juarna and V. Vajnovszki, Isomorphism between classes counted by Fibonacci numbers, *Words*, 36 (2005), 51–62, September 13 - 17.

[95] S. Kitaev, Introduction to partially ordered patterns, *Discrete Applied Mathematics*, 155 (2007), 929–944.

[96] S. Kitaev, T. Mansour, Partially ordered generalized patterns and k-ary words, *Ann. Comb.*, 7(2) (2003), 191–200.

[97] S. Kitaev and A. Pyatkin, On avoidance of V- and Λ-patterns in permutations, *To appear in Ars Combinatoria*, 2009.

[98] D.E. Knuth, *The art of computer programming, combinatorial algorithms*, volume 4. Pre-fascicle 2b, (Generating all permutations), 2002.

[99] D.E. Knuth, Fondamental Algorithms, vol. 1., Addison-Wesley, 1997.

[100] J.F. Korsh, Loopless generation of up-down permutations, *Discrete Math.*, 240(1-3) (2001), 97–122.

[101] J.F. Korsh, Constant time generation of derangements, *Information Process. Letters,* 90(4) (2004), 181–186.

[102] D. Kremer, Permutations with forbidden subsequences and a generalized SchrÍoder number, *Disc. Math.*, 218(1-3) (2000), 121–130.

[103] G.J. Kreweras, Sur les partitions noncroisées d'un cycle, *Discrete Math.*, 1 (1972), 333–350.

[104] J.P.S. Kung, Matching and Radon transforms in lattices, *Order*, 2 (1985), 105–112.

[105] A. Labarre, New bounds and tractable instances for the transposition distance, *IEEE/ACM Transactions on Computational Biology and Bioinformatics (TCBB)*, 3(4) (2006), 380–394.

[106] J.C. Lagarias and D.P. Weisser, Fibonacci and Lucas cubes, *Fibonacci Quart.*, 19(1) (1981), 39–43.

[107] J. Liu, W-J. Hsu and M.J. Chung, Generalized Fibonacci cubes are mostly Hamiltonian, *Journal of Graph Theory*, 18(8) (1994), 817–829.

[108] M. Lothaire, *Combinatorics on words. Encyclopedia of Mathematics and its applications*, 17, Addison-Wesley, 1983.

[109] J.M. Lucas, Untangling binary trees via rotations, *Comput. J.*, 47 (2004), 259–269.

[110] J.M. Lucas, A direct algorithm for restricted rotation distance, *Inform. Process. Lett.*, 90 (2004), 129–134.

[111] F. Luccio, L. Pagli, On the upper bound of the rotation distance of binary trees, *Inform. Process. Lett.*, 31 (1989), 57–60.

[112] T. Mansour, Restricted 132-avoiding k-ary words, Chebyshev polynomials, and continued fractions, *Adv. in Appl. Math.*, 36(2) (2006), 175–193.

[113] T. Mansour, Permutations avoiding a pattern from S_k and at least two patterns from S_3, *Ars Combin.*, 62 (2002), 227–239.

[114] T. Mansour, V. Vajnovszki, Restricted 123-avoiding Baxter permutations and the Padovan numbers, *Discrete Applied Mathematics*, 155(11) (2007), 1430–1440.

[115] T. Mansour, S.H.F. Yan, Minimal permutations with d descents, *European journal of combinatorics*, 31 (2010), 1445–1460.

[116] R. Mantaci and F. Rakotondrajao, A permutation representation that knows what 'Eulerian' means, *Discrete Math. and Theo. Comput. Science*, 4 (2001), 101–108.

[117] B. Monjardet,Metrics on partially ordered sets-A survey, *Discrete Math.*, 35 (1981), 173–184.

[118] E. Munarini, C. Perelli Cippo and N. Zagaglia Salvi, On the Lucas cubes,*Fibonacci Quart.*, 39(1) (2001), 12–21.

[119] A. Nijenhuis and H.S. Wilf, *Combinatorial Algorithms for Computers and Calculators*, Academic Press, 1978.

[120] J. Noonan, The number of permutations containing exactly one increasing subsequence of length three, *Discrete Math.*, 152 (1996), 307–313.

[121] R. Nussinov, Some indications for inverse DNA duplication, *Journal of theoretical biology*, 95(4) (1982), 783.

[122] J.M. Pallo, Enumerating, ranking and unranking binary trees, *The Computer Journal*, 29 (1986), 171–175.

[123] J.M Pallo, On the rotation distance in the lattice of binary trees, *Inform. Process. Lett.*, 25 (1987), 369–373.

[124] J.M. Pallo,A distance metric on binary trees using lattice-theoretic measures, *Inform. Process. Lett.*, 34 (1990), 113–116.

[125] J.M. Pallo, An algorithm to compute the Möbius function of the rotation lattice of binary trees, *RAIRO Informatique Théorique et Applications*, 27 (1993), 341–348.

[126] J.M. Pallo, An efficient upper bound of the rotation distance of binary trees, *Inform. Process. Lett.*, 73 (2000), 87–92.

[127] J.M. Pallo, Generating binary trees by Glivenko classes on Tamari lattices, *Inform. Process. Lett.*, 85 (2003), 235–238.

[128] J.M. Pallo, Right-arm rotation distance between binary trees, *Inform. Process. Lett.*, 87 (2003), 173–177.

[129] J.M. Pallo, Enumerating, ranking and unranking binary trees, *The Computer Journal*, 29 (1986), 171–175.

[130] J.M. Pallo, On the rotation distance in the lattice of binary trees, *Inform. Process. Lett.*, 25 (1987), 369–373.

[131] J.M. Pallo, An algorithm to compute the Möbius function of the rotation lattice of binary trees, R̂AIRO Informatique Théorique et Applications, 27 (1993), 341–348.

[132] J.M. Pallo,An efficient upper bound of the rotation distance of binary trees, *Inform. Process. Lett.*, 73 (2000), 87–92.

[133] J.M. Pallo and V. Vajnovszki, Parallel algorithms for listing well-formed parentheses strings, *Parallel Process. Lett.*, 8 (1998), 19–28.

[134] D.S. Parker, P. Ram, The construction of Huffman codes is a submodular ("convex") optimization problem over a lattice of binary trees, *SIAM J. on Computing*, 28 (1999), 1875–1905.

[135] M. Poneti, V. Vajnovszki, Generating restricted classes of involutions, Bell and Stirling permutations, *European Journal of Combinatorics*, 31 (2010), 553–564.

[136] G. Pruesse and F. Ruskey, Generating linear extensions fast, *SIAM J. Comput.*, 23 (1994), 373–386.

[137] V. Reiner, Noncrossing partitions for classical reflection groups, *Discrete Math.*, 177 (1997), 195–222.

[138] K. Reuter, Matching for linearly indecomposable modular lattices, *Discrete Math.*, 63 (1987), 245–247.

[139] R. Rieper and M. Zeleke, Valleyless sequences, *ArXiv :math/0005180*, 2000.

[140] J. Riordan. An introduction to combinatorial analysis. *New York, J. Wiley*, 1958.

[141] R.O. Rogers, On finding shortest paths in the rotation graph of binary trees, *Congr. Numer.*, 137 (1999), 75-95.

[142] R.O. Rogers, R.D. Dutton, On distance in the rotation graph of binary trees, *Congr. Numer.*, 120 (1996), 103–113.

[143] D. Roelants van Baronaigien. Constant time generation of involutions. *Congressus Numerantium*, 90 (1992), 87–96.

[144] D. Roelants van Baronaigien and F. Ruskey, Generating permutations with given ups and downs, *Disc. Appl. Math.*, 36(1) (1992), 57–65.

[145] D.P. Roselle, Permutations by number of rises and successions, *Proc. Amer. Math. Soc.*, 20 (1968), 8–16.

[146] A.A. Ruiz, F. Luccio, A.M. Enriquez, L. Pagli, k-restricted rotation with an application to search tree rebalancing, *9th* WADS, *Lecture Notes in Computer Science, Springer*, 3608 (2005), 2–13.

[147] F. Ruskey and T.C. Hu, Generating binary trees lexicographically, *SIAM J. on Computing*, 6(4) (1977), 745–758.

[148] F. Ruskey and U. Taylor, Fast generation of restricted classes of permutations, *Manuscript*, 1995.

[149] F. Ruskey, http ://www.theory.cs.uvic.ca/~cos/gen/perm.html.

[150] F. Ruskey, Simple combinatorial Gray codes constructed by reversing sublists in ISAAC conference, *Lecture notes in computer Science*, 762 (1993), 201–208, Berlin Springer.

[151] F. Ruskey and S. Effler, A CAT algorithm for generating permutations with a fixed number of inversions, *Inform. Process. Lett.*, 86(2) (2003), 107–112.

[152] F. Ruskey and D. Roelants van Baronaigien, Generating permutations with given ups and downs, *Discrete Appl. Math.*, 36(1) (1992), 57–67.

[153] D. Sankoff, Gene and genome duplication, *Current Opinion in Genetics and Development*, 11 (2001), 681–684.

[154] C. Savage, A Survey of Combinatorial Gray Codes, *Society of Industrial and Applied Mathematics*, 39 (1997), 605–629.

[155] R. Sedgewick, Permutation generation methods, *Comput. Surveys*, 9(2) (1977), 137–164.

[156] L.W. Shapiro, A Catalan triangle, *Discrete Math.*, 14 (1976), 83–90.

[157] L. W. Shapiro, W.-J. Woan, and S. Getu, Runs, slides and moments, *SIAM J. Algebraic and Discrete Methods*, 4(461) (1983).

[158] R. Simion and D. Ullman, On the structure of lattice of noncrossing partitions, *Discrete Math.*, 98 (1991), 193–206.

[159] R. Simion, Noncrossing partitions, *Discrete Math.*, 217 (2000), 367–409.

[160] R. Simion and F.W. Schmidt, Restricted permutations, *Eur. J. Comb.*, 6 (1985), 383–406.

[161] D.D. Sleator, R.E. Tarjan and W.P. Thurston, Rotation distance, triangulations and hyperbolic geometry, *J. Amer. Math. Soc.*, 1 (1988), 647–681.

[162] N.J.A. Sloane, The On-line Encyclopedia of Integer Sequences, available electronically at http :\\ www.research.att.com\ ∼njas\sequences.

[163] Z. Stankova, Forbidden subsequences, *Discrete Math.*, 132(1-3) (1994), 291–316.

[164] R.P. Stanley, *Enumerative Combinatorics*, volume 2. Cambridge University Press, 1999.

[165] R. Stanley. *Enumerative combinatorics*, volume 1. Cambridge University Press, Cambridge, England, 1997.

[166] H.F. Trotter, *PERM (Algorithm 115)*, Communications of ACM, 5(8) (1962), 434–435.

[167] V. Vajnovszki, A loopless generation of bitstrings without p consecutive ones, *Discrete Mathematics and Theoretical Computer Science*, Springer (2001), 227–240.

[168] V. Vajnovszki, Gray visiting Motzkins, *Acta Informatica*, 38 (2002), 793–811.

[169] V. Vajnovszki, A loopless algorithm for generating the permutations of a multiset, *Theoretical Computer Science*, 307 (2003), 415–431.

[170] V. Vajnovszki, More restrictive Gray codes for necklaces and Lyndon words, *Information Processing Letters*, 106 (3) (2008), 96–99.

[171] V. Vajnovszki, Gray code order for Lyndon words, *Discrete Mathematics and Theoretical Computer Science*, 9 (2) (2007), 145-152.

[172] V. Vajnovszki, A Loopless Algorithm for Generating the Permutations of a Multiset, *Theoretical Computer Science*, 307 (2003), 415–431.

[173] V. Vajnovszki, Generating a Gray Code for P-sequences, *The Journal of Mathematical Modelling and Algorithms*, 1 (2002), 31–41.

[174] T.R. Walsh, Gray code for involutions, *J. Combin. Math. Combin. Comput.*, 36 (2001), 95–118, 2001.

[175] T.R. Walsh, Generating Gray codes in $O(1)$ worst-case time per word, *DMTCS03, Lect. Notes Comput. Sci. 2731*, Springer (2003), 72–88.

[176] J. West, Generating trees and the Catalan and SchrÍoder numbers, *Disc. Math.*, 146 (1994), 247–262.

[177] J. West, Generating trees and forbidden subsequences, *Discrete Mathematics*, 157(1-3) (1996), 363–374.

[178] H.S. Wilf, Ascending subsequences of permutations and the shapes of tableaux, *J. Combin. Theory Ser. A*, 60(1) (1992), 155–157.

[179] R.M. Wilson and J.H. van Lint. *A course in Combinatorics*. Cambridge University Press, 2002.

[180] J. Wu, Extended Fibonacci cubes, *IEEE Transactions on Parallel and Distributed Systems*, 8(12) (1997), 1203–1210.

MATCHINGS IN THREE CATALAN LATTICES

J.L. BARIL and J.M. PALLO
Université de Bourgogne. L.E.2.I.
B.P 47870
21078 DIJON-Cedex FRANCE

March 19, 2004

Abstract

In this note we consider a series of lattices that are enumerated by the well-known Catalan numbers. For each of these lattices, we exhibit a matching in a constructive way.

KEY WORDS: Lattices, matchings, well-formed parentheses strings, binary trees.

C.R. CATEGORIES: G.2.1. G.2.2.

1 INTRODUCTION

Given a lattice L, we denote the zero (resp. unit) by $\mathbf{0}$ (resp. $\mathbf{1}$) if it exists. The meet and the join of (x, y) are denoted respectively $x \wedge y$ and $x \vee y$. $x \in L$ is a join (resp. meet)-irreducible element if $x = a \vee b$ (resp. $x = a \wedge b$) implies $x = a$ or $x = b$. In other words, join (resp. meet)-irreducible elements have a unique lower (resp. upper) cover. Given a finite lattice L, let $J(L)$ (resp. $M(L)$) denote the set of nonzero join-irreducible (resp. nonunit meet-irreducible) elements of L. We say that L has a matching σ if σ is a map of $J(L) \cup \{\mathbf{0}\}$ to $M(L) \cup \{\mathbf{1}\}$ which is one-to-one and verifies $j \leq \sigma(j)$ for each join-irreducible j [2, 8, 16]. Kung has proved that every consistent lattice has a matching [8]. In this note, we exhibit matchings for three Catalan lattices, i.e. lattices which are enumerated by the well-known Catalan numbers. The three sets of combinatorial objects which are endowed with a lattice structure are respectively the well-formed parentheses strings, the binary trees and the noncrossing partitions. We adopt a *constructive* point of view. Indeed we build *explicit* matchings by giving precise constructions.

For the first two lattices often used in computer science [4, 13, 22], the main idea follows [6, p. 83]. Let C be a particular maximal chain in the lattice L. Let assume that the length of C is equal to $|J(L)|$. For $j \in J(L)$, let $s(j)$ be the smallest member of C such that $j \leq s(j)$. We thereby build a one-to-one map s of $J(L)$ to $C - \{\mathbf{0}\}$. For $m \in M(L)$, let $t(m)$ be the greatest member of C such that $t(m) \leq m$. We thereby build a one-to-one map of $M(L)$ to $C - \{\mathbf{1}\}$. Using the above bijections, a matching in L can be constructed.

2 MATCHINGS IN LATTICES OF WELL-FORMED PARENTHESES STRINGS

A well-formed parentheses string (w.f.p. string in short) is a word on the alphabet $\{(,)\}$ generated by the grammar $S \to (S)|SS|\lambda$ where λ is the empty word. We denote by P_n the set of w.p.f. strings with n open and n close parentheses. It is well-known that $|P_n| = c_n$ where $c_n = \binom{2n}{n}/(n+1)$ is the nth Catalan number. Let $\to$ denote the adjacent parentheses interchange, i.e. we write $w \to w'$ $(w, w' \in P_n)$ if there exist x and $y \in \{(,)\}^+$ such that $w = x)(y$ and $w' = x()y$. Thus we obtain w' from w by interchanging two adjacent parentheses. Let $\overset{*}{\to}$ be the reflexive transitive closure of $\to$.
In order to characterize $\overset{*}{\to}$, we use the following coding introduced in [15]. Let define the P-sequence of $w \in P_n$ as the integer sequence $(p_w(1), \dots, p_w(n))$ where $p_w(i)$ is the number of open parentheses written before the ith close parenthesis of w. For example, if $w = ((()())))(()))() \in P_8$ then $p_w = (4, 5, 5, 5, 7, 7, 7, 8)$.
An n-integer sequence p is the P-sequence of a w.f.p. string of P_n iff $p(n) = n$ and for all $i \in [1, n-1]$: $i \leq p(i) \leq p(i+1)$ [15]. We have shown in [4] the following characterization:

Theorem 1 *For all $w, w' \in P_n$ we have $w \overset{*}{\to} w'$ iff for all $i \in [1, n] : p_w(i) \leq p_{w'}(i)$.*

$(P_n, \overset{*}{\to})$ is a distributive lattice with $\mathbf{0}$ and $\mathbf{1}$ for all n, which is graded by the rank function $r(w) = \sum_{i=1}^{n} p_w(i)$. It is well-known that every distributive lattice has a matching [2].
We have $p_{\mathbf{0}} = (1, 2, 3, \dots, n)$ and $p_{\mathbf{1}} = (n, n, n, \dots, n)$. The P-sequences of the meet and the join of w and $w' \in P_n$ are respectively computed by $p_{w \wedge w'}(i) = min(p_w(i), p_{w'}(i))$ and $p_{w \vee w'}(i) = max(p_w(i), p_{w'}(i))$ for all $i \in [1, n]$. In short, we write $p_{w \wedge w'} = min(p_w, p_{w'})$ and $p_{w \vee w'} = max(p_w, p_{w'})$.

Theorem 2 *w is a non-zero join-irreductible element of P_n iff there exist $k \in [1, n]$ and $l \in [2, n]$ such that $p_w = (1, 2, \dots, k-1, \underbrace{k+l-1, \dots, k+l-1}_{l}, k+l, k+l+1, \dots, n)$.*

Proof. Let $w \neq \mathbf{0}$ be a join-irreducible element of P_n. Thus we have $\{i \in [1, n] | p_w(i) > i\} \neq \emptyset$ since $p_w \neq (1, 2, 3, \dots, n)$. Let denote $i_1 = min\{i \in [1, n] | p_w(i) > i\}$ and $i_2 = max\{i \in [1, n] | p_w(i) > i\}$. Suppose that $p_w(i_1) < p_w(i_2)$. Let us denote $i = max\{j \in [i_1, i_2[| p_w(j) < p_w(i_2)\}$. Then we obtain the following decomposition: $p_w = max(p', p'')$ with $p' = (p_w(1), \dots, p_w(i_1) - 1, \dots, p_w(i_2), \dots, p_w(n))$ and $p'' = (p_w(1), \dots, p_w(i_1), \dots, p_w(i), p_w(i_2) - 1, \dots, p_w(i_2) - 1, p_w(i_2 + 1), \dots, p_w(n))$ which contradicts the fact that w has a unique lower cover. Therefore $p_w(i_1) = p_w(i_2) = i_2 + 1$ and we can write $p_w = (1, \dots, i_1 - 1, i_2 + 1, \dots, i_2 + 1, \dots, n)$. The result holds with $i_1 = k$ and $l = i_2 - i_1 + 2$. Conversely, if the P-sequence of w is $p_w = (1, \dots, k-1, k+l-1, \dots, k+l-1, k+l, \dots, n)$, only one w' satisfies $r(w') = r(w) - 1$. This w' verifies $p_{w'} = (1, \dots, k-1, k+l-2, k+l-1, \dots, k+l-1, k+l, \dots, n)$.

Theorem 3 *w is a non-unit meet-irreducible element of P_n iff there exist k and l with $1 \leq l \leq k \leq n-1$ such that $p_w = (\underbrace{k, k, \dots, k}_{l}, n, \dots, n)$.*

Proof. Let $w \neq \mathbf{1}$ be a meet-irreducible element of P_n. Thus we have $\{i \in [1, n-1] | p_w(i) < p_w(i+1)\} \neq \emptyset$ since $p_w \neq (n, n, \dots, n)$. Let us denote $i_1 = min\{i \in [1, n-1] | p_w(i) < p_w(i+1)\}$ and $i_2 = max\{i \in [1, n-1] | p_w(i) < p_w(i+1)\}$. In the case where $i_1 < i_2$, p_w could be written as $p_w = min(p', p'')$ with $p' = (p_w(1), \dots, p_w(i_1 - 1), p_w(i_1) + 1, \dots, p_w(i_2), \dots, p_w(n))$ and $p'' = (p_w(1), \dots, p_w(i_1), \dots, p_w(i_2 - 1), p_w(i_2) + 1, p_w(i_2 + 1), \dots, p_w(n))$, contradicting the existence of

a unique upper cover. Thus $i_1 = i_2$ holds and denoting $p_w(i_2) = k$, we have $p_w = (k, \dots, k, n, \dots, n)$. Conversely if the P-sequence of w is $p_w = (k, k, \dots, k, n, n, \dots, n)$, only one w' satisfies $r(w') = r(w) + 1$. Thus w' verifies $p_{w'} = (k, \dots, k, k+1, n, \dots, n)$.

Remark. The number of join-irreductibles is equal to the number of meet-irreductibles, namely $\frac{n(n-1)}{2}$. Furthermore, G. Gratzer [6, p. 83] shows that in a distributive lattice, any maximal chain has length $|J(L)|$. The tool of this proof is using for constructing a matching of P_n.

Theorem 4 *The map $\sigma : J(L) \cup \{\mathbf{0}\} \longrightarrow M(L) \cup \{\mathbf{1}\}$ defined by:*

for $k \in [1, n]$: $\sigma((1, 2, \cdots, k-1, n, \cdots, n)) = (\underbrace{k-1, \cdots, k-1}_{k-1}, n, \cdots, n)$

and if $k + l - 1 < n$ with $l \in [2, n]$:

$$\sigma((1, 2, \dots, k-1, \underbrace{k+l-1, \dots, k+l-1}_{l}, k+l, \dots, n)) = (\underbrace{k+l-1, \dots, k+l-1}_{k}, \underbrace{n, \dots, n}_{n-k})$$

is a matching in L.

Proof. The construction method consists in choosing a maximal chain C of P_n, then in constructing a non-decreasing one-to-one map of $J(L) \cup \{\mathbf{0}\}$ to C and finally in constructing a non-increasing one-to-one map of $M(L) \cup \{\mathbf{1}\}$ to C.
We choose a maximal chain $C = \{\mathbf{0} = c_0 \to c_1 \to \dots \to c_{n(n-1)/2} = \mathbf{1}\}$ in P_n in the following way. We start with $c_0 = \mathbf{0}$ and for each w.f.p. string w of C, we apply the adjacent parentheses interchange $)(\to ()$ on the rightmost occurrence $)($ in w. For example in P_5: $()()()()() \to ()()()(()) \to ()()(()()) \to ()()((())) \to ()(()()()) \to ()(()()()) \to ()((()())) \to (()(()())) \to ((()()())) \to (((()()))) \to ((((()))))$. See also the bold path in Figure 1.
A non-unit w.f.p. string of this chain C is characterized by a P-sequence of the form $(1, 2, \dots, k-1, p_k, n, \dots, n)$ with $1 \le k \le n-1$ and $k \le p_k \le n-1$.
For $j \in J(L) \cup \{\mathbf{0}\}$, let $s(j)$ be the smallest member of C such that $j \le s(j)$. Thus $s(\mathbf{0}) = \mathbf{0}$ and $s((1, 2, \dots, k-1, \underbrace{k+l-1, \dots, k+l-1}_{l}, k+l, \dots, n)) = (1, 2, \dots, k-1, k+l-1, n, \dots, n)$. s is obviously injective and thus is a one-to-one map. For $m \in M(L) \cup \{\mathbf{1}\}$, let $t(m)$ be the greatest member of C such that $t(m) \le m$. Thus $t(\mathbf{1}) = \mathbf{1}$ and if $k \ne n$, $l \ge 1$ then
$t((\underbrace{k, k, \dots, k}_{l}, n, \dots, n)) = (1, \dots, l-1, k, \underbrace{n, \dots, n}_{n-l})$. t is also injective and thus is a one-to-one map. t^{-1} is defined by $t^{-1}(\mathbf{1}) = \mathbf{1}$ and if $k + l - 1 \ne n$, $t^{-1}((1, 2, \dots, k-1, k+l-1, n, \dots, n)) = (\underbrace{k+l-1, \dots, k+l-1}_{k}, n, \dots, n)$ and $t^{-1}((1, 2, \cdots, k-1, n, \cdots, n)) = (\underbrace{k-1, \cdots, k-1}_{k-1}, n, \cdots, n)$.
Therefore the map $\sigma = t^{-1} \circ s$ defined previously is a matching in P_n.

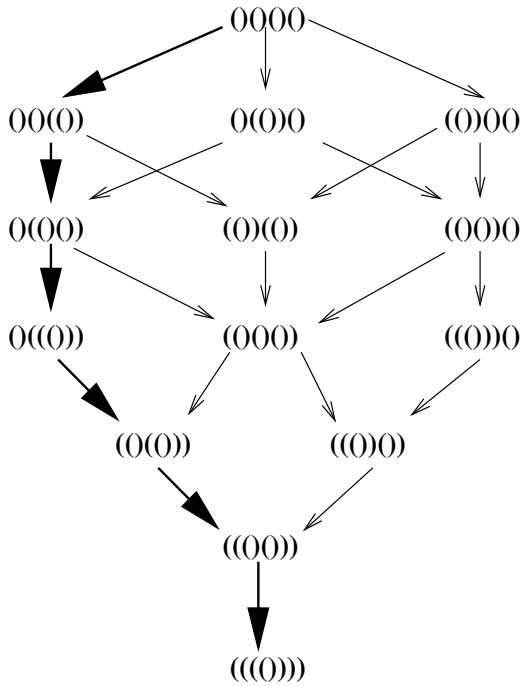


Figure 1: The distributive lattice P_4

3 MATCHINGS IN TAMARI LATTICES

The so-called Tamari lattices are orderings of w.f.p. words that were introduced by Tamari fifty years ago [20] and later shown to be lattices [1, 5, 10, 11]. The Tamari lattices can be described in many ways via the known bijections between families of Catalan combinatorial objects [3]. A system that is isomorphic to Tamari lattices is that of triangulations of a polygon related by the diagonal flip operation [19]. Another system that is isomorphic to Tamari lattices too is that of binary trees related by rotations [11, 12, 14].

A binary tree is a rooted, ordered tree in which every internal node $\bigcirc$ has exactly two sons. External nodes or leaves have no children and are denoted $\Box$. We denote by B the set of binary trees. We denote by T_L and T_R the left and right subtrees of $T \in B$ if $T \neq \Box$. In Polish notation, we can write $T = \bigcirc T_L T_R$.

We denote by B_n the set of binary trees with n internal nodes (and thus with $n+1$ leaves). It is well-known that $|B_n| = c_n$.

The leaves of a binary tree T are numbered by a preorder traversal of T (i.e. from left to right). The weight $|T|$ of a tree T is the number of leaves of T. Given $T \in B_n$, the weight sequence of T is the integer sequence $w_T = (w_T(1), \ldots, w_T(n))$ where $w_T(i)$ is the weight of the largest subtree of T whose last leaf is i [11, 14]. Rotation is a transformation $\rightarrow$ on B_n such that a subtree $\bigcirc T_1 \bigcirc T_2 T_3$ of a tree of B_n is replaced by the subtree $\bigcirc \bigcirc T_1 T_2 T_3$ where $T_1, T_2, T_3 \in B$. We have proved in [11] the following characterization:

Theorem 5 *Given T and $T' \in B_n$, we have $T \xrightarrow{*} T'$ iff $w_T(i) \leq w_{T'}(i)$ for all $i \in [1, n]$.*

$(B_n, \overset{*}{\rightarrow})$ is a semidistributive lattice for all n, called Tamari lattice, with $\mathbf{0}$ and $\mathbf{1}$ [1, 5, 10, 11]. Thus it is a consistent lattice [21] and therefore it has a matching [8]. We have $w_{\mathbf{0}} = (1, 1, \dots, 1)$ and $w_{\mathbf{1}} = (1, 2, 3, \dots, n)$. The weight sequence of the meet of T and T' is easy to compute: $w_{T \wedge T'}(i) = min(w_T(i), w_{T'}(i))$. See [13] for computing the join. The following characterization can be shown easily:

Theorem 6 *A n-integer sequence w is the weight sequence of a tree of B_n iff for all $i \in [1, n]$:*

(i) $1 \leq w_i \leq i$ *and*

(ii) if $j \in [i - w_i + 1, i]$ *then* $i - w_i \leq j - w_j$.

Theorem 7 *T is a join-irreducible element of B_n iff there exist i and $k \in [2, n]$ such that $w_T(i) = k$ with $k \leq i$ and $w_T(j) = 1$ for all $j \neq i$.*

Proof. All elements of a weight sequence of a tree T are equal to 1 except one iff there exists a unique occurrence $\bigcirc\bigcirc$ of two consecutive internal nodes in the Polish notation of T. Thus T has a unique lower cover.

Theorem 8 *T is a meet-irreducible element of B_n iff there exist $k, l \in [1, n]$ such that $w_T = (1, 2, \dots, k, 1, 2, \dots, l, k + l + 1, \dots, n)$.*

Proof. A tree T has a unique upper cover iff there exists a unique occurrence $\square\bigcirc$ of a leaf followed by an internal node in the Polish notation of T. If k denotes the number of this leaf, then $w_T(i) = i$ for $1 \leq i \leq k$. This internal node is the root of a subtree T'. If $l = |T'_L|$, then $w_T(i) = i - k$ for $k + 1 \leq i \leq k + l$.

Remark. The number of join-irreductibles is equal to the number of the meet-irreductibles [1, 10], namely $\frac{n(n-1)}{2}$. Furthermore, B_n has a maximal chain of length $|J(L)|$ (see [10, 13]). The previous tool is now applied once again.

Theorem 9 *The map $\sigma : J(L) \cup \{\mathbf{0}\} \longrightarrow M(L) \cup \{\mathbf{1}\}$ defined by :*

$$\sigma(0) = h(1, \cdots, 1) = (1, 2, \cdots, n - 1, 1)$$

for $i \geq 2$:

$$\sigma((\underbrace{1, 1, \cdots, 1}_{i-1}, 2, 1, \cdots, 1)) = (1, 2, 3, \cdots, i - 2, 1, 2, \cdots, n - i + 2)$$

and if $3 \leq k \leq i$:

$$\sigma((\underbrace{1, 1, \cdots, 1}_{i-1}, k, 1, \cdots, 1)) = (1, 2, 3, \cdots, i - k + 1, 1, 2, \cdots, k - 2, i, i + 1, \cdots, n)$$

is a matching in L.

Proof. First we choose a chain C of maximal length ($|J(L)|$). The smallest element of C is $\mathbf{0}$. In order to obtain the successor of a tree T, we apply the rotation transformation on the rightmost occurrence of $\square\bigcirc$ in the Polish notation of T. For example in B_5, we obtain the maximal chain: $11111 \to 11112 \to 11113 \to 11123 \to 11124 \to 11134 \to 11234 \to 11235 \to 11245 \to 11345 \to 12345$. See also the bold path in Figure 2. The non-unit trees of this maximal chain can be characterized by a weight sequence of the form:

$$c_{k,l} = (\underbrace{1, \cdots, 1}_{k}, \underbrace{1, 2, \cdots, l}_{l}, l+2, l+3, \cdots, n-k+1) \tag{1}$$

with $k \geq 1$, $l \geq 1$ *and* $k + l \leq n$. Let denote $c_{0,n} = (1, 2, \cdots, n) = w_{\mathbf{1}}$.

As in the case of the previous distributive lattice, we construct a non-decreasing one-to-one map f between $J(L) \cup \{\mathbf{0}\}$ and the chain C and then a non-increasing one-to-one map g between $M(L) \cup \{\mathbf{1}\}$ and this chain C. For the first bijection s, we associate to an element j of $J(L) \cup \{\mathbf{0}\}$ the smallest element of the chain such that $j \leq s(j)$. This allows us to define f between $J(L) \cup \{\mathbf{0}\}$ and C by $f(\mathbf{0}) = \mathbf{w_0} = \mathbf{c_{n-1,1}}$, if $2 < k \leq i$ then $f((\underbrace{1, 1, \cdots, 1, 1}_{i-1}, k, 1, 1, \cdots, 1)) = (\underbrace{1, \cdots, 1}_{i-k+1}, \underbrace{1, 2, \cdots, k-2}_{k-2}, k, k+1, \cdots, n-i+k) = c_{i-k+1,k-2}$

and $f((\underbrace{1, 1, \cdots, 1}_{i-1}, 2, 1, \cdots, 1)) = (\underbrace{1, 1, \cdots, 1}_{i-1}, 2, 3, \cdots, n-i+2) = c_{i-2,n-i+2}$

Similarly, we associate to an element of $M(L) \cup \{\mathbf{1}\}$ the greatest element of the chain which is lower than it. The second non-increasing bijection g is therefore defined between $M(L) \cup \{\mathbf{1}\}$ and C by:

$g(\mathbf{1}) = \mathbf{1}$ and if $k \geq 1$, $l \geq 1$ $(k + l \leq n)$ then $g((1, 2, \cdots, k, 1, 2, \cdots, l, k+l+1, \cdots, n)) = (\underbrace{1, \cdots, 1}_{k}, \underbrace{1, 2, \cdots, l}_{l}, l+2, l+3, \cdots, n-k+1) = c_{k,l}$.

The inverse function is therefore a non-decreasing one-to-one map and is defined by : $g^{-1}(\mathbf{1}) = \mathbf{1}$ and $g^{-1}(c_{k,l}) = (\underbrace{1, 2, \cdots, k-1, k}_{k}, \underbrace{1, 2, \cdots, l}_{l}, k+l-1, \cdots, n)$. Thus $\sigma = g^{-1} \circ f$ constitutes a matching in L.

4 MATCHINGS IN LATTICES OF NONCROSSING PARTITIONS

A partition $B_1/B_2/\ldots/B_k$ of $\{1, 2, \ldots, n\}$ is called noncrossing if there do not exist four numbers $a < b < c < d$ such that $a, c \in B_i$ and $b, d \in B_j$ with $i \neq j$. For example 12579/34/6/8 is a noncrossing partition of $\{1, 2, \ldots, 9\}$ (ncp in short) while 13568/2479 is crossing. We denote NC_n the set of all ncp of $\{1, 2, \ldots, n\}$. We have $|NC_n| = c_n$. The refinement order $\leq$ is defined on NC_n in the following manner: two ncp p and p' satisfy $p \leq p'$ if every block of p is a subset of some block of p'. NC_n is a graded lattice under refinement with $\mathbf{0}$ and $\mathbf{1}$ [3, 7, 17, 18]. We have $\mathbf{0} = 1/2/3/\ldots/n$ and $\mathbf{1} = 123\ldots n$. The following characterizations can be proved easily:

Theorem 10 *p is a join-irreducible element of NC_n iff $p = B_1/B_2/\ldots/B_{n-1}$ with $|B_1| = 2$ and $|B_i| = 1$ for $2 \leq i \leq n-1$. p is a meet-irreducible element of NC_n iff $p = B_1/B_2$.*

Theorem 11 *The map $\sigma : J(L) \longrightarrow M(L)$ defined by :*
$\sigma(\{i, n\} \underset{j \notin \{i,n\}}{/} \{j\}) = \{i+1\}/\{j, j \neq i+1\}$ *for* $i \in [1, n-2]$

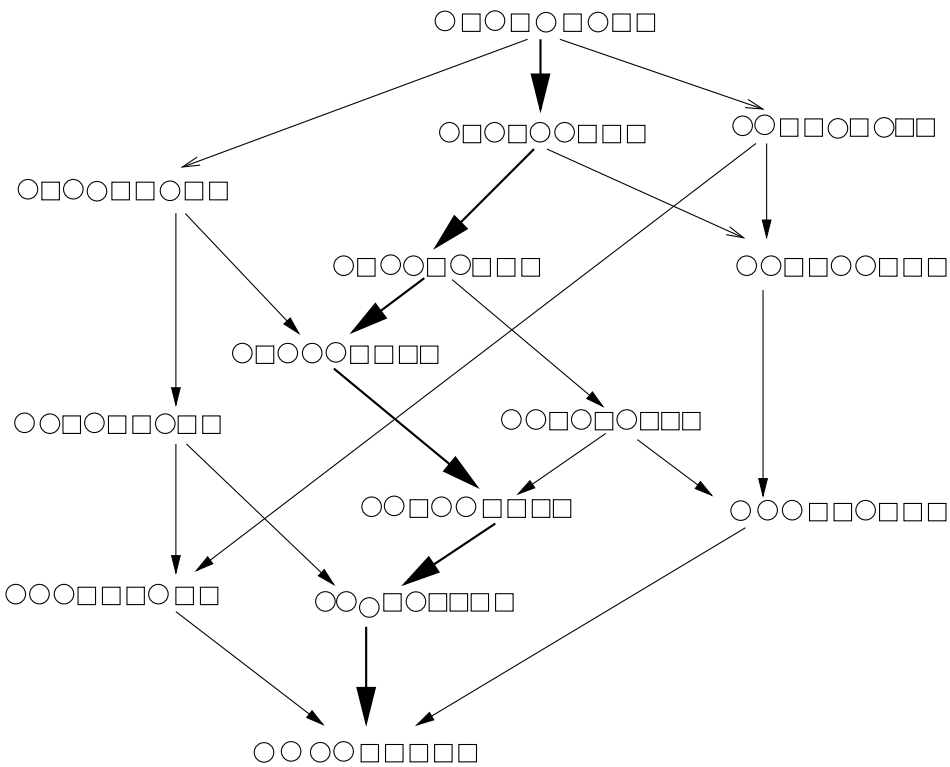


Figure 2: The Tamari lattice B_4

$\sigma(\{n-1,n\} \underset{j\notin\{n-1,n\}}{/} \{j\}) = \{1\}/\{j, j \neq 1\}$
$\sigma(\{i,j\} \underset{k\notin\{i,j\}}{/} \{k\}) = \{i, i+1, \cdots, j\}/\{k, k \notin [i,j]\}$ *for* $1 \leq i < j \leq n-1$
is a matching in NC_n.

Proof. We just need to verify the surjectivity of σ. Let $A = A_1/A_2$ be a ncp in $M(L)$. If there exists $i \in \{1,2\}$ such that $|A_i| = 1$, by definition of σ, A is in the range of σ. Let us suppose now that there does not exist $i \in \{1,2\}$ such that $|A_i| = 1$ and that n is in the block A_2. Let $i = \min\{k \in A_1\}$ and $j = \max\{k \in A_1\}$. Thus, we obtain $i < j < n$. Now, according to the definition of a ncp, there is no element c in A_2 such that $i < c < j < n$. This implies that all integers between i and j are in A_1. So, we can write $A_1 = \{i, i+1, \cdots, j\}$. Therefore, we have $\sigma(\{i,j\} \underset{k\notin\{i,j\}}{/} \{k\}) = \{i, i+1, \cdots, j\}/\{k, k \notin [i,j]\}$ and thus σ is a bijection.

References

[1] M.K. Bennet, G. Birkhoff, Two families of Newman lattices, *Algebra Universalis*, **32** (1994) 115-144.

[2] D. Duffus, Matching in modular lattices, *J. Combin. Theory*, **32** (1982) 303-314.

[3] C. Germain, J. Pallo, The number of coverings in four Catalan lattices, *Intern. J. Computer Math.*, **61** (1996) 19-28.

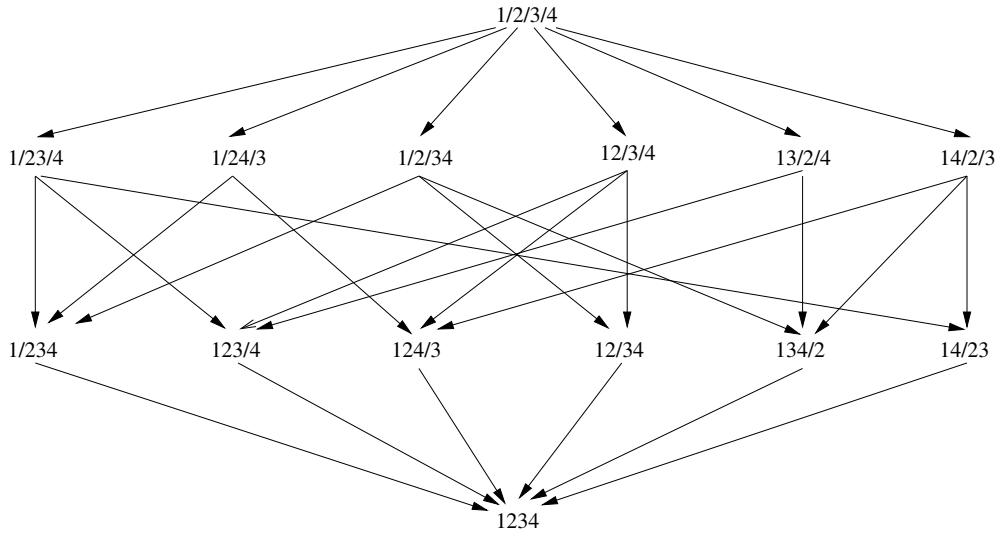


Figure 3: The lattice NC_4

[4] C. Germain, J. Pallo, Two shortest path metrics on well-formed parentheses strings, *Inform. Process. Lett.*, **60** (1996) 283-287.

[5] W. Geyer, On Tamari lattices, *Discrete Math.*, **133** (1994) 99-122.

[6] G. Grätzer, General Lattice Theory, Second Edition, Birkhäuser (1998).

[7] G. Kreweras, Sur les partitions noncroisées d'un cycle, *Discrete Math.*, **1** (1972) 333-350.

[8] J.P.S. Kung, Matchings and Radon transforms in lattices. I. Consistent lattices, *Order*, **2** (1985) 105-112.

[9] J.P.S. Kung, Matchings and Radon transforms in lattices. II. Concordant sets, *Math. Proc. Camb. Phil. Soc.*, **101** (1987) 221-231.

[10] G. Markowski, Primes, irreducibles and extremal lattices, *Order*, **9** (1992) 265-290.

[11] J.M. Pallo, Enumerating, ranking and unranking binary trees, *Computer J.*, **29** (1986) 171-175.

[12] J. Pallo, On the rotation distance in the lattice of binary trees, *Inform. Process. Lett.*, **25** (1987) 369-373.

[13] J.M. Pallo, A distance metric on binary trees using lattice-theoretic measures, *Inform. Process. Lett.*, **34** (1990) 113-116.

[14] J.M. Pallo, An algorithm to compute the Möbius function of the rotation lattice of binary trees, *Theoretical Informatics and Applications*, **27** (1993) 341-348.

[15] J.M. Pallo, R. Racca, A note on generating binary trees in A-order and B-order, *Intern. J. Computer Math.*, **18** (1985) 27-39.

[16] K. Reuter, Matchings for linearly indecomposable modular lattices, *Discrete Math.*, **63** (1987) 245-247.

[17] R. Simion, Noncrossing partitions, *Discrete Math.*, **217** (2000) 367-409.

[18] R. Simion, D. Ullman, On the structure of the lattice of noncrossing partitions, *Discrete Math.*, **98** (1991) 193-206.

[19] D.D. Sleator, R.E. Tarjan, W.P. Thurston, Rotation distance, triangulations and hyperbolic geometry, *J. Amer. Math. Soc.*, **1** (1988) 647-681.

[20] D. Tamari, Monoïdes préordonnés et chaînes de Malcev, *Bull. Soc. Math. France*, **82** (1954) 53-96.

[21] A. Walendziak, On consistent lattices, *Acta Sci. Math. (Szeged)*, **59** (1994) 49-52.

[22] T.R. Walsh, Generation of well-formed parenthesis strings in constant worst-case time, *J. Algorithms*, **29** (1998) 165-173.

Computer Science Journal of Moldova, vol.**11**, no.2(32), 2003

Gray Code for Cayley Permutations

J.-L. Baril

Abstract

A length-n Cayley permutation p of a total ordered set S is a length-n sequence of elements from S, subject to the condition that if an element x appears in p then all elements $y < x$ also appear in p. In this paper, we give a Gray code list for the set of length-n Cayley permutations. Two successive permutations in this list differ at most in two positions.

Keywords : Weak-order, Gray Code, Permutations, Combinations.

1 Introduction and definitions

A Gray code for a class of combinatorial objects is an *ordered* list for the objects of the class such that two successive objects differ in a '*small prespecified way*', see for example Carla Savage [Sav89].

Let $S = \{a_1, \ldots, a_n\}$ be a total ordered set with $a_1 < a_2 < \ldots < a_n$. A length-n Cayley permutation (C-permutation) p of S is a length-n sequence $p = p_1\ p_2\ \ldots p_n$ of elements from S satisfying the following property: if for any i, a_i appears in p, then all elements a_j, $j < i$, also appear in p. In fact, S may contain more than n elements, but in any C-permutation of length-n only the first n elements of S can appear. For example, if S is the set of natural numbers, then there are thirteen C-permutations of length three: 000, 001, 011, 012, 010, 021, 101, 102, 100, 201, 110, 120, 210. Without any loss of generality we will consider only the C-permutations of the set of natural numbers.

Cayley permutations have many interesting combinatorial interpretations; some of the more natural ones are the weak-orders. Recall that a weak-order is a relation $\leq$ that is transitive (if $x \leq y$ and $y \leq$

©2003 by J.-L. Baril

z then $x \leq z$) and complete ($x \leq y$ or $y \leq x$ always holds). We can write $x \equiv y$ if $x \leq y$ and $y \leq x$, and we note $x < y$ if $x \leq y$ and $y \not\leq x$. There exists a one-to-one map between C-permutations of length-n and the weak-orders on n elements. Indeed, a C-permutation $p = p_1\ p_2\ \ldots\ p_n$ of length-n can represent the weak-order on the set $\{1, 2, \ldots, n\}$ defined as follows: j is preceded by exactly p_j signs $<$. For example, the thirteen weak-orders on three elements $\{1, 2, 3\}$ are: $1 \equiv 2 \equiv 3$, $1 \equiv 2 < 3$, $1 < 2 \equiv 3$, $1 < 2 < 3$, $1 \equiv 3 < 2$, $1 < 3 < 2$, $2 < 1 \equiv 3$, $2 < 1 < 3$, $2 \equiv 3 < 1$, $2 < 3 < 1$, $3 < 1 \equiv 2$, $3 < 1 < 2$, $3 < 2 < 1$.

We may also regard C-permutations as certain classes of trees called Cayley trees [Cay91], as multipartite compositions, or as the different ways in which n different things can be distributed into an unknown number of different parcels without blank lot [Gro62].

If we note W_n the set of all C-permutations of length-n, for $w_n = card(W_n)$, this gives [Gro62]:

$$w_n = \sum_{i=1}^{\infty} 2^{-(i+1)} \times i^n,$$

or recursively

$$w_n = \sum_{i=0}^{n-1} \binom{n}{i} \times w_i, \text{ for } n \geq 1 \text{ with } w_0 = 1 \tag{1}$$

where $\binom{n}{i}$ represent the cardinality of the set $C_{n,i}$ of all i-combinations of $[n] = \{0, 1, \ldots, n-1\}$. Moreover $\frac{w_n}{n!}$ is the coefficient of x^n in the series of $(2 - e^x)^{-1}$ [Cay91]. M. More and A.S. Fraenkel [FM84] gave lexicographic generating and ranking algorithms for C-permutations.

Various studies have been made on Gray codes and generation algorithms for permutations and their restrictions (with given ups and downs [vBR92], [Kor01], involutions [Wal01], and derangements [BVar]) or their generalizations (multiset permutations [Vaj]). In this paper, we give a Gray code list for the set of C-permutations of length-n verifying that two successive elements in the list differ at most in two positions. The aim of this article is twofold. One is to propose the first Gray code

for Cayley permutations and thus provide new insights into the combinatorics of particular classes of permutations. The other is to show how the shuffle constructor enables us to obtain new Gray codes and generating algorithms from similar results for simpler objects.

For a set L of length-n sequences we denote by $\mathcal{L}$ an ordered list of all sequences in L. We note $first(\mathcal{L})$ and $last(\mathcal{L})$ the first and the last element of the list $\mathcal{L}$ respectively. The *rank* of an element of $\mathcal{L}$ is the number of elements which precede it, and so the rank of $first(\mathcal{L})$ is 0. $\overline{\mathcal{L}}$ is the list obtained by reversing $\mathcal{L}$, and more generally $\mathcal{L}^{(i)}$ is the list $\mathcal{L}$ if i is even and $\overline{\mathcal{L}}$ if i is odd ; if $\mathcal{L}_1$, ..., $\mathcal{L}_n$ are n ($n > 1$) lists then $\bigcirc_{i=1}^{n}\mathcal{L}_i = \mathcal{L}_1 \circ \ldots \circ \mathcal{L}_n$ is their concatenation.

2 The Gray code

Our construction of a Gray code for the set W_n of length-n C-permutations is based on the combinatorial proof of the relation above (1). Indeed, if we assume that W_0 contains only the empty word λ then each element u of W_n ($n \geq 1$) can be recursively constructed from a length-i C-permutation v ($0 \leq i \leq n-1$) and an i-combination c of n objects. In addition, v and c are unique.

More formally, let $0 \leq i \leq n-1$ and

• $c = c_1\ c_2\ \ldots\ c_n$ be the binary representation of an i-combination of n objects,

• $v = v_1\ v_2\ \ldots\ v_i$ be a C-permutation of length-i.

We define a C-permutation of length-n, $u = u_1\ u_2 \ldots u_n$, denoted by $\sqcup\!\sqcup(c, v)$, where each u_k, $1 \leq k \leq n$, is defined as:

$$u_k = \begin{cases} 0 & \text{if } c_k = 0 \\ v_j + 1 & \text{if } c_k = 1 \text{ is the } j\text{th 1 in } c. \end{cases}$$

Note that the number of 1s in c equals the length of v, and in particular if there are no 1s in c then $\sqcup\!\sqcup(c, \lambda) = 0\ 0 \ldots 0 = c$.
For example, if $i = 5$, $n = 8$, $c = 0\ \mathbf{1}\ \mathbf{1}\ 0\ 0\ \mathbf{1}\ \mathbf{1}\ 0\ \mathbf{1}\ 0$ and $v = 1\ 3\ 0\ 2\ 1$ then $\sqcup\!\sqcup(c, v) = 0\ \mathbf{2}\ \mathbf{4}\ 0\ 0\ \mathbf{1}\ \mathbf{3}\ 0\ \mathbf{2}\ 0$.

We call $\sqcup\!\sqcup(c, v)$ the *shuffle* of c by v, and c is the *trajectory* of v in $\sqcup\!\sqcup(c, v)$; see also [Vaj] where Vajnovszki gives a more general definition for the shuffle operator over combinatorial objects.

If $C_{n,i}$ denotes the set of all i-combinations of n objects in binary sequence representation, we have:

Theorem 1 *For $n \geq 1$,*

(1) $\sqcup\!\sqcup : C_{n,i} \times W_i \hookrightarrow W_n$ *is a one-to-one map for* $0 \leq i \leq n-1$

(2) $\bigcup_{i=0}^{n-1} C_{n,i} \times W_i$ *is isomorphic to* W_n.

Proof : (1) Let $c, c' \in C_{n,i}$; $v, v' \in W_i$; and $u = \sqcup\!\sqcup(c, v)$, $u' = \sqcup\!\sqcup(c', v')$. If $u = u'$ then $u_k = 0$ iff $u'_k = 0$ and so $c = c'$. By a simple translation we have $v = v'$.

(2) it is sufficient to prove that each C-permutation of length-n can be uniquely constructed by the shuffle operation from an appropriate combination c and a C-permutation of length smaller than n.

Let $u \in W_n$, and we construct the binary sequence $c = c_1 \ \ldots c_n$ as:

$$c_k = \begin{cases} 0 & \text{if } u_k = 0 \\ 1 & \text{otherwise} \end{cases} \quad \forall 1 \leq k \leq n.$$

Let i be the weight of c (the number of ones in c), and we define $v = v_1 \ \ldots v_i \in W_i$ as: for $1 \leq k \leq i$, $v_k = u_j - 1$ if u_j is the kth non zero element in u. We can see that $u = \sqcup\!\sqcup(c, v)$. The uniqueness of the decomposition is obtained by the first point of the present theorem.

□

Now, we extend the shuffle operation to lists of C-permutations and lists of combinations. If $\mathcal{L} = \ell_1, \ell_2, \ell_3, \ldots$ is a list of length-i C-permutations ($i \geq 1$) and $c \in C_{n,i}$ then $\sqcup\!\sqcup(c, \mathcal{L})$ is the ordered list $\sqcup\!\sqcup(c, \ell_1) \circ \sqcup\!\sqcup(c, \ell_2) \circ \sqcup\!\sqcup(c, \ell_3) \circ \ldots = \bigcirc_{\ell \in \mathcal{L}} \sqcup\!\sqcup\,(c, \mathcal{L})$ and obviously we have: $\sqcup\!\sqcup(c, \overline{\mathcal{L}}) = \bigcirc_{\ell \in \overline{\mathcal{L}}} \sqcup\!\sqcup\,(c, \ell) = \overline{\bigcirc_{\ell \in \mathcal{L}} \sqcup\!\sqcup\,(c, \ell)}$.

Moreover, if $\mathcal{C} = c_1, c_2, \ldots$ is a list of combinations in $C_{n,i}$ then $\sqcup\!\sqcup(\mathcal{C}, \mathcal{L})$ is the list $\sqcup\!\sqcup(c_1, \mathcal{L}) \circ \sqcup\!\sqcup(c_2, \overline{\mathcal{L}}) \circ \sqcup\!\sqcup(c_3, \mathcal{L}) \ldots$. More formally:

$$\sqcup(\mathcal{C},\mathcal{L}) = \bigcirc_{c\in\mathcal{C}} \sqcup\,(c,\mathcal{L}^{(s)}) = \bigcirc_{c\in\mathcal{C}} \bigcirc_{\ell\in\mathcal{L}^{(s)}} \sqcup(c,\ell) \tag{2}$$

where s is the rank of c in $\mathcal{C}$.

Lemma 1

$$\sqcup(\mathcal{C},\mathcal{L}) = \begin{cases} \overline{\sqcup(\overline{\mathcal{C}},\overline{\mathcal{L}})} \text{ if } card(\mathcal{C}) \text{ is odd} \\ \\ \overline{\sqcup(\overline{\mathcal{C}},\mathcal{L})} \text{ if } card(\mathcal{C}) \text{ is even} \end{cases}$$

Proof : Suppose that $card(\mathcal{C})$ is odd. Then:

$$\begin{aligned} \overline{\sqcup(\overline{\mathcal{C}},\overline{\mathcal{L}})} &= \overline{\bigcirc_{c\in\overline{\mathcal{C}}} \bigcirc_{\ell\in\overline{\mathcal{L}}^{(s)}} \sqcup(c,\ell)} \\ &= \bigcirc_{c\in\mathcal{C}} \bigcirc_{\ell\in\mathcal{L}^{(s)}} \sqcup(c,\ell), \end{aligned}$$

where s is the rank of c in $\overline{\mathcal{C}}$. If we denote by r the rank of c in $\mathcal{C}$ then $r = card(\mathcal{C}) - s + 1$. Since $card(\mathcal{C})$ is odd, $r = s \bmod 2$ and

$$\begin{aligned} \overline{\sqcup(\overline{\mathcal{C}},\overline{\mathcal{L}})} &= \bigcirc_{c\in\mathcal{C}} \bigcirc_{\ell\in\mathcal{L}^{(r)}} \sqcup(c,\ell) \\ &= \sqcup(\mathcal{C},\mathcal{L}). \end{aligned}$$

When $card(\mathcal{C})$ is even the proof is similar. □

Now, let d be the Hamming distance on length-n sequences. A 2-Gray code for a set S is a list $\mathcal{S}$ for this set where any two successive sequences s and s' verify $d(s,s') \leq 2$. When S is a 2-Gray code for a set of combinations in binary sequence representation, we say that $\mathcal{S}$ is homogeneous if $\mathcal{S}$ is a 2-Gray code, all pairs of successive sequences of $\mathcal{S}$ differ by a transposition of two bits, and bits between those transposed are 0s.

Lemma 2 *Let $\mathcal{C}$ be a homogeneous Gray code for a set $C \subset C_{n,k}$ of combinations in binary sequence representation and $\mathcal{L}$ a 2-Gray code for a set $L \subset W_k$ of Cayley permutations. Then the list $\sqcup(\mathcal{C},\mathcal{L})$ is a 2-Gray code.*

Proof : Let u, u' be two successive elements of the list $\sqcup\!\sqcup(\mathcal{C}, \mathcal{L})$. There are as a whole two cases:

1) u and u' are defined from the same combination $c \in \mathcal{C}$ and two successive elements v and v' of $\mathcal{L}$ that are $u = \sqcup\!\sqcup(c, v)$, $u' = \sqcup\!\sqcup(c, v')$. Since $\mathcal{L}$ is a 2-Gray code ($d(v, v') \leq 2$) and u and u' are defined along the same trajectory c, $d(u, u') \leq 2$.

2) The two successive elements u and u' are defined by two combinations c and c' and only one C-permutation v that is $u = \sqcup\!\sqcup(c, v)$, $u' = \sqcup\!\sqcup(c', v)$. Since c and c' are successive in $\mathcal{C}$ and $\mathcal{C}$ is homogeneous, c and c' differ in exactly two positions, say i and j, and $c_k = c'_k = 0$ for $i < k < j$. Thus $u_k = u'_k = 0$ for $i < k < j$, $u_i = u'_j$, and $u_j = u'_i$. $\square$

In order to define our Gray code for C-permutations, we need a homogeneous Gray code for the set $C_{n,k}$ of combinations in binary sequence representation.

Various Gray codes are given for combinations, but one of them, as defined by Ruskey, is more interesting for our purpose. It's crucial that the code is homogeneous which is the case when the Gray code is two-close. More precisely, we use the following Gray code, which is a slight variation of Ruskey's Gray code, where each binary sequence is reversed (see Vajnovszki and Walsh [VW02]):

$$\mathcal{C}_{n,k} =$$

$$= \begin{cases} 1^n & \text{if } k = n \\ 0^n & \text{if } k = 0 \\ 0.1^{n-1} \circ 1\ \mathcal{C}_{n-1,k-2} & \text{if } k = n-1 \\ 0\ \overline{\mathcal{C}_{n-1,1}} \circ 1\ 0^{n-1} & \text{if } k = 1 \\ 0\ \overline{\mathcal{C}_{n-1,k}} \circ 1\ 0\ \mathcal{C}_{n-2,k-1} \circ 1\ 1\ \mathcal{C}_{n-2,k-2} & \text{if } 1 < k < n-1\ . \end{cases} \tag{3}$$

Property 1 *[Rus93] The list $\mathcal{C}_{n,k}$ defined by (3) satisfies the properties:*

(1) $\mathcal{C}_{n,k}$ is a list of all k-combinations of $[n]$

(2) $first(\mathcal{C}_{n,k}) = 0\ 1^k\ 0^{n-k-1}$

(3) $last(\mathcal{C}_{n,k}) = 1^k\ 0^{n-k}$

(4) The list $\mathcal{C}_{n,k}$ *is a two-close 2-Gray code.*

Let $\mathcal{W}_n$ be the list for the set W_n and $\phi_{n,k}$ the sequence defined by $\phi_{n,n-1} = 0$ and for $1 \leq k \leq n-1$

$$\phi_{n,n-k-1} = \phi_{n,n-k} + \binom{n}{n-k}.$$

For $k \geq 0$, we denote by $\mathcal{V}_k$ the list $\mathcal{W}_k^{(\phi_{n,k})}$ and define recursively the list of all C-permutations by $\mathcal{W}_0 = \lambda$ and for $n \geq 1$

$$\begin{aligned} \mathcal{W}_n &= \sqcup(\mathcal{C}_{n,n-1}^{(0)}, \mathcal{V}_{n-1}) \circ \sqcup(\mathcal{C}_{n,n-2}^{(1)}, \mathcal{V}_{n-2}) \circ \dots \\ &\dots \circ \sqcup(\mathcal{C}_{n,1}^{(n-2)}, \mathcal{V}_1) \circ \sqcup(\mathcal{C}_{n,0}^{(n-1)}, \mathcal{V}_0) \\ &= \bigcirc_{k=0}^{n-1} \sqcup\ (\mathcal{C}_{n,n-1-k}^{(k)}, \mathcal{V}_{n-1-k}) = \\ &= \bigcirc_{k=0}^{n-1} \sqcup\ (\mathcal{C}_{n,n-1-k}^{(k)}, \mathcal{W}_{n-1-k}^{(\phi_{n,n-1-k})}) \end{aligned} \tag{4}$$

In table 1 we give the list for the set W_4:

<u>Table 1</u>: The list
$\mathcal{W}_4 = \sqcup(\mathcal{C}_{4,3}^{(0)}, \mathcal{W}_3^{(0)}) \circ \sqcup(\mathcal{C}_{4,2}^{(1)}, \mathcal{W}_2^{(4)}) \circ \sqcup(\mathcal{C}_{4,1}^{(2)}, \mathcal{W}_1^{(10)}) \circ \sqcup(\mathcal{C}_{4,0}^{(3)}, \mathcal{W}_0^{(14)})$.

The first fifty two elements form the sublist $\sqcup(\mathcal{C}_{4,3}^{(0)}, \mathcal{W}_3^{(0)})$; the sublist $\sqcup(\mathcal{C}_{4,2}^{(1)}, \mathcal{W}_2^{(4)})$ is in **boldface**; the sublist $\sqcup(\mathcal{C}_{4,1}^{(2)}, \mathcal{W}_1^{(10)})$ is in *italic* . The last element is the single element list $\sqcup(\mathcal{C}_{4,0}^{(3)}, \mathcal{W}_0^{(14)})$.

1. 0123	14. 1011	27. 1203	40. 1110	53. **1200**	66. **0201**
2. 0132	15. 1021	28. 1302	41. 1210	54. **2100**	67. **0101**
3. 0122	16. 1012	29. 1202	42. 1120	55. **1100**	68. **0110**
4. 0212	17. 2011	30. 2102	43. 2110	56. **1010**	69. **0210**
5. 0312	18. 2021	31. 3102	44. 2210	57. **2010**	70. **0120**
6. 0213	19. 3021	32. 2103	45. 3210	58. **1020**	71. *0100*
7. 0231	20. 2031	33. 2301	46. 2310	59. **1002**	72. *0001*
8. 0321	21. 2013	34. 3201	47. 2130	60. **2001**	73. *0010*
9. 0221	22. 3012	35. 2201	48. 3120	61. **1001**	74. *1000*
10. 0211	23. 2012	36. 2101	49. 2120	62. **0011**	75. **0000**
11. 0112	24. 1022	37. 1102	50. 1220	63. **0021**	
12. 0121	25. 1032	38. 1201	51. 1320	64. **0012**	
13. 0111	26. 1023	39. 1101	52. 1230	65. **0102**	

Property 2 *For* $n \geq 1$

(1) $first(\mathcal{W}_n) = 0\ 1\ 2\ \ldots\ n-1$

(2) $last(\mathcal{W}_n) = 0\ 0\ \ldots\ 0$

(3) Two successive elements of the list $\mathcal{W}_n$ *differ in at most two positions.*

Proof : By Definition 2 of the list $\mathcal{W}_n$, we have $first(\mathcal{W}_1) = last(\mathcal{W}_1) = 0$ and for $n \geq 1$, $last(\mathcal{W}_n) = 0\ 0 \ldots 0$. The recurrence on $n \geq 1$ completes the proof:

$$\begin{aligned} first(\mathcal{W}_n) &= first(\sqcup(\mathcal{C}_{n,n-1}, \mathcal{W}_{n-1})) \\ &= first(\sqcup(\mathcal{C}_{n,n-1}, first(\mathcal{W}_{n-1})) \\ &= \sqcup(0\ 1\ 1 \ldots 1, 0\ 1\ 2 \ldots n-2) \\ &= 0\ 1\ 2\ 3 \ldots n-1. \end{aligned}$$

For the third point of the property, Lemma 2 proves that each block $\sqcup(\mathcal{C}^{(k)}_{n,n-1-k}, \mathcal{W}^{(\phi_{n,n-1-k})}_{n-1-k})$ is a 2-Gray code.

In order to prove this, we must verify that the last element of the list $\sqcup(\mathcal{C}^{(k+1)}_{n,n-k}, \mathcal{W}^{(\phi_{n,n-k})}_{n-k})$ and the first element of the list $\sqcup(\mathcal{C}^{(k)}_{n,n-1-k}, \mathcal{W}^{(\phi_{n,n-1-k})}_{n-1-k})$ differ in at most two positions.

There are four cases :

• k is even and $\phi_{n,n-1-k}$ is even. Then we have:

$$first(\sqcup(\mathcal{C}^{(k)}_{n,n-1-k}, \mathcal{W}^{(\phi_{n,n-1-k})}_{n-1-k})) = \sqcup(first(\mathcal{C}_{n,n-1-k}), first(\mathcal{W}_{n-1-k}))$$
$$= 0\ 1\ 2 \ldots (n-1-k)\ 0^k$$

and since $\phi_{n,n-k} = \phi_{n,n-1-k} - \binom{n}{n-k}$ then

$$\begin{aligned} last(\sqcup(\mathcal{C}^{(k+1)}_{n,n-k}, \mathcal{W}^{(\phi_{n,n-k})}_{n-k})) &= \sqcup(last(\overline{\mathcal{C}_{n,n-k}}), last(\overline{\mathcal{W}_{n-k}})) \\ &= 0\ 1\ 2\ \ldots (n-1-k)\ (n-k)\ 0^{k-1}. \end{aligned} \tag{5}$$

• k is odd and $\phi_{n,n-1-k}$ is even:

$$first(\sqcup(\mathcal{C}^{(k)}_{n,n-1-k}, \mathcal{W}^{(\phi_{n,n-1-k})}_{n-1-k})) = \sqcup(first(\overline{\mathcal{C}_{n,n-1-k}}), first(\mathcal{W}_{n-1-k}))$$
$$= 1\ 2 \ldots (n-1-k)\ 0^{k+1}$$

and

$$\begin{aligned} last(\sqcup(\mathcal{C}^{(k+1)}_{n,n-k}, \mathcal{W}^{(\phi_{n,n-k})}_{n-k})) &= \sqcup(last(\mathcal{C}_{n,n-k}), last(\overline{\mathcal{W}_{n-k}})) \\ &= 1\ 2\ \ldots (n-1-k)\ (n-k)\ 0^k. \end{aligned} \tag{6}$$

• k is even and $\phi_{n,n-1-k}$ is odd:

$$first(\sqcup(\mathcal{C}^{(k)}_{n,n-1-k}, \mathcal{W}^{(\phi_{n,n-1-k})}_{n-1-k})) = \sqcup(first(\mathcal{C}_{n,n-1-k}), first(\overline{\mathcal{W}_{n-1-k}}))$$
$$= 0\ 1^{n-1-k}\ 0^k$$

and

$$\begin{aligned} last(\sqcup(\mathcal{C}^{(k+1)}_{n,n-k}, \mathcal{W}^{(\phi_{n,n-k})}_{n-k})) &= \sqcup(last(\overline{\mathcal{C}_{n,n-k}}), last(\mathcal{W}_{n-k})) \\ &= 0\ 1^{n-k}\ 0^{k-1}. \end{aligned} \tag{7}$$

• k is odd and $\phi_{n,n-1-k}$ is odd:

$$first(\sqcup(\mathcal{C}^{(k)}_{n,n-1-k}, \mathcal{W}^{(\phi_{n,n-1-k})}_{n-1-k})) = \sqcup(first(\overline{\mathcal{C}_{n,n-1-k}}), first(\overline{\mathcal{W}_{n-1-k}}))$$

$$= 1^{n-1-k}\ 0^{k+1}$$

and

$$\begin{aligned} last(\sqcup(\mathcal{C}^{(k+1)}_{n,n-k}, \mathcal{W}^{(\phi_{n,n-k})}_{n-k})) &= \sqcup(last(\mathcal{C}_{n,n-k}), last(\mathcal{W}_{n-k})) \\ &= 1^{n-k}\ 0^{k}. \end{aligned} \quad . \tag{8}$$

□

3 Algorithmic considerations

In this part we explain how the recursive definition (4) can be implemented into efficient generating algorithms. Such algorithms already exist for combinations [EM][RP90], so we will just give the main difficulties inimplementing ours.

Successive C-permutations are stored in an array w, and a loop statement generates the sublists $\mathcal{L}_k = \sqcup(\mathcal{C}^{(k)}_{n,n-1-k}, \mathcal{W}^{(\phi_{n,n-1-k})}_{n-1-k})$ for $0 \leq k \leq n-1$. For each list $\mathcal{L}_k$, we use an algorithm from Vajnovszki and Walsh [VW02] to generate the two-close list $\mathcal{C}^{(k)}_{n,n-1-k}$ such that there is a constant number of computations between successive combinations. So, for each $(n-1-k)$-combination c of length-n we produce recursively the list $\mathcal{W}_{n-1-k}$ or $\overline{\mathcal{W}_{n-1-k}}$ according to the parity of $\phi_{n,n-1-k}$ and to the rank of c in $\mathcal{C}^{(k)}_{n,n-1-k}$.

In order to store the different combinations c for each level of recursivity, we use $(n-1)$ global arrays T_i $(1 \leq i \leq n-1)$. More precisely, if $c = c_1\ c_2 \ldots c_k$ is a combination in integer sequence representation, the array T_i is defined by: $T_i[0] = c_1$, for $1 \leq j \leq k-1$ $T_i[c_j] = c_{j+1}$ and $T_i[c_k] = n+1$. In fact, $T_i[c_j]$ is the position of the $(j+1)$th entry greater than $(i-1)$ in the current C-permutation w. Notice that T_i corresponds to a combination on the trajectory defined by T_{i-1}. The interest of this representation of combinations is that between two consecutive combinations on the level i we need to change at most one value in the array T_i, and we don't modify the others. Likewise, between two lists $\mathcal{L}_k$, we need to modify only one entry on one array T_i. This structure allows us to assure that this algorithm transforms an object into its successor in a constant amortized time.

More precisely, this can be implemented as follows:
- *initialize a global array* w *by* $first(\mathcal{W}_n) = 0\ 1\ 2 \dots\ n-1$
- *initialize* $(n-1)$ *global arrays* $T_1, T_2, \dots, T_n$ according to $first(\mathcal{W}_n)$
- *For* k *from* $n-1$ *downto* 0
    - *while* $T_{n-k} \neq last(\mathcal{C}^{(k)}_{n,n-1-k})$ *do*
        - $T_{n-k} = succ(T_{n-k})$
        - *call resursively the algorithm on the trajectory* T_{n-k}
    - *modify the array* T_{n-k} *according to the relations* (5) (6) (7) (8)

The time complexity of this algorithm is proportional to the total number of recursive calls. The number of calls of degree 1 is at most w_n, and the others calls have a degree of at least 2. So, we have the following inequality:

$$\frac{number\ of\ recursive\ calls}{number\ of\ generated\ objects} \leq 3,$$

which proves that the complexity is linear in the number of generated words.

4 Concluding remarks

Our Gray code verifies that two successive elements differ in at most two positions i and j $(i < j)$ without more restrictions on these indices. Are there a similar Gray code $\mathcal{W}_n$ and constant c (independent of n) such that i and j verify the $|i-j| < c$? Moreover, we do not find a Gray code verifying that two successive elements of the list differ in only one position. However, if we denote by $even(W_n)$ the number of C-permutations $w = w_1 \dots w_n \in W_n$ verifying $\sum_{i=1}^{n} c_i$ is even, and $odd(W_n) = w_n - even(w_n)$, we verify easily that for $n \geq 3$: $even(W_n) < odd(W_n) - 2$. This inequality proves that there is no Gray code such that two successive elements differ in only one position and the entry on this position differs by one.

References

[BVar] J.L. Baril and V. Vajnovszki. Gray code for derangements. *Discrete Applied Math.*, (to appear).

[Cay91] A. Cayley. On the analytical forms called trees. In *Collected Mathematical Papers*, volume 4. Cambridge University Press, 1891.

[EM] G. Eades and B. McKay. An algorithm for generating subsets of fixed size with a strong minimal change property. *I.P.L.*, 19:131–133.

[FM84] A.S. Fraenkel and M. Mor. Cayley permutations. *Discrete Mathematics*, 48:101–112, 1984.

[Gro62] O.A. Gross. Preferential arrangements. *Amer. Math.*, Monthly 69, 1962.

[Kor01] J.F. Korsh. Loopless generation of up-down permutations. *Discrete Math.*, 240:97–122, 2001.

[RP90] F. Ruskey and A. Proskurowski. Generating binary trees by transpositions. *J. Algorithms*, 11:68–84, 1990.

[Rus93] F. Ruskey. Simple combinatorial Gray codes constructed by reversing sublists. *L.N.C.S*, 762:201–208, 1993.

[Sav89] C.S. Savage. Gray code sequences of partitions. *J. Algorithms*, (10):577–595, 1989.

[Vaj] V. Vajnovszki. Generating multiset permutations. *To appear T.C.S.*

[vBR92] D. Roelants van Baronaigien and F. Ruskey. Generating permutations with given ups and downs. *Discrete Appl. Math.*, 1 (36), 1992.

[VW02] V. Vajnovski and T. Walsh. A loopless two-close Gray code algorithm for listing k-ary Dick words. *Submited*, 2002.

[Wal01] T. Walsh. Gray codes for involutions. *J. Combin. Math. Combin. Comput.*, 36:95–118, 2001.

J.-L. Baril,

Received June 26, 2003

LE2I, UMR-CNRS 5158, Université de Bourgogne
BP 47870, 21078 Dijon cedex, France
E–mail: $barjl@u-bourgogne.fr$

Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
Discrete Applied Mathematics 140 (2004) 207–221
DISCRETE APPLIED MATHEMATICS
www.elsevier.com/locate/dam

Gray code for derangements

Jean-Luc Baril, Vincent Vajnovszki*

LE2I-UMR CNRS 5158, Université de Bourgogne, B.P. 47 870, 21078 Dijon Cedex, France

Received 24 May 2002; received in revised form 27 May 2003; accepted 7 June 2003

Abstract

We give a Gray code and constant average time generating algorithm for *derangements*, i.e., permutations with no fixed points. In our Gray code, each derangement is transformed into its successor either via one or two transpositions or a rotation of three elements. We generalize these results to permutations with number of fixed points bounded between two constants.
© 2003 Elsevier B.V. All rights reserved.

Keywords: Restricted permutations; Gray codes; Generating algorithms

1. Introduction

Various studies have been made on Gray codes and generating algorithms for permutations and their restrictions (with given *ups* and *downs* [9,11] or inversions [6,17], involutions, and fixed-point free involutions [18]) or their generalizations (multiset permutations [8,16]). See [7,13] for surveys of permutation generation methods.

A length-n *derangement* (or *rencontre* or *coincidence*) is a permutation $\pi \in S_n$ with no fixed points, i.e., $\pi(i) \neq i$ for all $i \in [n] = \{1, 2, \ldots, n\}$. If D_n is the set of all length-n derangements, then a recurrence relation for $d_n = card(D_n)$ is given by

$$d_n = (n-1)(d_{n-1} + d_{n-2}) \tag{1}$$

for $n \geqslant 2$, with $d_1 = 0$ and $d_2 = 1$; see for instance [4, p. 180] or [14, p. 67]. There are sequential [1] and parallel [2, p. 650] algorithms for generating derangements in lexicographic order. However, we know of no published algorithms for derangements in Gray code order. Here, we present such an algorithm which is based on the combinatorial proof of relation (1) above.

* Corresponding author.
E-mail addresses: barjl@u-bourgogne.fr (J-L. Baril), vvajnov@u-bourgogne.fr (V. Vajnovszki).

0166-218X/$ - see front matter © 2003 Elsevier B.V. All rights reserved.
doi:10.1016/j.dam.2003.06.002

We represent permutations in one-line notation; i.e., $\pi = (i_1, i_2, \ldots, i_n)$ iff $\pi(k) = i_k$, and if $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ is a length-n integer sequence, then $\sigma \cdot \pi$ is the sequence $(\sigma_{\pi(1)}, \sigma_{\pi(2)}, \ldots, \sigma_{\pi(n)})$. As a particular case, when $\sigma \in S_n$ then $\sigma \cdot \pi \in S_n$ is their composition (or product).

Let $i_1, i_2, \ldots, i_k$ be k different values in $[n] = \{1, 2, \ldots, n\}$, $1 \leqslant k \leqslant n$. The *cycle* $\gamma = \langle i_1, i_2, \ldots, i_k \rangle$ is the following permutation: $\gamma(i_1) = i_2$, $\gamma(i_2) = i_3, \ldots, \gamma(i_{k-1}) = i_k$, $\gamma(i_k) = i_1$, and $\gamma(j) = j$ for all $j \neq i_\ell$, $1 \leqslant \ell \leqslant k$. A length-two cycle is a *transposition*, and each cycle can be written as a product of transpositions: $\langle i_1, i_2, \ldots, i_k \rangle = \langle i_1, i_k \rangle \cdot \langle i_1, i_{k-1} \rangle \cdot \ldots \cdot \langle i_1, i_2 \rangle$ for $k \geqslant 2$. Also, the composition of two cycles with disjoint domains is commutative, and each permutation is the product of cycles with disjoint domains. In a permutation $\sigma \in S_n$, transposing the *positions* i and j corresponds to the product $\sigma \cdot \langle i, j \rangle$ and transposing the *values* x and y corresponds to the product $\langle x, y \rangle \cdot \sigma$.

For a length-n integer sequence $\alpha = (\alpha(1), \alpha(2), \ldots, \alpha(n))$ and a permutation π in S_n, we say that π is the *normal form* of α if α is *order-isomorphic* to π, i.e., $\alpha(i) < \alpha(j)$ if and only if $\pi(i) < \pi(j)$ for all $1 \leqslant i, j \leqslant n$. In this case, all the elements of α are distinct.

In the Gray code we give in the next section, a derangement is obtained from the previous one via one or two transpositions, and, as a particular case when the domains of the two transpositions are not disjoint, via a length-three cycle. In Section 3 this code is implemented as a generating algorithm and in Section 4 it is extended for permutations with a given number of fixed points and for permutations with the number of fixed points between two bounds.

A list $\mathscr{L}$ for a set L of integer sequences is an ordered list of the elements of L. $first(\mathscr{L})$ is the first element and $last(\mathscr{L})$ the last element on the list $\mathscr{L}$; $\overline{\mathscr{L}}$ is the list obtained by reversing $\mathscr{L}$, and obviously $first(\mathscr{L}) = last(\overline{\mathscr{L}})$ and $first(\overline{\mathscr{L}}) = last(\mathscr{L})$; $\mathscr{L}^{(i)}$ is the list $\mathscr{L}$ if i is even, and $\overline{\mathscr{L}}$ if i is odd; if $\mathscr{L}_1$ and $\mathscr{L}_2$ are two lists, then $\mathscr{L}_1 \circ \mathscr{L}_2$ is their concatenation, and generally $\bigcirc_{i=1}^{n} \mathscr{L}_i$ is the list $\mathscr{L}_1 \circ \mathscr{L}_2 \circ \cdots \circ \mathscr{L}_n$.

2. The Gray code

In this section, first we show how the set D_n can by recursively constructed from D_{n-1} and D_{n-2}, and then we extend this construction to lists of derangements in order to obtain a Gray code.

Let τ be a length-$(n-1)$ derangement, $n \geqslant 3$, and let i be an integer such that $1 \leqslant i \leqslant n-1$. If we denote by σ the permutation in S_n obtained from τ by replacing the entry with *value* i by n and appending i in the last position, then σ is a length-n derangement with n not belonging to a transposition.

Similarly, let τ be a length-$(n-2)$ derangement, $n \geqslant 4$, and let i be an integer such that $1 \leqslant i \leqslant n-1$. If σ denotes the permutation in S_n obtained from τ by: (1) adding one to each entry greater than or equal to i, (2) inserting n in position i, and finally, (3) appending i in the last position, then σ is a length-n derangement with n belonging to a transposition (the transposition $\langle i, n \rangle$). Moreover, each length-n derangement, $n \geqslant 4$, can be uniquely obtained by one of these constructions.

More formally, for $n > 0$, let D'_n be the set of length-n derangements where n does not belong to a transposition and D''_n its complement, i.e., n belongs to a transposition. Clearly, $D'_1 = D''_1 = D'_2 = \emptyset$, and $D_2 = D''_2$ is the single derangement list $(2,1)$; and $D'_n \cup D''_n$ is a two-partition of the set D_n of all length-n derangements. The functions ϕ and ψ defined below give a bijection between $[n-1] \times D_{n-1}$ and D'_n on the one hand and between $[n-1] \times D_{n-2}$ and D''_n on the other.

Definition 1. (1) For $n \geqslant 3$, an integer $i \in [n-1]$ and a derangement $\tau \in D_{n-1}$, we define a length-n permutation $\sigma = \phi_n(i,\tau)$ by

$$\sigma(j) = \begin{cases} n & \text{if } \tau(j) = i, \\ i & \text{if } j = n, \\ \tau(j) & \text{otherwise.} \end{cases}$$

(2) For $n \geqslant 4$, an integer $i \in [n-1]$ and a derangement $\tau \in D_{n-2}$, we define a length-n permutation $\sigma = \psi_n(i,\tau)$ by

$$\sigma(j) = \begin{cases} i & \text{if } j = n, \\ n & \text{if } j = i, \\ \tau(j) & \text{if } j < i \text{ and } \tau(j) < i, \\ \tau(j)+1 & \text{if } j < i \text{ and } \tau(j) \geqslant i, \\ \tau(j-1) & \text{if } j > i \text{ and } \tau(j-1) < i, \\ \tau(j-1)+1 & \text{if } j > i \text{ and } \tau(j-1) \geqslant i. \end{cases}$$

With i and τ as above, it is easy to see that

- $\phi_n(i,\tau) \in D'_n$ and $\phi_n : [n-1] \times D_{n-1} \to D'_n$ is a bijection; and
- $\psi_n(i,\tau) \in D''_n$ and $\psi_n : [n-1] \times D_{n-2} \to D''_n$ is a bijection.

So, for $d_n = card(D_n)$ we have $d_n = card(D'_n) + card(D''_n) = (n-1)d_{n-1} + (n-1)d_{n-2}$ which is a combinatorial proof of (1).

Conversely, we have.

Remark 2. If $\sigma \in D_n$, $n \geqslant 4$, and $i = \sigma(n)$, then

(1) if $\sigma(i) \neq n$ (n is not in a transposition in σ) then $\sigma = \phi_n(i,\tau)$ with τ the permutation represented by the first $n-1$ entries of $\langle i,n\rangle \cdot \sigma$;
(2) if $\sigma(i) = n$ (n is in a transposition in σ) then $\sigma = \psi_n(i,\tau)$ with τ the permutation represented by the normal form of the sequence $(\sigma(1),\sigma(2),\ldots,\sigma(i-1),\sigma(i+1),\ldots,\sigma(n-1))$.

In the following we will omit the subscript n for the functions ϕ and ψ, and it should be clear by context. Also, we extend the functions ϕ and ψ in a natural way to sets and lists of derangements. For $i \in [n-1]$ and $\mathscr{L}$ a list of length-$(n-1)$ derangements we

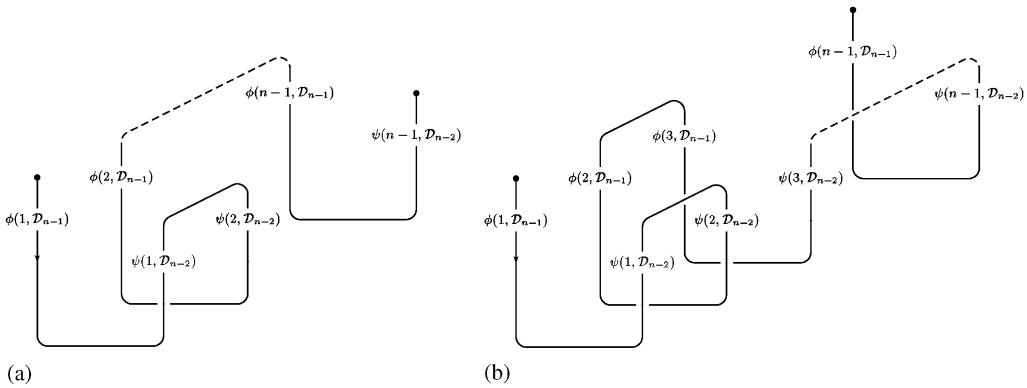


Fig. 1. The list $\mathscr{D}_n$: (a) n is even, (b) n is odd.

have $\phi(i,\overline{\mathscr{L}})=\overline{\phi(i,\mathscr{L})}$, $\phi(i,first(\mathscr{L}))=first(\phi(i,\mathscr{L}))$, and $\phi(i,last(\mathscr{L}))=last(\phi(i,\mathscr{L}))$. Similar results hold for the function ψ.

Let $\mathscr{D}_n$ be the list for the set D_n defined by

$$\begin{aligned}\mathscr{D}_n &= \phi(1,\mathscr{D}_{n-1})\circ\psi(1,\overline{\mathscr{D}}_{n-2}),\\ &\quad\circ\psi(2,\mathscr{D}_{n-2})\circ\phi(2,\overline{\mathscr{D}}_{n-1}),\\ &\quad\circ\phi(3,\mathscr{D}_{n-1})\circ\psi(3,\overline{\mathscr{D}}_{n-2}),\\ &\quad\vdots\\ &= \bigcirc_{i=1}^{n-1}(\phi(i,\mathscr{D}_{n-1})\circ\psi(i,\overline{\mathscr{D}}_{n-2}))^{(i+1)}\end{aligned}\tag{2}$$

for $n\geqslant 3$, anchored by $\mathscr{D}_1=\psi(1,\emptyset)=\psi(2,\emptyset)=\emptyset$ and $\mathscr{D}_2=(2,1)$.

In Fig. 1 below, the list $\mathscr{D}_n$ is illustrated for even and odd n by a path, where going down means generating a sublist in direct order and going up means generating it in reverse order.

Let f_n denote the first derangement in the list $\mathscr{D}_n$ and ℓ_n denote the last one. The following lemma evaluates f_n and ℓ_n for all n.

Lemma 3. *If $n\geqslant 3$ then*

(1) $f_n=(2,3,\ldots,n-1,n,1)$;

(2) $\ell_n=\begin{cases}(2,3,\ldots,n-2,n,1,n-1) & \text{if } n \text{ is odd},\\ (2,3,\ldots,n-2,1,n,n-1) & \text{if } n \text{ is even}.\end{cases}$

Proof. (1) $f_n=\phi(1,f_{n-1})$, and by the induction hypothesis, $f_n=\phi(1,(2,3,\ldots,n-1,1))=(2,3,\ldots,n-1,n,1)$.

(2) If n is odd, then

$$\begin{aligned}\ell_n &= last(\overline{\phi(n-1,\mathscr{D}_{n-1})\circ\psi(n-1,\overline{\mathscr{D}}_{n-2})})\\ &= first(\phi(n-1,\mathscr{D}_{n-1}))\\ &= \phi(n-1,f_{n-1})\\ &= (2,3,\ldots,n-2,n,1,n-1).\end{aligned}$$

If n is even, then

$$\begin{aligned}\ell_n &= last(\phi(n-1,\mathscr{D}_{n-1})\circ\psi(n-1,\overline{\mathscr{D}}_{n-2}))\\ &= last(\psi(n-1,\overline{\mathscr{D}}_{n-2}))\\ &= \psi(n-1,f_{n-2})\\ &= (2,3,\ldots,n-2,1,n,n-1). \qquad \square\end{aligned}$$

Note that $f_n(j)=\ell_n(j)=j+1$ for all $j=1,2,\ldots,n-3$.

The next lemma ensures a smooth transition between the sublists in relation (2), namely between: (i) the list $\psi(i,\overline{\mathscr{D}}_{n-2})$ and $\psi(i+1,\mathscr{D}_{n-2})$, with i odd; (ii) the list $\phi(i,\overline{\mathscr{D}}_{n-1})$ and $\phi(i+1,\mathscr{D}_{n-1})$ with i even; and (iii) the list $\phi(i,\mathscr{D}_{n-1})$ and $\psi(i,\overline{\mathscr{D}}_{n-2})$, or equivalently, the list $\psi(i,\mathscr{D}_{n-2})$ and $\phi(i,\overline{\mathscr{D}}_{n-1})$. More precisely, successive derangements in $\mathscr{D}_n$ differ either by one or two transpositions or by a circular shift of three elements (See Table 1 for two examples).

Lemma 4. (i) *If* $n\geqslant 4$ *then*

$$\psi(i+1,f_{n-2})=\begin{cases}\psi(i,f_{n-2})\cdot\langle i-1,n\rangle\cdot\langle i,i+1\rangle & \text{if } 1<i\leqslant n-2,\\ \psi(i,f_{n-2})\cdot\langle 1,2\rangle\cdot\langle n-1,n\rangle & \text{if } i=1\end{cases} \tag{3}$$

Table 1
The lists $\mathscr{D}_4$ and $\mathscr{D}_5$

$\mathscr{D}_4$		$\mathscr{D}_5$							
1	2341	1	**23451**	12	*35412*	23	**25413**	34	*23154*
2	3421	2	**34251**	13	*45132*	24	**54213**	35	*31254*
3	4321	3	**43251**	14	**51432**	25	**45213**	36	**21534**
4	3412	4	**34521**	15	**41532**	26	**54123**	37	**51234**
5	3142	5	**35421**	16	**54132**	27	**51423**	38	**25134**
6	4312	6	**43521**	17	**43152**	28	**45123**	39	**53124**
7	2413	7	**24531**	18	**31452**	29	**24153**	40	**31524**
8	4123	8	**45231**	19	**34152**	30	**41253**	41	**35124**
9	2143	9	**25431**	20	**43512**	31	**21453**	42	**53214**
		10	*54231*	21	**34512**	32	*41523*	43	**35214**
		11	*53421*	22	**53412**	33	*24513*	44	**23514**

In $\mathscr{D}_5$ the sublists $\phi(i,\mathscr{D}_4)$ and $\psi(i,\mathscr{D}_3)$, in direct or reverse order, for some i, $1\leqslant i\leqslant 4$, are in bold-face and italic, respectively.

or, conversely, by reading this relation left-to-right and replacing i by $i-1$,

$$\psi(i-1, f_{n-2}) = \begin{cases} \psi(i, f_{n-2}) \cdot \langle i-2, n\rangle \cdot \langle i-1, i\rangle & \text{if } 2 < i \leqslant n-1, \\ \psi(i, f_{n-2}) \cdot \langle 1, 2\rangle \cdot \langle n-1, n\rangle & \text{if } i = 2. \end{cases} \tag{4}$$

(ii) *If $n \geqslant 3$ then*

$$\phi(i+1, f_{n-1}) = \begin{cases} \phi(i, f_{n-1}) \cdot \langle n, i, i-1\rangle & \text{if } 1 < i \leqslant n-2, \\ \phi(i, f_{n-1}) \cdot \langle 1, n-1, n\rangle & \text{if } i = 1 \end{cases} \tag{5}$$

or, conversely and by replacing i by $i-1$,

$$\phi(i-1, f_{n-1}) = \begin{cases} \phi(i, f_{n-1}) \cdot \langle i-2, i-1, n\rangle & \text{if } 2 < i \leqslant n-1, \\ \phi(i, f_{n-1}) \cdot \langle n, n-1, 1\rangle & \text{if } i = 2. \end{cases} \tag{6}$$

(iii) *If $n = 4$*

$$\psi(i, \ell_2) = \begin{cases} \phi(i, \ell_3) \cdot \langle i, i+1\rangle & \text{if } i \neq 3, \\ \phi(i, \ell_3) \cdot \langle 1, 3\rangle & \text{if } i = 3. \end{cases} \tag{7}$$

If $n \geqslant 5$

$$\psi(i, \ell_{n-2}) = \begin{cases} \phi(i, \ell_{n-1}) \cdot \langle n-2, n-3, 1\rangle & \text{if } i = 1,\ n \text{ even}, \\ \phi(i, \ell_{n-1}) \cdot \langle 1, n-3, n-2\rangle & \text{if } i = 1,\ n \text{ odd}, \\ \phi(i, \ell_{n-1}) \cdot \langle i-1, i\rangle \cdot \langle n-3, n-2\rangle & \text{if } 2 \leqslant i \leqslant n-4, \\ \phi(i, \ell_{n-1}) \cdot \langle n-2, n-3, n-4\rangle & \text{if } i = n-3, \\ \phi(i, \ell_{n-1}) \cdot \langle n-4, n-3, n-1\rangle & \text{if } i = n-2, \\ \phi(i, \ell_{n-1}) \cdot \langle n-4, n-2\rangle \cdot \langle n-3, n-1\rangle & \text{if } i = n-1,\ n \text{ even}, \\ \phi(i, \ell_{n-1}) \cdot \langle n-1, n-2, n-4\rangle & \text{if } i = n-1,\ n \text{ odd}, \end{cases} \tag{8}$$

or conversely

$$\phi(i, \ell_{n-1}) = \psi(i, \ell_{n-2}) \cdot \pi^{-1} \tag{9}$$

with π the permutation which occurs in the corresponding case in relation (7) or (8). So, for example, if $n \geqslant 5$, $i = n-1$ then:

- $\phi(i, \ell_{n-1}) = \psi(i, \ell_{n-2}) \cdot \langle n-4, n-2\rangle \cdot \langle n-3, n-1\rangle$ *if n is even (in this case $\pi = \pi^{-1}$),*
- $\phi(i, \ell_{n-1}) = \psi(i, \ell_{n-2}) \cdot \langle n-1, n-4, n-2\rangle$ *if n is odd.*

Proof. The proof is direct, and consists essentially of checking each case. For brevity, we do not give the proof of (iii) which is similar to the first two cases.

(i) If $1 < i \leqslant n-2$ then

$$\psi(i+1, f_{n-2}) = (2, \ldots, i-1,\ \mathbf{i},\ \mathbf{i+2},\ \mathbf{n},\ i+3, \ldots, 1,\ \mathbf{i+1}),$$

$$\psi(i, f_{n-2}) = (2, \ldots, i-1,\ \mathbf{i+1},\ \mathbf{n},\ \mathbf{i+2},\ i+3, \ldots, 1,\ \mathbf{i})$$

and if $i = 1$ then

$$\psi(2, f_{n-2}) = (\mathbf{3}, \mathbf{n}, 4, \ldots, n-1, \mathbf{1}, \mathbf{2}),$$

$$\psi(1, f_{n-2}) = (\mathbf{n}, \mathbf{3}, 4, \ldots, n-1, \mathbf{2}, \mathbf{1}),$$

where the little numbers are indices of array elements.

(ii) If $1 < i \leqslant n-2$ then

$$\phi(i+1, f_{n-1}) = (2, \ldots, i-1,\ \mathbf{i},\ \mathbf{n},\ i+2, \ldots, 1,\ \mathbf{i+1}),$$

$$\phi(i, f_{n-1}) = (2, \ldots, i-1,\ \mathbf{n},\ \mathbf{i+1},\ i+2, \ldots, 1,\ \mathbf{i})$$

and if $i = 1$ then,

$$\phi(2, f_{n-1}) = (\mathbf{n}, 3, \ldots, n-1, \mathbf{1}, \mathbf{2}),$$

$$\phi(1, f_{n-1}) = (\mathbf{2}, 3, \ldots, n-1, \mathbf{n}, \mathbf{1}). \qquad \square$$

Corollary 5. *Successive derangements in $\mathscr{D}_n$ differ at most in four positions.*

Note that $\mathscr{D}_n$ is a cyclic Gray code.

3. Generating algorithm

The definition given by (2) says that $\mathscr{D}_n$ is the concatenation of many lists, which are all similar in some sense to $\mathscr{D}_{n-1}$ or $\mathscr{D}_{n-2}$. This result is formalized in Lemma 9 below, and our generating algorithm for $\mathscr{D}_n$ is based on it. Now we give some technical definitions.

Two lists are *isomorphic* if, in the first list, a sequence is transformed into its successor via the same permutation as the corresponding sequence in the second list is transformed into its successor; and two lists are *similar* if after erasing the constant entries in the first list, and possibly reversing it, the lists become isomorphic. More formally.

Definition 6. Let $\mathscr{L}$ and $\mathscr{S}$, respectively, be a list of length-n integer sequences and a list of permutations in S_n. We say that $\mathscr{L}$ is isomorphic to $\mathscr{S}$ if:

(1) the lists contain the same number of sequences, say p,

Table 2
$\mathscr{A}$ is isomorphic to $\mathscr{D}_4$ and $\mathscr{B} = \phi(2, \overline{\mathscr{D}}_4)$ is $\{1,2,3,4\}$-similar to $\mathscr{D}_4$

$\mathscr{D}_4$	$\mathscr{A}$	$\mathscr{B}$
2341	5341	51432
3421	3451	41532
4321	4351	54132
3412	3415	43152
3142	3145	31452
4312	4315	34152
2413	5413	43512
4123	4153	34512
2143	5143	53412

Note that $\mathscr{A}$ is the reverse of the list obtained by erasing the last entry of each sequence in $\mathscr{B}$.

(2) for all $1 \leqslant j < p$, if σ (resp. τ) is the jth sequence in $\mathscr{L}$ (resp. permutation in $\mathscr{S}$) and σ' (resp. τ') is its successor in $\mathscr{L}$ (resp. $\mathscr{S}$) then $\sigma' = \sigma \cdot \pi$, where π is such that $\tau' = \tau \cdot \pi$.

Definition 7. For a length-n integer sequence list $\mathscr{L}$, a set $T \subseteq [n]$ with $card(T) = m \leqslant n$, and a length-m permutation list $\mathscr{S}$, we say that $\mathscr{L}$ is T-similar to $\mathscr{S}$ if:

(1) for all $i \in [n] \setminus T$, the entry in position i has constant value throughout the list $\mathscr{L}$,
(2) after erasing all entries in positions $i \in [n] \setminus T$ in each sequence in $\mathscr{L}$ the obtained list is isomorphic to $\mathscr{S}$ or to $\overline{\mathscr{S}}$.

In the list $\mathscr{L}$, indices in T and their corresponding entries are called *active* (relative to $\mathscr{S}$).

Clearly, if $\mathscr{L}$ is isomorphic to $\mathscr{S}$ then $\mathscr{L}$ is $[n]$-similar to $\mathscr{S}$. See Table 2 for an example of isomorphic and similar lists.

Lemma 8. *If $n \geqslant 2$ then*

(1) *for $1 \leqslant i \leqslant n$ the list $\phi(i, \mathscr{D}_n)$ is $\{1,2,\ldots,n\}$-similar to $\mathscr{D}_n$,*
(2) *for $1 \leqslant i \leqslant n+1$ the list $\psi(i, \mathscr{D}_n)$ is $\{1,2,\ldots,i-1,i+1,\ldots,n+1\}$-similar to $\mathscr{D}_n$.*

Proof. Clearly, each derangement in $\phi(i, \mathscr{D}_n)$ has its last position, the $(n+1)$th one, equal to i. Consider σ in $\phi(i, \mathscr{D}_n)$ and τ in $\mathscr{D}_n$ with $\sigma = \phi(i, \tau)$. Then $\sigma' = \phi(i, \tau')$ with σ' and τ' the successor of σ and τ, respectively, and the result holds by considering the form of τ and τ' given in point (1) of Remark 2. (2) This proposition is proved similarly, by considering the point (2) of Remark 2. $\square$

For $U \subset T \subseteq [n]$ we say that U is a *child-subset* (or *c-subset*) of T if: (1) the largest element of T is not in U, and (2) $1 \leqslant card(T \setminus U) \leqslant 2$.

```
procedure gen_up(n,t,r)
var i,run;
begin
tail := pred[tail];
if t = ψ then remove(r); endif
if n = 3
then d := d · ⟨head, succ[head], tail⟩;
else run := head
    for i := 1 to n − 1 do
        if i is odd
        then gen_up(n − 1, φ, run);    // produce a φ(i, D_{n−1})-like list
             update d as in (7) or (8);
             if n > 4 then
             gen_down(n − 2, ψ, run);  // produce a ψ(i, D̄_{n−2})-like list
             endif
             if i ≠ n − 1
             then update d as in (3); run := succ[run];
             endif
        else if n > 4 then
             gen_up(n − 2, ψ, run);    // produce a ψ(i, D_{n−2})-like list
             endif
             update d as in (9);
             gen_down(n − 1, φ, run);  // produce a φ(i, D̄_{n−1})-like list
             if i ≠ n − 1
             then update d as in (5); run := succ[run];
             endif
        endif
    enddo
endif
if t = ψ then append(r); endif
tail := succ[tail];
end
```

Fig. 2. Generating derangements in Gray code order.

Lemma 9. *Let $\mathscr{L}$ be a length-n sequence list, $T \subseteq [n]$, and $m = card(T) \geqslant 4$. If $\mathscr{L}$ is T-similar to the derangement list $\mathscr{D}_m$ then $\mathscr{L} = \mathscr{L}_1 \circ \mathscr{L}_2 \circ \cdots \circ \mathscr{L}_{2(m-1)}$ where each sublist $\mathscr{L}_j$ is U_j-similar to the derangement list $\mathscr{D}_{card(U_j)}$, with U_j a c-subset of T.*

Proof. By Lemma 8 above and applying recursively relation (2). □

Obviously, $\mathscr{D}_n$ is $\{1,2,\ldots,n\}$-similar to itself and the procedure *gen_up* in Fig. 2 generates the list $\mathscr{D}_n$ according to the lemma above: the lists $\mathscr{L}_j$ are produced iteratively, and each of them is generated recursively. So, each call of this procedure fills up entries with indices in an active set $T \subseteq [n]$ associated with it, and in a recursive call T is replaced by a c-subset of T.

In our algorithm, the set T of active indices is represented by four global variables: the integers *head* and *tail* and the arrays *succ* and *pred*, defined as follows. If at a computational step $T = \{i_1, i_2, \ldots, i_k\}$, then $head = i_1$, $tail = i_k$, $succ[i_j] = i_{j+1}$ and $pred[i_j] = i_{j-1}$.

If the active set associated with the current call is T, then the call of $gen\_up(j,t,run)$ initiated by the current call generates a sublist that is U-similar to $\mathscr{D}_j$, where

$$U = \begin{cases} T \setminus \{tail\} & \text{if } t = \phi, \\ T \setminus \{run, tail\} & \text{if } t = \psi \end{cases}$$

(recall that $tail = \max(T)$).

Less formally, $gen\_up(j,\phi,run)$ produces a '$\phi(i,\mathscr{D}_j)$-*like*' list, and $gen\_up(j,\psi,run)$ produces a '$\psi(i,\mathscr{D}_j)$-*like*' list (see relation (2)), where *run* is the ith element in the set T. The call of *gen_down* works as *gen_up* except $\mathscr{D}_j$ is replaced by $\overline{\mathscr{D}}_j$.

For a simpler expression of the generating algorithm we consider initially the active set $T = [n+1]$, and each call begins by removing *tail*, the largest element in T. Thus, before the first call of the generating procedure, the variables which correspond to T are: $head = 1$, $tail = n+1$, $succ[i] = i+1$ for $1 \leqslant i \leqslant n$, and $pred[i] = i-1$ for $2 \leqslant i \leqslant n+1$. The current derangement is stored in a global variable d, initialized by $d = first(\mathscr{D}_n)$.

The main call $gen\_up(n,\phi,0)$ produces the list $\mathscr{D}_n$, $n \geqslant 3$, and the value $r = 0$ is for convenience; in fact when $t = \phi$ the value of r is not required. The procedure which generates the reverse list $\overline{\mathscr{D}}_n$ is called *gen_down*, shown in the appendix, and essentially executes the statements of *gen_up* in reverse order and replaces the calls of *gen_up* by *gen_down* and vice versa. Procedures $remove(r)$ and $append(r)$, also shown in the appendix, remove and append r in the current active set (given by the variables *head*, *tail*, *succ* and *pred*).

Between any successive calls at least one update statement is performed, and after each update statement (including the case $n = 3$) a new derangement is produced and printed out. The current derangement d is transformed into its successor according to relations (3), (5), (7), (8) or (9) in Lemma 4. More precisely, the current derangement is subject to the transformation given in the appropriate case of Lemma 4, and it acts on the active indices.

For example, in our algorithm relation (5) becomes:

if $i = 1$
then $d := d \cdot \langle head, pred[tail], tail \rangle$;
else $d := d \cdot \langle tail, run, pred[run] \rangle$;
endif

Clearly, the time complexity of *gen_up* is proportional to the total number of recursive calls. Since each call produces at least one new derangement the time complexity of $gen\_up(n,t,r)$ is in $\mathcal{O}(d_n)$. A *C* implementation of our algorithm is available at http://www.u-bourgogne.fr/v.vincent/AA/.

4. Permutations with a given number of fixed points

Here, we generalize the Gray code in the previous sections to permutations with a given number of fixed points and permutations with a bounded number of fixed points.

Let $c=(c(1),c(2),\ldots,c(n))$ be an n-combination of m, $n\leqslant m$, in integer sequence representation, so that $1\leqslant c(i)<c(i+1)\leqslant m$ for $i=1,2,\ldots,n-1$. Also, let $t_c=t$ be the binary representation of c, i.e., $t=(t(1),t(2),\ldots,t(m))$ with $t(i)=1$ if there exists a j such that $c(j)=i$, and $t(i)=0$ elsewhere. With those notations, for a derangement $d=(d(1),d(2),\ldots,d(n))\in D_n$ we define the length-m sequence $u=(u(1),u(2),\ldots,u(m))$, denoted by $\sqcup\!\sqcup(c;d)$, as

$$u(i)=\begin{cases} i & \text{if } t(i)=0,\\ c(d(j)) & \text{if } t(i) \text{ is the } j\text{th } 1 \text{ in } t\end{cases} \tag{10}$$

and we call $u=\sqcup\!\sqcup(c;d)$ the *shuffle* of c by d on the trajectory t.

In other words, u acts on indices $c(1),c(2),\ldots,c(n)$ as d, and fixes the other indices. The shuffle operator over combinatorial objects was formally defined in a larger context in [15,16]. It is not hard to show that $\sqcup\!\sqcup(c;d)$ is a permutation of $[m]$ with exactly n "deranged" points (i.e. with exactly $m-n$ fixed points), and in addition, each such permutation can be uniquely constructed by shuffle operation from an appropriate combination and a derangement. More formally, if $u=(u(1),u(2),\ldots,u(m))$ is a permutation of $[m]$ with exactly $m-n$ fixed points then $u=\sqcup\!\sqcup(c;d)$, where

- $c=(c(1),c(2),\ldots,c(n))$ is the n-combination of m corresponding to the subset of $[m]$ where $u(i)\neq i$, and
- d is the normal form of the sequence $(u(c(1)),u(c(2)),\ldots,u(c(n)))$.

Example. If $n=3$, $m=6$, $c=(2,5,6)$, and $d=(2,3,1)$ then $t=(0,\mathbf{1},0,0,\mathbf{1},\mathbf{1})$ and $\sqcup\!\sqcup(c;d)=(1,\mathbf{5},3,4,\mathbf{6},\mathbf{2})$; or if $n=4$, $m=6$, $c=(1,2,4,5)$, and $d=(2,4,1,3)$ then $t=(\mathbf{1},\mathbf{1},0,\mathbf{1},\mathbf{1},0)$ and $\sqcup\!\sqcup(c;d)=(\mathbf{2},\mathbf{5},3,\mathbf{1},\mathbf{4},6)$.

See also [18] for a similar approach. To summarize, we have:

Lemma 10. *If $C_{m,n}$ is the set of all n-combinations of $[m]$ and $S_{m,n}$ the set of all permutations of $[m]$ with exactly $m-n$ fixed points then*

$$\sqcup\!\sqcup : C_{m,n}\times D_n\to S_{m,n}$$

defined by (10) *is a bijection.*

Also, we extend the shuffle operation in a natural way to lists of derangements: if $\mathscr{D}=d_1,d_2,\ldots$ is a sublist of $\mathscr{D}_n$ and $c\in C_{m,n}$ then $\sqcup\!\sqcup(c;\mathscr{D})$ is the list $\sqcup\!\sqcup(c;d_1),\sqcup\!\sqcup(c;d_2),\ldots$, and $\sqcup\!\sqcup(c;\overline{\mathscr{D}})=\overline{\sqcup\!\sqcup(c;\mathscr{D})}$.

A *strong* Gray code for the set $C_{m,n}$ of n-combinations of m, in integer sequence representation, is a list for $C_{m,n}$ where two successive sequences, say $c=(c(1),c(2),\ldots,c(n))$ and $c'=(c'(1),c'(2),\ldots,c'(n))$, are such that, for some $1\leqslant j\leqslant m$, $c(i)=c'(i)$ for all $i\neq j$; see [3,5,12] for such a Gray code.

Lemma 11. *If $\mathscr{C}_{m,n}$ is a strong Gray code for the set $C_{m,n}$ then the list $\mathscr{S}_{m,n}$ defined by*

$$\mathscr{S}_{m,n}=\bigcirc_{c \text{ in } \mathscr{C}_{m,n}} \sqcup\!\sqcup(c;\mathscr{D}_n^{(r)}) \tag{11}$$

is a Gray code for the set $S_{m,n}$, where r is the rank of c in $\mathscr{C}_{m,n}$ (the first combination in $\mathscr{C}_{m,n}$ has rank zero) and $\mathscr{D}_n^{(r)}$ is $\mathscr{D}_n$ or $\overline{\mathscr{D}}_n$ according as r is even or odd.

Proof. The list $\mathscr{S}_{m,n}$ has no repetitions, and, disregarding the order, it equals the set $S_{m,n}$. Moreover, for a fixed c in $C_{m,n}$, the Hamming distance between two derangements in D_n, say d and d', equals the Hamming distance between $\sqcup(c;d)$ and $\sqcup(c;d')$. So, any successive permutations in $\sqcup(c;\mathscr{D}_n)$—or equivalently in $\sqcup(c;\overline{\mathscr{D}}_n)$—differ in at most four positions. If c' is the successor of c in $\mathscr{C}_{m,n}$ then $t'=t_{c'}$ and $t=t_c$, the binary representations of c' and c, differ in exactly two positions, say k and ℓ, with $t(k)=t'(\ell)=0$ and $t'(k)=t(\ell)=1$. Since $\mathscr{C}_{m,n}$ is a strong Gray code, the permutations $\sigma=\sqcup(c;d)$ and $\sigma=\sqcup(c';d)$ differ in exactly three positions, namely k,ℓ and i, where i is such that $\sigma(i)=\ell$ and $\sigma'(i)=k$. Moreover, the index i can be computed in constant time if d is the first or last derangement in $\mathscr{D}_n$. □

The next lemma extends the result of the previous one to permutations where the number of fixed points is bounded between two constants. In this case $\mathscr{C}_{m,n}$ denotes the Eades–McKay Gray code for combinations, and it has (see [5,12])

- $first(\mathscr{C}_{m,n})=(1,2,\ldots,n)$, and
- $last(\mathscr{C}_{m,n})=(m-n+1,m-n+2,\ldots,m)$.

Lemma 12. *Let $1\leqslant k\leqslant \ell\leqslant m$ and $S_{m,k,\ell}$ be the set of all permutations in S_m with i "deranged" points, $k\leqslant i\leqslant \ell$. Then the list*

$$\mathscr{S}_{m,k,\ell}=\bigcirc_{i=k}^{\ell}\mathscr{S}_{m,i}^{(k-i)} \tag{12}$$

is a Gray code for the set $S_{m,k,\ell}$.

Proof. It is sufficient to prove that the last permutation in $\mathscr{S}_{m,i}^{(k-i)}$ and the first one in $\mathscr{S}_{m,i+1}^{(k-i+1)}$ differ in at most four positions. But $last(\mathscr{S}_{m,i}^{(k-i)})=\sqcup(c,e_i)$, and $first(\mathscr{S}_{m,i+1}^{(k-i+1)})=\sqcup(c',e_{i+1})$, with

(i) $c=last(\mathscr{C}_{m,i})$ and $c'=last(\mathscr{C}_{m,i+1})$ if $k-i$ is even, or
(ii) $c=first(\mathscr{C}_{m,i})$ and $c'=first(\mathscr{C}_{m,i+1})$ if $k-i$ is odd,

and $e_j=f_j$ or $e_j=\ell_j$; see Lemma 3 and the remark that follows. If $u=\sqcup(c,e_i)$ and $u'=\sqcup(c',e_{i+1})$ then: in case (i), $u(j)=u'(j)=j$ for all $j=1,2,\ldots,m-i-1$ and $u(j)=u'(j)=j+1$ for all $j=m-i+1,m-i+2,\ldots,m-3$; in case (ii), $u(j)=u'(j)=j+1$ for all $j=1,2,\ldots,i-3$ and $u(j)=u'(j)=j$ for all $j=i+2,i+3,\ldots,m$. In both cases u differs from u' in at most four positions. □

Algorithmic considerations

The lists $\sqcup(c;\mathscr{D}_n^{(r)})$ in (11) is c-similar to $\mathscr{D}_n$ (c is regarded as a subset of $[m]$) and the procedure *gen_up* and *gen_down* can easily be transformed to generate these

lists. In addition, with an efficient algorithm to compute the successor of c in $\mathscr{C}_{m,n}$ and with appropriate initial values for the variables and transition statements between lists, the iterative call of *gen_up* and *gen_down* produces $\mathscr{S}_{m,n}$ in constant average time. See [16,18] for loopless generating algorithms for $\mathscr{C}_{m,n}$. Similar considerations hold for the list $\mathscr{S}_{m,k,\ell}$ defined in relation (12). The loopless generation of those lists remains an open problem.

Acknowledgements

The authors thank the referees for various comments whose suggestions greatly improved the present paper.

Note added in proof. We recently learned of Korsh' and LaFollette's [10] algorithm for generating derangements. Their algorithm has the remarkable properties (a) that successive permutations differ by only one transposition or one rotation of three elements and (b) it is loopless. Our algorithm is based on a recursive counting relation and so has the advantage of being simpler to describe.

Appendix.

```
procedure gen_down(n,t,r)
var i,run;
begin
tail := pred[tail];
if t = ψ then remove(r); endif
if n = 3 then d := d · ⟨head,tail,succ[head]⟩;
else run := pred[tail]
   for i := n − 1 downto 1 do
      if i is odd
      then if n > 4 then
         gen_up(n − 2,ψ,run);
         endif
         update d as in (9);
         gen_down(n − 1,ϕ,run);
         if i ≠ 1
         then update d as in (6); run := pred[run];
         endif
      else gen_up(n − 1,ϕ,run);
         update d as in (7) or (8);
         if n > 4 then
         gen_down(n − 2,ψ,run);
         endif
         update d as in (4);
         run := pred[run];
```

```
      endif
   enddo
endif
if t = ψ then append(r); endif
tail := succ[tail];
end

procedure remove(r)
begin
if r = head
then head := succ[r];
else if r = tail
   then tail := pred[tail]
   else succ[pred[r]] := succ[r];
      pred[succ[r]] := pred[r];
   endif
endif
end

procedure append(r)
begin
if r < head
then head := pred[head];
else if r > tail
   then tail := succ[tail]
   else succ[pred[r]] := r;
      pred[succ[r]] := r;
   endif
endif
end
```

References

[1] S.G. Akl, A new algorithm for generating derangements, BIT 20 (1980) 2–7.

[2] S.G. Akl, The Design and Analysis of Parallel Algorithms, Prentice-Hall, Englewood Cliffs, NJ, 1989.

[3] P.J. Chase, Combination generation and graylex ordering, Congr. Numer. 69 (1989) 215–242.

[4] L. Comtet, Advanced Combinatorics: The Art of Finite and Infinite Expansions, Reidel, Dordrecht, Holland, 1974.

[5] P. Eades, B. McKay, An algorithm for generating subsets of fixed size with a strong minimal change property, IPL 19 (1984) 131–133.

[6] S. Effler, F. Ruskey, A CAT algorithm for generating permutations with a fixed number of inversions, Inform. Process. Lett. 86 (2) (2003) 107–112.

[7] D.E. Knuth, The Art of Computer Programming, Combinatorial Algorithms, Vol. 4. Pre-fascicle 2b (Generating all permutations), http://www-cs-staff.Stanford.EDU/knuth/taocp.html#vol4,2002.

[8] C.W. Ko, F. Ruskey, Generating permutations of a bag by interchanges, IPL 41 (5) (1992) 263–269.

[9] J.F. Korsh, Loopless generation of up-down permutations, Discrete Math. 240 (1–3) (2001) 97–122.

[10] J. Korsh, P.S. LaFollette, Constant time generation of derangements, Department Technical Report, Temple University CIS, December 2002.
[11] D. Roelants van Baronaigien, F. Ruskey, Generating permutations with given ups and downs, Discrete Appl. Math. 36 (1) (1992) 57–65.
[12] F. Ruskey, Simple combinatorial Gray codes constructed by reversing sublists in ISAAC Conference, Lecture Notes in Computer Science, Vol. 762, Springer, Berlin, 1993, pp. 201–208.
[13] R. Sedgewick, Permutation generation methods, Comput. Surveys 9 (2) (1977) 137–164.
[14] R. Stanley, Enumerative Combinatorics, Vol. 1, Cambridge University Press, Cambridge, England, 1997.
[15] V. Vajnovszki, Gray visiting Motzkins, Acta Inform. 38 (2002) 793–811.
[16] V. Vajnovszki, A loopless algorithm for generating the permutations of a multiset, TCS, 307 (2003) 415–431.
[17] T. Walsh, Loop-free sequencing of bounded integer compositions, J. Combin. Math. Combin. Comput. 33 (2000) 323–345.
[18] T. Walsh, Gray codes for involutions, J. Combin. Math. Combin. Comput. 36 (2001) 95–118.

Available online at www.sciencedirect.com
SCIENCE DIRECT®

Theoretical Computer Science 346 (2005) 189 – 199

Theoretical Computer Science

www.elsevier.com/locate/tcs

Minimal change list for Lucas strings and some graph theoretic consequences

Jean-Luc Baril, Vincent Vajnovszki*

LE2I– CNRS UMR 5158, Université de Bourgogne B.P. 47 870, 21078 Dijon-Cedex, France

Abstract

We give a minimal change list for the set of order p length-n Lucas strings, i.e., the set of length-n binary strings with no p consecutive 1's nor a 1^ℓ prefix and a 1^m suffix with $\ell+m \geqslant p$. The construction of this list proves also that the order p n-dimensional Lucas cube has a Hamiltonian path if and only if n is not a multiple of $p+1$, and its second power always has a Hamiltonian path.
© 2005 Elsevier B.V. All rights reserved.

Keywords: Minimal change list; Gray code; Fibonacci and Lucas string; Lucas cube; Hamiltonian path

1. Introduction

The Hamming distance between two strings in $\{0, 1\}^n$ is the number of positions in which they differ. A *k-Gray code* for a set of binary strings $B \subset \{0, 1\}^n$ is an *ordered* list $\mathcal{B}$ for B, such that the Hamming distance d between any two consecutive strings in $\mathcal{B}$ is at most k. In addition, if the list $\mathcal{B}$ minimizes both $k = \max d(x, x')$ and $\sum d(x, x')$, where (x, x') ranges over all pairs of successive strings in $\mathcal{B}$, then it is called *minimal change list* or *minimal Gray code*. Obviously, a 1-Gray code is a minimal change list.

We call Hamiltonian a graph having a Hamiltonian path and a k-Gray code is a Hamiltonian path in $Q_n^k|B$, the restriction of the kth power of the hypercube Q_n to the set B.

The set $F_{n,p}$, of order p length-n Fibonacci strings, is the set of length-n binary strings such that there are no p consecutive 1's. The set $L_{n,p}$, of order p length-n Lucas strings, is the set of all strings in $F_{n,p}$ which do not begin by 1^ℓ and end by 1^m with $\ell + m \geqslant p$.

* Corresponding author.
E-mail addresses: barjl@u-bourgogne.fr (J.-L. Baril), vvajnov@u-bourgogne.fr (V. Vajnovszki).

0304-3975/$ - see front matter © 2005 Elsevier B.V. All rights reserved.
doi:10.1016/j.tcs.2005.08.020

In other words, $L_{n,p}$ is the set of length-n binary strings such that there are no 1^p factors if strings are regarded circularly, i.e., the last entry of a string is followed by the first one.

A number of papers concerning the Fibonacci and Lucas strings have been published [3,6–9,13,15]. In the present one we introduce an order relation on $\{0, 1\}^n$ which induces a minimal change list on $L_{n,p}$. Note that this order relation yields a 1-Gray code on the set $F_{n,p}$, of order p length-n Fibonacci strings [13].

This paper is the extended version of [2] and the remaining is organized as follows. In the next section we prove that a 1-Gray code for $L_{n,p}$ is possible only if $(p+1)$ does not divide n. In Section 3 we give such a Gray code and a 2-Gray code when $(p+1)$ divides n; both of them are minimal change lists. Few graph theoretic consequences are presented in Section 4 and in the final part some algorithmic considerations are given.

2. Parity difference relation

For a binary string set B we denote by B' (resp. B'') the subset of B of strings with an odd (resp. even) number of 1's. Let $Q(L_{n,p})$ be the order p n-dimensional Lucas cube, i.e., the restriction of the hypercube Q_n to the set $L_{n,p}$. The graph $Q(L_{n,p})$ is bipartite, and with the notations above $\{L'_{n,p}, L''_{n,p}\}$ is a bipartition. No Hamiltonian path is possible in $Q(L_{n,p})$ (or equivalently, 1-Gray code for $L_{n,p}$) if $|card(L''_{n,p}) - card(L'_{n,p})| > 1$, i.e., the number of vertices in the two bipartitions differs by more than one.

The main result of this section is Theorem 5. In the following we suppose $p > 1$ fixed and more often in this section we will omit the subscript p for the sets $F_{n,p}$, $L_{n,p}$ and other related things.

Let $\{\phi_n\}_{n \geqslant 0}$ and $\{\lambda_n\}_{n \geqslant 0}$ be the parity difference integer sequences corresponding to Fibonacci and Lucas strings defined by

- $\phi_n = card(F''_n) - card(F'_n)$, and
- $\lambda_n = card(L''_n) - card(L'_n)$.

Lemma 1. (1) ϕ_n *satisfies*

$$\phi_n = \phi_{n-1} - \phi_{n-2} + \cdots + (-1)^{p+1}\phi_{n-p} \quad \textit{for } n \geqslant p+1. \tag{1}$$

(2) λ_n *is related to* ϕ_n *by*

$$\lambda_n = \phi_{n-2} - 2 \cdot \phi_{n-3} + \cdots + (-1)^{p+1} p \cdot \phi_{n-p-1} \quad \textit{for } n \geqslant p+2. \tag{2}$$

Proof. 1. The recursive definition in [13]

$$F_n = 0 \cdot F_{n-1} \cup 10 \cdot F_{n-2} \cup 110 \cdot F_{n-3} \cup \cdots \cup 1^{p-1}0 \cdot F_{n-p} \quad \text{for } n \geqslant p+1$$

can be expanded as

$$F'_n = 0 \cdot F'_{n-1} \cup 10 \cdot F''_{n-2} \cup 110 \cdot F'_{n-3} \cup \cdots,$$

and

$$F''_n = 0 \cdot F''_{n-1} \cup 10 \cdot F'_{n-2} \cup 110 \cdot F''_{n-3} \cup \cdots,$$

and (1) holds.

2. When $n \geqslant p+2$ the set L_n is the union of the sets in the table below, where the strings with prefix $1^{i-1}0$ are in the ith line and those with the suffix 01^{j-1} are in the jth column, $1 \leqslant i, j \leqslant p$.

$0 \cdot F_{n-2} \cdot 0,$	$0 \cdot F_{n-3} \cdot 01,$	$\cdots$	$0 \cdot F_{n-p} \cdot 01^{p-2},$	$0 \cdot F_{n-p-1} \cdot 01^{p-1}$
$10 \cdot F_{n-3} \cdot 0,$	$10 \cdot F_{n-p-4} \cdot 01,$	$\cdots$	$10 \cdot F_{n-p-1} \cdot 01^{p-2}$	
$\vdots$	$\vdots$	$\vdots$		
$1^{p-2}0 \cdot F_{n-p} \cdot 0,$	$1^{p-2}0 \cdot F_{n-p-1} \cdot 01$			
$1^{p-1}0 \cdot F_{n-p-1} \cdot 0$				

In this decomposition, each F_{n-k-1} occurs exactly k times, and reading this table diagonally one has

$$card(L_n) = \sum_{j=1}^{p} j \cdot card(F_{n-j-1}),$$

and F_{n-k-1} appears as $1^s 0 \cdot F_{n-k-1} \cdot 01^t$, with $s+t = k-1$, thus

$$card(L'_n) = card(F'_{n-2}) + 2 \cdot card(F''_{n-3}) + 3 \cdot card(F'_{n-4}) \ldots,$$

and

$$card(L''_n) = card(F''_{n-2}) + 2 \cdot card(F'_{n-3}) + 3 \cdot card(F''_{n-4}) \ldots,$$

so (2) holds. □

The next proposition gives the generating function for the sequences $\{\phi_n\}_{n \geqslant 0}$ (except its first term) and $\{\lambda_n\}_{n \geqslant 0}$ (except its $p+2$ first terms).

Proposition 2. *If $\phi(z)$ and $\lambda(z)$ denote the generating functions for the sequences $\{\phi_n\}_{n \geqslant 0}$ and $\{\lambda_n\}_{n \geqslant 0}$, respectively, then*
(1)

$$\phi(z) = \phi_0 + z \cdot (-z)^{p-1} \cdot \frac{1+z}{1-(-z)^{p+1}}, \tag{3}$$

(2)

$$\lambda(z) = \sum_{j=0}^{p+1} \lambda_j z^j + z^{p+2} \cdot (-1)^{p+1} \cdot \frac{1-(p+1)(-z)^p + p(-z)^{p+1}}{(1-(-z)^{p+1}) \cdot (1+z)}. \tag{4}$$

Proof. 1. For $i = 1, 2, \ldots, p-1$ all strings in $\{0,1\}^i$ are in F_i, and half of them are in F'_i and other half are in F''_i, so $\phi_1 = \phi_2 = \cdots = \phi_{p-1} = 0$. If $i = p$, then a single string in $\{0,1\}^p$ does not belong to F_p, namely 1^p, so $\phi_p = (-1)^{p+1}$. By relation (1) we have

$$\phi(z) = \phi_0 + z \cdot \frac{f(z)}{(1-(-z)^{p+1})/(1+z)},$$

where $f(z) = (-z)^{p-1}$ is given by the values of $\phi_1, \phi_2, \ldots, \phi_p$. See for instance Flajolet and Sedgewick's seminal book [4, p. 79].

2. If $\lambda^*(z) = \lambda(z) - \sum_{j=0}^{p+1} \lambda_j z^j$, then relation (2) gives (see again [4])

$$\lambda^*(z) = (\phi(z) - \phi_0) \cdot (z^2 - 2z^3 + \cdots + (-1)^{p+1} p z^{p+1})$$

and finally

$$\lambda^*(z) = z^{p+2} \cdot (-1)^{p+1} \cdot \frac{1 - (p+1)(-z)^p + p(-z)^{p+1}}{(1 - (-z)^{p+1}) \cdot (1+z)}. \qquad \square \tag{5}$$

For $n \geqslant p+2$, λ_n is given by the coefficient of z^n in (4). The next corollary shows that the sequence $\{\lambda_n\}_{n \geqslant 0}$ is periodic from $n \geqslant p+1$ and gives its generating function if it is extended by periodicity to $\lambda_0, \lambda_1, \ldots, \lambda_{p+1}$.

Corollary 3. *The sequence $\{\lambda_n\}_{n \geqslant p+2}$ has the period $2(p+1)$. In addition, if one defines $\lambda_n = \lambda_{n+2(p+1)}$ for all $n = 0, 1, \ldots, p+1$ then its generating function becomes*

$$\lambda(z) = \frac{(-z)^{p+1} + (p+1)z + p}{(1 - (-z)^{p+1}) \cdot (1+z)}. \tag{6}$$

Proof. For $\lambda^*(z)$ given by relation (5) it is easy to show that $(\lambda^*(z)/z^{p+2})(1 - z^{2(p+1)})$ is a polynomial of degree less than $2p+2$. Thus $\lambda^*(z)/z^{p+2}$ is the generating function for a periodic integer sequence with the period $2(p+1)$, and so is $\lambda(z)$ assuming it is extended by periodicity.

For (6) it is enough to find a polynomial $g(z)$ of degree less than $p+2$ such that $\lambda^*(z) - g(z)$ is divisible by $z^{2(p+1)}$, and in this case $\lambda(z) = (\lambda^*(z) - g(z))/z^{2(p+1)}$. It is not hard to check that $g(z) = (-z)^{p+2}((-z)^p - 1)/(z+1)$ satisfies this and we obtain (6). $\square$

Corollary 4. *The parity difference integer sequence corresponding to Lucas strings satisfies*

$$\lambda_{n,p} = \begin{cases} (-1)^{n+1} & \text{if } (p+1) \nmid n, \\ (-1)^n \cdot p & \text{if } (p+1) | n. \end{cases} \tag{7}$$

Proof. $\lambda(z)$ in relation (6) can be expressed as

$$\begin{aligned} \lambda(z) &= \frac{p+1}{1 - (-z)^{p+1}} - \frac{1}{1+z} \\ &= (p+1) \cdot \sum_{k \geqslant 0} (-z)^{k(p+1)} - \sum_{n \geqslant 0} (-z)^n \end{aligned}$$

and $\lambda_{n,p}$ is the coefficient of z^n in $\lambda(z)$. $\square$

Observe that the choice of $\lambda_{i,p}$, $i = 1, 2, \ldots, p+1$, in Corollary 3 seems arbitrary since extended by periodicity. In fact, for $i = 1, 2, \ldots, p$ all the 2^i strings in $\{0, 1\}^i$

are in $L_{i,p}$, except 1^i which contains p consecutive 1's if strings are regarded circularly. In this case $\lambda_{i,p} = (-1)^{i+1}$, and similarly, $\lambda_{p+1,p} = (-1)^{p+1}p$, which are in accord with (7).

Theorem 5. *If $Q(L_{n,p})$ is Hamiltonian then $(p+1) \nmid n$.*

Proof. If $Q(L_{n,p})$ is Hamiltonian then $|\lambda_{n,p}| \leqslant 1$, so $(p+1) \nmid n$. $\square$

The Hamiltonism of $Q(L_{n,p})$, when $(p+1) \nmid n$, is shown constructively in the next section.

3. The Gray codes

We adopt the convention that lower case bold letters represent length-n binary strings, e.g., $\boldsymbol{x} = x_1x_2\ldots x_n$; and we use the same group of letters to denote a set A and an *ordered list* $\mathcal{A}$ for a set A. A list $\mathcal{A}$ for the set $A \subset \{0,1\}^n$ is equivalent to an order relation on A: $\boldsymbol{x} < \boldsymbol{y}$ iff $\boldsymbol{x}$ precedes $\boldsymbol{y}$ in $\mathcal{A}$. For example, if $\boldsymbol{x} \neq \boldsymbol{y} \in \{0,1\}^n$ and i is the leftmost position with $x_i \neq y_i$ then:

- the lexicographic order is given by: $\boldsymbol{x} < \boldsymbol{y}$ iff x_i is even $(=0)$, and y_i is odd $(=1)$;
- the *reflected Gray code order* due to Frank Gray in 1953 [5] is given by: $\boldsymbol{x} < \boldsymbol{y}$ iff $\sum_{j=1}^{i} x_j$ is even (and $\sum_{j=1}^{i} y_j$ is odd).

We say that an order relation $<$ on a set of strings induces a *k-Gray code* if the set listed in $<$ order yields a k-Gray code, i.e., successive strings differ in at most k positions. So, the reflected Gray code order above induces a 1-Gray code on $\{0,1\}^n$; and its restriction to the strings with fixed density (i.e., strings in $\{0,1\}^n$ with a constant number of 1's) induces a 2-Gray code, called *revolving door code* by Nijenhuis and Wilf [10].

According to [13] we recall the following definition which gives another order relation on binary strings and all of those presented here are particular cases of *genlex order* [14], that is, any set of strings listed in such an order has the property that strings with a common prefix are contiguous.

Definition 6. We say that $\boldsymbol{x}$ is less than $\boldsymbol{y}$ in *local reflected order*, denoted by $\boldsymbol{x} \prec \boldsymbol{y}$, if $\sum_{j=1}^{i}(1-x_j)$ is odd and $\sum_{j=1}^{i}(1-y_j)$ is even, where i is the leftmost position with $x_i \neq y_i$.

Remark 7.

(1) $\boldsymbol{x} \prec \boldsymbol{y}$ iff the prefix $x_1x_2\ldots x_i$ contains an odd number of 0's.
(2) $\boldsymbol{x} \prec \boldsymbol{y}$ iff $\overline{\boldsymbol{x}} > \overline{\boldsymbol{y}}$ in reflected Gray code order, with $\overline{\boldsymbol{x}}$ and $\overline{\boldsymbol{y}}$ the bitwise complement of $\boldsymbol{x}$ and $\boldsymbol{y}$.
(3) As the reflected Gray code order, the local reflected order $\prec$ induces a 1-Gray code on $\{0,1\}^n$ and a 2-Gray code on length-n binary strings with fixed density.
(4) In [13] it is shown that, unlike the reflected Gray code order, the local reflected order $\prec$ induces a 1-Gray code on the set $F_{n,p}$.

The main result of this section is Corollary 12 and Theorem 13 which say that $\prec$ induces also a minimal Gray on the set $L_{n,p}$, of order p length-n Lucas strings.

If z is a length-k string, we denote by $z^{n/k}$ the length-n prefix of the infinite string $zzz\ldots$, or equivalently,

$$z^{n/k} = \underbrace{zz\ldots z}_{\lfloor n/k \rfloor} z_r,$$

where $r = n \bmod k$, and z_r is the length-r prefix of z. In the following, the length-$(p+1)$ binary string

$$\chi = \underbrace{1\ldots 1}_{p-1}00$$

plays a central role for our purposes.

Let $\mathcal{F}_{n,p}$ and $\mathcal{L}_{n,p}$ be the lists obtained by ordering the sets $F_{n,p}$ and $L_{n,p}$, respectively, by the relation $\prec$. In [13] it is proved that the first and the last strings of $\mathcal{F}_{n,p}$ are $first(\mathcal{F}_{n,p}) = 0\chi^{(n-1)/(p+1)}$ and $last(\mathcal{F}_{n,p}) = \chi^{n/(p+1)}$. The next lemma gives similar results for $\mathcal{L}_{n,p}$.

Lemma 8.
(1) $first(\mathcal{L}_{n,p}) = 0\chi^{(n-1)/(p+1)}$.
(2) $last(\mathcal{L}_{n,p}) = \chi^{(n-1)/(p+1)}0$.

Proof. 1. Let $f_1 f_2 \ldots f_n = 0\chi^{(n-1)/(p+1)}$ and $1 \leqslant j \leqslant n$ such that $\sum_{i=1}^{j}(1-f_i)$ is even, then: (1) $j > 0$ and $f_j = 0$, and (2) f_{j-1} is the rightmost 1 bit in a contiguous 1's sequence of length $p-1$ and, by Definition 6, $0\chi^{(n-1)/(p+1)}$ has no predecessor in $L_{n,p}$ in $\prec$ order. The proof of 2 is similar, the string $\chi^{(n-1)/(p+1)}0$ has no successor in $\mathcal{L}_{n,p}$. $\square$

Now we describe how we compute the successor of a string in the lists $\mathcal{F}_{n,p}$ and $\mathcal{L}_{n,p}$. Since in the lists $\mathcal{F}_{n,p}$ and $\mathcal{L}_{n,p}$ strings with a common prefix are contiguous, the successor of $\boldsymbol{x} \in \mathcal{F}_{n,p}$ (resp. of $\boldsymbol{x} \in \mathcal{L}_{n,p}$) is given by changing the rightmost bit in $\boldsymbol{x}$ such that the obtained string remains in $\mathcal{F}_{n,p}$ (resp. in $\mathcal{L}_{n,p}$), and it is greater than $\boldsymbol{x}$ in $\prec$ order. More formally we have:

Lemma 9. *Let $\boldsymbol{x} \neq last(\mathcal{F}_{n,p})$ and $s(\boldsymbol{x})$ its successor in $\mathcal{F}_{n,p}$, then either* 1, 2 *or* 3 *below holds.*
(1) *$\boldsymbol{x}$ contains an odd number of* 0*'s and it has not a suffix of the form* $1^{p-1}0$. *In this case* $s(\boldsymbol{x}) = x_1 \ldots x_{n-1}(1-x_n)$.
(2) *$\boldsymbol{x}$ contains an even number of* 0*'s and ends by* $1^{p-1}0$. *Then* $s(\boldsymbol{x}) = x_1 \ldots x_{n-2}0x_n$.
(3) *$\boldsymbol{x}$ contains an even number of 0's and does not end by* $1^{p-1}0$, *or it contains an odd number of 0's and ends by* $1^{p-1}0$. *Let* $x_1x_2\ldots x_{k-1}x_k$ *be the length minimal prefix of $\boldsymbol{x}$ with an odd number of* 0*'s and such that* $\boldsymbol{x} = x_1x_2\ldots x_{k-1}x_k0\chi^{(n-k-1)/(p+1)}$ (*with* $\chi^{(n-k-1)/(p+1)}$ *possibly empty*). *In this case* $s(\boldsymbol{x}) = x_1x_2\ldots x_{k-1}(1-x_k)0\chi^{(n-k-1)/(p+1)}$.

Note that if $\boldsymbol{x}$ is like described in point 3 of Lemma 9 then the required prefix $x_1x_2\ldots x_{k-1}x_k$ always exists. For example, in $\mathcal{F}_{6,3}$ we have: by point 1, $s(01001\mathbf{1}) = 01001\mathbf{0}$; by point 2, $s(00011\mathbf{0}) = 00010\mathbf{0}$; and by point 3, $s(10010\mathbf{0}) = 10011\mathbf{0}$ and $s(1\mathbf{0}0110) = 1\mathbf{1}0110$. In

[13] a similar idea is used to compute, in constant time, the successor $s(\boldsymbol{x})$ of a Fibonacci string $\boldsymbol{x}$.

Obviously, if $\boldsymbol{x}$ is a Lucas string then $s(\boldsymbol{x})$ is a Fibonacci string but not necessarily a Lucas string too. The next proposition states that if we denote by $succ(\boldsymbol{x})$ the successor of $\boldsymbol{x} \in L_{n,p}$ in the list $\mathcal{L}_{n,p}$ then $succ(\boldsymbol{x})$ is either $s(\boldsymbol{x})$, $s^2(\boldsymbol{x}) = s(s(\boldsymbol{x}))$ or $s^3(\boldsymbol{x}) = s(s(s(\boldsymbol{x})))$.

Proposition 10. *Let $\boldsymbol{x} \in L_{n,p}$, $\boldsymbol{x} \neq last(L_{n,p})$ and $s(\boldsymbol{x}) \notin L_{n,p}$; then either* 1 *or* 2 *below holds.*

(1) $(p+1)|n$ *and* $\boldsymbol{x} = 1^k 00\chi^{(n-k-2)/(p+1)}$ *with* $0 \leqslant k < p-1$. *In this case*

$$\begin{aligned} succ(\boldsymbol{x}) &= 1^{k+1}0\chi^{(n-k-3)/(p+1)}0 \\ &= s^2(\boldsymbol{x}). \end{aligned}$$

(2) $\boldsymbol{x} = 1^k z 1^\ell 0$ *with* $0 < k, \ell < p-1$ *and* $k + \ell \geqslant p-1$. *In this case*

$$\begin{aligned} succ(\boldsymbol{x}) &= 1^k s(z) 1^\ell 0 \\ &= s^3(\boldsymbol{x}). \end{aligned}$$

Proof. When $s(\boldsymbol{x})$ is not a Lucas string then it is obtained from $\boldsymbol{x}$ by changing the 0 bit which either follows a 1's prefix or is in the last position. □

Remark 11.

(1) If $(p+1)|n$ then there exist exactly $p-1$ Lucas strings as in point 1 of Proposition 10, one for each k, $0 \leqslant k < p-1$. In addition, if $\boldsymbol{x}$ is such a string and d denotes the Hamming distance then

(a)

$$\begin{aligned} d(\boldsymbol{x}, succ(\boldsymbol{x})) &= d(\boldsymbol{x}, s^2(\boldsymbol{x})) \\ &= 2, \end{aligned}$$

(b) the string $\boldsymbol{v} = 1^k 00\chi^{(n-k-3)/(p+1)}0 \in L_{n,p}$ is the predecessor of $\boldsymbol{x}$ in $\prec$ order (i.e., $succ(\boldsymbol{v}) = \boldsymbol{x}$) and $d(\boldsymbol{v}, succ(\boldsymbol{x})) = 1$. (In fact $\boldsymbol{v} = \boldsymbol{x} \cdot succ(\boldsymbol{x})$, the bitwise product of $\boldsymbol{x}$ and $succ(\boldsymbol{x})$.)

(2) If $\boldsymbol{x}$ is a Lucas string as in point 2 of Proposition 10 then

$$\begin{aligned} d(\boldsymbol{x}, succ(\boldsymbol{x})) &= d(1^k z 1^\ell 0, 1^k z' 1^\ell 0) \\ &= d(z, z') \\ &= 1. \end{aligned}$$

This remark proves the following:

Corollary 12.

(1) *If* $(p+1) \nmid n$ *then* $\mathcal{L}_{n,p}$ *is a* 1*-Gray code for* $L_{n,p}$.

(2) *If* $(p+1)|n$ *then* $\mathcal{L}_{n,p}$ *is a* 2*-Gray code for* $L_{n,p}$ *and there are exactly* $p-1$ *strings with* $d(\boldsymbol{x}, succ(\boldsymbol{x})) = 2$.

Table 1
The lists $\mathcal{L}_{4,2}$ and $\mathcal{L}_{4,3}$

$\mathcal{L}_{4,2}$	$\mathcal{L}_{4,3}$
0 1 0 **0**	0 1 **1** 0
0 **1** 0 1	0 1 0 **0**
0 0 0 **1**	0 **1** 0 1
0 0 **0** 0	0 0 0 **1**
0 0 1 0	0 0 **0** 0
1 0 **1** 0	0 0 1 **0**
1 0 0 0	**0** 0 1 **1**
	1 0 **1** 0
	1 0 0 **0**
	1 **0** 0 **1**
	1 1 0 0

Changed bits are in bold-face.

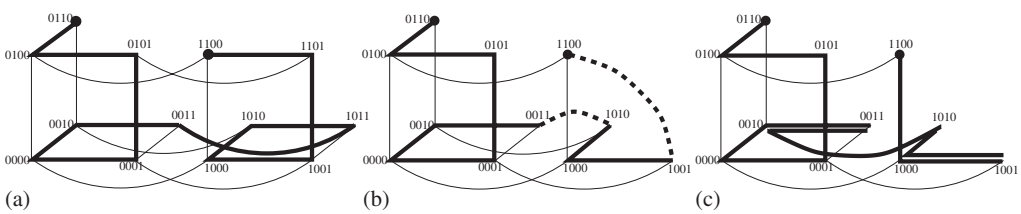


Fig. 1. (a) The Hamiltonian path $\mathcal{F}_{4,3}$ in $Q(F_{4,3})$. (b) The *'path'* $\mathcal{L}_{4,3}$ in $Q(L_{4,3})$, dashed arcs connect distance-2 vertices. (c) An $\mathcal{H}(1,2)$-path in $Q(L_{4,3})$.

Theorem 13. *$\mathcal{L}_{n,p}$ is a minimal change list for the set $L_{n,p}$.*

Proof. By Corollary 4, there are no more restrictive Gray code as $\mathcal{L}_{n,p}$. □

See Table 1 and Fig. 1(b) for the list $\mathcal{L}_{4,2}$ and $\mathcal{L}_{4,3}$.

4. Graph theoretic issues

Here we present some graph theoretic consequences of the previous results.

4.1. Hamiltonicity

For a graph G let G^k be its kth power, where edges connect vertices which are linked by a path in G of length at most k. A k-Gray code for $B \subset \{0,1\}^n$ is a Hamiltonian path for $Q_n^k|B$, the restriction of the kth power of the hypercube to the set B. Generally, $(Q_n|B)^k$ is a subgraph of $Q_n^k|B$ and equality holds if Hamming distance and the shortest path length coincide on $Q_n|B$; this is the case for $B = F_{n,p}$ or $L_{n,p}$, but not for Dyck words for instance, since the restriction of Q_n to length-n Dyck words is not a connected graph.

As in Section 2, let $Q(L_{n,p}) = Q_n|L_{n,p}$ denote the Lucas cube. Corollary 12 says that $\mathcal{L}_{n,p}$ is a Hamiltonian path in:

- the Lucas cube iff $(p+1) \not| n$;
- the second power of the Lucas cube elsewhere. (In general, $Q(L_{n,p})$ is not 2-connected as it may be checked for $L_{3,2}$, and so, the Hamiltonicity of its second power cannot be obtained trivially from a well-known result in graph theory).

Corollary 14. *If $(p+1)|n$ then*
(1) *the minimal number of paths covering $Q(L_{n,p})$ is p;*
(2) *the length of the maximal path in $Q(L_{n,p})$ is $card(L_{n,p}) - p + 1$.*

Proof. Point 1 follows from Corollary 12. For point 2, from Remark 11, it results that by bypassing in $\mathcal{L}_{n,p}$ the $p-1$ strings $\boldsymbol{y} = succ(\boldsymbol{x})$, with $d(\boldsymbol{x}, \boldsymbol{y}) = 2$, we obtain a maximal length path in $Q(L_{n,p})$. □

When a graph does not have a Hamiltonian path it may be desirable to visit each vertex but not necessarily once, such that the Hamming distance between two successive vertices is one. Following [12], a graph is in the class $\mathcal{H}(s,t)$ if it has a path that visits every vertex at least s times and at most t times, and such a path is called $\mathcal{H}(s,t)$-path. Thus a graph is in $\mathcal{H}(1,1)$ exactly if it is Hamiltonian. In this context, we have:

Corollary 15. *If $(p+1)|n$ then $Q(L_{n,p})$ is in $\mathcal{H}(1,2)$.*

Proof. When $(p+1)|n$, by point 1(a) of Remark 11, in the list $\mathcal{L}_{n,p}$ there are strings $\boldsymbol{x}$ which differ from $succ(\boldsymbol{x})$ in two positions. For each such string, we insert $\boldsymbol{v}$—the predecessor of $\boldsymbol{x}$, see point 1(b) of Remark 11—between $\boldsymbol{x}$ and $succ(\boldsymbol{x})$ and one obtains an $\mathcal{H}(1,2)$ path in $Q(L_{n,p})$. □

Notice that in the $\mathcal{H}(1,2)$ paths above exactly $p-1$ strings are visited twice, and this is optimal. See Fig. 1(c) for an $\mathcal{H}(1,2)$ path in $Q(L_{4,3})$.

4.2. Some structural properties

Let us now recall some definitions concerning a connected graph G with vertices set V:

- the eccentricity of a vertex v is $e(v) = \max_{u \in V} d(u,v)$,
- the diameter of G is $diam(G) = \max_{u,v \in V} d(u,v) = \max_{v \in V} e(v)$,
- the radius of G is $rad(G) = \min_{v \in V} e(v)$, and
- the center of G is $Z(G) = \{u \in G \mid e(u) = rad(G)\}$.

The next proposition generalizes similar results presented in [9] for $L_{n,2}$.

Proposition 16. *Let $n \geqslant 1$, $p \geqslant 2$.*
(1) $diam(Q(L_{n,p})) = \begin{cases} n-1 & \text{if } p = 2 \text{ and } n \text{ odd}, \\ n & \text{otherwise}, \end{cases}$
(2) $e(0^n) = n - \lceil \frac{n}{p} \rceil$,
(3) $rad(Q(L_{n,p})) = n - \lceil \frac{n}{p} \rceil$ *and* $Z(Q(L_{n,p})) = \{0^n\}$.

Proof. 1. Let $\boldsymbol{x} = 1010\ldots$ ($x_i = 1$ iff i is odd) and $\boldsymbol{y} = 0101\ldots$ ($y_i = 1$ iff i is even). If $p = 2$ and n is even or if $p > 2$ then $\boldsymbol{x}, \boldsymbol{y} \in L_{n,p}$ and they are at maximal distance equal to n. If $p = 2$ and n is odd there does not exist a pair of strings in $L_{n,p}$ at distance n and it is easy to find two strings at distance $n - 1$.

2. Let $\boldsymbol{v} = 1^{p-1}0\ldots1^{p-1}01^k0$ with $0 \leqslant k \leqslant p-1$. Clearly, $\boldsymbol{v} \in L_{n,p}$ and it has a minimum number of 0's, namely $\lceil n/p \rceil$, or equivalently, a maximal number of 1's, equal to $n - \lceil n/p \rceil$, So, $e(0^n) = d(0^n, \boldsymbol{v}) = n - \lceil n/p \rceil$. The same result can be obtained by replacing $\boldsymbol{v}$ by any of its circular shifts.

3. For any Lucas string $\boldsymbol{u} \neq 0^n$ we construct another Lucas string at distance greater than $n - \lceil n/p \rceil$ to $\boldsymbol{u}$. The bitwise product $\boldsymbol{v} \cdot \boldsymbol{x}$ of a Lucas string $\boldsymbol{v}$ by a binary (not necessarily Lucas) string $\boldsymbol{x}$ is a Lucas string and $d(\boldsymbol{x}, \boldsymbol{v} \cdot \boldsymbol{x})$ equals the number of positions i where $v_i = 0$ and $x_i = 1$.

Let $\boldsymbol{u} \in L_{n,p}$, $\boldsymbol{u} \neq 0^n$, $\overline{\boldsymbol{u}}$ its bitwise complement and $\boldsymbol{v}$ a circular permutation of the string $1^{p-1}0\ldots1^{p-1}01^k0$ given in point 2, such that $\overline{\boldsymbol{u}}$ and $\boldsymbol{v}$ have at least one 0 in the same position. $\boldsymbol{v} \cdot \overline{\boldsymbol{u}} \in L_{n,p}$ and since $\boldsymbol{v}$ has exactly $\lceil n/p \rceil$ 0's and at least one of them corresponds to a 0 in $\overline{\boldsymbol{u}}$ one has $d(\overline{\boldsymbol{u}}, \boldsymbol{v} \cdot \overline{\boldsymbol{u}}) < \lceil n/p \rceil$. By the triangle inequality

$$d(\boldsymbol{u}, \boldsymbol{v} \cdot \overline{\boldsymbol{u}}) + d(\boldsymbol{v} \cdot \overline{\boldsymbol{u}}, \overline{\boldsymbol{u}}) \geqslant d(\boldsymbol{u}, \overline{\boldsymbol{u}}) = n$$

and so

$$d(\boldsymbol{u}, \boldsymbol{v} \cdot \overline{\boldsymbol{u}}) \geqslant n - d(\overline{\boldsymbol{u}}, \boldsymbol{v} \cdot \overline{\boldsymbol{u}}) > n - \left\lceil \frac{n}{p} \right\rceil. \qquad \square$$

A *stable set* of a graph is a subset of vertices such that there are not two adjacent vertices and the *stability number*, denoted by $\alpha(G)$, is the number of vertices in a stable set of maximum cardinality.

Proposition 17. $\alpha(Q(L_{n,p})) = \max(card(L'_{n,p}), card(L''_{n,p}))$ *where* $\{L'_{n,p}, L''_{n,p}\}$ *is the bipartition of* $L_{n,p}$.

Proof. Let A be the set of maximum cardinality between $L'_{n,p}$ and $L''_{n,p}$, and B the other set. Clearly, A is stable and we will prove that any stable set has at most the same cardinality as A. By Corollaries 4 and 12, the successor $succ(\boldsymbol{x})$ of $\boldsymbol{x}$ in the list $\mathcal{L}_{n,p}$ induces an injective function $succ : B \to A$; in addition, $\boldsymbol{x} \in B$ and $succ(\boldsymbol{x}) \in A$ are connected in $Q(L_{n,p})$. Thus, if a stable set contains $\boldsymbol{x} \in B$ then it must not contain $succ(\boldsymbol{x}) \in A$, and so the cardinality of a stable set does not exceed that of A. $\square$

5. Algorithmic considerations

In [13] is given an exhaustive generating algorithm for the list $\mathcal{F}_{n,p}$ which runs with constant delay between any two successive strings. Now we show how one can modify this algorithm in order to produce efficiently the list $\mathcal{L}_{n,p}$.

Proposition 10 guarantees that at most two Fibonacci strings exist between any two consecutive Lucas string in $\mathcal{L}_{n,p}$; and by point 2 of Lemma 8 the last string in $\mathcal{L}_{n,p}$ is followed by at most one Fibonacci string. So, the algorithm in [13] can be modified to

generate the list $\mathcal{L}_{n,p}$ by simply bypassing the Fibonacci strings which are not Lucas strings. The obtained algorithm inherits the constant delay property if one can decide, in constant time, if the current generated Fibonacci string is also a Lucas string. Additional variables, as (1) the length of the contiguous prefix of 1's and (2) the length and the first position of the rightmost contiguous sequence of 1's, can be used to distinguish in constant time Fibonacci from Lucas strings.

Acknowledgements

We thank Olivier Togni who presented us the problem we deal with in this paper and for some technical suggestions.

References

[2] J.-L. Baril, V. Vajnovszki, Gray codes for order p Lucas strings, in: T. Harju, J. Karhumäki (Eds.), Proc. 4th Internat. Conf. on Combinatorics on Words, September 2003, Turku, Finland, TUCS General Publication, No. 27, August 2003, pp. 149–158.
[3] E. Dedó, D. Torri, N. Zagaglia Salvi, The observability of the Fibonacci and the Lucas cubes, Combinatorics '98 (Palermo), Discrete Math. 255 (1–3) (2002) 55–63.
[4] P. Flajolet, R. Sedgewick, Introduction à l'analyse des algorithmes, Thomson Publishing, 1996.
[5] F. Gray, Pulse code communication, US Patent 2632058, 1953.
[6] W.-J. Hsu, Fibonacci cubes—a new interconnection topology, IEEE Trans. Parallel Distrib. Systems 4 (1) (1993) 3–12.
[7] J.C. Lagarias, D.P. Weisser, Fibonacci and Lucas cubes, Fibonacci Quart. 19 (1) (1981) 39–43.
[8] J. Liu, W.-J. Hsu, M.J. Chung, Generalized Fibonacci cubes are mostly Hamiltonian, J. Graph Theory 18 (8) (1994) 817–829.
[9] E. Munarini, C. Perelli Cippo, N. Zagaglia Salvi, On the Lucas cubes, Fibonacci Quart. 39 (1) (2001) 12–21.
[10] A. Nijenhuis, H.S. Wilf, Combinatorial Algorithms for Computers and Calculators, Academic Press, New York, 1978.
[12] G. Pruesse, F. Ruskey, Generating linear extensions fast, SIAM J. Comput. 23 (1994) 373–386.
[13] V. Vajnovszki, A loopless generation of bitstrings without p consecutive ones, Discrete Mathematics and Theoretical Computer Science, Springer, Berlin, 2001, pp. 227–240.
[14] T.R. Walsh, Generating Gray codes in $O(1)$ worst-case time per word, DMTCS03, Lecture Notes in Computer Science, Vol. 2731, Springer, Berlin, 2003, pp. 72–88.
[15] J. Wu, Extended Fibonacci cubes, IEEE Trans. Parallel Distrib. Systems 8 (12) (1997) 1203–1210.

Available online at www.sciencedirect.com
SCIENCE DIRECT®

Discrete Mathematics 304 (2005) 1 – 10

DISCRETE MATHEMATICS

www.elsevier.com/locate/disc

The irregularity strength of circulant graphs

J.-L. Baril, H. Kheddouci, O. Togni

LE2I, UMR CNRS 5158, Université de Bourgogne, BP 47870, 21078 Dijon cedex, France

Received 13 July 2002; received in revised form 21 September 2005; accepted 27 September 2005

Abstract

The irregularity strength of a simple graph is the smallest integer k for which there exists a weighting of the edges with positive integers at most k such that all the weighted degrees of the vertices are distinct. In this paper we study the irregularity strength of circulant graphs of degree 4. We find the exact value of the strength for a large family of circulant graphs.
© 2005 Elsevier B.V. All rights reserved.

Keywords: Graph; Labeling; Irregularity strength; Circulant graph

1. Introduction and definitions

All the graphs we deal with are undirected, simple and connected.

Let $G = (V, E)$ be a graph with vertex set V and edge set E.

A function $w : E \to \mathbb{Z}^+$ is called a *weighting* of G, and for an edge $e \in E$, $w(e)$ is called the *weight* of e. The *strength* $s(w)$ of w is defined as $s(w) = \max_{e\in E} w(e)$. The *weighted degree* of a vertex $x \in V$ is the sum of the weights of its incident edges: $d_w(x) = \sum_{e\ni x} w(e)$. The *irregularity strength* $s(G)$ of G is defined as $s(G) = \min\{s(w), w \text{ is an irregular weighting of } G\}$.

The study of $s(G)$ was initiated by Chartrand et al. [3] and has proven to be difficult in general. There are not many graphs for which the irregularity strength is known. For an overview of the subject, the reader is referred to the survey of Lehel [8] and recent papers [7,6,1,9,5].

E-mail addresses: barjl@u-bourgogne.fr (J.-L. Baril), kheddouc@u-bourgogne.fr (H. Kheddouci), otogni@u-bourgogne.fr (O. Togni).

0012-365X/$ - see front matter © 2005 Elsevier B.V. All rights reserved.
doi:10.1016/j.disc.2005.09.006

The irregularity strength of regular graphs was considered by several authors. For a regular graph G of order n and degree r, let $\lambda(G) = \lceil (n + r - 1)/r \rceil$. A simple counting argument gives $s(G) \geqslant \lambda(G)$.

On the other hand, an upper bound of $(n/2) + 9$ was given for regular graphs of order n in [4]. This result was improved recently in [5]. Nevertheless, there is still a great gap between lower and upper bounds on the irregularity strength of regular graphs.

Moreover, for all the connected regular graphs for which the irregularity strength is known, we have $s(G) \leqslant \lambda(G) + 1$.

The following conjecture is due to Jacobson (see [8,4]):

Conjecture 1. There exists an absolute constant c such that for each regular graph G, $s(G) \leqslant \lambda(G) + c$.

Circulant graphs are a large family of regular graphs which are studied in the context of interconnection networks [2] and have good properties such as symmetry and vertex-transitivity.

Definition 1. Let n be an integer and let $s_1, s_2, \ldots, s_k$ be a sequence of integers, with $1 \leqslant s_1 < s_2 < \cdots < s_k \leqslant n/2$. The circulant graph $G = C_n(s_1, s_2, \ldots, s_k)$ of order n is a graph with vertex set $V(G) = \{0, 1, \ldots, n-1\}$ and edge set $E(G) = \{(x, x \pm s_i \bmod n),\ x \in V(G),\ 1 \leqslant i \leqslant k\}$. It is a Cayley graph on the additive group of integers modulo n, with set of generators $\{s_1, s_2, \ldots, s_k\}$.

In this paper, we study the irregularity strength of circulant graphs of degree 4. We find the exact value for the irregularity strength of circulant graphs of the form $C_n(1, k)$ with $n \geqslant 4k + 1$. These graphs consists in a cycle of length n plus chords joining vertices at distance k on the cycle.

We will use the following notations (see Fig. 1): for any t, the vertices of $C_{4t+1}(1, k)$ are labeled $y_0, y_1, \ldots, y_{2t}$ going clockwise on the cycle and $x_0 = y_0,\ x_1, x_2, \ldots, x_{2t}$ going counterclockwise.

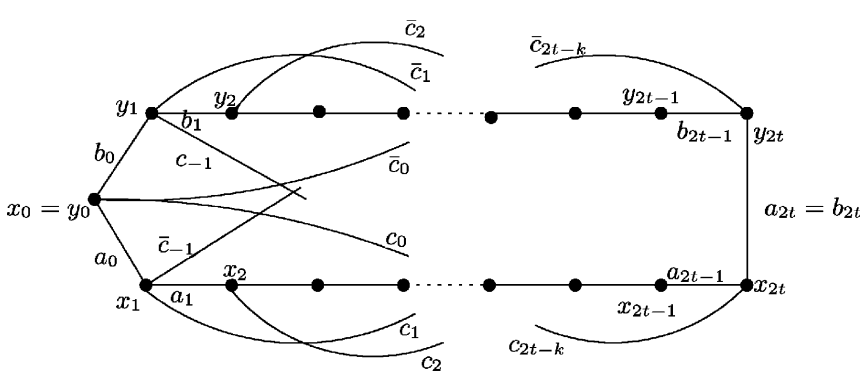


Fig. 1. Notations.

For a weighting w on $C_{4t+1}(1,k)$, denote by a_i (resp. b_i) the weight of the edge (x_i, x_{i+1}) (resp. (y_i, y_{i+1})) for $0 \leqslant i \leqslant 2t-1$, and by $a_{2t} = b_{2t}$ the weight of the edge (x_{2t}, y_{2t}), and let

$$\begin{cases} c_i = w(x_i, x_{i+k}) & 0 \leqslant i \leqslant 2t-k, \\ \bar{c}_i = w(y_i, y_{i+k}) & 0 \leqslant i \leqslant 2t-k, \\ c_i = w(x_i, y_{4t-i-k+1}) & 2t-k+1 \leqslant i \leqslant 2t, \\ \bar{c}_i = w(y_i, x_{4t-i-k+1}) & 2t-k+1 \leqslant i \leqslant 2t, \\ c_{-i} = w(y_i, x_{k-i}) & 1 \leqslant i \leqslant k-1, \\ \bar{c}_{-i} = w(x_i, y_{k-i}) & 1 \leqslant i \leqslant k-1. \end{cases}$$

Let us give a construction of $C_{4(t+1)+1}(1,k)$ from $C_{4t+1}(1,k)$ useful for the proof of main results. Let $G = C_{4t+1}(1,k)$ and let $P_4 = (x_{2t+1}, x_{2t+2}, y_{2t+2}, y_{2t+1})$ be a path of length 3 where $t \geqslant k \geqslant 2$. We define the graph $G.P_4$ as follows:

$$V(G.P_4) = V(G) \cup V(P_4),$$

$$\begin{aligned} E(G.P_4) = E(G) &\setminus \left(\bigcup_{i=1}^{k} \{(x_{2t-k+i}, y_{2t-i+1})\} \cup \{(x_{2t}, y_{2t})\} \right) \\ &\cup E(P_4) \bigcup_{i=1}^{k} \{(x_{2t-k+2+i}, y_{2t+3-i})\} \\ &\cup \{(x_{2t-k+1}, x_{2t+1}), (x_{2t-k+2}, x_{2t+2}), (y_{2t-k+1}, y_{2t+1}), \\ &\quad (y_{2t-k+2}, y_{2t+2})\}. \end{aligned}$$

Observe that $G.P_4 = C_{4t+5}(1,k)$. Thus for any $t \geqslant k \geqslant 2$, $C_{4t+1}(1,k)$ can be obtained by iterating $(t-k)$ times this composition starting from $C_{4k+1}(1,k)$.

2. Preliminaries

Definition 2. For $n = 4t+1$, with $t \geqslant k \geqslant 2$, $C_n(1,k)$ verifies the property $\mathscr{P}_t$ if it admits an irregular weighting of strength $s = \lambda(C_n(1,k)) = t+1$ such that:

(i) $a_{2t} = b_{2t} = a_{2t-1} = b_{2t-1} + 1 = s$,
(ii) $c_{2t-i} = \bar{c}_{2t-i} = s$, for each i, $0 \leqslant i \leqslant k$,
(iii) $a_{2t-2i+1} \geqslant b_{2t-2i+1} \geqslant \lceil k/2 \rceil + 2 - i$, for each i, $0 \leqslant i \leqslant \lceil k/2 \rceil - 1$,
(iv) when k is even, $a_{2t-2k+2i+1} \geqslant i+1$, for each i, $0 \leqslant i \leqslant k-1$.

Lemma 1. *For $k \geqslant 2$, $C_{4k+1}(1,k)$ verifies the property $\mathscr{P}_k$.*

Proof. In order to prove that $C_{4k+1}(1,k)$ verifies the property $\mathscr{P}_k$, we construct an irregular weighting w as follows:

$$\begin{cases} a_0 = b_0 = 1, & \\ a_i = \left\lfloor \dfrac{i+1}{2} \right\rfloor + 1 & \text{for each } i, \quad 1 \leqslant i \leqslant 2k, \\ b_i = a_{i-1} & \text{for each } i, \quad 1 \leqslant i \leqslant 2k, \\ c_i = \bar{c}_i = i+1 & \text{for each } i, \quad 0 \leqslant i \leqslant k-1, \\ c_i = \bar{c}_i = k+1 & \text{for each } i, \quad k \leqslant i \leqslant 2k, \\ c_i = \bar{c}_i = 1 & \text{for each } i, \quad 1-k \leqslant i \leqslant -1. \end{cases}$$

We can easily verify that we have $d_w(x_0) = d_w(y_0) = 4$, and for each $1 \leqslant i \leqslant 2k$:

$$\begin{cases} d_w(x_i) = a_i + a_{i-1} + c_i + c_{i-k} = 4 + 2i, \\ d_w(y_i) = b_i + b_{i-1} + c_i + c_{i-k} = 3 + 2i. \end{cases}$$

Consequently, w is irregular and of strength equal to $\lambda(C_{4k+1}(1,k)) = k+1$ (i.e., statements (i) and (ii) are verified). Moreover, by construction the remaining statements of property $\mathscr{P}_k$ are also verified: the third statement is verified since $a_{2k-2i+1} \geqslant b_{2k-2i+1} = \lfloor (2k-2i+1)/2 \rfloor + 1 \geqslant \lceil k/2 \rceil - i + 2$. The fourth statement is also verified since $a_{2i+1} = \lfloor (2i+2)/2 \rfloor + 1 = i+2 \geqslant i+1$. See part A of Appendix for an example when $k = 4$. □

3. Results

In this section, we present two propositions. The first one construct an irregular weighting for $C_{4t+1}(1,k)$ for any $t \geqslant k$. The second proposition allows to give an irregular weighting for the remaining graphs $C_n(1,k)$, with $n = 4t+2, 4t+3, 4t+4$ for each $t \geqslant k$.

Proposition 1. *For any t with $t \geqslant k \geqslant 2$, there exists an irregular weighting of $C_{4t+1}(1,k)$ which verifies the property $\mathscr{P}_t$.*

Proof. The proof is done by induction on t. For $t = k$, the result holds by Lemma 1.

Assume that there exists an irregular weighting of $C_{4t+1}(1,k)$, which verifies the property $\mathscr{P}_t$. We will construct a weighting of $C_{4t+5}(1,k)$ which verifies property $\mathscr{P}_{t+1}$.

Let w be an irregular weighting of $G = C_{4t+1}(1,k)$ which verifies the property $\mathscr{P}_t$. Let $G' = G.P_4 = C_{4t+5}(1,k)$ and w' be the weighting of G' constructed from the weighting w of G as follows:

$$w'(e) = \begin{cases} w(e) & \text{for } e \in E(G) \backslash M, \\ w(e) - 1 & \text{for } e \in M, \\ t+2 & \text{for } e \in A \cup B, \\ t+1 & \text{for } e \in C, \end{cases}$$

where

$$M = \bigcup_{i=1}^{\lceil k/2\rceil - 1} \{(x_{2t-2i+1}, x_{2t-2i+2}),\ (y_{2t-2i+1}, y_{2t-2i+2})\},$$

$$A = \bigcup_{i=1}^{k} \{(x_{2t-k+i}, y_{2t-i+3})\},$$

$$B = \{(x_{2t+2}, y_{2t+2}), (y_{2t-k+2}, y_{2t+2}), (x_{2t-k+2}, x_{2t+2}), (x_{2t+1}, x_{2t+2})\},$$

$$C = \{(x_{2t}, x_{2t+1}), (y_{2t}, y_{2t+1}), (x_{2t-k+1}, x_{2t+1}), (y_{2t+1}, y_{2t+2}), (y_{2t-k+1}, y_{2t+1})\}.$$

Note that, by statement (iii) of property $\mathscr{P}_t$ of G, $w(e) \geqslant 2$ if $e \in M$. Thus, $w'(e) \geqslant 1$ for each $e \in E(G')$. Moreover, the degrees of the vertices given by the weighting w' are as follows:

$$\begin{cases} d_{w'}(x_i) = d_w(x_i) = 2i + 4 & \text{for } 0 \leqslant i \leqslant 2t - k + 1, \\ d_{w'}(y_i) = d_w(y_i) = 2i + 3 & \text{for } 1 \leqslant i \leqslant 2t - k + 1. \end{cases}$$

Observe that for each of the following vertices, the weight of one adjacent edge has been decreased by one, and the weight of another adjacent edge has been increased by one. So the weighted degrees are unchanged: for $0 \leqslant i \leqslant \lceil k/2\rceil - 2$,

$$\begin{cases} d_{w'}(x_{2t-2i}) = (a_{2t-2i-1} - 1) + a_{2t-2i} + c_{2t-2i-k} + (c_{2t-2i} + 1) = d_w(x_{2t-2i}), \\ d_{w'}(y_{2t-2i}) = (b_{2t-2i-1} - 1) + b_{2t-2i} + \bar{c}_{2t-2i-k} + (\bar{c}_{2t-2i} + 1) = d_w(y_{2t-2i}), \\ d_{w'}(x_{2t-2i-1}) = a_{2t-2i-1} + (a_{2t-2i} - 1) + c_{2t-2i-k} + (c_{2t-2i} + 1) \\ \qquad\qquad = d_w(x_{2t-2i-1}), \\ d_{w'}(y_{2t-2i-1}) = b_{2t-2i-1} + (b_{2t-2i} - 1) + \bar{c}_{2t-2i-k} + (\bar{c}_{2t-2i} + 1) \\ \qquad\qquad = d_w(y_{2t-2i-1}). \end{cases}$$

Moreover, when k is even, the weighted degrees of vertices x_{2t-k+2} and y_{2t-k+2} are not given above. Their weighted degrees are:

$$\begin{cases} d_{w'}(x_{2t-k+2}) = a_{2t-k+1} + a_{2t-k+2} + c_{2t-2k+2} + c_{2t-k+1} = d_w(x_{2t-k+2}) + 1, \\ d_{w'}(y_{2t-k+2}) = b_{2t-k+1} + b_{2t-k+2} + c_{2t-2k+2} + c_{2t-k+1} = d_w(y_{2t-k+2}) + 1. \end{cases}$$

We can see that, in this case, $d_{w'}(x_{2t-k+2}) = d_{w'}(y_{2t-k+3})$ and there does not exist a vertex $u \in V(G')$ such that $d_{w'}(u) = d_w(y_{2t-k+2})$ (see part B of Appendix).

For the vertices of P_4 we have

$$\begin{cases} d_{w'}(x_{2t+1}) = 4t + 6, \\ d_{w'}(x_{2t+2}) = 4t + 8, \\ d_{w'}(y_{2t+1}) = 4t + 5, \\ d_{w'}(y_{2t+2}) = 4t + 7. \end{cases}$$

Let a'_i, b'_i, c'_i and $\bar{c}'_i$ be the values of edges of G' given by the weighting w' as defined in Section 1. We distinguish two cases depending on the parity of k.

Case 1: k is odd.

In order to verify the property $\mathscr{P}_{t+1}$ for G', one can see that $1 \leqslant w'(e) \leqslant t+2$ for each $e \in E(G')$ and for each $u, v \in V(G')$, $d_{w'}(u) \neq d_{w'}(v)$ when $u \neq v$.

Thus w' is irregular and the statements (i) and (ii) hold.

The third one (iii) is also verified since

$$a'_{2(t+1)-2i+1} \geqslant b'_{2(t+1)-2i+1} = t+1 \geqslant \left\lceil \frac{k}{2} \right\rceil + 2 - i \quad \text{for } i = 1 \text{ or } 2$$

and

$$a'_{2(t+1)-2i+1} \geqslant b'_{2(t+1)-2i+1} \geqslant b_{2t-2(i-1)+1} - 1 \geqslant \left\lceil \frac{k}{2} \right\rceil + 2 - (i-1) - 1$$
$$\geqslant \left\lceil \frac{k}{2} \right\rceil + 2 - i \quad \text{for each } i,\ 3 \leqslant i \leqslant \left\lceil \frac{k}{2} \right\rceil - 1.$$

Case 2: k is even (a complete example is given in Appendix). We use a cycle C induced by the following vertices $C = (x_{2t-k+2}, x_{2t-k+1}, \ldots, x_{2t-2k+2})$. In order for the weighting to be irregular, it suffices to decrease by two the weighted degree of x_{2t-k+2} in w' (i.e., to obtain $d_{w'}(x_{2t-k+2}) = d_w(y_{2t-k+2})$) without changing degrees of other vertices.

To do this, we decrease and increase alternately by one the weight of the edges of C starting from x_{2t-k+2}. As C is odd, the two edges of C which are adjacent to x_{2t-k+2} are both decreased by one. Thus the degree of x_{2t-k+2} is decreased by two. Moreover, for any other vertex of C, the degree is unchanged.

As G verified statement (iv) of property $\mathscr{P}_t$, then the weights of edges of C which are decreased are greater or equal to one. In other hand, as the strength of G is equal to $t+1$ and edges of C are in G, so by increasing some edges of C by one, the strength of w' remains $t+2 = \lambda(G')$.

Observe that statements (i)–(iii) of property $\mathscr{P}_{t+1}$ are proved similarly than in case k odd. So we now prove statement (iv) for G'. Note that

$$a'_{2(t+1)-2k+2i+1} = a'_{2t-2k+2(i+1)+1} = a_{2t-2k+2(i+1)+1} - 1 \geqslant i+1$$
$$\text{for } 1 \leqslant i \leqslant k-2.$$

Moreover, for $i = k-1$, $a'_{2(t+1)-2k+2i+1} = t+2 \geqslant k-1+1 = k$. $\square$

The following proposition extends the irregular weighting to cases $n = 4t+2, 4t+3$ and $4t+4$, for $t \geqslant k$.

Proposition 2. *If $C_{4t+1}(1,k)$ verifies property $\mathscr{P}_t$ then there exists an irregular weighting of $C_{4t+1+i}(1,k)$, for any i, $1 \leqslant i \leqslant 3$, of strength $s' = \lambda(C_{4t+1+i}(1,k)) = t+2$.*

Proof. Observe that $\lambda(C_{4t+1+i}(1,k)) = \lceil (4t+1+i+3)/4 \rceil = t+2$. We start from $G = C_{4t+1}(1,k)$ with an irregular weighting verifying property $\mathscr{P}_t$ and we add i vertices,

$1 \leqslant i \leqslant 3$, by subdividing the edge (x_{2t}, y_{2t}) in $i+1$ edges (see Figs. 2–4, where α_1, α_2, β_1 and β_2 are the weights of some edges of the irregular weighting of $C_{4t+1}(1, k)$).

The new edges and the edges whose endpoints have been modified are weighted as shown below:

Case $i = 1$: a weighting is obtained simply by subdividing the edge (x_{2t}, y_{2t}) in two and then by giving the weight $t+2$ to the two created edges (see Fig. 2).

Case $i = 2$: we distinguish between the case k odd and k even. The process for both cases is given in Fig. 3, where dashed edges have weight $t+1$ and bold edges are edges whose weights have been modified.

Case $i = 3$: we distinguish between the case k odd and k even. The process for both cases is given in Fig. 4. □

From Propositions 1 and 2, we deduce the following result:

Theorem 1. *For* $n \geqslant 4k+1$ *and* $k \geqslant 2$,

$$s(C_n(1, k)) = \left\lceil \frac{n+3}{4} \right\rceil.$$

Corollary 1. *For* $n \geqslant 4s_k + 1$,

$$s(C_n(1, s_1, s_2, \ldots, s_k)) \leqslant \left\lceil \frac{n+3}{4} \right\rceil.$$

Proof. Starting from an irregular weighting of $C_n(1, s_k)$ given by Theorem 1, assign the weight 1 to the other edges of $C_n(1, s_1, s_2, \ldots, s_k)$ and the weighting will remain irregular. □

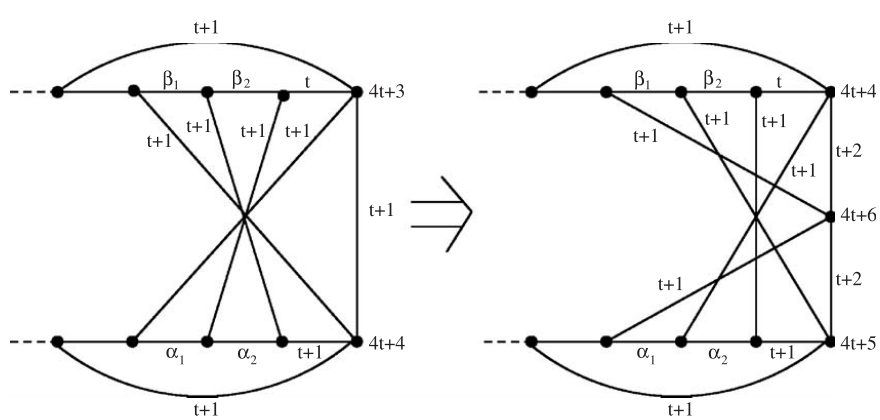


Fig. 2. An irregular weighting of $C4t+2(1, k)$ on the right is obtained from the irregular weighting of $C4t+1(1, k)$ on the left.

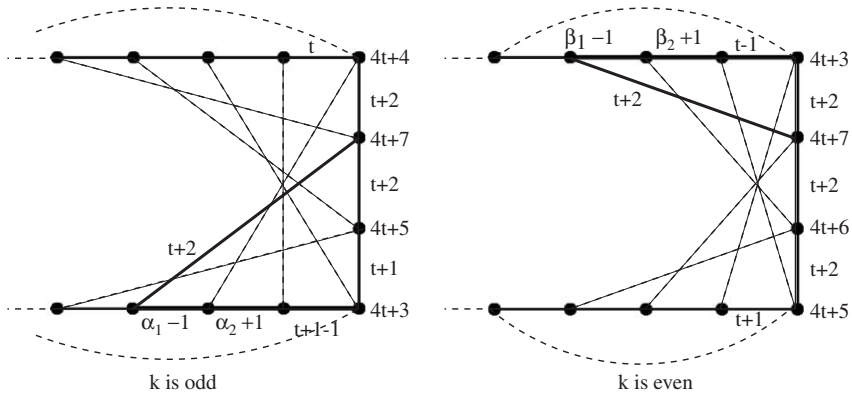


Fig. 3. An irregular weighting of $C_{4t+3}(1,k)$, with k odd ($k=5$) on the left and k even ($k=4$) on the right.

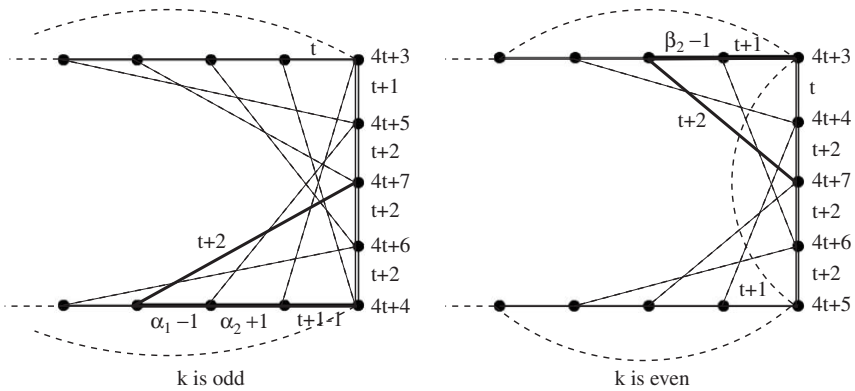


Fig. 4. An irregular weighting of $C_{4t+4}(1,k)$, with k odd ($k=5$) on the left and k even ($k=4$) on the right.

4. Concluding remarks

We studied the irregularity strength of circulant graphs $C_n(1,k)$ when $k \leqslant (n-1)/4$ and found the exact value. Our method does not seem to work for $(n/4) \leqslant k \leqslant (n/2)$, because we have not found an irregular weighting to start the induction process.

But, the isomorphisms of circulant graphs allow us to come back to some cases when $k \leqslant (n/4)$. For instance, the following isomorphisms are easy to see:

$$\begin{cases} C_{2k-1}(1,k) \cong C_{2k-1}(1,2), \\ C_{3k-1}(1,k) \cong C_{3k-1}(1,3), \\ C_{4k-1}(1,k) \cong C_{4k-1}(1,4). \end{cases}$$

More generally, if $\gcd(n,k)=1$ then we have $C_n(1,k) \cong C_n(1,k^{-1} \bmod n)$.

Moreover, our study is also generalized to some circulant graphs $C_n(s_1,s_2)$ with $s_1 \neq 1$ and $s_2 > s_1$. If $\gcd(n,s_1)=1$ or $\gcd(n,s_2)=1$ then $C_n(s_1,s_2) \cong C_n(1,k)$ for some k.

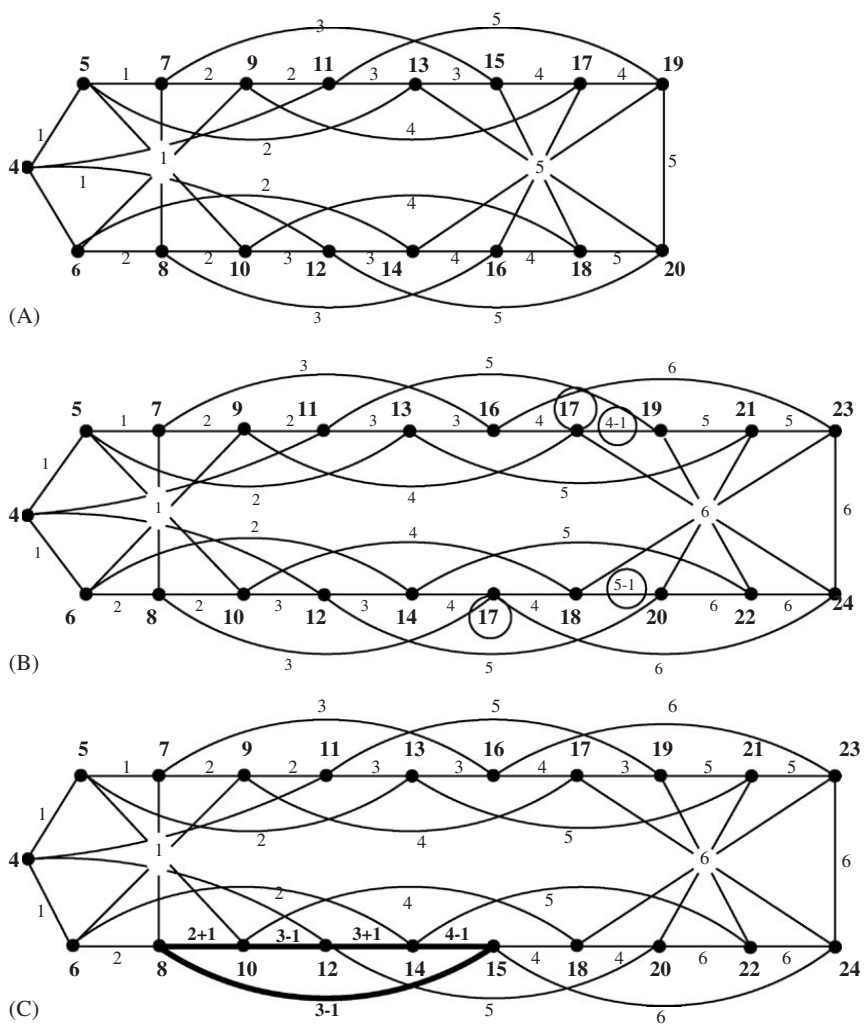
Appendix: a complete example when $k = 4$

Let us give a complete example of the construction of an irregular weighting of $C_{21}(1, 4)$ from $C_{17}(1, 4)$. The values on the edges represent their weights and the bold values on vertices represent their weighted degrees.

Part A: an irregular weighting of $C_{17}(1, 4)$ verifying property $\mathscr{P}_4$.

Part B: the first step of the extension of the weighting of $C_{17}(1, 4)$ to the graph $C_{21}(1, 4)$ by composition with P_4. Notice that the weighting is not irregular as there is two vertices of degree 17.

Part C: the weighting of part B is modified to become irregular by using a cycle of length 5 (in bold on the figure) to decrease the weighted degree of one of the two vertices of degree 17 to 15.



References

[1] D. Amar, O. Togni, Irregularity strength of trees, Discrete Math. 190 (1–3) (1998) 15–38.
[2] J.C. Bermond, F. Comellas, D.F. Hsu, Distributed loop computer-networks: a survey, J. Parallel Distributed Computing 24 (1995) 2–10.
[3] G. Chartrand, M. Jacobson, J. Lehel, O. Oellerman, S. Ruiz, F. Saba, Irregular networks, Congr. Numer. 64 (1988) 187–192.
[4] R.J. Faudree, J. Lehel, Bound on the irregularity strength of regular graphs, Colloq. Math. Soc. János Bolya 52 (1987) 247–256.
[5] A.M. Frieze, R. Gould, M. Karonski, F. Pfender, On graph irregularity strength, J. Graph Theory 41 (2) (2002) 120–137.
[6] S. Jendroľ, M. Tkáč, The irregularity strength of tK_p, Discrete Math. 145 (1995) 301–305.
[7] S. Jendroľ, V. Žoldák, The irregularity strength of generalized Petersen graphs, Math. Slovaca 2 (1995) 107–113.
[8] J. Lehel, Facts and quests on degree irregular assignments, Graph Theory, Combin. Appl. (1991) 765–782.
[9] T. Nierhoff, A tight bound on the irregularity strength of graphs, SIAM J. Discrete Math. 13 (3) (2000) 313–323.

AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **35** (2006), Pages 89–102

Adjacent vertex distinguishing edge-colorings of meshes

J.-L. Baril

Laboratoire LE2I, Université de Bourgogne
BP 47870, 21078 Dijon cedex
France
barjl@u-bourgogne.fr

H. Kheddouci

Laboratoire PRISMa, Université Claude Bernard Lyon1
843, Bd. du 11 novembre 1918
69622 Villeurbanne Cedex
France
hkheddou@bat710.univ-lyon1.fr

O. Togni

Laboratoire LE2I, Université de Bourgogne
BP 47870, 21078 Dijon cedex
France
otogni@u-bourgogne.fr

Abstract

This paper studies the minimum number of colors for an edge-coloring of a graph such that adjacent vertices are distinguished by their color sets (colors of edges that are incident to them). Such a coloring is called adjacent vertex distinguishing. We find optimal adjacent vertex distinguishing colorings of the multidimensional mesh (toroidal or not) and of the hypercube. We show that for both graphs, this number of colors is equal to the maximum degree of the graph plus one.

1 Introduction

All graphs we deal with are undirected, simple and connected.

Let $G = (V, E)$ be a graph with vertex set V and edge set E.

A *proper edge coloring* c is a mapping from E to $\mathbb{N}$ satisfying $c(xy) \neq c(yz)$, for any $xy, yz \in E$. For any vertex $x \in V$, let $S(x)$ denote the set of the colors of all edges incident to x.

As defined in [2, 3], a proper edge coloring c is said to be *vertex distinguishing* if $S(x) \neq S(y)$, for any $x, y \in V$, $x \neq y$. The minimum number of colors among all vertex distinguishing colorings of a graph is called the *observability* of the graph in [4].

In this note, we study a relaxed version of this parameter in which only adjacent vertices have to be distinguished by their color sets (see [1, 5]). Formally, an *adjacent vertex distinguishing* edge coloring is a proper edge-coloring satisfying $S(x) \neq S(y)$, for any x, y with $xy \in E$. Let $\chi'_a(G)$ denote the minimum number of colors of any AVD-coloring of G.

Little is known about AVD-colorings. In [1], the authors prove that $\chi'_a(G) \leq 5$ for graphs of maximum degree 3 and $\chi'_a(G) \leq \Delta(G) + 2$ for bipartite graphs.

The following conjecture was made in [5]:

Conjecture 1 *Let G be a connected graph of maximum degree Δ and different from C_5 and K_2. Then*

$$\Delta \leq \chi'_a(G) \leq \Delta + 2.$$

Notice that $\chi'_a(G)$ is obviously greater than Δ and if G has two adjacent vertices of degree Δ then $\chi'_a(G) \geq \Delta + 1$.

The purpose of this paper is to find AVD-colorings of (toroidal or simple) multidimensional meshes and of hypercubes, with as few colors as possible. In fact we shall present stronger results that bound the number of colors needed for an AVD-coloring of the Cartesian product of a graph by a path or by a cycle. These results allow us to compute the exact value of the parameter χ'_a.

The *d-dimensional mesh* $M_{n_1,n_2,\ldots,n_d}$ is the Cartesian product of d paths: $M_{l_1,l_2,\ldots,l_d} = P_{l_1} \times P_{l_2} \times \ldots \times P_{l_d}$.

The *d-dimensional toroidal mesh* $TM_{l_1,l_2,\ldots,l_d}$ is the Cartesian product of d cycles: $TM_{l_1,l_2,\ldots,l_d} = C_{l_1} \times C_{l_2} \times \ldots \times C_{l_d}$.

The *d-dimensional hypercube* H_d is the Cartesian product of K_2 by itself d times: $H_d = H_{d-1} \times K_2$, $H_1 = K_2$.

We will use the following notation throughout the paper: The 2-dimensional mesh $M_{m,n}$ has vertex set $V = \{x_{i,j}, 0 \leq i \leq m-1, 0 \leq j \leq n-1\}$ and edge set $E = \{x_{i,j}x_{i,j+1}, 0 \leq i \leq m-1, 0 \leq j \leq n-2\} \cup \{x_{i,j}x_{i+1,j}, 0 \leq i \leq m-2, 0 \leq j \leq n-1\}$.

If G is the mesh $M_{m,n}$ then we denote by $\mathcal{V}_k$ the sequence of vertical edges $\mathcal{V}_k = (x_{k-1,0}x_{k,0}, \ldots, x_{k-1,n-1}x_{k,n-1})$ with $1 \leq k \leq m-1$ and for an edge-coloring c of G, we will write $c(\mathcal{V}_k) = (c_1, c_2, \ldots, c_{n-1})$ to mean that the first edge of $\mathcal{V}_k$ is assigned the color c_1, the second the color c_2, and so on.

Similarly, we denote by $\mathcal{H}_k$ the sequence of horizontal edges

$\mathcal{H}_k = (x_{k-1,0}x_{k-1,1}, \ldots, x_{k-1,n-2}x_{k-1,n-1})$ with $1 \leq k \leq m$ and for an edge-coloring c of G, we will write $c(\mathcal{H}_k) = (c_1, c_2, \ldots, c_{n-1})$ to denote the colors of the edges of $\mathcal{H}_k$.

We define the sequences of edges of the toroidal mesh $TM_{m,n}$ as we did for the mesh $M_{m,n}$ but in that case, we have the sequence $\mathcal{V}_0 = (x_{m-1,0}x_{0,0}, \ldots, x_{m-1,n-1}x_{0,n-1})$ and one edge more at the beginning of each sequence $\mathcal{H}_k$: $\mathcal{H}_k = (x_{k-1,n-1}x_{k-1,0}, \ldots)$.

2 AVD-colorings of 2-dimensional meshes

Let us begin with the following lemmas:

Lemma 1 *Let $M_{m,n}$ be a 2-dimensional mesh, with $n, m \geq 3$, $n + m \geq 7$. Then $\chi'_a(M_{m,n}) = 5$.*

Proof : We define the coloring c of $\mathcal{H}_k$ $(1 \leq k \leq m)$ of the mesh $M_{m,n}$ by:

$$\begin{cases} c(\mathcal{H}_1) & = (4, 3, 4, 3, \ldots), \\ c(\mathcal{H}_i) & = (c(\mathcal{H}_{i-1}) + 1) \bmod 5 \text{ for } 2 \leq i \leq m, \end{cases}$$

and the coloring of $\mathcal{V}_k$ $(1 \leq k \leq m-1)$ by:

$$\begin{cases} c(\mathcal{V}_1) & = (1, 2, 1, 2, \ldots), \\ c(\mathcal{V}_i) & = (c(\mathcal{V}_{i-1}) + 1) \bmod 5 \text{ for } 2 \leq i \leq m-1. \end{cases}$$

See Figure 1 for an illustration of this coloring.

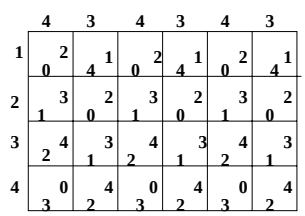


Figure 1: An AVD-coloring for the mesh $M_{5,7}$.

First, one can see that, by construction, c is a proper coloring. It remains to show that c is an AVD-coloring. Notice that for any two vertices x and y, if $d(x) \neq d(y)$ then $S(x) \neq S(y)$. Thus, we only have to compare the sets of colors of adjacent vertices of the same degree.

- **Case 1**: degree 2 vertices. Since there are only four vertices of degree 2 and since they form an independent set in $M_{m,n}$, there is no problem with these vertices.

- **Case 2**: degree 3 vertices. Note that for any i, $1 \leq i \leq m-3, S(x_{i,0}) = \{\alpha, \alpha+1, \beta\}$ and $S(x_{i+1,0}) = \{\alpha+1, \alpha+2, \beta+1\}$ for some $\alpha, \beta \in \{0,1,2,3,4\}$, where addition is taken modulo 5. This proves that $S(x_{i,0}) \neq S(x_{i+1,0})$. A similar argument yields $S(x_{i,n-1}) \neq S(x_{i+1,n-1})$. For two adjacent vertices u and v on the horizontal border, the two horizontal edges incident to u have the same color as the two ones incident to v. But the colors of the verticals edges of u and v are different. Thus the sets of colors are different.

- **Case 3**: degree 4 vertices. By construction, we have $S(x_{i,j}) = \{\alpha, \alpha+1, \beta, \beta+1\}$ for some $\alpha, \beta \in \{0,1,2,3,4\}$, where addition is taken modulo 5. Moreover, $S(x_{i,j+1}) = \{\alpha+1, \alpha+2, \beta, \beta+1\}$ if j is even and $S(x_{i,j+1}) = \{\alpha-1, \alpha, \beta, \beta+1\}$ if j is odd. Therefore, $S(x_{i,j}) \neq S(x_{i,j+1})$.

  On the other hand, $S(x_{i+1,j}) = \{\alpha+1, \alpha+2, \beta+1, \beta+2\}$. Thus, $S(x_{i,j}) = S(x_{i+1,j})$ if $\{\alpha, \beta\} = \{\alpha+2, \beta+2\}$ which is impossible in $\mathbb{Z}_5$.

Hence c is an AVD-coloring.

□

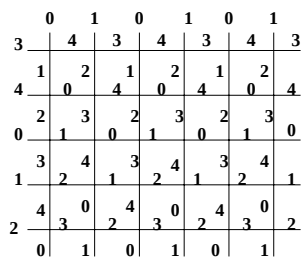


Figure 2: An AVD-coloring for $TM_{5,6}$.

In the two following propositions, we give the exact values of $\chi'_a(TM_{m,n})$ for any m and n.

Proposition 1 *Let $m, n \in \mathbb{Z}$, $m, n \geq 3$. If m or n is even, then $\chi'_a(TM_{m,n}) = 5$.*

Proof : Without loss of generality, we assume that n is even. We discuss five cases depending on values of m:

- **Case 1**: $m \equiv 0 \bmod 5$. Let us construct the following edge-coloring for any $TM_{m,n}$ (see Figure 2 for $TM_{5,6}$) :

  $c(\mathcal{H}_i) = (i+2, i+3, i+2, i+3, \ldots, i+3, i+2)$, for each $1 \leq i \leq m$, where $i+j$ is taken modulo 5 with $2 \leq j \leq 3$.

  $c(\mathcal{V}_i) = (i, i+1, i, i+1, \ldots, i, i+1)$, for each $0 \leq i \leq m-1$, where $i+j$ is taken modulo 5 with $0 \leq j \leq 1$.

  Observe that in such a coloring, for each $uv \in E(TM_{m,n})$, there always exists one color $j \in S(v)$ such that $j \notin S(u)$. Therefore c is an AVD-coloring.

- **Case 2**: $m \equiv 3 \bmod 5$. An AVD-coloring of $TM_{3,n}$ is illustrated in B_2 of Figure 3. For $m > 3$, by Case 1, we obtain an AVD-coloring of $TM_{m-3,n}$ such that $\chi'_a(TM_{m-3,n}) = 5$. To construct an AVD-coloring of $TM_{m,n}$, we extend the AVD-coloring of $TM_{m-3,n}$ by joining to it the AVD-coloring of $TM_{3,n}$ (see Figure 3).

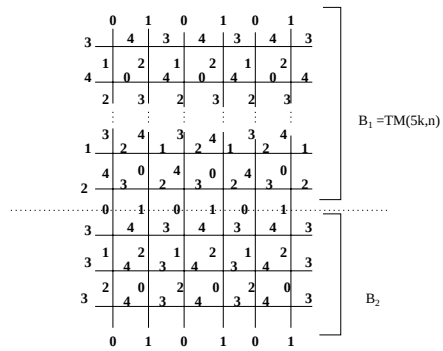


Figure 3: An AVD-coloring for $TM_{5k+3,6}$, with $k \geq 1$.

- **Case 3**: $m \equiv 4 \bmod 5$. An AVD-coloring of $TM_{4,n}$ is illustrated in Figure 4. For $m > 4$, by Case 1, we obtain an AVD-coloring of $TM_{m-4,n}$ such that $\chi'_a(TM_{m-4,n}) = 5$. To construct an AVD-coloring of $TM_{m,n}$, we extend the AVD-coloring of $TM_{m-4,n}$ by joining to it the AVD-coloring of $TM_{4,n}$ (it is obtained by replacing in Figure 3 the block B_2 by the coloring illustrated in Figure 4).

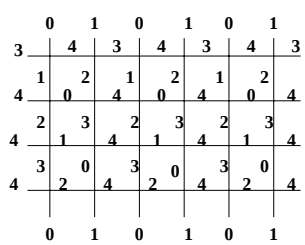


Figure 4: An AVD-coloring for $TM_{4,6}$.

- **Case 4**: $m \equiv 1 \bmod 5$. An AVD-coloring of $TM_{6,n}$ is given in Figure 5. For $m > 6$, by Case 1, we obtain an AVD-coloring of $TM_{m-6,n}$ such that $\chi'_a(TM_{m-6,n}) = 5$. To construct an AVD-coloring of $TM_{m,n}$, we extend the AVD-coloring of $TM_{m-6,n}$ by joining to it (as is done in above cases) the AVD-coloring of $TM_{6,n}$.

- **Case 5**: $m \equiv 2 \bmod 5$. Let $m = m' + m''$ such that $m' \equiv 3 \bmod 5$ and $m'' \equiv 4 \bmod 5$. By Cases 2 and 3, we obtain the AVD-colorings of $TM_{m',n}$ and $TM_{m'',n}$ respectively such that $\chi'_a(TM_{m',n}) = 5$ et $\chi'_a(TM_{m'',n}) = 5$. As in

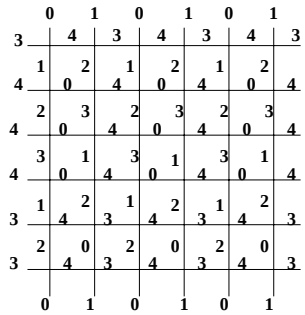


Figure 5: An AVD-coloring for $TM_{6,6}$.

the preceding cases, one can construct an AVD-coloring for $TM_{m,n}$ by joining together those of $TM_{m',n}$ and $TM_{m'',n}$.

In the following proposition, we study the 2-dimensional toroidal mesh when m and n are odd.

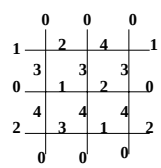


Figure 6: An AVD-coloring for $TM_{3,3}$.

Proposition 2 *Let $m, n \in \mathbb{Z}$, $m, n \geq 3$. If m and n are odd then $\chi'_a(TM_{m,n}) = 5$.*

Proof : See Figure 6 for $m = n = 3$. The case $m = 9$ and $n = 3$ is obtained by joining three copies of $TM_{3,3}$ colored according to Figure 6. The case $m = n = 9$ is obtained by joining three copies of $TM_{9,3}$.

Now, we decompose the rest of the proof into five cases:

- **Case 1:** $m \equiv 0 \bmod 5$. The AVD-coloring of $TM_{m,n}$ is obtained as follows, for $1 \leq i \leq m-1$ and $1 \leq j \leq m$:

$$\begin{cases} c(\mathcal{V}_0) & = (0,1,0,1,0,\ldots,1,0) \\ c(\mathcal{V}_i) & = (c(\mathcal{V}_{i-1})+1) \bmod 5 \\ c(\mathcal{H}_j) & = (j+1, j+3, j+2, \ldots, j+3, j+2) \bmod 5 \end{cases}$$

  See Figure 7 for an example.

- **Case 2:** $m \equiv 1 \bmod 5$. For $m > 6$, the AVD-coloring of $TM_{m,n}$ results from joining one copy of $TM_{m-6,n}$ given by Case 1 with one copy of $TM_{6,n}$ colored according to Figure 8.

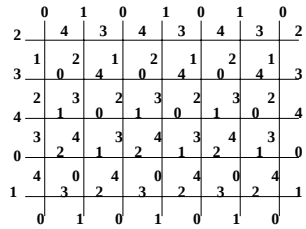


Figure 7: An AVD-coloring of $TM_{5,7}$.

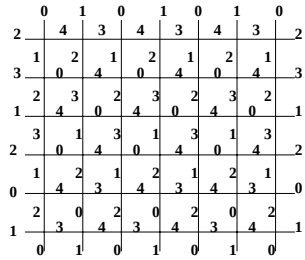


Figure 8: An AVD-coloring of $TM_{6,7}$.

- **Case 3:** $m \equiv 2 \bmod 5$. For $m = 7$, an AVD-coloring is illustrated in Figure 9. For $m > 7$, we join one copy $TM_{m-7,n}$ with the coloring given by Case 1 with one copy of $TM_{7,n}$ colored according to Figure 9.

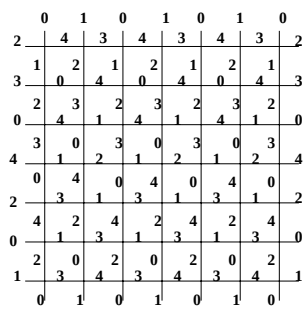


Figure 9: An AVD-coloring of $TM_{7,7}$.

- **Case 4:** $m \equiv 3 \bmod 5$. The cases $m = n = 3$ and $m = 3$, $n = 9$ are solved in the beginning of the present proof. The case $m = 3$, $n = 5$ can be solved with Case 1 if we take $m = 5$ and $n = 3$. The case $m = 3$, $n = 7$ is similar to Case 3 after permuting m and n. Similarly, when $m = 3$, $n = 11$ by permuting m and n, we solve the case as it is given in Case 2.

  So, for $m \geq 13$, we join one copy of $TM_{m-13,n}$ (Case 1) with one copy of $TM_{6,n}$ (Case 2) and one copy of $TM_{7,n}$ (Case 3).

- **Case 5:** $m \equiv 4 \bmod 5$. The beginning of the present proof gives the case

$m = n = 9$ and the other cases when $m = 9$ are solved in the preceding cases. So, for $m \geq 14$, we join one copy of $TM_{m-14,n}$ (Case 1) with one copy of $TM_{14,n}$ (Case 3).

□

3 AVD-colorings of n-dimensional meshes and hypercubes

To find the minimum number of colors for an AVD-coloring of multidimensional meshes and hypercubes, we shall prove more general results concerning AVD-colorings of the Cartesian product of a graph by a path or a cycle.

Theorem 1 *Let $d \geq 2$ be an integer and let G be a graph of maximum degree $\Delta \leq d-1$. If there exists an AVD-coloring of G with d colors, then $\chi'_a(G \times P_2) \leq d+1$.*

Proof : Let $d = \chi'_a(G)$. The graph $G' = G \times P_2$ consists of two copies G_0 and G_1 of G and a perfect matching between them. Take an AVD-coloring c of G with colors $0, 1, \ldots, d-1$. For an edge e (a vertex u, respectively) of G, let e_i (u_i respectively) be the corresponding edge (vertex respectively) of G_i for $i = 0, 1$. A coloring c' of G' is defined as follows:

for each $e \in E(G)$,

$$c'(e_0) = c(e) \text{ and } c'(e_1) = (c(e) + 1) \bmod d,$$

and for each $u \in V(G)$,

$$c'(u_0 u_1) = d.$$

It is routine to see that c' is an AVD-coloring. □

Theorem 2 *Let $d \geq 2$ be an integer and let G be a graph of maximum degree $\Delta \leq d-1$ and of minimum degree $\delta \geq 2$. If there exists an AVD-coloring of G with d colors, then $\chi'_a(G \times C_k) \leq d+2$.*

Proof : Let c be an AVD-coloring of G with the d colors $0, 1, \ldots, d-1$. Let $G' = G \times C_k$. The graph G' consists of k copies of G, say $G_0, G_1, \ldots, G_{k-1}$; with for each $i, 0 \leq i \leq k-1$, a perfect matching linking each vertex of copy G_i with the corresponding vertex of G_{i+1} (addition modulo k).] For an edge e (a vertex u, respectively) of G, let e_i (u_i respectively) be the corresponding edge (vertex respectively) in G_i.

Case 1: k is even. The coloring c' of G' is described below:

for each $e \in E(G)$, for each $i, 0 \leq i \leq \frac{k-2}{2}$, set

$$\begin{cases} c'(e_{2i}) = c(e), \\ c'(e_{2i+1}) = (c(e) + 1) \bmod d. \end{cases}$$

For the edges of the perfect matchings, we add two new colors d and $d+1$: for each $u \in V(G)$, for each $j, 0 \leq j \leq \frac{k-2}{2}$, set

$$\begin{cases} c'(u_{2j}u_{2j+1}) = d+1, \\ c'(u_{2j+1}u_{2j+2}) = d. \end{cases}$$

An illustration is given in Figure 10.

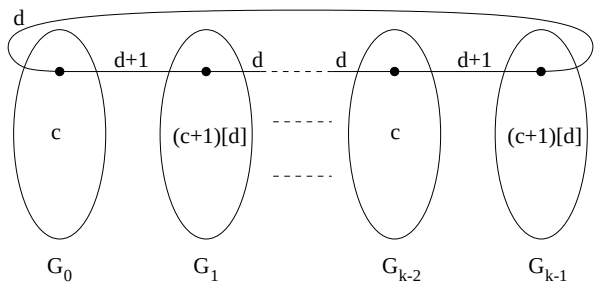


Figure 10: The coloring c' of G' when k is even.

Now, let us prove that c' is an AVD-coloring: First notice that as c is an AVD-coloring of G, c' is an AVD-coloring of each copy G_i. Next, let $j \in \{0, 1, \ldots d-1\}$ be a color such that $j \notin S(u_{2i})$ and $(j-1) \bmod d \in S(u_{2i})$ (there always exists one). Then, by the definition of the coloring c', we have $j \in S(u_{2i+1})$, which proves that these two vertices have distinct sets of colors.

Case 2: k is odd. The coloring c' of G' is constructed first from the coloring given in Case 1 for the copies G_0, G_1, $\ldots$, G_{k-3}, i.e.:

for each $e \in E(G)$, set

$$\begin{cases} c'(e_{2i}) = c(e), & 0 \leq i \leq \frac{k-3}{2} \\ c'(e_{2i+1}) = (c(e)+1) \bmod d, & 0 \leq i \leq \frac{k-5}{2}, \end{cases}$$

and for each $u \in V(G)$, set

$$\begin{cases} c'(u_{2j}u_{2j+1}) = d+1, & 0 \leq j \leq \frac{k-3}{2} \\ c'(u_{2j+1}u_{2j+2}) = d, & 0 \leq j \leq \frac{k-5}{2}. \end{cases}$$

Then for the edges of G_{k-2}, set

$$\begin{cases} c'(e_{k-2}) = c(e) + 1 \text{ (without modulo) for } d \text{ even} \\ c'(e_{k-2}) = \tau(c(e)) \quad \text{for } d \text{ odd} \end{cases}$$

where τ is a bijection from $\{0, 1, \ldots d-1\}$ onto $\{1, 2, \ldots d-1, d\}$ defined by:

$$\tau(i) = \begin{cases} i-1, & \text{for } 4 \leq i \leq d-3 \text{ or } i = 2 \\ d-2, & \text{for } i = 0 \\ d-1, & \text{for } i = 1 \\ d-3, & \text{for } i = 3 \\ 2, & \text{for } i = d-2 \\ d, & \text{for } i = d-1 \end{cases} \qquad \text{for } d \geq 5,$$

and $\tau(0) = 2$, $\tau(1) = 3$, $\tau(2) = 1$ for $d = 3$.

For G_{k-1}, set

$$c'(e_{k-1}) = \sigma(c(e)),$$

where σ is a bijection from $\{0, 1, \ldots d-1\}$ onto $\{1, 2, \ldots d-1, d+1\}$ defined by:

$$\sigma(i) = \begin{cases} i+2, & \text{for } 0 \leq i \leq d-3 \\ 1, & \text{for } i = d-2 \\ d+1, & \text{for } i = d-1 \end{cases} \qquad \text{for } d \text{ even and,}$$

$$\sigma(i) = \begin{cases} i+1, & \text{for } 1 \leq i \leq d-5 \text{ or } i = d-3 \\ d-3, & \text{for } i = 0 \\ d-1, & \text{for } i = d-4 \\ d+1, & \text{for } i = d-1 \\ 1, & \text{for } i = d-2 \end{cases} \qquad \text{for } d \text{ odd, } d \neq 3,$$

and $\sigma(0) = 1$, $\sigma(1) = 2$, $\sigma(2) = 4$ for $d = 3$.

For each edge between copies G_{k-2}, G_{k-1} and G_0, set

$$\begin{cases} c'(u_{k-2}u_{k-1}) = 0, \\ c'(u_{k-1}u_0) = d. \end{cases}$$

An illustration is given in Figure 11 when d is even.

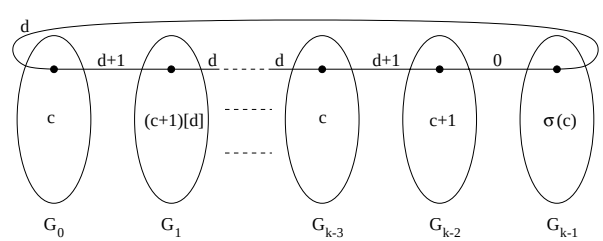


Figure 11: The coloring c' of G' when k is odd and d is even.

Now, let us prove that c' is an AVD-coloring when d is even. First, as for the previous case, it is not difficult to see that since c is an AVD-coloring of G, then c' is an AVD-coloring of each copy G_i (in particular for the copy G_{k-1}, as σ is a bijection). Next, notice that a vertex u_i of copy G_i is distinguished from a vertex u_{i+1} of copy G_{i+1}, for $0 \leq i \leq k-4$ since the coloring for this part of the graph G' is similar to the one defined for even k. Thus we only have to check that two adjacent vertices from copies G_{k-3}, G_{k-2}, G_{k-1} and G_0 receive distinct sets of colors: let u be a vertex of G and let $S = S_c(u)$ denote its set of colors with the coloring c.

Let $j \in \{0, 1, \ldots d-1\}$ be a color such that $j \notin S(u_{k-3})$ and $j+1 \bmod d \in S(u_{k-3})$. Then, by the definition of the coloring c' we have, $S(u_{k-2}) \neq S(u_{k-3})$ since $j + 1 \bmod d \in S(u_{k-3})$ but $j+1 \notin S(u_{k-2})$.

By contradiction, we are going to show that a vertex u_{k-2} is distinguished from its neighbor u_{k-1}. Let $S' = \{s+1 \mid s \in S\}$ and $S_\sigma = \{\sigma(s) \mid s \in S\}$. Assume that $S(u_{k-2}) = S(u_{k-1})$, i.e. $S' \cup \{0, d+1\} = S_\sigma \cup \{0, d\}$. Then we have $d+1 \in S_\sigma$ and thus $d-1 \in S$ i.e. $d \in S'$. Now, since $\delta(G) \geq 2$ by hypothesis, there exists $j, 0 \leq j \leq d-2$ such that $j \in S$. If $j = d-2$ then $1 \in S_\sigma$ and

$$1 \in S_\sigma \Rightarrow 1 \in S' \Rightarrow 0 \in S \Rightarrow \sigma(0) = 2 \in S_\sigma \Rightarrow 2 \in S' \Rightarrow \ldots \Rightarrow d-1 \in S.$$

We obtain $S = \{0, 1, \ldots, d-1\}$, which contradicts the hypothesis that $\Delta(G) \leq d-1$. If $j < d-2$, the argument is similar.

To show that a vertex u_{k-1} is distinguished from its neighbor u_0, we proceed in a similar way. Assume that $S(u_{k-1}) = S(u_0)$, i.e. that $S_\sigma \cup \{0, d\} = S \cup \{d, d+1\}$. This implies $0 \in S$ and when d is even $0 \in S \Rightarrow \sigma(0) = 2 \in S \Rightarrow \sigma(2) = 4 \in S \Rightarrow \ldots \Rightarrow \sigma(d-4) = d-2 \in S \Rightarrow \sigma(d-2) = 1 \in S \Rightarrow \sigma(1) = 3 \in S \Rightarrow \ldots \Rightarrow \sigma(d-3) = d-1 \in S$.

Thus $S = \{0, 1, \ldots, d-1\}$, a contradiction.

If d is odd the argument is similar.

In both cases, we have that c' is an AVD-coloring of G' with $d+2$ colors. □

Theorem 3 *Let $d \geq 2$ be an integer and let G be a graph of maximum degree $\Delta \leq d-1$. If there exists an AVD-coloring of G with d colors, then $\chi'_a(G \times P_k) \leq d+2$.*

Proof : It is easy to see that the coloring c' given in the proof of Theorem 2 for even k also works for $G \times P_{k'}$ ($k' \in \{k, k-1\}$) since vertices are distinguished in each copy G_i, and $G \times P_{k'}$ is obtained from $G \times C_k$ by deleting the edges (which have the same color) of the perfect matching between copies G_0 and G_{k-1}; and if k' is odd we also delete the copy G_{k-1} with all incident edges. □

Corollary 1 *For the hypercube H_d of dimension $d \geq 3$, we have*

$$\chi'_a(H_d) = d+1.$$

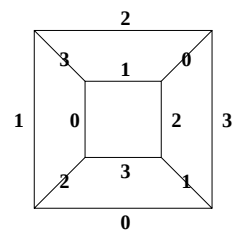


Figure 12: An AVD-coloring of H_3.

Proof : Figure 12 gives an AVD-coloring of H_3 with 4 colors. Since $H_d = H_{d-1} \times P_2$, from Theorem 1 we obtain the result for $d \geq 4$. □

If we start with the AVD-coloring of the 2-dimensional toroidal mesh given by Proposition 1 or Proposition 2 and if we apply inductively Theorem 2, we obtain the following result for the toroidal mesh.

Corollary 2 *For the k-dimensional toroidal mesh $TM_{m_1,m_2,\ldots,m_k}$, with $k \geq 2$ and $m_i \geq 3$, we have*

$$\chi'_a(TM_{m_1,m_2,\ldots,m_k}) = 2k + 1.$$

Remark 1 *Observe that $\chi'_a(C_{3m}) = 3$. Therefore, in this case, one can start the induction from only the cycle (instead of the 2-dimensional toroidal mesh).*

Theorem 3 together with Lemma 1 give the following result for the mesh.

Corollary 3 *For the k-dimensional mesh $M_{m_1,m_2,\ldots,m_k}$, with $k \geq 2$ and $m_i > 3$, we have*

$$\chi'_a(M_{m_1,m_2,\ldots,m_k}) = 2k + 1.$$

Finally one can give a more general result from the above theorems.

Theorem 4 *Let $G = G_1 \times G_2 \times \ldots \times G_k$ be a general mesh with $k \geq 2$, where each G_i is either a path or a cycle. Then*

(i) $\chi'_a(G) = \Delta(G)$ if and only if $G_i = P_3$ for $i = 1, 2, \ldots, k$;

(ii) $\chi'_a(G) = \Delta(G) + 2$ if and only if $k = 2$ and $(G_1, G_2) \in \{(C_5, P_2), (P_2, P_2)\}$;

(iii) $\chi'_a(G) = \Delta(G) + 1$ in all remaining cases.

Proof : We prove this theorem by induction on k using Theorems 1, 2 and 3. The starting points of this induction are Lemma 1, Propositions 1 and 2 and the following facts.

(a) $\chi'_a(M_{3,3}) = \chi'(M_{3,3}) = 4$ since in $M_{3,3}$ only vertices of distincts degrees are adjacent.

(b) $\chi'_a(C_5 \times P_2) = 5$: The inequality $\chi'_a(C_5 \times P_2) \leq 5$ follows from the colorings illustrated in Figure 13.

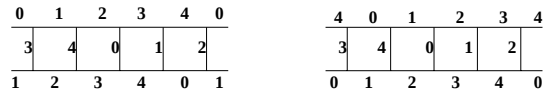


Figure 13: Two AVD-colorings of $C_5 \times P_2$.

Assume that $\chi'_a(C_5 \times P_2) \leq 4$, i.e. $\chi'_a(C_5 \times P_2) = 4$ (the graph $C_5 \times P_2$ is cubic). Since $\lfloor |E(C_5 \times P_2)|/4 \rfloor = \lfloor 15/4 \rfloor = 3$, there is a color set C of cardinality at most 3. Observe that the vertex independence number of $C_5 \times P_2$ is 4 and that a 4-vertex independent set is unique (up to an automorphism of $C_5 \times P_2$). If $|C| \leq 2$, the vertex set $W := V(C_5 \times P_2) \backslash V(C)$ is of cardinality at least 6, and so there is an edge $w_1 w_2$ with $w_1, w_2 \in W$. Then, however, w_1 and w_2 are not distinguished by their color sets (the color of the edges of C is missing at both w_1 and w_2), a contradiction. On the other hand, if $|C| = 3$, then $|W| = 4$ and W must be an independent set (otherwise we have a contradiction again). The graph $(C_5 \times P_2) \backslash W$ consists (see the observation above) of two components P_1 and one component P_4; therefore, its vertex set cannot be covered by three independent edges of C, a contradiction.

(c) $\chi'_a(C_m \times P_2) = 4$ for any $m \neq 5$: This can be seen by induction on m. Indeed, for $m = 3, 4, 8$ an appropriate coloring of $C_m \times P_2$ is illustrated in Figure 14, sharing the "block" B of the first three "columns".

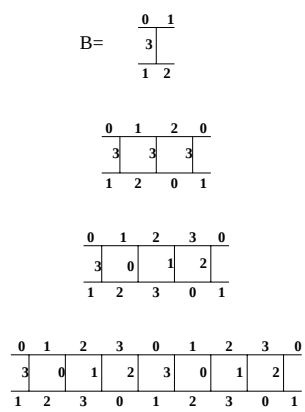


Figure 14: AVD-colorings for $C_m \times P_2$, $m = 3, 4, 8$.

A coloring of $C_{m+3} \times P_2$ is obtained from a coloring of $C_m \times P_2$ (containing B) when replacing B by the "block" illustrated in Figure 15 containing B as a xi""subblock" on both its left and right end.

(d) $\chi'_a(P_2 \times P_2) = \chi'_a(C_4) = 4$ (according to [5]).

(e) $\chi'_a(C_5 \times P_2 \times P_2) = 5$: Use for the two vertex-disjoint copies of $C_5 \times P_2$ present in $C_5 \times P_2 \times P_2$ as induced subgraphs the two colorings illustrated in Figure 13.

Corresponding pairs of vertices joined by a "vertical" edge of $C_5 \times P_2$ have color sets $\{\alpha, \alpha+1, \alpha+3\}$ (top) and $\{\alpha+1, \alpha+2, \alpha+3\}$ (bottom) in the copy colored by

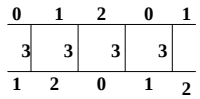


Figure 15: An AVD-coloring for $C_4 \times P_2$.

the coloring on the left of Figure 13 with α increasing from 0 to 4 when going from the left to the right. Therefore, "horizontal" edges between corresponding vertices of the two copies of $C_5 \times P_2$ can be colored by $\alpha + 2$ (top) and $\alpha + 4$ (bottom). It is easy to check that an appropriate coloring of $C_5 \times P_2 \times P_2$ is obtained.

□

Acknowledgements

The authors thank the anonymous referee for various comments whose suggestions greatly improved the present paper and particularly for providing Theorem 4.

References

[1] P. N. Balister, E. Győri, J. Lehel and R. H. Schelp, Adjacent vertex distinguishing edge-colorings, Preprint.

[2] Cristina Bazgan, Amel Harkat-Benhamdine, Hao Li and Mariusz Woźniak, On the vertex-distinguishing proper edge-colorings of graphs, *J. Combin. Theory Ser. B* 75(2) (1999), 288–301.

[3] P. N. Balister, O. M. Riordan and R. H. Schelp, Vertex-distinguishing edge colorings of graphs, *J. Graph Theory* 42(2) (2003), 95–109.

[4] Ján Černý, Mirko Horňák and Roman Soták, Observability of a graph, *Math. Slovaca* 46(1) (1996), 21–31.

[5] Zhongfu Zhang, Linzhong Liu and Jianfang Wang, Adjacent strong edge coloring of graphs, *Appl. Math. Lett.* 15(5) (2002), 623–626.

(Received 5 Nov 2004)

Order
DOI 10.1007/s11083-006-9034-0

The Phagocyte Lattice of Dyck Words

J. L. Baril · J. M. Pallo

Received: 26 September 2005 / Accepted: 25 May 2006
© Springer Science + Business Media B.V. 2006

Abstract We introduce a new lattice structure on Dyck words. We exhibit efficient algorithms to compute meets and joins of Dyck words.

Key words lattice · Dyck words · noncrossing partitions

1. Introduction

A large number of various classes of combinatorial objects are enumerated by the well-known Catalan numbers. It is the case, among others, of ballot sequences, planar trees, Young tableaux, nonassociative products, stack sortable permutations, and so on. A list of over 60 types of such combinatorial classes of independent interest has been compiled by Stanley [26]. A certain number of explicit bijections between these Catalan classes can also be found in the literature.

From an order theoretic point of view, there are some partial ordering relations on Catalan sets which endow them with a lattice structure. Of much interest are the following lattices. First, the so-called Tamari lattice can be obtained equivalently in three different ways. The coverings correspond to elementary transformations in three Catalan classes, namely to reparenthesizations of letters products [6], to rotations on binary trees [17, 24], and to diagonal flips in triangulations [10, 24]. See references in [18]. Second, the lattice of noncrossing partitions is equipped with the refinement order [11]. Under this partial ordering $\leq$, two partitions π and π' satisfy $\pi \leq \pi'$ if every block of π is a subset of some block of π'. See [3, 4, 20, 22] and numerous references in the exhaustive survey [23]. More recently, a link between

J. L. Baril ( ) · J. M. Pallo
LE2I, UMR 5158, Université de Bourgogne, B.P. 47870, F21078 Dijon-Cedex, France
e-mail: barjl@u-bourgogne.fr

J. M. Pallo
e-mail: pallo@u-bourgogne.fr

Dyck paths, noncrossing partitions and 321-avoiding permutations allows to define a new distributive lattice structure on these Catalan sets [1, 5].

In this paper, we introduce a new lattice structure on Dyck words (also called well-formed parentheses strings). We define a 'phagocyte' mutation on Dyck words in the following way. When in a Dyck word a Dyck subword w occurs immediately at the left of a word of the form $((\cdots(())\cdots))$, w is inserted at the center of this word. Thus we obtain $((\cdots((w))\cdots))$. Thereby, when $w((\cdots(())\cdots))$ occurs in a Dyck word, then w is 'phagocyted' by $((\cdots(())\cdots))$, and so is changed to $((\cdots((w))\cdots))$. w has been 'gobbled' by a series of nested parentheses. This elementary mutation endows the Catalan set of Dyck words with a lattice structure. We show how to compute meets and joins of Dyck words efficiently. After Grätzer, which cites Tamari lattices "to dispel the impression that it is always easy to prove that a poset is a lattice" [9, p. 18], we should like to say that the currently studied lattice falls into the same category.

2. Dyck Words and Noncrossing Partitions

The set D of Dyck words over $\{(,)\}$ is the language generated by the grammar $S \to \lambda|(S)|SS$, i.e. the set of legal strings of matched parentheses. Let us denote D_n the set of Dyck words with n open and n close parentheses. The cardinality of D_n is the *n-th* Catalan number: $card(D_n) = (2n)!/(n!(n+1)!)$. The open (resp. close) parentheses of $w \in D_n$ are numbered from 1 to n by traversing w from left to right.

A partition $b_1/b_2/\cdots/b_k$ of $[1,n] = \{1,2,\cdots,n\}$ into blocks b_i is called noncrossing if there do not exist four numbers $p < q < r < s$ such that $p, r \in b_i$ and $q, s \in b_j$ with $i \neq j$. Thus 146/23/5 is a noncrossing partition of $[1,6]$ (ncp in short) while 135/2/46 is crossing. Let us denote NC_n the set of all ncp of $[1,n]$. We have $card(NC_n) = card(D_n)$. A ncp will be written by listing the elements in each block in increasing order, and the blocks in increasing order of their minima. There are various ways to represent a ncp. For example, in the linear representation, $[1,n]$ appears as usual on the real line and successive elements in the same block are joined by an arc in the first quadrant. In this paper, we will use what we call the canonical representation. Given a ncp $\pi \in NC_n$, let us define the relation R_π on $[1,n]$ by $i\,R_\pi\,j$ iff i is the smallest element of the block of π which contains j ($i \leq j$). The canonical representation of π is obtained by drawing the directed graph of R_π on the first quadrant. In these two representations, the noncrossing property of the partition corresponds to the fact that arcs do not intersect.

Since $card(D_n) = card(NC_n)$, let us exhibit an explicit bijection between D_n and NC_n [7, 8, 22, 23]. Given a ncp $\pi \in NC_n$ and its canonical representation, for each visited element $i \in [1,n]$ (for i going from 1 to n), we write r open parentheses ($r \geq 0$) if there are r arcs starting in i and next we write one close parenthesis since there is always exactly one arc ending in i. Figure 1 illustrates this bijection.

$$w = ((()(()))(()())())$$

$$\pi_w = 149/23/57/6/8$$

$$c_w = (1,2,2,1,5,6,5,8,1)$$

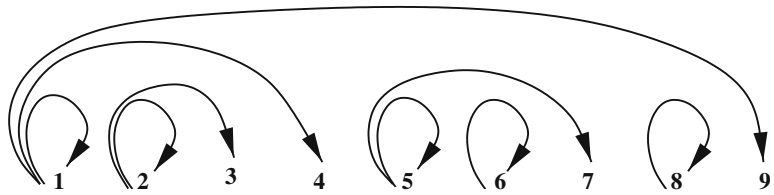


Figure 1 The canonical representation of a Dyck word w of D_9.

We denote by π_w the ncp corresponding to the Dyck word w. Let us remark that the number $nbl(\pi_w)$ of blocks of π_w is equal to the number of occurrences of '()' in w. Given $w \in D_n$, we define the vector c_w where $c_w[i]$ is the smallest element of the block of π_w which contains i ($1 \leq i \leq n$).

In the sequel, the strict (respectively, large) inclusion relation will be denoted by the symbol $\subset$ (respectively, $\subseteq$).

3. The Phagocyte Transformation

DEFINITION 1. *The phagocyte transformation* $\longrightarrow$ *on* D_n *is defined by* $w \longrightarrow w'$ *if we have* $w = u'v(^n)^n u''$ *and* $w' = u'(^n v)^n u''$, *where* v *is a non-empty Dyck word,* $n \geq 1$ *and* $u', u'' \in \{(,)\}^*$. *Let* $\overset{*}{\longrightarrow}$ *denote the reflexive transitive closure of* $\longrightarrow$.

LEMMA 1. *For all* $w, w' \in D_n$, *we have* $w \longrightarrow w'$ *iff* π_w *and* $\pi_{w'}$ *hold:*

- *one among the blocks of* $\pi_{w'}$ *is the union of two blocks* b_1 and b_2 of π_w where $\min(b_1) < \min(b_2)$ and $b_2 = [\min(b_2), max(b_2)]$ *is an interval,*
- *every block of* π_w *different from* b_1 *and* b_2 *is also a block of* $\pi_{w'}$.

Proof. This result is a simple application of Definiton 1. Remark that in Figure 2, the covering $1/24/3 \rightarrow 124/3$ is in NC_4 but is not in D_4 since 24 is not an interval. □

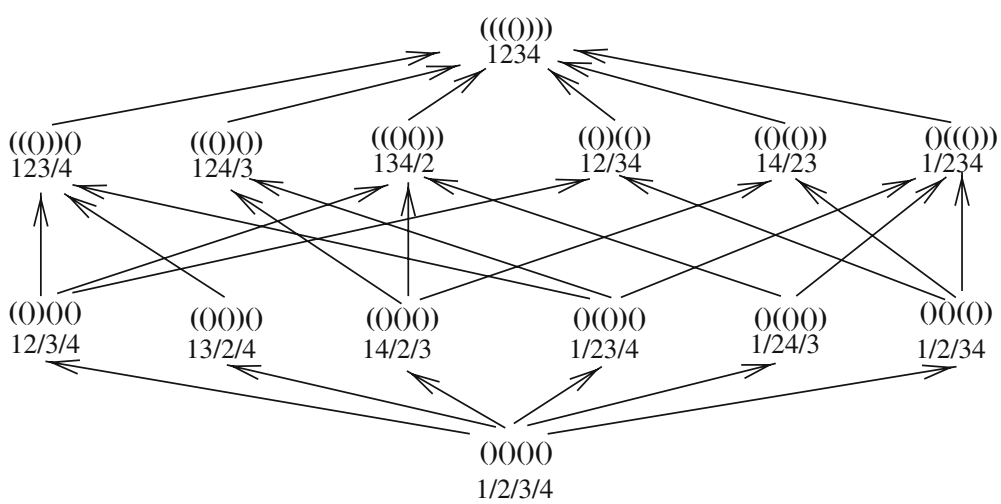


Figure 2 The phagocyte lattice D_4.

PROPOSITION 1. *$(D_n, \overset{*}{\longrightarrow})$ is a poset with zero $\mathbf{0} = ()()\ldots()$ and unit $\mathbf{1} = ((\ldots()\ldots))$, which is graded by the rank function $r(w) = n - nbl(\pi_w)$.*

Proof. The integer valued function r defined by $r(w) = n - nbl(\pi_w)$ verifies obviously $r(w') = r(w) + 1$ if $w \longrightarrow w'$. □

See the diagram of D_4 in Figure 2. In order to prove that the poset $(D_n, \overset{*}{\longrightarrow})$ is a lattice, we first give a Lemma on the structure of noncrossing partitions, and then we characterize the reflexive transitive closure of the phagocyte transformation.

LEMMA 2. *Let $\pi \in NC_n$. If b is a block of π which is not an interval, then there exists a block b' of π verifying the following two conditions:*

- $b' \subset [\min(b), \max(b)]$
- *b' is an interval possibly reduced to a singleton.*

Proof. Given a block b of π, let us consider in $[\min(b), \max(b)]$ the leftmost largest interval $I = [x, y]$ such that $I \cap b = \emptyset$. If I is a block of π, the result holds. If I is not a block of π, then let us consider the block b_1 of π such that $x = \min(b_1)$. Remark that b_1 exists by the noncrossing property. We have $b_1 \subset I$ and we repeat this process with the new block b_1 of π. So we construct a finite sequence of non-empty blocks b_i $(1 \leq i \leq r)$ such that $b_{i+1} \subset [\min(b_i), \max(b_i)]$ for each i. The last block b_r of the sequence is an interval possibly reduced to a singleton. Moreover b_r verifies the condition: $b_r \subset [\min(b), \max(b)]$. □

THEOREM 1. *For all $w, w' \in D_n$, we have $w \overset{*}{\longrightarrow} w'$ iff the following two conditions $(C1)$ and $(C2)$ hold:*

- *$(C1)$ every block of π_w is a subset of some block of $\pi_{w'}$,*
- *$(C2)$ for all blocks $b_1, b_2 \in \pi_w$, $b' \in \pi_{w'}$ such that $b_1 \cup b_2 \subset b'$ and $\min(b_1) < \min(b_2)$ then $[\min(b_2), \max(b_2)] \subset b'$.*

Remark 1. Condition (C1) is the well-known refinement order on partitions [11, 23]. Condition (C2) means that any block $b' \in \pi_{w'}$ is made up of one block $b_i \in \pi_w$ and possibly disjoint intervals $[k, l]$ for which one block $b_j \in \pi_w$ $(j > i)$ exists such that $k = \min(b_j)$ and $l = \max(b_j)$.

Proof. First let us suppose that $w \overset{*}{\longrightarrow} w'$ and let us apply the phagocyte transformation on the Dyck subword $v((\ldots()\ldots))$ of $w \in D_n$ in order to obtain the Dyck word w_1. Let $i \geq 0$ be the number of close parentheses of w before the first open parenthesis of v and let $j > i$ be the number of close parentheses of w before the first open parenthesis of the subword $((\ldots()\ldots))$ defined above. So there are in π_w a block b containing $i + 1$ and a block $[j + 1, j + k]$ which is an interval with $k \geq 1$. The result is a Dyck word w_1 which contains the Dyck subword $((\ldots(v)\ldots))$. Thus

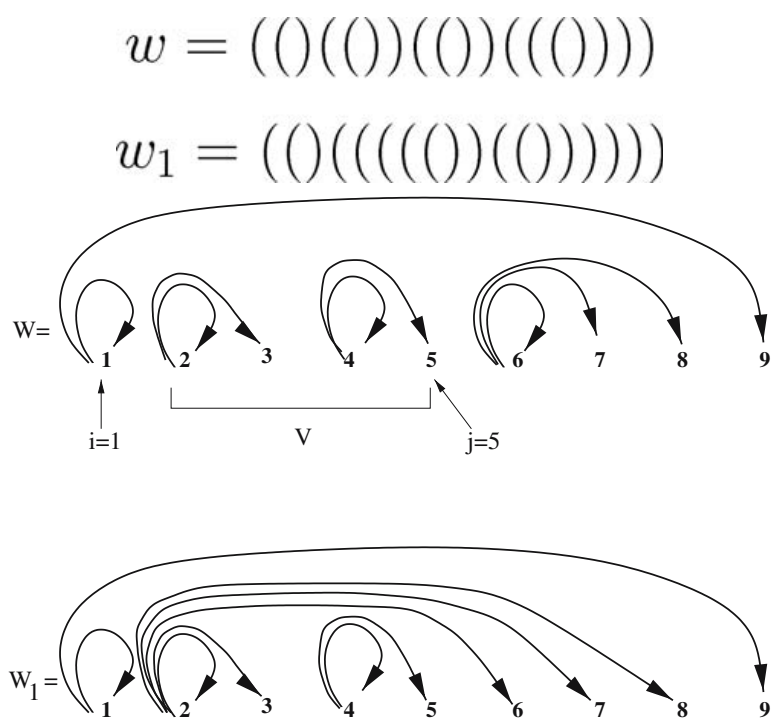


Figure 3 A phagocyte mutation $w \longrightarrow w_1$; $\pi_w = 19/23/45/678$ and $\pi_{w_1} = 19/23678/45$.

π_{w_1} has a block which contains b and $[j+1, j+k]$. See Figure 3. So conditions $(C1)$ and $(C2)$ are satisfied for w and w_1 such that $w \longrightarrow w_1$. By applying the phagocyte mutation several times, conditions $(C1)$ and $(C2)$ are also satisfied for w and w' such that $w \xrightarrow{*} w'$. □

Conversely, suppose that w and w' verify the two conditions $(C1)$ and $(C2)$, and denote it by $w \prec w'$. Let us consider $w, w' \in D_n$ such that $w \neq w'$ and $w \prec w'$. Consider also $\pi_w = b_1/b_2/\ldots$ and $\pi_{w'} = b'_1/b'_2/\ldots$ with $\min(b_i) < \min(b_{i+1})$ and $\min(b'_i) < \min(b'_{i+1})$ for all $i \geq 1$. Let us denote by m the smallest integer such that $b_m \neq b'_m$. Since $b_1 = b'_1, \ldots, b_{m-1} = b'_{m-1}$, we have $\min(b_m) = \min(b'_m) = i_0$. By condition $(C1)$, we have $b_m \subset b'_m$ since $i_0 \in b_m \cap b'_m$. Now let j_0 be the smallest integer such that $j_0 \notin b_m$ and $j_0 \in b'_m$. Let b_p $(p \geq m+1)$ be the block of π_w which contains j_0. Since $j_0 \in b_p \cap b'_m$, we have $b_p \subseteq b'_m$ and by the condition $(C2)$ we have $[j_0 = \min(b_p), \max(b_p)] \subseteq b'_m$. Then there are two cases to consider:

- (a): b_p is an interval.
  Let denote $j_0 = \min(b_p)$. Let v be the largest Dyck subword of w, the last close parenthesis of which has the number $j_0 - 1$ and the first open parenthesis of which has the number $y \geq i_0$. So there exists in w a Dyck subword $v(\ldots()\ldots)$ such that the first open (resp. last close) parenthesis of $(\ldots()\ldots)$ has the number $j_0 = \min(b_p)$ (resp. $\max(b_p)$). If we apply on w the phagocyte transformation $v(\ldots()\ldots) \longrightarrow (\ldots(v)\ldots)$, we obtain a Dyck word w_1. We have $w_1 \prec w'$. Indeed π_w and π_{w_1} have the same blocks except one block which is the union of b_m and b_p.

– (b): b_p is not an interval, i.e. $b_p \subset \left[\min(b_p), \max(b_p)\right]$. Let denote $j_0 = \min(b_p)$. According to Lemma 2, there exists a block b_l ($l \geq p$) of π_w which is an interval $b_l = [r, s]$ included in $\left[\min(b_p), \max(b_p)\right]$. Observe that $r = s$ can occur. We choose the rightmost block b_l. Let b be the block of π_w which contains $r - 1$ and $x = min(b)$. Let v be the largest Dyck subword of w the last close parenthesis of which has the number $r - 1$ and the first open parenthesis of which has the number $y \geq x$. Then, we apply on w the phagocyte mutation $v(\ldots()\ldots) \longrightarrow (\ldots(v)\ldots)$ where the first open (resp. last close) parentheses of the nested parenthesis $((\ldots()\ldots))$ has number r (resp. s). We obtain the Dyck word w_1 which verifies $w \longrightarrow w_1$ and $w_1 \prec w'$ as above.

By repeating this process, we find a finite sequence of Dyck words w_k such that $w_k \prec w'$ for all k and $w \longrightarrow w_1 \longrightarrow w_2 \ldots \longrightarrow w_l = w'$. By transitivity $w \xrightarrow{*} w'$ follows.

Remark 2. This proof is constructive since we exhibit a minimal length path between w and w' if $w \xrightarrow{*} w'$.

THEOREM 2. *For all n, the poset* $(D_n, \xrightarrow{*})$ *is a lattice.*

Proof. It suffices to show that any two elements of D_n have a greatest lower bound. The existence of least upper bounds then follows automatically since D_n is finite. Given $w, w' \in D_n$ with $w \neq w'$, let us consider $b \in D_n$ such that $b \xrightarrow{*} w$, $b \xrightarrow{*} w'$ and b maximal, i.e. there does not exist $b' \in D_n$ ($b' \neq b$) such that $b \xrightarrow{*} b'$, $b' \xrightarrow{*} w$, $b' \xrightarrow{*} w'$. Assume that there exists $a \in D_n$ maximal such that $a \xrightarrow{*} w$, $a \xrightarrow{*} w'$ and $a \neq b$. We denote:

$$\pi_a = A_1/A_2/\ldots,$$

$$\pi_b = B_1/B_2/\ldots,$$

$$\pi_w = W_1/W_2/\ldots,$$

$$\pi_{w'} = W'_1/W'_2/\ldots.$$

Let m be the smallest integer such that $A_m \neq B_m$. Obviously there exists x such that $x \in A_m$ and $x \notin B_m$ or $x \notin A_m$ and $x \in B_m$. Let us define W_k (respectively, W'_l) as the block of π_w (respectively, $\pi_{w'}$) which contains x. We have $A_m \subseteq W_k$ and $A_m \subseteq W'_l$. Moreover, $B_m \subset W_k$ and $B_m \subset W'_l$ since $A_m \cap B_m \neq \emptyset$ and $\min(A_m) = \min(B_m)$. By Remark 1 we have:

$$W_k = A_{i_0} \cup \bigcup_{i \in I} [r_i, \sim s_i] \text{ and}$$

$$W_k = B_{i_0} \cup \bigcup_{j \in J} [u_j, \sim v_j]$$

where $i_0 \leq m$, and for each $i \in I$ (resp. $j \in J$) r_i and s_i (resp. u_j and v_j) are the minimum and the maximum of the same block of π_a (resp. π_b). We have as well: $W'_l = A_{i'_0} \cup \bigcup_{i' \in I'} [r'_{i'}, \sim s'_{i'}]$ and $W'_l = B_{i'_0} \cup \bigcup_{j' \in J'} [u'_{j'}, \sim v'_{j'}]$.

Since $x \notin B_m$ and $A_{i_0} = B_{i_0}$ for all $i_0 < m$, x belongs to an interval $[u_{j_0}, v_{j_0}]$, $j_0 \in J$, where u_{j_0} and v_{j_0} are in a same block B_x of π_b ($x \in B_x$). Similarly x belongs to an interval $[u'_{j'_0}, v'_{j'_0}]$ with $j'_0 \in J'$.

Since b verifies the noncrossing property, we have necessarily

$$\left[u_{j_0}, v_{j_0}\right] \subseteq \left[u'_{j'_0}, v'_{j'_0}\right] \text{or}$$

$$\left[u'_{j'_0}, v'_{j'_0}\right] \subseteq \left[u_{j_0}, v_{j_0}\right]$$

Assume that the first case holds.

If $[u'_{j'_0}, v'_{j'_0}] \neq [u_{j_0}, v_{j_0}]$ then $[u'_{j'_0}, v'_{j'_0}] \setminus [u_{j_0}, v_{j_0}]$ has some elements below and above the interval $[u_{j_0}, v_{j_0}]$. Let us remark that the block containing $u'_{j'_0}$ surrounds the block B_x.

In order to transform b into w, we must merge the block B_x with the block B_{i_0}. So the only possibility is either that the block which contains $u'_{j'_0}$ is in B_{i_0} or that we also must merge B_x with B_{i_0}. This proves that the interval $[u'_{j'_0}, v'_{j'_0}]$ is included into W_k.

Hence, each block included in $[u'_{j'_0}, v'_{j'_0}]$ is also included into W_k and W'_l, which is trivially verified if $[u'_{j'_0}, v'_{j'_0}] = [u_{j_0}, v_{j_0}]$. We can built $b' \in D_n$ such that $\pi_{b'}$ and π_b have the same blocks except the block containing $u'_{j'_0}$ which is replaced by its union with each block of π_b included in $[u_{j_0}, v_{j_0}]$. So $b' \in D_n$ verifies $b \neq b'$ and $b \xrightarrow{*} b'$. By construction $b' \xrightarrow{*} w$ and $b' \xrightarrow{*} w'$, which contradicts the maximality of b. □

In the following propositions, we enumerate the join- and meet-irreducible elements of D_n. Recall that $x \in D_n$ is a join (resp. meet)-irreducible element if $x = a \vee b$ (resp. $x = a \wedge b$) implies $x = a$ or $x = b$. In other words, join (resp. meet)-irreducible elements are the elements that have an unique lower (resp. upper) cover. The proofs are easily obtained.

PROPOSITION 2. *The number of join-irreducible elements in the phagocyte lattice D_n is $n(n-1)/2$.*

Proof. It is enough to count all ncps consisting of one block with two elements and all other blocks having one element. □

PROPOSITION 3. *The number of meet-irreducible elements in the phagocyte lattice D_n is $2^{n-1} - 1$.*

Proof. Let denote $X = \{(^p)^p, p \geq 1\}$ and $Y = \{(^q, q \geq 1\}$. Then elements we are interested in are exactly alternative sequences of element in X and Y followed by two elements in X and some), i.e. the sequences of the form $x_1 y_2 x_3 \ldots y_{k-2} x_{k-1} x_k)^r$ where $x_i \in X$, $y_i \in Y$ and $r \geq 0$. Thus the number of these sequences equals to the number of non-trivial compositions of n i.e. $2^{n-1} - 1$. □

4. Computing Meets and Joins

Using Theorem 1, we exhibit algorithms for computing the meet $w \wedge w'$ and the join $w \vee w'$ of two Dyck words $w, w' \in D_n$.

```
Meet algorithm:
for i := 1 to n do c_{w∧w'}[i] := 0 enddo
for i := 1 to n do
    if c_{w∧w'}[i] = 0 then
        if c_w[i] < i or c_{w'}[i] < i then
            j_0 := i;
            for j := i to n do
                if c_{w∧w'}[j] = 0 and c_w[j] = c_w[i] and c_{w'}[j] = c_{w'}[i] then
                    c_{w∧w'}[j] := j_0;
                else j_0 := j; endif
            endo
        else
            for j := i to n do
                if c_w[j] = i and c_{w'}[j] = i then
                    c_{w∧w'}[j] := i; endif
            enddo
        endif
    endif
enddo
```

Proof and analysis of the meet algorithm. Suppose the current element $c_{w\wedge w'}[i]$ has been computed for all $i < i_0$. For computing $c_{w\wedge w'}[i_0]$ there are two cases to consider:

- (1): $c_w[i_0] = i_0$ and $c_{w'}[i_0] = i_0$.
  So there exist two blocks $b \in \pi_w$ and $b' \in \pi_{w'}$ such that $i_0 = \min(b) = \min(b')$. We put $c_{w\wedge w'}[i_0] = i_0$. Now for $j \geq i_0$, let $b_j \in \pi_w$ (respectively, $b'_j \in \pi_{w'}$) be the block of π_w (respectively, $\pi_{w'}$) which contains j. If we have $i_0 \in b_j$ and $i_0 \in b'_j$ then we put also $c_{w\wedge w'}[j] = i_0$. We repeat this process for all $j \geq i_0$.
- (2): $c_w[i_0] < i_0$ or $c_{w'}[i_0] < i_0$.
  Assume that $c_w[i_0] < i_0$. Thus there exists a block $b \in \pi_w$ such that $\min(b) = c_w[i_0] < i_0$ and $i_0 \in b$. Since $c_{w\wedge w'}[i_0] = 0$ is not determined by the algorithm, i_0 does not belong to the block b' of $\pi_{w\wedge w'}$ which contains $c_w[i_0] < i_0$. To construct a path between $w \wedge w'$ and w, we must merge the block b'' of $\pi_{w\wedge w'}$ which contains i_0 with the block b' since i_0 and $c_w[i_0]$ are in the same block $b \in \pi_w$. Using Theorem 1, we obtain $[\min(b''), \max(b'')] \subseteq b$. So b'' is the greatest interval $b'' = [i_0, k]$ with $k \geq i_0$ verifying $b'' \subseteq b$ and b'' is also included in a block of $\pi_{w'}$. We repeat this process for all integers greater than i_0.

Let σ be the noncrossing partition corresponding to the vector $c_{w\wedge w'}$ obtained by the above algorithm. Then σ verifies the followings properties:

First by construction, each block of σ is included into a block of π_w and a block of $\pi_{w'}$.

Second if there exist a block b_σ in σ and a block b_π in π_w (respectively, $\pi_{w'}$) such that $\min(b_\sigma) = \min(b_\pi)$, then $\pi_w \backslash b_\sigma$ is by construction an union of intervals I verifying (i) I is also a block of σ and $\min(I) \geq \min(\sigma)$, (ii) I is included into a block of π_w (respectively, $\pi_{w'}$). From Theorem 1, we obtain $\sigma \xrightarrow{*} \pi_w$ and $\sigma \xrightarrow{*} \pi_{w'}$.

Moreover, we determine at each step of algorithm the greatest block verifying the two previous properties. This means that σ is the greatest noncrossing partition such that $\sigma \xrightarrow{*} \pi_w$ and $\sigma \xrightarrow{*} \pi_{w'}$. Indeed in the case (1), we consider the greatest set containing i_0 which is included into a block of π_w and $\pi_{w'}$ and in the case (2), we consider the greatest interval containing i_0 which is also included into a block of π_w and $\pi_{w'}$. So, σ corresponds to the noncrossing partition of $w \wedge w'$.

The space complexity is $O(n)$. The time complexity is $O(n^2)$ because computing $nbl(\pi_w)$ requires $O(n)$ time and the meet algorithm requires two nested loops.

We can easily define a recursive algorithm to compute the join $\pi \sqcup \pi'$ of two ncp π and π' in the classical lattice of noncrossing partitions. So we can suppose that $c_{w \sqcup w'}$ has been computed for all $w, w' \in D_n$. Assume that c_w and $c_{w'}$ are determined. Let d_w (respectively, $d_{w'}$) be the vector defined by $d_w[i]$ (respectively, $d_{w'}[i]$) is the greatest element of the block of π_w (respectively, $\pi_{w'}$) which contains i.

```
Join algorithm.
for i := 1 to n do
    if c_{w∨w'}[i] = 0 then
        for j := i to n if
            if c_{w⊔w'}[j] = c_{w⊔w'}[i] then
                c_{w∨w'}[j] := c_{w⊔w'}[i];
                if c_w[j] ≠ c_w[i] then
                    for k := j + 1 to d_w[j] do
                        c_{w∨w'}[k] := c_{w⊔w'}[i]; enddo
                endif
                if c_{w'}[j] ≠ c_{w'}[i] then
                    for k := j + 1 to d_{w'}[j] do
                        c_{w∨w'}[k] := c_{w⊔w'}[i]; enddo
                endif
            endif
        enddo
    endif
enddo
```

Proof of the join algorithm. Suppose the current element $c_{w \vee w'}[i]$ has been computed for all $i < i_0$. So we put $c_{w \vee w'}[i_0] = i_0$ and for all $j > i_0$ such that j and i_0 are in the same block in $\pi_{w \sqcup w'}$, we put $c_{w \vee w'}[j] = c_{w \sqcup w'}[i_0]$. We discuss the two following cases (1) and (2):

- (1): If $c_w[j] \neq c_w[i_0]$: i_0 and j do not belong to the same block in π_w. To construct a path between w and $w \vee w'$, we must include the interval $\left[j+1, d_w[j]\right]$ in the block of $\pi_{w \vee w'}$ which contains i_0. So if $k \in \left[j+1, d_w[j]\right]$, we put $c_{w \vee w'}[k] = c_{w \sqcup w'}[i_0]$.
- (2): If $c_{w'}[j] \neq c_{w'}[i_0]$. The proof is similar to the proof of (1).

Let σ (respectively, τ) be the noncrossing partition obtained by $c_{w \vee w'}$ (respectively, $c_{w \sqcup w'}$). By construction, every block of σ is included into a block of τ, so every block of π_w and $\pi_{w'}$ is included into a block of σ.

Moreover, if there exist a block b_τ in τ and a block b_π in π_w (respectively, $\pi_{w'}$) such that $\min(b_\tau) = \min(b_\pi)$ then $b_\tau \backslash b_\pi$ is by construction an union of blocks b in π_w (respectively, $\pi_{w'}$) with $\min(b) \geq \min(b_\tau)$. In order to obtain σ from τ, it is necessary to merge the block b_π with the intervals $[\min(b), \max(b)]$ (by Theorem 1). So at the end of the algorithm, σ verifies the conditions (C_1) and (C_2) of Theorem 1. Thus we obtain $\pi_w \xrightarrow{*} \sigma$ and $\pi_{w'} \xrightarrow{*} \sigma$.

The minimality of σ holds since at each step of this algorithm we run only the necessary operations. Thus $\sigma = \pi_{w \vee w'}$.

For example, if $\pi_w = 149/23/57/6/8$ and $\pi_{w'} = 147/23/56/89$, the algorithms give $\pi_{w \wedge w'} = 14/23/5/6/7/8/9$ and $\pi_{w \vee w'} = 1456789/23$.

5. Conclusion

In this paper, a new lattice structure has been defined on the Catalan sets of Dyck words via a natural transformation. The simple and natural definition of the phagocyte transformation is unfortunately at odds with the rather complex theorem which characterizes this transformation.

The greedy transformation $v() \to (v)$ defined in [7] is a particular case of the phagocyte transformation. However the poset obtained by the greedy transformation is not an effective lattice. Nevertheless, this poset is a graded lower semi-modular meet-semilattice. This property allows to compute the corresponding shortest path metric [12].

Some problems remain to be solved.

Is there an algorithm to compute the Möbius function of the phagocyte lattice of Dyck words as in [16]?

Computer experiments show that the number of coverings $cov(n)$ of D_n is equal to $3(2n+2)!/((n+1)!(n+4)!)$ for small values of n ($n \leq 6$). We obtain the first terms of the sequence **A003517**=$[1, 6, 27, 110, \cdots]$ included in the Sloane Encyclopedia [25]. This sequence enumerates two combinatorial classes of objects: (a) The set PIS_n of permutations on $[n]$ with exactly one increasing subsequence of length three [13] and (b) the set NIP_n of pairs of non-intersecting paths of length n and distance three [21]. In order to prove the equality $cov(n) = card(PIS_{n+1}) = card(NIP_{n+1})$, can we exhibit two explicit bijections between the coverings of D_n and two above combinatorial classes? Is there a polynomial time algorithm to compute the minimal path length distance between Dyck words in the phagocyte lattice [12]? If so, a new shortest-path-type metric could be obtained, and could be added to the existing metrics on Catalan sets [2, 7, 14, 15, 17, 19, 24]. Let us recall that we still do not know if the rotation distance on binary trees can be computed in polynomial time.

Acknowledgments We would like to thank the anonymous referees and Jean-Paul Auffrand for their constructive remarks which have greatly improved this paper.

References

1. Barcucci, E., Bernini, A., Ferrari, L., Poneti, M.: A distributive lattice structure connecting Dyck paths, noncrossing partitions and 312-avoiding permutations. Order (2005), available online

2. Bonnin, A., Pallo, J.M.: A shortest path metric on unlabeled binary trees. Pattern Recogn. Lett. **13**, 411–415 (1992)
3. Edelman, P.H.: Chain enumeration and noncrossing partitions. Discrete Math. **31**, 171–180 (1980)
4. Edelman, P.H., Simion, R.: Chains on the lattice of noncrossing partitions. Discrete Math. **126**, 107–119 (1994)
5. Ferrari, L., Pinzani, R.: Lattices of lattice paths. J. Stat. Plan. Inference **135**, 77–92 (2005)
6. Friedman, H., Tamari, D.: Problèmes d'associativité: une structure de treillis fini induite par une loi demi-associative. J. Comb. Theory **2**, 215–242 (1967)
7. Germain, C., Pallo, J.M.: Two shortest path metrics on well-formed parentheses strings. Inf. Process. Lett. **60**, 283–287 (1996)
8. Germain, C., Pallo, J.M.: The number of coverings in four Catalan lattices. Int. J. Comput. Math. **61**, 19–28 (1996)
9. Grätzer, G.: General Lattice Theory. Birkäuser Verlag, Basel (1998)
10. Hanke, S., Ottmann, T., Schuierer, S.J.: The edge-flipping distance of triangulations. J. Univers. Comput. Sci. **2**, 570–579 (1996)
11. Kreweras, G.J.: Sur les partitions noncroisées d'un cycle. Discrete Math. **1**, 333–350 (1972)
12. Monjardet, B.: Metrics on partially ordered sets – A survey. Discrete Math. **35**, 173–184 (1981)
13. Noonan, J.: The number of permutations containing exactly one increasing subsequence of length three. Discrete Math. **152**, 307–313 (1996)
14. Pallo, J.M.: On the rotation distance in the lattice of binary trees. Inf. Process. Lett. **25**, 369–373 (1987)
15. Pallo, J.M.: A distance metric on binary trees using lattice-theoretic measures. Inf. Process. Lett. **34**, 113–116 (1990)
16. Pallo, J.M.: An algorithm to compute the Möbius function of the rotation lattice of binary trees. RAIRO Informatique Théorique et Applications **27**, 341–348 (1993)
17. Pallo, J.M.: An efficient upper bound of the rotation distance of binary trees. Inf. Process. Lett. **73**, 87–92 (2000)
18. Pallo, J.M.: Generating binary trees by Glivenko classes on Tamari lattices. Inf. Process. Lett. **85**, 235–238 (2003)
19. Pallo, J.M.: Right-arm rotation distance between binary trees. Inf. Process. Lett. **87**, 173–177 (2003)
20. Reiner, V.: Noncrossing partitions for classical reflection groups. Discrete Math. **177**, 195–222 (1997)
21. Shapiro, L.W.: A Catalan triangle. Discrete Math. **14**, 83–90 (1976)
22. Simion, R., Ullman, D.: On the structure of lattice of noncrossing partitions. Discrete Math. **98**, 193–206 (1991)
23. Simion, R.: Noncrossing partitions. Discrete Math. **217**, 367–409 (2000)
24. Sleator, D.D., Tarjan, R.E., Thurston, W.P.: Rotation distance, triangulations and hyperbolic geometry. J. Am. Math. Soc. **1**, 647–681 (1988)
25. Sloane, N.J.A.: The On-Line Encyclopedia of Integer Sequences, published electronically at http://www.research.att.com/~njas/sequences/
26. Stanley, R.P.: Enumerative Combinatorics, vol. 2. Cambridge University Press, Cambridge (1999)

Available online at www.sciencedirect.com
ScienceDirect
Information Processing Letters 100 (2006) 131–136
Information Processing Letters
www.elsevier.com/locate/ipl

Efficient lower and upper bounds of the diagonal-flip distance between triangulations

Jean-Luc Baril, Jean-Marcel Pallo *

LE2I, UMR 5158, Université de Bourgogne, BP 47870, F21078 Dijon cedex, France

Received 29 May 2006; received in revised form 7 July 2006; accepted 10 July 2006

Available online 2 August 2006

Communicated by L. Boasson

Abstract

There remains today an open problem whether the rotation distance between binary trees or equivalently the diagonal-flip distance between triangulations can be computed in polynomial time. We present an efficient algorithm for computing lower and upper bounds of this distance between a pair of triangulations.
© 2006 Elsevier B.V. All rights reserved.

Keywords: Triangulations; Diagonal-flip distance; Approximation algorithms

1. Introduction

Culik and Wood defined in 1982 the rotation distance between a pair of binary trees as the minimum number of rotations needed to convert one tree into the other [5]. There exists a well-known explicit bijection between binary trees and triangulations. Thus a system that is isomorphic to binary trees related by rotations is that of triangulations of a convex polygon related by the diagonal-flip transformation. A diagonal-flip transformation is an operation that converts one triangulation into another by removing a diagonal in the triangulation and adding the diagonal that subdivides the resulting quadrilateral in the opposite way. Thus rotation distance of binary trees and diagonal-flip distance of triangulations are equivalent.

An open problem is the complexity status of computing the rotation distance between two binary trees or equivalently the diagonal-flip distance between two triangulations. Lucas has presented a quadratic time algorithm for computing the rotation distance between binary trees of restricted form [8]. But in the general case, there remains the open problem whether these distances can be computed in polynomial time.

Some upper bounds of these distances have been exhibited [6,9,14]. Some authors approach the problem by limiting the reshaping primitive to a restricted version of the general rotation operation [1–4,7,13,16]. Obviously these restricted rotation distances will be bounded below by the usual rotation distance. Another approach for computing an upper bound uses a "flexion" operation on binary trees [12].

In [11], a rough lower bound of the rotation distance is computed using ordinal tools. But to our knowledge, efficient lower bounds are not found in literature.

In this paper we present a polynomial time algorithm for computing lower and upper bounds. Computer ex-

* Corresponding author.
E-mail addresses: barjl@u-bourgogne.fr (J.-L. Baril), pallo@u-bourgogne.fr (J.-M. Pallo).

0020-0190/$ – see front matter © 2006 Elsevier B.V. All rights reserved.
doi:10.1016/j.ipl.2006.07.001

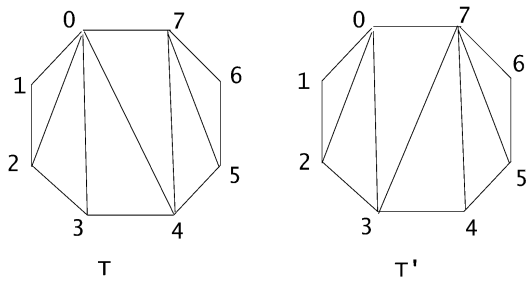


Fig. 1. A diagonal flip in $\mathcal{T}_8$.

periments show that these bounds are efficient. For simplification reasons, we prefer stating our results in terms of triangulations rather than binary trees.

2. Notations and definitions

Let us consider n-gons, i.e., convex polygons with n sides and with a distinguished side as the top. We label the vertices from 0 to $n-1$ counterclockwise such that the top has 0 and $n-1$ as two vertices. Any triangulation of the n-gon has $n-2$ triangles and $n-3$ non-crossing diagonals. Let $\mathcal{T}_n$ denote the set of triangulations of the n-gon. There is an explicit bijection between $\mathcal{T}_n$ and the set of binary trees with $n-2$ internal nodes (and thus $n-1$ leaves) [12,17]. The diagonal-flip operation on $\mathcal{T}_n$ is defined as follows. A diagonal inside the polygon is removed, creating a face with four sides. The opposite diagonal of this quadrilateral is inserted in place of the one removed, restoring the diagram to a triangulation of the polygon [17]. See Fig. 1. If $T, S \in \mathcal{T}_n$, let $\mathrm{dist}(T, S)$ be the minimum number of diagonal-flips needed to transform T into S.

The internal degree $d_i(T)$ of a vertex i of $T \in \mathcal{T}_n$ is the number of diagonals incident to i ($0 \leqslant i \leqslant n-1$). In a pair (T, S) of triangulations of $\mathcal{T}_n$, the composite degree $cd_i(T, S)$ is the total number of diagonals incident to vertex i in *both* triangulations. If $T, S \in \mathcal{T}_n$ have a diagonal $\{i, j\}$ in common, we say that $\{i, j\}$ is a $(2, 2)$-diagonal if $cd_i(T, S) = cd_j(T, S) = 2$. Given two triangulations of $\mathcal{T}_n$, a flip-to-match diagonal is a diagonal in one of the triangulations which can be flipped to make it match a diagonal in the other [15]. Given $T, S \in \mathcal{T}_n$, we define the type of vertex i by $type_i(T, S) = (k : l)$ where $k = d_i(T)$ and $l = d_i(S)$.

Definition 1. [17] Given $T \in \mathcal{T}_n$, we define the normalized triangulation $N_T(i, j)$ (respectively $N'_T(i, j)$) with respect to the diagonal $\{i, j\}$ as follows:

(1) $N_T(i, j)$ and $N'_T(i, j)$ contain the diagonal $\{i, j\}$;
(2) $N_T(i, j)$ and $N'_T(i, j)$ contain every diagonal of T that does not cross the diagonal $\{i, j\}$;
(3) if T contains a diagonal $\{a, b\}$ that crosses the diagonal $\{i, j\}$, then $N_T(i, j)$ (respectively $N'_T(i, j)$) contains the diagonals $\{a, j\}$ and $\{b, j\}$ (respectively $\{a, i\}$ and $\{b, i\}$).

Definition 2. In case where $j = i + 2 \bmod n$, we say that $N_T(i, i+2)$ (respectively $N'_T(i, i+2)$) is the counterclockwise (respectively clockwise) normalization with respect to the vertex $i+1$ and we denote $N_T(i, i+2) = N_T(i+1)$ (respectively $N'_T(i, i+2) = N'_T(i+1)$).

Definition 3. Let $\{i, j\}$ be a diagonal of $T \in \mathcal{T}_n$ such that $d_i(T) = d_j(T) = 2$. Then T can be double-normalized with respect to the diagonal $\{i, j\}$ to create a new triangulation $N''_T(i, j)$ as follows:

(1) we flip in T the diagonal $\{i, j\}$,
(2) then we flip the diagonal adjacent to vertex i,
(3) and we flip the diagonal adjacent to vertex j.

Definition 4. Let i be a vertex of $T \in \mathcal{T}_n$ such that $d_i(T) = 3$. We denote $\{a, i\}$, $\{b, i\}$ and $\{c, i\}$ the three diagonals incident to vertex i with the vertices a, b, c in clockwise order. Then T can be pseudo-normalized with respect to vertex i to create a new triangulation $N'''_T(i)$ as follows: we flip the two diagonals $\{a, i\}$ and $\{c, i\}$, then we flip the middle diagonal $\{b, i\}$.

It is worth noting that each of these normalizations creates at least one diagonal of the form $\{i-1, i+1\} \bmod n$. Then the edges $\{i-1, i\}$ and $\{i, i+1\}$ will be "nibbled" away by the algorithm of Section 8.

3. Preliminaries

Lemma 1. [17] *Given $T, S \in \mathcal{T}_n$, if it is possible to flip one diagonal of T creating T_1 so that T_1 has one more diagonal in common with S than does T, then there exists a shortest path from T to S in which the first flip creates T_1.*

Lemma 2. [17] *Given $T, S \in \mathcal{T}_n$, if T and S have a diagonal in common, then a shortest path from T to S never flips this diagonal.*

Lemma 3. [8,17] *If $T, S \in \mathcal{T}_n$ share a common diagonal, this diagonal splits T (respectively S) into two subtriangulations T' and T'' (respectively S' and S'')*

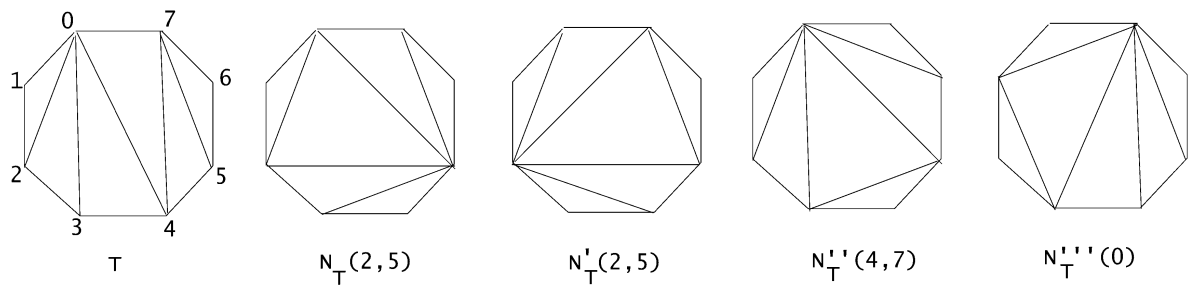


Fig. 2. Four normalizations of $T \in \mathcal{T}_6$.

in such a way that T' and S' (and thus T'' and S'') have the same vertices. Then we have the following formula:

$$\mathrm{dist}(T, S) = \mathrm{dist}(T', S') + \mathrm{dist}(T'', S'').$$

Lemma 4. *If $T, S \in \mathcal{T}_n$, then there exists a vertex i such that $cd_i(T, S) \leqslant 3$.*

Proof. Let us consider the graph which is obtained by matching T and S. We have $\sum_{i=1}^{n} cd_i(T, S) = 4(n-3)$. Thus the average verifies the inequality

$$\frac{1}{n}\sum_{i=1}^{n} cd_i(T, S) = \frac{4(n-3)}{n} < 4$$

and there exists at least a vertex i such that $cd_i(T, S) \leqslant 3$. □

According to Lemma 4, we study in the sequel the different types of vertices which can occur, i.e. $(2:0)$, $(1:1)$, $(3:0)$ and $(2:1)$. The case $(1:0)$ is a particular case of Lemma 1.

It is important to notice that, in Sections 4–7 below, we assume that the hypotheses of Lemmas 1 and 2 are not verified.

4. Properties of vertices of type (2 : 0)

Lemma 5. *Given $T, S \in \mathcal{T}_n$ with a vertex i such that $type_i(T, S) = (2:0)$, there exists a shortest path from T to S in which the two first flips create either $N_T(i)$ or $N'_T(i)$.*

Proof. Let $\mathcal{P} : T = T_0, T_1, T_2, \ldots, T_p = S$ be a shortest path connecting T and S in which T_l and T_{l+1} differ only by one flip. Since $d_i(S) = 0$, S contains the diagonal $\{i-1, i+1\} \bmod n$. Necessarily there exist in $\mathcal{P}$ two pairs (T_j, T_{j+1}) and (T_k, T_{k+1}) such that $d_i(T_j) = 2$, $d_i(T_{j+1}) = 1$, $d_i(T_k) = 1$ and $d_i(T_{k+1}) = 0$. There are two cases to consider:

Case 1: $d_{i+1}(T_{j+1}) > d_{i+1}(T_j)$. Then we have $N_{T_j}(i) = N_{T_{j+1}}(i)$ and $N_{T_k}(i) = N_{T_{k+1}}(i)$. Consider the sequence of triangulations: $N_{\mathcal{P}}(i) : N_T(i) = N_{T_0}(i)$, $N_{T_1}(i), N_{T_2}(i), \ldots, N_{T_p}(i) = N_S(i)$. Successive triangulations of $N_{\mathcal{P}}(i)$ are either identical or differ only by one flip. The length of $N_{\mathcal{P}}(i)$ is at most $p-2$. Since $\mathrm{dist}(T, N_T(i)) = 2$, we have built a new shortest path from T to S which contains $N_T(i)$.

Case 2: $d_{i-1}(T_{j+1}) > d_{i-1}(T_j)$. Then we have $N'_{T_j}(i) = N'_{T_{j+1}}(i)$ and $N'_{T_k}(i) = N'_{T_{k+1}}(i)$. The proof concludes *mutatis mutandis*. □

Theorem 1. *Given $T, S \in \mathcal{T}_n$ with a vertex i such that $type_i(T, S) = (2:0)$, then we have*:

$$\mathrm{dist}(T, S) = \min\bigl(\mathrm{dist}\bigl(N_T(i), S\bigr), \mathrm{dist}\bigl(N'_T(i), S\bigr)\bigr) + 2$$

and

$$\mathrm{dist}(T, S) \geqslant \max\bigl(\mathrm{dist}\bigl(N_T(i), S\bigr), \mathrm{dist}\bigl(N'_T(i), S\bigr)\bigr) + 1.$$

Proof. The equality is a straightforward consequence of Lemma 5. It is worth noting that $\mathrm{dist}(N_T(i), N'_T(i)) = 1$. Thus

$$\begin{aligned}\mathrm{dist}(T, S) &\\ &\geqslant \min\bigl(\mathrm{dist}\bigl(N_T(i), S\bigr), \mathrm{dist}\bigl(N_T(i), S\bigr) - 1\bigr) + 2\\ &= \mathrm{dist}\bigl(N_T(i), S\bigr) + 1\end{aligned}$$

and

$$\mathrm{dist}(T, S) \geqslant \mathrm{dist}\bigl(N'_T(i), S\bigr) + 1$$

by symmetry. □

Theorem 2. *Given $T, S \in \mathcal{T}_n$ with a $(2,2)$-diagonal $\{i, j\}$ such that $d_T(i) = 2$, $d_S(i) = 0$, $d_T(j) = 2$ and $d_S(j) = 0$, we have*:

$$\mathrm{dist}(T, S) = \mathrm{dist}\bigl(N''_T(i, j), S\bigr) + 3.$$

Proof. According to Theorem 1: $\mathrm{dist}(T, S) = 2 + \min(\mathrm{dist}(N_T(i), S), \mathrm{dist}(N'_T(i), S))$. Thus there are two cases to consider (see Fig. 3):

Case 1: $d_j(N_T(i)) = 1$. Then we have: $\mathrm{dist}(N_T(i), S) = \mathrm{dist}(N''_T(i, j), S) + 1$ and $\mathrm{dist}(N'_T(i), S) \geqslant \mathrm{dist}(N''_T(i,$

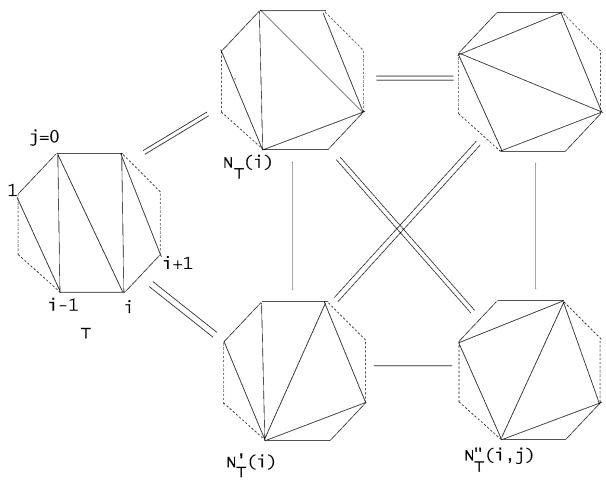


Fig. 3. Illustration of the proof of Theorem 2. A simple line is for distance one and a double line for distance two.

$j), S) + 1$. Thus there is a shortest path from T to S which contains $N''_T(i, j)$.

Case 2: $d_j(N'_T(i)) = 1$. Then we have $\mathrm{dist}(N'_T(i), S) = \mathrm{dist}(N''_T(i, j), S) + 1$ and $\mathrm{dist}(N_T(i), S) \geqslant \mathrm{dist}(N''_T(i, j), S) + 1$ proving that there exists a shortest path from T to S which contains $N''_T(i, j)$.

The proof concludes with $\mathrm{dist}(T, N''_T(i, j)) = 3$. □

5. Properties of vertices of type (1 : 1)

Theorem 3. *Given $T, S \in \mathcal{T}_n$ with a vertex i such that $type_i(T, S) = (1 : 1)$ then the following inequalities hold*:

$$\begin{aligned}&\mathrm{dist}\big(N_T(i), N_S(i)\big) + 1\\ &\quad\leqslant \mathrm{dist}(T, S)\\ &\quad\leqslant \mathrm{dist}\big(N_T(i), N_S(i)\big) + 2.\end{aligned}$$

Proof. Let $\mathcal{P} : T = T_0, T_1, T_2, \ldots, T_p = S$ be a shortest path connecting T and S in which T_l and T_{l+1} differ only by one flip. Let us consider the path: $T = T_0$, $N_{T_0}(i), N_{T_1}(i), N_{T_2}(i), \ldots, N_{T_p}(i), T_p = S$ of length $p + 2$. Two successive trees differ only by one flip, proving the right inequality. Since $type_i(T, S) = (1 : 1)$, $d_i(T) = d_i(S) = 1$, there are two cases to consider:

Case 1: there exists in $\mathcal{P}$ a tree T_q such that $d_i(T_q) = 0$. Thus $\{i - 1, i + 1\}$ mod n is a diagonal of T_q. By Lemma 1, there exists a shortest path connecting T and T_q which contains $N_T(i)$. Similarly there exists a shortest path connecting T_q and S which contains $N_S(i)$. We obtain $\mathrm{dist}(T, S) = \mathrm{dist}(N_T(i), N_S(i)) + 2 > \mathrm{dist}(N_T(i), N_S(i)) + 1$.

Case 2: there exists in $\mathcal{P}$ a tree T_r such that $d_i(T_r) = 1$ and $d_i(T_{r+1}) = 2$. We necessarily have either $N_{T_r}(i) = N_{T_{r+1}}(i)$ or $N'_{T_r}(i) = N'_{T_{r+1}}(i)$. In the first case, the length of the path $N_{\mathcal{P}}(i) : N_T(i), N_{T_1}(i), N_{T_2}(i), \ldots, N_S(i)$ connecting $N_T(i)$ and $N_S(i)$ is $p - 1$, proving $\mathrm{dist}(N_T(i), N_S(i)) \leqslant \mathrm{dist}(T, S) - 1$. The second case is similar, thus completing the proof of the left inequality. □

6. Properties of vertices of type (3 : 0)

Theorem 4. *Given $T, S \in \mathcal{T}_n$ with a vertex i such that $type_i(T, S) = (3 : 0)$ then the following inequalities hold*:

$$\begin{aligned}\mathrm{dist}\big(N'''_T(i), S\big) + 1 &\leqslant \mathrm{dist}(T, S)\\ &\leqslant \mathrm{dist}\big(N'''_T(i), S\big) + 3.\end{aligned}$$

Proof. The right inequality follows trivially from $\mathrm{dist}(T, N'''_T(i)) = 3$. Let $\mathcal{P} : T = T_0, T_1, T_2, \ldots, T_p = S$ be a shortest path connecting T and S in which T_l and T_{l+1} differ only by one flip. Since $d_i(T) = 3$ and $d_i(S) = 0$, necessarily there exist in $\mathcal{P}$ three pairs of trees (T_q, T_{q+1}), (T_r, T_{r+1}), (T_s, T_{s+1}) such that $d_i(T_q) = 3$, $d_i(T_{q+1}) = 2$, $d_i(T_r) = 2$, $d_i(T_{r+1}) = 1$, $d_i(T_s) = 1$, $d_i(T_{s+1}) = 0$. We have either $N_{T_r}(i) = N_{T_{r+1}}(i)$ or $N'_{T_r}(i) = N'_{T_{r+1}}(i)$. But we have $N_{T_s}(i) = N_{T_{s+1}}(i)$ and $N'_{T_s}(i) = N'_{T_{s+1}}(i)$. There are two cases to consider:

Case 1: there exist $u, v \in \{q, r, s\}$ with $u \neq v$ such that $N_{T_u}(i) = N_{T_{u+1}}(i)$ and $N_{T_v}(i) = N_{T_{v+1}}(i)$. Let us consider the path $N_{\mathcal{P}}(i) : N_T(i) = N_{T_0}(i), N_{T_1}(i), N_{T_2}(i), \ldots, N_{T_p}(i) = N_S(i) = S$ connecting $N_T(i)$ and S. Indeed $N_S(i) = S$ since $d_i(S) = 0$. The length of $N_{\mathcal{P}}(i)$ is $p - 2$ proving $\mathrm{dist}(N_T(i), S) + 2 \leqslant \mathrm{dist}(T, S)$.

Case 2: there exist $u', v' \in \{q, r, s\}$ with $u' \neq v'$ such that $N'_{T_{u'}}(i) = N'_{T_{u'+1}}(i)$ and $N'_{T_{v'}}(i) = N'_{T_{v'+1}}(i)$. Similarly $\mathrm{dist}(N'_T(i), S) + 2 \leqslant \mathrm{dist}(T, S)$ holds.

It is enough to observe that $\mathrm{dist}(N_T(i), N'''_T(i)) = 1$ and $\mathrm{dist}(N'_T(i), N'''_T(i)) = 1$. In particular we deduce $\mathrm{dist}(N'''_T(i), S) \leqslant 1 + \mathrm{dist}(N_T(i), S) \leqslant 1 + \mathrm{dist}(T, S) - 2$ proving the left inequality. □

7. Properties of vertices of type (2 : 1)

Theorem 5. *Given $T, S \in \mathcal{T}_n$ with a vertex i such that $type_i(T, S) = (2 : 1)$ then the following inequalities hold*:

$$\begin{aligned}&\mathrm{dist}(T, S)\\ &\quad\geqslant \max\big(\mathrm{dist}\big(N_T(i), N_S(i)\big), \mathrm{dist}\big(N'_T(i), N'_S(i)\big)\big) + 1,\\ &\mathrm{dist}(T, S)\\ &\quad\leqslant \min\big(\mathrm{dist}\big(N_T(i), N_S(i)\big), \mathrm{dist}\big(N'_T(i), N'_S(i)\big)\big) + 3.\end{aligned}$$

Proof. Since $d_i(T) = 2$ and $d_i(S) = 1$, we have $\mathrm{dist}(T, N_T(i)) = 2$ and $\mathrm{dist}(S, N_S(i)) = 1$ proving the second inequality. For proving the first inequality, let $\mathcal{P} : T = T_0, T_1, T_2, \ldots, T_p = S$ be a shortest path connecting T and S in which two consecutive trees differ only by one flip. We denote $\{i, a\}$ and $\{i, b\}$ the two diagonals of T incident to vertex i with a and b in clockwise order. Now let $f \geqslant 1$ be the smallest $s \in [1, p]$ such that the diagonal $\{i, a\}$ is not the first diagonal (in clockwise order) adjacent to i in T_s. Similarly let $\ell \geqslant 1$ be the smallest $r \in [1, p]$ such that the diagonal $\{i, b\}$ is not the last diagonal (in clockwise order) adjacent to i in T_r. Thus we necessarily have $N_{T_{f-1}}(i) = N_{T_f}(i)$ and $N'_{T_{\ell-1}}(i) = N'_{T_\ell}(i)$. Therefore we obtain $\mathrm{dist}(N_T(i), N_S(i)) \leqslant \mathrm{dist}(T, S) - 1$ and $\mathrm{dist}(N'_T(i), N'_S(i)) \leqslant \mathrm{dist}(T, S) - 1$ proving the first inequality. □

8. Computing lower and upper bounds

The goal of the following algorithm is to "nibble" the edges (and consequently the vertices) of the two triangulations by applying Lemma 1 and then Theorems 1–5. It is necessary to check whether there is a flip-to-match diagonal first. In this case, the corresponding flip is carried out. Otherwise, the existence of type $(1:0)$, $(2:0)$, $(1:1)$, $(3:0)$ and $(2:1)$ diagonals (in this precise order) should be checked. In fact, whatever the order, the algorithm is correct. However we choose this precise order for obvious statistical reasons. It will lead statistically to minimum number of operations.

Now we provide a recursive algorithm (according to Lemma 4) to find lower and upper bounds (in our algorithm *low* and *up*) for the rotation distance between two triangulations $T, S \in \mathcal{T}_n$. It should be pointed out that, in this algorithm, trees T, S and therefore types $(i:j)$, $(j:i)$ play the same role.

Algorithm (Computing *low* and *up*)
Given $T, S \in \mathcal{T}_n$

procedure low-up(T, S, n)
if $n \geqslant 3$ **then**
if T and S have a common diagonal **then**
  $T := (T_1, T_2)$; $S := (S_1, S_2)$;
  $n_1 := \mathrm{size}(T_1) = \mathrm{size}(S_1)$; $n_2 := \mathrm{size}(T_2) = \mathrm{size}(S_2)$;
  low-up(T_1, S_1, n_1);
  low-up(T_2, S_2, n_2);
else
  if T and S verify Lemma 1 **then**
    flip the diagonal in T or in S
    $low := low + 1$; $up := up + 1$
    low-up(T, S, n);
  else
    if T and S have a vertex i of type $(2:0)$ **then**
      if T and S have a $(2,2)$-diagonal $\{i, j\}$ **then**
        $T := N''_T(i, j)$ or $S := N''_S(i, j)$
        $low := low + 3$; $up := up + 3$
        low-up$(T, S, n-2)$;
      else
        $T := N_T(i)$ or $S := N_S(i)$
        $low := low + 1$; $up := up + 2$
        low-up$(T, S, n-1)$;
    else
      if T and S have a vertex i of type $(1:1)$ **then**
        $T := N_T(i)$ and $S := N_S(i)$
        $low := low + 1$; $up := up + 2$
        low-up$(T, S, n-1)$;
      else
      if T and S have a vertex i of type $(3:0)$ **then**
        $T := N'''_T(i)$ or $S := N'''_S(i)$
        $low := low + 1$; $up := up + 3$
        low-up$(T, S, n-1)$;
      else T and S have a vertex i of type $(2:1)$
        $T := N_T(i)$ and $S := N_S(i)$
        $low := low + 1$; $up := up + 3$
        low-up$(T, S, n-1)$;
end low-up

Examples. If (T, S) are the two triangulations $\in \mathcal{T}_{23}$ of Rogers [15, p. 88], then the above algorithm provides $20 \leqslant \mathrm{dist}(T, S) \leqslant 29$. The exact rotation distance is 21.

If (T, S) are the two triangulations $\in \mathcal{T}_{20}$ of Lucas [8, p. 261], then the above algorithm provides $17 \leqslant \mathrm{dist}(T, S) \leqslant 25$. The exact rotation distance is 20.

An applet for computing lower and upper bounds is available on the web site: http://www.u-bourgogne.fr/jl.baril/titi.html. This applet will choose the better of the two upper bounds, either the result of this paper, or the result of [12]. Let us remark that the upper bound provided in [12] is often, but not necessarily, better than the upper bound computed in this paper. This applet uses the coding of binary trees (or equivalently triangulations) by weight sequences [10] and represents triangulations by adjacency matrices. Recall that the adjacency matrix M_T of T is such that $M_T(i, j) = 1$ if the vertices i and j are connected in T and $M_T(i, j) = 0$ otherwise. Between two recursive calls we carry out a flip-to-match diagonal and some computations in order to obtain the adjacency matrices of T_1 and T_2 from that of T. This step requires a time complexity $\mathcal{O}(n^2)$. Since the triangulations have $n - 3$ diagonals the time complexity

is given by $\sum_{i=1}^{n-3} i^2$ and thus the time complexity is $\mathcal{O}(n^3)$.

9. Computer experiments

We present here some exhaustive and statistical results. In the first array we obtain exhaustive values for $n \leqslant 13$ about the ratio of pairs (T, S) verifying $(up - low) \leqslant k$ with $k = 0, 1, 2, n/2$. In the second array we provide probabilistic results for some $n \in [14, 100]$ by generating a random huge number of pairs (T, S). We notice that our algorithm gives efficient lower and upper bounds for large values of n: 99% of pairs (T, S) have upper and lower bounds verifying $(up - low) \leqslant n/3$.

Exhaustive results

Table 1

n-gons	5	6	7	8	9	10	11	12	13
$Up = Low$	100(%)	96	88	78	67	57	47	38	33
$Up - Low \leqslant 1$	100	100	99	99	95	90	82	73	54
$Up - Low \leqslant 2$	100	100	100	100	100	99	96	91	64
$Up - Low \leqslant n/2$	100	100	100	100	100	100	99.9	99.9	99.9

Statistical results

Table 2

n-gons	14	15	16	17	18	20	30	50	100
$Up - Low \leqslant n/10$	81.5 (%)	77	75	71	69	65	59	45	39
$Up - Low \leqslant n/5$	98	97	97	96	95	95	95	94	93
$Up - Low \leqslant n/3$	99.9	99.9	99.9	99.9	99.8	99.8	99.8	99.7	99.7
$Up - Low \leqslant n/2$	99.9	99.9	99.9	99.9	99.9	99.9	99.9	99.9	99.9

Acknowledgements

We are grateful to an anonymous referee for the improvement of the proofs of Theorems 4 and 5.

References

[1] A. Bonnin, J. Pallo, A shortest path metric on unlabeled binary trees, Pattern Recognition Lett. 13 (1992) 411–415.

[2] S. Cleary, Restricted rotation distance between binary trees, Inform. Process. Lett. 84 (2002) 333–338.

[3] S. Cleary, J. Taback, Bounding restricted rotation distance, Inform. Process. Lett. 88 (2003) 251–256.

[4] Y.J. Chen, J.M. Chang, Y.L. Wang, An efficient algorithm for estimating rotation distance between two binary trees, Intern. J. Comput. Math. 82 (2005) 1095–1106.

[5] K. Culik, D. Wood, A note on some tree similarity measures, Inform. Process. Lett. 15 (1982) 39–42.

[6] S. Hanke, T. Ottmann, S. Schuierer, The edge-flipping distance of triangulations, J. UCS 2 (1996) 570–579.

[7] J.M. Lucas, A direct algorithm for restricted rotation distance, Inform. Process. Lett. 90 (2004) 129–134.

[8] J.M. Lucas, Untangling binary trees via rotations, Comput. J. 47 (2004) 259–269.

[9] F. Luccio, L. Pagli, On the upper bound of the rotation distance of binary trees, Inform. Process. Lett. 31 (1989) 57–60.

[10] J.M. Pallo, Enumerating, ranking and unranking binary trees, Comput. J. 29 (1986) 171–175.

[11] J.M. Pallo, On the rotation distance in the lattice of binary trees, Inform. Process. Lett. 25 (1987) 369–373.

[12] J.M. Pallo, An efficient upper bound of the rotation distance of binary trees, Inform. Process. Lett. 73 (2000) 87–92.

[13] J.M. Pallo, Right-arm rotation distance between binary trees, Inform. Process. Lett. 87 (2003) 173–177.

[14] R.O. Rogers, R.D. Dutton, On distance in the rotation graph of binary trees, Congr. Numer. 120 (1996) 103–113.

[15] R.O. Rogers, On finding shortest paths in the rotation graph of binary trees, Congr. Numer. 137 (1999) 75–95.

[16] A.A. Ruiz, F. Luccio, A.M. Enriquez, L. Pagli, k-restricted rotation with an application to search tree rebalancing, in: 9th WADS, in: Lecture Notes in Computer Science, vol. 3608, Springer, Berlin, 2005, pp. 2–13.

[17] D.D. Sleator, R.E. Tarjan, W. Thurston, Rotation distance, triangulations and hyperbolic geometry, J. Amer. Math. Soc. 1 (1988) 647–681.

ARTICLE IN PRESS

Available online at www.sciencedirect.com

ScienceDirect

Discrete Mathematics ▮▮▮ (▮▮▮▮) ▮▮▮–▮▮▮

DISCRETE MATHEMATICS

www.elsevier.com/locate/disc

Gray code for permutations with a fixed number of cycles

Jean-Luc Baril

LE2I UMR-CNRS 5158, Université de Bourgogne, B.P. 47 870, 21078 DIJON-Cedex, France

Received 12 January 2006; received in revised form 22 August 2006; accepted 2 September 2006

Abstract

We give the first Gray code for the set of n-length permutations with a given number of cycles. In this code, each permutation is transformed into its successor by a product with a cycle of length three, which is optimal. If we represent each permutation by its transposition array then the obtained list still remains a Gray code and this allows us to construct a constant amortized time (CAT) algorithm for generating these codes. Also, Gray code and generating algorithm for n-length permutations with fixed number of left-to-right minima are discussed.
© 2006 Elsevier B.V. All rights reserved.

Keywords: Restricted permutations; Gray codes; Generating algorithms; Transposition array

1. Introduction

Various study have been made on generating algorithms for permutations and their restrictions (with given *ups* and *downs* [5,10], derangements [1], involutions [6], and fixed-point free involutions [16]) or their generalizations (multiset permutations [15]). See [4,12] for surveys of permutation generation methods. At [7] is given an implementation of Taylor and Ruskey [11] generating algorithm for n-length permutations with k cycles. However, the generating order is neither lexicographic nor Gray code order. On the other hand, these codes also called permutation codes or arrays have some applications for power line communication [2].

Let S_n be the set of all permutations of length $n \geqslant 1$. We represent permutations in one-line notation, i.e. if $i_1, i_2, \ldots, i_n$ are n distinct values in $[n] = \{1, 2, \ldots, n\}$, we denote the permutation σ by the sequence $(i_1, i_2, \ldots, i_n)$ if $\sigma(k) = i_k$ for $1 \leqslant k \leqslant n$. Moreover, if $\gamma = (\gamma(1), \gamma(2), \ldots, \gamma(n))$ is an n-length permutation then the *composition* (*or product*) $\gamma \cdot \sigma$ is the permutation $(\gamma(\sigma(1)), \gamma(\sigma(2)), \ldots, \gamma(\sigma(n)))$. In S_n, a *k-cycle* $\sigma = \langle i_1, i_2, \ldots, i_k \rangle$ is an n-length permutation verifying $\sigma(i_1) = i_2, \sigma(i_2) = i_3, \ldots, \sigma(i_{k-1}) = i_k, \sigma(i_k) = i_1$ and $\sigma(j) = j$ for $j \in [n] \backslash \{i_1, \ldots, i_k\}$; and the *domain* of σ is the set $\{i_1, \ldots, i_k\}$. In particular, a *transposition* (i.e. a 2-cycle) has domain of cardinality two. It is well known that each n-length permutation can be uniquely decomposed in a product of cycles with disjoint domains. For $1 \leqslant k \leqslant n$, we denote by $S_{n,k}$ the set of all n-length permutations which admit a decomposition in a product of k (disjoint) cycles and $\{S_{n,k}\}_{1 \leqslant k \leqslant n}$ forms a partition for S_n. The cardinality of $S_{n,k}$ is given by the signless Stirling numbers of the first kind $s(n, k)$ satisfying:

$$s(n, k) = (n - 1) \cdot s(n - 1, k) + s(n - 1, k - 1) \tag{1}$$

with the initial conditions $s(n, k) = 0$ if $n \leqslant 0$ or $k \leqslant 0$, except $s(0, 0) = 1$. See for instance [13,18].

E-mail address: barjl@u-bourgogne.fr.

0012-365X/$ - see front matter © 2006 Elsevier B.V. All rights reserved.
doi:10.1016/j.disc.2006.09.007

Please cite this article as: Jean-Luc Baril, Gray code for permutations with a fixed number of cycles, Discrete Mathematics (2006), doi: 10.1016/j.disc.2006.09.007

A *Gray code* for a sequence set S is an ordered list for S in which the Hamming distance between any two consecutive sequences in the list (the number of positions in which they differ) is bounded by a constant, independent on the size of the sequences. If this constant is minimal then the code is called *optimal*. A Gray code is *cyclic* if the Hamming distance between the first and the last sequence is also bounded by this constant.

Now we introduce some notations concerning lists.

If $\mathscr{S}$ is a list then $\overline{\mathscr{S}}$ is the list obtained by reversing $\mathscr{S}$, and first($\mathscr{S}$) (last($\mathscr{S}$), respectively) is the first (last respectively) element of the list, and obviously first($\mathscr{S}$) = last($\overline{\mathscr{S}}$) and first($\overline{\mathscr{S}}$) = last($\mathscr{S}$); $\mathscr{S}^{(i)}$ is the list $\mathscr{S}$ if i is even, and $\overline{\mathscr{S}}$ if i is odd; if $\mathscr{S}_1$ and $\mathscr{S}_2$ are two lists, then $\mathscr{S}_1 \circ \mathscr{S}_2$ is their concatenation, and generally $\bigcirc_{i=1}^{m} \mathscr{S}_i$ is the list $\mathscr{S}_1 \circ \mathscr{S}_2 \circ \cdots \circ \mathscr{S}_m$. Similarly, $\bigcirc_{i=m}^{1} \mathscr{S}_i$ is the list $\mathscr{S}_m \circ \mathscr{S}_{m-1} \circ \cdots \circ \mathscr{S}_1$.

In the following, $\mathscr{S}_{n,k}$ will denote our Gray code for the set $S_{n,k}$, $f_{n,k} = \text{first}(\mathscr{S}_{n,k})$ and $\ell_{n,k} = \text{last}(\mathscr{S}_{n,k})$.

Remark that if $\sigma \in S_n$ has k cycles in its (unique) decomposition in cycles with pairwise disjoint domains, and $\tau = \langle i, j \rangle$, $i \neq j$ a transposition in S_n, then the permutation $\sigma \cdot \tau$ has $k-1$ or $k+1$ cycles in its decomposition. Indeed, if i and j belong to the domain of the same cycle in σ then this cycle is splited into two cycles in $\sigma \cdot \tau$; otherwise two cycles of σ merge into a single cycle in $\sigma \cdot \tau$. This shows that there does not exist a Gray code for $S_{n,k}$ such that two successive permutations differ in less than three positions.

In this paper, we give the first Gray code for the set $S_{n,k}$. In this code successive permutations differ in three positions (or equivalently, by a product with a three-length cycle) and so, by the remark above it is optimal. By representing each permutation by its transposition array we provide an other Gray code which allows us to construct a constant amortized time (CAT) algorithm for generating these codes. Actually, this second Gray code lists n-length permutations with exactly k left-to-right minima in inversion array representation.

2. Preliminaries

In this section we show how $S_{n,k}$ can be recursively constructed from $S_{n-1,k}$ and $S_{n-1,k-1}$ which also gives a constructive proof of the counting relation (1). We also give three lemmas, crucial in our construction of the code.

Let $\gamma \in S_{n-1,k}$ be an $(n-1)$-length permutation with k cycles, $n \geqslant 2$, $n-1 \geqslant k \geqslant 1$; let also i be an integer, $1 \leqslant i < n$. If we denote by σ the permutation in S_n obtained from γ by replacing the entry in position i by n and appending this entry in the nth position, then σ is an n-length permutation with k cycles.

Similarly, if $\gamma \in S_{n-1,k-1}$ is an $(n-1)$-length permutation with $(k-1)$ cycles, $n \geqslant k \geqslant 2$, and if σ denotes the permutation in S_n obtained from γ by appending n in the nth position, then σ is an n-length permutation with k cycles. Moreover, each permutation in $S_{n,k}$, $n \geqslant 2$, can uniquely be obtained by one of these two constructions.

For $n, k \geqslant 1$, let $S'_{n,k}$ be the set of n-length permutations with k cycles where n is a fixed point (i.e. $\gamma(n) = n$) and $S''_{n,k} = S_{n,k} \backslash S'_{n,k}$ is its complement. The next definition formalizes the two constructions above.

Then the functions ϕ_n and ψ_n defined below induce a bijection between $S_{n-1,k-1}$ and $S'_{n,k}$ on one hand and between $[n-1] \times S_{n-1,k}$ and $S''_{n,k}$ on the other.

Definition 1.

(1) For $1 \leqslant k < n$, an integer $i \in [n-1]$ and a permutation $\gamma \in S_{n-1,k}$, we define an n-length permutation $\sigma = \psi_n(i, \gamma)$ by

$$\sigma(j) = \begin{cases} \gamma(i) & \text{if } j = n, \\ n & \text{if } j = i, \\ \gamma(j) & \text{otherwise.} \end{cases}$$

(2) For $n \geqslant k \geqslant 2$ and a permutation $\gamma \in S_{n-1,k-1}$, we define an n-length permutation $\sigma = \phi_n(\gamma)$ by

$$\sigma(j) = \begin{cases} n & \text{if } j = n, \\ \gamma(j) & \text{otherwise.} \end{cases}$$

Remark that with i and γ as above, it is easy to see that

- if $\gamma \in S_{n-1,k}$, $\psi_n(i, \gamma) \in S''_{n,k}$ and $\psi_n : [n-1] \times S_{n-1,k} \to S''_{n,k}$ is a bijection; and
- if $\gamma \in S_{n-1,k-1}$, $\phi_n(\gamma) \in S'_{n,k}$ and $\phi_n : S_{n-1,k-1} \to S'_{n,k}$ is a bijection.

Please cite this article as: Jean-Luc Baril, Gray code for permutations with a fixed number of cycles, Discrete Mathematics (2006), doi: 10.1016/j.disc.2006.09.007

So, the cardinality of $S_{n,k}$ is given by $s(n,k) = \mathrm{card}(S'_{n,k}) + \mathrm{card}(S''_{n,k}) = s(n-1,k-1) + (n-1)\cdot s(n-1,k)$ which is a combinatorial proof of (1).

In the following, we will omit the subscript n for the functions ϕ_n and ψ_n, and it should be clear from context. Also, we extend the functions ϕ and ψ in a natural way to sets and lists of permutations. Moreover for $i \in [n-1]$ and $\mathscr{S}$ a list of $(n-1)$-length permutations we have $\psi(i, \overline{\mathscr{S}}) = \overline{\psi(i, \mathscr{S})}$, $\psi(i, \mathrm{first}(\mathscr{S})) = \mathrm{first}(\psi(i, \mathscr{S}))$, and $\psi(i, \mathrm{last}(\mathscr{S})) = \mathrm{last}(\psi(i, \mathscr{S}))$. If we do not consider the parameter i, we obtain similar results for the function ϕ.

Now, we give some elementaries results which are crucial in the construction of our Gray code.

Lemma 2. *Let γ be an $(n-1)$-length permutation, if $n \geqslant 3$ and $1 \leqslant i, j \leqslant n-1$, $i \neq j$ then $\psi(i,\gamma) = \psi(j,\gamma)\cdot\langle i, j, n\rangle$.*

Proof. For each $(n-1)$-length permutation $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_{n-1})$, we have $\psi(j,\gamma) = (\gamma_1, \ldots, \gamma_{j-1}, n, \gamma_{j+1}, \ldots, \gamma_{n-1}, \gamma_j)$. Thus, we obtain

$$\begin{aligned}\psi(j,\gamma)\cdot\langle i, j, n\rangle &= (\gamma_1, \ldots, \gamma_{i-1}, n, \gamma_{i+1}, \ldots, \gamma_{n-1}, \gamma_i)\\ &= \psi(i,\gamma). \qquad \square\end{aligned}$$

Lemma 3. *Let γ and σ be two $(n-1)$-length permutations satisfying $\sigma = \gamma\cdot\langle i, j, k\rangle$ with i, j, k pairwise different, $1 \leqslant i, j, k \leqslant n-1$, $n \geqslant 3$ then $\psi(j,\sigma) = \psi(k,\gamma)\cdot\langle i, j, k\rangle$.*

Proof.

$$\begin{aligned}\psi(j,\sigma) &= \psi(j, \gamma\cdot\langle i, j, k\rangle)\\ &= \psi(j, (\gamma_1, \ldots, \gamma_{i-1}, \gamma_j, \gamma_{i+1}, \ldots \gamma_{j-1}, \gamma_k, \gamma_{j+1}, \ldots, \gamma_{k-1}, \gamma_i, \gamma_{k+1}, \ldots, \gamma_{n-1}))\\ &= (\gamma_1, \ldots, \gamma_{i-1}, \gamma_j, \gamma_{i+1}, \ldots \gamma_{j-1}, n, \gamma_{j+1}, \ldots, \gamma_{k-1}, \gamma_i, \gamma_{k+1}, \ldots, \gamma_{n-1}, \gamma_k)\end{aligned}$$

and

$$\begin{aligned}\psi(j,\sigma)\cdot\langle i, k, j\rangle &= (\gamma_1, \ldots, \gamma_{i-1}, \gamma_i, \gamma_{i+1}, \ldots \gamma_{j-1}, \gamma_j, \gamma_{j+1}, \ldots, \gamma_{k-1}, n, \gamma_{k+1}, \ldots, \gamma_{n-1}, \gamma_k)\\ &= \psi(k,\gamma). \qquad \square\end{aligned}$$

Lemma 4. *Let γ be an $(n-1)$-length permutation and $\sigma = \gamma\cdot\langle i, j\rangle$ with $i \neq j$, $1 \leqslant i, j \leqslant n-1$, $n \geqslant 3$ then $\psi(i,\sigma) = \phi(\gamma)\cdot\langle i, n, j\rangle$.*

Proof.

$$\begin{aligned}\psi(i,\sigma) &= \psi(i, \gamma\cdot\langle i, j\rangle)\\ &= \psi(i, (\gamma_1, \ldots, \gamma_{i-1}, \gamma_j, \gamma_{i+1}, \ldots, \gamma_{j-1}, \gamma_i, \gamma_{j+1}, \ldots, \gamma_{n-1}))\\ &= (\gamma_1, \ldots, \gamma_{i-1}, n, \gamma_{i+1}, \ldots, \gamma_{j-1}, \gamma_i, \gamma_{j+1}, \ldots, \gamma_{n-1}, \gamma_j)\end{aligned}$$

and

$$\begin{aligned}\psi(i,\sigma)\cdot\langle i, j, n\rangle &= (\gamma_1, \ldots, \gamma_{i-1}, \gamma_i, \gamma_{i+1}, \ldots, \gamma_{j-1}, \gamma_j, \gamma_{j+1}, \ldots, \gamma_{n-1}, n)\\ &= \phi(\gamma). \qquad \square\end{aligned}$$

3. The gray code

From the remark following Definition 1 results that the set $S_{n,k}$ can be written as:

$$S_{n,k} = \bigcup_{i=1}^{n-1} \psi(i, S_{n-1,k}) \cup \phi(S_{n-1,k-1}) \tag{2}$$

with $\phi(S_{n,0})$ and $\psi(i, S_{n,n+1})$ empty.

If $\mathscr{S}$ is a list of permutations where any two consecutive permutations differ in three positions then so is the image of $\mathscr{S}$ by ψ or ϕ. Therefore, it is natural to look for a Gray code for the set $S_{n,k}$ of the form

$$\mathscr{S}_1 \circ \mathscr{S}_2 \circ \cdots \circ \mathscr{S}_\ell \circ \mathscr{T} \circ \mathscr{S}_{\ell+1} \circ \cdots \circ \mathscr{S}_{n-1}, \tag{3}$$

Please cite this article as: Jean-Luc Baril, Gray code for permutations with a fixed number of cycles, Discrete Mathematics (2006), doi: 10.1016/j.disc.2006.09.007

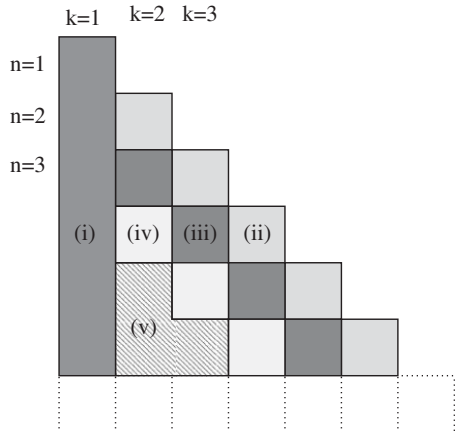


Fig. 1. The five cases for our Gray code.

where

- $\mathscr{T}$ is the list $\phi(\mathscr{S}_{n-1,k-1})$ or its reverse, and
- $\mathscr{S}_i$ is $\psi(j, \mathscr{S}_{n-1,k})$ or its reverse, for some j.

Notice that (3) is the ordered counterpart of (2).

Primarily, there are three difficulties to construct such a Gray code:

- for each $\mathscr{S}_i$, as in the last point above, we must determine a $j \leqslant n-1$ to apply ψ to $\mathscr{S}_{n-1,k}$,
- decide for each list if it is reversed or not,
- find the place where $\mathscr{T}$ must be inserted.

In the next we construct a Gray code of the form given by (3) according to the following cases (see Fig. 1):

(i) $k = 1 \leqslant n$,
(ii) $2 \leqslant k = n$,
(iii) $2 \leqslant k = n-1$,
(iv) $2 \leqslant k = n-2$,
(v) other cases,

and computer tests enable us to think that there is not simpler expression of such a Gray code.

For each case above, we give a recursive definition for $\mathscr{S}_{n,k}$, an ordered list for the set $S_{n,k}$ and we provide its first $f_{n,k}$ and last element $\ell_{n,k}$. $\mathscr{S}_{n,k}$ is the concatenation of n lists as in (3) and we prove that it is a Gray code by showing that there is a 'smooth' transition between successive sublists, that is, the last permutation of a sublist and the first one of the next sublist differ by a product with a three-length cycle. Recall that, by the remark in introduction, the Gray code will be optimal.

3.1. The case $k = 1$, $n \geqslant 1$

For $n \geqslant 1$ we define,

$$\mathscr{S}_{n,1} = \begin{cases} (1) & \text{if } n = 1, \\ (2, 1) & \text{if } n = 2, \\ (3, 1, 2), (2, 3, 1) & \text{if } n = 3, \\ \psi(1, \mathscr{S}_{3,1}) \circ \psi(2, \overline{\mathscr{S}_{3,1}}) \circ \psi(3, \overline{\mathscr{S}_{3,1}}) & \text{if } n = 4, \\ \psi(1, \mathscr{S}_{n-1,1}) \circ \bigcirc_{i=n-1}^{4} \psi(i, \mathscr{S}_{n-1,1})^{(i+1)} \circ \psi(2, \mathscr{S}_{n-1,1}) \circ \psi(3, \overline{\mathscr{S}_{n-1,1}}) & \text{if } n \geqslant 5. \end{cases} \tag{a}$$

Remark that the function ϕ does not appear in relation (a).

The lemma below gives the first and the last permutation of the list $\mathscr{S}_{n,1}$.

Please cite this article as: Jean-Luc Baril, Gray code for permutations with a fixed number of cycles, Discrete Mathematics (2006), doi: 10.1016/j.disc.2006.09.007

Lemma 5. *If $n \geqslant 4$ then*

(1) $f_{n,1} = (n, 1, 2, 3, \ldots, n-2, n-1)$,
(2) $\ell_{n,1} = (n-1, 1, n, 3, \ldots, n-2, 2)$.

Proof.

(1) $\mathscr{S}_{2,1}$ is the list (2, 1) and more generally relation (a) gives $f_{n,1} = \psi(1, f_{n-1,1})$, for $n \geqslant 3$. The recurrence on n completes the proof.
(2) Similarly, we have for $n \geqslant 4$, $\ell_{n,1} = \mathrm{last}(\overline{\psi(3, \mathscr{S}_{n-1,1})}) = \psi(3, f_{n-1,1}) = (n-1, 1, n, 3, \ldots, n-2, 2)$. □

Proposition 6. *The list $\mathscr{S}_{n,1}$ defined by* (a), *$n \geqslant 1$, is a Gray code.*

Proof. By Lemma 2, we have for $n \geqslant 3$ and $i \neq j$:

$$\begin{aligned}\mathrm{last}(\psi(i, \mathscr{S}_{n-1,1})) &= \psi(i, \mathrm{last}(\mathscr{S}_{n-1,1}))\\ &= \psi(j, \mathrm{last}(\mathscr{S}_{n-1,1})) \cdot \langle i, j, n\rangle\\ &= \mathrm{last}(\psi(j, \mathscr{S}_{n-1,1})) \cdot \langle i, j, n\rangle\\ &= \mathrm{first}(\psi(j, \overline{\mathscr{S}_{n-1,1}})) \cdot \langle i, j, n\rangle.\end{aligned}$$

Similarly, $\mathrm{last}(\psi(i, \overline{\mathscr{S}_{n-1,1}})) = \mathrm{first}(\psi(j, \mathscr{S}_{n-1,1})) \cdot \langle i, j, n\rangle$.

By Lemma 3,

$$\begin{aligned}\mathrm{last}(\psi(1, \mathscr{S}_{n-1,1})) &= \psi(1, \mathrm{last}(\mathscr{S}_{n-1,1}))\\ &= \psi(1, (n-2, 1, n-1, 3, \ldots, n-3, 2))\\ &= \psi(1, \mathrm{first}(\mathscr{S}_{n-1,1}) \cdot \langle 1, n-1, 3\rangle)\\ &= \mathrm{first}(\psi(n-1, \mathscr{S}_{n-1,1})) \cdot \langle 1, n-1, 3\rangle.\end{aligned}$$

Notice that the transition between $\psi(1, \mathscr{S}_{n-1,1})$ and $\overline{\psi(n-1, \mathscr{S}_{n-1,1})}$ is given in the first point of the proof by setting $i = 1$ and $j = n-1$. □

3.2. The case $k = n$, $n \geqslant 2$

Obviously, $\mathscr{S}_{n,n}$ is the single element list $(1, 2, \ldots, n-1, n)$ for $n \geqslant 2$, (b)

and in this case, there is nothing to do.

3.3. The case $k = n-1$, $n \geqslant 3$

For $n \geqslant 3$ we define

$$\mathscr{S}_{n,n-1} = \begin{cases} \psi(1, \mathscr{S}_{2,2}) \circ \psi(2, \mathscr{S}_{2,2}) \circ \phi(\mathscr{S}_{2,1}) & \text{if } n = 3,\\ \psi(1, \mathscr{S}_{n-1,n-1}) \circ \bigcirc_{\substack{i=n-1\\ i\neq n-2}}^{2} \psi(i, \mathscr{S}_{n-1,n-1}) \circ \psi(n-2, \mathscr{S}_{n-1,n-1}) \circ \phi(\overline{\mathscr{S}_{n-1,n-2}}) & \text{otherwise.}\end{cases} \quad \text{(c)}$$

Lemma 7. *If $n \geqslant 3$ then*

(1) $f_{n,n-1} = (n, 2, 3, \ldots, n-2, n-1, 1)$,
(2) $\ell_{n,n-1} = (n-1, 2, 3, \ldots, n-2, 1, n)$.

Please cite this article as: Jean-Luc Baril, Gray code for permutations with a fixed number of cycles, Discrete Mathematics (2006), doi: 10.1016/j.disc.2006.09.007

Proof.

(1) By relation (c), $\mathscr{S}_{3,2}$ is the list $(3, 2, 1)$, $(1, 3, 2)$, $(2, 1, 3)$ and more generally $f_{n,n-1} = \text{first}(\psi(1, \mathscr{S}_{n-1,n-1}))$. So, we obtain $f_{n,n-1} = \psi(1, f_{n-1,n-1}) = (n, 2, 3, \ldots, n-2, n-1, 1)$ with a recurrence on n.
(2) Similarly, $\ell_{n,n-1} = \text{last}(\overline{\phi(\mathscr{S}_{n-1,n-2})}) = \phi(f_{n-1,n-2}) = (n-1, 2, 3, \ldots, n-2, 1, n)$. □

Proposition 8. *The list $\mathscr{S}_{n,n-1}$ defined by* (c), $n \geqslant 3$, *is a Gray code.*

Proof. By Lemma 2, and since $\psi(i, \mathscr{S}_{n-1,n-1})$ has only one element, the transition between $\psi(i, \mathscr{S}_{n-1,n-1})$ and $\psi(j, \mathscr{S}_{n-1,n-1})$, $1 \leqslant i, j \leqslant n-1$ is a product with a three-length cycle. So, it remains to verify that $\text{last}(\psi(n-2, \mathscr{S}_{n-1,n-1}))$ differs from $\text{first}(\phi(\overline{\mathscr{S}_{n-1,n-2}}))$ by a three-cycle:

$$\begin{aligned}\text{last}(\psi(n-2, \mathscr{S}_{n-1,n-1})) &= \psi(n-2, (1, 2, 3, \ldots, n-1)) \\ &= \psi(n-2, (n-2, 2, 3, \ldots, n-3, 1, n-1) \cdot \langle 1, n-2\rangle),\end{aligned}$$

and by Lemma 4,

$$\begin{aligned}\text{last}(\psi(n-2, \mathscr{S}_{n-1,n-1})) &= \phi((n-2, 2, 3, \ldots, n-3, 1, n-1)) \cdot \langle 1, n, n-2\rangle \\ &= \phi(\text{first}(\overline{\mathscr{S}_{n-1,n-2}})) \cdot \langle 1, n, n-2\rangle \\ &= \text{first}(\phi(\overline{\mathscr{S}_{n-1,n-2}})) \cdot \langle 1, n, n-2\rangle. \qquad \square\end{aligned}$$

3.4. The case $k = n-2$, $n \geqslant 4$

In this case we define for $n \geqslant 4$,

$$\mathscr{S}_{n,n-2} = \psi(1, \mathscr{S}_{n-1,n-2}) \circ \bigcirc_{i=3}^{n-1} \psi(i, \mathscr{S}_{n-1,n-2})^{(i)} \circ \phi(\mathscr{S}_{n-1,n-3}) \circ \psi(2, \overline{\mathscr{S}_{n-1,n-2}}). \tag{d}$$

Lemma 9. *If $n \geqslant 4$ then*

(1) $f_{n,n-2} = (n, 2, 3, \ldots, n-2, 1, n-1)$,
(2) $\ell_{n,n-2} = (n-1, n, 3, 4, \ldots, n-2, 1, 2)$.

Proof.

(1) By relation (d), $\mathscr{S}_{4,2}$ is the list $(4, 2, 1, 3)$, $(4, 3, 2, 1)$, $(4, 1, 3, 2)$, $(2, 1, 4, 3)$, $(1, 3, 4, 2)$, $(3, 2, 4, 1)$, $(3, 1, 2, 4)$, $(2, 3, 1, 4)$, $(2, 4, 3, 1)$, $(1, 4, 2, 3)$, $(3, 4, 1, 2)$ and more generally $f_{n,n-2} = \text{first}(\psi(1, \mathscr{S}_{n-1,n-2}))$. So, $f_{n,n-2} = \psi(1, f_{n-1,n-2})$ and the recurrence on n completes the proof.
(2) Similarly,

$$\begin{aligned}\ell_{n,n-2} &= \text{last}(\psi(2, \overline{\mathscr{S}_{n-1,n-2}})) \\ &= \psi(2, f_{n-1,n-2}) \\ &= (n-1, n, 3, \ldots, n-2, 1, 2). \qquad \square\end{aligned}$$

Proposition 10. *The list $\mathscr{S}_{n,n-2}$ defined by* (d), $n \geqslant 4$, *is a Gray code.*

Proof. By Lemma 2, the transition between $\psi(i, \mathscr{S}_{n-1,n-2})$ and $\psi(j, \overline{\mathscr{S}_{n-1,n-2}})$, $1 \leqslant i, j \leqslant n-1$, (or respectively between $\psi(j, \overline{\mathscr{S}_{n-1,n-2}})$ and $\psi(i, \mathscr{S}_{n-1,n-2})$) is a product by a three-length cycle. It remains to examine the transitions between $\psi(n-1, \mathscr{S}_{n-1,n-2})^{(n-1)}$ and $\phi(\mathscr{S}_{n-1,n-3})$, and between $\phi(\mathscr{S}_{n-1,n-3})$ and $\psi(2, \overline{\mathscr{S}_{n-1,n-2}})$

$$\text{last}(\psi(n-1, \mathscr{S}_{n-1,n-2})^{(n-1)}) = \begin{cases}(n-1, 2, 3, \ldots, n-3, n-2, n, 1) & \text{if } n \text{ even}, \\ (n-2, 2, 3, \ldots, n-3, 1, n, n-1) & \text{if } n \text{ odd}\end{cases}$$

Please cite this article as: Jean-Luc Baril, Gray code for permutations with a fixed number of cycles, Discrete Mathematics (2006), doi: 10.1016/j.disc.2006.09.007

and

$$\begin{aligned}\text{first}(\phi(\mathscr{S}_{n-1,n-3})) &= (n-1, 2, 3, \ldots, n-3, 1, n-2, n)\\ &= \begin{cases}\text{last}(\psi(n-1, \mathscr{S}_{n-1,n-2})^{n-1}) \cdot \langle n-2, n-1, n\rangle & \text{if } n \text{ even},\\ \text{last}(\psi(n-1, \mathscr{S}_{n-1,n-2})^{n-1}) \cdot \langle 1, n-1, n\rangle & \text{if } n \text{ odd}.\end{cases}\end{aligned}$$

Moreover,

$$\begin{aligned}\text{last}(\phi(\mathscr{S}_{n-1,n-3})) &= (n-2, n-1, 3, 4, \ldots, n-3, 1, 2, n)\\ &= (n-2, n, 3, 4, \ldots, n-3, 1, n-1, 2) \cdot \langle 2, n-1, n\rangle\\ &= \psi(2, \text{last}(\mathscr{S}_{n-1,n-2})) \cdot \langle 2, n-1, n\rangle\\ &= \text{first}(\psi(2, \overline{\mathscr{S}_{n-1,n-2}})) \cdot \langle 2, n-1, n\rangle. \qquad \square\end{aligned}$$

3.5. The case $2 \leqslant k \leqslant n-3$

If $2 \leqslant k \leqslant n-3$, we define

$$\mathscr{S}_{n,k} = \psi(1, \mathscr{S}_{n-1,k}) \circ \bigcirc_{i=n-1}^{k+1} \psi(i, \mathscr{S}_{n-1,k})^{(i)} \circ \phi(\mathscr{S}_{n-1,k-1})^{(k)} \circ \bigcirc_{i=k}^{2} \psi(i, \mathscr{S}_{n-1,k})^{(i-1)}. \tag{e}$$

Lemma 11. *If* $2 \leqslant k \leqslant n-3$ *then*

(1) $f_{n,k} = (n, 2, 3, \ldots, k, 1, k+1, k+2, \ldots, n-1)$

(2) $\ell_{n,k} = \begin{cases}(n-1, n, 3, \ldots, k, 1, k+1, k+2, \ldots, n-2, 2) & \textit{if } k \neq 2,\\ (n-1, n, 1, 3, \ldots, n-2, 2) & \textit{otherwise}.\end{cases}$

Proof. By recurrence on n, we have $f_{n,k}$=first($\psi(1, \mathscr{S}_{n-1,k})$)=$\psi(1, f_{n-1,k})$ anchored by $f_{k+2,k}$=$(k+2, 2, 3, \ldots, k, 1, k+1)$. Similarly, $\ell_{n,k} = \text{last}(\psi(2, \overline{\mathscr{S}_{n-1,k}})) = \psi(2, f_{n-1,k})$ also anchored by $f_{k+2,k} = (k+2, 2, 3, \ldots, k, 1, k+1)$. $\square$

Proposition 12. *The list* $\mathscr{S}_{n,k}$ *defined by* (e), $2 \leqslant k \leqslant n-3$ *is a Gray code.*

Proof. We use Lemma 2 for the transitions from $\psi(i, S_{n-1,k})$ to $\psi(j, \overline{S_{n-1,k}})$ and from $\psi(j, \overline{S_{n-1,k}})$ to $\psi(i, S_{n-1,k})$. It remains to examine the three transitions (i) $\psi(1, S_{n-1,k})$ and $\psi(n-1, S_{n-1,k})$; (ii) $\psi(k+1, \mathscr{S}_{n-1,k})^{(k+1)}$ and $\phi(\mathscr{S}_{n-1,k-1})^{(k)}$; (iii) $\phi(\mathscr{S}_{n-1,k-1})^{(k)}$ and $\psi(k, \mathscr{S}_{n-1,k})^{(k-1)}$.

Case (i): If $k \neq 2$,

$$\begin{aligned}\text{last}(\psi(1, \mathscr{S}_{n-1,k})) &= \psi(1, \text{last}(\mathscr{S}_{n-1,k}))\\ &= \psi(1, (n-2, n-1, 3, \ldots, k, 1, k+1, \ldots, n-3, 2))\\ &= \psi(1, (n-1, 2, 3, \ldots, k, 1, k+1, \ldots, n-3, n-2) \cdot \langle 1, n-1, 2\rangle).\end{aligned}$$

By Lemma 3,

$$\begin{aligned}\text{last}(\psi(1, \mathscr{S}_{n-1,k})) &= \psi(n-1, (n-1, 2, 3, \ldots, k, 1, k+1, \ldots, n-3, n-2)) \cdot \langle 2, n-1, 1\rangle\\ &= (n-1, 2, 3, \ldots, k, 1, k+1, \ldots, n-3, n, n-2) \cdot \langle 2, n-1, 1\rangle\\ &= \psi(n-1, \text{first}(\mathscr{S}_{n-1,k})) \cdot \langle 2, n-1, 1\rangle.\end{aligned}$$

The case $k = 2$ is similar.

Case (ii): Let $\sigma = \text{first}(\mathscr{S}_{n-1,k})$ (resp. $\text{last}(\mathscr{S}_{n-1,k})$) and $\gamma = \text{first}(\mathscr{S}_{n-1,k-1})$ (resp. $\text{last}(\mathscr{S}_{n-1,k-1})$). We have $\sigma = \gamma \cdot \langle k, k+1\rangle$. By Lemma 4,

$$\begin{aligned}\psi(k+1, \sigma) &= \psi(k+1, \gamma \cdot \langle k, k+1\rangle)\\ &= \phi(\gamma) \cdot \langle k+1, n, k\rangle.\end{aligned}$$

Case (iii): The proof is similar as for the case (ii). $\square$

Please cite this article as: Jean-Luc Baril, Gray code for permutations with a fixed number of cycles, Discrete Mathematics (2006), doi: 10.1016/j.disc.2006.09.007

Table 1
The lists $\mathscr{S}_{3,k}$, $1 \leqslant k \leqslant 3$, and $\mathscr{S}_{4,k}$ for $1 \leqslant k \leqslant 4$

$\mathscr{S}_{3,1}$		$\mathscr{S}_{3,2}$		$\mathscr{S}_{3,3}$		$\mathscr{S}_{4,1}$		$\mathscr{S}_{4,2}$		$\mathscr{S}_{4,3}$		$\mathscr{S}_{4,4}$	
1	**312**	1	**321**	1	**123**	1	**4123**	1	**4213**	1	**4231**	1	**1234**
2	*231*	2	*132*			2	**4312**	2	**4321**	2	*1243*		
		3	**213**			3	*2413*	3	**4132**	3	**1432**		
						4	*3421*	4	*2143*	4	*2134*		
						5	**2341**	5	*1342*	5	*1324*		
						6	**3142**	6	*3241*	6	*3214*		
								7	**3124**				
								8	**2314**				
								9	*2431*				
								10	*1423*				
								11	*3412*				

For instance, in $\mathscr{S}_{4,2}$ the sublists of relation (d), $\psi(1, \mathscr{S}_{3,2})$, $\overline{\psi(3, \mathscr{S}_{3,2})}$, $\phi(\mathscr{S}_{3,1})$ and $\overline{\psi(2, \mathscr{S}_{3,2})}$, are alternatively in bold-face and italic.

Note that our Gray code $\mathscr{S}_{n,k}$ is cyclic and optimal for all n, k. See Table 1 for some examples.

Definition 13. For a list of permutations $\mathscr{L}$ let denote $\mathscr{L}^{-1}$ the list obtained from $\mathscr{L}$ by replacing each permutation in $\mathscr{L}$ by its group theoretical inverse.

By a simple calculation we have:

Theorem 14. *The list $\mathscr{S}_{n,k}^{-1}$ is also an optimal cyclic Gray code for n-length permutations with k cycles*; *i.e. two successive permutations differ by a product with a three-cycle*.

4. Transposition array representation

Here we show that replacing each permutation in the list $\mathscr{S}_{n,k}$ by its transposition array (defined below) the obtained list still remains an optimal Gray code. Its interest is twofold. First, the recursive definition of $\mathscr{S}_{n,k}$ and the resulting generating algorithm in the next section express subsequently permutations in transposition array representation. Secondly, this shows how a map, which generally does not preserve closeness, transforms a Gray code into another one. See for instance [3] for more about closeness preserving bijections.

Any permutation $\pi \in S_n$ has a unique decomposition as a product of transpositions

$$\pi = \langle p_1, 1\rangle \cdot \langle p_2, 2\rangle \cdot \langle p_3, 3\rangle \cdot \ldots \cdot \langle p_n, n\rangle = \prod_{i=1}^{n} \langle p_i, i\rangle \tag{4}$$

with $1 \leqslant p_i \leqslant i$ and obviously any such decomposition provides a permutation. So (4) yields a bijection $S_n \longrightarrow T_n$ with T_n the product set $[1] \times [2] \times \cdots \times [n]$, and a string $p_1 p_2 p_3 \ldots p_n \in T_n$ is yet another way to represent a permutation called *transposition array*. Alternatively, a string $p_1 p_2 p_3 \ldots p_n \in T_n$ can be viewed as the *inversion table* of a permutation $\sigma \in S_n$: p_i is the number of entries $\sigma_j > \sigma_i$, $j < i$, plus 1.

The relation between the two permutations, one obtained by regarding a given string in T_n as its transposition array and the other as its inversion table, has never been studied and this might be an interesting direction of research.

Let now $T_{n,k}$ be the set of strings in T_n with exactly k fixed points, that is, k entries p_i with $p_i = i$. Since the number of 'identity transpositions' $\langle i, i\rangle$ in (4) equals the number of cycles in π, $T_{n,k}$ is the set of transposition arrays of permutations in $S_{n,k}$ and relation (4) induces a bijection $S_{n,k} \longrightarrow T_{n,k}$.

On the other hand, $T_{n,k}$ is the set of inversion tables of permutations in S_n having k *left-to-right minima*; a left-to-right minimum in a permutation is an entry less than all the entries to its left. As a consequence, we obtain the following enumerative result: the number of permutations in S_n with k left-to-right minima is the signless Stirling number $s(n, k)$. See also [13].

Please cite this article as: Jean-Luc Baril, Gray code for permutations with a fixed number of cycles, Discrete Mathematics (2006), doi: 10.1016/j.disc.2006.09.007

Table 2
The lists $\mathscr{S}_{4,2}$ and $\mathscr{T}_{4,2}$

$\mathscr{S}_{4,2}$	$\mathscr{T}_{4,2}$
4213	1211
4321	1221
4132	1131
2143	1133
1342	1223
3241	1213
3124	1114
2314	1124
2431	1132
1423	1222
3412	1212

Each string in $\mathscr{T}_{4,2}$ is the transposition array of the corresponding permutation in $\mathscr{S}_{4,2}$.

Generally, a bijection between two sets can magnify small changes between objects and this is the case with the bijection $S_n \longrightarrow T_n$ defined in (4). For example, if π and π' are two permutations such that $\pi' = \pi \cdot \langle a, b, c\rangle$ then the decomposition of π' differ from π by possibly many transpositions: take $\pi = (7, 1, 2, 3, 4, 5, 6) = \langle 1, 1\rangle \cdot \langle 1, 2\rangle \cdot \langle 1, 3\rangle \cdot \langle 1, 4\rangle \cdot \langle 1, 5\rangle \cdot \langle 1, 6\rangle \cdot \langle 1, 7\rangle$ and $\pi' = (2, 1, 4, 3, 7, 5, 6) = \pi \cdot \langle 1, 3, 5\rangle = \langle 1, 1\rangle \cdot \langle 1, 2\rangle \cdot \langle 3, 3\rangle \cdot \langle 3, 4\rangle \cdot \langle 5, 5\rangle \cdot \langle 5, 6\rangle \cdot \langle 5, 7\rangle$. Conversely, if two decompositions differ by at most two transpositions, then the corresponding one-line permutations can possibly differ by many entries: consider $\pi = (2, 3, 4, 5, 6, 1) = \langle 1, 1\rangle \cdot \langle 1, 2\rangle \cdot \langle 2, 3\rangle \cdot \langle 3, 4\rangle \cdot \langle 4, 5\rangle \cdot \langle 5, 6\rangle$ and $\pi' = (2, 4, 1, 6, 3, 5) = \langle 1, 1\rangle \cdot \langle 1, 2\rangle \cdot \langle 2, 3\rangle \cdot \langle 2, 4\rangle \cdot \langle 4, 5\rangle \cdot \langle 4, 6\rangle$.

Let now $\mathscr{T}_{n,k}$ be the list for the set $T_{n,k}$ obtained by replacing each permutation in the list $\mathscr{S}_{n,k}$ by its transposition array. Surprisingly, in $\mathscr{T}_{n,k}$ two consecutive sequences differ in at most two positions and so it is a Gray code (see Table 2). This result is shown in the next theorem and despite its very similarity with Theorem 14 they are not a simple consequence each other.

Definition 15. With the notations above, the permutation

$$\tau = \prod_{i=n-j+1}^{n} \langle p_i, i\rangle$$

in S_n is called the *jth* characteristic of π. The nth characteristic of an n-length permutation is the permutation itself, and it is convenient to consider that the 0th characteristic of any permutation is the identity.

Remark 16. Let τ be the *jth* characteristic of π as above.

(1) The number of cycles in τ is $(n - \ell)$, where ℓ is the number of p_i, $n - j + 1 \leqslant i \leqslant n$, with $p_i < i$,
(2) If τ and π have the same number of cycles then $\tau = \pi$.

Remark 17. For π and σ, the following are equivalent:

(1) π and σ have the same *jth* characteristic,
(2) π and σ have the same *ith* characteristic for i from 0 to j,
(3) π and σ have the entries from $n - j + 1$ to n in the same respective positions,
(4) π^{-1} and σ^{-1} have the same entries for each position from $n - j + 1$ to n.

Now let consider the transformations ψ and ϕ in Definition 1. If γ is a permutation in S_{n-1} and $\pi = \psi(i, \gamma) \in S_n$ then the first characteristic of π is $\langle i, n\rangle$; and if $\pi = \phi(\gamma) \in S_n$ then the first characteristic of π is the identity (actually, the 'transposition' $\langle n, n\rangle$). Similar results hold if we replace γ and π by sets (or lists) of permutations. Generally, if $\mathscr{A}$ is a list of permutations in S_{n-j} and

$$\mathscr{B} = \alpha_1(\alpha_2(\ldots \alpha_j(\mathscr{A}) \ldots)) \in S_n$$

with $\alpha_i(\cdot)$ of the form $\phi(\cdot)$ or $\psi(x_i, \cdot)$ then each permutation in $\mathscr{B}$ has the jth characteristic

$$\prod_{i=n-j+1}^{n} \langle p_i, i\rangle,$$

where $p_i = x_{n-i+1}$ if $\alpha_i(\cdot) = \psi(p_i, \cdot)$, and $x_i = i$ if $\alpha_i(\cdot) = \phi(\cdot)$.

By recursivity of relations (a)–(e) we have:

Remark 18. For any $\tau \in S_{n,k}$, all permutations with the same *jth* characteristic as τ, $0 \leqslant j \leqslant n$, form a contiguous sublist of the list $\mathscr{S}_{n,k}$ defined by the appropriate relation (a)–(e). Actually, j is the depth of recursivity involved to reach this sublist.

Theorem 19. *The list $\mathscr{T}_{n,k}$ is a cyclic Gray code for n-length permutations with k cycles; i.e. two successive permutations differ by one or two transpositions in their decompositions.*

Proof. Let π and σ be two successive permutations in $\mathscr{S}_{n,k}$. We will show that their transposition arrays differ in at most two positions. π and σ belong to the same sublist of $\mathscr{S}_{n,k}$ as in relation (3) iff they have the same first characteristic. By induction, it is enough to show that for two successive sublists $\alpha(\mathscr{A})$ and $\beta(\mathscr{B})$ in the definition of $\mathscr{S}_{n,k}$ (with α, β as $\phi(\cdot)$ or $\psi(i, \cdot)$) the transposition arrays of last($\mathscr{A}$) and first($\mathscr{B}$) differ in at most one position. Indeed, we give below the transposition array representations of the first element $f_{n,k}$ and the last element $\ell_{n,k}$ of the list $\mathscr{S}_{n,k}$ and with relations (a)–(e) we can verify that last($\mathscr{A}$) and first($\mathscr{B}$) differ in at most one position

$$f_{n,k} = \begin{cases} 1\ 1 \ldots 1 & \text{if } k = 1, \\ 1\ 2 \ldots (n-1)n & \text{if } k = n, \\ 1\ 2 \ldots (n-1)1 & \text{if } k = n-1, \\ 1\ 2 \ldots (n-2)11 & \text{if } k = n-2, \\ 1\ 2 \ldots k11 \ldots 1 & \text{otherwise,} \end{cases}$$

$$\ell_{n,k} = \begin{cases} 11 \ldots 13 & \text{if } k = 1, \\ 12 \ldots (n-1)n & \text{if } k = n, \\ 12 \ldots (n-2)1n & \text{if } k = n-1, \\ 12 \ldots (n-2)12 & \text{if } k = n-2, \\ 12 \ldots k11 \ldots 12 & \text{otherwise.} \end{cases} \qquad \square$$

Remark 20.

- $\mathscr{T}_{n,k}$ is at the same time a Gray code for the set of n-length permutations with k cycles in transposition array representation, and for the set of n-length permutations with k left-to-right minima in inversion table representation.
- $\mathscr{T}_{n,k}$ is suffix partitioned and satisfies Walsh's desiderata, so looplessly implementable [17] .

5. Algorithmic considerations

In this part, we explain how the recursive definitions (a)–(e) can be implemented into an efficient algorithm, i.e. in a CAT algorithm. Such algorithms already exist for derangements or involutions [1,9,10], so we will just give here the main difficulties to implement our one.

Before the main call of procedure *gen_up(n,k)* given in Appendix, τ is initialized by the identity, so it has n cycles and is the 0th characteristic of the permutations in $\mathscr{S}_{n,k}$. Before each recursive call of *gen_up*(n, k) or *gen_down*(n, k), τ is, for some j, the jth characteristic of a sublist of permutations in $\mathscr{S}_{n,k}$.

(1) If τ has k cycles, then τ is printed, and no recursive call is produced. This corresponds to point 2 of Remark 16 and to the recursive definition (b).

Please cite this article as: Jean-Luc Baril, Gray code for permutations with a fixed number of cycles, Discrete Mathematics (2006), doi: 10.1016/j.disc.2006.09.007

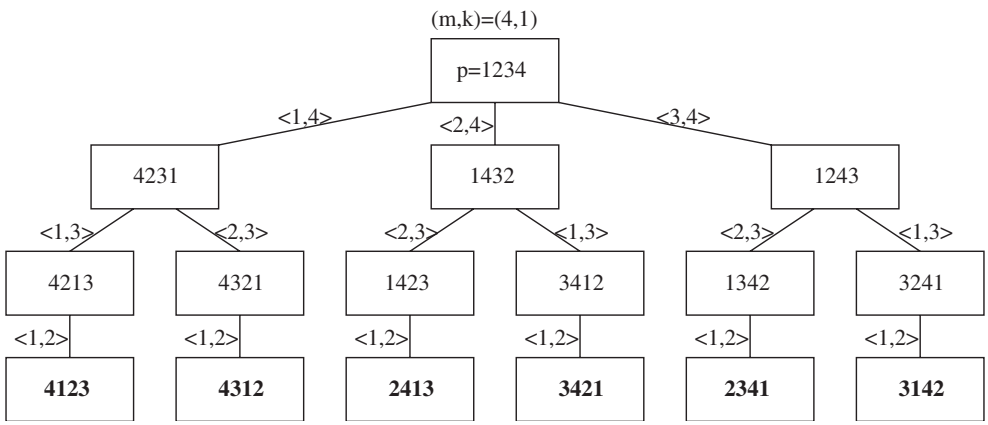


Fig. 2. The generating tree for the list $\mathscr{S}_{4,1}$ with the decomposition in a product of transpositions of each element. The list $\mathscr{S}_{4,1}$ appears in bold.

(2) If τ has $(k + (n - j))$ cycles then $(n - j - 1)$ calls are produced and before each of them τ is updated as $\tau := \langle i, n - j\rangle \cdot \tau$, with $i = 1, 2, \ldots, n - j - 1$ (not necessarily in this order). Each τ updated like that is a $(j + 1)$th characteristic obtained by a premultiplication by a (not dummy) transposition of the previous jth characteristic. The number of cycles in τ decreases by one. This corresponds to the recursive definition (a) which gives the order of calls.
(3) Similarly, if the number of cycles in τ is more than k but less than $(k + (n - j))$, then $(n - j)$ calls are produced, $(n - j - 1)$ of them are those of the point above. Before the additional call, τ, the new $(j + 1)$th characteristic is unchanged (τ is 'updated' as $\tau := \langle n - j, n - j\rangle \cdot \tau$). This corresponds to the recursive definitions (c)–(e), and again they give the order of calls.

In our algorithm j is the depth of the recursive call and the order of successive calls directly produced by a given call is determined by the appropriate definition (a), (c) and (d). See Fig. 2 for the generating tree of $\mathscr{S}_{4,1}$.

This algorithm enable us to ensure that this transforms an object into its successor in CAT. Indeed, excepted for the calls where $n = 2$, $k = 1$, all calls have a degree 0 or at least 2 (see Fig. 2 for an example). Between two recursive calls, we execute two transpositions, and moreover at least one permutation is generated in each recursive call. This means that the total amount of computation divided by the number of objects is bounded by a constant (see [14]). Thus the complexity of this algorithm is $\mathcal{O}(s_{n,k})$. A Java implementation of our algorithm is available from the author on request.

6. Uncited reference

[8].

Acknowledgments

I thank V. Vajnovszki and the anonymous referees for their constructive remarks which have greatly improved this paper.

Appendix A.

The call of *gen_up(n,k)* generates the list $\mathscr{S}_{n,k}$. In order to produce $\overline{\mathscr{S}_{n,k}}$ we consider also the procedure *gen_down(n,k)* which has the same instructions of *gen_up(n,k)* in the inverse order. The notation $gen\_up/down(n-1, k)$ means that we use *gen_up(n,k)* or *gen_down(n,k)* according to the sense of each sublist in relations (a)–(e). To multiplicate (on the left) a jth characteristic σ by a transposition $\langle x, n\rangle$, we conserve the inverse σ^{-1} at each level of the recursivity, and we calculate the product $\sigma \cdot \langle \sigma^{-1}(x), \sigma^{-1}(n)\rangle$, i.e we just need to exchange the positions $\sigma^{-1}(x)$ and $\sigma^{-1}(n) = n$ in σ.

Please cite this article as: Jean-Luc Baril, Gray code for permutations with a fixed number of cycles, Discrete Mathematics (2006), doi: 10.1016/j.disc.2006.09.007

```
procedure gen_up(n, k)
var i,j;
begin
if n = k
then print(σ);
else if n = 4 and k = 1
    then σ := ⟨1, n⟩ · σ; gen_up(n − 1, k); σ := ⟨1, n⟩ · σ;
         σ := ⟨2, n⟩ · σ; gen_down(n − 1, k); σ := ⟨2, n⟩ · σ;
         σ := ⟨3, n⟩ · σ; gen_down(n − 1, k); σ := ⟨3, n⟩ · σ;
    else
        if k = n − 1
        then σ := ⟨1, n⟩ · σ; gen_up(n − 1, k); σ := ⟨1, n⟩ · σ;
             if n − 1 > 1 then
                σ := ⟨n − 1, n⟩ · σ; gen_up(n − 1, k); σ := ⟨n − 1, n⟩ · σ;
             endif
             for j = n − 3 downto 2 do
                 σ := ⟨j, n⟩ · σ; gen_up/down(n − 1, k); σ := ⟨j, n⟩ · σ;
             enddo
             if n − 2 > 1 then
                σ := ⟨n − 2, n⟩ · σ; gen_up(n − 1, k); σ := ⟨n − 2, n⟩ · σ;
             endif
             if k ⩾ 2 then
                σ := ⟨n, n⟩ · σ; gen_down(n − 1, k − 1);
             endif
        else
            if k = n − 2
            then σ := ⟨1, n⟩ · σ; gen_up(n − 1, k); σ := ⟨1, n⟩ · σ;
                 for j = 3 to n − 1 do
                     σ := ⟨j, n⟩ · σ; gen_up/down(n − 1, k); σ := ⟨j, n⟩ · σ;
                 endo
                 if k ⩾ 2 then
                    σ := ⟨n, n⟩ · σ; gen_up(n − 1, k − 1);
                 σ := ⟨2, n⟩ · σ; gen_down(n − 1, k); σ := ⟨2, n⟩ · σ;
                 endif
            else
                if k = 1
                then σ := ⟨1, n⟩ · σ; gen_up(n − 1, k); σ := ⟨1, n⟩ · σ;
                     for j = n − 1 downto 4 do
                         σ := ⟨j, n⟩ · σ;gen_up/down(n − 1, k); σ := ⟨j, n⟩ · σ;
                     endo
                     σ := ⟨2, n⟩ · σ; gen_up(n − 1, k); σ := ⟨2, n⟩ · σ;
                     σ := ⟨3, n⟩ · σ; gen_down(n − 1, k); σ := ⟨3, n⟩ · σ;
                else
                     σ := ⟨1, n⟩ · σ; gen_up(n − 1, k); σ := ⟨1, n⟩ · σ;
                     for j = n − 1 downto k + 1 do
                         σ := ⟨j, n⟩ · σ; gen_up/down(n − 1, k); σ := ⟨j, n⟩ · σ;
                     endo
                     if k ⩾ 2 then σ := ⟨n, n⟩ · σ; gen_up/down(n − 1, k − 1); endif
                     for j = k downto 2 do
                         σ := ⟨j, n⟩ · σ;  gen_up/down(n − 1, k); σ := ⟨j, n⟩ · σ;
                     endo
                endif
            endif
        endif
    endif
end
```

References

[1] J.L. Baril, V. Vajnovszki, Gray code for derangements, Discrete Appl. Math. 140 (1–3) (2004) 207–221.

[2] W. Chu, Ch. Colbourn, P. Dukes, Constructions for permutation codes in powerline communications, Designs, Codes and Cryptography 32 (2004) 51–64.

Please cite this article as: Jean-Luc Baril, Gray code for permutations with a fixed number of cycles, Discrete Mathematics (2006), doi: 10.1016/j.disc.2006.09.007

[3] A. Juarna, V. Vajnovszki, Isomorphism between classes counted by Fibonacci numbers, Words 2005, September 2005, pp. 51–62.
[4] D.E. Knuth, The Art of Computer Programming, Combinatorial Algorithms, vol. 4, Pre-fascicle 2b, Generating all permutations, 2002.
[5] J.F. Korsh, Loopless generation of up–down permutations, Discrete Math. 240 (1–3) (2001) 97–122.
[6] D. Roelants van Baronaigien, Constant time generation of involutions, Congr. Numer. 90 (1992) 87–96.
[7] F. Ruskey, ⟨http://www.theory.cs.uvic.ca/∼cos/gen/perm.html⟩.
[8] F. Ruskey, Simple combinatorial Gray codes constructed by reversing sublists in ISAAC conference, Lecture Notes in Computer Science, vol. 762, Springer, Berlin, 1993, pp. 201–208.
[9] F. Ruskey, S. Effler, A CAT algorithm for generating permutations with a fixed number of inversions, Inform. Process. Lett. 86 (2) (2003) 107–112.
[10] F. Ruskey, D. Roelants van Baronaigien, Generating permutations with given ups and downs, Discrete Appl. Math. 36 (1) (1992) 57–67.
[11] F. Ruskey, U. Taylor, Fast generation of restricted classes of permutations, Manuscript, 1995.
[12] R. Sedgewick, Permutation generation methods, Comput. Surveys 9 (2) (1977) 137–164.
[13] R. Stanley, Enumerative Combinatorics, vol. 1, Cambridge University Press, Cambridge, England, 1997.
[14] V. Vajnovszki, Gray visiting Motzkins, Acta Inform. 38 (2002) 793–811.
[15] V. Vajnovszki, A loopless algorithm for generating the permutations of a multiset, Theoret. Comput. Sci. 307 (2003) 415–431.
[16] T. Walsh, Gray code for involutions, J. Combin. Math. Combin. Comput. 36 (2001) 95–118.
[17] T. Walsh, Generating Gray codes in O(1) worst-case time per word, Lecture Notes in Computer Science, vol. 2731, 2003, pp. 73–88.
[18] R.M. Wilson, J.H. van Lint, A Course in Combinatorics, Cambridge University Press, Cambridge, 2002.

Please cite this article as: Jean-Luc Baril, Gray code for permutations with a fixed number of cycles, Discrete Mathematics (2006), doi: 10.1016/j.disc.2006.09.007

Journal of Discrete Mathematical Sciences and Cryptography

ISSN: 0972-0529 (Print) 2169-0065 (Online) Journal homepage: https://www.tandfonline.com/loi/tdmc20

Hamiltonian paths for involutions in the square of a Cayley graph

Jean-Lue Baril

To cite this article: Jean-Lue Baril (2007) Hamiltonian paths for involutions in the square of a Cayley graph, Journal of Discrete Mathematical Sciences and Cryptography, 10:4, 473-484, DOI: 10.1080/09720529.2007.10698133

To link to this article: https://doi.org/10.1080/09720529.2007.10698133

Published online: 03 Jun 2013.

Submit your article to this journal

Article views: 14

View related articles

Full Terms & Conditions of access and use can be found at
https://www.tandfonline.com/action/journalInformation?journalCode=tdmc20

Hamiltonian paths for involutions in the square of a Cayley graph

Jean-Lue Baril *

LE21 UMR-CNRS 5158
Université de Bourgogne
B.P. 47 870
21078 DIJON-Cedex
France

Abstract

In this paper we give a new construction of a Hamiltonian cycle for the restriction to involutions of the graph $G(S_n)^2$, where $G(S_n)$ is the Cayley graph of S_n with respect to the transpositions. Then we construct a Hamiltonian path in the restriction to *fixed-point-free* involutions of $G(S_n)^2$ and by extension we obtain a Hamiltonian path for the restriction of $G(S_n)^2$ to involutions with a given number of fixed points.

Keywords and phrases : *Hamiltonian path, Cayley graph, permutations, involutions.*

1. Introduction

Let S_n be the set of n-length permutations. We represent permutations in one-line notation, i.e., $\pi = (i_1, i_2, \ldots, i_n)$ iff $\pi(k) = i_k$, $1 \leq k \leq n$. If $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ is an n-length permutation then $\sigma \cdot \pi$ is the permutation $(\sigma_{\pi(1)}, \sigma_{\pi(2)}, \ldots, \sigma_{\pi(n)})$. Let $i_1, i_2, \ldots, i_k$ be k different values in $[n]$, $1 \leq k \leq n$. The cycle $\gamma = \langle i_1, i_2, \ldots, i_k \rangle$ is the following permutation: $\gamma(i_1) = i_2$, $\gamma(i_2) = i_3, \ldots, \gamma(i_{k-1}) = i_k$, $\gamma(i_k) = i_1$, and $\gamma(j) = j$ for all $j \neq i_\ell$, $1 \leq \ell \leq k$. A two length cycle is a *transposition*. We let by T_n be the set of transpositions of S_n. For a permutation $\sigma \in S_n$, transposing the *positions* i and j correspond to the product $\sigma \cdot \langle i, j \rangle$ and transposing the *values* x and y correspond to the product $\langle x, y \rangle \cdot \sigma$.

E-mail: barjl@u-bourgogne.fr

Journal of Discrete Mathematical Sciences & Cryptography
Vol. 10 (2007), No. 4, pp. 473–484
© Taru Publications

An n-length *involution* is a permutation $\pi \in S_n$ such that π is equal to its inverse π^{-1}. An n-length *fixed-point-free* involution, or $fpfi$, is an involution with no fixed point, i.e., an involution σ such that there exists no i verifying $\sigma(i) = i$. Note that the number of fixed points in an n-length involution has the same parity as n. If I_n is the set of all n-length involutions, a recurrence relation for $i_n = \mathrm{card}(I_n)$ is given by:

$$i_n = i_{n-1} + (n-1) \cdot i_{n-2} \tag{1}$$

for $n \geq 3$, with $i_1 = 1$ and $i_2 = 2$, see for instance [4]. Moreover, if I'_n is the set of all n-length $fpfi$, a recurrence relation for $i'_n = \mathrm{card}(I'_n)$ is given by:

$$i'_n = (n-1) \cdot i'_{n-2} \tag{2}$$

for $n \geq 3$, with $i'_1 = 0$ and $i'_2 = 1$, see [11]. Note that I'_{2n+1} is empty. In the following, we will let $I_{n,2m}$ be the set of n-length involutions with $(n-2m)$ fixed points.

A path $\mathcal{P}$ for a graph is a sequence (or an ordered list) of vertices of G such that two successive vertices on the path are connected by an edge. $\mathrm{first}(\mathcal{P})$ is the first vertex and $\mathrm{last}(\mathcal{P})$ the last vertex of the path $\mathcal{P}$; $\mathcal{P}$ is Hamiltonian if each vertex of the graph appears exactly once. Moreover, if the first and the last vertex are connected by an edge, we say that $\mathcal{P}$ is a, Hamiltonian cycle. $\overline{\mathcal{P}}$ is the path obtained by reversing $\mathcal{P}$, and obviously $\mathrm{first}(\mathcal{P}) = \mathrm{last}(\overline{\mathcal{P}})$ and $\mathrm{first}(\overline{\mathcal{P}}) = \mathrm{last}(\mathcal{P})$; $\mathcal{P}^{(i)}$ is the path $\mathcal{P}$ if i is even, and $\overline{\mathcal{P}}$ if i is odd; if $\mathcal{P}_1$ and $\mathcal{P}_2$ are two paths such that $\mathrm{first}(\mathcal{P}_2)$ and $\mathrm{last}(\mathcal{P}_1)$ are connected by an edge in the graph, then we define the path $\mathcal{P}_1 \circ \mathcal{P}_2$ as the concatenation of $\mathcal{P}_1$ and $\mathcal{P}_2$; more generally, if $\mathrm{first}(\mathcal{P}_{i+1})$ and $\mathrm{last}(\mathcal{P}_i)$ are connected, $1 \leq i \leq n-1$, the path $\mathcal{P}_1 \circ \mathcal{P}_2 \circ \ldots \circ \mathcal{P}_n$ is denoted $\bigcirc_{i=1}^{n} \mathcal{P}_i$.

The *Cayley graph* $G(S_n)$ of S_n with respect to the set of transpositions T_n is defined in the following way: the vertices are the elements of S_n and an edge connects two permutations π and π' iff they differ by a product of only one transposition, i.e., $\pi = \pi' \cdot \tau$ where $\tau \in T_n$. See [8, 5, 2] for instance.

Since the restriction of involutions has no Hamiltonian path, in the following we study the second power $G(S_n)^2$ which is induced from $G(S_n)$ by adding the edges between two vertices of distance two in $G(S_n)$.

In the following, we denote by $\frac{G(S_n)^2}{I_n}$ the restriction to involutions of the graph $G(S_n)^2$. Note immediately that $\frac{G(S_n)^2}{I_n}$ is not isomorphic to $\left(\frac{G(S_n)}{I_n}\right)^2$ since we have strictly $\left(\frac{G(S_n)}{I_n}\right)^2 \subset \frac{G(S_n)^2}{I_n}$, for $n \geq 4$. For instance, the two involutions $(2,1,4,3)$ and $(4,3,2,1)$ are connected in $\frac{G(S_4)^2}{I_4}$ via the common neighbours $(2,3,4,1)$, which is not an involution, but not in $\left(\frac{G(S_4)}{I_n}\right)^2$ because none of their common neighbours are involutions.

In the next sections, we construct new Hamiltonian paths in $\frac{G(s_n)^2}{I_n}$; more precisely, this means that each involution in the path is obtained from the previous one via a product of one or two transpositions. We will define this as $\mathfrak{I}_n$ ($\mathfrak{I}'_n$ and $\mathfrak{I}_{n,2m}$ respectively) a Hamiltonian path for the graph $\frac{G(S_n)^2}{I_n}$ ($\frac{G(S_n)^2}{I'_n}$ and $\frac{G(S_n)^2}{I'_{n,2m}}$, respectively).

Various studies have been conducted looking at implementing algorithms for generating involutions [6, 11, 12] by using the terminology of list or Gray codes. Recall that a k-Gray code for a set is an exhaustive ordered list such that two consecutive elements in the list differ in at most k positions. Obviously, a Hamiltonian path $\mathfrak{I}_n$ of $\frac{G(S_n)^2}{I_n}$ can also be considered as a 4-Gray code for the set of involutions.

T. Walsh [11, 12] exhibits a 4-Gray code for involutions (or Hamiltonian path for $\frac{G(S_n)^2}{I_n}$), but the advantage of this paper is that we present by induction a simple construction of an isomorphic Hamiltonian path. We also give a new Hamiltonian path for $\frac{G(S_n)^2}{I'_n}$ and we extend it to obtain another Hamiltonian path for $\frac{G(S_n)^2}{I_{n,2m}}$. The improvement of this paper is that we obtain a recursive construction and derivations simpler than in [11].

2. Hamiltonian cycle for $\frac{G(S_n)^2}{I_n}$

In this section, we show how the set I_n can be recursively constructed from I_{n-1} and I_{n-2}, and then we extend this construction to lists of involutions in order to obtain a Hamiltonian path of $\frac{G(S_n)^2}{I_n}$.

Let τ be an $(n-1)$-length involution, $n \geq 2$. If we denote by σ the permutation in S_n obtained from τ by appending n in the last position, then σ is an n-length involution. Similarly, if τ is an $(n-2)$-length involution, $n \geq 3$, $1 \leq i \leq n-1$, and if σ denotes the permutation in S_n obtained from τ by: (1) adding one to each entry greater than or

equal to i, (2) appending n in position i, and finally (3) appending i in the last position, then σ is an n-length involution. Moreover, each n-length involution, $n \geq 3$, can be uniquely obtained by one of these constructions.

More formally, we define below two functions ϕ_n and ψ_n.

Definition 1. For $n \geq 2$ and an involution $\tau \in I_{n-1}$, we define an n-length permutation $\sigma = \phi_n(\tau)$ by

$$\sigma(j) = \begin{cases} n\,, & \text{if } j = n\,, \\ \tau(j)\,, & \text{otherwise.} \end{cases}$$

Definition 2. For $n \geq 3$, an integer $i \in [n-1]$ and an involution $\tau \in I_{n-2}$, we define an n-length permutation $\sigma = \psi_n(i, \tau)$ by

$$\sigma(j) = \begin{cases} i & \text{if } j = n \\ n & \text{if } j = i \\ \tau(j)\,, & \text{if } j < i \text{ and } \tau(j) < i\,, \\ \tau(j) + 1\,, & \text{if } j < i \text{ and } \tau(j) \geq i\,, \\ \tau(j-1)\,, & \text{if } j > i \text{ and } \tau(j-1) < i\,, \\ \tau(j-1) + 1\,, & \text{if } j > i \text{ and } \tau(j-1) \geq i\,. \end{cases}$$

With i and τ as above, it is easy to see that:

- $\phi_n(\tau) \in I_n$ and $\phi_n : I_{n-1} \to I_n$ is a one-to-one map; and
- $\psi_n(i, \tau) \in I_n$ and $\psi_n : [n-1] \times I_{n-2} \to I_n$ is a one-to-one map.

Moreover, for each $i \in [n-1]$, we have $\psi_n(i, I_{n-2}) \cap \phi_n(I_{n-1}) = \emptyset$. Conversely, if $v \in I_n$, we have two cases: (i) $v(n) = n$ then $v \in \phi_n(I_{n-1})$ and (ii) $v(n) \neq n$ then let $i = v(n)$ then $n = v(i)$ and we have $v \in \psi_n(i, I_{n-2})$. So, for $i_n = \text{card}(I_n)$ we have $i_n = \text{card}(I_{n-1}) + (n-1) \cdot \text{card}(I_{n-2}) = i_{n-1} + (n-1) \cdot i_{n-2}$ which is a combinatorial proof of (1).

In the following, we will omit the subscript n for the functions ϕ_n and ψ_n as it should be clear from context. Also, we extend the functions ϕ and ψ in a natural way to sets, lists and paths of involutions. For each $i \in [n-1]$ and $\mathcal{L}$ a list (or path) of $(n-1)$-length involutions we have $\phi(\overline{\mathcal{L}}) = \overline{\phi(\mathcal{L})}$, $\phi(\text{first}(\mathcal{L})) = \text{first}(\phi(\mathcal{L}))$, and $\phi(\text{last}(\mathcal{L})) = \text{last}(\phi(\mathcal{L}))$. Similar results hold for the function ψ.

Let $\mathfrak{I}_n$ be the list of vertices of $\frac{G(S_n)^2}{I_n}$ recursively denned by :

$$\mathfrak{I}_n = \phi(\mathfrak{I}_{n-1}) \circ \bigcirc_{i=1}^{n-1} \psi(i, \mathfrak{I}_{n-2})^{(n+i)} \tag{3}$$

for $n \geq 3$, with $\mathfrak{I}_1 = (1)$ and $\mathfrak{I}_2 = (1,2), (2,1)$.

If f_n denotes the first involution of the list $\mathfrak{I}_n$ and ℓ_n denotes the last one, then we have:

Lemma 1. *If $n \geq 2$ then*

- $f_n = (1, 2, 3, \ldots, n-1, n)$,
- $\ell_n = (1, 2, 3, \ldots, n-2, n, n-1)$.

Proof.

1. By relation (3), $\mathfrak{I}_2$ is the list $(1,2)$, $(2,1)$ and in general $f_n = \phi(f_{n-1})$ $= (1, 2, 3, \ldots, n-1, n)$. A recurrence on n completes the proof.
2. We have $\ell_2 = (2,1)$, and for $n \geq 3$

$$\begin{aligned} \ell_n &= \text{last}\left(\overline{\psi(n-1, \mathfrak{I}_{n-2})}\right) \\ &= \text{first}(\psi(n-1, \mathfrak{I}_{n-2})) \\ &= \psi(n-1, f_{n-2}) \\ &= (1, 2, 3, \ldots, n-2, n, n-1)\,. \end{aligned}$$

Table 1

The lists of vertices $\mathfrak{I}_3$, $\mathfrak{I}_4$ and $\mathfrak{I}_5$. In $\mathfrak{I}_5$ the sublists $\phi(\mathfrak{I}_4)$ and $\psi(2, \psi_3)$, $\psi(4, \mathfrak{I}_3)$ are in bold-face respectively and font

$\mathfrak{I}_3$		$\mathfrak{I}_4$		$\mathfrak{I}_5$					
1	123	1	**1234**	1	**12345**	11	52341	21	42513
2	213	2	**2134**	2	**21345**	12	53241	22	14523
3	321	3	**3214**	3	**32145**	13	54321	23	**13254**
4	132	4	**1324**	4	**13245**	14	52431	24	**32154**
		5	4321	5	**43215**	15	**15432**	25	**21354**
		6	4231	6	**42315**	16	**45312**	26	**12354**
		7	**1432**	7	**14325**	17	**35142**		
		8	**3412**	8	**34125**	18	**15342**		
		9	2143	9	**21435**	19	12543		
		10	1243	10	**12435**	20	21543		

Remark that $f_n(j) = \ell_n(j) = j$ for all $j = 1, 2, \ldots, n-2$ and f_n differs from l_n by a product of one transposition.

The next lemma ensures a smooth transition between the sublists in relation (3), namely between: (i) the list $\psi(i, \overline{\mathfrak{I}}_{n-2})$ and $\psi(i+1, \mathfrak{I}_{n-2})$, (ii) the list $\phi(\mathfrak{I}_{n-1})$ and $\psi(1, \overline{\mathfrak{I}}_{n-2})$ with n even; and (iii) the list $\phi(i, \mathfrak{I}_{n-1})$ and $\psi(1, \mathfrak{I}_{n-2})$ with n odd. More precisely, successive involutions in $\mathfrak{I}_n$ differ either by a product of one or two transpositions or by a circular shift of three elements.

Lemma 2.

(i) *If $n \geq 3$ then*

$$\psi(i+1, f_{n-2}) = \psi(i, f_{n-2}) \cdot \langle i, n, i+1 \rangle \ \ \textit{if} \ \ 1 \leq i \leq n-2. \tag{4}$$

(ii) *If $n \geq 3$ then*

$$\psi(i+1, \ell_{n-2}) = \psi(i, \ell_{n-2}) \cdot \langle i, n, i+1 \rangle \ \ \textit{if} \ \ 1 \leq i \leq n-3 \tag{5}$$

and

$$\psi(n-1, \ell_{n-2}) = \psi(n-2, \ell_{n-2}) \cdot \langle n-2, n-1 \rangle \cdot \langle n-3, n \rangle. \tag{6}$$

(iii) *If $n \geq 3$ then*

$$\phi(\ell_{n-1}) = \begin{cases} \psi(1, f_{n-2}) \cdot \langle 1, n \rangle, & \textit{if } n \textit{ odd}, \\ \psi(1, \ell_{n-2}) \cdot \langle 1, n \rangle \cdot \langle n-1, n-2 \rangle, & \textit{if } n \textit{ even}. \end{cases} \tag{7}$$

Proof. The proof is direct, and consists essentially in checking each case.

(i) If $1 \leq i \leq n-2$ then

$$\begin{array}{rcl} & & \begin{array}{cccccccc} 1 & & i & i+1 & & & & n \end{array} \\ \psi(i{+}1, f_{n-2}) & = & (1, \ldots, i{-}1, \ \mathbf{i}, \ \ \mathbf{n}, \ \ i{+}2, \ i{+}3, \ \ldots, \ n{-}2 \ \ n{-}1, \ \mathbf{i{+}1}) \\ \psi(i, f_{n-2}) & = & (1, \ldots, i{-}1, \ \mathbf{n}, \ \mathbf{i+1}, \ i{+}2, \ i{+}3, \ \ldots, \ n{-}2 \ \ n{-}1, \ \ \mathbf{i}) \end{array}$$

(ii) If $1 \leq i \leq n-3$ then

$$\begin{array}{rcl} & & \begin{array}{cccccccc} 1 & & i & i+1 & & & & n \end{array} \\ \psi(i{+}1, \ell_{n-2}) & = & (1, \ \ldots, \ i{-}1, \ \ \mathbf{i}, \ \ \mathbf{n}, \ \ \ i{+}2, \ \ldots \ n{-}1, \ n{-}2, \ \ \mathbf{i{+}1}) \\ \psi(i, f_{n-2}) & = & (1, \ \ldots, \ i-1, \ \mathbf{n}, \ \mathbf{i+1}, \ i+2, \ \ldots \ n-1, \ n-2, \ \ \mathbf{i}) \end{array}$$

and

$$\begin{array}{rcl} \psi(n-1, \ell_{n-2}) & = & (1, \ \ldots, \ n-4, \ \mathbf{n-2}, \ \mathbf{n-3}, \ \ \mathbf{n}, \ \ \mathbf{n-1}) \\ \psi(n-2, \ell_{n-2}) & = & (1, \ \ldots, \ n-4, \ \mathbf{n-1}, \ \ \mathbf{n}, \ \ \mathbf{n-3}, \ \mathbf{n-2}) \end{array}$$

(iii)

$$\begin{array}{lcl} & & \quad 1 \qquad\qquad\qquad\qquad\qquad\qquad n-2 \quad n-1 \quad n \\ \phi(\ell_{n-1}) & = & (\mathbf{1},\ \ldots,\ i-1,\ i,\ i+1,\ \ldots,\ \mathbf{n-1},\ \mathbf{n-2},\ \mathbf{n}) \\ \psi(1,\ell_{n-2}) & = & (\mathbf{n},\ \ldots,\ i-1,\ i,\ i+1,\ \ldots,\ \mathbf{n-1},\ \mathbf{n-2},\ \mathbf{1}) \\ \psi(1,f_{n-2}) & = & (\mathbf{n},\ \ldots,\ i-1,\ i,\ i+1,\ \ldots,\ \mathbf{n-2},\ \mathbf{n-1},\ \mathbf{1}). \end{array}$$

□

Theorem 1. *The list $\mathfrak{I}_n$ is a Hamiltonian cycle in $\frac{G(S_n)^2}{I_n}$.*

Proof. In virtue of Lemma 2, two consecutive vertices of the list $\mathfrak{I}_n$ differ by a product of one or two transpositions. Thus this list corresponds to a Hamiltonian path in $\frac{G(S_n)^2}{I_n}$. □

3. Hamiltonian path for $\frac{G(S_n)^2}{I'_n}$

In this section, we define a list of $fpfi$ and we prove that it is also a Hamiltonian path for $\frac{G(S_{2n})^2}{I'_{2n}}$.

Let $\mathfrak{I}'_{2n}$ be the list for the set I'_{2n} recursively defined by:

$$\mathfrak{I}'_{2n} = \bigcirc_{i=1}^{2n-1} \psi(i, \mathfrak{I}'_{2n-2})^{(i+1)} \tag{8}$$

for $n \geq 2$, with $\mathfrak{I}'_2 = (2,1)$.

If f'_{2n} denotes the first involution in the list $\mathfrak{I}'_{2n}$ and ℓ'_{2n} denotes the last one then we have the following Lemma.

Lemma 3. *If $n \geq 1$ then*

(1) $f'_{2n} = (2n, 2n-1, \ldots, 2, 1)$.
(2) $\ell'_{2n} = (2, 1, 4, 3, \ldots 2n-2, 2n-3, 2n, 2n-1)$.

Proof.

(1) By relation (8), $\mathfrak{I}'_2$ is the list $(2,1)$ and more generally $f'_{2n} = \psi(1, f'_{2n-2}) = (2n, 2n-1, 2n-2, \ldots, 3, 2, 1)$.

Moreover, $\ell'_{2n} = \psi(2n-1, \ell'_{2n-2}) = (2, 1, 4, 3, \ldots, 2n-2, 2n-3, 2n, 2n-1)$; a recurrence on n completes the proof. □

The next lemma ensures a smooth transition between the sublists in relation (3), namely between: (i) the list $\psi(i, \overline{\mathfrak{I}'_{2n-2}})$ and $\psi(i+1, \mathfrak{I}'_{2n-2})$, (ii) the list $\psi(i, \psi'_{2n-2})$ and $\psi(i+1, \overline{\mathfrak{I}}_{2n-2})$. More precisely, successive involutions in $\mathfrak{I}_{2n}$ differ either by a product of one or two transpositions or by a circular shift of three elements.

Table 2

The lists of vertices $\mathfrak{I}'_4$ and $\mathfrak{I}'_6$. In $\mathfrak{I}'_6$, the sublists $\psi(1,\mathfrak{I}'_4)$, $\psi(3,\mathfrak{I}'_4)$, $\psi(5,\mathfrak{I}'_4)$ are in bold-face respectively, and $\psi(2,\mathfrak{I}'_4)$, $\psi(4,\mathfrak{I}'_4)$ are in classic font

$\mathfrak{I}'_4$		$\mathfrak{I}'_6$			
1	4321	1	**654321**	9	**216543**
2	3412	2	**645231**	10	215634
3	2143	3	**632541**	11	351624
		4	361542	12	532614
		5	465132	13	**432165**
		6	564312	14	**341265**
		7	**546213**	15	**214365**
		8	**456123**		

Lemma 4.

(i) *If* $2n \geq 4$ *then*

$$\psi(i+1, f_{2n-2}) = \psi(i, f_{2n-2}) \cdot \langle i, 2n, i+1 \rangle \quad \textit{if } 1 \leq i \leq 2n-2. \tag{9}$$

(ii) *If* $2n \geq 4$ *then*

$$\psi(i+1, \ell_{2n-2}) = \psi(i, \ell_{2n-2}) \cdot \langle i, 2n, i+1 \rangle \quad \textit{if } 1 \leq i \leq 2n-2 \tag{10}$$

and

$$\psi(2n-1, \ell_{2n-2}) = \psi(2n-2, \ell_{2n-2}) \cdot \langle 2n-2, 2n, 2n-1 \rangle. \tag{11}$$

(iii) *If* $n \geq 3$ *then*

$$\phi(\ell_{n-1}) = \begin{cases} \psi(1, f_{n-2}) \cdot \langle 1, n \rangle \cdot \langle n-1, n-2 \rangle, & \textit{if } n \textit{ odd}, \\ \psi(1, \ell_{n-2}) \cdot \langle 1, n \rangle, & \textit{if } n \textit{ even}. \end{cases} \tag{12}$$

Proof. Directly as the previous proof. □

Theorem 2. *The list* $\mathfrak{I}'_{2n}$ *is a Hamiltonian path of* $\frac{G(S_{2n})^2}{I'_{2n}}$.

Proof. In vertue of Lemma 4, two consecutive vertices of the list $\mathfrak{I}'_{2n}$ differ by a product of one or two transpositions. Thus, it is also a Hamiltonian path for $\frac{G(S_n)^2}{I'_{2n}}$. □

4. Involutions with a given number of fixed points

In this section, we exhibit a Hamiltonian path for the restriction of the graph $G(S_n)^2$ to the set $I_{n,2m}$ of involutions with a given number of fixed points equals to $(n-2m)$.

We will use the set $C_{n,2m}$ of n-length combinations which have $2m$ ones in binary sequence representation. We let $G(C_{n,2m})$ be the graph defined as follows: the vertices are the elements of $C_{n,2m}$ and two vertices are connected by an edge iff their binary representations differ exactly on two positions. Various studies have been conducted on Hamiltonian paths on this graph, but one of them, defined by Eades and McKay in the terminology of Gray code [3,7], is more interesting for our purposes. Indeed, it is crucial that the path verify that two consecutive vertices c_1 and c_2 on the path differ on two positions i and j such that $c_1(k) = 0$, for all k, $i < k < j$. Following the Gray code terminology, we say that the path is homogeneous if it verifies the previous property.

F. Ruskey [7] gives a recursive definition for the list $\mathcal{C}_{n,2m}$ already constructed by P. Eades and B. McKay.

Definition 3 ([7]).

$$\mathcal{C}_{n,2m} = \begin{cases} 0^n\,, & \text{if } k=0\,,\\ 10^{n-1}\circ 010^{n-2}\circ\ldots\circ 0^{n-1}1\,, & \text{if } k=1\,,\\ \mathcal{C}_{n-1,2m}\cdot 0\circ\overline{\mathcal{C}_{n-2,2m-1}}\cdot 01\circ\mathcal{C}_{n-2,2m-2}\cdot 11\,, & \text{if } 1<k<n\,,\\ 1^n\,, & \text{if } k=n\,. \end{cases}$$

This list has a number of interesting properties which are reflected in the proposition below.

Proposition 1 ([3,7]). *The list $\mathcal{C}_{n,2m}$ of all $2m$-combinations of $[n]$ satisfies the properties:*

(1) $\text{first}(\mathcal{C}_{n,2m}) = (1^{2m}0^{n-2m})$,

(2) $\text{last}(\mathcal{C}_{n,2m}) = (0^{n-2m}1^{2m})$,

(3) *the list $\mathcal{C}_{n,2m}$ is a homogeneous Hamiltonian path for $G(C_{n,2m})$.*

Let $c = (c(1), c(2), \ldots, c(2m))$ be a $2m$-combination of n, $2m \leq n$, in integer sequence representation, i.e., the entry $c(i)$ is the position of the ith one in the binary representation. So, $1 \leq c(i) < c(i+1) \leq n$ for $i = 1, 2, \ldots, 2m-1$ and let $t_c = t$ be the binary representation of c, i.e., $t = (t(1), t(2), \ldots, t(n))$ with $t(i) = 1$ if there exists a j such

that $c(j) = i$, and $t(i) = 0$ elsewhere. With these notations, for an involution $v = (v(1), v(2), \ldots, v(2m)) \in I'_{2m}$ we define the n-length sequence $u = (u(1), w(2), \ldots, u(n))$, denoted by $\sqcup\!\sqcup(c; v)$, as

$$u(i) = \begin{cases} i\,, & \text{if } t(i) = 0\,, \\ c(v(j))\,, & \text{if } t(i) \text{ is the } j\text{th 1 in } t, \end{cases} \tag{13}$$

and we call $u = \sqcup\!\sqcup(c; v)$ the *shuffle* of c by v.

In other words, u acts on indices $c(1), c(2), \ldots, c(2m)$ as v, and fixes the other indices. It is not difficult to show that $\sqcup\!\sqcup(c; v)$ is an involution of I_n with exactly $2m$ "deranged" points (i.e., with exactly $n - 2m$ fixed points). Conversely, each such involution can be uniquely constructed by a shuffle operation from an appropriate combination and an involution with no fixed point. The shuffle operator over combinatorial objects was formally defined in a larger context in [1, 10]. To summarize, we have:

Lemma 5. *If $C_{n,2m}$ is the set of all $2m$-combinations of $[n]$ and $I_{n,2m}$ the set of all permutations of $[n]$ with exactly $(n - 2m)$ fixed points then*

$$\sqcup\!\sqcup : C_{n,2m} \times I'_{2m} \to I_{n,2m}$$

defined by (13) *is a bijection.*

Also, we extend the shuffle operation in a natural way to lists of involutions as follows: if $\mathfrak{I}' = v_1, v_2, \ldots$ is a sublist of $\mathfrak{I}'_{2m}$ and $c \in C_{n,2m}$ then $\sqcup\!\sqcup(c; \mathfrak{I}')$ is the list $\sqcup\!\sqcup(c; v_1), \sqcup\!\sqcup(c; v_2), \ldots$, and $\sqcup\!\sqcup(c; \overline{\mathfrak{I}'}) = \overline{\sqcup\!\sqcup(c; \mathfrak{I}')}$.

Theorem 3. *If $\mathcal{C}_{n,2m}$ is the previous Hamiltonian path for the graph $G(C_{n,2m})$ then the list $\mathfrak{I}_{n,2m}$ defined by*

$$\mathfrak{I}_{n,2m} = \bigcirc_{c \in \mathcal{C}_{n,2m}} \sqcup\!\sqcup \left(c; \mathfrak{I}'^{(r(c))}_{2m}\right) \tag{14}$$

is a Hamiltonian path for $\frac{G(S_n)^2}{I_{n,2m}}$, where $r(c)$ is the rank of c in $\mathcal{C}_{n,2m}$ (the first combination in $\mathcal{C}_{n,2m}$ has rank zero) and $\mathfrak{I}'^{(r(c))}_{2m}$ is $\mathfrak{I}'_{2m}$ or $\overline{\mathfrak{I}'}_{2m}$ depending upon whether if $r(c)$ is even or odd.

Proof. The list $\mathfrak{I}_{m,2n}$ has no repetitions, and disregarding the order, it equals the set $I_{m,2n}$. Moreover, for a combination c in $C_{m,2n}$ and two involutions v and v' in I_{2n} such that $v' = v \cdot \tau$ where τ is a k-cycle of $S_{2n}, k \geq 2$, then $\sqcup\!\sqcup(c; v)$ and $\sqcup\!\sqcup(c; v')$ differ by a product of a k-cycle. So, any successive permutations in $\sqcup\!\sqcup(c; \mathfrak{I}'_{2m})$ — or equivalently in $\sqcup\!\sqcup(c; \overline{\mathfrak{I}'}_{2m})$ — differ by a product of one or two transpositions. If c' is the successor

of c in $\mathcal{C}_{n,2m}$ then $t' = t_{c'}$ and $t = t_c$, the binary representations of c' and c, differ in exactly two positions, say k and ℓ, with $t(k) = t'(\ell) = 0$ and $t'(k) = t(\ell) = 1$. According to Proposition 1 the permutations $\sigma = \sqcup(c; d)$ and $\sigma = \sqcup(c'; d)$ differ in exactly three positions, namely k, ℓ and i, where i is such that $\sigma(i) = \ell$ and $\sigma'(i) = k$. □

We have, consequently:

Proposition 2.

(1) $\mathrm{first}(\mathfrak{I}_{n,2m}) = (2m, 2m-1, \ldots 2, 1, 2m+1, 2m+2, \ldots n-1, n)$, *and*

(2) $\mathrm{last}(\mathfrak{I}_{n,2m}) = \begin{cases} (1, 2, 3, \ldots n-2m, n-2m+2, n-2m+1, \ldots n, n-1) \\ \textit{or} \\ (1, 2, 3, \ldots n-2m, n, n-1, \ldots n-2m+2, n-2m+1) \end{cases}$

depending on the parity of the cardinality of $C_{n,2m}$.

Proof. By recurrence. □

Table 3

The path $C(5,4)$ and the path $\mathfrak{I}_{55,4}$ with the sublists of the recursive definition (14) illustrated in bold-face and classic font respectively

$C(5,4)$		$\mathfrak{I}_{5,4}$			
1	11110	1	**43215**	9	35142
2	11101	2	**53241**	10	34125
3	11011	3	**54321**	11	**21435**
4	10111	4	**52431**	12	**21543**
5	01111	5	**15432**	13	**21354**
		6	14523	14	**32154**
		7	42513	15	**13254**
		8	45312		

5. Concluding remarks

In this paper we give a construction of Hamiltonian paths for $\frac{G(S_n)^2}{I_n}$, $\frac{G(S_n)^2}{I'_n}$ and $\frac{G(S_n)^2}{I_{n,2m}}$. Some of these paths are already known in [11] but their construction is recursive that can be written more simply. Two successive vertices on the cycle differ by a product of one or two transpositions. We do not explain here how to implement an algorithm to generate these

paths. Indeed there exist in the literature similar algorithms (see [1] for instance). In our case, we can easily see that the time complexity of our algorithm is proportional to the total number of recursive calls, i.e., in $\mathcal{O}(i_n)$ for the path in $\frac{G(S_n)^2}{I_n}$.

References

[1] J. L. Baril and V. Vajnovszki, Gray code for derangements, *Discrete Appl. Math.*, Vol. 140 (1–3) (2004), pp. 207–221.

[2] S. Curran and J. Gallian, Hamiltonian cycles and paths in Cayley graphs and diagraphs – survey, *Discrete Math.*, Vol. 156 (1996), pp. 1–18.

[3] P. Eades and B. McKay, An algorithm for generating subsets of fixed size with a strong minimal change property, *IPL*, Vol. 19 (1984), pp. 131–133.

[4] E. Foundas and A. Sapounakis, Algorithms on involutions, *Journal of Information and Optimization Sciences*, Vol. 12 (1991), pp. 339–361.

[5] M. C. Golumbic, *Algorithmic Graph Theory and Perfect Graphs*, Academic Press, 1980, p. 157.

[6] D. Roelants van Baronaigien, Constant time generation of involutions, *Congressus Numerantium*, Vol. 90 (1992), pp. 87–96.

[7] F. Ruskey, Simple combinatorial Gray codes constructed by reversing sublists, in *ISAAC Conference*, *Lecture notes in computer Science*, Vol. 762 (1993), pp. 201–208.

[8] F. Ruskey, *Combinatorial Generation*, Manuscript, 2004.

[9] V. Vajnovszki and T. Walsh, A loopless two-close Gray code algorithm for listing k-ary Dick words, *Research report, UQAM*, Vol. 140 (2003).

[10] V. Vajnovszki, A loopless algorithm for generating the permutations of a multiset, *Theoretical Computer Science*, Vol. 307 (2003), pp. 415–431.

[11] T. Walsh, Gray code for involutions, *J. Combin. Math. Combin. Comput.*, Vol. 36 (2001), pp. 95–118.

[12] T. Walsh, A Gray code for involutions with a given number of fixed points, *Research Report 10*, Department of Computer Science, Université du Quebec, Montreal, July 1997.

Received July, 2006

PU.M.A. Vol. **17** (2006), No. 1–2, pp. 19–37

ECO-Generation for p-generalized Fibonacci and Lucas permutations

Jean-Luc Baril
LE2I UMR-CNRS 5158, Université de Bourgogne
B.P. 47 870, 21078 DIJON-Cedex France
e-mail: barjl@u-bourgogne.fr

and
Phan-Thuan Do
LE2I UMR-CNRS 5158, Université de Bourgogne
B.P. 47 870, 21078 DIJON-Cedex France
e-mail: phan-thuan.do@u-bourgogne.fr

(Received: October 31, 2006)

Abstract. Using the ECO method, we give succession rules for p-generalized Fibonacci and Lucas sequences. We provide sets of pattern-avoiding permutations which are enumerated by the p-generalized Fibonacci and Lucas sequences. Finally we give generating algorithms in Constant Amortized Time for these sets.

Mathematics Subject Classifications (2000). 05A05, 05A15, 69R05

Keywords: Fibonacci and Lucas permutations, ECO method, pattern-avoiding permutation.

1 Introduction

Barcucci, Del Lungo, Pergola and Pinzani have developed an Enumerating Combinatorial Method (ECO) [6] for the enumeration and the recursive construction of classes of combinatorial objects. The method is used for *enumeration* [4, 5, 6], *algebraic characterization* such as operations on succession rules [9, 13, 14] or production matrices [10], *generating functions* associated with an ECO operator [2, 11], *random and exhaustive generation* [1, 3, 18]. The ECO method consists in producing succession rules to describe certain combinatorial object classes. A *succession rule* (Ω) is a system consisting of an axiom (b), $b \in \mathbb{N}^+$, and a set of productions:

$$\{(k) \rightsquigarrow (e_1(k))(e_2(k))\dots(e_k(k)), k \in \mathbb{N}\},$$

where $e_i : \mathbb{N}^+ \longrightarrow \mathbb{N}^+$, which explains how to derive the successors $(e_1(k))$, $(e_2(k))$,..., $(e_k(k))$ of any given (k), $k \in \mathbb{N}^+$. The positive integers (b), (k), $(e_i(k))$ are called *labels* of (Ω). The root can be represented by means of a *generating tree* where (b) is the label of the root and each node labeled (k) has k sons labeled $(e_1(k))$, $(e_2(k))$,..., $(e_k(k))$. A succession rule (Ω) induces a sequence of positive integers $(a_n)_{n\geq 0}$ where a_n is the number of nodes at level n in the generating tree.

The succession rules for traditional Fibonacci sequences are given by Banderier et al. [2]. In this paper, we extend this result to p-generalized Fibonacci

and Lucas sequences. Moreover, we are interested in families of pattern-avoiding permutations which can be enumerated in terms of p-generalized Fibonacci or Lucas sequences. In this regard, a number of papers concerning Fibonacci sequences have been published [7, 12, 17, 19, 20, 21] but none have been forthcoming on Lucas. For instance, Mansour described general results (see Theorem 3 in [19]) such that these families are particular cases. However, in [8], an efficient algorithm is given to generate generalized Lucas binary strings in Gray code order.

This paper is organized as follows. In the next section, we provide the succession rules for p-generalized Fibonacci sequences and their connections with pattern-avoiding permutations. Then we investigate p-generalized Lucas sequences by giving succession rules and two sets of pattern-avoiding permutations which are counted by these sequences. In the last section, we produce constant amortized time algorithms to generate these different classes of p-generalized Fibonacci and Lucas permutations.

2 p-generalized Fibonacci sequences

In this part of the study we only consider the results obtained from p-generalized Fibonacci sequences. Some of the results in this section have already been presented in [7]. Recall that the p-generalized Fibonacci sequences verify $f_{p,n} = \sum_{i=1}^{p} f_{p,n-i}$ anchored by $f_{p,n} = 0$ if $n \leq 0$, $f_{p,1} = 1$ and let us denote $F_{p,n}$ be the set of n-length binary strings without p consecutive ones. Obviously $|F_{p,n}| = f_{p,n+2}$.

2.1 Succession rules

In order to construct generating trees for the p-generalized Fibonacci sequences we use the ECO method, providing the label of its root and a set of succession rules which describe, for each node, the label set of its successors. There are many ways to label fields of succession rules. Here, each node has at most two successors. So we label each node by (2_i) or (1) to encode information concerning the number of successors (2 or 1); the index i $(0 < i < p)$ allows us to distinguish nodes having two successors.

For example, the succession rules for the well-known Fibonacci sequences $(p = 2)$, which appear in [2], can be written as:

$$\begin{cases} (2_1) \\ (2_1) \rightsquigarrow (2_1)(1) \\ (1) \rightsquigarrow (2_1). \end{cases}$$

The following theorem extends this finding to p-generalized Fibonacci sequences.

THEOREM 1 *For $p \geq 2$, a system of succession rules (Ω_p) for p-generalized*

Fibonacci sequences is given by:

$$(\Omega_p)\begin{cases}(2_{p-1})\\(2_{p-1})\rightsquigarrow(2_{p-1})(2_{p-2})\\(2_{p-2})\rightsquigarrow(2_{p-1})(2_{p-3})\\\dots\\(2_1)\rightsquigarrow(2_{p-1})(1)\\(1)\rightsquigarrow(2_{p-1}).\end{cases}$$

Proof. By translating the concept of succession rules into matrix notation [10], we obtain the *production matrix* of size $p\times p$, as shown below:

$$M_p=\begin{pmatrix}1&1&0&\dots&0&0\\1&0&1&\dots&0&0\\\dots&\dots&\dots&\ddots&\dots&\dots\\1&0&0&\dots&1&0\\1&0&0&\dots&0&1\\1&0&0&\dots&0&0\end{pmatrix}.$$

Here the i-th row (or column) of the production matrix corresponds to the label (2_{p-i}) for $1\le i\le p-1$ and (1) for $i=p$. For example, the entry in the first row and the second column is the number of labels (2_{p-2}) produced from the label (2_{p-1}).

Let d_p be the determinant of $(x\cdot I-M_p)$ where x is a variable and I the unitary matrix of size $p\times p$. It is easy to prove that d_p verifies the recurrence relation: $d_p=x\cdot d_{p-1}-1$ anchored by $d_1=x-1$. So we obtain the characteristic polynomial $d_p=x^p-x^{p-1}-x^{p-2}-\dots-x-1$.

Using the Hamilton–Cayley theorem we replace x by M and:

$$M_p^p-M_p^{p-1}-M_p^{p-2}-\dots-M_p-I=0.$$

Since M_p is invertible, it is equivalent to

$$M_p^n=\sum_{i=n-p}^{n-1}M_p^i,\qquad\forall n\ge p.$$

The succession rules (Ω_p) induce a sequence $(a_n)_{n\ge0}$ where a_n is the number of nodes at level n ($a_0=1$ since the tree has only one root). By [10], $a_n=u^T\cdot M_p^n\cdot e$ where u^T is the row vector $(1\ 0\ 0\ \dots)$ of length p and e is the column vector $(1\ 1\ 1\ \dots)^T$ of length p. So the previous equality implies:

$$a_n=\sum_{i=n-p}^{n-1}a_i,\quad\forall n\ge p.$$

Moreover each node on level $0\le n\le p-2$ has exactly two successors. Thus the number of nodes on level $n\le p-1$ is obviously 2^n.

So

$$a_n = \begin{cases} 2^n & \text{if } 0 \leq n \leq p-1 \\ \sum_{i=n-p}^{n-1} a_i & \text{if } p \leq n, \end{cases}$$

which implies that a_n is equal to the well-known p-generalized Fibonacci number $f_{p,n+2}$. □

By means of simple calculations, we produce the formula of the well-known generating function of p-generalized Fibonacci [12]:

$$f_p(z) = \frac{1 + z + z^2 + \ldots + z^{p-1}}{1 - z - z^2 - \ldots - z^p} = \frac{1}{z} \cdot \left(\frac{1}{1 - \sum_{i=1}^{p} z^i} - 1 \right).$$

Since $f_p(z) \underset{p \to \infty}{\longrightarrow} \frac{1}{1-2z}$, these succession rules produce a kind of discrete continuity between the p-generalized Fibonacci sequences ($p \geq 2$) and the sequences of binary strings.

The following proposition demonstrates how the p-generalized Fibonacci strings can be attached to the generating tree of (Ω_p).

Proposition 1 *The generating tree of (Ω_p) can be encoded by the p-generalized Fibonacci binary strings. An n-length binary string is obtained from one of length $(n-1)$ by inserting 0 or 1 into its last position (see Figure 1).*

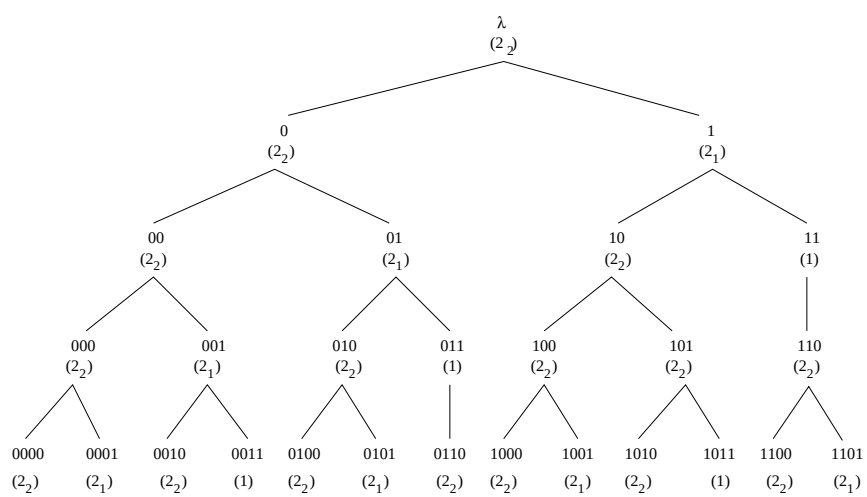


Figure 1: The first five levels of the generating tree encoded by the 3-generalized Fibonacci binary strings.

Proof. We will prove this by recurrence. Obviously, this is true for the root which is encoded by the empty binary string λ (for convenience let the level of the root be 0). Assume that this holds until the level k, i.e. each level $i \leq k$

is encoded by the elements of $F_{p,i}$. We will show that this is also true for the level $k+1$.

Let τ be the $(k+1)$-length binary string of one node at the level $k+1$ and let δ be its predecessor on the level k.

Firstly, if τ is obtained from δ by inserting 0 on the right then τ also belongs to $F_{p,k+1}$. Secondly, if τ is obtained by inserting 1 on the right of δ then δ has two children, thus its label is (2_i), $(1 \leq i \leq p-1)$.

We distinguish between two cases.

- δ is labeled (2_{p-1}). Here we obtain $\delta = \delta'0$ and $\tau = \delta'01$ with $\delta' \in F_{p,k-1}$ via the recurrence hypothesis. Thus τ belongs to $F_{p,k+1}$.
- δ is labeled (2_i), $1 \leq i \leq p-2$. Its predecessor δ_1 is labeled (2_{i+1}). We repeat the process with δ_1 until we find a label (2_{p-1}) (notice that the process always finishes). We have thus constructed a path $\delta_1, \ldots, \delta_{p-1-i}$ where δ_j is the successor of δ_{j+1} for $(1 \leq j \leq p-2-i)$ and δ_j is labeled (2_{i+j}). In particular δ_{p-1-i} is labeled (2_{p-1}). So we deduce that δ is of the form $\delta = \delta'01^{p-i-1}$ and $\tau = \delta'01^{p-i}$ where $\delta' \in F_{p,k-p+i}$ by the recurrence hypothesis. Since the length $p-1-i$ of the path verifies $p-1-i \leq p-2$, we conclude that $\tau \in F_{p,k+1}$.

Moreover, using Theorem 1, each level n of the tree contains exactly $f_{p,n+2} = |F_{p,n}|$ nodes, which shows that the tree is encoded by the p-generalized Fibonacci binary strings. □

2.2 Generating trees using pattern avoiding permutations

Let S_n be the set of permutations on $[n] = \{1, 2, \ldots, n\}$. We represent a permutation $\pi \in S_n$ in online notation: i.e. $\pi = \pi(1)\pi(2)\ldots\pi(n)$. A permutation $\pi \in S_n$ contains the pattern $\tau \in S_k$ if and only if a sequence of indices $1 \leq i_1 < i_2 < \ldots < i_k \leq n$ exists such that $\pi(i_1)\pi(i_2)\ldots\pi(i_k)$ is ordered as τ. We denote by $S_n(\tau)$ the set of permutations of S_n avoiding the pattern τ.

In this subsection, we encode the generating tree of (Ω_p) by permutations. The root is encoded by the identity of length one. Let π be an n-length permutation in the generating tree; each successor is obtained from π by inserting $n+1$ into certain positions also known as the *active sites* of π.

THEOREM 2 *The generating tree of (Ω_p) can be encoded by the permutations π in $S_n(321, 312, 234\ldots(p+1)1)$. The active sites of π are the two last positions if the label of π is (2_i); otherwise only the last position is active (see Figure 2).*

Proof. We will prove this by recurrence. Obviously, this is true for the root which is encoded by the permutation (1) (for convenience let the level of the root take the value 1). Assume that this holds until the level $k \geq 1$, i.e. each level $i \leq k$ is encoded by the permutations in $S_i(321, 312, 234\ldots(p+1)1)$. We will show that this also holds for the level $k+1$.

Let τ be the $(k+1)$-length permutation of one node of the level $k+1$ and let δ be its predecessor on the level k.

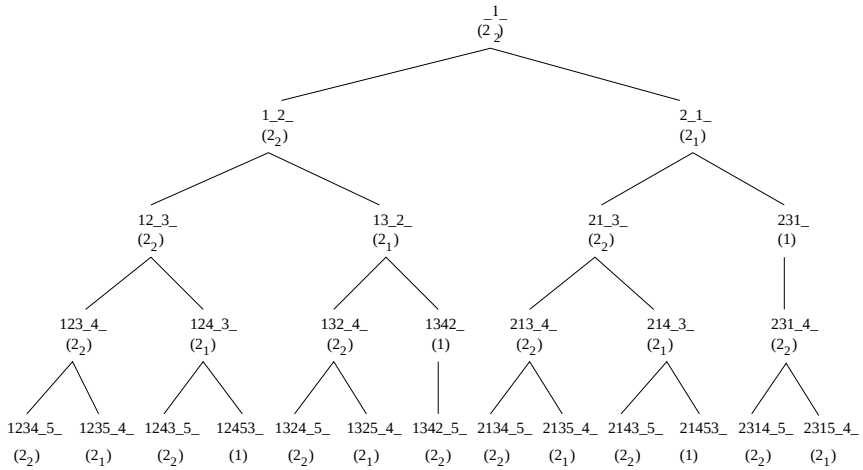


Figure 2: The first five levels of the generating tree of (Ω_3) are encoded by permutations in $S_n(321, 312, 2341)$. The active sites are represented by underscores.

Firstly, if τ is obtained from δ by inserting $k+1$ into the last position, then τ does not contain any occurrence of the forbidden subsequences 321, 312 and $234\ldots(p+1)1$.

Secondly, if τ is obtained by inserting $k+1$ into the site (k) of δ (i.e. the site just before the last entry of δ) then τ does not contain any occurrence of the forbidden subsequences 321, 312. We will now show that τ also belongs to $S_{k+1}(234\ldots(p+1)1)$. Notice that in this case δ has two children, thus its label is (2_i), $(1 \leq i \leq p-1)$.

We can distinguish two cases.

- δ is labeled (2_{p-1}). Here we obtain $\delta = \delta' k$ and $\tau = \delta'(k+1)k$ with $\delta' \in S_{k-1}(321, 312, 234\ldots(p+1)1)$ by the recurrence hypothesis. Thus τ belongs to $S_{k+1}(321, 312, 234\ldots(p+1)1)$.

- δ is labeled (2_i), $1 \leq i \leq p-2$. Its predecessor δ_1 is labeled (2_{i+1}). We repeat the process with δ_1 until we find a label (2_{p-1}). Remark that the process always finishes. So we have constructed a path $\delta_1, \ldots, \delta_{p-1-i}$ where δ_j is successor of δ_{j+1} for $(1 \leq j \leq p-2-i)$ and δ_j is labeled (2_{i+j}). In particular δ_{p-1-i} is labeled (2_{p-1}). So we deduce that δ is of the form $\delta = \delta'(k-p+i+2)\ldots(k-1)k(k-p+i+1)$ and $\tau = \delta'(k-p+i+2)\ldots(k-1)k(k+1)(k-p+i+1)$ where $\delta' \in S_{k-p+i}(321, 312, 234\ldots(p+1)1)$ via the recurrence hypothesis. Since the length $p-1-i$ of the path verifies $p-1-i \leq p-2$, we conclude that τ does not contain any occurrence of the forbidden subsequence $23\ldots(p+1)1$. Hence $\tau \in S_{k+1}(321, 312, 234\ldots(p+1)1)$.

Moreover, Mansour (Theorem 3, [19]) found the cardinality of $S_{k+1}(321, 312, \tau)$ for any $\tau \in S_{p+1}(321, 312)$. In particular, he obtains that the cardinality of $S_{k+1}(321, 312, 234...(p+1)1)$ is equal to that $F_{p,k}$. □

By considering other active sites, we obtain an other set of pattern-avoiding permutations which is associated with the same generating tree.

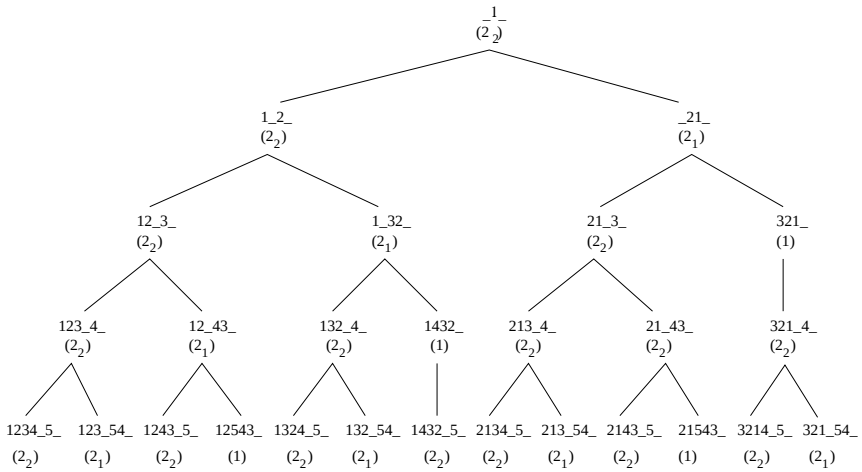


Figure 3: The first five levels of the generating tree of 3-generalized Fibonacci sequences encoded by permutations $S_n(231, 312, 4321)$. The active sites are represented by underscores.

THEOREM 3 *The generating tree of* (Ω_p) *can be encoded by the permutations* π *in* $S_n(231, 312, (p+1)p \dots 321)$. *A permutation* π *always has an active site in the last position; moreover, in the case where the label is* (2_i), *there is another one just before the maximal entry of* π *(see Figure 3).*

Proof. In fact, an insertion into the last active site or into active site just before the site of the maximal entry does not produce any patterns of the form 231 and 312. The rest of the proof is similar to the demonstration of Theorem 2. □

3 p-generalized Lucas sequences

In this section, we consider p-generalized Lucas sequences. Recall that the p-generalized Lucas sequences verify: $\ell_{p,n} = \sum_{i=1}^{p} \ell_{p,n-i}$ anchored by $\ell_{p,n} = 2^n - 1$ if $0 < n \leq p$, and let $L_{p,n}$ be the set of n-length binary strings without p consecutive ones by considering the strings circularly. Notice that for convenience we assume here that the string 1^n $(n < p)$ belongs to $L_{p,n}$. So $|L_{p,n}| = \ell_{p,n} + 1$

if $n < p$ and $|L_{p,n}| = \ell_{p,n}$ otherwise. We thus exhibit succession rules for p-generalized Lucas sequences and provide a generating tree for two new associated classes of pattern-avoiding permutations.

3.1 Succession rules

In Section 2 each node was labeled (1) or (2_i) where $1 \leq i \leq p-1$. In the following we consider the label (1) or $(1')$ for the nodes having one successor and the labels (2_i) or $(2'_i)$ $(1 \leq i \leq p-1)$ for the nodes having two successors.

THEOREM 4 *For $p \geq 2$, a system of succession rules (Φ_p) for p-generalized Lucas sequences is expressed by:*

$$(\Phi_p)\begin{cases} (2'_{p-1}) \\ (2'_{p-1}) \rightsquigarrow (2_{p-1})(2'_{p-2}) \\ (2'_{p-2}) \rightsquigarrow (2_{p-2})(2'_{p-3}) \\ \dots \\ (2'_1) \rightsquigarrow (2_1)(1') \\ (1') \rightsquigarrow (1) \\ (2_{p-1}) \rightsquigarrow (2_{p-1})(2_{p-2}) \\ (2_{p-2}) \rightsquigarrow (2_{p-1})(2_{p-3}) \\ \dots \\ (2_1) \rightsquigarrow (2_{p-1})(1) \\ (1) \rightsquigarrow (2_{p-1}), \end{cases}$$

which means that (Φ_p) produces $\ell_{p,n}$ if $n > p$ and $\ell_{p,n} + 1 = 2^n$ otherwise.

Proof. By translating the concept of succession rules into matrix notation [10], we obtain the following production matrix of size $2p \times 2p$. The i-th column (or row) corresponds to the labels $(2'_i)$ for $i \leq p-1$, $(1')$ for $i = p$, (2_i) for $p+1 \leq i \leq 2p-1$ and (1) for $i = 2p$.

$$M_p = \left(\begin{array}{c|c} A_p & B_p \\ \hline C_p & D_p \end{array}\right),$$

$$A_p = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \ddots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \qquad B_p = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \ddots & \dots \\ 0 & 0 & \dots & 1 \end{pmatrix},$$

$$C_p = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{pmatrix}, \qquad D_p = \begin{pmatrix} 1 & 1 & 0 & \dots & 0 \\ 1 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \ddots & \dots \\ 1 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

It is easy to see that the number a_n of nodes on a level $n \leq p-1$ (the root being at the zero level) verifies: $u^T \cdot M_p^n \cdot e = 2^n$ (see Section 2.1 for the definition of u^T and e) and the number of nodes on the level p is $u^T \cdot M_p^p \cdot e = 2^p - 1$. Moreover by [10] the distribution on the level $p+1$ of the labels $(2'_i)$ for $1 \leq i \leq p-1$, $(1')$, (2_i) for $1 \leq i \leq p-1$ and (1) is the row vector

$$r_{p+1} = u^T \cdot M_p^{p+1} = (0, 0, \ldots, 0, 2^p - 1, 2^{p-1} - 1, \ldots, 2^2 - 1, 1).$$

Thus

$$r_{p+1} = (0, 0, \ldots, 0, \ell_{p,p}, \ell_{p,p-1}, \ldots, \ell_{p,1})$$

and

$$u^T \cdot M_p^{p+1} \cdot e = \ell_{p,p+1}.$$

By recurrence on $i \geq 1$,

$$\begin{aligned} r_{p+i+1} &= u^T \cdot M_p^{p+i} \cdot M_p \\ &= (0, 0, \ldots, 0, \ell_{p,p+i-1}, \ell_{p,p+i-2}, \ldots, \ell_{p,i}) \cdot M_p \\ &= (0, 0, \ldots, 0, \sum_{j=i}^{p+i-1} \ell_{p,j}, \ell_{p,p+i-1}, \ell_{p,p+i-2}, \ldots, \ell_{p,i+1}) \\ &= (0, 0, \ldots, 0, \ell_{p,p+i}, \ell_{p,p+i-1}, \ell_{p,p+i-2}, \ldots, \ell_{p,i+1}), \end{aligned}$$

and

$$u^T \cdot M_p^{p+i+1} \cdot e = \ell_{p,p+i+1}, \qquad \forall i \geq 1,$$

thus the sequence $(a_n)_{n \geq 0}$ induced by (Φ_p) verifies:

$$a_n = u^T \cdot M_p^n \cdot e = \begin{cases} 0 & \text{if } n < 0 \\ \ell_{p,n} + 1 & \text{if } 0 \leq n \leq p-1 \\ \ell_{p,n} & \text{if } n \geq p. \end{cases}$$

We deduce that a_n is equal to the well known p-generalized Lucas sequence $\ell_{p,n}$ (except for $n \leq p-1$ where $a_n = \ell_{p,n} + 1$). □

By means of simple calculations, we can deduce the generating function of the sequences corresponding to (Φ_p) as follows:

$$\ell_p(z) = \frac{\sum_{i=0}^{p-1} (z^i - i \cdot z^{2p-i})}{1 - z - z^2 - \ldots - z^p}.$$

As for the case of Fibonacci and since $\ell_p(z) \underset{p \to \infty}{\longrightarrow} \frac{1}{1-2z}$, these succession rules produce a kind of discrete continuity between the p-generalized Lucas sequences $(p \geq 2)$ and the sequences of binary strings.

Here the generating tree of (Φ_p) cannot be encoded by the set $L_{p,n}$. However, it is encoded by another binary string set $K_{p,n}$ defined as follows: $K_{p,n} \subseteq F_{p,n}$ and each binary string of $K_{p,n}$ contains at least two 0s in theirs length $p+1$ prefix. For example, the only binary string in $F_{2,4}$ but not in $K_{2,4}$ is 1010 and $F_{3,5} \backslash K_{3,5} = \{10110, 11011, 11010\}$. In [16] the authors provide a constructive bijection ϕ between $K_{p,n}$ and $L_{p,n}$ defined as follows: if $x \in K_{p,n}$, x can be written $x = 1^i 0 y$ with $0 \leq i < p$ and $\phi(x) = y 0 1^i \in L_{p,n}$.

Proposition 2 *The generating tree of* (Φ_p) *is encoded by the strings of the set* $K_{p,n}$. *An* n*-length binary string is obtained from one of length* $(n-1)$ *by inserting 0 or 1 into its last position (see Figure 4).*

Proof. Obviously, this is true for the root which is encoded by the empty binary string λ (for convenience let the level of the root be 0). Notice that the predecessor of each node on a level $k \leq p-1$ has two successors. This means that each level k $(k \leq p-1)$ is encoded by the set of all binary strings, therefore by $K_{p,k}$. For the level $k = p$, the nodes are encoded by the binary strings different from 1^p which proves the results for the level p.

Now we will prove the result for $k \geq p+1$ by recurrence. Assume that this holds until the level $k \geq p$, i.e. each level $i \leq k$ is encoded by the elements of $K_{p,i}$. We will show that this is also true for the level $k+1$.

Let τ be the $(k+1)$-length binary string of one node of the level $k+1$ and let δ be its predecessor on level k.

Firstly, if τ is obtained from δ by inserting 0 on the right, then τ also belongs to $K_{p,k+1}$. Secondly, if τ is obtained by inserting 1 on the right of δ, then δ has two children, hence its label is (2_i), $(1 \leq i \leq p-1)$.

We distinguish between two cases.

- δ is labeled (2_{p-1}). Here we obtain $\delta = \delta'0$ and $\tau = \delta'01$ with $\delta' \in K_{p,k-1}$ by the recurrence hypothesis. Remark that $\delta' \neq 1^{k-1}$, i.e. it contains at least one zero. Indeed, since $k-1 \geq p-1$, if $\delta' = 1^{k-1}$ then its label is $(1')$ and the label of δ is (1) which is a contradiction. Thus τ belongs to $K_{p,k+1}$.

- δ is labeled (2_i), $1 \leq i \leq p-2$. Since $k \geq p$, its predecessor δ_1 is labeled (2_{i+1}). We repeat the process with δ_1 until we find a label (2_{p-1}). Note that since $k \geq p$ the process always finishes. Indeed if the process does not reach the label (2_{p-1}), it inevitably meets $(2'_j)$ with $j \geq i$. This would mean that $\delta = 1^{p-1-j}01^{j-i}$ and δ would be on the level $k = p-i$ which contradicts $k \geq p$. So we have constructed a path $\delta_1, \ldots, \delta_{p-1-i}$ where δ_j is successor of δ_{j+1} for $(1 \leq j \leq p-2-i)$. Moreover δ_j is labeled (2_{i+j}) and in particular δ_{p-1-i} by (2_{p-1}). So we deduce that δ is of the form $\delta = \delta'01^{p-i-1}$ and $\tau = \delta'01^{p-i}$ where $\delta' \in K_{p,k-p+i}$ using the recurrence hypothesis. As δ_{p-1-i} is labeled (2_{p-1}), $\delta' = 1^{k-p+i}$ implies that δ' is labeled $(2'_{p-1})$. Thus $\delta' = \lambda$ and $\tau = 01^{p-i}$ which contradicts $k+1 \geq p+1$. So, τ has at least two zeros in its length $(p+1)$ prefix and we conclude that $\tau \in K_{p,k+1}$.

Via Theorem 4, each level n of the tree contains exactly $\ell_{p,n}$ nodes for $n > p$ and $\ell_{p,n}+1$ nodes for $n \geq p$. which is the same cardinality as $K_{p,n}$ (see [16]). This shows that the tree is encoded by $K_{p,n}$, for $n \geq 1$. □

3.2 Generating trees using pattern avoiding permutations

In this subsection, we consider generalized patterns. A *barred* pattern $\overline{\tau}$ is a permutation in S_k having a bar over one of its elements. Let τ be a permutation

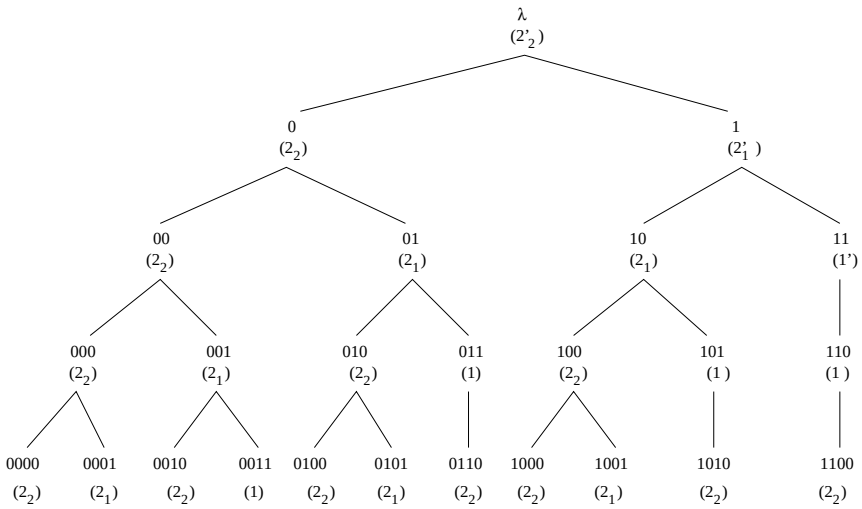


Figure 4: The first five levels of the generating tree of 3-generalized Lucas sequences encoded by $K_{3,n}$.

on $[k]$ identical to $\overline{\tau}$ but unbarred and $\hat{\tau}$ be the permutation on $[k-1]$ made up of the $(k-1)$ unbarred elements of $\overline{\tau}$, rewritten to be a permutation on $[k-1]$. Then $\pi \in S_n$ contains the pattern $\overline{\tau}$ if π contains the pattern $\hat{\tau}$ that cannot be expanded into a pattern τ in π. For example, if $\overline{\tau} = 4\bar{1}32$ then $\pi = 58132674 \in S_8(4\bar{1}32)$.

Let us also define other patterns as permutations possibly with added colons between entries (e.g. $\tau = 13:24$). If two consecutive entries in τ are separated by a *colon*, then the permutation π contains the pattern τ if the entries of π corresponding to those elements are adjacent in π (and in the same order given by τ). For example, 13524 fails to be $13:24$ avoiding but is 1324 avoiding. Notice that this is a particular case of the *distanced patterns* defined in [15].

In the following, we provide two new sets of pattern-avoiding permutations which are also enumerated by the succession rules (Φ_p).

Theorem 5 *Let us assume*

$$T_p = \begin{cases} \overline{1}34\ldots(p+1)2:(p+3)(p+2) \\ \overline{1}34\ldots p2:(p+2)(p+3)(p+1) \\ \ldots \\ \overline{1}32:56\ldots(p+3)4. \end{cases}$$

The succession rule (Φ_p) gives a generating tree encoded by the permutations in $S_n(321, 312, 234\ldots(p+1)1, T_p)$. The active sites are the two last positions if the label is (2_i) or $(2'_i)$; otherwise only the last position is active (see Figure 5).

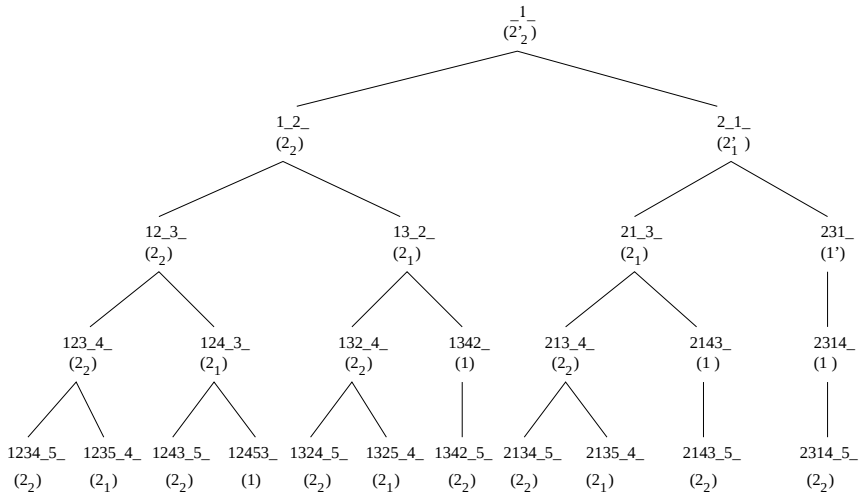


Figure 5: The first five levels of the generating tree of (Φ_3)-generalized Lucas sequences. Each node is encoded by a permutation in $S_n(321, 312, 2341, \overline{1}342 : 65, \overline{1}32 : 564)$.

Proof. Obviously, this is true for the root which is encoded by the permutation of length one (for convenience assume the level of the root is 1). Notice that the predecessor of each node on a level $k \leq p$ has two successors. This means that each level k $(k \leq p)$ is encoded by the set of permutations avoiding 321 and 312 which is also the sets $S_k(321, 312, 234 \ldots (p+1)1, T_p)$ for $k \leq p$. Note also that each node of level p has two successors with one exception which is labeled $(1')$ and encoded by $234 \ldots p1$. This means that the permutation $234 \ldots (p+1)1$ cannot belong to the level $p+1$. We therefore obtain the result for the level $p+1$ as well.

Now we will prove the result for $k \geq p+1$ by recurrence. Assume that this holds up to the level $k \geq p+1$, i.e. each level $i \leq k$ is encoded by the elements of $S_i(321, 312, 234 \ldots (p+1)1, T_p)$. We will show that this is also true for the level $k+1$.

Let τ be the $(k+1)$-length permutation of one node of the level $k+1$ and let δ be its predecessor on the level k.

Firstly, if τ is obtained from δ by inserting $k+1$ on the right, then τ also belongs to $S_{k+1}(321, 312, 234 \ldots (p+1)1, T_p)$. Secondly, if τ is obtained by inserting $k+1$ just before the last entry of δ then δ has two children, hence its label is (2_i), $(1 \leq i \leq p-1)$.

We distinguish between two cases.

- δ is labeled (2_{p-1}). Here we obtain $\delta = \delta' k$ and $\tau = \delta'(k+1)k$ with $\delta' \in S_{k-1}(321, 312, 234 \ldots (p+1)1, T_p)$ via the recurrence hypothesis. Notice that $\delta' \neq 23 \ldots (k-2)(k-1)1$ since $k-1 \geq p$. Indeed if $\delta' = 23 \ldots (k-$

$2)(k-1)1$ then the node of δ' is labeled $(1')$ and thus the label of δ would be (1) which is a contradiction. Thus τ belongs to $S_{k+1}(321, 312, 234\ldots(p+1)1, T_p)$.

- δ is labeled (2_i), $1 \leq i \leq p-2$. Since $k \geq p+1$, its predecessor δ_1 is labeled (2_{i+1}). We repeat the process with δ_1 until we find a label (2_{p-1}). Note that since $k \geq p+1$ the process always finishes. Indeed, if the process does not reach the label (2_{p-1}), it inevitably meets $(2'_j)$ where $j \geq i$. This would mean that $\delta = 23\ldots(p-1-j)(p-j)1(p-j+2)(p-j+3)\ldots(p-i)(p+1-i)(p-j+1)$ and δ would be on the level $k = p+1-i$ which contradicts $k \geq p+1$. So we have constructed a path $\delta_1, \ldots, \delta_{p-1-i}$ where δ_j is the successor of δ_{j+1} for $(1 \leq j \leq p-2-i)$. Moreover δ_j is labeled (2_{i+j}) and in particular δ_{p-1-i} is labeled (2_{p-1}). So we deduce that δ is of the form $\delta = \delta'(k-p+i+2)(k-p+i+3)\ldots(k-1)k(k-p+i+1)$ and $\tau = \delta'(k-p+i+2)(k-p+i+3)\ldots k(k+1)(k-p+i+1)$ where $\delta' \in S_{k-p+i}(321, 312, 234\ldots(p+1)1, T_p)$ by the recurrence hypothesis. As δ_{p-1-i} is labeled (2_{p-1}), $\delta' = 23\ldots(k-p+i)1$ implies that δ' is labeled $(2'_{p-1})$. Thus $\delta' = 1$ and $\tau = 13\ldots(p-i+1)(p-i+2)2$ which contradicts $k+1 = p-i+2 \geq p+2$. So $\delta' \neq 23\ldots(k-p+i)1$ and we conclude that $\tau \in S_{k+1}(321, 312, 234\ldots(p+1)1, T_p)$.

According to [16] $S_{k+1}(321, 312, 234\ldots(p+1)1, T_p)$ and $K_{p,k+1}$ have the same cardinality which shows that the tree is encoded by the permutations in $S_n(321, 312, 234\ldots(p+1)1, T_p)$ for $n \geq 1$. $\square$

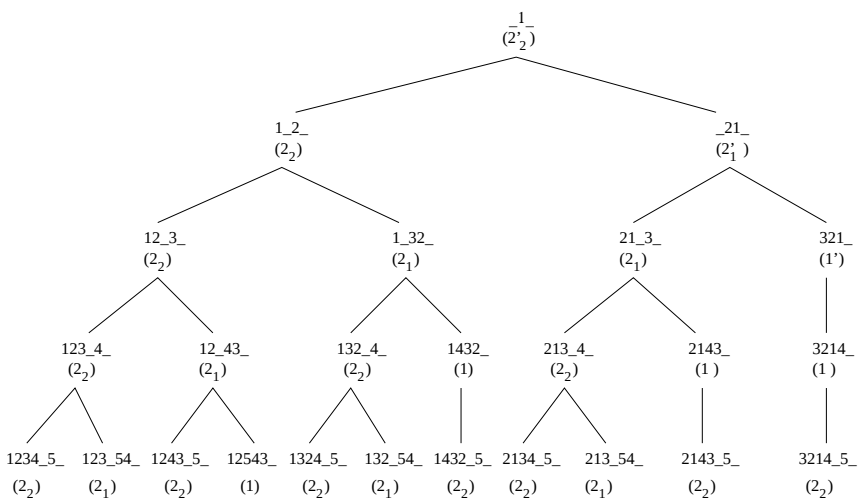


Figure 6: The first five levels of the generating tree of 3-generalized Lucas sequences. Each node is encoded by a permutation in $S_n(231, 312, 4321, \overline{1}432 : 65, \overline{1}32 : 654)$.

Now we will obtain similar results for another permutation set.

THEOREM 6 *Let us assume*

$$T'_p = \begin{cases} \overline{1}(p+1)p\ldots 32 : (p+3)(p+2) \\ \overline{1}p(p-1)\ldots 32 : (p+3)(p+2)(p+1) \\ \ldots \\ \overline{1}32 : (p+3)(p+2)\ldots 54. \end{cases}$$

The generating tree of (Φ_p) *can be encoded by the permutations* π *in* $S_n(231, 312, (p+1)p\ldots 321, T'_p)$. *A permutation* π *always has an active site on the last position; moreover in the case where the label is* (2_i) *or* $(2'_i)$, *it has another one just before the maximal entry of* π *(see Figure 6).*

Proof. We will prove this using the same method as we did in the proof of Theorem 5. □

4 Algorithmic considerations

In this section, we provide efficient algorithms to generate the different permutation sets obtained in Sections 1 and 2. More precisely, the complexity of our algorithms is always in Constant Amortized Time (CAT). Recall that an algorithm is CAT if the number of computations after a small amount of preprocessing is proportional to the number of objects generated.

4.1 Generation for Fibonacci permutations

Consider now the generating tree of $S_n(321, 312, 234\ldots(p+1)1)$. We provide below a simple recursive procedure for the generation of $S_n(321, 312, 234\ldots(p+1)1)$ using the succession rules (Ω_p) defined in Section 1. The generation is anchored by $f1gen(1, p-1)$ where σ is initialized by the identity permutation $12\ldots n$.

In the procedure $f1gen(i, k)$, i is the level of the recursive call (i.e. the level in the tree) and the index k corresponds to the label (2_k) for $1 \leq k \leq p-1$ and (1) for $k = 0$. Between two recursive calls of $f1gen$, the current permutation σ is updated by transposing its entry $\sigma(i)$ with $\sigma(i+1)$, i.e. $\sigma = \sigma \cdot \langle i, i+1 \rangle$.

```
f1gen(i, k)
        if i = n then output σ;
        else
            if k ≠ 0 then
                f1gen(i + 1, p − 1);
                σ = σ·⟨i, i+1⟩; {transpose the entries σ(i) and σ(i+1)}
                f1gen(i + 1, k − 1);
                σ = σ · ⟨i, i + 1⟩;
            else f1gen(i + 1, p − 1);
```

end ;

By comparing (in the generating tree) the number of nodes having one successor with the number of nodes with two successors and the number of objects generated, we show that the generation of Fibonacci permutations sets is CAT.

PROPOSITION 3 *The procedure $f1gen$ generates $S_n(321, 312, 234\ldots(p+1)1)$ in constant amortized time.*

Proof. Let n,$p \in \mathbb{N}$ and T be the tree induced by the recursive calls of the procedure $f1gen$ and m the number of leaves of T. T has the following properties: (a) the root has 2 successors, (b) if a node x has a single successor, say y, then y is a terminal node or it has two successors.

By using these properties, we deduce

- by merging in T each one successor node with its child, we obtain a complete binary tree with $(m-1)$ internal nodes which is the number of two successor nodes.
- the number of one successor nodes is less than the number of two successor nodes.

So T has less than $2(m-1)$ internal nodes.
We have:

$$\frac{\#recursive\ calls}{\#generated\ objects} = \frac{\#internal\ nodes\ of\ T}{m} \leq \frac{2(m-1)}{m} \leq 2.$$

Moreover, the number of operations in each $f1gen(i,k)$ is negligible, thus *the average cost* of $f1gen$ is a constant. Therefore, this algorithm is CAT. □

For the generation of $S_n(231, 312, (p+1)p\ldots 321)$, we give the procedure $f2gen(i,k,t)$ and we run $f2gen(1, p-1, 1)$ anchored by $\sigma = 12\ldots n$. The first two parameters are the same as above, and t corresponds to the position of the entry i in the permutation σ. Here, between two recursive calls, we must insert $i+1$ (which is on the $(i+1)$-th position of σ) just before the entry i (which is in the position t). So we apply to σ the product of successive transpositions $\langle i, i+1\rangle$, $\langle i-1, i\rangle$, …, $\langle t, t+1\rangle$; i.e. $\sigma = \sigma \cdot \langle i, i+1\rangle \cdot \langle i-1, i\rangle \cdot \ldots \cdot \langle t, t+1\rangle$.

```
f2gen(i,k,t)
        if i = n then output σ;
        else
            if k ≠ 0 then
                f2gen(i+1, p-1, i+1);
                for j = i downto t
                  σ = σ · ⟨j, j+1⟩;
                f2gen(i+1, k-1, t);
```

```
            for j = t to i
                σ = σ · ⟨j, j+1⟩;
        else f2gen(i+1, p−1, i+1);
end ;
```

PROPOSITION 4 *The procedure $f2gen$ generates $S_n(231, 312, (p+1)p \ldots 321)$ in constant amortized time.*

Proof. Clearly, according to the algorithm, the number of operations between two recursive calls in $f2gen(i,k)$ is $\mathcal{O}(p)$. Thus, similarly to the proof above, we have the average cost of $f2gen$ in $\mathcal{O}(p)$. So, this algorithm is also CAT. □

REMARK 1 A simple adaptation of the procedure $f1gen$ for the binary strings gives the generation of the p-generalized Fibonacci strings in $F_{p,n}$ in constant amortized time.

4.2 Generation of Lucas permutations

As for p-generalized Fibonacci permutations, we also obtain CAT algorithms for the generation of Lucas permutation sets in Section 3.2. More precisely, the number of computations divided by the number of generated objects is in $\mathcal{O}(1)$ (respectively $\mathcal{O}(p)$) for $S_n(321, 312, 234 \ldots (p+1)1, T_p)$ (respectively $S_n(231, 312, (p+1)p \ldots 321, T'_p)$). The procedures are constructed in the same way as for $f1gen$ and $f2gen$ in the previous subsection, thus we just exhibit here the procedure $l1gen$ which generates the set $S_n(321, 312, 234 \ldots (p+1)1, T_p)$ and the produce $l2gen$ which generates the set $S_n(231, 312, (p+1)p \ldots 321, T'_p)$. Notice that these procedures use $f1gen$ and $f2gen$ and are anchored by $l1gen(1, p-1)$ and $l2gen(1, p-1, 1)$. Moreover the set $K_{p,n}$ can be generated in constant average cost $\mathcal{O}(1)$ by a simple adapted algorithm for binary strings. And by means of an ad hoc data structure we obtain the generation of $L_{p,n}$ in constant average cost $\mathcal{O}(p)$.

```
l1gen(i, k)

if i = n then output σ;
else
    if k ≠ 0 then
        f1gen(i+1, k);
        σ = σ · ⟨i, i+1⟩;
        l1gen(i+1, k−1);
        σ = σ · ⟨i, i+1⟩;
    else f1gen(i+1, 0);
end;
```

```
l2gen(i, k, t)

if i = n then output σ;
else
    if k ≠ 0 then
        f2gen(i+1, k, i+1);
        for j = i downto t
            σ = σ · ⟨j, j+1⟩;
        l2gen(i+1, k−1, t);
        for j = t to i
            σ = σ · ⟨j, j+1⟩;
    else f2gen(i+1, 0, i+1);
end;
```

An applet for the generation of theses permutation sets is available on the web site http://www.u-bourgogne.fr/ jl.baril/applet.html.

5 Concluding remarks

In this paper, we use the ECO method to show several algorithms for the generation of different sets of binary strings or pattern-avoiding permutations enumerated by the p-generalized Fibonacci or Lucas sequences. Each given algorithm runs in constant amortized time. We summarize our results in Table 1 where, for each studied set, we give the average complexity of our algorithm.

p-generalized Fibonacci and Lucas sets	Average complexity
$F_{p,n}$	$\mathcal{O}(1)$
$S_n(321, 312, 234\ldots(p+1)1)$	$\mathcal{O}(1)$
$S_n(231, 312, (p+1)p\ldots 321)$	$\mathcal{O}(p)$
$K_{p,n}$	$\mathcal{O}(1)$
$L_{p,n}$	$\mathcal{O}(p)$
$S_n(321, 312, 234\ldots(p+1)1, T_p)$	$\mathcal{O}(1)$
$S_n(231, 312, (p+1)p\ldots 321, T'_p)$	$\mathcal{O}(p)$

Table 1: Average complexity for each generating algorithm.

However, some problems are still left open. Indeed, can one generate these permutation sets in Gray code order? Some results concerning Fibonacci permutation sets are given in [16] but none are for Lucas permutations. Can one find other pattern-avoiding permutation sets enumerated by p-generalized Fibonacci and Lucas sequences? This will also be interesting to exhibit constructive bijection between the permutation sets found in this paper.

References

[1] S. Bacchelli, E. Barcucci, E. Grazzini and E. Pergola, *Exhaustive generation of combinatorial objects by ECO*, Acta Informatica, **40** (2004), 585–602.

[2] C. Banderier, P. Flajolet, D. Gardy, M. Bousquet-Melou, A. Denise, and D. Gouyou-Beauchamps, *Generating functions for generating trees*, Discrete Mathematics, **246** (2002), 29–55.

[3] E. Barcucci, A. Del Lungo and E. Pergola, *Random generation of trees and other combinatorial objects*, Theoretical Computer Science, **218** (2) (1999), 218–232.

[4] E. Barcucci, A. Del Lungo, E. Pergola and R. Pinzani, *A methodology for plane tree enumeration*, Discrete Mathematics, **180** (1998), 45–64.

[5] E. Barcucci, A. Del Lungo, E. Pergola and R. Pinzani, *Directed animals, forests of trees and permutations*, Discrete Mathematics, **204** (1999), 41–71.

[6] E. Barcucci, A. Del Lungo, E. Pergola and R. Pinzani, *ECO: a methodology for the enumeration of combinatorial objects*, Journal of Difference Equations and Applications, **5** (1999), 435–490.

[7] J.L. Baril and P.T. Do, *p-Generalized Fibonacci permutations generation by the ECO method*, CANT06, May 8-19, 2006.

[8] J.L. Baril and V. Vajnovszki, *Minimal change list for Lucas strings and some graph theoretic consequences*, Theoretical Computer Science, **346** (2005), 189–199.

[9] S. Brlek, E. Duchi, E. Pergola and S. Rinaldi, *On the equivalence problem for succession rules*, Discrete Math., **298** (2005), 142–154.

[10] E. Deutsch, L. Ferrari and S. Rinaldi, *Production matrices*, Advances in Applied Mathematics, **34** (2005), 101–122.

[11] E. Duchi, A. Frosini, R. Pinzani and S. Rinaldi, *A note on rational succession rules*, Journal of Integer Sequences, Article 03.1.7, 6, 2003.

[12] E. Egge and T. Mansour, *Restricted permutations, Fibonacci numbers, and k-generalized Fibonacci numbers*, Integers Electronic Journal of Combinatorial Number Theory, Article A01, **5** (2005).

[13] L. Ferrari, E. Pergola, R. Pinzani and S. Rinaldi, *An algebraic characterization of the set of succession rules*, Theoretical Computer Science, **281** (2002), 351–367.

[14] L. Ferrari, E. Pergola, R. Pinzani and S. Rinaldi, *Jumping succession rules and their generating functions*, Discrete Math., **271** (2003), 29–50.

[15] G. Firro and T. Mansour, *Restricted permutations and polygons*, The Third International Conference on Permutations Patterns, Gainsville, Florida., March 7–11, 2005.

[16] A. Juarna and V. Vajnovszki, *More about a Simion-Schmidt's bijection*, (Preprint Submitted).

[17] A. Juarna and V. Vajnovszki, *Isomorphism between classes counted by Fibonacci numbers*, Words, **36** (2005), 51–62, September 13–17.

[18] A. Del Lungo, A. Frosini and S. Rinaldi, *ECO method and the exhaustive generation of convex polyominoes*, DMTCS, pages 103–116, 2003.

[19] T. Mansour, *Permutations avoiding a pattern from S_k and at least two patterns from S_3*, Ars Combin., **62** (2002), 227–239.

[20] R. Simion and F.W. Schmidt, *Restricted permutations*, European J. Combin., **6** (1985), 383–406.

[21] J. West, *Generating trees and forbidden subsequences*, Discrete Mathematics, **157** (1-3) (1996), 363–374.

Theoretical Computer Science 409 (2008) 382–393

Contents lists available at ScienceDirect

Theoretical Computer Science

journal homepage: www.elsevier.com/locate/tcs

The pruning-grafting lattice of binary trees

J.L. Baril *, J.M. Pallo

LE2I, UMR 5158, Université de Bourgogne, B.P. 47870, F21078 DIJON-Cedex, France

ARTICLE INFO

Article history:
Received 6 April 2007
Received in revised form 4 June 2008
Accepted 25 August 2008
Communicated by M. Crochemore

Keywords:
Lattices
Binary trees
Feasible sequences
Distribution sequences
Catalan numbers

ABSTRACT

We introduce a new lattice structure B_n on binary trees of size n. We exhibit efficient algorithms for computing meet and join of two binary trees and give several properties of this lattice. More precisely, we prove that the length of a longest (resp. shortest) path between **0** and **1** in B_n equals to the Eulerian numbers $2^n - (n+1)$ (resp. $(n-1)^2$) and that the number of coverings is $\binom{2n}{n-1}$. Finally, we exhibit a matching in a constructive way. Then we propose some open problems about this new structure.

© 2008 Elsevier B.V. All rights reserved.

1. Introduction

This paper introduces a new operation on rooted, ordered, binary trees called the pruning-grafting transformation. We show that this transformation endows the set of binary trees with a new lattice structure. This is a new attempt to define a lattice structure on a Catalan set, i.e. a set of combinatorial objects enumerated by Catalan numbers.

The set B_n of binary trees with n internal nodes related by rotations is a lattice, the so-called n-th Tamari lattice [24]. These lattices have been extensively studied for algebraic and combinatorial purposes. A number of references on this subject are available in [28]. Like rotation, our pruning-grafting transformation is a simple and natural operation on binary trees. But as for rotation, the characterization of this transformation is unfortunately rather complex.

The idea developed in this paper is both similar to, and different from the tool introduced by Parker and Ram in [30]. Similar because our pruning-grafting operation resembles their balancing exchange. But different because binary trees used in [30] for constructing Huffman codes are *unordered*, whereas the binary trees in our paper are *ordered*.

Ordered binary trees of B_n are enumerated by the well-known n-th Catalan number $\binom{2n}{n}/(n+1)$. A large number of various classes of combinatorial objects are Catalan sets. It is the case, among others, of ballot sequences, planar trees, Young tableaux, nonassociative products, stack sortable permutations, and so on. A list of over 60 types of such combinatorial classes of independent interest has been compiled by Stanley [39, p. 219]. A certain number of explicit bijections between these Catalan classes can also be found in the literature.

Of much of interest are the following lattices. First the uppermentioned Tamari lattices can be obtained equivalently in two other ways. The coverings correspond to reparenthesizations of letters products [14] and to diagonal flips in triangulations [4,19,37]. Then the lattice of noncrossing partitions, the so-called Kreweras lattice, is equipped with the refinement order [21]. See [11,12,31,35] and numerous references in the exhaustive survey [36]. Also, Dyck words [3,17] have been endowed by lattice structures. In [1,13] the authors studied the Stanley lattices in term of Dyck paths.

The paper is organized as follows. First we recall some classical definitions on binary trees. Then we show that B_n equipped with the pruning-grafting transformation is a lattice. We exhibit an efficient algorithm for computing the meet and join of two binary trees. We study some properties of this lattice and conclude by some open problems.

* Corresponding author. Tel.: +33 0380393801.
E-mail addresses: barjl@u-bourgogne.fr (J.L. Baril), pallo@u-bourgogne.fr (J.M. Pallo).

0304-3975/$ – see front matter © 2008 Elsevier B.V. All rights reserved.
doi:10.1016/j.tcs.2008.08.031

2. Distribution sequences of binary trees

A *binary tree* is a rooted, ordered, unlabeled tree in which every node is either a leaf $\square$ (i.e. a node without child) or an internal node $\bigcirc$ having two children. We denote by B_n the set of binary trees with n internal nodes (and thus with $n+1$ leaves). It is well-known that $\mathrm{card}(B_n) = |B_n|$ equals the Catalan number $\binom{2n}{n}/(n+1)$. The set B of binary trees is recursively defined by $B = \square + \bigcirc BB$ in Polish or linear notation. It is well known that a sequence with n circles and $(n+1)$ squares corresponds to the Polish notation of a tree of B_n iff in every proper prefix of this sequence, the number of circles is greater or equal to the number of squares, the last one being a square. Let T_L and T_R be the left and right subtrees of $T \in B_n$. Thus, we can write $T = \bigcirc T_L T_R$. The leaves of a binary tree T are numbered by a preorder traversal (i.e. from left to right). The order of leaves is *significant*. The *feasible sequence* of T (also called *path-length sequence*) is the integer sequence $\ell_T = (\ell_T(1), \ell_T(2), \ldots, \ell_T(n+1))$ where $\ell_T(i)$ is the *level number* of the i-th leaf, i.e. the length of unique path between the i-th leaf and the root [34]. Let also $\bar{\ell}$ be the mirror sequence $\bar{\ell} = (\ell_T(n+1), \ell_T(n), \ldots, \ell_T(1))$ which is the feasible sequence of the mirror tree $\bar{T}$ recursively defined by $\bar{T} = \bigcirc \bar{T}_R \bar{T}_L$ and $\bar{\square} = \square$. For example the tree defined in Polish notation by $\bigcirc\square\bigcirc\bigcirc\square\bigcirc\square\bigcirc\square\square\bigcirc\bigcirc\square\square\square$ has the feasible sequence $(1, 3, 4, 5, 5, 4, 4, 3)$. Its mirror is $\bigcirc\bigcirc\bigcirc\square\bigcirc\square\square\bigcirc\bigcirc\bigcirc\square\square\square\square\square$ and has the feasible sequence $(3, 4, 4, 5, 5, 4, 3, 1)$. Now let us consider a sequence $(a(1), a(2), \ldots, a(n+1))$ with an integer $k \in [2, n+1]$ such that $a(k-1) = a(k) = q$. The process of replacing the pair $(a(k-1), a(k)) = (q, q)$ by $q-1$ in order to get the new sequence $(a(1), a(2), \ldots, a(k-2), q-1, a(k+1), \ldots, a(n+1))$ is called a *reduction*. Ruskey and Hu [34] give the following necessary and sufficient condition so that an $(n+1)$-length sequence represents a binary tree of B_n. A sequence $(a(1), a(2), \ldots, a(n+1))$ is feasible iff a series of n reductions from the left or right (in any order) reduce the original sequence to the single integer 0.

Moreover, the feasible sequence of a binary tree obeys what we call Kraft equality, a special case of the Kraft inequality of noiseless coding theory (see [15], p. 45). A necessary condition [20, p. 404, ex. 3] for a sequence $(a(1), a(2), \ldots, a(n+1))$ to be feasible is that:

$$\sum_{i=1}^{n+1} 2^{-a(i)} = 1.$$

Lemma 1 ([20, p. 404, ex. 3]). *Let $\ell_T = (\ell_T(1), \ell_T(2), \ldots, \ell_T(n+1))$ be a feasible sequence of a binary tree T and let T' be a subtree of T where the root of T' is at level k in T. If $m_1 < m_2$ and $(\ell_T(m_1), \ell_T(m_1+1), \ldots, \ell_T(m_2-1), \ell_T(m_2))$ is the subsequence of ℓ_T corresponding to the leaves of T' then*

$$\sum_{i=m_1}^{m_2} 2^{-\ell_T(i)} = 2^{-k}.$$

The sequence $2^{-\ell_T} = (2^{-\ell_T(1)}, 2^{-\ell_T(2)}, \ldots, 2^{-\ell_T(n+1)})$ is called the *density sequence* of the binary tree (the sum of its entries is one). So we define its associated *distribution sequence* as the ascending sequence:

$$L_T = \left(2^{-\ell_T(1)}, 2^{-\ell_T(1)} + 2^{-\ell_T(2)}, \ldots, \sum_{j=1}^{i} 2^{-\ell_T(j)}, \ldots, 1\right).$$

In the sequel of this paper, the feasible sequences are denoted by lowercase letters (ℓ for instance) and their distribution sequences by uppercase letters (L). We assume that $\leq$ is the usual ordering on two n-length sequences, i.e. if $\ell = (\ell(1), \ldots, \ell(n+1))$ and $\ell' = (\ell'(1), \ldots, \ell'(n+1))$, we say that $\ell \leq \ell'$ if $\ell(i) \leq \ell'(i)$ for all $i \in [1, n+1]$.

3. The pruning-grafting transformation

In this section, we define a new transformation on binary trees which can be characterized using distribution sequences. This induces a new structure lattice on B_n.

Definition 1. The *pruning-grafting transformation* $\rightarrow$ on B_n with $n \geq 2$ is defined by the covering $T \rightarrow T'$ if there exist $k \geq 1$, $\tau_1 \in \bigcirc\{\bigcirc, \square\}^*$ and $\tau_2 \in \{\bigcirc, \square\}^*$ such that $T = \tau_1 \square \bigcirc^k \square\square\tau_2$ and $T' = \tau_1 \bigcirc \square\square \bigcirc^{k-1} \square\tau_2$. Let $\overset{*}{\rightarrow}$ be the reflexive transitive closure of $\rightarrow$ on B_n.

The pruning-grafting transformation prunes by replacing a "little" subtree $T_1 = \bigcirc\square\square$ of T by a leaf and grafts this subtree $T_1 = \bigcirc\square\square$ instead of the leaf $\square$ just before T_1 in the Polish notation of T. See Fig. 1 for instance.

Proposition 1. *Let $T, T' \in B_n$ and $\ell_T, \ell_{T'}$ be their corresponding feasible sequences. Then we have $T \rightarrow T'$ iff there exist $p \geq 1$, $q \geq 2$ and $2 \leq i \leq n$ such that*

$$\begin{aligned} \ell_T &= (\ell_T(1), \ell_T(2), \ldots, \ell_T(i-2) \quad , p \quad , q \quad , q \quad , \ell_T(i+2), \ldots, \ell_T(n+1)) \text{ and} \\ \ell_{T'} &= (\ell_T(1), \ell_T(2), \ldots, \ell_T(i-2) \quad , p+1 \quad , p+1 \quad , q-1 \quad , \ell_T(i+2), \ldots, \ell_T(n+1)). \end{aligned}$$

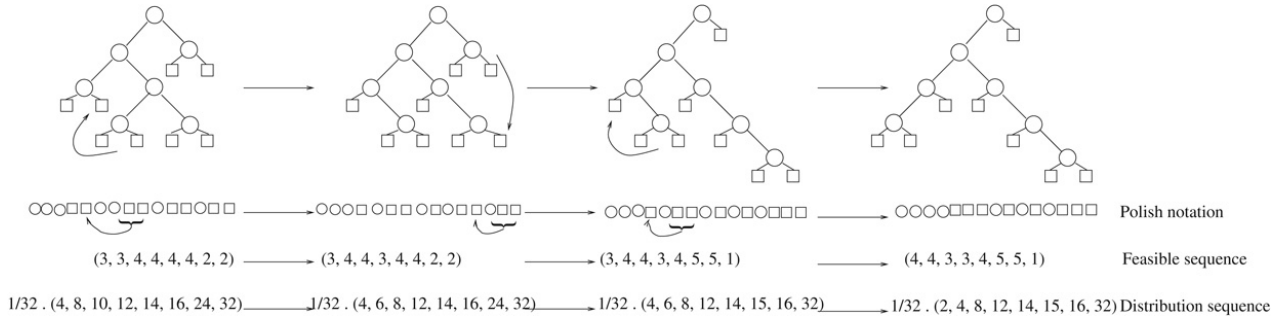


Fig. 1. Three pruning-grafting transformations in B_7.

Proof. One can easily prove that the condition is necessary. Conversely, let us assume that the feasible sequences of T and T' are $\ell_T = (\ell_T(1), \ell_T(2), \ldots, \ell_T(i-2), p, q, q, \ell_T(i+2), \ldots, \ell_T(n+1))$ and $\ell_{T'} = (\ell_T(1), \ell_T(2), \ldots, \ell_T(i-2), p+1, p+1, q-1, \ell_T(i+2), \ldots, \ell_T(n+1))$.
We distinguish between two cases.

(1) The i-th leaf of T is a left leaf. Since the i-th and $(i+1)$-th leaves have the same level number q, the Polish notation of T is $\tau\,\square\,\bigcirc^k \underbrace{\square\quad\square}_{i\text{-th}(i+1)\text{-th}} \ldots$ where $\tau \in \bigcirc\{\bigcirc, \square\}^*$ and $k \geq 1$. Thus, T' is obtained from T by a pruning-grafting transformation.

(2) Assume that the i-th leaf of T is a right leaf. Since the i-th and $(i+1)$-th leaves have the same level number q, T can be written $T = \tau\,\underbrace{\square\,\square}_{i\text{-th}}\,\bigcirc^k \underbrace{\square}_{(i+1)\text{-th}} \ldots$ where $\tau \in \bigcirc\{\bigcirc, \square\}^*$ and $k \geq 1$. Since the $(i-1)$-th and i-th leaves of T' have level $p+1$, T' is of the form $\tau\,\bigcirc\,\underbrace{\square\,\square}_{i\text{-th}} \ldots$. Moreover, the $(i+1)$-th leaf of T' is attached to a node of τ that also is the parent of the i-th leaf in T. This would mean that the level number of the $(i+1)$-th leaf of T' is at least q which contradicts the hypothesis. □

Corollary 1. *Let $T, T' \in B_n$ and L_T, $L_{T'}$ their distribution sequences. Then we have $T \to T'$ iff there exist $p \geq 1$, $q \geq 2$ and $2 \leq i \leq n$ such that $L_T = (L_T(1), \ldots, L_T(i-2), L_T(i-2) + 2^{-p}, L_T(i-2) + 2^{-p} + 2^{-q}, L_T(i+1), \ldots, 1)$ and $L_{T'} = (L_T(1), \ldots, L_T(i-2), L_T(i-2) + 2^{-(p+1)}, L_T(i-2) + 2^{-p}, L_T(i+1), \ldots, 1)$.*

Definition 2. Let T and T' be two different trees of B_n. Let $\sigma_1 \in \bigcirc\{\bigcirc, \square\}^*$ be the longest common prefix of T and T' in their Polish notation. Let us assume that

$$T = \sigma_1\sigma_2\,\square\,\bigcirc^j \underbrace{\bigcirc\,\square\,\square}_{\tau}\,\sigma_3$$

where

- $j \geq 0$, and
- σ_2 is the empty word or $\sigma_2 \in \square\{\bigcirc, \square\}^*$ and $\sigma_2\,\square$ does not contain any occurrence of $\tau = \bigcirc\,\square\,\square$ (i.e. $\tau = \bigcirc\,\square\,\square$ is the leftmost occurrence of $\bigcirc\,\square\,\square$ in T after σ_1), and
- $\sigma_3 \in \{\bigcirc, \square\}^*$.

Then we necessarily have $T' = \sigma_1\sigma_2'$ with $\sigma_2' \in \bigcirc\{\bigcirc, \square\}^*$.

We define the tree $U(T, T') \in B_n$ (U for up) such that $T \to U(T, T')$ and

$$U(T, T') = \sigma_1\sigma_2 \underbrace{\bigcirc\,\square\,\square}_{\tau}\,\bigcirc^j\,\square\,\sigma_3.$$

Moreover, we set $U(T, T) = T$.

For example, if:

$$T = \bigcirc\,\bigcirc\,\square\,\square\,\bigcirc\,\square\,\bigcirc\,\bigcirc\,\square\,\bigcirc\,\bigcirc\,\square\,\square\,\square\,\square$$
$$T' = \bigcirc\,\bigcirc\,\square\,\square\,\bigcirc\,\bigcirc\,\square\,\square\,\bigcirc\,\square\,\bigcirc\,\bigcirc\,\square\,\square\,\square$$

then $\sigma_1 = \bigcirc\,\bigcirc\,\square\,\square\,\bigcirc$ $\sigma_2 = \square\,\bigcirc\,\bigcirc$, $\sigma_3 = \square\,\square$ and $j = 1$. We obtain $U(T, T') = \bigcirc\,\bigcirc\,\square\,\square\,\bigcirc\,\square\,\bigcirc\,\bigcirc\,\bigcirc\,\square\,\square\,\bigcirc\,\square\,\square\,\square$.

Lemma 2. *Let T and T' be two trees in B_n such that their distribution sequences L_T and $L_{T'}$ verify $L_T > L_{T'}$. Thus, $U(T, T')$ exists and its distribution sequence $L_{U(T,T')}$ verifies $L_T > L_{U(T,T')} \geq L_{T'}$.*

Proof. Via Definition 2, $U(T, T')$ exists since $L_T > L_{T'}$ implies $T = \sigma\,\square\,\nu$ and $T' = \sigma\,\bigcirc\,\nu'$ with $\sigma \in \bigcirc\{\bigcirc, \square\}^*$ and $\nu, \nu' \in \{\bigcirc, \square\}^*$. Let us prove the inequality $L_U \geq L_{T'}$ where $L_U = L_{U(T,T')}$. Let $r = \min\{i \geq 1 | L_T(i) > L_{T'}(i)\}$ and let t be the smallest $i > r$ such that T is of the form $T = \ldots \bigcirc \underbrace{\square\quad\square}_{i\text{-th}(i+1)\text{-th}} \ldots$. Let T_0 be the smallest subtree in T containing the r-th and t-th leaves. In Polish notation let $T_0 = \bigcirc T_1 T_2$. The r-th leaf belongs to T_1 and the t-th belongs to T_2. The rightmost leaf of T_1 is numbered by $s \geq r$ in T. With these hypotheses, we necessarily have the following properties:

(a) $\ell_T(r) \geq \ell_T(r+1)$ and $\ell_T(i) > \ell_T(i+1)$ for $r+1 \leq i \leq s-1$;
(b) $\ell_T(i) < \ell_T(i+1)$ for $s+1 \leq i \leq t-1$.

As ℓ_T and $\ell_{T'}$ have the same entries for $i \leq r-1$, we have $L_T(r-1) = L_{T'}(r-1)$. Moreover, $L_T(r) > L_{T'}(r)$ implies that $\ell_T(r) < \ell_{T'}(r)$ and Lemma 1 shows that there exists r_0, $r_0 \geq r+1$, such that $L_T(r-1) + 2^{-\ell_T(r)} = L_T(r-1) + \sum_{i=r}^{r_0} 2^{-\ell_{T'}(i)}$, i.e. $L_T(r) = L_{T'}(r_0)$.

In the case where r verifies $r+1 \leq s$, then the $(r+1)$-th leaf of T is attached to a node of the common prefix of T and T'. Thus, we necessarily have $\ell_T(r+1) \leq \ell_{T'}(r_0+1)$. Therefore, there exists r_1 with $r < r_0 < r_1$, such that $L_T(r+1) = L_{T'}(r_1)$. By repeating this process until reaching the s-th leaf, we prove that there exists $s' \geq s+1$ such that $L_T(s) = L_{T'}(s')$.

Now we first want to show that there exists $s' \geq t$ such that $L_{T'}(s') = L_T(s)$ and $s \leq t-1$.

If $s' < t$, the hypothesis gives us $L_{T'}(s'+1) \leq L_T(s'+1)$. This means that $2^{-\ell_{T'}(s'+1)} \leq 2^{-\ell_T(s+1)} + \cdots + 2^{-\ell_T(s'+1)}$. Since $s+1 \leq s' < t$, the inequalities follow $\ell_T(s+1) < \ell_T(s+2) < \cdots < \ell_T(s'+1)$ and we obtain $2^{-\ell_{T'}(s'+1)} \leq 2^{-\ell_T(s+1)} + 2^{-\ell_T(s+2)-1} < 2^{-\ell_T(s+1)+1}$. Thus, $\ell_{T'}(s'+1) \geq \ell_T(s+1)$ and there exists $s'' \geq s'+1$ such that $L_{T'}(s'') = L_T(s+1)$. If $s'' \leq t-1$, we repeat this process for the index $s+2$ and so on, and we stop it when s'' holds $s'' \geq t$.

So, we have obtain s' such that $s \leq t-1$ and $s' \geq t$ with $L_{T'}(s') = L_T(s)$. Notice that if $s = t-1$, then we necessarily have $L_T(t-2) = L_{T'}(s)$.

Since $L_{T'}(i) \leq L_T(i)$ for all i, we obtain $L_{T'}(i) \leq L_T(i) = L_U(i)$ for $i \leq t-2$ and $i \geq t+1$.

In the case where $i = t-1$, we obtain

$$\begin{aligned} L_U(t-1) - L_{T'}(t-1) &= L_T(t-1) - 2^{-\ell_T(t-1)+1} - L_{T'}(t-1) \\ &= L_T(t-1) - L_T(t-2) + L_T(t-2) - 2^{-\ell_T(t-1)+1} - L_{T'}(t-1) \\ &= 2^{-\ell_T(t-1)+1} + L_T(t-2) - L_{T'}(t-1) \\ &\geq 2^{-\ell_T(t-1)+1} + L_T(s) - L_{T'}(t-1) \\ &\geq 2^{-\ell_T(t-1)+1} + L_T(s) - L_{T'}(s') \geq 2^{-\ell_T(t-1)+1}. \end{aligned}$$

If $i = t$, we have

$$\begin{aligned} L_U(t) - L_{T'}(t) &= L_T(t-1) - L_{T'}(t) \\ &\geq L_T(s) - L_{T'}(t) \\ &\geq L_T(s) - L_{T'}(s') \geq 0. \quad \square \end{aligned}$$

Definition 3. Let T and T' be two different trees of B_n with $n \geq 2$. Let $\sigma_1 \in \bigcirc\{\bigcirc, \square\}^*$ be their longest common prefix in their Polish notation. Let us assume that $T = \sigma_1 \square \sigma_2$ with $\sigma_2 \in \{\bigcirc, \square\}^*$. Then we necessarily have

$$T' = \sigma_1 \sigma_2' \underbrace{\bigcirc \square \square}_{\tau} \bigcirc^j \square \sigma_3'$$

where

- $j \geq 0$, and
- σ_2' is the empty word or $\sigma_2' \in \bigcirc\{\bigcirc, \square\}^*$ such that σ_2' does not contain any occurrence $\tau = \bigcirc \square \square$ (i.e. τ is the leftmost occurrence of $\bigcirc \square \square$ in T' after σ_1), and
- $\sigma_3' \in \{\bigcirc, \square\}^*$.

We define the tree $D(T, T') \in B_n$ (D for down) such that $D(T, T') \to T'$ and

$$D(T, T') = \sigma_1 \sigma_2' \square \bigcirc^j \underbrace{\bigcirc \square \square}_{\tau} \sigma_3'.$$

Moreover, we set $D(T, T) = T$.

For example, if:

$$\begin{aligned} T &= \bigcirc \bigcirc \square \square \bigcirc \square \bigcirc \square \bigcirc \bigcirc \square \bigcirc \square \square \square \\ T' &= \bigcirc \bigcirc \square \square \bigcirc \bigcirc \square \bigcirc \square \square \bigcirc \square \bigcirc \square \square \end{aligned}$$

then $\sigma_1 = \bigcirc \bigcirc \square \square \bigcirc$, $\sigma_2' = \bigcirc \square$, $\sigma_3' = \bigcirc \square \square$ and $j = 1$. We obtain $D(T, T') = \bigcirc \bigcirc \square \square \bigcirc \bigcirc \square \square \bigcirc \bigcirc \square \square \bigcirc \square \square$.

Lemma 3. *Let T and T' be two trees in B_n such that their distribution sequences L_T and $L_{T'}$ verify $L_T > L_{T'}$. Thus, $D(T, T')$ exists and its distribution sequence $L_{D(T,T')}$ verifies $L_T \geq L_{D(T,T')} > L_{T'}$.*

Proof. Via Definition 3, $D(T, T')$ exists since $L_T > L_{T'}$ implies $T = \sigma \square \nu$ and $T' = \sigma \bigcirc \nu'$ with $\sigma \in \bigcirc\{\bigcirc, \square\}^*$ and $\nu, \nu' \in \{\bigcirc, \square\}^+$. Let us prove the inequality $L_D \leq L_T$ where $L_D = L_{D(T,T')}$. Let $r = \min\{i \geq 1 | L_T(i) > L_{T'}(i)\}$ and let s be the smallest $i \geq r$ such that T' is of the form $T' = \sigma \bigcirc \ldots \bigcirc \underset{i\text{-th}}{\square} \underset{(i+1)\text{-th}}{\square} \ldots$ where σ is the common prefix of T and T'. Since L_D differs from $L_{T'}$ at indices s and $s+1$, the inequality $L_D(i) = L_{T'}(i) \leq L_T(i)$ holds for $i \notin \{s, s+1\}$. Now let us examine $L_D(i) - L_T(i)$ for $i \in \{s, s+1\}$. Using Lemma 1, we deduce easily that $L_T(r) \geq L_{T'}(s+1)$. This provides $L_D(s) - L_T(s) = L_{T'}(s+1) - L_T(s) \leq L_T(r) - L_T(s) \leq 0$.

For the index $s+1$, $L_D(s+1) - L_T(s+1) = L_{T'}(s+2) - L_T(s+1) - 2^{-(\ell_{T'}(s+2)+1)}$. We distinguish two cases:

(1) if there exists $k \le s+1$ such that $L_{T'}(s+2) \le L_T(k)$ then we clearly have $L_D(s+1) - L_T(s+1) \le 0$;

(2) otherwise (i.e. $k \ge s+2$), $L_{T'}(s+2) > L_T(s+1)$. Thus, $2^{-\ell_{T'}(s+2)} > \sum_{i=r+1}^{s+1} 2^{-\ell_T(i)}$ and $\forall\, i \in [r+1, s+1]$, $2^{-\ell_T(i)} < 2^{-\ell_{T'}(s+2)}$ which means $\ell_T(i) \ge \ell_{T'}(s+2)+1$, $\forall\, i \in [r+1, s+1]$. This implies that there exists $k \ge r+1$ such that $L_T(k) = L_{T'}(s+2)$. The hypothesis $L_{T'} < L_T$ induces $k \le s+2$ and with $k \ge s+2$ we obtain $k = s+2$, i.e. $L_{T'}(s+2) = L_T(s+2)$.

Therefore, $L_{T'}(s+1) + 2^{-\ell_{T'}(s+2)} = L_T(s+1) + 2^{-\ell_T(s+2)}$ and since $L_{T'}(s+1) \le L_T(r) \le L_T(s) < L_T(s+1)$, we obtain $2^{-\ell_T(s+2)} < 2^{-\ell_{T'}(s+2)}$ (i.e. $\ell_T(s+2) \ge \ell_{T'}(s+2)+1$). Thus,

$$\begin{aligned} L_D(s+1) - L_T(s+1) &= L_{T'}(s+2) - L_T(s+1) - 2^{-(\ell_{T'}(s+2)+1)} \\ &= L_T(s+2) - L_T(s+1) - 2^{-(\ell_{T'}(s+2)+1)} \\ &= 2^{-\ell_T(s+2)} - 2^{-(\ell_{T'}(s+2)+1)} \le 0. \quad \square \end{aligned}$$

Theorem 1. *Giving T and $T' \in B_n$. Let L_T and $L_{T'}$ be their distribution sequences. Then we have $T \xrightarrow{*} T'$ iff $L_T \ge L_{T'}$.*

Proof. Let us assume that $T \xrightarrow{*} T'$, i.e. there exists a path $T = T_0 \to T_1 \to T_2 \to \cdots \to T_k = T'$ ($k \ge 1$). Via Corollary 1, the distribution sequences L_{T_i} of T_i verify: $L_T \ge L_{T_1} \ge L_{T_2} \ge \cdots \ge L_{T_{k-1}} \ge L_{T'}$. Thus, we obtain $L_T \ge L_{T'}$.

Conversely, assume that L_T and $L_{T'}$ verify $L_T \ge L_{T'}$ with $T \ne T'$. Let $i_0 \ge 1$ be the smallest integer i such that $L_T(i) > L_{T'}(i)$. There are two cases to consider:

(1) there doesn't exist any internal node $\bigcirc$ after the i_0-th leaf in T. Since $\ell_T(i) = \ell_{T'}(i)$ for $i < i_0$ and $\ell_T(i_0) < \ell_{T'}(i_0)$, the Polish notation of T and T' can be written as $T = \sigma\, \square^k$ for $k \ge 1$, $\sigma \in \bigcirc\{\bigcirc, \square\}^*$ and $T' = \sigma\, \bigcirc \ldots$ where σ is the longest common prefix of T and T'. Thus, the number of nodes of T' is greater than those of T which is a contradiction

(2) there exists an internal node $\bigcirc$ after the i_0-th leaf in T. This means that there exists an occurrence of $\bigcirc\, \square\, \square$ after the i_0-th leaf in T. Via Lemma 2, we have $T \to U(T, T')$ and if L_U is the distribution sequence of $U(T, T')$ then $L_T > L_U \ge L_{T'}$.

By iterating this process, we obtain $T \xrightarrow{*} T'$. $\square$

Theorem 2. *For all n, the poset $(B_n, \xrightarrow{*})$ is a lattice.*

Proof. It suffices to show that any two elements of B_n have a least upper bound. The existence of greatest lower bound then follows automatically since B_n is finite with as least element $\mathbf{0}$ and as greatest element $\mathbf{1}$ defined by $\ell_{\mathbf{0}} = (1, 2, \ldots, n-1, n, n)$ and $\ell_{\mathbf{1}} = (n, n, n-1, \ldots, 2, 1)$. Let T and T' be two different trees in B_n and L_T and $L_{T'}$ their distribution sequences. In this proof, we construct a tree $S \in B_n$ that is candidate to be the join element of T and T' and we show that this element is really the join.

Given τ and τ' in B_n, we define the following function **join**(τ,τ'):

Function join(τ,τ')
begin
  while $\tau \ne \tau'$ **do**
    $i_0 := \min\{i \in [1, n] | L_\tau(i) \ne L_{\tau'}(i)\}$
    if $L_\tau(i_0) < L_{\tau'}(i_0)$ **then**
      $\nu' := U(\tau', \tau)$ (thus $\tau' \to \nu'$)
      $\tau' := \nu'$
    else
      $\nu := U(\tau, \tau')$ (thus $\tau \to \nu$)
      $\tau := \nu$
    endif
  enddo
  return τ
end

For example, if we perform this function for $\tau = (2, 2, 3, 3, 3, 3)$ and $\tau' = (1, 5, 5, 4, 3, 2)$ (i.e. $L_\tau = \frac{1}{32}(8, 16, 20, 24, 28, 32)$ and $L_{\tau'} = \frac{1}{32}(16, 17, 18, 20, 24, 32)$), then $i_0 = 1$ and $L_\tau(1) < L_{\tau'}(1)$; we replace τ' by $U(\tau', \tau) = (2, 2, 4, 4, 3, 2)$ that have the distribution sequence $\frac{1}{32}(8, 16, 18, 20, 24, 32)$; $i_0 = 3$ and $L_\tau(3) > L_{\tau'}(3)$, we replace τ by $U(\tau, \tau') = (2, 2, 3, 4, 4, 2)$ whose distribution sequence is $\frac{1}{32}(8, 16, 20, 22, 24, 32)$. Yet, $i_0 = 3$, and $L_\tau(3) > L_{\tau'}(3)$, we replace τ by $U(\tau, \tau') = (2, 2, 4, 4, 3, 2)$ which implies that $\tau = \tau'$. The function returns $\tau = (2, 2, 4, 4, 3, 2)$.

In order to construct the tree $S = T \vee T'$, we apply the function **join**(τ,τ') with $\tau = T$ and $\tau' = T'$. At the end of this process, τ and τ' are the same tree: we obtain $\tau = \tau' = S$. So we have constructed a path $\mathcal{P}$ between T and S such that $\mathcal{P} : T \to \tau_1 \to \cdots \to \tau_k = S$ and a path $\mathcal{P}'$ between T' and S such that $\mathcal{P}' : T' \to \tau'_1 \to \cdots \to \tau'_{k'} = S$. Moreover, if L_S is the distribution sequence of S, we obviously have $L_S(i) = L_T(i) = L_{T'}(i)$ for $i < i_0$ and $L_S(i_0) = \min(L_T(i_0), L_{T'}(i_0))$.

Now, we show that the tree S is really the least upper bound of T and T'. For this, let T'' be a tree in B_n such that $T \xrightarrow{*} T''$ and $T' \xrightarrow{*} T''$ and prove us that $S \xrightarrow{*} T''$ or equivalently $L_{T''} \le L_S$ in term of distribution sequences.

In the case where there exists $i_1 \in [1, i_0[$ such that $L_{T''}(i) = L_S(i)$ for $1 \le i < i_1$ and $L_{T''}(i_1) \ne L_S(i_1)$, we necessarily have $L_{T''}(i_1) < L_T(i_1) = L_{T'}(i_1)$. Using Lemma 3, there exists $S^{(1)} = D(T'', T) = D(T'', T') \in B_n$ such that $S^{(1)} \to T''$, $T \xrightarrow{*} S^{(1)}$ and

$T' \xrightarrow{*} S^{(1)}$. If we denote by $L_{S^{(1)}}$ the distribution sequence of $S^{(1)}$ then we have $L_{S^{(1)}} \leq L_T$ and $L_{S^{(1)}} \leq L_{T'}$. By repeating this process with $S^{(1)}$, we obtain a path $S^{(0)} = T'' \leftarrow S^{(1)} \leftarrow \cdots \leftarrow S^{(m)}$ such that $T \xrightarrow{*} S^{(m)}$, $T' \xrightarrow{*} S^{(m)}$ and the distribution sequence $L_{S^{(m)}}$ of $S^{(m)}$ verifies $L_{S^{(m)}}(i) = L_S(i) = L_T(i) = L_{T'}(i)$ for $1 \leq i < i_0$. Thus, $L_{S^{(m)}} \leq L_T$ and $L_{S^{(m)}} \leq L_{T'}$.

Moreover, if we have $L_T(i_0) < L_{T'}(i_0)$ (resp. $L_T(i_0) > L_{T'}(i_0)$) then, via Lemma 2, $\tau_1 \xrightarrow{*} S^{(m)}$ (resp. $\tau_1' \xrightarrow{*} S^{(m)}$).

We repeat all this process as follows: if $L_T(i_0) < L_{T'}(i_0)$, we replace T by τ_1 and T'' by $S^{(m)}$; or, if $L_T(i_0) > L_{T'}(i_0)$, we replace T' by τ_1' and T'' by $S^{(m)}$, and so on.

At the end of this process, $S \xrightarrow{*} S^{(m)}$ which proves that S corresponds to the least upper bound of T and T'. □

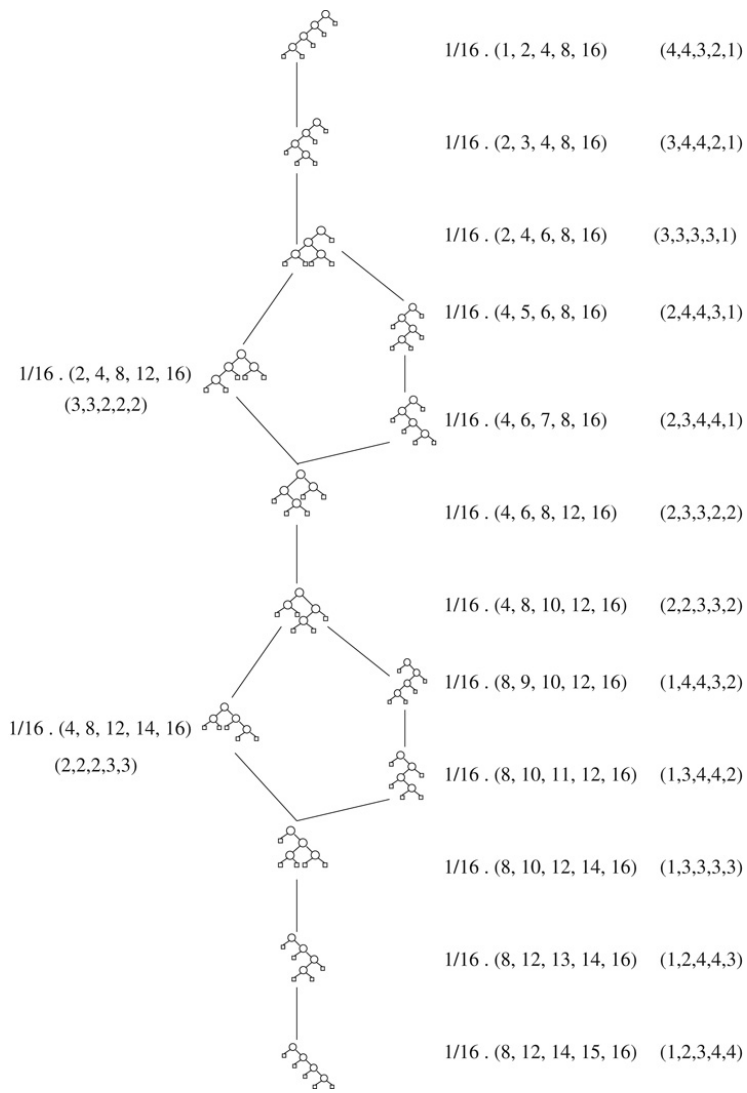


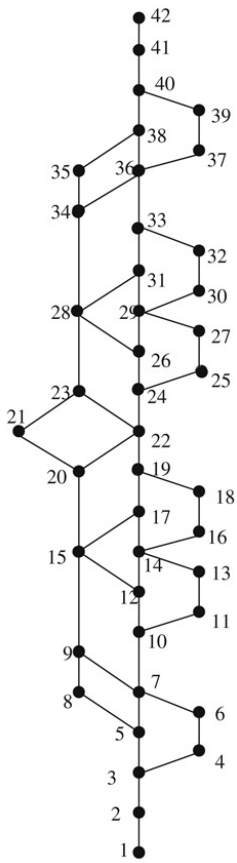
Fig. 2. The pruning-grafting lattice B_4. Each tree is encoded with its feasible and distribution sequences.

Notice that some relations do exist between the already known Catalan lattices. Indeed, recall that if T and T' are two elements in the Kreweras lattice such that $T \leq T'$, then $T \leq T'$ occurs in the Tamari lattice. If $T \leq T'$ in the Tamari lattice, then $T \leq T'$ occurs in the Stanley lattice. If $T \leq T'$ in the phagocyte lattice defined in [3], then $T \leq T'$ occurs in the Kreweras lattice. The covering set of the phagocyte lattice is included in the covering set of Kreweras lattice. However, the pruning-grafting lattice does not appear to have some similar relations with these lattices, see Figs. 2 and 3.

4. Some properties of $(B_n, \xrightarrow{*})$

The lattice $(B_n, \xrightarrow{*})$ has a greatest $\mathbf{1}$ and a least element $\mathbf{0}$ and their feasible sequences are respectively, $\ell_{\mathbf{1}} = (n, n, n-1, \ldots, 2, 1)$ and $\ell_{\mathbf{0}} = (1, 2, \ldots, n-1, n, n)$. $(B_n, \xrightarrow{*})$ is symmetric because $T \xrightarrow{*} T'$ iff $\bar{T}' \xrightarrow{*} \bar{T}$. The lattice $(B_n, \xrightarrow{*})$ is not modular since it contains pentagons (see Figs. 2 and 3).

1.	(1, 2, 3, 4, 5, 5)	1/32.(16,24,28,31,31,32)
2.	(1, 2, 3, 5, 5, 4)	1/32.(16,24,28,29,30,32)
3.	(1, 2, 4, 4, 4, 4)	1/32.(16,24,26,28,30,32)
4.	(1, 2, 4, 5, 5, 3)	1/32.(16,24,26,27,28,32)
5.	(1, 3, 3, 3, 4, 4)	1/32.(16,20,24,28,30,32)
6.	(1, 2, 5, 5, 4, 3)	1/32.(16,24,25,26,28,32)
7.	(1, 3, 3, 4, 4, 3)	1/32.(16,20,24,26,28,32)
8.	(2, 2, 2, 3, 4, 4)	1/32.(8,16,24,28,30,32)
9.	(2, 2, 2, 4, 4, 3)	1/32.(8,16,24,26,28,32)
10.	(1, 3, 4, 4, 3, 3)	1/32.(16,20,22,24,28,32)
11.	(1, 3, 4, 5, 5, 2)	1/32.(16,20,22,23,24,32)
12.	(1, 4, 4, 3, 3, 3)	1/32.(16,18,20,24,28,32)
13.	(1, 3, 5, 5, 4, 2)	1/32.(16,20,21,22,24,32)
14.	(1, 4, 4, 4, 4, 2)	1/32.(16,18,20,22,24,32)
15.	(2, 2, 3, 3, 3, 3)	1/32.(8,16,20,24,28,32)
16.	(1, 4, 5, 5, 3, 2)	1/32.(16,18,19,20,24,32)
17.	(2, 2, 3, 4, 4, 2)	1/32.(8,16,20,22,24,32)
18.	(1, 5, 5, 4, 3, 2)	1/32.(16,17,18,20,24,32)
19.	(2, 2, 4, 4, 3, 2)	1/32.(8,16,18,20,24,32)
20.	(2, 3, 3, 2, 3, 3)	1/32.(8,12,16,24,28,32)
21.	(3, 3, 2, 2, 3, 3)	1/32.(4,8,16,24,28,32)



22.	(2, 3, 3, 3, 3, 2)	1/32.(8,12,16,20,24,32)
23.	(3, 3, 2, 3, 3, 2)	1/32.(4,8,16,20,24,32)
24.	(2, 3, 4, 4, 2, 2)	1/32.(8,12,14,16,24,32)
25.	(2, 3, 4, 5, 5, 1)	1/32.(8,12,14,15,16,32)
26.	(2, 4, 4, 3, 2, 2)	1/32.(8,10,12,16,24,32)
27.	(2, 3, 5, 5, 4, 1)	1/32.(8,12,13,14,16,32)
28.	(3, 3, 3, 3, 2, 2)	1/32.(4,8,12,16,24,32)
29.	(2, 4, 4, 4, 4, 1)	1/32.(8,10,12,14,16,32)
30.	(2, 4, 5, 5, 3, 1)	1/32.(8,10,11,12,16,32)
31.	(3, 3, 3, 4, 4, 1)	1/32.(4,8,12,14,16,32)
32.	(2, 5, 5, 4, 3, 1)	1/32.(8,9,10,12,16,32)
33.	(3, 3, 4, 4, 3, 1)	1/32.(4,8,10,12,16,32)
34.	(3, 4, 4, 2, 2, 2)	1/32.(4,6,8,16,24,32)
35.	(4, 4, 3, 2, 2, 2)	1/32.(2,4,8,16,24,32)
36.	(3, 4, 4, 3, 3, 1)	1/32.(4,6,8,12,16,32)
37.	(3, 4, 5, 5, 2, 1)	1/32.(4,6,7,8,16,32)
38.	(4, 4, 3, 3, 3, 1)	1/32.(2,4,8,12,16,32)
39.	(3, 5, 5, 4, 2, 1)	1/32.(4,5,6,8,16,32)
40.	(4, 4, 4, 4, 2, 1)	1/32.(2,4,6,8,16,32)
41.	(4, 5, 5, 3, 2, 1)	1/32.(2,3,4,8,16,32)
42.	(5, 5, 4, 3, 2, 1)	1/32.(1,2,4,8,16,32)

Fig. 3. The pruning-grafting lattice B_5. Each tree is encoded with its feasible and distribution sequences.

Proposition 2. *Let T and T' be two trees of B_n such that $T \overset{*}{\to} T'$, then*

(i) $\overline{U(\bar{T}, \bar{T}')} \to T'$,

(ii) $T \to \overline{D(\bar{T}, \bar{T}')}$,

(iii) *the path* $T' \leftarrow D(T, T') \leftarrow D(T, D(T, T')) \leftarrow D(T, D(T, D(T, T'))) \leftarrow \cdots \leftarrow T$ *is a longest path between T and T',*

(iv) *the path* $T \to U(T, T') \to U(U(T, T'), T') \to U(U(U(T, T'), T'), T') \to \cdots \to T'$ *is a shortest path between T and T'.*

Proof. The two first items are deduced from the symmetry of the lattice.

Let us prove (iii). First, notice that if T' is obtained from T by one pruning-grafting transformation, T' cannot be obtained from T by two or more consecutive pruning-grafting transformations (we call this remark R_1).

We proceed by induction on the length of a longest path between T and T'. The property holds when the length of the longest path is two. Indeed, let $T \to T_1 \to T'$ be a longest path between T and T'. If T' has not two lower covers then $T_1 = D(T, T')$ and we obtain directly the result. Otherwise, T' has at least two lower covers, T_1 and T_2 and assume that $T_1 \neq D(T, T')$. As the longest length between T and T' is two, we easily see that there exists a two length path between T and T' such that $T \to D(T, T') \to T'$ which gives the results. This is true because we do not have $T = D(T, T')$ with the remark R_1. Now, let us assume that the property holds when the length of the longest path between T and T' is $\ell \geq 2$ and let us examine the case for a longest path of length $\ell + 1$. Let also $T = T_0 \to T_1 \to \cdots \to T_{\ell+1} = T'$ be a longest path of length $\ell + 1$ between T and T'.

If $T_\ell = D(T, T')$ and using the recurrence hypothesis for the path $T \to \cdots \to T_\ell$ we conclude directly.

Now, we assume that $T_\ell \neq D(T, T')$. This induces that T' has at least two lower covers T_ℓ and $D(T, T')$. We apply the recurrence hypothesis for the path between T and T_ℓ. This means that the path can be of the form $T \to T_1 \to \cdots \to T_{\ell-1} = D(T, T_\ell) \to T_\ell \to T'$.

Moreover, the remark R_1 allows us to prove that $D(T, T') \neq T_{\ell-1}$. Then we distinguish two cases: (I) $D(T, D(T, T')) = T_{\ell-1}$, (II) $D(T, D(T, T')) \neq T_{\ell-1}$ and we will prove that the second case does not occur.

Case (I). In this case, we can conclude directly that there exists a longest path of the form $T \to T_1 \to \cdots \to T_{\ell-1} \to D(T, T') \to T'$. See the following diagram for an illustration of this case.

$$\begin{tikzcd}
 & T' & \\
D(T,T') \arrow[ur] & & T_\ell \arrow[ul] \\
 & T_{\ell-1} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

with

$$\begin{aligned}
T' &= \sigma_1 \bigcirc \square\square \bigcirc^r \square\sigma_2 \bigcirc \square\square \bigcirc^s \square\sigma_3 \\
D(T,T') &= \sigma_1 \square \bigcirc^{r+1} \square\square\sigma_2 \bigcirc \square\square \bigcirc^s \square\sigma_3 \\
T_\ell &= \sigma_1 \bigcirc \square\square \bigcirc^r \square\sigma_2 \square \bigcirc^{s+1} \square\square\sigma_3 \\
T_{\ell-1} &= \sigma_1 \square \bigcirc^{r+1} \square\square\sigma_2 \square \bigcirc^{s+1} \square\square\sigma_3
\end{aligned}$$

where $r, s \geq 0, \sigma_1 \in \bigcirc\{\square, \bigcirc\}^*$ and $\sigma_2, \sigma_3 \in \{\square, \bigcirc\}^*$.
Case (II). This induces the following diagram which gives a contradiction with the hypothesis that $T \to T_1 \to \cdots \to T_{\ell-1} \to T_\ell \to T'$ is a longest path between T and T'.

$$\begin{tikzcd}
 & T' & \\
D(T,T') \arrow[ur] & & T_\ell \arrow[ul] \\
S \arrow[u] & & \\
 & T_{\ell-1} \arrow[ul] \arrow[uur] &
\end{tikzcd}$$

with

$$\begin{aligned}
T' &= \sigma_1 \bigcirc \square\square \bigcirc^r \square\square \bigcirc^s \square\sigma_2 \\
D(T,T') &= \sigma_1 \square \bigcirc^{r+1} \square\square\square \bigcirc^s \square\sigma_2 \\
S &= \sigma_1 \square \bigcirc^r \square \bigcirc \square\square \bigcirc^s \square\sigma_2 \\
T_\ell &= \sigma_1 \bigcirc \square\square \bigcirc^{r-1} \square \bigcirc^{s+1} \square\square\sigma_2 \\
T_{\ell-1} &= \sigma_1 \square \bigcirc^r \square\square \bigcirc^{s+1} \square\square\sigma_2
\end{aligned}$$

where $r \geq 1, s \geq 0, \sigma_1 \in \bigcirc\{\square, \bigcirc\}^*$ and $\sigma_2 \in \{\square, \bigcirc\}^*$.

This case does not occur since this would mean that the path $T \to T_1 \to \cdots \to T_{\ell-1} \to T_\ell \to T'$ is not a longest path between T and T'.

In conclusion, only the first case appears and we have constructed a path of length $(\ell + 1)$ between T and T' such that $T \to \cdots \to D(T, T_\ell) \to D(T, T') \to T'$ which proves the results by induction.

The item (iv) is obtained *mutatis mutandis*. Indeed, we also proceed by induction on the length of the path between T and T'. With the remark R_1, the result holds when the length of the shortest path is less or equal than two. Let us assume that the property holds when the length of the shortest path between T and T' is less or equal than $\ell \geq 2$ and let us examine the case for a shortest path of length $\ell + 1$. Let also $T = T_0 \to T_1 \to \cdots \to T_{\ell+1} = T'$ be a shortest path of length $\ell + 1$ between T and T'.

If $T_1 = U(T, T')$ and using the recurrence hypothesis for the path $T_1 \to \cdots \to T_\ell \to T'$, we conclude directly.

Now, let us assume that $T_1 \neq U(T, T')$. This induces that T has at least two upper covers T_1 and $U(T, T')$. We apply the recurrence hypothesis for the path between T_1 and T'. This means that the path can be of the form $T \to T_1 \to U(T_1, T') = T_2 \to \cdots \to T'$.

Thus, we distinguish two cases: (I) $T_2 = U(U(T, T'), T')$, (II) $T_2 \neq U(U(T, T'), T')$ and we will prove that the second case does not occur.
Case (I). $T_2 = U(U(T, T'), T')$; as previously, we can conclude directly that there exists a shortest path of the form $T \to U(T, T') \to U(T_1, T') = T_2 \to \cdots \to T'$.

Case (II). $T_2 \neq U(U(T, T'), T')$; we have the following diagram:

$$\begin{tikzcd}
 & T_3 & \\
U(T,T') \arrow[ur] & & T_2 \arrow[ul] \\
 & & T_1 \arrow[u] \\
 & T \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

with

$$\begin{aligned}
T &= \sigma_1 \square\, \bigcirc^r\, \square\square\, \bigcirc^s\, \square\square \sigma_2 \\
T_1 &= \sigma_1 \square\, \bigcirc^r\, \square\, \bigcirc\, \square\square\, \bigcirc^{s-1}\, \square \sigma_2 \\
T_2 &= \sigma_1 \square\, \bigcirc^{r+1}\, \square\square\square\, \bigcirc^{s-1}\, \square \sigma_2 \\
T_3 &= \sigma_1 \bigcirc\, \square\square\, \bigcirc^r\, \square\square\, \bigcirc^{s-1}\, \square \sigma_2 \\
U(T,T') &= \sigma_1 \bigcirc\, \square\square\, \bigcirc^{r-1}\, \square\, \bigcirc^s\, \square\square \sigma_2
\end{aligned}$$

where $r, s \geq 1$ $\sigma_1 \in \bigcirc\{\square, \bigcirc\}^*$ and $\sigma_2 \in \{\square, \bigcirc\}^*$. This case produces a contradiction with the hypothesis that the path $T \to T_1 \to T_2 \to \cdots \to T'$ is a shortest path between T and T'. Thus, only the case (I) occurs and there is a shortest path between T and T' of the form $T \to U(T, T') \to \cdots \to T'$ which gives the result by induction. □

Corollary 2. *Let $\mathcal{C}$ be the path defined by:*

$$\mathbf{1} \leftarrow D(\mathbf{0}, \mathbf{1}) \leftarrow D(\mathbf{0}, D(\mathbf{0}, \mathbf{1})) \leftarrow D(\mathbf{0}, D(\mathbf{0}, D(\mathbf{0}, \mathbf{1}))) \leftarrow \cdots \leftarrow \mathbf{0}.$$

T belongs to $\mathcal{C}$ if and only if T verifies the condition (A) defined by: T has either a unique occurrence of $\bigcirc\square\square$, or two occurrences of $\bigcirc\square\square$ such that there does not exist any leaf between them in the Polish notation of T. Moreover, if T and T' belong to the path $\mathcal{C}$ and verify $T \to T'$, then T' is obtained from T by a pruning-grafting transformation of the rightmost occurrence of $\bigcirc\square\square$ in T.

Proof. $\mathbf{1}$ has only one occurrence of $\bigcirc\square\square$ thus $\mathbf{1}$ verifies condition (A). Assume that $T' \in \mathcal{C}$ verifies (A). We put $T = D(\mathbf{0}, T')$ (thus $T' \leftarrow T$). If T' has only one occurrence of $\bigcirc\square\square$ then $T = D(\mathbf{0}, T')$ has at most two occurrences of $\bigcirc\square\square$, and there does not exist any leaf between them. Thus, T' is necessarily obtained from T by a pruning-grafting of the rightmost occurrence of $\bigcirc\square\square$. If T' has exactly two occurrences of $\bigcirc\square\square$ such that there does not exist any leaf between them. Then $T = D(\mathbf{0}, T')$ has either a unique occurrence of $\bigcirc\square\square$ or two occurrences such that there does not exist any leaf between them. By induction all trees $T \in \mathcal{C}$ hold (A). Conversely, let T be a tree verifying (A). It is straightforward to obtain T from $\mathbf{0}$ by successive pruning-grafting transformations of rightmost occurrences of $\bigcirc\square\square$. □

Corollary 3. *Let $\mathcal{C}'$ be the path defined by:*

$$\mathbf{0} \to U(\mathbf{0}, \mathbf{1}) \to U(U(\mathbf{0}, \mathbf{1}), \mathbf{1}) \to U(U(U(\mathbf{0}, \mathbf{1}), \mathbf{1}), \mathbf{1}) \to \cdots \to \mathbf{1}.$$

If $T \to T'$ is on the path $\mathcal{C}'$ then T' is obtained from T by the leftmost possible pruning-grafting transformation in T.

Proof. Indeed, if T and T' belong to $\mathcal{C}'$ such that $T \to T'$, then $T' = U(T, \mathbf{1})$ and T' is clearly obtained from T by the leftmost possible pruning-grafting transformation on T. □

Remark 1. Via Corollary 2, the longest path between $\mathbf{0}$ and $\mathbf{1}$ has length $2^n - (n + 1)$. These are the Eulerian numbers given by the sequence **A000295** of [38]. Indeed, if $T \to T'$ belongs to this path then T' is obtained from T by a pruning-grafting transformation of the rightmost occurrence $\bigcirc\square\square$ in T. Let us denote by $\ell(n)$ the length of the longest path in B_n between $\mathbf{1}$ and $\mathbf{0}$. Then there are $\ell(n-1)$ pruning-grafting transformations between $\mathbf{0}$ and T_1 with the feasible sequence $(1, n, n, n-1, \ldots, 2)$; there are n transformations between T_1 and T_2 with feasible sequence $(2, 3, 4, \ldots, n-1, n, n, 1)$; there are $\ell(n-1)$ transformations between T_2 and $\mathbf{1}$. This means that $\ell(n)$ verifies the recurrence relation $\ell(n) = 2 \cdot \ell(n-1) + n$ with $\ell(2) = 1$, which implies that $\ell(n) = 2^n - (n+1)$.

On the other hand, the shortest path has the length $(n-1)^2$. Indeed, with Corollary 3 we obtain the successor (in the shortest path) of a tree T by the leftmost possible pruning-grafting transformation in T. Thus, there are $(n-1)$ pruning-grafting transformations between $\mathbf{0}$ and the tree T_1 with feasible sequence $(2, 2, 2, 3, 4, \ldots, n-2, n-1, n-1)$. After iterating this process $(n-1)$ times from T_1, we obtain directly that $\ell(n) = (n-1)^2$.

This result allows us to compute the complexity of the above **join** algorithm for computing the least upper bound $T \vee T'$ of two binary trees $T, T' \in B_n$. The *while* loop is performed at most twice the length of a shortest path between $\mathbf{0}$ and $\mathbf{1}$, i.e. $2(n-1)^2$. A loop iteration takes $\mathcal{O}(n)$ time. Thus, the time complexity is $\mathcal{O}(n^3)$ in the worst case.

Now, we prove that the join (resp. meet) irreducible elements in B_n are enumerated by the Eulerian numbers. Recall that $x \in B_n$ is a join (resp. meet)-irreducible element if $x = a \vee b$ (resp. $x = a \wedge b$) implies $x = a$ or $x = b$. In other words, join (resp. meet)-irreducible elements are the elements that have a unique lower (resp. upper) cover.

Theorem 3. *The number of join (resp. meet)-irreducible elements in the pruning-grafting lattice* $(B_n, \overset{*}{\rightarrow})$ *is given by the Eulerian number* $2^n - (n+1)$.

Proof. A join-irreducible element T ($T \neq \mathbf{0}$) of B_n can be characterized as follows: (1) it contains exactly one node having two leaves as children and these leaves are not the last two of the tree; or (2) it contains exactly two nodes with two leaves as children and one of these nodes is the parent of the two last leaf of T. There exists a one-to-one map between theses elements and the Dyck paths of semi-length n having exactly one long descent (i.e. descent of length at least two). Indeed, we associate with the Polish notation of T the Dyck path defined as follows: a $\bigcirc$ is replaced by $U = (1, 1)$ and a $\square$ is replaced by $D = (1, -1)$ except the last leaf $\square$ which is not considered. For example, if $n = 4$, the Dyck path associated with the join-irreducible element $\bigcirc\square\bigcirc\bigcirc\square\square\bigcirc\square\square$ is $UDUU\mathbf{DD}UD$. If $T \neq \mathbf{0}$ verifies (1), then its Polish notation verifies $\sigma\bigcirc\square\square\square^k$ where $k \geq 1$ and $\sigma \in \bigcirc\{\bigcirc, \square\}^*$ does not contain any occurrence of $\bigcirc\square\square$. Thus, its corresponding Dyck path has a unique long descent (i.e. a unique maximal sequence $D \ldots D$ of length at least two). If T verifies (2), then its Polish notation verifies $\sigma\bigcirc\square\square\sigma'\bigcirc\square\square$ where $\sigma, \sigma' \in \bigcirc\{\bigcirc, \square\}^*$ do not contain any occurrence of $\bigcirc\square\square$. Thus, its corresponding Dyck path also has a unique long descent. Conversely, a Dyck path with a unique long descent can be associated with a Polish notation of a tree verifying (1) or (2). Thus, this map induces a one-to-one map between join-irreducible elements of B_n and Dyck paths of semi-length n having exactly one long descent. These elements are enumerated by the Eulerian numbers [20,33]. Using the symmetry of the lattice, we deduce the same result for the meet-irreducible elements. □

Theorem 4. *The number* $cov(n)$ *of coverings in* B_n *is equal to* $\binom{2n}{n-1}$.

Proof. Recall that a covering of B_n corresponds to an edge of B_n, i.e. a pruning-grafting transformation in B_n. Thus, there exists a bijection between the covering set of B_n and the long descents (i.e. descents of length at least two) in all Dyck paths of semilength n. Indeed, we associate with the Polish notation of a tree $T \in B_n$ a Dyck path $\mathcal{P}$ as follows: each $\bigcirc$ is replaced by $U = (1, 1)$ and each $\square$ is replaced by $D = (1, -1)$ except the last $\square$ which is not considered. Thus, the number of possible pruning-grafting transformations from T is equal to the number of occurrences $\bigcirc\square\square$ that are not on the left of T. By symmetry, if $T' = \overline{T}$, it is also the number of occurrences $\bigcirc\square\square$ that are not on the right of T'. Thus, this corresponds to the number of long descents of the Dyck path $\mathcal{P}$. Then $cov(n)$ is the number of long descents in all Dyck paths of semi-length n. Deutsch [9,38] proved that this number is $\binom{2n}{n-1}$. We obtain the sequence **A001791** of [38]. For example, $cov(3) = 4$ because in the five Dyck paths of semilength 3, namely $UDUDUD$, $UDUU\mathbf{DD}$, $UU\mathbf{DD}UD$, $UUDU\mathbf{DD}$, and $UUU\mathbf{DDD}$, we have four long descents (shown in bold-face). □

Let $J(B_n)$ (resp. $M(B_n)$) denote the set of nonzero join-irreducible (resp. nonunit meet-irreducible) elements of B_n. We say that B_n has a matching g if g is a map of $J(B_n) \cup \{\mathbf{0}\}$ to $M(B_n) \cup \{\mathbf{1}\}$ which is one-to-one and verifies $\tau \leq g(\tau)$ for each join-irreducible τ [2,10,22,32]. Kung [22] has proved that every consistent lattice has a matching. In the sequel, we build a matching of B_n in a *constructive* way.

Theorem 5. *For all n, there exists a matching in* $(B_n, \overset{*}{\rightarrow})$.

Proof. Let $\mathcal{C}$ be the particular maximal chain defined as follows: $T_0 = \mathbf{1}$ and $T_i = D(\mathbf{0}, T_{i-1})$. Following Remark 1, we have $T_{2^n-(n+1)} = \mathbf{0}$. By Corollary 2, a tree on the path $\mathcal{C}$ verifies: $T \in \mathcal{C}$ iff in the Polish notation of T we have: (a) there are at most two occurrences of $\bigcirc\square\square$, and (b) there does not exist any leaf between two different occurrences $\bigcirc\square\square$.

We will build an one-to-one map f between $M(B_n) \cup \{\mathbf{1}\}$ and $\mathcal{C}$ such that $\sigma \geq f(\sigma)$ for $\sigma \in M(B_n) \cup \{\mathbf{1}\}$. The symmetry of the lattice induces a one-to-one map $\overline{f}$ between $J(B_n) \cup \{\mathbf{0}\}$ and $\mathcal{C}$ such that $\tau \leq \overline{f}(\tau)$ for $\tau \in J(B_n) \cup \{\mathbf{0}\}$. Using these two bijections, we obtain a matching.

Recall that (cf. proof of Theorem 3) an element σ belongs to $M(B_n) \cup \{\mathbf{1}\}$ if and only if (1) it contains exactly one node having two leaves as children; or (2) it contains exactly two nodes with two leaves as children and one of these nodes is the parent of the two first leaf of T. Let M_1 (resp. M_2) be the set of meet-irreducible in the case (1) (resp. (2)). We have $M(B_n) \cup \{\mathbf{1}\} = M_1 \cup M_2$ and we define the map f as follows:

- if $\sigma \in M_1, f(\sigma) = \sigma$;
- if $\sigma \in M_2, \sigma$ can be written in terms of feasible sequences:

$\sigma = (\sigma_2, \sigma_2, \sigma_3, \ldots, \sigma_{k-1}, \sigma_k, \sigma_k, \sigma_{k+2}, \ldots, \sigma_n, \sigma_{n+1})$ with $k \geq 3$ where there exists r, $2 \leq r \leq k-1$, such that $\sigma_i \geq \sigma_{i+1}$ for $i \leq r-1$ and $\sigma_i \leq \sigma_{i+1}$ for $r \leq i \leq k$. Thus, if $k \geq 4$, we set $f(\sigma) = (\sigma_r - 1, \sigma_{r+1}, \ldots, \sigma_{k-2}, \sigma_{k-1} + 1, \ldots, \sigma_{k-1} + r - 2, \sigma_{k-1} + r - 1, \sigma_{k-1} + r - 1, \sigma_k, \sigma_k, \sigma_{k+2}, \ldots, \sigma_n, \sigma_{n+1})$; if $k = 3$ then $\sigma \in \mathcal{C}$ and we set $f(\sigma) = \sigma$. For example, if $\sigma = (4, 4, 3, 2, 3, 4, 7, 7, 6, 5, 2)$ then $k = 7, r = 4$ and $f(\sigma) = (1, 3, 5, 6, 7, 7, 7, 7, 6, 5, 2)$.

Obviously $f(\sigma) \in \mathcal{C}, f(\sigma) \leq \sigma$ and by construction $f(\sigma) = f(\sigma')$ implies $\sigma = \sigma'$. Notice that $f(\sigma)$ is the smallest element of $\mathcal{C}$ such that $\sigma \geq f(\sigma)$. Via the symmetry of B_n, we obtain a bijection $\overline{f}$ from $J(B_n) \cup \{\mathbf{0}\}$ to $\mathcal{C}$ defined by: for $\tau \in J(B_n) \cup \{\mathbf{0}\}, \overline{f}(\tau) = \overline{f(\overline{\tau})}$, and such that $\tau \leq \overline{f}(\tau)$. Therefore, the composition $g = f^{-1} \circ \overline{f}$ is a matching of B_n since g is a map from $J(B_n) \cup \{\mathbf{0}\}$ to $M(B_n) \cup \{\mathbf{1}\}$ which is one-to-one and verifies $\tau \leq g(\tau)$ for every $\tau \in J(B_n) \cup \{\mathbf{0}\}$. □

5. Conclusion and open problems

In this paper, a new lattice structure has been defined on the Catalan sets of binary trees via a natural transformation. The simple and natural definition of the pruning-grafting transformation is unfortunately at odds with the rather complex theorem which characterizes this transformation.

Some problems remain to be solved.

Is there an efficient and nonrecursive algorithm to compute the Möbius function of the pruning-grafting lattice of binary trees as in [26]? We conjecture that this Möbius function takes its values in $\{-1, 0, +1\}$.

Is there a polynomial time algorithm to compute the minimal path length distance between two binary trees in the pruning-grafting lattice [23]? If so, a new shortest-path-type metric could be obtained, and could be added to the existing metrics on Catalan sets [17,25,27,29,37]. Let us recall that we still do not know if the rotation distance on binary trees can be computed in polynomial time.

Let us define the integer-valued function v on $(B_n, \overset{*}{\rightarrow})$ by $v(T)$ be the length of a longest path between T and $\mathbf{1}$. We conjecture that v is an anti-monotone supervaluation, i.e. v verifies for all T and T': $v(T) + v(T') \leq v(T \vee T') + v(T \wedge T')$ and $T < T' \Longrightarrow v(T) > v(T')$. Were it the case, therefore $d(T, T') = v(T) + v(T') - 2v(T \vee T')$ would be a metric on $(B_n, \overset{*}{\rightarrow})$ [5,7].

Guttmann, Krattenthaler and Viennot [18] enumerate the k-chains in the Stanley lattices, i.e, the number of k-noncrossing Dyck paths. Kreweras [21] enumerates the k-chains in the lattices of noncrossing partitions. More generally, Chapoton proves in [8] that the sequence **A000260** of [38] enumerates the number of intervals in the Tamari lattices, i.e. the number of ordered pairs (T, T') such that $T \leq T'$: see also [16, p. 27]. Bernardi and Bonichon [6] construct bijections between the set of intervals of these lattices and the realizers of triangulations. Here, we obtain experimentally the numbers of intervals for the pruning-grafting lattices for small sizes:

$$1, 3, 15, 101, 818, 7486, 74648, 793005, 8843056, 102464586, \ldots$$

This sequence does not appear in [38]. Is it possible to obtain the generating function of this sequence?

Acknowledgements

We would like to thank the anonymous referees and Emeric Deutsch for their constructive remarks which have greatly improved this paper.

References

[1] E. Barcucci, A. Bernini, L. Ferrari, M. Poneti, A distributive lattice structure connecting Dyck paths, noncrossing partitions and 312-avoiding permutations, Order 22 (4) (2005) 311–328.
[2] J.L. Baril, J.M. Pallo, Matchings in three Catalan lattices, Internat. J. Comput. Math. 8 (2003) 193–200.
[3] J.L. Baril, J.M. Pallo, The phagocyte lattice of Dyck words, Order 23 (2–3) (2006) 97–107.
[4] J.L. Baril, J.M. Pallo, Efficient lower and upper bounds of the diagonal-flip distance between triangulations, Inform. Process. Lett. 100 (2006) 131–136.
[5] J.P. Barthélemy, B. Leclerc, B. Monjardet, On the use of ordered sets on problems of comparison and consensus of classifications, J. Classification 3 (1986) 187–224.
[6] O. Bernardi, N. Bonichon, Catalan's intervals and realizers of triangulations, in: FPSAC, Tianjin, China, 2007.
[7] S.A. Boorman, D.C. Olivier, Metrics on spaces of finite trees, J. Math. Psychol. 10 (1973) 26–59.
[8] F. Chapoton, Sur le nombre d'intervalles dans les treillis de Tamari, Séminaire Lotharingien de combinatoire 55 (2006).
[9] E. Deutsch, Private communication, 2008.
[10] D. Duffus, Matching in modular lattice, J. Combin. Theory 32 (1982) 303–314.
[11] P.H. Edelman, Chain enumeration and noncrossing partitions, Discrete Math. 31 (1980) 171–180.
[12] P.H. Edelman, R. Simion, Chains on the lattice of noncrossing partitions, Discrete Math. 126 (1994) 107–119.
[13] L. Ferrari, R. Pinzani, Lattices of lattice paths, J. Stat. Plan. Inference 135 (2005) 77–92.
[14] H. Friedman, D. Tamari, Problèmes d'associativité: une structure de treillis fini induite par une loi de demi-associative, J. Comb. Theory 2 (1967) 215–242.
[15] R.G. Gallager, Information Theory and Reliable Communications, J. Wiley, New York, 1968.
[16] C. Germain, J. Pallo, The number of coverings in four Catalan lattices, Internat. J. Comput. Math. 61 (1996) 19–28.
[17] C. Germain, J.M. Pallo, Two shortest path metrics on well-formed parentheses strings, Inform. Process. Lett. 60 (1996) 283–287.
[18] A.J. Guttmann, C. Krattenthaler, X.G. Viennot, Vicious walkers, friendly walkers and Young tableaux II: Between two walls, J. Statist. Phys. 110 (2003) 1069–1086.
[19] S. Hanke, T. Ottmann, S.J. Schuierer, The edge-flipping distance of triangulations, J. Univers. Comp. Sci. 2 (1996) 570–579.
[20] D.E. Knuth, Fondamental Algorithms, vol. 1, Addison-Wesley, 1997.
[21] G.J. Kreweras, Sur les partitions noncroisées d'un cycle, Discrete Math. 1 (1972) 570–579.
[22] J.P.S. Kung, Matching and Radon transforms in lattices, Order 2 (1985) 105–112.
[23] B. Monjardet, Metrics on partially ordered sets — A survey, Discrete Math. 35 (1981) 173–184.
[24] J.M. Pallo, Enumerating, ranking and unranking binary trees, The Computer J. 29 (1986) 171–175.
[25] J.M. Pallo, On the rotation distance in the lattice of binary trees, Inform. Process. Lett. 25 (1987) 369–373.
[26] J.M. Pallo, An algorithm to compute the Möbius function of the rotation lattice of binary trees, RAIRO Inform. Théorique et Appl. 27 (1993) 341–348.
[27] J.M. Pallo, An efficient upper bound of the rotation distance of binary trees, Inform. Process. Lett. 73 (2000) 87–92.
[28] J.M. Pallo, Generating binary trees by Glivenko classes on Tamari lattices, Inform. Process. Lett. 85 (2003) 235–238.
[29] J.M. Pallo, Right-arm rotation distance between binary trees, Inform. Process. Lett. 87 (2003) 173–177.
[30] D.S. Parker, P. Ram, The construction of Huffman codes is a submodular ("convex") optimization problem over a lattice of binary trees, SIAM J. Comput. 28 (1999) 1875–1905.
[31] V. Reiner, Noncrossing partitions for classical reflection groups, Discrete Math. 177 (1997) 195–222.

[32] K. Reuter, Matching for linearly indecomposable modular lattices, Discrete Math. 63 (1987) 245–247.
[33] D.P. Roselle, Permutations by number of rises and successions, Proc. Amer. Math. Soc. 20 (1968) 8–16.
[34] F. Ruskey, T.C. Hu, Generating binary trees lexicographically, SIAM J. Comput. 6 (4) (1977) 745–758.
[35] R. Simion, D. Ullman, On the structure of lattice of noncrossing partitions, Discrete Math. 98 (1991) 193–206.
[36] R. Simion, Noncrossing partitions, Discrete Math. 217 (2000) 367–409.
[37] D.D. Sleator, R.E. Tarjan, W.P. Thurston, Rotation distance, triangulations and hyperbolic geometry, J. Am. Math. Soc. 1 (1988) 647–681.
[38] N.J.A. Sloane, The On-line Encyclopedia of Integer Sequences, Available electronically at http://www.research.att.com/~njas/sequences.
[39] R.P. Stanley, Enumerative Combinatorics, vol. 2, Cambridge University Press, 1999.

Advances and Applications in Discrete Mathematics
Volume 3, Number 1, 2009, Pages 67-84
Published Online: March 30, 2009
This paper is available online at http://www.pphmj.com
© 2009 Pushpa Publishing House

GRAY CODE FOR COMPOSITIONS OF n WITH PARTS 1 AND p

JEAN-LUC BARIL and CÉLINE MOREIRA DOS SANTOS

LE2I UMR-CNRS 5158
Université de Bourgogne
B.P. 47 870, 21078 DIJON-Cedex
France
e-mail: barjl@u-bourgogne.fr

Abstract

Let $F_{n,p}$ be the set of all n-length binary strings such that there are at least p consecutive 0s between two 1s. Chinburg et al. [3] give a Gray code for the set $F_{n,p}$ such that two consecutive elements differ in at most two positions which are optimal for n large. In this paper, we construct a more restrictive Gray code for $F_{n,p}$ in the sense where our Gray code contains more transitions with one move than theirs. Moreover, using a combinatorial isomorphism between the set of n-length binary strings and the set of n-length permutations avoiding 123 and 132, we provide a bijection from $F_{n,p}$ to $S_{n+1}(123, 132, 213, (p+2)(p+3)\overline{(p+1)\cdots 4312})$ that induces a Gray code for the set $S_{n+1}(123, 132, 213, (p+2)(p+3)\overline{(p+1)p\cdots 4312})$. All these codes can be generated by implementing a recursive algorithm in constant amortized time.

2000 Mathematics Subject Classification: 11B39.

Keywords and phrases: composition of an integer, Fibonacci number, Gray code, permutation avoiding pattern.

Received October 17, 2008

1. Introduction and Definitions

Let $p \geq 2$ and $n \geq 1$ be two positive integers. A $(1, p)$-*composition* of n consists of an ordered sequence of the integers 1 and p whose sum is n. For instance, there are three $(1, 3)$-compositions of 4: $1+1+1+1$, $1+3$ and $3+1$. Let $C_{n,p}$ be the set of all $(1, p)$-compositions of n and $c_{n,p}$ its cardinality. In [4], the authors derive the following recurrence relation for the total number of $(1, p)$-compositions of n:

$$c_{n,p} = c_{n-1,p} + c_{n-p,p},$$

anchored by $c_{n,p} = 1$ for $0 \leq n \leq p$. The generating function of this sequence is $\dfrac{1}{1-x-x^p}$. Notice here that for $p = 2$, we obtain the well-known Fibonacci numbers [7, 11, 16, 18]. In the literature, there exist several combinatorial interpretations of this recurrence relation. Indeed, the sequence $c_{n,p}$ also counts the number of tilings of p-by-n rectangles with straight polyominoes, i.e., 1-by-p tiles (see Figure 1). Moreover, Di Cera and Kong [5] consider that $c_{n,p}$ is the number of ways to cover a linear lattice of n sites with molecules that are p sites wide, where there is no overlap of molecules, but gaps are allowed. In this representation, each molecule of size p corresponds to a summand p in a $(1, p)$-composition of n, and an empty site on the lattice corresponds to the summand 1. So we obtain a straightforward interpretation in terms of binary strings of length $(n-p+1)$ with at least $p-1$ zeros between two ones: by reading the lattice from left to right, each empty site corresponds to the bit 0 and each molecule of size p is represented by 10^{p-1}, except for a molecule touching the last site which is represented by 1 (see Figure 1). In the sequel, we set $n \geq 1$, $p \geq 1$ and $F_{n,p}$ will be the set of binary strings of length n with at least p zeros between two ones. So there is a bijection between the sets $F_{n,p}$ and $C_{n+p,p+1}$ for $n \geq 1$ and $p \geq 1$.

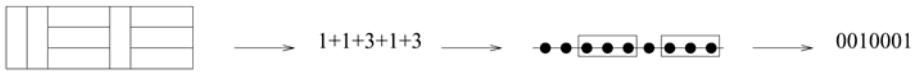


Figure 1. Combinatorial interpretations for a $(1, 3)$-composition of 9.

We will use the following definitions and notations throughout this paper. The *Hamming distance* d between two strings in $B_n = \{0, 1\}^n$ is the number of positions in which they differ. A *k-Gray code* for a binary string set $L_n \subset B_n$ is an *ordered list* $\mathcal{L}_n$ for L_n such that the Hamming distance between any two consecutive strings in $\mathcal{L}_n$ is at most k where k is independent of n. The code is called *optimal* if k is minimal. For instance, the list 000, 001, 011, 010, 110, 111, 101, 100 is an optimal 1-Gray code for binary strings of length three. We will denote by $first(\mathcal{L}_n)$ ($last(\mathcal{L}_n)$, respectively) the first (last, respectively) element of $\mathcal{L}_n$; $\overline{\mathcal{L}}_n$ is the list obtained by reversing $\mathcal{L}_n$. Thus, if $\mathcal{L}_n = l_1, l_2, ..., l_r$, with $r \geq 1$, then we get the following identities $\overline{\mathcal{L}}_n = l_r, l_{r-1}, ..., l_2, l_1$, $first(\overline{\mathcal{L}}_n) = last(\mathcal{L}_n)$ and $last(\overline{\mathcal{L}}_n) = first(\mathcal{L}_n)$. Let i be an integer, we set $\mathcal{L}_n^i = \mathcal{L}_n$ whether i is even and $\mathcal{L}_n^i = \overline{\mathcal{L}}_n$ otherwise. Let α be a non empty binary string and let $\mathcal{L}_n$ be a non empty list. Then α^n is the string which consists in the concatenation of n copies of α, while $\alpha \cdot \mathcal{L}_n$ ($\mathcal{L}_n \cdot \alpha$, respectively) is the list obtained by concatening α in front of (behind, respectively) each element of $\mathcal{L}_n$. Let $\mathcal{L}_n$ and $\mathcal{L}'_n$ be two lists. Then $\mathcal{L}_n \circ \overline{\mathcal{L}}_n$ is the list obtained by concatening $\mathcal{L}_n$ and $\mathcal{L}'_n$. For instance, if $\mathcal{L}_3 = 000, 010$ and $\mathcal{L}'_3 = 101, 111$, then we get $\mathcal{L}_3 \circ \mathcal{L}'_3 = 000, 010, 101, 111$. All these definitions can also be considered for lists of n-length permutations.

A number of papers concerning Gray codes for Fibonacci or Lucas strings have been published [1, 2, 18]. Some Gray codes are given for compositions of a positive integer n [10, 19] or for compositions of n with parts of size at most k [18]. Chinburg et al. [3] provide Gray codes for $F_{n,p}$ such that two consecutive elements differ in at most two positions, and they prove the optimality for n large.

In the second section, we give another Gray code for the set $F_{n,p}$ of n-length binary strings with at least $p \geq 2$ zeros between two ones. This Gray code is more complicated than [3] but it has the advantage that it contains more transitions with one move than theirs. Moreover in the third section, using a combinatorial isomorphism between $B_n = \{0, 1\}^n$ and the set $S_{n+1}(123, 132)$ of permutations avoiding the patterns 123 and 132, we construct a bijection from $F_{n,p}$ to

$$S_{n+1}(123, 132, 213, (p+2)(p+3)\overline{(p+1)p\cdots 43}12).$$

This induces a Gray code for

$$S_{n+1}(123, 132, 213, (p+2)(p+3)\overline{(p+1)\cdots 43}12).$$

In the last section, we show how these codes can be generated by implementing a recursive algorithm in constant amortized time.

Notice that if $p = 1$, then Vajnovszki [18] gives a 1-Gray code for $F_{n,1}$, $n \geq 1$.

In order to construct our Gray code we decompose $F_{n,p}$ in the following way.

Theorem 1. *Let n and p be positive integers. Then*

$$F_{n,p} = \begin{cases} 0^n \cup \bigcup_{k=0}^{n-1} 0^{n-k-1}10^k, & \text{if } 0 < n \leq p+1, \\ F_{n-p,p} \cdot 0^p \cup \bigcup_{k=0}^{n-p-2} F_{n-1-p-k,p} \cdot 0^p 10^k \cup & \\ \qquad \bigcup_{k=n-p-1}^{p-1} 0^{n-k-1}10^k, & \text{if } p+1 < n \leq 2p-1, \\ F_{n-p,p} \cdot 0^p \cup \bigcup_{k=0}^{p-1} F_{n-1-p-k,p} \cdot 0^p 10^k, & \text{if } n \geq 2p. \end{cases}$$

Proof. In the recurrence relation (1) we replace the term $F_{n-i,p}$ by $F_{n-i-1,p} \cdot 0 \cup F_{n-i-p-1,p} \cdot 0^p 1$ for i from 1 to $(p-1)$.

2. Gray Code for $F_{n,p}$

First we define our Gray code $\mathcal{F}_{n,p}$ for the set $F_{n,p}$ with $n \leq 2p-1$,

$$\mathcal{F}_{n,p} = \begin{cases} 10^{n-1} \circ 0^n \circ \bigcirc_{k=0}^{n-2} 0^{n-k-1}10^k, & \text{if } 0 < n \leq p+1 \\ \mathcal{F}_1 \cdot 0^p 10^{n-p-2} \circ \bigcirc_{k=n-p-1}^{p-1} 0^{n-k-1}10^k \circ \mathcal{F}_{n-p,p} \cdot 0^p & \\ \qquad \circ \bigcirc_{k=0}^{n-p-3} \mathcal{F}_{n-1-p-k,p}^{n-k-1} \cdot 0^p 10^k, & \text{if } p+1 < n \leq 2p-1. \end{cases}$$

In the following, the length-$(p+1)$ binary string $\chi = 10^p$ is central. Therefore we denote by $\chi^{n/(p+1)}$ the prefix of length n of the infinite string $\chi\chi\chi\ldots$, or equivalently, $\chi^{n/(p+1)} = \underbrace{\chi\chi\ldots\chi}_{\lfloor n/(p+1)\rfloor}\chi_r$, where $r = n \bmod (p+1)$, and χ_r is the length-r prefix of χ.

With these notations, we easily obtain $first(\mathcal{F}_{n,p}) = \chi^{\frac{n}{p+1}}$, $last(\mathcal{F}_{n,p}) = 0 \cdot \chi^{\frac{n-1}{p+1}}$ and two successive elements in $\mathcal{F}_{n,p}$ differ in at most two positions.

In the sequel, we define the Gray code for $n \geq 2p$ by distinguishing on the parity of p.

2.1. For $n \geq 2p$ and p even

Definition 1. Let n, p and $k \geq 1$ be *integers* such that $n \geq 2p \geq 2$ and p even. Then

$$\mathcal{F}_{n,p} = \begin{cases} \mathcal{F}_{n-p,p} \cdot 0^p \circ \bigcirc_{i=1}^{p-1} \mathcal{F}_{n-(p+1)-i,p}^{i+1} \cdot 0^p 10^i, & \text{if } n = 0 \bmod (p+1) \\ \bigcirc_{i=k-1}^{p-1} \mathcal{F}_{n-(p+1)-i,p}^{k-1-i} \cdot 0^p 10^i \circ \mathcal{F}_{n-p,p}^{k+1} \cdot 0^p & \\ \qquad \circ \bigcirc_{i=0}^{k-2} \mathcal{F}_{n-(p+1)-i,p}^{k-i} \cdot 0^p 10^i, & \text{if } n = k \bmod (p+1). \end{cases}$$

Proposition 1. *Let n and p be integers such that $n \geq 2p \geq 2$ and p even. Then*

- $first(\mathcal{F}_{n,p}) = \chi^{\frac{n}{p+1}} = \chi^{\left\lfloor \frac{n}{p+1} \right\rfloor} \cdot \chi_n \bmod(p+1)$,
- $last(\mathcal{F}_{n,p}) = 0 \cdot \chi^{\frac{n-1}{p+1}} = 0 \cdot \chi^{\left\lfloor \frac{n-1}{p+1} \right\rfloor} \cdot \chi(n-1) \bmod(p+1)$.

Proof. We will prove this result by induction on $n \geq 2p$. Recall that we have $first(\mathcal{F}_{n,p}) = \chi^{\frac{n}{p+1}}$ and $last(\mathcal{F}_{n,p}) = 0 \cdot \chi^{\frac{n-1}{p+1}}$ for $n \leq 2p-1$.

Suppose that $n+1 = 0 \bmod(p+1)$. Then by induction, the following identities hold:

$$first(\mathcal{F}_{n+1,p}) = first(\mathcal{F}_{n+1-p,p}) \cdot 0^p = \chi^{\left\lfloor \frac{n+1-p}{p+1} \right\rfloor} \cdot \chi_{(n+1-p) \bmod(p+1)} \cdot 0^p$$

thus

$$first(\mathcal{F}_{n+1,p}) = \chi^{\left\lfloor \frac{n+1}{p+1} \right\rfloor - 1} \cdot \chi \cdot \chi_{(n+1) \bmod(p+1)} = \chi^{\frac{n+1}{p+1}}.$$

Suppose that $n+1 = k \bmod(p+1)$ with $1 \leq k \leq p$, then

$$\begin{aligned} first(\mathcal{F}_{n+1,p}) &= first(\mathcal{F}_{n+1-(p+1)-(k-1),p}) \cdot 0^p 10^{k-1} \\ &= \chi^{\frac{n+1-(p+1)-k}{p+1}} \cdot 1 \cdot 0^p 10^{k-1} \end{aligned}$$

thus

$$first(\mathcal{F}_{n+1,p}) = \chi^{\frac{n+1-k}{p+1}-1} \cdot \chi \cdot 10^{k-1} = \chi^{\frac{n+1-k}{p+1}} \cdot \chi_k = \chi^{\frac{n+1}{p+1}}.$$

We proceed in the same way for the last element.

Suppose that $n+1 = 0 \bmod(p+1)$, then

$$last(\mathcal{F}_{n+1,p}) = last(\mathcal{F}_{n+1-(p+1)-(p-1),p}) \cdot 0^p 10^{p-1} = 0 \cdot \chi^{\frac{n+1-p}{p+1}-1} \cdot 0^p 10^{p-1}$$

thus

$$last(\mathcal{F}_{n+1,p}) = 0 \cdot \chi^{\frac{n+1-p-1}{p+1}-1} \cdot 1 \cdot 0^p 10^{p-1} = 0 \cdot \chi^{\frac{n+1-p-1}{p+1}} 10^{p-1} = 0 \cdot \chi^{\frac{n+1}{p+1}}.$$

Suppose that $n+1 = k \bmod (p+1)$, $1 \le k \le p$, then

$$\begin{aligned} last(\mathcal{F}_{n+1,p}) &= last(\mathcal{F}_{n+1-(p+1)-(k-2),p}) \cdot 0^p 10^{k-2} \\ &= 0 \cdot \chi^{\frac{n-(p+1)-(k-2)}{p+1}} \cdot 0^p 10^{k-2} \\ &= 0 \cdot \chi^{\frac{n+1-(k-1)}{p+1}-1} \cdot 0^p 10^{k-2} \end{aligned}$$

thus

$$\begin{aligned} last(\mathcal{F}_{n+1,p}) &= 0 \cdot \chi^{\frac{n+1-k}{p+1}-1} \cdot 1 \cdot 0^p 10^{k-2} \\ &= 0 \cdot \chi^{\frac{n+1+(p+1)-k}{p+1}} 10^{k-2} \\ &= 0 \cdot \chi^{\frac{n+1}{p+1}}. \end{aligned}$$

Theorem 2. *$\mathcal{F}_{n,p}$ is a Gray code for any integers n and p such that $n \ge 2p$ and p even. With* [3], *it is optimal for n sufficiently large. Two consecutive elements differ in at most two positions.*

Proof. We prove this theorem for $n = 0 \bmod (p+1)$. By induction, it suffices to verify that every transitions between consecutive sublists modify at most two bits.

For the transition between $\mathcal{F}_{n-p,p} \cdot 0^p$ and $\overline{\mathcal{F}}_{n-(p+1),p} \cdot 0^p 1$, we have:

$$\begin{aligned} &d(last(\mathcal{F}_{n-p,p} \cdot 0^p),\ first(\overline{\mathcal{F}}_{n-(p+1),p} \cdot 0^p 1)) \\ &= d\left(0 \cdot \chi^{\frac{n-p-1}{p+1}} \cdot 0^p,\ 0 \cdot \chi^{\frac{n-p-2}{p+1}} \cdot 0^p 1\right) = 1 \end{aligned}$$

For the transition between $\mathcal{F}_{n-(p+1)-i,\,p} \cdot 0^p 1 0^i$ and $\overline{\mathcal{F}}_{n-(p+1)-(i+1),\,p} \cdot 0^p 1 0^{i+1}$, we have:

$$d(last(\mathcal{F}_{n-(p+1)-i,\,p} \cdot 0^p 1 0^i),\ first(\overline{\mathcal{F}}_{n-(p+1)-(i+1),\,p} \cdot 0^p 1 0^{i+1}))$$

$$= d\left(0 \cdot \chi^{\frac{n-(p+1)-i-1}{p+1}} \cdot 0^p 1 0^i,\ 0 \cdot \chi^{\frac{n-(p+1)-(i+1)-1}{p+1}} \cdot 0^p 1 0^{i+1}\right) = 2.$$

Similarly, for the transition between $\overline{\mathcal{F}}_{n-(p+1)-i,\,p} \cdot 0^p 1 0^i$ and $\mathcal{F}_{n-(p+1)-(i+1),\,p} \cdot 0^p 1 0^{i+1}$, we have:

$$d(last(\overline{\mathcal{F}}_{n-(p+1)-i,\,p} \cdot 0^p 1 0^i),\ first(\mathcal{F}_{n-(p+1)-(i+1),\,p} \cdot 0^p 1 0^{i+1}))$$

$$= d\left(\chi^{\frac{n-(p+1)-i}{p+1}} \cdot 0^p 1 0^i,\ \chi^{\frac{n-(p+1)-(i+1)}{p+1}} \cdot 0^p 1 0^{i+1}\right) = 2.$$

When $n = k \bmod (p+1),\ k \neq 0,$ we just provide, without details, the different Hamming distances between the last and the first elements of two consecutive sublists. We have:

$$d(last(\mathcal{F}^{k+1}_{n-p,\,p} \cdot 0^p),\ first(\mathcal{F}^{k}_{n-(p+1),\,p} \cdot 0^p 1)) = 1$$

and the other transitions permute two opposite bits (i.e., the Hamming distance between the last and the first elements of two consecutive lists is 2).

2.2. For $n \geq 2p$ and p odd

In this part we decompose the set $F_{n,\,p},\ n \geq 2p,$ in the following way:

$$F_{n,\,p} = F_{n-1-p,\,p} \cdot 0^{p+1} \cup \bigcup_{k=0}^{p} F_{n-1-p-k,\,p} \cdot 0^p 1 0^k,$$

where we consider that the set $F_{-1,\,p} \cdot 0^p 1 0^p$ contains a single element $0^{p-1} 1 0^p$. This definition is needed to obtain a recursive definition for our Gray code for $n \geq 2p$.

Definition 2. Let n, p and $k \neq 1$, $0 \leq k \leq p$, be integers such that $n \geq 2p \geq 2$ and p odd. Then

$$\mathcal{F}_{n,p} =$$

$$\begin{cases} \mathcal{F}_{n-2p-1,p} \cdot 0^p 1 0^p \circ \overline{\mathcal{F}}_{n-p-1,p} \cdot 0^{p+1} \\ \qquad\qquad \circ \bigcirc_{i=1}^{p-1} \mathcal{F}_{n-p-i-1,p}^{i+1} \cdot 0^p 1 0^i, & \text{if } n = 0 \bmod (p+1), \\ \mathcal{F}_{n-p-1,p} \cdot 0^p 1 \circ \overline{\mathcal{F}}_{n-p-1,p} \cdot 0^{p+1} \\ \qquad\qquad \circ \bigcirc_{i=1}^{p} \mathcal{F}_{n-p-i-1,p}^{i+1} \cdot 0^p 1 0^i, & \text{if } n = 1 \bmod (p+1), \\ \bigcirc_{i=k-1}^{p} \mathcal{F}_{n-p-i-1,p}^{k-1-i} \cdot 0^p 1 0^i \circ \mathcal{F}_{n-p-1,p}^{k-1} \cdot 0^{p+1} \\ \qquad\qquad \circ \bigcirc_{i=0}^{k-2} \mathcal{F}_{n-p-i-1,p}^{k-i} \cdot 0^p 1 0^i, & \text{if } n = k \bmod (p+1). \end{cases}$$

Proposition 2. *Let n and p be integers such that $n \geq 2p \geq 2$ and p odd. Then*

- $first(\mathcal{F}_{n,p}) = \chi^{\frac{n}{p+1}} = \chi^{\left\lfloor \frac{n}{p+1} \right\rfloor} \cdot \chi_{n \bmod (p+1)}$,
- $last(\mathcal{F}_{n,p}) = 0 \cdot \chi^{\frac{n-1}{p+1}} 0 \cdot \chi^{\left\lfloor \frac{n-1}{p+1} \right\rfloor} \cdot \chi_{(n-1) \bmod (p+1)}$.

Proof. We omit the proof of Proposition 2 since it is similar to the p even case.

Theorem 3. *$\mathcal{F}_{n,p}$ is a Gray code for any integers n and p such that $n \geq 2p \geq 2$ and p odd. With* [3], *it is optimal for n sufficiently large. Two consecutive elements differ in at most two positions.*

Proof. We prove this theorem for $n = 0 \bmod (p+1)$. By induction, it suffices to verify that each transition between consecutive sublists modifies at most two bits.

For the transition between $\mathcal{F}_{n-2p-1,p} \cdot 0^p 1 0^p$ and $\overline{\mathcal{F}}_{n-p-1,p} \cdot 0^{p+1}$, we have:

$$d(last(\mathcal{F}_{n-2p-1,p} \cdot 0^p 1 0^p),\ first(\overline{\mathcal{F}}_{n-p-1,p} \cdot 0^{p+1}))$$

$$= d\left(0 \cdot \chi^{\frac{n-2p-2}{p+1}} \cdot 0^p 1 0^p,\ 0 \cdot \chi^{\frac{n-p-2}{p+1}} \cdot 0^{p+1}\right) = 2.$$

For the transition between $\overline{\mathcal{F}}_{n-p-1,\,p} \cdot 0^{p+1}$ and $\mathcal{F}_{n-p-2,\,p} \cdot 0^p 10$, we have:

$$d(last(\overline{\mathcal{F}}_{n-p-1,\,p} \cdot 0^{p+1}),\ first(\mathcal{F}_{n-p-2,\,p} \cdot 0^p 10))$$

$$= d\left(\chi^{\frac{n-p-1}{p+1}} \cdot 0^{p+1},\ \chi^{\frac{n-p-2}{p+1}} .0^p 10\right) = 1.$$

For the transition between $\mathcal{F}_{n-(p+1)-i,\,p} \cdot 0^p 10^i$ and $\overline{\mathcal{F}}_{n-(p+1)-(i+1),\,p} \cdot 0^p 10^{i+1}$, we have:

$$d(last(\mathcal{F}_{n-(p+1)-i,\,p} \cdot 0^p 10^i),\ first(\overline{\mathcal{F}}_{n-(p+1)-(i+1),\,p} \cdot 0^p 10^{i+1}))$$

$$= d\left(0 \cdot \chi^{\frac{n-(p+1)-i-1}{p+1}} \cdot 0^p 10^i,\ 0 \cdot \chi^{\frac{n-(p+1)-(i+1)-1}{p+1}} \cdot 0^p 10^{i+1}\right) = 2.$$

Similarly, for the transition between $\overline{\mathcal{F}}_{n-(p+1)-i,\,p} \cdot 0^p 10^i$ and $\mathcal{F}_{n-(p+1)-(i+1),\,p} \cdot 0^p 10^{i+1}$, we have:

$$d(last(\overline{\mathcal{F}}_{n-(p+1)-i,\,p} \cdot 0^p 10^i),\ first(\mathcal{F}_{n-(p+1)-(i+1),\,p} \cdot 0^p 10^{i+1}))$$

$$= d\left(\chi^{\frac{n-(p+1)-i}{p+1}} \cdot 0^p 10^i,\ \chi^{\frac{n-(p+1)-(i+1)}{p+1}} \cdot 0^p 10^{i+1}\right) = 2.$$

When $n = k \bmod (p+1)$, $k \neq 0$, we just provide, without details, the different Hamming distances between the last and the first elements of two consecutive sublists. We have:

$$d(last(\mathcal{F}^{k+1}_{n-p-1,\,p} \cdot 0^p),\ first(\mathcal{F}^{k}_{n-p-1,\,p} \cdot 0^p 1)) = 1$$

and the other transitions permute two opposite bits (i.e., the Hamming distance between the last and the first elements of two consecutive lists is 2).

By converting each binary string of $\mathcal{F}_{n,\,p}$ in its corresponding

$(1, p+1)$-composition in $C_{n+p-1, p+1}$, we obtain a list $\mathcal{C}_{n+p-1, p+1}$. Moreover, an element is obtained from its predecessor either by transposing one part of size $p+1$ with an adjacent part of size 1; or by gathering $p+1$ adjacent parts of size 1 in one of size $p+1$; or by dividing a part of size $p+1$ in $p+1$ parts of size one (see Table 1). Moreover it is straightforward to prove that our Gray code contains more transitions of one move than this one of [3].

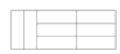
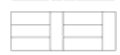
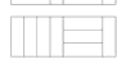
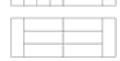
3. Gray Code for $S_{n+1}(123, 132, 213, (p+2)(p+3)\overline{(p+1)p\cdots 4312})$

Let S_n be the set of permutations on $[n] = \{1, 2, ..., n\}$. We represent a permutation $p \in S_n$ in one-line notation: i.e., $p = p(1)p(2)\cdots p(n)$. A permutation $p \in S_n$ contains the pattern $\tau \in S_k$ if and only if a sequence of indices $1 \le i_1 < i_2 < \cdots < i_k \le n$ exists such that $p(i_1)p(i_2)\cdots p(i_k)$ is ordered as τ. For example, the permutation **4132** contains (in bold-face) the pattern 321 but avoids the pattern 231. We denote by $S_n(\tau)$ the set of permutations of S_n avoiding the pattern τ. A *barred* pattern $\bar{\tau}$ is a permutation in S_k having a bar over certain of its elements. Put $1 \le r \le k-1$. Let τ be a permutation on $[k]$ identical to $\bar{\tau}$ but unbarred and $\hat{\tau}$ be the permutation on $[k-r]$ made up of the $(k-r)$ unbarred elements of $\bar{\tau}$, rewritten to be a permutation on $[k-r]$. Then $p \in S_n$ does not contain the pattern $\bar{\tau}$ if every pattern $\hat{\tau}$ in p can be expanded into a pattern τ in p. For example, if $\bar{\tau} = 4\bar{1}32$, then $p = 58132674 \in S_8(4\bar{1}32)$ (see, [6, 12, 15]).

In this part, we will construct a bijection from $F_{n,p}$ to $S_{n+1}(123, 132, 213, (p+2)(p+3)\overline{(p+1)p\cdots 4312})$ that induces a Gray code for $S_{n+1}(123, 132, 213, (p+2)(p+3)\overline{(p+1)p\cdots 4312})$.

We first give definitions about isomorphisms between two combinatorial classes.

Table 1. The lists $\mathcal{F}_{6,2}$, $\mathcal{C}_{8,3}$ and two combinatorial interpretations. For instance, in $\mathcal{F}_{6,2}$ the sublists of Gray code, $\mathcal{F}_{\mathbf{4,2}} \cdot \mathbf{00}$, $\mathcal{F}_{3,2} \cdot \mathit{001}$ and $\mathcal{F}_{\mathbf{2,2}} \cdot \mathbf{0010}$ are alternatively in bold-face and italic

	$\mathcal{F}_{6,2}$	$\mathcal{C}_{8,3}$	Tilings	Lattice
1	**100100**	**3+3+1+1**		
2	**000100**	**1+1+1+3+1+1**		
3	**001000**	**1+1+3+1+1+1**		
4	**100000**	**3+1+1+1+1+1**		
5	**000000**	**1+1+1+1+1+1+1+1**		
6	**010000**	**1+3+1+1+1+1**		
7	*010001*	*1+3+1+3*		
8	*001001*	*1+1+3+3*		
9	*000001*	*1+1+1+1+1+3*		
10	*100001*	*3+1+1+3*		
11	**100010**	**3+1+3+1**		
12	**000010**	**1+1+1+1+3+1**		
13	**010010**	**1+3+3+1**		

Two objects are *close* if they differ in some pre-specified way, usually a small Hamming distance (see [14]). A combinatorial *isomorphism* between two combinatorial classes is a bijection between them such that two objects in a class are close if and only if their images under this bijection are also close. Juarna and Vajnovszki [9] found an isomorphism φ between B_n and $S_{n+1}(123, 132)$ which is found as a generalization of the Simion-Schmidt injection [15].

Definition 3. For any binary string $b = b_1 b_2 \cdots b_n \in \{0, 1\}^n$ we construct a *permutation* $p = \phi(b) \in S_{n+1}$ which has its i-th entry p_i given by the following rule. If $X_i = [n+1] \setminus \{p_1, p_2, \ldots, p_{i-1}\}$, we set

$$p_i = \begin{cases} \text{the largest element in } X_i \text{ if } b_i = 0, \\ \text{the second largest element in } X_i \text{ if } b_i = 1. \end{cases}$$

For example, if $b = 1000100$, then $\phi(b) = 78653421$. Thus, the map ϕ is an isomorphism from B_n to $S_{n+1}(123, 132)$ in the sense that the closeness as defined in the article [9], i.e., two binary strings in B_n are close if they differ in a single position; and two transpositions in $S_{n+1}(123, 132)$ are close if they differ by the transposition of two entries.

In the sequel of the paper, we consider the following definitions:

Definition 4. (1) Two binary strings in B_n are *close* if they differ in a single position, or by a transposition of two bits.

(2) Two permutations in $S_n(123, 132)$ are *close* if they differ by a transposition, or by a product of two transpositions.

With these definitions, we obtain the following lemma:

Lemma 1. *Let* $b, b' \in B_n$ *and* $p = \phi(b)$, $p = \phi(b') \in S_{n+1}(123, 132)$. *If* b *and* b' *are close in* B_n, *then* p *and* p' *are close in* $S_n(123, 132)$.

Proof. In [9], the authors prove that if b and b' differ by one position, then $p = \phi(b)$ and $p' = \phi(b')$ differ by two positions and conversely.

Let us suppose that b is obtained from b' by transposing two bits (the i-th and the j-th). We have:

$$b = b_1 b_2 \cdots b_{i-1} 1 b_{i+1} \cdots b_{j-1} 0 b_{j+1} \cdots b_n$$

and

$$b' = b_1 b_2 \cdots b_{i-1} 0 b_{i+1} \cdots b_{j-1} 1 b_{j+1} \cdots b_n.$$

We set $b'' = b_1 b_2 \cdots b_{i-1} 0 b_{i+1} \cdots b_{j-1} 0 b_{j+1} \cdots b_n$. Since b'' differs from b and b' by a single position, $\phi(b'')$ differs from $\phi(b)$ and $\phi(b')$ by a transposition. Therefore $\phi(b)$ differs from $\phi(b')$ by two transpositions.

Lemma 2. $\phi : F_{n,p} \to S_{n+1}(123, 132, 213, (p+2)(p+3)\overline{(p+1)p \cdots 43}12)$ *is a bijection.*

Proof. Recall that $\phi(B_n) \subset S_{n+1}(123, 132)$. Let us prove that

$$\phi(F_{n,p}) = S_{n+1}(123, 132, 213, (p+2)(p+3)\overline{(p+1)p \cdots 43}12).$$

Each permutation $p \in S_{n+1}(123, 132)$ can be decomposed in the following way (see [18]). There exist indices $0 = k_0 < k_1 < \cdots < k_r < \cdots < k_m = n$ such that p is a sequence of m blocks:

$$p = \boxed{p_1 p_2 \cdots p_{k_1}} \cdots \boxed{p_{k_{r-1}+1} p_{k_{r-1}+2} \cdots p_{k_r}} \cdots \boxed{p_{k_{m-1}+1} p_{k_{m-1}+2} \cdots p_{k_m}}$$

such that: (i) the rightmost elements of each block are in decreasing order (i.e., $n = p_{k_1} > p_{k_2} > \cdots > p_{k_m}$); and (ii) in each block containing at least two elements, the first element equals the last one minus one (i.e., $p_{k_{r-1}+1} = p_{k_r} - 1$), and all elements, except the last one, are consecutive integers in decreasing order (i.e., $p_\ell = p_{\ell+1} + 1$ for $k_{r-1} + 1 \leq \ell < k_r - 1$).

Notice that the blocks of permutation $p \in S_{n+1}(123, 132)$ avoiding 213 are of size at most two. This means that $b = \phi^{-1}(p)$ does not contain the sequence 11. Conversely, if $b \in B_n$ does not contain a sequence 11, then $\phi(b)$ avoids 213. Moreover, if p avoids $(p+2)(p+3)\overline{(p+1)p \cdots 4312}$, then p contains at least $(p-1)$ blocks of size one between two blocks of size two. This means that $b = \phi^{-1}(p)$ contains at least p 0s between two ones, thus $b \in F_{n,p}$. Conversely, if $b \in F_{n,p}$, then we easily obtain that $\phi(b)$ avoids $(p+2)(p+3)\overline{(p+1)p \cdots 4312}$. Therefore

$$\phi(F_{n,p}) = S_{n+1}(123, 132, 213, (p+2)(p+3)\overline{(p+1)p \cdots 4312}).$$

Theorem 4. $\phi(\mathcal{F}_{n,p})$ *is a Gray code for the set*

$$S_{n+1}(123, 132, 213, (p+2)(p+3)\overline{(p+1)p \cdots 4312})$$

such that two successive elements differ by a transposition, or a product of two transpositions (*i.e., by at most four positions*).

Proof. The proof is deduced directly from Lemma 1 and Lemma 2.

In Table 2, we show our Gray code for the sets $\mathcal{F}_{7,3}$, $\mathcal{C}_{10,4}$ and $S_8(123, 132, 213, 67\overline{5432}12)$.

Table 2. The lists $\mathcal{F}_{7,3}, \mathcal{C}_{10,4}$ and $S_8(123, 132, 213, 67\overline{54}312)$. In $\mathcal{F}_{7,3}$ the sublists $\mathcal{F}_{\mathbf{1,3}} \cdot \mathbf{000100}$, $\mathcal{F}_{0,3} \cdot 0001000$, $\mathcal{F}_{\mathbf{3,3}} \cdot \mathbf{0000}$, $\overline{\mathcal{F}}_{3,3} \cdot 0001$ and $\mathcal{F}_{\mathbf{2,3}} \cdot \mathbf{00010}$ are alternatively in bold-face and italic

	$\mathcal{F}_{7,3}$	$\mathcal{C}_{10,4}$	$S_8(123, 132, 213, 67\overline{54}312)$
1	**1000100**	**4+4+1+1**	**78653421**
2	**0000100**	**1+1+1+1+4+1+1**	**87653421**
3	*0001000*	*1+1+1+4+1+1+1*	*87645321*
4	**1000000**	**4+1+1+1+1+1+1**	**78654321**
5	**0000000**	**1+1+1+1+1+1+1+1+1+1**	**87654321**
6	**0010000**	**1+1+4+1+1+1+1**	**87564321**
7	**0100000**	**1+4+1+1+1+1+1**	**86754321**
8	*0100001*	*1+4+1+4*	*86754312*
9	*0010001*	*1+1+4+4*	*87564312*
10	*0000001*	*1+1+1+1+1+1+4*	*87654312*
11	*1000001*	*4+1+1+4*	*78654312*
12	**1000010**	**4+1+4+1**	**78654231**
13	**0000010**	**1+1+1+1+1+4+1**	**87654231**
14	**0100010**	**1+4+4+1**	**86754231**

4. Algorithmic Considerations and Concluding Remarks

In this section, we just give the main difficulties to implement an algorithm generating the Gray code $\mathcal{F}_{n,p}$ obtained in Section 2. More precisely, the complexity of our algorithm will be in Constant Amortized Time (CAT). Recall that an algorithm is CAT if the number of computations after a small amount of preprocessing is proportional to the number of objects generated.

Our procedure gen(n, p, 1) generates all elements of the list $\mathcal{F}_{n,p}$ in an array t anchored by the first element of the list. On the other hand, gen(n, p, 1) generates the list $\overline{\mathcal{F}}_{n,p}$. It produces iteratively recursive calls for generating each sublist of our Gray code. Between any successive calls, at least one update statement is performed (according to the transitions of the different sublists), and after each update statement, a new permutation is produced and printed out. Clearly, the time

complexity of $\mathrm{gen}(n, p, 0)$ is proportional to the total number of recursive calls. Since each call produces at least one new element, the time complexity is $\mathcal{O}(|F_{n,p}|)$, (i.e., an average complexity $\mathcal{O}(1)$). A simple adaptation of this algorithm allows us to generate our Gray code for $S_{n+1}(123, 132, 213, (p+2)(p+3)\overline{(p+1)p\cdots 43}12)$ with the average complexity $\mathcal{O}(1)$. A java implementation of our algorithm is available at `http://www.u-bourgogne.fr/LE2I/jl.baril/compoapplet.html`. One can obtain the java code by sending a request to the authors.

In this paper, a new Gray code for $F_{n,p}$ is provided. It is optimal for n sufficiently large (see [3]). Moreover it contains more transitions of one move than this one in [3]. This induces a Gray code for $(1, p)$-compositions. Moreover, we construct a bijection between $F_{n,p}$ and the set of $(n+1)$-length permutation avoiding 123, 132, 213 and $(p+2)(p+3)\overline{(p+1)p\cdots 43}12$. This bijection induces a Gray code for

$$S_{n+1}(123, 132, 213, (p+2)(p+3)\overline{(p+1)p\cdots 43}12)$$

such that two successive permutations differ by a transposition or a product of two transpositions. Can we find a 3 or 2-Gray code for

$$S_{n+1}(123, 132, 213, (p+2)(p+3)\overline{(p+1)p\cdots 43}12)?$$

Can we find other pattern set T such that there is a one-to-one map between $S_{n+1}(T)$ and $F_{n,p}$?

References

[1] J.-L. Baril and P.-T. Do, ECO-generation for p-generalized Fibonacci and Lucas permutations, Pure Math. Appl. (PU.M.A.) 17(1-2) (2006), 19-37.

[2] J.-L. Baril and V. Vajnovszki, Minimal change list for Lucas strings and some graph theoretic consequences, Theoret. Comput. Sci. (In memoriam: Alberto Del Lungo (1965-2003)) 346(2-3) (2005), 189-199.

[3] T. Chinburg, C. D. Savage and H. S. Wilf, Combinatorial families that are exponentially far from being listable in Gray code sequence, Trans. AMS 351 (1999), 379-402.

[4] P. Chinn and S. Heubach, (1, k)-compositions, Cong. Numer. 164 (2003), 183-194.

[5] E. Di Cera and Y. Kong, Theory of multivalent binding in one and two-dimensional lattices, Biophys. Chem. 61 (1996), 107-124.

[6] Eric S. Egge and T. Mansour, Restricted permutations, Fibonacci numbers, and k-generalized Fibonacci numbers, Integers 5(1) (2005), A1, 12 pp. (electronic).

[7] I. Flores, k-generalized Fibonacci numbers, Fibonacci Quart. 5 (1967), 258-266.

[8] J. T. Joichi, D. E. White and S. D. Williamson, Combinatorial Gray codes, SIAM J. Comput. 9(1) (1980), 130-141.

[9] A. Juarna and V. Vajnovszki, Some generalizations of a Simion-Schmidt bijection, The Computer Journal 50(5) (2007), 574-580.

[10] P. Klingsberg, A Gray code for compositions, J. Algorithms 3 (1982), 41-44.

[11] D. E. Knuth, The Art of Computer Programming, Combinatorial Algorithms, Vol. 4., Pre-fascicle 2b (generating all permutations), 2002.

[12] T. Mansour, Permutations avoiding a pattern from S_k and at least two patterns from S_3, Ars. Combin. 62 (2002), 227-239.

[13] F. Ruskey. http://www.theory.cs.uvic.ca/~cos/gen/perm.html.

[14] F. Ruskey, Combinatorial Generation, Book in preparation, 2004.

[15] R. Simion and F. W. Schmidt, Restricted permutations, European J. Combin. 6 (1985), 383-406.

[16] N. J. A. Sloane, The on-line encyclopedia of integer sequences, Published electronically at: http://www.research.att.com/~njas/sequences.

[17] R. Stanley, Enumerative Combinatorics, Vol. 1, Cambridge Univ. Press, Cambridge, England, 1997.

[18] V. Vajnovszki, A loopless generation of bitstring without p consecutive ones, Proceedings of the Third Discrete Math. Theor. Comput. Sci., Springer, 2001, pp. 227-240.

[19] T. R. Walsh, Loop-free sequencing of bounded integer compositions, J. Combin. Math. Combin. Comput. 33 (2000), 323-345.

[20] R. M. Wilson and J. H. Van Lint, A Course in Combinatorics, Cambridge Univ. Press, 2002.

■

Information Processing Letters 109 (2009) 799–804

Contents lists available at ScienceDirect

Information Processing Letters

www.elsevier.com/locate/ipl

More restrictive Gray codes for some classes of pattern avoiding permutations

Jean-Luc Baril

LE2I UMR-CNRS 5158, Université de Bourgogne, B.P. 47 870, 21078 Dijon Cedex, France

ARTICLE INFO

Article history:
Received 8 July 2008
Received in revised form 26 January 2009
Accepted 22 March 2009
Available online 27 March 2009
Communicated by D. Pointcheval

Keywords:
Algorithms
Gray code
Pattern avoiding permutation
Generating algorithm
Catalan numbers
Schröder numbers
Central binomial coefficients

ABSTRACT

In a recent article [W.M.B. Dukes, M.F. Flanagan, T. Mansour, V. Vajnovszki, Combinatorial Gray codes for classes of pattern avoiding permutations, Theoret. Comput. Sci. 396 (2008) 35–49], Dukes, Flanagan, Mansour and Vajnovszki present Gray codes for several families of pattern avoiding permutations. In their Gray codes two consecutive objects differ in at most four or five positions, which is not optimal. In this paper, we present a unified construction in order to refine their results (or to find other Gray codes). In particular, we obtain more restrictive Gray codes for the two Wilf classes of Catalan permutations of length n; two consecutive objects differ in at most two or three positions which is *optimal* for n odd. Other refinements have been found for permutation sets enumerated by the numbers of Schröder, Pell, even index Fibonacci numbers and the central binomial coefficients. A general efficient generating algorithm is also given.

© 2009 Elsevier B.V. All rights reserved.

1. Introduction

Let S_n be the set of all permutations of length n ($n \geqslant 1$). We represent permutations in one-line notation, *i.e.* if $i_1, i_2, \ldots, i_n$ are n distinct values in $[n] = \{1, 2, \ldots, n\}$, we denote the permutation $\sigma \in S_n$ by the sequence $i_1 i_2 \ldots i_n$ if $\sigma(k) = i_k$ for $1 \leqslant k \leqslant n$. For instance, the identity permutation of length n, id_n, will be written $12\ldots(n-1)n$. Moreover, if $\gamma = \gamma(1)\gamma(2)\ldots\gamma(n)$ is an n-length permutation then the *composition* (*or product*) $\gamma \cdot \sigma$ is the permutation $\gamma(\sigma(1))\gamma(\sigma(2))\ldots\gamma(\sigma(n))$. In S_n, a *k-cycle* $\sigma = \langle i_1, i_2, \ldots, i_k \rangle$ is an n-length permutation satisfying $\sigma(i_1) = i_2, \sigma(i_2) = i_3, \ldots, \sigma(i_{k-1}) = i_k$, $\sigma(i_k) = i_1$ and $\sigma(j) = j$ for $j \in [n] \backslash \{i_1, \ldots, i_k\}$; in particular, a *transposition* is a 2-cycle. A permutation $\sigma = \sigma_1\sigma_2\ldots\sigma_n \in S_n$ contains the pattern $\gamma \in S_k$ ($k \geqslant 2$) if and only if a sequence of indices $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n$ exists such that $\sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_k}$ is ordered as γ. We denote by $S_n(\gamma)$ the set of permutations in S_n avoiding the pattern γ. For example, $\mathbf{25314} \notin S_5(123)$ but $43152 \in S_5(123)$.

Several algorithms for generating permutation classes have been published (generated in lexicographic order or in constant time per permutation, in the amortized sense). For example, efficient algorithms are known for derangements [1], up-down permutations [7,19], permutations with a fixed number of inversions [12] and for permutations avoiding a pattern [9]. On the other hand, algorithms for generating permutation classes in Gray code order are given for permutations [14,22] and their restrictions [17,19,20], derangements [3,16], with a fixed number of cycles [2], involutions and fixed-point free involutions [25] or their generalizations (multiset permutations [24]). Juarna and Vajnovszki [15] presented Gray codes and generating algorithms for two classes of pattern avoiding permutations: $S_n(123, 132)$, $S_n(123, 132, p(p-1)\ldots1(p+1))$. In a recent paper, Dukes et al. [11] give Gray codes and generating algorithms for classes of pattern avoiding permutations enumerated by Catalan, Schröder, Pell, even index Fibonacci numbers and the central binomial coefficients: two consecutive elements in their codes differ in at most four or five positions. Inspired by their article, we give here Gray codes for Catalan permutations and for

E-mail address: barjl@u-bourgogne.fr.

0020-0190/$ – see front matter © 2009 Elsevier B.V. All rights reserved.
doi:10.1016/j.ipl.2009.03.025

some other classes: two consecutive elements differ in at most three (or two) positions. This induces optimal Gray codes for some of these classes: $S_n(321)$ and $S_n(312)$ for n odd. Notice that in [8] or [23], Gray codes are given for some classes previously mentioned, but they do not provide them in term of classes of pattern avoiding permutations. Despite the fact that some well-known bijections exist between these classes they do not conserve the Gray code property.

This paper is organized as follow. In Section 2, we give the construction for our Gray codes based on the ECO method [4]. We obtain sufficient conditions showing that our construction produces Gray codes. Thus we investigate several classes of avoiding pattern permutation sets in Table 2. In the third and last sections, we shortly explain our generating algorithm to list our Gray codes. We present an implementation of our algorithm, and we discuss its complexity.

2. Construction for our Gray codes

In this part, we develop a general construction that provides Gray codes for several classes of pattern avoiding permutations. This construction is a refinement of that given in [11] and, in some cases, yields codes that are optimal. The construction is based on the method ECO [4]. It consists in determining a recursive construction for the considered class based on local expansion performed on the objects. Each object is produced exactly once by an object of lower size.

Let $\sigma \in S_n$; the *sites* of σ are the positions between two consecutives entries, before the first and after the last entry. We suppose that the sites are numbered, from right to left, from 1 to $n+1$. For example, the 3rd site of the permutation $\sigma = 463512$ is between the entries 5 and 1. Moreover, let $\sigma \in S_n(T)$, where T is a set of forbidden patterns, the ith site of σ is said *active* if the permutation γ obtained from σ by inserting $(n+1)$ into this site belongs to $S_{n+1}(T)$. We will denote this permutation γ by $\phi(i,\sigma)$ and we will say that γ is a *successor* (or *son*) of σ. The active sites of a permutation $\sigma \in S_n(T)$ are *right-justified* (see [11]) if all sites to the right of the leftmost active site are also active. We denote by $\chi_T(i,\sigma)$ the number of active sites of $\phi(i,\sigma)$. A set of patterns T is called *regular* [11] if for any $n \geqslant 1$ and $\gamma \in S_n(T)$,

- γ has at least two active sites, and the active sites of γ are right-justified,
- there exist $\sigma \in S_{n-1}(T)$ and an integer i, $1 \leqslant i \leqslant n$, such that $\gamma = \phi(i,\sigma)$,
- $\chi_T(i,\sigma)$ does not depend on σ but only on the number k of active sites of σ; in this case, we will denote $\chi_T(i,\sigma)$ by $\chi_T(i,k)$ if σ has k active sites.

For the remainder of this paper, we only consider regular set T of patterns. Now, for each $n \geqslant 1$ and for any permutation $\sigma \in S_n(T)$, we associate:

- a *direction*: *up* or *down*; a permutation σ with the direction *up* (*down* respectively) will be denoted by σ^1 (σ^0 respectively). Such a permutation will be called a *directed* permutation,
- a list of its sons considered with their directions, *i.e.* a list consisting of some $\phi(i,\sigma)^j$ where $j \in \{0,1\}$ and i are the active sites of σ in an order that we will describe below.

The list of successors of σ^0 will be obtained by reversing the list of successors of σ^1 and by reversing the direction of each element of the list.

Let σ^1 be a directed permutation in $S_n(T)$ with k active sites and let L be a unimodal sequence of integers where $L(i)$ denotes the ith $(1 \leqslant i \leqslant k)$ integer of the following sequence L:

$$L = \begin{cases} 3,5,\ldots,(k-3),(k-1),k,(k-2), \\ \quad (k-4),\ldots,4,2,1 & \text{if } k \text{ is even}, \\ 3,5,\ldots,(k-4),(k-2),k,(k-1), \\ \quad (k-3),\ldots,4,2,1 & \text{if } k \text{ is odd}. \end{cases}$$

We also consider the sequence L' defined as follows:

$$L' = \begin{cases} 2,4,\ldots,(k-4),(k-2),k,(k-1), \\ \quad (k-3),\ldots,5,3,1 & \text{if } k \text{ is even}, \\ 2,4,\ldots,(k-3),(k-1),k,(k-2), \\ \quad (k-4),\ldots,5,3,1 & \text{if } k \text{ is odd}. \end{cases}$$

Let $\Pi(\sigma^1)$ be the list of successors of σ^1 consisting of k directed permutations in $S_{n+1}(T)$ such that: the jth element is $\phi(L(j),\sigma)^{(i-1) \bmod 2}$ for $1 \leqslant j \leqslant k-1$ and the kth element is $\phi(L(k),\sigma)^1$. Moreover, if we replace the sequence L by L' in this definition, we obtain a list that we denote $\Pi'(\sigma^1)$. Figs. 1 and 2 illustrate the different cases for the lists $\Pi(\sigma)$ and $\Pi'(\sigma)$ for σ having $k=7$ or $k=8$ active sites.

Now, we set $d_n = \mathrm{card}(S_n(T))$ and we recursively define our Gray code $\mathcal{S}_n(T)$ for $S_n(T)$ as follows:

$$\mathcal{S}_n(T) = \begin{cases} 1^0 & \text{if } n=1, \\ \bigoplus_{i=1}^{d_{n-1}} \tilde{\Pi}(\sigma[i]) & \text{otherwise}, \end{cases}$$

where

- $\sigma[i]$ is the ith directed permutation of the list $\mathcal{S}_{n-1}(T)$,
- $\tilde{\Pi}(\sigma[i]) = \Pi'(\sigma[i])$ if: the direction of $\sigma[i]$ is 0 and the direction of $\sigma[i+1]$ is 1; $\sigma[i]$ and $\sigma[i+1]$ differ in three positions and have respectively at least three sons and exactly two sons,
- $\tilde{\Pi}(\sigma[i]) = \Pi'(\sigma[i])$ if: the direction of $\sigma[i]$ is 1 and the direction of $\sigma[i-1]$ is 0; $\sigma[i]$ and $\sigma[i-1]$ differ in three positions and have respectively at least three sons and exactly two sons,
- $\tilde{\Pi}(\sigma[i]) = \Pi(\sigma[i])$ otherwise.

Table 1 illustrates our construction for $\mathcal{S}_5(321)$ and $S_5(312)$. Notice that the lists Π and Π' appear in the construction of the Gray code $\mathcal{S}_5(321)$; but only the lists of the form Π are used for $\mathcal{S}_5(312)$.

For the remainder of this paper, we study several properties that allow us to prove this definition gives almost all Gray codes such that two consecutive elements in the list differ in at most three positions. If $\mathcal{L}$ is a list, we denote by first($\mathcal{L}$) and last($\mathcal{L}$) the first and last element of $\mathcal{L}$.

Fig. 1. If $k=7$ then $\Pi(\sigma^1)$ is the list $\phi(3,\sigma)^0, \phi(5,\sigma)^1, \phi(7,\sigma)^0, \phi(6,\sigma)^1, \phi(4,\sigma)^0, \phi(2,\sigma)^1, \phi(1,\sigma)^1$; if $k=8$, $\Pi(\sigma^1)$ is $\phi(3,\sigma)^0, \phi(5,\sigma)^1, \phi(7,\sigma)^0, \phi(8,\sigma)^1, \phi(6,\sigma)^0, \phi(4,\sigma)^1, \phi(2,\sigma)^0, \phi(1,\sigma)^1$. Each point (resp. encircled point) numbered by $\ell \in [1..k]$ represents $\phi(\ell,\sigma)^1$ (resp. $\phi(\ell,\sigma)^0$).

Fig. 2. If $k=7$ then $\Pi'(\sigma^1)$ is the list $\phi(2,\sigma)^0, \phi(4,\sigma)^1, \phi(6,\sigma)^0, \phi(7,\sigma)^1, \phi(5,\sigma)^0, \phi(3,\sigma)^1, \phi(1,\sigma)^1$; if $k=8$, $\Pi'(\sigma^1)$ is $\phi(2,\sigma)^0, \phi(4,\sigma)^1, \phi(6,\sigma)^0, \phi(8,\sigma)^1, \phi(7,\sigma)^0, \phi(5,\sigma)^1, \phi(3,\sigma)^0, \phi(1,\sigma)^1$. Each point (resp. encircled point) numbered by $\ell \in [1..k]$ represents $\phi(\ell,\sigma)^1$ (resp. $\phi(\ell,\sigma)^0$).

Table 1
The optimal cyclic Gray codes for $\mathcal{S}_5(312)$ and $\mathcal{S}_5(321)$. Two consecutive elements differ in at most three positions.

$\mathcal{S}_5(312)$			$\mathcal{S}_5(321)$		
12345	**321**45	**24**315	12345	1**324**5	**231**45
123**54**	3215**4**	2435**1**	123**54**	1325**4**	2315**4**
12**543**	32**541**	**2543**1	1**523**4	135**24**	235**14**
1245**3**	324**51**	2453**1**	**51**234	13**452**	234**51**
1243**5**	3241**5**	2354**1**	**125**34	1342**5**	2341**5**
14325	**432**15	2345**1**	1245**3**	**341**25	2413**5**
1435**2**	4325**1**	2341**5**	1243**5**	3415**2**	2415**3**
1**543**2	4**532**1	231**54**	**412**35	3451**2**	2451**3**
1**45**32	**54**321	2314**5**	4125**3**	3145**2**	2145**3**
13**542**	**435**21	21**43**5	45**12**3	3142**5**	2143**5**
134**52**	**34**521	2145**3**	4152**3**	3152**4**	2153**4**
1342**5**	3**54**21	2154**3**	**14**523	3**512**4	2**51**34
13**254**	3**425**1	21**354**	1425**3**	31**254**	21**354**
1324**5**	3421**5**	2134**5**	1423**5**	3124**5**	2134**5**

Proposition 1. *Let T be a regular set of patterns. The list $\mathcal{S}_n(T)$ contains exactly once all permutations in $S_n(T)$. The first element of the list is $123\ldots(n-1)n$ and the last element is $2134\ldots(n-1)n$.*

Proof. This holds for $n=1$ and $n=2$ since $\mathcal{S}_1(T)=\{1\}$ and $\mathcal{S}_2(T)=\{12,21\}$ where T is a regular pattern set that does not contain the pattern 21. By induction, let us suppose that this is true for all k, $2 \leqslant k < n$. Then, $\text{first}(\mathcal{S}_n(T)) = \phi(1, \text{first}(\mathcal{S}_{n-1}(T))) = 1234\ldots(n-1)n$, and $\text{last}(\mathcal{S}_n(T)) = \phi(1, \text{last}(\mathcal{S}_{n-1}(T))) = 2134\ldots(n-1)n$. The fact that the list $\mathcal{S}_n(T)$ contains each of the permutations exactly once is deduced directly from the above recursive definition of $\mathcal{S}_n(T)$. □

Lemma 2. *Let T be a regular set of patterns. Let σ be a permutation in $\mathcal{S}_n(T)$, and let $r \in \{0,1\}$ be its direction, then two successive permutations in $\Pi(\sigma^r)$ differ in at most three positions.*

Proof. Let α and β the two successive permutations in $\Pi(\sigma^r)$. Thus if k is the number of active sites of σ, there exists i and j, $i \neq j$, $1 \leqslant i, j \leqslant k$, such that $\alpha = \phi(i,\sigma)$ and $\beta = \phi(j,\sigma)$. Considering the definitions of the sequences L or L', we distinguish two cases: (a) if $i = j-1$ then $\alpha = \beta \cdot \langle i, j\rangle$ which induces the result; (b) if $i = j-2$ then $\alpha = \beta \cdot \langle i, j, i+1\rangle$ and the two permutations α and β differ in three positions. The other cases (i.e. $j = i-1$ and $j = i-2$), are similar. □

Lemma 3. *Let T be a regular set of patterns. Let σ^1 and γ^0 be two consecutive directed permutations in the list $\mathcal{S}_n(T)$. Then $d(\sigma,\gamma) = d(\text{last}(\Pi(\sigma^1)), \text{first}(\Pi(\gamma^0)))$.*

Proof. Indeed, let us assume that σ has k sons and γ has ℓ sons. Then we have:

$$\begin{aligned} &d\big(\text{last}\big(\Pi\big(\sigma^1\big)\big), \text{first}\big(\Pi\big(\gamma^0\big)\big)\big) \\ &\quad = d\big(\phi\big(L(k),\sigma\big), \phi\big(L(\ell),\gamma\big)\big) \\ &\quad = d\big(\phi(1,\sigma), \phi(1,\gamma)\big) = d(\sigma,\gamma). \qquad \square \end{aligned}$$

Lemma 4. *Let T be a regular set of patterns. Let σ^0 and γ^1 be two consecutive directed permutations in the list $\mathcal{S}_n(T)$. If both permutations σ and γ have at least three successors then we have: $d(\sigma,\gamma) = d(\text{last}(\Pi(\sigma^0)), \text{first}(\Pi(\gamma^1)))$.*

Proof. Since each permutation σ and γ has at least three sons, their corresponding sequences L satisfy $L(1) = 3$. Thus, we have:

$$\begin{aligned} &d\big(\text{last}\big(\Pi\big(\sigma^0\big)\big), \text{first}\big(\Pi\big(\gamma^1\big)\big)\big) \\ &\quad = d\big(\phi\big(L(1),\sigma\big), \phi\big(L(1),\gamma\big)\big) \\ &\quad = d\big(\phi(3,\sigma), \phi(3,\gamma)\big) = d(\sigma,\gamma). \qquad \square \end{aligned}$$

Lemma 5. *Let T be a regular set of patterns. Let σ^0 and γ^1 be two consecutive directed permutations in the list $\mathcal{S}_n(T)$. If both permutations σ and γ have two successors then we have: $d(\sigma,\gamma) = d(\text{last}(\Pi(\sigma^0)), \text{first}(\Pi(\gamma^1)))$.*

Proof. Since each permutation σ and γ has two sons, their corresponding sequences L verify $L(1) = 2$. Thus, we have:

$$\begin{aligned} &d\big(\text{last}\big(\Pi\big(\sigma^0\big)\big), \text{first}\big(\Pi\big(\gamma^1\big)\big)\big) \\ &\quad = d\big(\phi\big(L(1),\sigma\big), \phi\big(L(1),\gamma\big)\big) \\ &\quad = d\big(\phi(2,\sigma), \phi(2,\gamma)\big) = d(\sigma,\gamma). \qquad \square \end{aligned}$$

Lemma 6. *Let T be a regular set of patterns. Let σ^0 and γ^1 be two consecutive directed permutations in the list $\mathcal{S}_n(T)$ such that $\sigma = \gamma \cdot \langle n-1, n\rangle$. If σ has at least three sons and γ has two sons then we have*: $d(\mathrm{last}(\Pi(\sigma^0)), \mathrm{first}(\Pi(\gamma^1))) = 3$.

Proof. Since the permutation σ has at least three sons, it has a corresponding sequence L such that $L(1) = 3$; as γ has two sons its corresponding sequence L is such that $L(1) = 2$. Thus, we have:

$$\begin{aligned}
&d\big(\mathrm{last}\big(\Pi\big(\sigma^0\big)\big), \mathrm{first}\big(\Pi\big(\gamma^1\big)\big)\big)\\
&\quad = d\big(\phi\big(L(1), \sigma\big), \phi\big(L(1), \gamma\big)\big)\\
&\quad = d\big(\phi(3, \sigma), \phi(2, \gamma)\big)\\
&\quad = d\big(\phi\big(3, \gamma \cdot \langle n-1, n\rangle\big), \phi(2, \gamma)\big)\\
&\quad = d\big(\phi\big(2, \gamma \cdot \langle n-1, n\rangle\big) \cdot \langle n-1, n\rangle, \phi(2, \gamma)\big)\\
&\quad = d\big(\phi(2, \gamma) \cdot \langle n-1, n, n+1\rangle, \phi(2, \gamma)\big) = 3. \qquad \square
\end{aligned}$$

Lemma 7. *Let T be a regular set of patterns. Let σ^0 and γ^1 be two consecutive directed permutations in the list $\mathcal{S}_n(T)$ such that σ and γ differ in three positions. If σ has two sons (resp. at least three sons) and γ has at least three sons (resp. exactly two sons), then we have*: $d(\mathrm{last}(\Pi(\sigma^0)), \mathrm{first}(\Pi(\gamma^1))) = 3$.

Proof. Since the permutation σ has two sons, it has a corresponding sequence L such that $L(1) = 2$; as γ has at least three sons its corresponding sequence L' is such that $L'(1) = 2$. Thus, we have:

$$\begin{aligned}
&d\big(\mathrm{last}\big(\Pi\big(\sigma^0\big)\big), \mathrm{first}\big(\Pi\big(\gamma^1\big)\big)\big)\\
&\quad = d\big(\phi\big(L(1), \sigma\big), \phi\big(L'(1), \gamma\big)\big)\\
&\quad = d\big(\phi(2, \sigma), \phi(2, \gamma)\big) = 3. \qquad \square
\end{aligned}$$

Lemma 8. *Let T be a regular set of patterns. Let σ^1 and γ^1 be two consecutive directed permutations in the list $\mathcal{S}_n(T)$ such that $\sigma = \gamma \cdot \langle n-1, n\rangle$. If γ has at least three sons, then we have: $d(\mathrm{last}(\Pi(\sigma^1)), \mathrm{first}(\Pi(\gamma^1))) = 2$. On the other hand if γ has two sons then we have*: $d(\mathrm{last}(\Pi(\sigma^1)), \mathrm{first}(\Pi(\gamma^1))) = 3$.

Proof. Let us assume that the permutation σ has k sons. Thus its corresponding sequence L is such that $L(k) = 1$. Let us assume that γ has at least three sons. It corresponding sequence L is such that $L(1) = 3$. Thus, we have:

$$\begin{aligned}
&d\big(\mathrm{last}\big(\Pi\big(\sigma^1\big)\big), \mathrm{first}\big(\Pi\big(\gamma^1\big)\big)\big)\\
&\quad = d\big(\phi\big(L(k), \sigma\big), \phi\big(L(1), \gamma\big)\big)\\
&\quad = d\big(\phi(1, \sigma), \phi(3, \gamma)\big)\\
&\quad = d\big(\phi\big(1, \gamma \cdot \langle n-1, n\rangle\big), \phi(3, \gamma)\big)\\
&\quad = d\big(\phi(1, \gamma) \cdot \langle n-1, n\rangle, \phi(3, \gamma)\big)\\
&\quad = d\big(\phi(1, \gamma) \cdot \langle n-1, n\rangle, \phi(1, \gamma) \cdot \langle n-1, n\rangle\langle n, n+1\rangle\big)\\
&\quad = d\big(\phi(1, \gamma), \phi(1, \gamma) \cdot \langle n-1, n+1\rangle\big) = 2.
\end{aligned}$$

The proof for γ with two sons is similar. $\square$

Remark that:

- if $\chi_T(i, k) = 2$ for all i and k, the list $\mathcal{S}_n(T)$ defined above is a cyclic Gray code for $S_n(T)$ such that two consecutive elements in the list differ in two positions. Indeed, it suffices to apply Lemmas 2, 3, and 5.
- If $\chi_T(i, k) \geqslant 3$ for all i and k, then the list $\mathcal{S}_n(T)$ defined above is a cyclic Gray code for $S_n(T)$. Two consecutive elements in the list differ in at most three positions. It suffices to apply Lemmas 2, 3, 4 and 8.

In Table 2, we give several avoiding pattern sets T for which we can apply our construction in order to obtain Gray codes for $S_n(T)$. Notice that this construction allows us to obtain optimal Gray code for $S_{2n+1}(321)$ and $S_{2n+1}(312)$, $n \geqslant 0$. Indeed the difference between the number of even and odd permutations in $S_{2n+1}(321)$ (resp. $S_{2n+1}(312)$) is at least two (see [21]). Thus we cannot have a Gray code such that two consecutive permutations differ by a transposition. Since the difference between the number of even and odd permutations in $S_{2n}(321)$ (resp. $S_{2n}(312)$) is zero, we cannot deduce the optimality of our Gray code for $S_{2n}(321)$ (resp. $S_{2n}(312)$) and for other classes. Besides the Gray code for $S_n(321, 312)$ is optimal and it corresponds to the one in [15] which is obtained by another method.

3. Algorithmic considerations

In this part, we provide the procedure *gen_up*$(size, k)$ for generating our Gray codes for $S_n(T)$ characterized by the succession functions $\chi_T(k, i)$ given in Table 2. We initialize $\sigma := 123\ldots(n-1)n$ and the call *gen_down*$(1, 2)$ produces the Gray code $\mathcal{S}_n(T)$ (the procedure *gen_down*$(size, k)$ consists in the statements of *gen_up*$(size, k)$ in reverse order). This algorithm is inspired from the one in [11]. Only the order of the statements are modified according to the two sequences L and L'. Notice that the sequence L' is used if and only if we have the hypotheses of Lemma 7, i.e. when the current permutation σ produces a list Π' obtained from L'. This algorithm enables us to ensure that we transform an object into its successor in constant amortized time. Indeed, the degree of each call is not 1; moreover, between two recursive calls we execute at most two transpositions when we update the current permutation σ. Moreover at least one permutation is generated in each recursive call. This means that the total amount of computation divided by the number of objects is bounded by a constant. Thus the complexity of this algorithm is $\mathcal{O}(|S_n(T)|)$. An applet for the generation of Gray codes $\mathcal{S}_n(T)$ for each set T cited in Table 2, is available on the web site http://www.u-bourgogne.fr/jl.baril/applet.html.

```
procedure gen_up(size, k)
dr := 0;
if size = n then output σ;
else
    for i := 1 to k do
    update σ
    if we have hypotheses of Lemma 7 then
        gen_down(size + 1, χ_T(k, L'(i)));
        dr := 1 − dr;
```

Table 2
Classes of permutations avoiding patterns in T with their corresponding succession functions $\chi_T(i,k)$. We give also the different lemmas that are used in order to obtain a Gray code and the maximum numbers such that two consecutive permutations differ in [11] and in our Gray code.

Classes	T	$\chi_T(i,k)$	Lemmas	[11], B
2^{n-1} [6]	$\{321, 312\}$	2	2, 3, 5	3, 2
Catalan [26]	$\{312\}$	$i+1$	2, 3, 4, 6, 8	5, 3
	$\{321\}$	$k+1$ if $i=1$ i otherwise	2, 3, 4, 6, 7, 8	5, 3
Even index Fibonacci [6]	$\{321, 3412\}$	$k+1$ if $i=1$ 2 otherwise	2, 3, 5, 6, 8	5, 3
	$\{321, 4123\}$	3 if $i=1$ i otherwise	2, 3, 4, 5, 6, 8	5, 3
	$\{321, 4321\}$	3 if $i=3$ and $k=3$ $i+1$ otherwise	2, 3, 4, 6, 8	·, 3
Pell [6]	$\{321, 3412, 4123\}$	3 if $i=1$ 2 otherwise	2, 3, 5, 6, 8	5, 3
	$\{312, 4321, 3421\}$	3 if $i=2$ 2 otherwise	2, 3, 5, 6, 8	·,3
Schröder [13]	$\{4321, 4312\}$	$k+1$ if $i=1$ or $i=2$ i otherwise	2, 3, 4, 6, 8	5, 3
	$\{4231, 4132\}$	$k+1$ if $i=1$ or $i=k$ $i+1$ otherwise	2, 3, 4, 6, 8	5, 3
	$\{4123, 4213\}$	$k+1$ if $k-1 \leqslant i \leqslant k$ $i+2$ otherwise	2, 3, 4, 6, 8	5, 3
Central binomial coefficient [13]	$\{4321, 4231, 4312, 4132\}$	$k+1$ if $i=1$ 3 if $i=2$ i otherwise	2, 3, 4, 6, 8	5, 3
	$\{4231, 4132, 4213, 4123\}$	3 if $i=1$ $i+1$ otherwise	2, 3, 4, 6, 8	5, 3
Generalized pattern [6,10]	$\{321, (p+1)12\ldots p\}$	$k+1$ if $i=1$ and $k<p$ p if $i=1$ and $k=p$ i otherwise	2, 3, 4, 6, 7, 8	5, 3
	$\{321, 3412, (p+1)12\ldots p\}$	$k+1$ if $i=1$ and $k<p$ p if $i=1$ and $k=p$ 2 otherwise	2, 3, 5, 6, 8	5, 3
Generalized Motzkin [5]	for $p \geqslant 2$, $\{321, (p+2)\bar{1}(p+3)2\ldots(p+1)\}$	$k+1$ if $i=1$ i if $2 \leqslant i \leqslant p$ $i-1$ otherwise	2, 3, 4, 6, 7, 8	·, 3
Generalized Schröder [18]	$\bigcup_{\tau \in S_{n-1}} \{(p+1)\tau p\}$	$k+1$ if $k \leqslant p$ or $p+i \geqslant k+1$ $i+p-1$ otherwise	2, 3, 4, 6, 8	5, 3

A dot in the last column means that there is no Gray code for these classes in the literature.

```
    else
      update σ
      if dr = 1 then gen_up(size + 1, χ_T(k, L(i)));
      else gen_down(size + 1, χ_T(k, L(i)));
      end if
      dr := 1 − dr;
    end if
end if
end procedure;
```

Acknowledgements

I thank the anonymous referees for their constructive remarks which have greatly improved this paper.

References

[1] S.G. Akl, A new algorithm for generating derangements, BIT 20 (1980) 2–7.
[2] J.-L. Baril, Gray code for permutations with a fixed number of cycles, Discrete Math. 30 (13) (2007) 1559–1571.
[3] J.-L. Baril, V. Vajnovszki, Gray code for derangements, Discrete Appl. Math. 140 (2004) 207–221.
[4] E. Barcucci, A. Del Lungo, E. Pergola, R. Pinzani, A methodology for the enumeration of combinatorial objects, Discrete J. Difference Equations Appl. 5 (1999) 435–490.
[5] E. Barcucci, A. Del Lungo, E. Pergola, R. Pinzani, From Motzkin to Catalan permutations, Discrete Math. 217 (1–3) (2000) 33–49.
[6] E. Barcucci, A. Bernini, M. Poneti, From Fibonacci to Catalan permutations, PuMA 17 (1–2) (2006) 1–17.
[7] B. Bauslaugh, F. Ruskey, Generating alternating permutations lexicographically, BIT 30 (1990) 17–26.
[8] A. Bernini, E. Grazzini, E. Pergola, R. Pinzani, A general exhaustive generation algorithm for Gray structures, Acta Inform. 44 (5) (2007) 361–376.
[9] M. Bona, Combinatorics of Permutations, Chapman & Hall, 2004.
[10] T. Chow, J. West, Forbidden sequences and Chebyshev polynomials, Discrete Math. 204 (1999) 119–128.
[11] W.M.B. Dukes, M.F. Flanagan, T. Mansour, V. Vajnovszki, Combinatorial Gray codes for classes of pattern avoiding permutations, Theoret. Comput. Sci. 396 (2008) 35–49.

[12] S. Effler, F. Ruskey, A CAT algorithm for generating permutations with a fixed number of inversions, Inform. Process. Lett. 86 (2003) 107–112.
[13] O. Guibert, Combinatoire des permutations à motifs exclus en liaison avec mots, cartes planaires et tableaux de Young, PhD thesis, Universit'e Bordeaux 1, 1995.
[14] S.M. Johnson, Generation of permutations by adjacent transpositions, Math. Comp. 17 (1963) 282–285.
[15] A. Juarna, V. Vajnovszki, Some generalizations of a Simion–Schmidt bijection, Comput. J. 50 (2007) 574–580.
[16] J.F. Korsh, Constant time generation of derangements, Inform. Process. Lett. 90 (4) (2004) 181–186.
[17] J.F. Korsh, Loopless generation of up-down permutations, Discrete Math. 240 (1–3) (2001) 97–122.
[18] D. Kremer, Permutations with forbidden subsequences and a generalized Schröder number, Discrete Math. 218 (1–3) (2000) 121–130.
[19] D. Roelants van Baronaigien, F. Ruskey, Generating permutations with given ups and downs, Discrete Appl. Math. 36 (1) (1992) 57–65.
[20] F. Ruskey, U. Taylor, Fast generation of restricted classes of permutations, Manuscript, 1995.
[21] R. Simion, F.W. Schmidt, Restricted permutations, European J. Combin. 6 (1985) 383–406.
[22] H.F. Trotter, PERM (Algorithm 115), Comm. ACM 5 (8) (1962) 434–435.
[23] V. Vajnovszki, Gray visiting Motzkins, Acta Inform. 38 (2002) 793–811.
[24] V. Vajnovszki, A loopless algorithm for generating the permutations of a multiset, Theoret. Comput. Sci. 307 (2003) 415–431.
[25] T. Walsh, Gray codes for involutions, J. Combin. Math. Combin. Comput. 36 (2001) 95–118.
[26] J. West, Generating trees and the Catalan and Schröder numbers, Discrete Math. 146 (1994) 247–262.

Discrete Mathematics 309 (2009) 5147–5157

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Neighbor-distinguishing k-tuple edge-colorings of graphs

Jean-Luc Baril, Olivier Togni *

LE2I UMR-CNRS 5158, Université de Bourgogne, B.P. 47 870, 21078 DIJON-Cedex, France

ARTICLE INFO

Article history:
Received 15 October 2008
Received in revised form 1 April 2009
Accepted 2 April 2009
Available online 7 May 2009

Keywords:
Graph
k-tuple edge-coloring
Neighbor-distinguishing
Adjacent vertex-distinguishing
Fractional coloring
Circular coloring

ABSTRACT

This paper studies proper k-tuple edge-colorings of graphs that distinguish neighboring vertices by their sets of colors. Minimum numbers of colors for such colorings are determined for cycles, complete graphs and complete bipartite graphs. A variation in which the color sets assigned to edges have to form cyclic intervals is also studied and similar results are given.

© 2009 Elsevier B.V. All rights reserved.

1. Introduction

The graphs considered in this paper are undirected and without loops. For a graph $G = (V, E)$ with vertex set V and edge set E, let $\Delta(G)$ (or simply Δ for short) be its maximum degree.

A *proper k-tuple edge-coloring* ϕ of a graph G is a mapping from E to $\mathbb{N}^k$ such that edges incident with the same vertex receive disjoint color sets. The k-tuple version of the vertex-coloring problem was first considered by Stahl [12]. For any vertex x of G, let $S_\phi(x)$ denote the union of the color sets of all edges incident to x (we will omit the subscript ϕ if no confusion is possible). A proper k-tuple edge-coloring is said to be *neighbor-distinguishing* (*ND*) if $S(x) \neq S(y)$ $\forall xy \in E$. The least number of colors needed for a k-tuple ND-coloring of a graph G will be called its *k-tuple ND-chromatic index* and will be denoted by $\chi'_a(G; k)$. Notice that a graph with an isolated edge does not admit a k-tuple ND-coloring for any k. Consequently, even if not specified, all graphs we deal with are assumed to have no single edge as a component.

The study of 1-tuple ND-colorings of graphs was initiated in [16] under the name of adjacent strong edge-colorings and with the notation $\chi'_a(G) = \chi'_a(G; 1)$. In [1], Balister et al. proved that $\chi'_a(G) \leq 5$ for graphs of maximum degree 3 and $\chi'_a(G) \leq \Delta + 2$ for bipartite graphs. In [2], the 1-tuple ND-chromatic index of multidimensional meshes was determined.

The bound $\chi'_a(G) \geq \Delta$ is trivial. Moreover if G contains two adjacent vertices of degree Δ then $\chi'_a(G) \geq \Delta + 1$. The following conjecture was made in [16]:

Conjecture 1 ([16]). *Let $G \neq C_5$ be a connected graph of maximum degree Δ; then*

$$\Delta \leq \chi'_a(G) \leq \Delta + 2.$$

* Corresponding author.
E-mail addresses: barjl@u-bourgogne.fr (J.-L. Baril), olivier.togni@u-bourgogne.fr (O. Togni).

0012-365X/$ – see front matter © 2009 Elsevier B.V. All rights reserved.
doi:10.1016/j.disc.2009.04.003

In relation with this conjecture, Hatami [7] has shown that $\chi'_a(G) \leq \Delta + 300$ for any graph G of maximum degree $\Delta > 10^{20}$. Greenhill and Ruciński [5] prove the conjecture for almost all 4-regular graphs. Edwards et al. [4] showed that $\Delta + 1$ colors are sufficient for planar bipartite graphs of maximum degree $\Delta \geq 12$.

Some extensions and variations were also considered: total ND-colorings [15,3], ND-colorings from lists [8] and non proper ND-colorings [6]. For other related distinguishing coloring parameters, see [13].

Not surprisingly, Conjecture 1 cannot be extended to k-tuple edge-colorings, in the light of Shannon's well-known chromatic index theorem. Hence, the aim of this paper is to study k-tuple ND-colorings of graphs. We also study a variation of k-tuple ND-coloring where the set of colors assigned to each edge has to form a cyclic interval: a k-tuple ND-coloring ϕ of a graph G with colors from $\{0, \ldots, N-1\}$ is said to be a *cyclic k-tuple ND-coloring* if the color set $\phi(xy)$ of the edge between x and y is an interval modulo N (of size k) which will be denoted by $\phi(xy) = [i, i+k-1]_N$ for some i, $0 \leq i \leq N-1$, where $[a, b]_N = [a \bmod N, (a+1) \bmod N, \ldots, b \bmod N]$. The least number of colors needed for a cyclic k-tuple ND-coloring of a graph G will be called its *cyclic k-tuple ND-chromatic index* and will be denoted by $\chi'_{ac}(G; k)$.

Observe that for any graph G and any $k \geq 1$, $\chi'_{ac}(G; k) \geq \chi'_a(G; k)$, and, as we will show later, there exist several classes of graphs for which $\chi'_{ac}(G; k) > \chi'_a(G; k)$.

Without the ND constraint, k-tuple edge-colorings lead to the definition of the fractional chromatic index $\chi'_f(G)$ which can be defined by $\chi'_f(G) = \inf_k \frac{\chi'(G;k)}{k} = \min_k \frac{\chi'(G;k)}{k}$ [14]. In the same vein, cyclic k-tuple edge-colorings of graphs lead to the definition of the circular chromatic index [10,11]: the circular chromatic index of G is the ratio minimum N/k for which there exists an edge-coloring of G by cyclic (or circular) intervals modulo N of size k. Notice that, as can be seen in the remainder of the paper, there are graphs G for which $\chi'_a(G; k)/k$ is a strictly decreasing function of k, and thus the infimum is never reached (which is not the case without the ND constraint).

The paper is organized as follows. Section 2 presents some general simple results about the k-tuple ND-chromatic and cyclic k-tuple ND-chromatic indices of graphs. In Section 3, we determine the k-tuple ND-chromatic and cyclic k-tuple ND-chromatic indices of paths and cycles. In Section 4, k-tuple ND-chromatic and cyclic k-tuple ND-chromatic indices of complete and complete bipartite graphs are determined. Section 5 concludes the paper by presenting a conjecture about the ND-chromatic index of a multigraph.

2. General observations

We begin with some simple observations that will be used throughout the rest of the paper.

Let $I^N_{j,k}$ be the cyclic interval $[j, j+k-1]_N$. If N and k are clear from the context, we will permit ourselves to write I_j instead of $I^N_{j,k}$. Let also $\mathcal{I}^N_k = \{I^N_{j,k}, 0 \leq j \leq N-1\}$.

Observation 1. *For any graph G,*

$$\chi'_{ac}(G; k) \leq k\chi'_a(G).$$

Proof. Starting from an ND-coloring of G with $\chi'_a(G)$ colors, a cyclic k-tuple ND-coloring of G with $N = k\chi'_a(G)$ colors can be obtained simply by replacing each color i by the color interval I_{ki}. □

Observation 2. *For any graph G and any integers k and k′ with* $1 \leq k' < k$,

$$\chi'_a(G; k) \leq \chi'_a(G; k') + \chi'(G; k-k').$$

In particular, $\chi'_a(G; k) \leq \chi'_a(G) + (k-1)\chi'(G)$.

Remember that a graph is *class* 1 if $\chi'(G) = \Delta(G)$ and *class* 2 if $\chi'(G) = \Delta(G) + 1$. Thus, derived from the above observation, we have the following:

Observation 3. *For any class* 1 *graph G,*

$$\chi'_a(G; k) \leq \chi'_a(G) + (k-1)\Delta(G).$$

As a corollary, we easily obtain the next proposition:

Proposition 2. *For a class* 1 *graph G,*

- *if G has two adjacent vertices of maximum degree and if* $\chi'_a(G) = \Delta(G) + 1$, *then* $\chi'_a(G; k) = k\Delta(G) + 1$;
- *if* $\chi'_a(G) = \Delta(G)$, *then* $\chi'_a(G; k) = k\Delta(G)$.

For instance, the cycle C_{6p} is class 1 and $\chi'_a(C_{6p}) = 3$; thus $\chi'_a(C_{6p}; k) = 2k$.

3. The path P_n and cycle C_n

Let P_n be the path of order n, with vertex set $V = \{0, 1, \ldots, n-1\}$ and edge set $E = \{e_i = i(i+1), 0 \le i \le n-2\}$ and let C_n be the cycle of order n with vertex set V and edge set $E = \{e_i = i(i+1) \bmod n, 0 \le i \le n-1\}$.

For the path, we easily obtain the following theorem:

Theorem 3. *For $n \ge 4$ and $k \ge 1$, $\chi'_{ac}(P_n; k) = \chi'_a(P_n; k) = 2k+1$.*

Proof. The theorem directly follows from Proposition 2 since P_n is class 1 and $\chi'_a(P_n) = 3$. However, a cyclic k-tuple ND-coloring ϕ can be simply constructed by setting $\phi(e_i) = I^N_{ki,k}$ for $0 \le i \le n-1$ and $N = 2k+1$. □

In order to determine the cyclic k-tuple ND-chromatic index of the cycle, we need some notation. Let the *gap* $\mathrm{gap}(I^N_{a,k}, I^N_{b,k})$ be the number of integers between the last element of $I^N_{a,k}$ and the first of $I^N_{b,k}$ considered modulo N, i.e. $\mathrm{gap}(I^N_{a,k}, I^N_{b,k}) = b-a-k \bmod N$. Notice that this definition is not symmetrical since there exist many cases such that $\mathrm{gap}(I^N_{a,k}, I^N_{b,k}) \ne \mathrm{gap}(I^N_{b,k}, I^N_{a,k})$.

Any cyclic k-tuple coloring of the cycle C_n can naturally be associated with the sequence of gaps between intervals of consecutive edges. For instance, for the cycle C_9 with $k = 4$ and $N = 10$ colors, the sequence $S = (2, 1, 0, 0, 0, 0, 0, 0, 1)$ corresponds with intervals $(I_0, I_6, I_1, I_5, I_9, I_3, I_7, I_1, I_5)$ along the edges $(e_0, e_1, \ldots, e_{n-1})$ of the cycle.

Notice also that in order for a sequence $S = (s_1, s_2, \ldots, s_n)$ of gaps to correspond with a cyclic k-tuple coloring of the cycle C_n with N colors, we must have $\sum_{i=1}^n s_i + kn \equiv 0 \bmod N$ (because the gap between the last interval and the first one is completely determined by the $n-1$ other gaps).

Lemma 4. *For $k \ge 1$ and $n \ge 4$ even, $\chi'_{ac}(C_n; k) \le 2k+2$.*

Proof. We define a cyclic k-tuple ND-coloring of C_n with $N = 2k+2$ colors by giving its sequence of gaps S:

$$S = (\underbrace{2, 2, \ldots, 2}_{\frac{n}{2}-1}, 1, \underbrace{0, 0, \ldots, 0}_{\frac{n}{2}-1}, 1).$$

The associated coloring is clearly proper since S does not contain any gap of size at least 3. The coloring is also ND since the sequence does not contain any subsequence of length 2 of the form 0, 2; 2, 0 or 1, 1 that are the only cases which prevent the coloring from being ND when $N = 2k+2$. □

Lemma 5. *For $k \ge 2$ and n odd, $n \ge 2k+3$, $\chi'_{ac}(C_n; k) \le 2k+2$.*

Proof. We define a cyclic k-tuple ND-coloring of C_n with $2k+2$ colors by giving its sequence of gaps S:

$$S = \begin{cases} (\underbrace{2, 2, \ldots, 2}_{\frac{n-k-3}{2}}, 1, 0, 0, 0, \ldots, 0, 1) & \text{if } k \text{ is even,} \\ (\underbrace{2, 2, \ldots, 2}_{\frac{n-k-4}{2}}, 1, 0, 1, 0, 0, \ldots, 0, 1) & \text{otherwise.} \end{cases}$$

As above, it can be shown that the associated coloring is ND. In addition, we just need to verify that $\frac{n-k-3}{2} \ge 1$ for k even, and $\frac{n-k-4}{2} \ge 1$ otherwise. Indeed, $n-k-3 \ge 2k+3-k-3 \ge k \ge 2$ when k is even; and $n-k-4 \ge 2k+3-k-4 = k-1 \ge 2$ for k odd and $k \ge 2$. □

Theorem 6. *For $n \ge 3$ and $k \ge 1$,*

$$\chi'_{ac}(C_n; k) = \begin{cases} 5 & \text{if } (n, k) = (5, 1), \\ 2k + \left\lceil \dfrac{2k}{n-1} \right\rceil & \text{if } n < 2k+1 \text{ is odd}, \\ 2k+1 & \text{if } n \equiv 0 \bmod (2k+1), \\ 2k+2 & \text{otherwise}. \end{cases}$$

Proof. The case $(n, k) = (5, 1)$ is trivial.

We shall now show that $\chi'_{ac}(C_n; k) = 2k+1$ if and only if $n \equiv 0 \bmod (2k+1)$.

Let $N = 2k+1$ and $n \equiv 0 \bmod (2k+1)$. The k-tuple coloring ϕ of C_n defined by $\phi(e_j) = I_{kj}, 0 \le j \le n-1$, or equivalently by the gap sequence $S = (0, \ldots, 0)$ is clearly proper and ND since $\phi(e_{n-1}) = I_{2k^2} = I_{k+1} \ne \phi(e_1) = I_k$. This coloring is exhibited in Fig. 1 for $(n, k) = (7, 3)$.

On the other hand, it is easily seen that any cyclic k-tuple ND-coloring of C_n with $2k+1$ colors is isomorphic to ϕ up to a renumbering of the colors since in that case the interval of colors that can be assigned to an edge e_i is completely determined by the intervals assigned to its two preceding edges e_{i-1} and e_{i-2}. Thus a cyclic k-tuple ND-coloring of C_n with $2k+1$ colors exists only if $n \equiv 0 \bmod (2k+1)$.

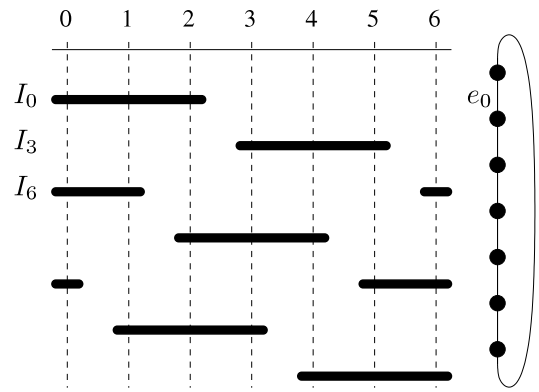


Fig. 1. A cyclic 3-tuple ND-coloring of C_7 with $N = 7$ colors.

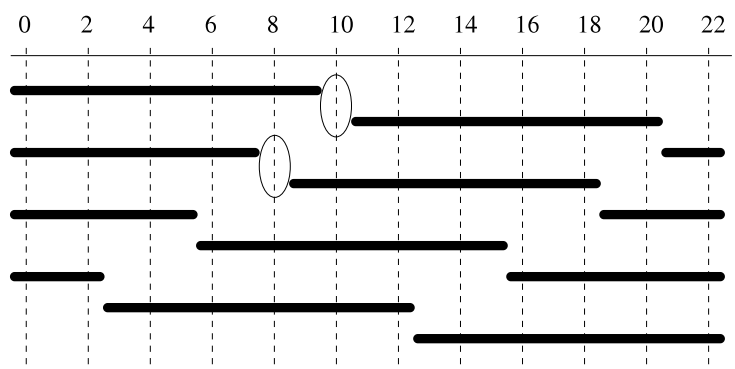


Fig. 2. A cyclic 10-tuple ND-coloring of C_9 with $N = 23$ colors and its associated sequence of gaps $(1, 0, 1, 0, 0, 0, 0, 0, 0)$.

We now show that if $n < 2k + 1$ and n is odd then $\chi'_{ac}(C_n; k) = 2k + \lceil \frac{2k}{n-1} \rceil$. It is sufficient to show that $\chi'_{ac}(C_n; k) \leq 2k + \lceil \frac{2k}{n-1} \rceil$ since $\chi'_{ac}(C_{2p+1}; k) \geq \chi'(C_{2p+1}; k) \geq k\frac{2p+1}{p} = k(2 + \frac{1}{p})$ (because at most p edges of C_{2p+1} can be given the same color).

Let $n = 2p + 1$, with $p \leq k - 1$ and let $s = p\lceil \frac{k}{p} \rceil - k$.

Consider the k-tuple coloring ϕ of C_{2p+1} with $N = 2k + \lceil \frac{2k}{n-1} \rceil$ colors defined by the following sequence of gaps:

$$S = (\underbrace{1, 0, 1, 0, \ldots, 1, 0}_{2s}, 0, 0, \ldots, 0, 0).$$

Remark that $0 \leq s \leq p\frac{k+p}{p} - k = p$.

As S does not contain (a) any gap of size at least 2, or (b) any subsequence of the form 1, 1, and since the sum of gaps plus nk equals 0 modulo $2k + \lceil \frac{2k}{n-1} \rceil$, then ϕ is a k-tuple ND-coloring of C_n with $2k + \lceil \frac{2k}{n-1} \rceil$ colors. An illustration is given in Fig. 2 for $(n, k) = (9, 10)$ (and thus $s = 2$).

We end the proof by showing that $\chi'_{ac}(C_n; k) = 2k + 2$ for the remaining cases. By the above, we know that $\chi'_{ac}(C_n; k) > 2k + 1$ if $n \not\equiv 0 \bmod (2k + 1)$. Thus, by virtue of Lemma 4 and Lemma 5, we have that $\chi'_{ac}(C_n; k) \leq 2k + 2$ for these cases, which completes the proof. □

Theorem 7. *For $n \geq 3$ and $k \geq 1$,*

$$\chi'_a(C_n; k) = \begin{cases} 5 & \text{if } (n, k) = (5, 1), \\ 2k + \left\lceil \dfrac{2k}{n-1} \right\rceil & \text{if } n < 2k + 1 \text{ is odd}, \\ 2k + 2 & \text{if } n = 4 \text{ or } (n, k) = (7, 2), \\ 2k + 1 & \text{otherwise}. \end{cases}$$

Proof. Recall that for any graph G and any $k \geq 1$, $\chi'_a(G; k) \leq \chi'_{ac}(G; k)$. Then, with Theorem 6, it remains to treat the following cases:

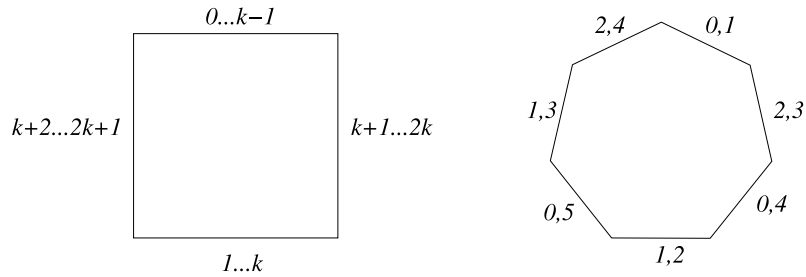


Fig. 3. The cases $n = 4$ and $(n, k) = (7, 2)$.

- $n = 4$ or $(n, k) = (7, 2)$. It is easily seen that a k-tuple ND-coloring with $2k + 1$ colors does not exist in these cases and a k-tuple ND-coloring with $2k + 2$ colors is illustrated in Fig. 3.
- When n is odd, we have that $\chi'_a(C_{2p+1}; k) \geq \chi'(C_{2p+1}; k) \geq k\frac{2p+1}{p} = k(2 + \frac{1}{p}) = 2k + \lceil \frac{2k}{n-1} \rceil$ (since at most p edges of C_{2p+1} can be given the same color).
- $k = 2, n \geq 5, n \neq 7$. In this part we consider five subcases depending on the residues of n modulo 5:

Subcase 0: $n \equiv 0 \mod 5$ and $n \geq 5$. We color the cycle by using the sequence of color sets $S_1 = (\{0, 1\}, \{2, 3\}, \{0, 4\}, \{1, 2\}, \{3, 4\})$ repetitively on each group of five consecutive edges of C_n.

Subcase 1: $n \equiv 1 \mod 5$ and $n \geq 6$. For $n = 5q + 1$, we color the cycle by using $(q - 1)$ times the sequence S_1 and one time the sequence $S_2 = (\{0, 1\}, \{2, 3\}, \{0, 4\}, \{1, 3\}, \{0, 2\}, \{3, 4\})$.

Subcase 2: $n \equiv 2 \mod 5$ and $n \geq 12$. for $n = 5q + 2$ with $q \geq 2$, we color the cycle by using $(q - 2)$ times the sequence S_1 and two times the sequence S_2.

Subcase 3: $n \equiv 3 \mod 5$ and $n \geq 8$. for $n = 5q + 3$, we color the cycle by using $(q - 1)$ times the sequence S_1 and one time the sequence $S_3 = (\{0, 1\}, \{2, 3\}, \{0, 4\}, \{1, 2\}, \{3, 4\}, \{0, 2\}, \{1, 3\}, \{2, 4\})$.

Subcase 4: $n \equiv 4 \mod 5$ and $n \geq 9$. For $n = 5q + 3$, we color the cycle by using $(q - 1)$ times the sequence S_1 and one time the sequence $S_4 = (\{0, 1\}, \{2, 3\}, \{0, 4\}, \{1, 2\}, \{0, 3\}, \{1, 4\}, \{0, 2\}, \{1, 3\}, \{2, 4\})$.

In each case we obtain a 2-tuple ND-coloring of C_n with $5 = 2k + 1$ colors.

- $k \geq 3$ and $n = 2p$ is even. By Observation 2, we have $\chi'_a(C_{2p}; k) \leq \chi'_a(C_{2p}; 2) + \chi'(C_{2p}; k - 2) = 5 + 2(k - 2) = 2k + 1$.
- $k \geq 3$ and n odd, $n \geq 2k + 1 \geq 7$. We set $N = 2k + 1$, $n = qN + r$ with $0 \leq r \leq N - 1$ and $q \geq 1$.

Subcase 1: $n \equiv 0 \mod N$. We color the cycle by using the periodic coloring E defined by $E_i = I^N_{ik,k}$, for $i \geq 0$.

Subcase 2: $n \not\equiv 0 \mod N$. We provide a k-tuple ND-coloring for each value of the residue of r modulo 6 by giving the sequence of color sets to assign to the edges in a consecutive manner (see Fig. 4 for an illustration of the construction).

– (i) $r \equiv 0 \mod 6$. Starting from the qN-first color sets of E, we append the r-first color sets of the following periodic coloring F:
  $F_0 = \{0, 1, 3, 4, \ldots, k\}, F_1 = \{2, k + 1, \ldots, 2k - 1\}, F_2 = \{1, 3, 4, \ldots, k, 2k\}, F_3 = \{0, k + 1, \ldots, 2k - 1\}, F_4 = I^N_1$, $F_5 = I^N_{k+1}$ with $F_i = F_{i-6}$ for $i \geq 6$.
– (ii) $r \equiv 1 \mod 6$. We discuss on the parity of k.
  For k odd, starting from the $((q-1)N+k)$-first color sets of E, we append the $k+1$ color sets defined for $0 \leq i \leq \lceil \frac{k}{2} \rceil - 2$ by

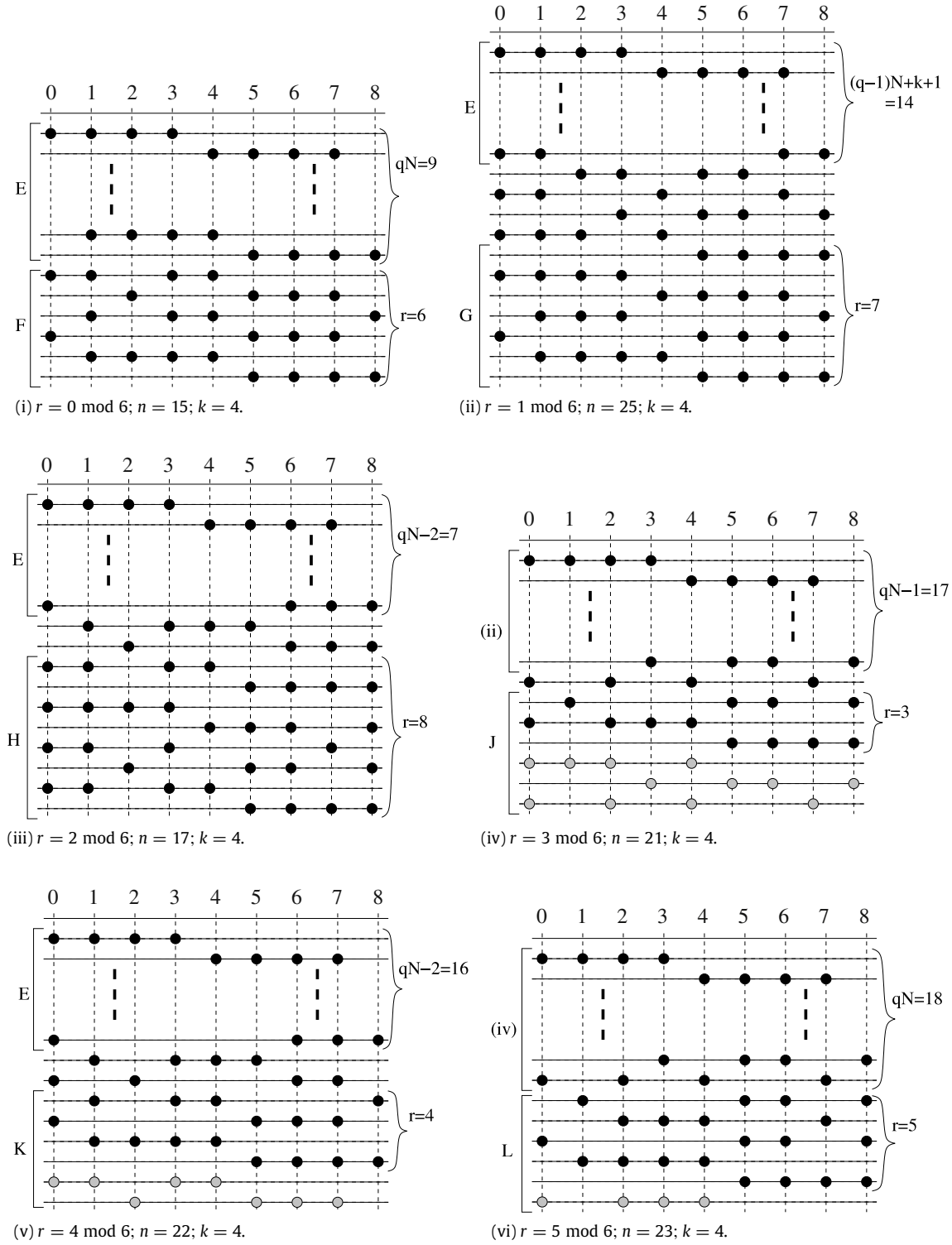
$$\begin{cases} \left\{\left\lceil \frac{k}{2} \right\rceil + i, \ldots, 2k - \left\lceil \frac{k}{2} \right\rceil, 2k + 1 - i, \ldots, 2k\right\} \\ \left\{0, 1, 2, \ldots, \left\lceil \frac{k}{2} \right\rceil - 1 + i, 2k + 1 - \left\lceil \frac{k}{2} \right\rceil, \ldots, 2k - 1 - i\right\} \end{cases} \quad \text{and by}$$

$$\begin{cases} \left\{k - 1, k + 1, \ldots, 2k - \left\lceil \frac{k}{2} \right\rceil, 2k + 2 - \left\lceil \frac{k}{2} \right\rceil, \ldots, 2k\right\} \\ \{0, 1, \ldots, k - 2, k\}, \end{cases}$$

  where a color set of the form $\{a, \ldots, b, c, \ldots, d\}$ with $c > d$ has to be understood as $\{a, \ldots, b\}$.
  Then, we complete this coloring by using the r-first color sets of G defined at the end of this subcase.
  For k even, we start by the $((q-1)N+k+1)$-first color sets of E; we append the k color sets defined for $0 \leq i \leq \frac{k}{2} - 1$ by

$$\begin{cases} \left\{\frac{k}{2} + i, \ldots, k - 1, k + 1, \ldots, 2k - \frac{k}{2}, 2k - i + 1, \ldots, 2k\right\} \\ \left\{0, 1, \ldots, \frac{k}{2} - 1 + i, k, 2k + 1 - \frac{k}{2}, \ldots, 2k - i - 1\right\}; \end{cases}$$

  and we complete this coloring by using the r-first color sets of the periodic coloring G defined by



(i) $r = 0 \bmod 6$; $n = 15$; $k = 4$.

(ii) $r = 1 \bmod 6$; $n = 25$; $k = 4$.

(iii) $r = 2 \bmod 6$; $n = 17$; $k = 4$.

(iv) $r = 3 \bmod 6$; $n = 21$; $k = 4$.

(v) $r = 4 \bmod 6$; $n = 22$; $k = 4$.

(vi) $r = 5 \bmod 6$; $n = 23$; $k = 4$.

Fig. 4. Structure of the k-tuple ND-colorings of C_n when $n \not\equiv 0 \bmod (2k+1)$, n odd.

$G_0 = I^N_{k+1}, G_1 = I^N_0, G_2 = I^N_k, G_3 = \{1, 2, \ldots, k-1, 2k\}, G_4 = \{0, k+1, k+2, \ldots, 2k-1\}, G_5 = I^N_1$ with $G_i = G_{i-6}$ for $i \geq 6$.

– (iii) $r \equiv 2 \bmod 6$. To the $qN - 2$-first sets of E, we add the two sets $\{1, 3, 4, \ldots, k+1\}$ and $\{2, k+2, k+3, \ldots, 2k\}$ and we append the r-first color sets of the following periodic coloring H defined by

$H_0 = \{0, 1, 3, 4, \ldots, k\}, H_1 = I^N_{k+1}, H_2 = I^N_0, H_3 = \{k, k+1, \ldots, 2k-2, 2k\}, H_4 = \{0, 1, 3, 4, \ldots, k-1, 2k-1\}$, $H_5 = \{2, k+1, k+2, \ldots, 2k-2, 2k\}$ with $H_i = H_{i-6}$ for $i \geq 6$.

– (iv) $r \equiv 3 \mod 6$. Starting from the $(qN-1)$-first sets of the coloring found in subcase (ii); we complete with $\{0, 2, \ldots, k-2, k, 2k+1-\lceil \frac{k}{2} \rceil\}$ ($\{0, 3, 5\}$ if $k = 3$) and we append the r-first color sets of the following periodic coloring J:
$J_0 = \{1, k+1, \ldots, 2k-\lceil \frac{k}{2} \rceil, 2k+2-\lceil \frac{k}{2} \rceil, \ldots, 2k\}$, $J_1 = \{0, 2, \ldots, k\}$, $J_2 = I^N_{k+1}, k$, $J_3 = \{0, \ldots, k-2, k\}$, $J_4 = \{k-1, k+1, \ldots, 2k-\lceil \frac{k}{2} \rceil, 2k+2-\lceil \frac{k}{2} \rceil, \ldots, 2k\}$, $J_5 = \{0, 2, \ldots, k-2, k, 2k+1-\lceil \frac{k}{2} \rceil\}$ with $J_i = J_{i-6}$ for $i \geq 6$.
– (v) $r \equiv 4 \mod 6$. Starting from the $(qN-2)$-first sets of the coloring E; we add the two sets $\{1, 3, 4, \ldots, k+1\}$ and $\{0, 2, k+2, k+3, \ldots, 2k-1\}$ and we append the r-first color sets of the following periodic coloring K:
$K_0 = \{1, 3, 4, \ldots, k, 2k\}$, $K_1 = \{0, k+1, \ldots, 2k-1\}$, $K_2 = I^N_1$, $K_3 = I^N_{k+1}$, $K_4 = \{0, 1, 3, 4, \ldots, k-1, k\}$, $K_5 = \{2, k+1, k+2, \ldots, 2k-2, 2k-1\}$ with $K_i = K_{i-6}$ for $i \geq 6$.
– (vi) $r \equiv 5 \mod 6$. To the qN-first sets of the coloring found in subcase (iv); we append the r-first color sets of the following periodic coloring L:
$L_0 = \{1, k+1, \ldots, 2k-\lceil \frac{k}{2} \rceil, 2k+2-\lceil \frac{k}{2} \rceil, \ldots, 2k\}$, $L_1 = \{2, \ldots, k, 2k+1-\lceil \frac{k}{2} \rceil\}$, $L_2 = \{0, k+1, \ldots, 2k-\lceil \frac{k}{2} \rceil, 2k+2-\lceil \frac{k}{2} \rceil, \ldots, 2k\}$, $L_3 = I^N_1$, $L_4 = I^N_{k+1}$, $L_5 = \{0, 2, \ldots, k\}$ with $L_i = L_{i-6}$ for $i \geq 6$. □

4. The complete graph K_n and complete bipartite graph $K_{n,n}$

Let $V(K_n) = \{0, 1, \ldots, n-1\}$ and $E(K_n) = \{e_{ij}, 0 \leq i, j \leq n-1\}$, with $e_{ij} = ij$ and let $V(K_{m,n}) = \{x_0, x_1, \ldots, x_{n-1}\} \cup \{y_0, y_1, \ldots, y_{n-1}\}$ and $E(K_{m,n}) = \{e_{ij}, 0 \leq i \leq m-1, 0 \leq j \leq n-1\}$, with $e_{ij} = x_i y_j$.

Theorem 8. *For any $n \geq 3$ and $k \geq 1$,*

$$\chi'_a(K_n; k) = \chi'_{ac}(K_n; k) = \begin{cases} k(n-1)+2 & \text{if } n \text{ is even,} \\ kn & \text{if } n \text{ is odd.} \end{cases}$$

Proof. First, remark that a k-tuple ND-coloring of K_n with N colors can exist only if there exist n distinct subsets of $\{0, \ldots, N-1\}$ of size $k(n-1)$ such that each number from $\{0, \ldots, N-1\}$ appears an even number of times.

Thanks to this remark, we can see that if n is even, then $\chi'_a(K_n; k) \geq k(n-1)+2$ since in any n sets of $k(n-1)$ elements among $k(n-1)+1$ (in each set, one color is not present and thus appears in the $(n-1)$ other sets), all elements appear an odd number of times, which is impossible.

If n is odd, $n = 2p+1$ for some p, the fact that $\chi'_a(K_{2p+1}; k) \geq k(2p+1)$ can be shown by contradiction: assume that $k(2p+1)-1$ colors suffice. Then, among any $2p+1$ sets of $k(2p)$ colors among $k(2p+1)-1$, there is at least one color that belongs to each set (assume that each color is present in at most $2p$ sets; then we would have $2p(k(2p+1)-1) \geq \sum_{i=1}^{2p+1} 2pk = 2pk(2p+1)$, a contradiction), thus an odd number of times, which is impossible. The fact that $\chi'_{ac}(K_{2p+1}; k) \leq k(2p+1)$ (and thus that $\chi'_a(K_{2p+1}; k) \leq k(2p+1)$) comes directly from Observation 1, as $\chi'_a(K_{2p+1}) = 2p+1$.

If n is even, then K_n is class 1 and $\chi'_a(K_n) = n+1$ [16]. Thus, by Observation 3, we have that $\chi'_a(K_n; k) \leq n+1+(k-1)(n-1) = k(n-1)+2$. It remains to show that $\chi'_{ac}(K_{2p}; k) \leq k(2p-1)+2$. In order to do that, we shall start with a cyclic edge-coloring of K_{2p} for which all vertices have the same set of colors and modify it by increasing or decreasing by one the color intervals of some edges in order vertices to have sets of colors different from each other.

Let ϕ be the cyclic k-tuple proper coloring of K_n with $N-2 = k(n-1)$ colors defined for $0 \leq i < j \leq n-1$ as follows:

$$\phi(e_{ij}) = \begin{cases} I^{N-2}_{k(i+j),k} & \text{if } 0 \leq i < j \leq n-2, \\ I^{N-2}_{2ki,k} & \text{otherwise.} \end{cases}$$

Notice that each vertex x has color set $S(x) = \{0, 1, \ldots, N-3\}$ and that each color interval can also be considered modulo $N = k(n-1)+2$ since $I^{N-2}_{k(i+j),k} = I^N_{k(i+j),k}$ and $I^{N-2}_{2ki,k} = I^N_{2ki,k}$.

Now we modify this coloring in order to obtain a cyclic k-tuple ND-coloring with N colors (all intervals are modulo N in the rest of the proof). We distinguish two cases depending on the residue of n modulo 4.

Case 1: $n \equiv 0 \mod 4$. We increase by one each color of the interval $\phi(e_{ij})$, $i < j$, for (i, j) such that

$$\begin{cases} \dfrac{3n}{4} \leq i < j \leq n-1 \\ j = n-1 \quad \text{and} \quad \dfrac{n}{4} \leq i \leq \dfrac{n}{2} - 1 \\ j+i < n-1 \quad \text{and} \quad 0 \leq i \leq \dfrac{n}{4} - 2 \quad \text{and} \quad \dfrac{3n}{4} \leq j \leq n-2 \\ j+i < n-1 \quad \text{and} \quad \dfrac{n}{4} \leq i < j \leq \dfrac{3n}{4} - 2, \end{cases}$$

and we decrease by one each color of the interval $\phi(e_{ij})$, $i < j$, for (i, j) such that

$$0 \leq i \leq \frac{n}{4} - 1 \quad \text{and} \quad j = n-1-i.$$

This construction is illustrated in Appendix A for the case $n = 12$ and $k = 3$.

Let ϕ' be this new coloring. The vertices can be classified into four groups depending on their two missing colors that are given by the next table.

Group	Missing colors for i	i
(a)	$k-1$ and $k(\frac{3n}{4}+i)$	$[0, \frac{n}{4}-1]$
(b)	$k(\frac{n}{4}+i)$ and $N-1$	$[\frac{n}{4}, \frac{3n}{4}-1]$
(c)	$k-1$ and $k(i+1-\frac{n}{4})$	$[\frac{3n}{4}, n-2]$
(d)	$k-1$ and $k\frac{n}{2}$	$n-1$

Now let us verify that ϕ' is a cyclic k-tuple ND-coloring for K_n when $n \equiv 0 \mod 4$. If we compare missing color sets in the same group, it is straightforward to see that they are different. Notice that a missing color set of the group (b) contains the color $N-1$ that does not belong to the sets in other groups. Moreover, the color $k-1$ belongs to each missing color set of the groups (a), (c) and (d). It then suffices to remark that the second missing colors of two such sets cannot be the same.

Case 2: $n \equiv 2 \mod 4$. The process is similar to that of Case 1 except that some color intervals are increased by two: we increase by one each color interval $\phi(e_{ij})$, $i < j$, for (i, j) such that

$$\begin{cases} \left\lfloor \dfrac{3n}{4} \right\rfloor \le i < j \le n-1 \\ j = n-1 \quad \text{and} \quad \left\lfloor \dfrac{n}{4} \right\rfloor \le i \le \dfrac{n}{2} - 1 \\ j + i < n-1 \quad \text{and} \quad 0 \le i \le \left\lfloor \dfrac{n}{4} \right\rfloor - 1 \quad \text{and} \quad \left\lfloor \dfrac{3n}{4} \right\rfloor \le j \le n-2 \\ j + i < n-1 \quad \text{and} \quad \left\lfloor \dfrac{n}{4} \right\rfloor \le i < j \le \left\lfloor \dfrac{3n}{4} \right\rfloor - 1, \end{cases}$$

we yet increase by one each color interval $\phi(e_{ij})$ for $(i, j) = (\lfloor \frac{n}{4} \rfloor, \lfloor \frac{3n}{4} \rfloor - 1)$ and $(i, j) = (\lfloor \frac{n}{4} \rfloor + 1, \lfloor \frac{3n}{4} \rfloor - 2)$, and we decrease by one each color interval $\phi(e_{ij})$, $i < j$, for (i, j) such that

$$0 \le i \le \left\lfloor \frac{n}{4} \right\rfloor - 1 \quad \text{and} \quad j = n - 1 - i.$$

Let ϕ' be this new coloring. This construction is illustrated in Appendix A for the case $n = 14$ and $k = 3$.

We can see that each vertex i, $0 \le i \le n-1$, has exactly two missing colors that are given by the next table.

Group	Missing colors for i	i
(a)	$k-1$ and $k(\lfloor \frac{3n}{4} \rfloor + i)$	$[0, \lfloor \frac{n}{4} \rfloor - 1]$
(b)	$k(\lfloor \frac{n}{4} \rfloor + i)$ and $k(n-2)+1$	$[\lfloor \frac{n}{4} \rfloor, \lfloor \frac{n}{4} \rfloor + 1]$
(c)	$k(\lfloor \frac{n}{4} \rfloor + i)$ and $k(n-1)+1$	$[\lfloor \frac{n}{4} \rfloor + 2, \lfloor \frac{3n}{4} \rfloor - 3]$
(d)	$k(\lfloor \frac{n}{4} \rfloor + i)$ and $k(n-2)+1$	$[\lfloor \frac{3n}{4} \rfloor - 2, \lfloor \frac{3n}{4} \rfloor - 1]$
(e)	$k(i - \lfloor \frac{n}{4} \rfloor)$ and $k(n-1)+1$	$\lfloor \frac{3n}{4} \rfloor$
(f)	$k-1$ and $k(i - \lfloor \frac{n}{4} \rfloor)$	$[\lfloor \frac{3n}{4} \rfloor + 1, n-2]$
(g)	$k-1$ and $2k\lfloor \frac{n}{4} \rfloor$	$n-1$

Like for the case $n \equiv 0 \mod 4$, it can be shown that ϕ' is a cyclic k-tuple ND-coloring for K_n when $n \equiv 2 \mod 4$. □

For complete bipartite graphs $K_{m,n}$, the only "interesting" case is when $m = n$ (if $m \ne n$ then adjacent vertices have different degrees).

Theorem 9. *For any $n \ge 2$ and $k \ge 1$,*

$$\chi'_a(K_{n,n}; k) = \chi'_{ac}(K_{n,n}; k) = kn + 2.$$

Proof. Let X and Y be the two sets of vertices of the bipartition. Firstly, it can be seen that $\chi'_a(K_{n,n}; k) \ge kn + 2$. Assume, to the contrary, that a k-tuple ND-coloring of $K_{n,n}$ with $kn + 1$ colors exists. Then, as each vertex of $K_{n,n}$ has degree kn, the set of missing colors $\bar{S}(v)$ on each vertex v consists of only one color and as $\{\bar{S}(x), x \in X\} = \{\bar{S}(y), y \in Y\}$, then for each vertex $x \in X$, there exists a vertex $y \in Y$ such that $\bar{S}(x) = \bar{S}(y)$ or equivalently such that $S(x) = S(y)$, a contradiction.

Secondly, we show that $\chi'_{ac}(K_{n,n}; k) \le kn + 2$ by constructing such a cyclic k-tuple ND-coloring, using an argument similar to the one of the proof of Theorem 8 for the complete graph. Indeed, our coloring distinguishes all vertices of $K_{n,n}$, not only adjacent ones.

As for the complete graph, we start with the cyclic k-tuple proper coloring ϕ of $K_{n,n}$ with $N-2=kn$ colors defined for $0 \le i, j \le n-1$ by

$$\phi(e_{ij}) = I^{N-2}_{k(n+i-j-1),k}.$$

Notice that each vertex x has color set $S(x) = \{0, 1, \ldots, N-3\}$. Now we modify this coloring in order to obtain a cyclic k-tuple ND-coloring with $N = kn+2$ colors. We distinguish two cases depending on the residue of n modulo 2.

Case 1: $n \equiv 0 \mod 2$. We increase by one each $\phi(e_{ij})$ for (i, j) such that $0 \le i \le j \le n-2$, and we decrease by one each $\phi(e_{i(i-1)})$ for $i \in [\frac{n}{2}, n-1]$, resulting in a coloring ϕ'. This construction is illustrated in Appendix B for the case $n = 12$ and $k = 3$. It can be seen that the missing color sets on two vertices x_i and y_j are always different since we have

$$\bar{S}_{\phi'}(x_i) = \begin{cases} \{k(i+1), N-1\} & \text{for } i \in \left[0, \dfrac{n}{2}-1\right], \\ \{k(i+1), k-1\} & \text{for } i \in \left[\dfrac{n}{2}, n-1\right]. \end{cases}$$

$$\bar{S}_{\phi'}(y_j) = \begin{cases} \{k(n-j-1), N-1\} & \text{for } j \in \left[0, \dfrac{n}{2}-2\right], \\ \{k(n-j-1), k-1\} & \text{for } j \in \left[\dfrac{n}{2}-1, n-2\right], \\ \{N-2, N-1\} & \text{for } j = n-1. \end{cases}$$

Thus ϕ' is a cyclic k-tuple ND-coloring of $K_{n,n}$ with $N+2 = kn+2$ colors when $n \equiv 0 \mod 2$.

Case 2: $n \equiv 1 \mod 2$. We also increase by one each $\phi(e_{ij})$ for (i, j) such that $0 \le i \le j \le n-2$; we decrease by one each $\phi(e_{i(i-1)})$ for $i \in [\frac{n+1}{2}, n-1]$ and we further increase by one $\phi(e_{ii})$ for $i = \lfloor \frac{n}{2} \rfloor - 1$, resulting in a coloring ϕ'. This construction is illustrated in Appendix B for the case $n = 11$ and $k = 3$. It can be seen that the missing colors on two vertices x_i and y_j are always different since we have

$$\bar{S}_{\phi'}(x_i) = \begin{cases} \{k(i+1), N-1\} & \text{for } i \in \left[0, \left\lfloor \dfrac{n}{2} \right\rfloor\right], i \ne \lfloor \dfrac{n}{2} \rfloor - 1, \\ \{k(i+1), N-k-1\} & \text{for } i = \left\lfloor \dfrac{n}{2} \right\rfloor - 1, \\ \{k(i+1), k-1\} & \text{for } i \in \left[\left\lfloor \dfrac{n}{2} \right\rfloor + 1, n-1\right]. \end{cases}$$

$$\bar{S}_{\phi'}(y_j) = \begin{cases} \{k(n-j-1), N-1\} & \text{for } j \in \left[0, \left\lfloor \dfrac{n}{2} \right\rfloor - 2\right], \\ \{k(n-j-1), N-k-1\} & \text{for } j = \left\lfloor \dfrac{n}{2} \right\rfloor - 1, \\ \{k(n-j-1), k-1\} & \text{for } j \in \left[\left\lfloor \dfrac{n}{2} \right\rfloor, n-2\right], \\ \{N-2, N-1\} & \text{for } j = n-1. \end{cases}$$

Thus ϕ' is a cyclic k-tuple ND-coloring of $K_{n,n}$ with $N = kn+2$ colors when $n \equiv 1 \mod 2$. □

5. Concluding remarks

We have turned our attention on k-tuple ND-colorings of graphs or, equivalently, to ND-colorings of k-uniform multigraphs (multigraphs where each edge has multiplicity k). However, it seems also interesting to study ND-colorings of non-uniform multigraphs. Going in this direction, we propose the following 'Vizing-like' conjecture for the ND-chromatic index of a (not necessarily uniform) multigraph. It extends the one given in [16] for graphs and is similar with the one given for the total chromatic number of a multigraph (see [9], Section 4.9):

Conjecture 10. *For any connected multigraph G of order at least 3, $G \ne C_5$, and of multiplicity $\mu(G)$,*

$$\chi'_a(G) \le \Delta(G) + \mu(G) + 1.$$

Appendix A. Cyclic 3-tuple ND-colorings of K_{12} and K_{14}

Construction of the cyclic 3-tuple ND-coloring ϕ' given in the proof of Theorem 8: below are the matrices of the color intervals on the edges of K_{12} and K_{14} (to simplify, only the first color of each cyclic interval is given). Values increased by one are in bold; values decreased by one are underlined and values increased by two are overlined (and in bold). The column-vectors on the right are the sets of the two missing colors $\bar{S}(i)$ of each vertex i.

$$\phi'(K_{12}) = \begin{bmatrix}
 & 3 & 6 & 9 & 12 & 15 & 18 & 21 & 24 & \mathbf{28} & \mathbf{31} & \underline{34} \\
3 & & 9 & 12 & 15 & 18 & 21 & 24 & 27 & \mathbf{31} & \underline{34} & 6 \\
6 & 9 & & 15 & 18 & 21 & 24 & 27 & 30 & \underline{34} & 3 & 12 \\
9 & 12 & 15 & & \mathbf{22} & \mathbf{25} & \mathbf{28} & \mathbf{31} & 0 & 3 & 6 & \mathbf{19} \\
12 & 15 & 18 & \mathbf{22} & & \mathbf{28} & \mathbf{31} & 0 & 3 & 6 & 9 & \mathbf{25} \\
15 & 18 & 21 & \mathbf{25} & \mathbf{28} & & 0 & 3 & 6 & 9 & 12 & \mathbf{31} \\
18 & 21 & 24 & \mathbf{28} & \mathbf{31} & 0 & & 6 & 9 & 12 & 15 & 3 \\
21 & 24 & 27 & \mathbf{31} & 0 & 3 & 6 & & 12 & 15 & 18 & 9 \\
24 & 27 & 30 & 0 & 3 & 6 & 9 & 12 & & 18 & 21 & 15 \\
\mathbf{28} & \mathbf{31} & \underline{34} & 3 & 6 & 9 & 12 & 15 & 18 & & \mathbf{25} & \mathbf{22} \\
\mathbf{31} & \underline{34} & 3 & 6 & 9 & 12 & 15 & 18 & 21 & \mathbf{25} & & \mathbf{28} \\
\underline{34} & 6 & 12 & \mathbf{19} & \mathbf{25} & \mathbf{31} & 3 & 9 & 15 & \mathbf{22} & \mathbf{28} &
\end{bmatrix}, \begin{bmatrix} \{2,27\} \\ \{2,30\} \\ \{2,33\} \\ \{18,34\} \\ \{21,34\} \\ \{24,34\} \\ \{27,34\} \\ \{30,34\} \\ \{33,34\} \\ \{2,21\} \\ \{2,24\} \\ \{2,18\} \end{bmatrix}$$

$$\phi'(K_{14}) = \begin{bmatrix}
 & 3 & 6 & 9 & 12 & 15 & 18 & 21 & 24 & 27 & \mathbf{31} & \mathbf{34} & \mathbf{37} & \underline{40} \\
3 & & 9 & 12 & 15 & 18 & 21 & 24 & 27 & 30 & \mathbf{34} & \mathbf{37} & \underline{40} & 6 \\
6 & 9 & & 15 & 18 & 21 & 24 & 27 & 30 & 33 & \mathbf{37} & \underline{40} & 3 & 12 \\
9 & 12 & 15 & & \mathbf{22} & \mathbf{25} & \mathbf{28} & \mathbf{31} & \mathbf{34} & \overline{\mathbf{38}} & 0 & 3 & 6 & \mathbf{19} \\
12 & 15 & 18 & \mathbf{22} & & \mathbf{28} & \mathbf{31} & \mathbf{34} & \overline{\mathbf{38}} & 0 & 3 & 6 & 9 & \mathbf{25} \\
15 & 18 & 21 & \mathbf{25} & \mathbf{28} & & \mathbf{34} & \mathbf{37} & 0 & 3 & 6 & 9 & 12 & \mathbf{31} \\
18 & 21 & 24 & \mathbf{28} & \mathbf{31} & \mathbf{34} & & 0 & 3 & 6 & 9 & 12 & 15 & \mathbf{37} \\
21 & 24 & 27 & \mathbf{31} & \mathbf{34} & \mathbf{37} & 0 & & 6 & 9 & 12 & 15 & 18 & 3 \\
24 & 27 & 30 & \mathbf{34} & \overline{\mathbf{38}} & 0 & 3 & 6 & & 12 & 15 & 18 & 21 & 9 \\
27 & 30 & 33 & \overline{\mathbf{38}} & 0 & 3 & 6 & 9 & 12 & & 18 & 21 & 24 & 15 \\
\mathbf{31} & \mathbf{34} & \mathbf{37} & 0 & 3 & 6 & 9 & 12 & 15 & 18 & & \mathbf{25} & \mathbf{28} & \mathbf{22} \\
\mathbf{34} & \mathbf{37} & \underline{40} & 3 & 6 & 9 & 12 & 15 & 18 & 21 & \mathbf{25} & & \mathbf{31} & \mathbf{28} \\
\mathbf{37} & \underline{40} & 3 & 6 & 9 & 12 & 15 & 18 & 21 & 24 & \mathbf{28} & \mathbf{31} & & \mathbf{34} \\
\underline{40} & 6 & 12 & \mathbf{19} & \mathbf{25} & \mathbf{31} & \mathbf{37} & 3 & 9 & 15 & \mathbf{22} & \mathbf{28} & \mathbf{34} &
\end{bmatrix}, \begin{bmatrix} \{2,30\} \\ \{2,33\} \\ \{2,36\} \\ \{18,37\} \\ \{21,37\} \\ \{24,40\} \\ \{27,40\} \\ \{30,40\} \\ \{33,37\} \\ \{36,37\} \\ \{21,40\} \\ \{2,24\} \\ \{2,27\} \\ \{2,18\} \end{bmatrix}$$

Appendix B. Cyclic 3-tuple ND-colorings of $K_{12,12}$ and $K_{11,11}$

Construction of the cyclic 3-tuple ND-coloring ϕ' given in the proof of Theorem 9: below are the matrices of the color intervals on the edges of $K_{12,12}$ and $K_{11,11}$ (to simplify, only the first color of each cyclic interval is given). Values increased by one are in bold; values decreased by one are underlined and values increased by two are overlined (and in bold). The two column-vectors on the right of each matrix are the sets of the two missing colors of the vertices x_i and y_j.

$$\phi'(K_{12,12}) = \begin{bmatrix}
\mathbf{34} & \mathbf{31} & \mathbf{28} & \mathbf{25} & \mathbf{22} & \mathbf{19} & \mathbf{16} & \mathbf{13} & \mathbf{10} & \mathbf{7} & \mathbf{4} & 0 \\
0 & \mathbf{34} & \mathbf{31} & \mathbf{28} & \mathbf{25} & \mathbf{22} & \mathbf{19} & \mathbf{16} & \mathbf{13} & \mathbf{10} & \mathbf{7} & 3 \\
3 & 0 & \mathbf{34} & \mathbf{31} & \mathbf{28} & \mathbf{25} & \mathbf{22} & \mathbf{19} & \mathbf{16} & \mathbf{13} & \mathbf{10} & 6 \\
6 & 3 & 0 & \mathbf{34} & \mathbf{31} & \mathbf{28} & \mathbf{25} & \mathbf{22} & \mathbf{19} & \mathbf{16} & \mathbf{13} & 9 \\
9 & 6 & 3 & 0 & \mathbf{34} & \mathbf{31} & \mathbf{28} & \mathbf{25} & \mathbf{22} & \mathbf{19} & \mathbf{16} & 12 \\
12 & 9 & 6 & 3 & 0 & \mathbf{34} & \mathbf{31} & \mathbf{28} & \mathbf{25} & \mathbf{22} & \mathbf{19} & 15 \\
15 & 12 & 9 & 6 & 3 & \underline{\mathit{37}} & \mathbf{34} & \mathbf{31} & \mathbf{28} & \mathbf{25} & \mathbf{22} & 18 \\
18 & 15 & 12 & 9 & 6 & 3 & \underline{\mathit{37}} & \mathbf{34} & \mathbf{31} & \mathbf{28} & \mathbf{25} & 21 \\
21 & 18 & 15 & 12 & 9 & 6 & 3 & \underline{\mathit{37}} & \mathbf{34} & \mathbf{31} & \mathbf{28} & 24 \\
24 & 21 & 18 & 15 & 12 & 9 & 6 & 3 & \underline{\mathit{37}} & \mathbf{34} & \mathbf{31} & 27 \\
27 & 24 & 21 & 18 & 15 & 12 & 9 & 6 & 3 & \underline{\mathit{37}} & \mathbf{34} & 30 \\
30 & 27 & 24 & 21 & 18 & 15 & 12 & 9 & 6 & 3 & \underline{\mathit{37}} & 33
\end{bmatrix}, \begin{bmatrix} \{3,37\} \\ \{6,37\} \\ \{9,37\} \\ \{12,37\} \\ \{15,37\} \\ \{18,37\} \\ \{2,21\} \\ \{2,24\} \\ \{2,27\} \\ \{2,30\} \\ \{2,33\} \\ \{2,36\} \end{bmatrix}, \begin{bmatrix} \{33,37\} \\ \{30,37\} \\ \{27,37\} \\ \{24,37\} \\ \{21,37\} \\ \{2,18\} \\ \{2,15\} \\ \{2,12\} \\ \{2,9\} \\ \{2,6\} \\ \{2,3\} \\ \{36,37\} \end{bmatrix}$$

$$\phi'(K_{11,11}) = \left[\begin{array}{cccccccccc|c}
\mathbf{31} & \mathbf{28} & \mathbf{25} & \mathbf{22} & \mathbf{19} & \mathbf{16} & \mathbf{13} & \mathbf{10} & \mathbf{7} & \mathbf{4} & 0 \\
0 & \mathbf{31} & \mathbf{28} & \mathbf{25} & \mathbf{22} & \mathbf{19} & \mathbf{16} & \mathbf{13} & \mathbf{10} & \mathbf{7} & 3 \\
3 & 0 & \mathbf{31} & \mathbf{28} & \mathbf{25} & \mathbf{22} & \mathbf{19} & \mathbf{16} & \mathbf{13} & \mathbf{10} & 6 \\
6 & 3 & 0 & \mathbf{31} & \mathbf{28} & \mathbf{25} & \mathbf{22} & \mathbf{19} & \mathbf{16} & \mathbf{13} & 9 \\
9 & 6 & 3 & 0 & \overline{\mathbf{32}} & \mathbf{28} & \mathbf{25} & \mathbf{22} & \mathbf{19} & \mathbf{16} & 12 \\
12 & 9 & 6 & 3 & 0 & \mathbf{31} & \mathbf{28} & \mathbf{25} & \mathbf{22} & \mathbf{19} & 15 \\
15 & 12 & 9 & 6 & 3 & \underline{34} & \mathbf{31} & \mathbf{28} & \mathbf{25} & \mathbf{22} & 18 \\
18 & 15 & 12 & 9 & 6 & 3 & \underline{34} & \mathbf{31} & \mathbf{28} & \mathbf{25} & 21 \\
21 & 18 & 15 & 12 & 9 & 6 & 3 & \underline{34} & \mathbf{31} & \mathbf{28} & 24 \\
24 & 21 & 18 & 15 & 12 & 9 & 6 & 3 & \underline{34} & \mathbf{31} & 27 \\
27 & 24 & 21 & 18 & 15 & 12 & 9 & 6 & 3 & \underline{34} & 30
\end{array}\right], \left[\begin{array}{c}
\{3,34\} \\ \{6,34\} \\ \{9,34\} \\ \{12,34\} \\ \{15,31\} \\ \{18,34\} \\ \{2,21\} \\ \{2,24\} \\ \{2,27\} \\ \{2,30\} \\ \{2,33\}
\end{array}\right], \left[\begin{array}{c}
\{30,34\} \\ \{27,34\} \\ \{24,34\} \\ \{21,34\} \\ \{18,31\} \\ \{2,15\} \\ \{2,12\} \\ \{2,9\} \\ \{2,6\} \\ \{2,3\} \\ \{33,34\}
\end{array}\right]$$

References

[1] P.N. Balister, E. Győri, J. Lehel, R.H. Schelp, Adjacent vertex distinguishing edge-colorings, SIAM J. Discrete Math. 21 (1) (2007) 237–250.
[2] J.-L. Baril, H. Kheddouci, O. Togni, Adjacent vertex distinguishing edge-colorings of meshes, Australas. J. Combin. 35 (2006) 89–102.
[3] J.-L. Baril, H. Kheddouci, O. Togni, Vertex distinguishing edge- and total-colorings of cartesian and other product graphs, Ars Combinatoria (2008) (in press).
[4] K. Edwards, M. Horňák, M. Woźniak, On the neighbour-distinguishing index of a graph, Graphs Combin. 22 (3) (2006) 341–350.
[5] C. Greenhill, A. Ruciński, Neighbour-distinguishing edge colourings of random regular graphs, Electron. J. Combin. 13 (1) (2006) Research Paper 77, 12 pp. (electronic).
[6] E. Győri, M. Horňák, C. Palmer, M. Woźniak, General neighbour-distinguishing index of a graph, Discrete Math. 308 (5–6) (2008) 827–831.
[7] H. Hatami, $\Delta + 300$ is a bound on the adjacent vertex distinguishing edge chromatic number, J. Combin. Theory Ser. B 95 (2) (2005) 246–256.
[8] M. Horňák, M. Woźniak, On neighbour-distinguishing colourings from lists. Preprinty–Seria Matematyka Dyskretna, Uniwersytet Jagielloński, Kraków, Preprint MD 029, Extended abstract in Electronic Notes in Discrete Mathematics 24, 2006, pp. 295–297. 2007.
[9] T.R. Jensen, B. Toft, Wiley-Interscience Series in Discrete Mathematics and Optimization, John Wiley & Sons Inc., New York, 1995, A Wiley-Interscience Publication.
[10] T. Kaiser, D. Král, R. Škrekovski, X. Zhu, The circular chromatic index of graphs of high girth, J. Combin. Theory Ser. B 97 (1) (2007) 1–13.
[11] A. Nadolski, The circular chromatic index of some class 2 graphs, Discrete Math. 307 (11–12) (2007) 1447–1454.
[12] S. Stahl, n-tuple colorings and associated graphs, J. Combin. Theory Ser. B 20 (2) (1976) 185–203.
[13] O. Togni, Coloration de graphes: Quelques contributions. HDR dissertation 2006. http://www.u-bourgogne.fr/o.togni/HDRTogni.pdf (in French).
[14] J. Wang, G. Liu, Some results on fractional edge coloring of graphs, Ars Combin. 83 (2007) 249–255.
[15] Z. Zhang, X.E. Chen, J. Li, B. Yao, X. Lu, J. Wang, On adjacent-vertex-distinguishing total coloring of graphs, Sci. China Ser. A 48 (3) (2005) 289–299.
[16] Z. Zhang, L. Liu, J. Wang, Adjacent strong edge coloring of graphs, Appl. Math. Lett. 15 (5) (2002) 623–626.

Efficient generating algorithm for permutations with a fixed number of excedances

J.-L. Baril

LE2I UMR-CNRS 5158, Université de Bourgogne
B.P. 47 870, 21078 DIJON-Cedex France
barjl@u-bourgogne.fr

Abstract

In this paper, we develop a constant amortized time (CAT) algorithm for generating permutations with a fixed number of excedances. We obtain a Gray code for permutations having one excedance. We also give a bijection between the set of n-length permutations with exactly one excedance and the set $S_n(321, 2413, 3412, 21534)\backslash\{123\ldots(n-1)n\}$. This induces a Gray code for the set $S_n(321, 2413, 3412, 21534)\backslash\{123\ldots(n-1)n\}$.
Keywords: Eulerian numbers, permutation, excedance, descent, pattern avoiding permutation, Gray code, generating algorithm.

1 Introduction

Let S_n be the set of all permutations of length n ($n \geq 1$). We represent permutations in one-line notation, *i.e.*, if $i_1,\ i_2, \ldots,\ i_n$ are n distinct values in $[n] = \{1,\ 2, \ldots,\ n\}$, we denote the permutation $\sigma \in S_n$ by the sequence $i_1 i_2 \ldots i_n$ if $\sigma(k) = i_k$ for $1 \leq k \leq n$. For instance, the identity permutation of length n, id_n, will be written $12\ldots(n-1)n$. Moreover, if $\gamma = \gamma(1)\gamma(2)\ldots\gamma(n)$ is an n-length permutation then the *composition* (*or product*) $\gamma \cdot \sigma$ is the permutation $\gamma(\sigma(1))\gamma(\sigma(2))\ldots\gamma(\sigma(n))$. In S_n, a *k-cycle* $\sigma = \langle i_1,\ i_2, \ldots,\ i_k\rangle$ is an n-length permutation verifying $\sigma(i_1) = i_2,\ \sigma(i_2) = i_3, \ldots,\ \sigma(i_{k-1}) = i_k,\ \sigma(i_k) = i_1$ and $\sigma(j) = j$ for $j \in [n]\backslash\{i_1, \ldots, i_k\}$; in particular, a *transposition* is a 2-cycle. Now, let us consider $\sigma \in S_n$. We say that $\sigma(i)$, $1 \leq i \leq n$, is a *strict excedance* (or *excedance* for short) of σ if $\sigma(i) > i$. For $0 \leq k \leq n-1$, we denote by $E_{n,k}$ the set of all n-length permutations with exactly k excedances. Obviously, $\{E_{n,k}\}_{0\leq k\leq n-1}$ forms a partition for S_n and it is well known (see for instance [9]) that the cardinality of $E_{n,k}$ is given by the Eulerian numbers $e(n,k)$ satisfying:

$$e(n,k) = (n-k)\cdot e(n-1,k-1) + (k+1)\cdot e(n-1,k) \tag{1}$$

anchored by $e(n,0) = 1$ and $e(n,n-1) = 1$ for $n \geq 1$. Notice that $e(n,k)$ also enumerates the n-length permutations with exactly k descents ($\sigma(i)$ is a *descent* of $\sigma \in S_n$ ($1 \leq i \leq n-1$), if $\sigma(i) > \sigma(i+1)$). Moreover, $e(n,1)$ is the number of Dyck paths of semi-length n having exactly one long ascent [23] (*i.e.*, ascent of length at least two).

Eulerian numbers have been widely studied in enumerative combinatorics [9, 11, 18, 19]. Foata and Schützenberger [12] give several fundamental properties of these numbers; some applications to analysis of algorithms are given by Knuth [14] and to combinatorics on words by Lothaire [17]. Many of these references study Eulerian numbers as a distribution statistic (called *Eulerian statistic*) in S_n. However, an efficient algorithm does not exist for generating all n-length permutations with a given number of excedances. Algorithms for generating other permutation classes have been published (generated in lexicographic order and in constant time per permutation, in amortized sense). For example, efficient algorithms are known for derangements [1], involutions [20], up-down permutations [8, 22], permutations with a fixed number of inversions [10], Fibonacci and Lucas permutations [5]. Also, several algorithms for generating

permutation classes in Gray code order are given for permutations [13, 24] and their restrictions [15, 20, 21], derangements [7, 16], with a fixed number of cycles [4], involutions and fixed-point free involutions [26] or their generalizations (multiset permutations [25]).

In this paper, we provide an algorithm for generating n-length permutations with a fixed number of excedances, in Constant Amortized Time (CAT). An improvement of this algorithm allows us to list permutations with exactly one excedance in Gray code order. This code is optimal and two consecutive elements in the list differ in at most three positions. We construct a bijection between permutations with exactly one excedance and the set of avoiding permutations $S_n(321, 3413, 3412, 21534)\backslash\{123\ldots(n-1)n\}$ and we also obtain a Gray code for this last set.

2 Generating permutations with a given number of excedances

In this section, we provide a CAT algorithm for generating the set $E_{n,k}$, *i.e.* the set of n-length permutations with k excedances.

2.1 Algorithm

Here, we show how one can recursively construct $E_{n,k}$ from $E_{n-1,k}$ and $E_{n-1,k-1}$ which will induce a constructive proof of the enumerating relation (1). We also give a lemma, crucial in the construction of our algorithm.

Let $\gamma \in E_{n-1,k}$ be an $(n-1)$-length permutation with k excedances, $n \geq 2$, $0 \leq k \leq n-2$; let i be an integer, $1 \leq i \leq n-1$. If we denote by σ the permutation in S_n obtained from γ by replacing $\gamma(i)$ by n and by appending $\gamma(i)$ on the right of γ, then we consider two cases:

(a) if $\gamma(i)$ is an excedance in γ, then $\sigma \in E_{n,k}$;

(b) otherwise, $\sigma \in E_{n,k+1}$.

Moreover, (c) if σ is obtained from γ by appending n on the right, then $\sigma \in E_{n,k}$. Conversely, each permutation in $E_{n,k}$, $n \geq 2$, can be uniquely obtained by one of these three constructions (a), (b) and (c), which gives a combinatorial proof of the relation (1).

Then we define below the functions ϕ_n and ψ_n as follow:

Definition 1. *For $0 \leq k \leq n-2$, an integer $i \in [n-1]$ and a permutation $\gamma \in E_{n-1,k}$, we define an n-length permutation $\sigma = \phi_{n-1}(i, \gamma)$ by*

$$\sigma(j) = \begin{cases} n & \text{if } j = i \\ \gamma(i) & \text{if } j = n \\ \gamma(j) & \text{otherwise.} \end{cases}$$

We also define the function ψ_{n-1} from $E_{n-1,k}$ to S_n that transforms $\gamma \in E_{n-1,k}$ in the permutation $\sigma \in E_{n,k}$ obtained from γ by appending n on its right.

For example, if $\gamma = 3142 \in E_{4,2}$, we have $\phi_4(3, \gamma) = 31524 \in E_{5,2}$; $\phi_4(2, \gamma) = 35421 \in E_{5,3}$ and $\psi_4(\gamma) = 31425 \in E_{5,2}$. More generally and with the same hypothesis of Definition 1, we can easily remark that:

$$\phi_{n-1}(i, \gamma) = \psi_{n-1}(\gamma) \cdot \langle i, n \rangle = \langle \gamma(i), n \rangle \cdot \psi_{n-1}(\gamma).$$

In the sequel of this paper, we will omit the index n in ϕ_n and ψ_n since it will be known by the context.

The above comments induce the following straightforward lemma that is the key of our algorithm.

Lemma 2.1. *Let n, m, ℓ, k be four natural numbers such that $1 \leq m < n$, $0 \leq \ell \leq m-1$, and $0 \leq k \leq n-1$. Let also $\gamma \in E_{m,\ell}$.*

- *If there exists i, $1 \leq i \leq m-1$, such that $\gamma(i)$ is an excedance of γ, then there is a permutation $\sigma \in E_{n,k}$ such that σ is obtained from $\phi(i, \gamma)$ (respectively $\psi(\gamma)$) by applying several times the functions ϕ and ψ if and only if: (i) $\ell \leq k$ and $m + 1 - \ell \leq n - k$.*

- *Now let us consider i, $1 \leq i \leq m$, such that $\gamma(i)$ is not an excedance of γ, then there is a permutation $\sigma \in E_{n,k}$ such that σ is obtained from $\phi(i,\gamma)$ by applying several times the functions ϕ and ψ if and only if: (ii) $\ell+1 \leq k$ and $m-\ell \leq n-k$.*
- *If $\sigma \in E_{n,k}$ is obtained from $\gamma \in E_{m,\ell}$ by the first previous construction and $\tau \in E_{n,k}$ from the second, then σ and τ are different.*

Now we explain the main difficulties in implementing our procedure $\mathbf{gen}(m,\ell)$ given in Figure 1. The procedure $\mathbf{gen}(1,0)$ generates recursively all permutations $\sigma \in E_{n,k}$. Indeed, let us consider that when we run the recursive call $\mathbf{gen}(m,\ell)$, the current permutation is $\sigma \in E_{m,\ell}$ where its excedances are in positions $i_1, i_2, \ldots, i_\ell$, (*i.e.* the excedances of σ are $\sigma(i_1), \ldots, \sigma(i_\ell)$). Then, an array t_1 contains these ℓ positions $i_1, i_2, \ldots, i_\ell$. On the other hand, the indices in $[m]\backslash\{i_1, i_2, \ldots, i_\ell\}$ are stored in an array t_2 of length $r = m-\ell$. By considering the necessary conditions of Lemma 2.1, the procedure $\mathbf{gen}(m,\ell)$ generates permutations in $E_{m+1,\ell}$ or $E_{m+1,\ell+1}$ by applying to σ the functions ϕ and/or ψ. In order to obtain:

- $\psi(\sigma) \in E_{m+1,\ell}$; we add the index $(m+1)$ on the right of array t_2 ($t_2[r+1] = m+1$) and we call $\mathbf{gen}(m+1,\ell)$.
- $\phi(i_j,\sigma)$ with $j \in [\ell]$; we add $(m+1)$ on the right of t_2 ($t_2[r+1] = m+1$). We update $\sigma = \sigma \cdot \langle t_1[j], m+1\rangle$ and we call $\mathbf{gen}(m+1,\ell)$.
- $\phi(i,\sigma)$ with $i \notin \{i_1, i_2, \ldots, i_\ell\}$; *i.e.*, for each $j \leq r$, we put $temp = t_2[j]$. We add $temp$ on the right of t_1 ($t_1[\ell+1] = temp$) and we replace $t_2[j]$ by $m+1$; then we update $\sigma = \sigma \cdot \langle temp, m+1\rangle$ and we call $\mathbf{gen}(m+1,\ell+1)$.

After each recursive call, we update t_1 and t_2 and σ in order to obtain the previous configuration before the call. These statements require a complexity in $\mathcal{O}(1)$ (*i.e.* independent of the permutation length). See Figure 1 for our generating algorithm and for some examples.

```
procedure gen(m, ℓ)
r := m − ℓ
if m = n then output σ;
else
    if ℓ ≤ k and m + 1 − ℓ ≤ n − k then
        t2[r + 1] := m + 1
        gen(m + 1, ℓ)
        for each v ∈ [ℓ]
            σ := σ · ⟨t1[v], m + 1⟩
            gen(m + 1, ℓ)
            σ := σ · ⟨t1[v], m + 1⟩
    if ℓ + 1 ≤ k and  m − ℓ ≤ n − k then
        for each v ∈ [r]
            t1[ℓ + 1] := t2[v]
            temp := t2[v]; t2[v] := m + 1
            σ := σ · ⟨temp, m + 1⟩
            gen(m + 1, ℓ + 1)
            σ := σ · ⟨temp, m + 1⟩
            t2[v] := temp;
end procedure
```

$E_{4,1}$		$E_{4,2}$		$E_{5,1}$			
1	**4231**	1	3**241**	1	**52**341	14	**52134**
2	1**432**	2	**3412**	2	1**5342**	15	1324**5**
3	1**243**	3	4**321**	3	12**543**	16	1**5243**
4	3**214**	4	**1342**	4	123**54**	17	1423**5**
5	**4213**	5	2**431**	5	4231**5**	18	**15234**
6	1**324**	6	**2143**	6	**52314**	19	**2**134**5**
7	**1423**	7	3**421**	7	1432**5**	20	**5**134**2**
8	**2**134	8	**3142**	8	1**5324**	21	**4**132**5**
9	**4132**	9	**2**314	9	1243**5**	22	**51324**
10	**3**124	10	**4312**	10	**12534**	23	**3**124**5**
11	4123	11	2413	11	**3**214**5**	24	**51243**
				12	**5**214**3**	25	**4**123**5**
				13	**4**213**5**	26	51234

Figure 1: Generating algorithm for permutations with a fixed number of excedances and the lists $E_{4,1}$, $E_{4,2}$ and $E_{5,1}$.

2.2 Complexity

Here we do not consider the trivial case $k = 0$ ($k = n-1$ resp.) where $E_{n,0} = \{12\ldots n\}$ ($E_{n,n-1} = \{2\ldots n1\}$ resp.). Let us assume $k \neq 0$ and $k \neq n-1$. Then our algorithm produces all permutations of $E_{n,k}$ in constant amortized time (CAT); *i.e.* the amount of computation is

proportional to the number of generated objects. Indeed, in our procedure the computation is proportional to the number of recursive calls performed by it, and each call generates a combinatorial object, or produces at least two calls. A Java implementation of this algorithm can be viewed at http://www.u-bourgogne.fr/jl.baril/applet.html.

3 Gray code for permutations with exactly one excedance

In this part, we will adapt the previous algorithm for generating $E_{n,1}$, $n \geq 1$, in a Gray code order. A *Gray code* is a family $\{\mathcal{L}_n\}_{n\geq 0}$ of lists of n-length sequences such that in each list the Hamming distance d between any two consecutive sequences (*i.e.* the number of positions in which they differ) is bounded by a constant independently of their length. If this constant is minimal then the code is called *optimal*.

Now we explain how one can modify the procedure of Figure 1 in order to produce a Gray code listing for $E_{n,1}$. Let us remark that the generating tree associated with the previous algorithm has the following properties:

- for each $k \leq n-1$, a permutation $\sigma \in E_{k,1}$ has two sons: $\phi(i,\sigma)$ if $\sigma(i)$ is the excedance in σ; and $\psi(\sigma)$;
- for each $k \leq n-2$, $\sigma = id_k = 12\ldots k$ has $(k+1)$ sons: $\psi(id_k) = id_{k+1}$, $\phi(1, id_k), \ldots, \phi(k-1, id_k), \phi(k, id_k)$;
- for $k = n-1$, $\sigma = id_{n-1} = 12\ldots(n-1)$ has $(n-1)$ sons: $\phi(1, id_{n-1}), \ldots, \phi(n-1, id_{n-1})$.

This means that each level $k \leq n-1$ of the generating tree contains all permutations of $E_{k,1} \cup \{id_k\}$ and the last level n contains those of $E_{n,1}$. Then for each $k \leq n-1$ and for a permutation $\sigma \in E_{k,1} \cup \{id_k\}$, we associate:

- a *direction*, *up* or *down*. A permutation σ with the direction *up* (*down* respectively) will be denoted by σ^1 (σ^0 respectively).
- a list of its sons in the generating tree considered with their directions. The list of successors of σ^0 is obtained by reversing the list of successors of σ^1 and by reversing the direction of each element of the list.

Now, let $\sigma^1 \in E_{k,1} \cup \{id_k\}$. We define the sequence L of integer where $L(i)$ denotes the ith integer of the sequence L:

$$L = \begin{cases} k-1, k-3, k-5, \ldots, 1, 2, 4, \ldots, k & \text{if } k \text{ is even,} \\ k-1, k-3, k-5, \ldots, 2, 1, 3, \ldots, k & \text{if } k \text{ is odd,} \end{cases}$$

and we distinguish two cases:

- if σ^1 has two sons where $\sigma(i)$ is an excedance, the list of its successors is $\phi(i,\sigma)^0, \psi(\sigma)^1$.
- if $\sigma^1 = id_k^1$, $k \leq n-1$, the list of its successors is
$$\phi(L(1),\sigma)^0, \phi(L(2),\sigma)^1, \ldots, \phi(L(k),\sigma)^{(k-1) \bmod 2}, \psi(\sigma)^1,$$

Recall that for σ^0, we take the reverse list of σ^1.

So we recursively define our Gray code $\mathcal{E}_{n,1} \cup \{id_n\}$ for $E_{n,1} \cup \{id_n\}$ as follow:

$$\mathcal{E}_{n,1} = \begin{cases} 1^1 & \text{if } n = 1 \\ \bigcirc_{i=1}^{e_{n-1,1}+1} \Pi(i) & \text{otherwise,} \end{cases}$$

where $\Pi(i)$ is the list of the ith permutation of the $\mathcal{E}_{n-1,1} \cup \{id_{n-1}\}$ considered with its direction, and $\bigcirc$ is the concatenation operator for lists. See Figure 2 for instance.

Theorem 2. *The list $\mathcal{E}_{n,1} \cup \{id_n\}$ defined above is an optimal Gray code for $E_{n,1} \cup \{id_n\}$. Two consecutive elements in the list differ in at most three positions. Consequently, the restricted list $\mathcal{E}_{n,1}$ is also an optimal Gray code for the set $E_{n,1}$.*

Proof. We proceed by induction on n. The property is true for $n \leq 3$. We assume that it holds for $k \leq n-1$ and we will prove that $\mathcal{E}_{n,1} \cup \{id_n\}$ is a Gray code.
First, we consider transitions between the successors of a permutation $\sigma \in \mathcal{E}_{n-1,1} \cup \{id_{n-1}\}$, *i.e.* the transitions of the form (i) $\phi(i,\sigma)$, $\phi(j,\sigma)$ for $1 \leq |i-j| \leq 2$, or (ii) $\phi(i,\sigma)$, $\psi(\sigma)$, with $1 \leq i,j \leq n-1$. For the transition (i), we easily have $\phi(i,\sigma) = \phi(j,\sigma) \cdot \langle j,i,n \rangle$. For (ii), we have $\phi(i,\sigma) = \psi(\sigma) \cdot \langle i,n \rangle$. Thus two successive permutations in $\mathcal{E}_{n,1} \cup \{id_n\}$, that have the same predecessors in $\mathcal{E}_{n-1,1} \cup \{id_{n-1}\}$, differ in at most three positions.

Now we assume that σ and τ are two successive permutations in $\mathcal{E}_{n,1} \cup \{id_n\}$ that do not have the same predecessor in $\mathcal{E}_{n-1,1} \cup \{id_{n-1}\}$. Let $\sigma_1 \in \mathcal{E}_{n-1,1} \cup \{id_{n-1}\}$ (respectively $\tau_1 \in \mathcal{E}_{n-1,1} \cup \{id_{n-1}\}$) be the predecessor of σ (respectively τ). Via the recurrence hypothesis, σ_1 and τ_1 differ by at most three positions. We distinsguish three cases:

- (a) If σ_1 has the direction 1 (*up*), and τ_1 the direction 0 (*down*), then $\sigma = \psi(\sigma_1)$ and $\tau = \psi(\tau_1)$ which implies that σ and τ differ by at most three positions.

- (b) If σ_1 has the direction 0, and τ_1 the direction 1, then they necessarily have the same predecessor $\gamma \in \mathcal{E}_{n-2,1} \cup \{id_{n-2}\}$.
  If $\gamma = id_{n-2}$ then $\sigma_1 = \phi(i, id_{n-2})$, $\tau_1 = \phi(j, id_{n-2})$ with $1 \leq |i-j| \leq 2$ and $1 \leq i,j \leq n-2$; or $\sigma_1 = \phi(n-2, id_{n-2})$ and $\tau_1 = \psi(id_{n-2}) = id_{n-1}$. Therefore, for the first case, $\sigma = \phi(i,\sigma_1)$, $\tau = \phi(j,\tau_1)$ and $\tau = \sigma \cdot \langle j,i,n-1 \rangle$; and for the second case, we have $\sigma = \phi(n-2,\sigma_1)$ and $\tau = \phi(n-2, id_{n-1})$ which means $\tau = \sigma \cdot \langle n-1, n \rangle$. Thus σ and τ differ in at most three positions.
  If $\gamma \neq id_{n-2}$ then we obtain $\sigma_1 = \phi(i,\gamma)$ and $\tau_1 = \psi(\gamma)$ (or conversely) where $\gamma(i)$ is the only one excedance of γ. Thus we have $\sigma = \phi(i,\sigma_1)$ and $\tau = \phi(i,\tau_1)$, which implies that $\tau = \sigma \cdot \langle n-1, n \rangle$. Thus σ and τ differ in at most three positions.

- (c) If σ_1 has the direction 1, and τ_1 the direction 1, then they necessarily have the same predecessor $id_{n-2} \in \mathcal{E}_{n-2,1} \cup \{id_{n-2}\}$. Thus, $\sigma_1 = \phi(n-2, id_{n-2})$, $\tau_1 = id_{n-1}$ and we have $\sigma = \phi(n-2,\sigma_1)$ and $\tau = \phi(n-2, id_{n-1})$ which also prove that σ and τ differ in at most three positions.

For the optimality, it suffices (for instance) to remark for instance that $\mathcal{E}_{4,1} \cup \{id_4\}$ contains seven odd permutations and five even permutations. This allows us to prove that we can not obtain Gray code for $E_{4,1} \cup \{id_4\}$ such that two consecutive permutations differ by two positions. □

Notice that, since id_n is the last permutation of the Gray code $\mathcal{E}_{n,1} \cup \{id_n\}$, the restricted list $\mathcal{E}_{n,1}$ is also an optimal Gray code. Figure 2 illustrates the generating tree for $\mathcal{E}_{4,1}$.

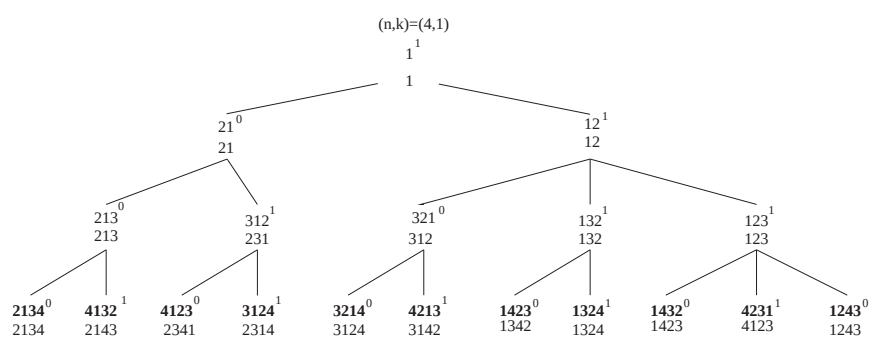


Figure 2: The generating tree for the Gray codes $\mathcal{E}_{4,1}$. Each permutation of $\mathcal{E}_{4,1}$ with its direction, lies on the first line of the leaves (in boldface). The second line generates permutations in $S_4(321, 2413, 3412, 21534) \backslash \{1234\}$ studied in Section 4.

Remark 3.1. *Notice that we can obtain the generating tree of $E_{n,1} \cup \{id_n\}$ using the method ECO [2, 3]. Indeed, the following succession rules*

$$\begin{cases} (2) & root \\ (k) & \to (2_1)^{k-1}(k+1) \\ (2_1) & \to (2_1)^2 \end{cases}$$

also provide $E_{n,1} \cup \{id_n\}$. Here, the rules for obtaining a successor of a permutation $\sigma \in E_{k,1}$ is not the classical insertion between two entries of σ; but we obtain the successor of σ by appending $(k+1)$ at the end of σ and by a product (on the right) of a transposition $\langle i, k+1 \rangle$ for some $i \in [k+1]$.

This rule does not satisfy the *stability property* introduced in [6]. However, the rule is almost *stable* since the label of the root is the only one that does not produce two labels (2). Consequently, the method in [6] allows to construct a Gray code in a particular representation of $E_{n,1} \cup \{id_n\}$ (which is not the classical representation of a permutation).

4 A bijection between $E_{n,1} \cup \{12 \ldots n\}$ and $S_n(T)$ with $T = \{321, 2413, 3412, 21534\}$

In this part we give a correspondence between $E_{n,1} \cup \{12 \ldots n\}$ and a set of avoiding permutations $S_n(321, 2413, 3412, 21534)$. A permutation $\sigma = \sigma_1\sigma_2 \ldots \sigma_n \in S_n$ contains the pattern $\gamma \in S_k$ $(k \geq 2)$ if and only if a sequence of indices $1 \leq i_1 < i_2 < \ldots < i_k \leq n$ exists such that $\sigma_{i_1}\sigma_{i_2} \ldots \sigma_{i_k}$ is ordered as γ. We denote by $S_n(\gamma)$ the set of permutations of S_n avoiding the pattern γ. For example, $\mathbf{234}1 \notin S_4(123)$ but $4312 \in S_4(123)$.

Now, we define the function ϕ' below:

Definition 2. *For an integer $i \in [n-1]$ and a permutation $\gamma \in S_{n-1}$, we define an n-length permutation $\sigma = \phi'_{n-1}(i, \gamma)$ by*

$$\sigma(j) = \begin{cases} n & \text{if } j = i \\ \gamma(j) & \text{if } j \leq i-1 \\ \gamma(j-1) & \text{otherwise.} \end{cases}$$

Thus the permutation $\phi'_{n-1}(i, \gamma)$ is obtained from γ by inserting n just before $\gamma(i)$.

For example, $\phi'_5(3, 52431) = 52\mathbf{6}431$. As for the function ϕ, we will omit the index in ϕ'_n.

Theorem 3. *If we replace ϕ by ϕ' in the definition of our Gray code $\mathcal{E}_{n,1} \cup \{id_n\}$, we also obtain an optimal Gray code for the set $S_n(321, 2413, 3412, 21534)$ such that two successive permutations differ in at most three positions.*

Proof. Let us prove that our generating tree considered with ϕ', creates all permutations of the set $S_n(321, 2413, 3412, 21534)$. We proceed by induction on n. This holds for $n \leq 5$. Let us assume that the result is true for $k \leq n$. Let σ be a permutation on the level $n+1$. Therefore, σ is obtained from a permutation τ in $S_n(321, 2413, 3412, 21534)$. If $\tau = id_n$ then $\sigma = \phi'(i, id_n)$ with $1 \leq i \leq n$, or $\sigma = \psi(id_n) = id_{n+1}$, which implies that σ belongs to the set $S_{n+1}(321, 2413, 3412, 21534)$. If $\tau \neq id_n$ then $\sigma = \phi'(n, \tau)$ or $\sigma = \psi(\tau)$, and σ is still in the set $S_{n+1}(321, 2413, 3412, 21534)$.
Conversely, let us prove that each permutation $\sigma \in S_{n+1}(321, 2413, 3412, 21534)$ belongs to the level $n+1$ of the generating tree. If we have $\sigma(n) = n+1$ or $\sigma(n+1) = n+1$ then there exists $\tau \in S_n(321, 2413, 3412, 21534)$ such that, either $\sigma = \phi'(n, \tau)$ or $\sigma = \psi(\tau)$. This proves that σ belongs to the level $n+1$ of the generating tree. For the other cases, there exists $i \leq n-1$, such that $\sigma(i) = n+1$. Since σ avoids 321, we necessarily have $\sigma(i+1) < \sigma(i+2) < \ldots < \sigma(n+1)$. Since σ avoids 2413 and 3412, we necessarily have $\sigma(j) < \sigma(i+1)$ for $j \leq i-1$. Since σ avoids 21534 then $\sigma(1) < \sigma(2) < \ldots < \sigma(i-1)$. For all these reasons, we obtain $\sigma = 123 \ldots (i-1)(n+1)i(i+1) \ldots n = \phi(i, id_n)$ which induces that σ belongs to the generating tree. This means that the generating tree with ϕ' allows us to generates all permutations in the set $S_n(321, 2413, 3412, 21534)$. It also provides a Gray code for the set $S_n(321, 2413, 3412, 21534)$

such that two successive permutations differ in at most three positions. This can be proved in the same way as Theorem 2. □

This theorem induces a constructive bijection between $E_{n,1} \cup \{id_n\}$ and $S_n(321, 2413, 3412, 21534)$.

Corollary 4.1. *Let $\sigma \in E_{k-1,1} \cup \{id_{k-1}\}$, we recursively define the maps f_k from $E_{k,1} \cup \{id_k\}$ to $S_k(321, 2413, 3412, 21534)$, $1 \leq k \leq n$ by:*

- $f_1(id_1) = id_1$,
- $f_k(\psi(\sigma)) = \psi(f_{k-1}(\sigma))$,
- $f_k(\phi(i, id_{k-1})) = \phi'(i, id_{k-1})$, *for* $k \leq n$,
- $f_k(\phi(i, \gamma)) = \phi'(k-1, f_{k-1}(\gamma))$, *if* $\gamma \neq id_{k-1}$.

Then f_k is a bijection such that $f_k(id_k) = id_k$. Moreover, $f_k(\sigma)$ can be obtained from σ in linear time.

Proof: The fact that f_k is a bijection from $E_{k,1} \cup \{id_k\}$ to $S_k(321, 2413, 3412, 21534)$ is directly deduced from Theorem 3. Now let us prove that $f_k(\sigma)$ can be obtained from σ in linear time. Indeed, for the trivial case where $\sigma(k) = id_k$ then $f_k(\sigma) = \sigma$ and there is nothing to do. Now let us assume that $\sigma \neq id_k$ and $\gamma \in E_{k-1,1} \cup \{id_{k-1}\}$ be its predecessor in the generating tree. Therefore, if $\sigma(k) = k$ then $f_k(\sigma) = \psi(f_{k-1}(\gamma))$. Now let us consider $\sigma(k) \neq k$. Thus, there exists i, $1 \leq i \leq k-1$ such that $\sigma(i) = k$ is the only one excedance of σ and we have $\gamma(i) = \sigma(k)$. We distinguish two cases. (a) If $\sigma(k) = i$ then we necessarily have $\gamma = id_{k-1}$ and $f_k(\sigma) = \phi'(i, id_{k-1})$. (b) If $\sigma(k) \neq i$ then $\gamma \neq id_{k-1}$ and $f_k(\sigma(k)) = \phi'(k-1, f_{k-1}(\gamma))$. The recursivity is stopped when γ reaches an identity permutation, *i.e.*, when we reach the trivial case explained above. This means that the case (a) is performed only one time in order to construct $f_k(\sigma)$; for the other levels of the recursivity, we update the current permutation by appending a value on its right and eventually by a product on the right of a transposition. This proves that $f_k(\sigma)$ can be obtained in linear time.

J

For example, $f_4(3214) = f_4(\psi(321)) = \psi(f_3(321)) = \psi(f_3(\phi(1, 12))) = \psi(\phi'(1, 12)) = \psi(312) = 3124$.

Remark 4.1. *The list $f_n(\mathcal{E}_{n,1})$ is also an optimal Gray code for the set of avoiding permutations $S_n(321, 2413, 3412, 21534) \backslash \{id_n\}$. Two consecutive elements in the list differ in at most three positions. Notice that despite this last result, this bijection is not an isomorphism in the sense where the images of two close elements are close. Indeed, if d is the Hamming distance, we have for instance $d(12345, 52143) = 3$ and $d(f_5(12345), f_5(52143)) = d(12345, 31254) = 5$. See Table 1 for an illustration of our bijection onto the Gray code $\mathcal{E}_{n,1}$.*

Now, let us consider the map ϕ'' defined as follow:

Definition 3. *For an integer $i \in [n-1]$ and a permutation $\gamma \in S_{n-1}$, we define an n-length permutation $\sigma = \phi''_{n-1}(i, \gamma)$ by*

$$\sigma(j) = \begin{cases} n & \text{if } j = \gamma^{-1}(i) \\ \gamma(j) & \text{if } j \leq \gamma^{-1}(i) - 1 \\ \gamma(j-1) & \text{otherwise.} \end{cases}$$

Thus the permutation $\phi''_{n-1}(i, \gamma)$ is obtained from γ by inserting n just before i.

For example, $\phi''_5(3, 52431) = 524\mathbf{6}31$.

Remark 4.2. *If we replace ϕ by ϕ'' in the definition of our Gray code $\mathcal{E}_{n,1} \cup \{id_n\}$, we also obtain a generating tree for the set $S_n(231, 4132, 4213, 21534)$ which can not be directly implemented in a CAT algorithm. Notice that the generating tree obtained does not list the set $S_n(231, 4132, 4213, 21534)$ in Gray code order. We do not provide the proof of this remark since it can be obtained mutatis mutandis from this one of Theorem 2.*

Table 2: The Gray codes $\mathcal{E}_{4,1}$, $\mathcal{E}_{5,1}$, and their images by the bijection of Corollary 4.1: $S_4(T)\backslash\{1234\}$ and $S_5(T)\backslash\{12345\}$ where $T = \{321, 2413, 3412, 21534\}$.

$\mathcal{E}_{4,1}$		$\mathcal{E}_{5,1}$			
1	**2**13**4**	1	**2**134**5**	14	152**34**
2	41**32**	2	513**42**	15	1**524**3
3	**4**12**3**	3	**5**132**4**	16	1**324**5
4	3**12**4	4	41**32**5	17	1432**5**
5	**3**21**4**	5	**4**123**5**	18	**1532**4
6	**42**13	6	512**34**	19	**5**2314
7	142**3**	7	**5**124**3**	20	**423**15
8	1**324**	8	3**12**45	21	124**35**
9	**143**2	9	**3**214**5**	22	125**34**
10	**423**1	10	521**43**	23	**1254**3
11	1243	11	**5**213**4**	24	**52**341
		12	**42**135	25	1534**2**
		13	142**35**	26	12354

$\mathcal{S}_4(T)\backslash\{Id_4\}$		$\mathcal{S}_5(T)\backslash\{Id_5\}$			
1	21**34**	1	213**45**	14	1345**2**
2	21**43**	2	21**354**	15	132**54**
3	234**1**	3	214**53**	16	1**324**5
4	**231**4	4	2143**5**	17	142**35**
5	31**24**	5	234**15**	18	**14**253
6	**31**42	6	2345**1**	19	412**53**
7	13**42**	7	231**54**	20	**412**35
8	1**324**	8	**231**45	21	124**35**
9	**14**23	9	312**45**	22	124**53**
10	**412**3	10	31**254**	23	**125**34
11	1243	11	314**52**	24	**51**234
		12	**31**425	25	1**523**4
		13	134**25**	26	12354

5 Final remarks

In this paper, we give an efficient (CAT) algorithm for generating permutations with a fixed number of excedances. An improvement of this algorithm allows us to obtain optimal Gray code for permutations with exactly one excedance. We provide a constructive bijection between $E_{n,1}$ and the set $S_n(321, 2413, 3412, 21534)\backslash\{12\ldots(n-1)n\}$ which induces an optimal Gray code for $S_n(321, 2413, 3412, 21534)\backslash\{12\ldots(n-1)n\}$. Moreover we remark that there is a one-to-one map from $S_n(321, 2413, 3412, 21534)$ to $S_n(231, 4132, 4213, 21534)$ that does not induces a Gray code for the second set. Can one find a Gray code for the set $S_n(231, 4132, 4213, 21534)$? More generally, can one find a Gray code for the set $E_{n,k}$ for $2 \leq k \leq n-1$? Is it possible to obtain a set of avoiding permutations which is in bijection with $E_{n,k}$ for $2 \leq k \leq n-1$? Eulerian numbers also enumerate permutations of length n with a fixed number of descents; can one develop an efficient (CAT) algorithm for generating these objects?

References

[1] S.G. Akl, A new algorithm for generating derangements, *BIT*, 20: 2–7, 1980.

[2] S. Bacchelli, E. Barcucci, E. Grazzini, E. Pergola, Exhaustive generation of combinatorial objects by ECO, *Acta Informatica*, 40:585 – 602, 2004.

[3] E. Barcucci, A. Del Lungo, E. Pergola, R. Pinzani, ECO: a methodology for the enumeration of combinatorial objects, *Journal of Difference Equations and Applications*, 5:435–490, 1999.

[4] J.L. Baril, Gray code for permutations with a fixed number of cycles, *Discrete Math.*, 307, 13(6): 1559–1571, 2007.

[5] J.L. Baril, P.T. Do, ECO-generation for p-generalized Fibonacci and Lucas permutations, *Journal of Pure Mathematics and Applications*, 17(1-2): 19–37, 2006.

[6] A. Bernini, E. Grazzini, E. Pergola, R. Pinzani, A general exhaustive generation algorithm for Gray structures, *Acta Inform.*, 44(5): 361–376, 2007.

[7] J.L. Baril, V. Vajnovszki, Gray code for derangements, *Discrete Appl. Math.*, 140(1-3): 207–221, 2004.

[8] B. Bauslaugh, F. Ruskey, Generating alternating permutations lexicographically, *BIT*, 30: 17–26, 1990.

[9] M. Bóna, Combinatorics of permutations, *Chapman & Hall/CRC*, 2004.

[10] S. Effler, F. Ruskey, A CAT algorithm for generating permutations with a fixed number of inversions, *Information Process. Letters*, 86: 107–112, 2003.

[11] R. Ehrenborg, E. Steingrímsson, The excedance set of a permutation, *Advances in Applied Mathematics* 24:284–299, 2000.

[12] D. Foata, M.P. Schützenberger, Théorie géométrique des polynômes Eulériens, *Lectures Notes in Mathematics*, 138, Springer-Verlag, Berlin, 1970.

[13] S.M. Johnson, Generation of permutations by adjacent transpositions, *Mathematics of computation*, 17: 282–285, September 1963.

[14] D.E. Knuth, *The art of computer programming, combinatorial algorithms*, vol. 4. Pre-fascicle 2b, *Generating all permutations*, 2002.

[15] J.F. Korsh, Loopless generation of up-down permutations, *Discrete Math.*, 240(1-3): 97–122, 2001.

[16] J.F. Korsh, Constant time generation of derangements, *Information Process. Letters*, 90(4): 181–186, 2004.

[17] M. Lothaire, Combinatorics on words, *Encyclopedia of Mathematics and its applications*, 17, Addison-Wesley, 1983.

[18] R. Mantaci, F. Rakotondrajao, A permutation representation that knows what "Eulerian" means, *Discrete Math. and Theo. Comput. Science*, 4: 101–108, 2001.

[19] J. Riordan, An introduction to combinatorial analysis, *New York, J. Wiley*, 1958.

[20] D. Roelants van Baronaigien, Constant time generation of involutions, *Congressus Numerantium*, 90: 87–96, 1992.

[21] F. Ruskey, U. Taylor, Fast generation of restricted classes of permutations, *Manuscript*, 1995.

[22] F. Ruskey, D. Roelants van Baronaigien, Generating permutations with given ups and downs, *Discrete Appl. Math.*, 36(1): 57–67, 1992.

[23] N.J.A Sloane, *The Encyclopedia of Integer Sequences*, http://www.research.att.com/∼njas/sequences.

[24] H.F. Trotter, *PERM (Algorithm 115)*. Communications of ACM, 5(8), 434–435, 1962.

[25] V. Vajnovszki, A loopless algorithm for generating the permutations of a multiset. *Theoretical Computer Science*, 307: 415–431, 2003.

[26] T. Walsh, Gray code for involutions. *J. Combin. Math. Combin. Comput.*, 36: 95–118, 2001.

Information Processing Letters 110 (2010) 474–480

Contents lists available at ScienceDirect

Information Processing Letters

www.elsevier.com/locate/ipl

Whole mirror duplication-random loss model and pattern avoiding permutations

Jean-Luc Baril *, Rémi Vernay

LE2I UMR-CNRS 5158, Université de Bourgogne, B.P. 47 870, 21078 Dijon Cedex, France

ARTICLE INFO

Article history:
Received 29 June 2009
Received in revised form 10 February 2010
Accepted 20 April 2010
Available online 21 April 2010
Communicated by A.A. Bertossi

Keywords:
Algorithms
Combinatorial problems
Pattern avoiding permutation
Whole duplication-random loss model
Genome
Generating algorithm
Binary reflected Gray code

ABSTRACT

In this paper we study the problem of the whole mirror duplication-random loss model in terms of pattern avoiding permutations. We prove that the class of permutations obtained with this model after a given number p of duplications of the identity is the class of permutations avoiding the alternating permutations of length $2^p + 1$. We also compute the number of duplications necessary and sufficient to obtain any permutation of length n. We provide two efficient algorithms to reconstitute a possible scenario of whole mirror duplications from identity to any permutation of length n. One of them uses the well-known binary reflected Gray code (Gray, 1953) [10]. Other relative models are also considered.

© 2010 Elsevier B.V. All rights reserved.

1. Introduction and notation

The well-known genome duplication consists in copying a part of the original genome inserted into itself, followed by the loss of one copy of each of the duplicated genes (see [2,9,11,14,17] for an explanation of different methods of duplication). From a formal point of view, a genome of n genes is represented by a permutation of length n. In a previous article, Chaudhuri et al. [7] investigated a variant called the tandem duplication-random loss model: the duplicated part (of size K) of the genome is inserted immediately after the original portion, followed by the loss procedure. This model comes from evolutionary biology where it has been applied to the vertebrate mitochondrial genomes. Chaudhuri et al. introduce a notion of distance between two genomes and they provide an algorithm to compute it efficiently for certain regions of the parameter space. Bouvel and Rossin [5] have also studied this model. They proved that the class of permutations obtained from the identity after p steps (of width K) is also a class of pattern avoiding permutations. More particularly, they investigate the restricted case of a *whole duplication* (*W-duplication* for short): the whole duplication consists in copying entirely the permutation on its right and the loss procedure consists to delete one of the two copies of each gene. Here, we give an example of the process of a W-duplication followed by the loss procedure on the permutation 123456:

$$\begin{aligned}
123456 &\rightsquigarrow 1\,2\,3\,4\,5\,6\underbrace{1\,2\,3\,4\,5\,6}_{\text{duplication}} \\
&\rightsquigarrow \underbrace{1\,2\,\cancel{3}\,4\,\cancel{5}\,6\,\cancel{1}\,\cancel{2}\,3\,\cancel{4}\,5\,\cancel{6}}_{\text{loss procedure}} \\
&\rightsquigarrow 124635.
\end{aligned}$$

So, they prove that the permutations obtained after p W-duplications is the class of permutations avoiding all minimal permutations with 2^p descents, minimal in the sense of pattern-involvement relation on permutations. Moreover, they computed the number of duplication-loss steps

* Corresponding author.
E-mail addresses: barjl@u-bourgogne.fr (J.-L. Baril), remi.vernay@u-bourgogne.fr (R. Vernay).

0020-0190/$ – see front matter © 2010 Elsevier B.V. All rights reserved.
doi:10.1016/j.ipl.2010.04.016

$$5421673 \rightsquigarrow 5\ 4\ 2\ 1\ 6\ 7\ 3\ \underbrace{3\ 7\ 6\ 1\ 2\ 4\ 5}_{\text{mirror duplication}} \rightsquigarrow \underbrace{5\ 4\ \cancel{2}\ \cancel{1}\ 6\ \cancel{7}\ \cancel{3}\ 3\ 7\ \cancel{6}\ 1\ 2\ \cancel{4}\ \cancel{5}}_{\text{loss procedure}} \rightsquigarrow 5463712.$$

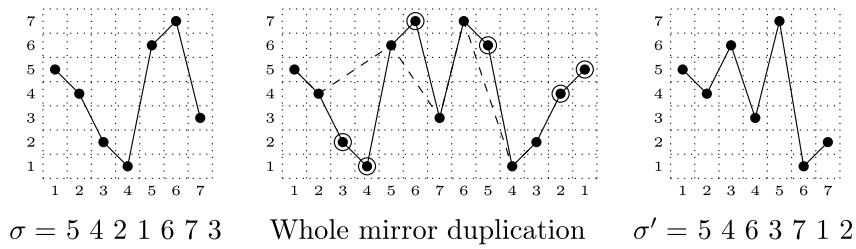


Fig. 1. A WM-duplication of the permutation 5421673. The encircled points are deleted by the loss procedure.

of width K necessary and sufficient to obtain any permutation. More recently, Bouvel and Pergola [4] showed a local and simpler characterization and several properties of the set of minimal permutations with 2^p descents.

In this article, we focus on the *whole mirror duplication-random loss model* (*WM-duplication* for short): it consists in copying the mirror of the permutation on its right followed by the loss procedure. This model very likely occurs in one half of eubacterial genomes, and possibly in most chromosomes [8,15]. Fig. 1 illustrates a *WM-duplication* for the permutation $\sigma = 5421673$:

$$\begin{aligned} 5421673 &\rightsquigarrow 5\ 4\ 2\ 1\ 6\ 7\ 3\ \underbrace{3\ 7\ 6\ 1\ 2\ 4\ 5}_{\text{mirror duplication}} \\ &\rightsquigarrow \underbrace{5\ 4\ \cancel{2}\ \cancel{1}\ 6\ \cancel{7}\ \cancel{3}\ 3\ 7\ \cancel{6}\ 1\ 2\ \cancel{4}\ \cancel{5}}_{\text{loss procedure}} \\ &\rightsquigarrow 5463712. \end{aligned}$$

Let $\mathcal{S}_n$ denote the set of n-length *sequences* $s = s_1 s_2 \dots s_n$ of positive integers. The *mirror* of s is $\bar{s} = s_n s_{n-1} \dots s_1$. A *subsequence* of s is a sequence $s_{i_1} s_{i_2} \dots s_{i_m}$ for $1 \leqslant i_1 < i_2 < \dots < i_m \leqslant n$. A subsequence is called a *substring* when the set $\{i_j,\ 1 \leqslant j \leqslant m\}$ is an interval, *i.e.* when the subsequence appears as consecutive elements in s. An *ascent* (resp. a *descent*) of s is any position i $(1 \leqslant i \leqslant n-1)$ with $s_i < s_{i+1}$ (resp. $s_i > s_{i+1}$). A *run up* (resp. *a run down*) of s is a substring (of length at least one) in which the elements are in increasing order (resp. decreasing order). More generally, a run up (or run down) is said to be *maximal* when it cannot be extended in a longer run up (resp. run down) in the sequence. We refer the reader to Rodney and Wilf [6] for results concerning the enumeration of permutations that have a given number of runs up and down. For instance, if $s = 5467312$ then position 4 is a descent, the substring 46 is a run up, and 467 is a maximal run up. Note that 5 is also a maximal run up. A *valley* of s is a substring which is a run down followed by a run up each of the length of at least two, *i.e.* a substring $s_k s_{k+1} \dots s_\ell$, $1 \leqslant k < \ell \leqslant n$, such that there is j $(k < j < \ell)$ verifying $s_k > s_{k+1} > \dots > s_j$ and $s_j < s_{j+1} < \dots < s_\ell$. A valley is *maximal* if the substring is maximal for this property. In the above example, the substrings 5467 and 7312 are the only maximal valleys of s. We denote by $\mathrm{val}(s)$ the number of maximal valleys in s, *i.e.* the cardinality of the set $\{j, s_j < \min\{s_{j-1}, s_{j+1}\}\}$. A recursive formula enumerating permutations with a given number of valleys can be found in [16].

On the other hand, a sequence s of length n is a *permutation* whenever each s_i is a distinct member of the $[n] = \{1, 2, \dots, n\}$. In the sequel, permutations will be denoted by Greek letters: $\sigma, \pi, \tau, \dots$. Let S_n be the set of all permutations of length n $(n \geqslant 1)$. In relation to the previous definition, any permutation σ contains at most $\lfloor \frac{n-1}{2} \rfloor$ valleys. A permutation $\sigma \in S_n$ is *alternating* if $\sigma_1 > \sigma_2 < \sigma_3 > \sigma_4 < \sigma_5 > \cdots$. In the literature [1], alternating permutations are also called *down-up* permutations and are enumerated by the Euler numbers (A000111 [19]). For instance, the permutation 324165 is alternating. Notice that an alternating permutation of length n contains exactly $\lfloor \frac{n-1}{2} \rfloor$ valleys.

A permutation π of length k, $k \leqslant n$, is a *pattern* of a permutation $\sigma \in S_n$ if there is a subsequence of σ which is order-isomorphic to π; *i.e.*, if there is a subsequence $\sigma_{i_1} \sigma_{i_2} \dots \sigma_{i_k}$ of σ (with $1 \leqslant i_1 < i_2 < \dots < i_k \leqslant n$) such that $\sigma_{i_\ell} < \sigma_{i_m}$ whenever $\pi_\ell < \pi_m$. We write $\pi \prec \sigma$ to denote that π is a pattern of σ. A permutation σ that does not contain π as a pattern is said to *avoid* π. For example, $\sigma = 1423$ contains the patterns 132, 312 and 123; but σ avoids the pattern 321. The class of all permutations avoiding the patterns $\pi_1, \pi_2, \dots, \pi_k$ is denoted $S(\pi_1, \pi_2, \dots, \pi_k)$, and $S_n(\pi_1, \pi_2, \dots, \pi_k)$ denotes the set of permutations of length n avoiding $\pi_1, \pi_2, \dots$ and π_k. We also say that $S(\pi_1, \pi_2, \dots, \pi_k)$ is a class of pattern-avoiding permutations of basis $\{\pi_1, \pi_2, \dots, \pi_k\}$. A class $\mathcal{C}$ of permutations is *stable* for $\prec$ if, for any $\sigma \in \mathcal{C}$, for any $\pi \prec \sigma$, then we also have $\pi \in \mathcal{C}$. We now formulate a remark that is crucial for the present study.

Remark 1. If a class $\mathcal{C}$ of permutations is stable for $\prec$ then $\mathcal{C}$ is also a class of pattern avoiding permutations of basis $\mathcal{B} = \{\sigma \notin \mathcal{C}, \forall \pi \prec \sigma \text{ with } \pi \neq \sigma, \pi \in \mathcal{C}\}$.

The paper is organized as follow. In Section 2, we prove that the class of permutations obtained from the identity after a given number p of whole mirror duplications is the class of permutations with at most $2^{p-1} - 1$ valleys. This is the class of permutations avoiding the alternating permutations of length $2^p + 1$. Moreover, we obtain the length of a shortest path between any permutation and the identity. In Section 3, we yield two algorithms (and their

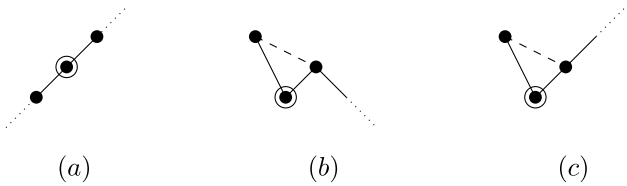


Fig. 2. The three non-isomorphic configurations in the proof of Lemma 1.

complexity) which construct such a shortest path. One of them uses an efficient algorithm for generating the well-known binary reflected Gray code [3,10]. In Section 4, we give some results about other models of duplications using W- and WM-duplications.

2. Pattern avoiding permutations and the mirror duplication-random loss model

In this section we study the WM-duplication random loss model in terms of pattern avoiding permutations. We establish that the class $\mathcal{C}(p)$ obtained from the identity after p WM-duplications is exactly the class of permutations avoiding all alternating permutations of length $2^p + 1$.

Lemma 1. *Let σ and π be two different permutations such that $\pi \prec \sigma$. Then σ contains at least many valleys as π does.*

Proof. It is sufficient to check the result for $\sigma \in S_n$ and $\pi \in S_{n-1}$ since a straightforward induction will complete the proof. Let $\sigma \in S_n$ and $\pi \in S_{n-1}$ such that $\pi \prec \sigma$, then π is order-isomorphic to a subsequence of σ obtained from σ by deleting only one entry of σ. We distinguish three non-isomorphic configurations illustrated in Fig. 2. More precisely, if we delete the encircled value in configuration (b), this reduces the number of valleys of σ. This does not occur for (a) and (c) where the number of valleys remains invariant after deletion of the encircled value. In all cases, a deletion of value in σ can not increase the number of valleys. □

Lemma 2. *A permutation obtained from the identity after a given number p of WM-duplications contains at most $2^{p-1} - 1$ valleys.*

Proof. We obtain the proof by induction. The result holds for $p = 1$; indeed a mirror duplication of the identity cannot create a valley. Now, let us assume that each permutation π obtained from the identity after $(p-1)$ mirror duplications contains at most $2^{p-2} - 1$ valleys. Let σ be a permutation obtained from the identity after p mirror duplications. Then σ is obtained from a permutation π with $2^{p-2} - 1$ valleys after exactly one mirror duplication. Therefore σ can be written as the concatenation of two subsequences of π and $\bar{\pi}$: *i.e.* $\sigma = \tau\tau'$ where τ (resp. τ') is a subsequence of π (resp. $\bar{\pi}$). According to Lemma 1, τ and τ' contains at most $2^{p-2} - 1$ valleys. As the concatenation of τ and τ' can (eventually) create one valley between them, σ contains at most $2 \cdot (2^{p-2} - 1) + 1 = 2^{p-1} - 1$ valleys which achieves the induction. □

Theorem 1. *The class $\mathcal{C}(p)$ of permutations obtained from the identity after a given number p of mirror duplications is the class of permutations with at most $2^{p-1} - 1$ valleys.*

Proof. After considering Lemma 2, it suffices to prove that any permutation σ with at most $2^{p-1} - 1$ valleys can be obtained from the identity after p mirror duplications. We proceed by induction on p. Indeed, let σ be a permutation with k valleys, $2^{p-2} - 1 < k \leqslant 2^{p-1} - 1$. Then σ can be written $\sigma = \tau\tau'$ where τ corresponds to the longest prefix of σ containing exactly $2^{p-2} - 1$ valleys and τ' the remaining suffix. We decompose the permutation $\tau = u_1 d_1 u_2 d_2 \ldots u_\ell d_\ell$, where u_i and d_i are respectively runs up and down defined as follows: u_1 is the first run up excepted its top value; d_1 the run down just after u_1; u_2 the run up just after d_1 excepted its top value, and so on. Notice that u_1 can be empty which does not occur for d_ℓ. Remark that we have $\ell = 2^{p-2}$ and thus $k < 2\ell$. For example, $\tau = 5421673$ has the decomposition: u_1 is empty, $d_1 = 5421$, $u_2 = 6$ and $d_2 = 73$. Let also $u_{\ell+1} d_{\ell+1} \ldots u_k d_k \ldots u_{2\ell} d_{2\ell}$ be the similar decomposition for τ'. In this decomposition, the runs u_i and d_i are empty for $i > k$. Now let us perform the following process: we sort in decreasing order the values that appear in d_ℓ or $u_{\ell+1}$ which creates a run down D_ℓ; we sort in increasing order the values in u_ℓ or $d_{\ell+1}$ which creates a run up U_ℓ; we construct the sequence $S = U_\ell D_\ell$. We sort in a decreasing order sequence $D_{\ell-1}$ the values in $d_{\ell-1}$ or $u_{\ell+2}$, and so on. At each step j, we insert the obtained ordered sequence U_j and D_j at the beginning of S. The permutation $S = U_1 D_1 \ldots U_{\ell-1} D_{\ell-1} U_\ell D_\ell$ obtained at the end of this process contains at most $2^{p-2} - 1$ valleys. See Fig. 3 for an example of construction of S. Thus, by induction, S can be obtained from the identity after $(p-1)$ mirror duplications. By construction σ is reached by one mirror duplication of S. Indeed, from any U_i (resp. D_i), $1 \leqslant i \leqslant \ell$, we can reconstitute the corresponding u_i and $d_{2\ell-i+1}$ (resp. d_i and $u_{2\ell-i+1}$). Therefore σ can be constructed from the identity with p mirror duplications.

Notice that the permutation σ is decomposed in a partition into runs up and down u_j and d_j. We will use this decomposition in Section 3.2 to reconstitute a path of WM-duplications from the identity to σ. □

Corollary 1. *Let σ be a permutation and* $\mathrm{val}(\sigma)$ *the number of its valleys. In the mirror duplication model, $\lceil \log_2(\mathrm{val}(\sigma) + 1) \rceil + 1$ steps are necessary and sufficient in order to obtain σ from the identity permutation.*

Proof. Let p be the integer such that $2^{p-2} - 1 < \mathrm{val}(\sigma) \leqslant 2^{p-1} - 1$, *i.e.* $p = \lceil \log_2(\mathrm{val}(\sigma) + 1) \rceil + 1$. According to Theorem 1, $p = \lceil \log_2(\mathrm{val}(\sigma) + 1) \rceil + 1$ steps are sufficient to obtain σ from the identity. It is also necessary since σ contains at least 2^{p-2} valleys which means that σ cannot be obtained from the identity in $(p-1)$ steps at most. □

In the next part we will provide algorithms in order to reconstitute a shortest path between the identity and a given permutation.

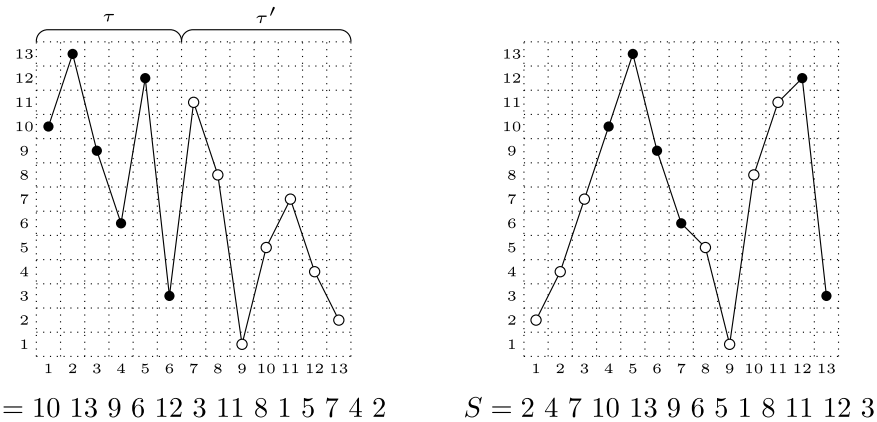


Fig. 3. Decomposition $\sigma = \tau\tau'$ and the permutation S obtained after the process in the proof of Theorem 1. We have $\tau = 10\ 13\ 9\ 6\ 12\ 3$, $\tau' = 11\ 8\ 1\ 5\ 7\ 4\ 2$, $u_1 = 10$, $d_1 = 13\ 9\ 6$, u_2 is empty, $d_2 = 12\ 3$, u_3 is empty, $d_3 = 11\ 8\ 1$, $u_4 = 5$, $d_4 = 7\ 4\ 2$ and, $D_2 = 12\ 3$, $U_2 = 1\ 8\ 11$; $D_1 = 13\ 9\ 6\ 5$; $U_1 = 2\ 4\ 7\ 10$. Thus $S = U_1D_1U_2D_2 = 2\ 4\ 7\ 10\ 13\ 9\ 6\ 5\ 1\ 8\ 11\ 12\ 3$.

Theorem 2. *The class $\mathcal{C}(p)$ of permutations obtained after a given number p of WM-duplications is the class of permutations avoiding the alternating permutations of length $2^p + 1$.*

Proof. Indeed, the permutations obtained after p mirror duplications is stable for the relation $\prec$, *i.e.* if $\sigma \in \mathcal{C}(p)$ and $\pi \prec \sigma$ then $\pi \in \mathcal{C}(p)$ (see Lemma 1). Thus (see Remark 1), $\mathcal{C}(p)$ is also a class of pattern avoiding permutations $S(B)$ where B is the set of minimal (relatively to $\prec$) permutations σ that are not in $\mathcal{C}(p)$. Such a minimal permutation σ contains exactly 2^{p-1} valleys. Indeed, $\sigma \notin \mathcal{C}(p)$ means $\mathrm{val}(\sigma) \geqslant 2^{p-1}$ and any permutation with at least $2^{p-1} + 1$ valleys is not minimal since it contains a pattern $\pi \notin \mathcal{C}(p)$ with 2^{p-1} valleys. Moreover, the configuration $\sigma_{i-1} < \sigma_i < \sigma_{i+1}$ and $\sigma_{i-1} > \sigma_i > \sigma_{i+1}$ can not occur since if we delete σ_i we do not decrease the number of valleys. Also we necessarily have $\sigma_1 > \sigma_2$. Thus we deduce that σ is an alternating permutation with 2^{p-1} valleys and its length is $2^p + 1$. □

For example, $\mathcal{C}(1) = S(213, 312)$ and $\mathcal{C}(2) = S(21435, 31425, 41325, 32415, 42315, 21534, 31524, 51324, 32514, 52314, 41523, 51423, 42513, 52413, 43512, 53412)$.

We also obtain a more general result (see Theorem 3) for the class of permutations having at most p valleys. Notice that, with [16,12,13], a bivariate generating function for this class is:

$$\frac{1}{1-y}\left(1 - \frac{1}{y} + \frac{1}{y}\sqrt{y-1} \times \tan\left(x\sqrt{y-1} + \arctan\left(\frac{1}{\sqrt{y-1}}\right)\right)\right),$$

where the coefficient of $x^n y^k$ is the number of permutations of length n with at most k valleys.

Theorem 3. *The class of permutations having at most p valleys is the class of permutations avoiding the alternating permutations of length $2p + 3$.*

Proof. The proof is obtained *mutatis mutandis*. □

3. Algorithmic considerations

In this section, we present two algorithms that provide a possible scenario of whole mirror duplications and losses for any $\sigma \in S_n$. Both construct a shortest path of WM-duplications from identity to σ. The first algorithm reconstitutes the path in reverse order, *i.e.*, we start from σ and at each step we compute the predecessor of the current permutation until the identity. The second algorithm computes the successor of the current permutation by starting from the identity until σ. These two algorithms give (generally) two different paths. The second method is interesting in the sense where it uses the special structure of the reflected binary Gray code [10]. Moreover, in the case where we want to get the first (resp. last) elements of the path, Algorithm 2 (resp. Algorithm 1) will be more efficient. Finally, we discuss the complexity of these algorithms.

Notice that if we have a shortest path $\mathcal{P}$ of WM-duplications from identity to σ, we can easily obtain a shortest path of WM-duplications from σ^{-1} to the identity by applying σ^{-1} on each permutation of $\mathcal{P}$.

3.1. *A path from $12\ldots n$ to $\sigma \in S_n$: Algorithm 1*

Here we explain how we can establish an algorithm in order to construct a scenario of WM-duplications from identity to $\sigma \in S_n$ by reconstructing this path from σ, *i.e.* at each step we find the predecessor of the current permutation and the process finishes when the identity is reached. We partition the permutation σ as follow: $\sigma = u'_1 d'_1 u'_2 d'_2 \ldots u'_k d'_k$ where $u'_1 = \sigma_1 \ldots \sigma_{i_1}$ is the maximal run up containing the first entry σ_1; d'_k is the maximal run down containing σ_n; d'_1 is the maximal run down containing σ_{i_1+1}; u'_k is the maximal run up just before d'_k, *i.e.* the value just before d'_k in σ belongs to u'_k; we continue this process by alternating runs up and down until the permutation σ is entirely partitioned. Notice that this decomposition is not the same as we have done in the proof of Theorem 1. For i from 1 to $\ell = \lfloor \frac{k+1}{2} \rfloor$, we define by U_i (resp. D_i) the increasing (resp. decreasing) order sorting of u'_i and d'_{k-i+1} (resp. d'_i and u'_{k-i+1}). Let $\pi = U_1 D_1 U_2 D_2 \ldots U_\ell D_\ell$ be the concatenation of all sorted

Algorithm 1 The procedure BackPath producing a scenario of WM-duplications from $12\ldots n$ to $\sigma \in S_n$.

procedure BackPath
while $\sigma \neq 12\ldots n$ **do**
- $\pi \leftarrow$ *empty*
- Let $\sigma = u'_1 d'_1 u'_2 d'_2 \ldots u'_k d'_k$ be the partition into runs up and runs down

for $i \leftarrow 1$ **to** $\lfloor \frac{k+1}{2} \rfloor$ **do**
- Sort in increasing order u'_i and d'_{k-i+1} and append the sorted substring on the right of π
- Sort in decreasing order d'_i and u'_{k-i+1} and append the sorted substring on the right of π

end for
- $\sigma \leftarrow \pi$

end while

substrings U_i and D_i where D_ℓ can be empty. Therefore π contains at most $(\ell - 1) = \lfloor \frac{k+1}{2} \rfloor - 1 = \lfloor \frac{\mathrm{val}(\sigma)+2}{2} \rfloor - 1 = \lfloor \frac{\mathrm{val}(\sigma)}{2} \rfloor$ valleys. This step requires $\mathcal{O}(n)$ computations since we can simultaneously detect and sort the runs up and down. By iterating this process with this new permutation π, Corollary 1 ensures that the procedure BackPath (see Algorithm 1) constructs a path from identity to σ in $(\lceil \log_2(\mathrm{val}(\sigma)+1) \rceil + 1)\mathcal{O}(n)$, *i.e.* $\mathcal{O}(n \cdot \log_2(n))$ in the worst case. For instance, this process applied to the permutation $\sigma = \mathbf{5}421\mathbf{6}73$ gives the path $\sigma \leftarrow \mathbf{357}6421 \leftarrow 1234567$ (the runs up are in boldface).

3.2. A path from $12\ldots n$ to $\sigma \in S_n$: Algorithm 2

In this section, we construct a path of WM-duplications from $12\ldots n$ to the permutation $\sigma \in S_n$, *i.e.* we start from the identity and at each step we find the successor of the current permutation; the process finishes when σ is reached. We decompose the permutation $\sigma = u_1 d_1 u_2 d_2 \ldots u_k d_k \ldots u_{2\ell} d_{2\ell}$ in runs up and down in the same way as we have done in the proof of Theorem 1. We recall this decomposition: u_i and d_i are respectively runs up and down defined as follows: u_1 is the first run up excepted its top value; d_1 the run down just after u_1; u_2 the run up just after d_1 excepted its top value, and so on. Notice that u_1 can be empty: such a decomposition is given as example in the proof of Theorem 1.

We now label each run in this decomposition with the reflected binary Gray code [10]: we can do this using a loopless algorithm introduced by Bitner, Ehrlich, and Reingold [3]. For instance, Fig. 4 shows such a labeling. The structure of the binary reflected Gray code B_n is crucial: $B_n = 0 \cdot B_{n-1} \cup 1 \cdot \overline{B_{n-1}}$ anchored by $B_1 = \{0, 1\}$ and where $\overline{B_n}$ is the list B_n considered in the reverse order. The first runs from u_1 to d_ℓ have 0 as their least significant bit, then those have 1. According to the proof of Theorem 1, we reconstitute the previous permutation π of σ by concatenating (on the left) the sorting in decreasing order D_ℓ of d_ℓ and $u_{\ell+1}$, the sorting in increasing order U_ℓ of u_ℓ and $d_{\ell+1}$, and so on.

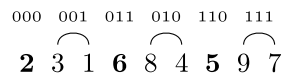
Conversely, at the step j of our algorithm we perform a WM-duplication of π by: (i) keeping in the first copy of π the elements labeled 0 on the jth least significant bit, and, (ii) keeping in the mirror the elements labeled 1 on the jth least significant bit (see Fig. 4 for an example). This step requires only $\mathcal{O}(n)$ computations.

Algorithm 2 The procedure Path producing a scenario of WM-duplications from $12\ldots n$ to $\sigma \in S_n$.

procedure Path
- $\pi = 12\ldots n$
- Partition $\sigma = u_1 d_1 u_2 d_2 \ldots u_k d_k$ into runs up and runs down
- Label the runs up and runs down with the reflected binary Gray code (loopless algorithm [3])

for $j = 1$ **to** $1 + \lfloor \log_2(k-1) \rfloor$ **do**
- Make a WM-duplication step on π that keeps in the first copy of π exactly the elements whose label has 0 in its jth least significant bit.

end for



$$123456789 \to 245897631 \to 231679854 \to 231684597$$

Fig. 4. Labeling of $\sigma = 231684597$ with the binary reflected Gray code according to the runs up and down. Example of a path of WM-duplications from the identity permutation to σ.

By iterating this process, Corollary 1 ensures that the procedure Path (see Algorithm 2) constructs a path from the identity to σ in $(\lceil \log_2(\mathrm{val}(\sigma) + 1) \rceil + 1)\mathcal{O}(n)$, *i.e.* $\mathcal{O}(n \cdot \log_2(n))$ in the worst case. Remark that Algorithm 2 is more efficient than Algorithm 1 in the case where we want to get the first permutations of the path. An implementation of our algorithm is available on http://www.u-bourgogne.fr/jl.baril/id2perm.php.

4. Other models of duplication

In this section, we investigate two variants of the whole mirror duplication: 1) we do one WM-duplication of the identity followed by several W-duplications and, 2) we do one WM- or/and W-duplication of the identity followed by several W-duplications. For these two cases we provide a characterization of the class of permutations obtained after p duplications.

4.1. One WM-duplication followed by several W-duplications

We have seen in the first section that one WM-duplication of the identity gives the class of permutations without valley. Thus, we obtain by induction that the class of permutations obtained after one WM-duplication of the identity followed by $(p-1)$ W-duplications is included in the class of permutations with at most $2^{p-1} - 1$ valleys.

Theorem 4. *The class of permutations obtained after one WM-duplication of the identity followed by $(p-1)$ W-duplications is the class of permutations with at most $2^{p-1} - 1$ valleys.*

Proof. Theorem 1 induces that the class of permutations obtained after one WM-duplication of the identity is the class of permutations without valley. Thus the result is true for $p = 1$. Now we proceed by induction. Let us assume that the class of permutations obtained after one WM-duplication of the identity followed by $(p-1)$ W-duplications is the class of permutations with at most $2^{p-1} - 1$ valleys. Let σ be a permutation in this last class. If we apply one

W-duplication on σ, we obtain a permutation π with at most $2(2^{p-1}-1)+1=2^p-1$ valleys. Now it suffices to prove that any permutation π with at most 2^p-1 valleys can be obtained after one W-duplication of a permutation σ with at most $2^{p-1}-1$ valleys. Indeed, let π be a permutation with k valleys, $2^{p-1}-1<k\leqslant 2^p-1$. As we have done in the proof of Theorem 2, we write $\pi=\tau\tau'$ where τ corresponds to the longest prefix of π containing exactly $2^{p-1}-1$ valleys and τ' the remaining suffix. We decompose the permutation $\tau=u_1d_1u_2d_2\dots u_\ell d_\ell$, where u_i and d_i are respectively runs up and down defined as follows: u_1 is the first run up excepted its top value; d_1 is the run down just after u_1; u_2 is the run up just after d_1 excepted its top value, and so on. We necessarily have $\ell=2^{p-1}$ and thus $k<2\ell$. Let also $u_{\ell+1}d_{\ell+1}\dots u_kd_k\dots u_{2\ell}d_{2\ell}$ be the similar decomposition for τ'. Now let us perform the following process: we sort in increasing order the values that appear in u_1 or $u_{\ell+1}$ which creates a run up U_1; we sort in decreasing order the values in d_1 or $d_{\ell+1}$ which creates a run down D_1; we construct the sequence $S=U_1D_1$. We sort in an increasing order sequence U_2 the values in u_2 or $u_{\ell+2}$, and so on. At each step j, we insert the obtained ordered sequences U_j and D_j on the right of S. The permutation $S=U_1D_1\dots U_{\ell-1}D_{\ell-1}U_\ell D_\ell$ obtained at the end of this process contains at most $2^{p-1}-1$ valleys. By construction π is reached by one W-duplication of S since we can reconstitute u_i and $u_{\ell+i}$ from U_i (resp. d_i and $d_{\ell+i}$ from D_i). We conclude by induction. □

The following corollary is directly deduced from Theorem 4 with the same proof as for Theorem 2.

Corollary 2. *The class of permutations obtained after one WM-duplication of the identity followed by $(p-1)$ W-duplications is the class of permutations avoiding the alternating permutations of length 2^p+1.*

4.2. One W- or WM-duplication followed by several W-duplications

In this part, we perform one W- or WM-duplication of the identity followed by several W-duplications. Let us recall that an ascent (resp. a descent) of an n-length permutation σ is a position i, $1\leqslant i\leqslant n-1$, such that $\sigma(i)<\sigma(i+1)$ (resp. $\sigma(i)>\sigma(i+1)$).

Theorem 5. *The class of permutations obtained after one W- or WM-duplication of the identity followed by p W-duplications is the union of the two classes*: (i) *permutations having at most 2^p-1 valleys and,* (ii) *permutations with at most $2^{p+1}-1$ descents.*

Proof. After one W- or WM-duplication of the identity we obtain the union of the class of permutations without valley with the class of permutations with at most one descent. A straightforward induction on p allows us to obtain the result using Theorems 1 and 4. □

Corollary 3. *The class $\mathcal{C}'(p)$ of permutations obtained after one W- or WM-duplication of the identity followed by p W-duplications is the class of permutations avoiding the permutations of length $3\cdot 2^p+1$ having exactly 2^p valleys and 2^{p+1} descents.*

Proof. Let σ be a minimal permutation which is not in $\mathcal{C}'(p)$. The minimality of σ induces that σ does not contain any run up of size at least three. By contradiction, if it is the case then by deleting the second value of the run up, the obtained sequence is isomorphic to a permutation which is also not in $\mathcal{C}'(p)$. Also σ cannot begin by an ascent for the same reasons. The permutation σ necessarily has 2^p valleys and 2^{p+1} descents. Indeed, if σ contains at least $2^{p+1}+1$ descents, it contains a run down of length at least three which contradicts the fact that σ is minimal. A simple calculation allows us to show that σ is necessarily of length $3\cdot 2^p+1$. □

For example, $\mathcal{C}'(0)=S(4132,3142,4312,3241,3214,4231,4213,2143)$; we have computed the cardinality of the basis of $\mathcal{C}'(1)$, which is 720. In the same way, we have the more general corollary:

Corollary 4. *The class of permutations having at most p valleys and at most $2p+1$ descents is the class of permutations avoiding the permutations of length $3p+4$ having exactly $p+1$ valleys and $2p+2$ descents.*

Notice that the set of permutations of length $3p+4$ having exactly $p+1$ valleys and $2p+2$ descents is also the set of permutations with the same length, the same number of valleys and where every ascent is immediately preceded by a descent (which defines a valley). This set was studied by Shapiro et al. [18] (see also the sequence A101280 in [19]). Indeed, they enumerate the permutations of length n having k peaks and with the additional property that every ascent is immediately followed by a descent. Thus, the cardinality of our set is obtained when $n=3p+4$ and $k=p+1$. For instance, the first cardinalities for $p=0,1,2,3,4$ are $8, 720, 230144, 179266560, 277662253056$.

5. Further research directions

In this paper, the whole mirror duplication model is studied. However, many relative open problems still need to be considered. For example, let σ_1 and σ_2 be two permutations in S_n. Can one exhibit an efficient algorithm to compute the permutation π which minimizes the sum $d(\pi,\sigma_1)+d(\pi,\sigma_2)$ where $d(\pi,\sigma)$ is the minimum of WM-duplication steps required to transform π into σ? Can one characterize the class of avoiding permutations corresponding to the permutations obtained from the identity after one W-duplication followed by p WM-duplications? More generally, can we characterize the permutations obtained after p steps of either W- or WM-duplications? Can one obtain similar results with the tandem mirror duplication model: the mirror of the duplicated part (of size K) is inserted immediately after the original portion and we apply the loss procedure.

Acknowledgements

We would like to thank the anonymous referee and Vincent Vajnovszki for their constructive remarks which have greatly improved this paper.

References

[1] B. Bauslaugh, F. Ruskey, Generating alternating permutations lexicographically, BIT Numerical Mathematics 30 (1) (1990) 17–26.
[2] S. Bérard, A. Bergeron, C. Chauve, C. Paul, Perfect sorting by reversals is not always difficult, IEEE ACM Transactions on Computational Biology and Bioinformatics 4 (1) (2007) 4–16.
[3] J.R. Bitner, G. Ehrlich, E.M. Reingold, Efficient generation of the binary reflected Gray code and its applications, Communications of the ACM 19 (9) (1976) 517–521.
[4] M. Bouvel, E. Pergola, Posets and permutations in the duplication-loss model, Pure Math. Appl. 19 (2–3) (2008) 71–80.
[5] M. Bouvel, D. Rossin, A variant of the tandem duplication-random loss model of genome rearrangement, Theoretical Computer Science 410 (8–10) (2009) 847–858.
[6] E.R. Canfield, H.S. Wilf, Counting permutations by their alternating runs, Journal of Combinatorial Theory 115 (2008) 213–225.
[7] K. Chaudhuri, K. Chen, R. Mihaescu, S. Rao, On the tandem duplication-random loss model of genome rearrangement, in: Proceedings of the Seventeenth Annual ACM–SIAM Symposium on Discrete Algorithm, ACM, New York, NY, USA, 2006, pp. 564–570.
[8] H.D. Chen, W.L. Fan, S.G. Kong, H. Lee, B. Zheng, N. Zhou, Inverse symmetry in genomes and whole-genome inverse duplication, in: International Bioinformatics Workshop (IBW2008), Yunnan University, Kunming, Yunnan, China, 2008.
[9] M.C. Chen, R.C.T. Lee, Sorting by transpositions based on the first increasing substring concept, in: BIBE'04, Proceedings of the 4th IEEE Symposium on Bioinformatics and Bioengineering, IEEE Computer Society, Washington, DC, USA, 2004, p. 553.
[10] F. Gray, Pulse code communication, U.S. Patent, 2,632,058, 1953.
[11] M.T. Hallett, J. Lagergren, New algorithms for the duplication-loss model, in: Proceedings of the Fourth Annual International Conference on Computational Molecular Biology, ACM, New York, NY, USA, 2000, pp. 138–146.
[12] S. Kitaev, Introduction to partially ordered patterns, Discrete Applied Mathematics 155 (2007) 929–944.
[13] S. Kitaev, A. Pyatkin, On avoidance of V- and Λ-patterns in permutations, Ars Combinatoria (2010), in press.
[14] A. Labarre, New bounds and tractable instances for the transposition distance, IEEE/ACM Transactions on Computational Biology and Bioinformatics (TCBB) 3 (4) (2006) 380–394.
[15] R. Nussinov, Some indications for inverse DNA duplication, Journal of Theoretical Biology 95 (4) (1982) 783.
[16] R. Rieper, M. Zeleke, Valleyless sequences, ArXiv:math/0005180, 2000.
[17] D. Sankoff, Gene and genome duplication, Current Opinion in Genetics and Development 11 (2001) 681–684.
[18] L.W. Shapiro, W.-J. Woan, S. Getu, Runs, slides and moments, SIAM J. Algebraic and Discrete Methods 4 (461) (1983).
[19] N.J.A. Sloane, The on-line encyclopedia of integer sequences, http://www.research.att.com/~njas/sequences/.

Discrete Mathematics 311 (2011) 51–57

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Cyclic and lift closures for $k \ldots 21$-avoiding permutations

Jean-Luc Baril

LE2I UMR-CNRS 5158, Université de Bourgogne, B.P. 47 870, 21078 DIJON-Cedex, France

ARTICLE INFO

Article history:
Received 11 January 2010
Received in revised form 25 August 2010
Accepted 18 September 2010
Available online 6 October 2010

Keywords:
Pattern avoiding permutation class
Cyclic, lift and toric closures

ABSTRACT

We prove that the cyclic closure of the permutation class avoiding the pattern $k(k-1)\ldots 21$ is finitely based. The minimal length of a minimal permutation is $2k-1$ and these basis permutations are enumerated by $(2k-1).c_k$ where c_k is the kth Catalan number. We also define lift operations and give similar results. Finally, we consider the toric closure of a class and we propose some open problems.

© 2010 Elsevier B.V. All rights reserved.

1. Introduction and notation

Let S (resp. S_n) be the class of permutations (resp. of length n). We represent permutations in one-line notation, i.e. if $i_1, i_2, \ldots, i_n$ are n distinct values in $[n] = \{1, 2, \ldots, n\}$, we denote the permutation $\sigma \in S_n$ by the sequence $i_1 i_2 \cdots i_n$ if $\sigma(k) = i_k$ for $1 \le k \le n$. For instance, the identity permutation of length n, Id_n, will be written $1\,2 \cdots n$. A *segment* of a permutation σ is a subsequence of the form $\sigma_i\sigma_{i+1}\cdots\sigma_{j-1}\sigma_j$ or $\sigma_j\sigma_{j+1}\cdots\sigma_n\sigma_1\cdots\sigma_{i-1}\sigma_i$ where $1 \le i \le j \le n$. The *cardinality* (or *length*) of a segment W will be denoted $|W|$. A permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in S_n$ contains the *pattern* $\pi \in S_k$, $k \ge 2$, if and only if a sequence of indices $1 \le i_1 < i_2 < \cdots < i_k \le n$ exists such that $\sigma(i_1)\sigma(i_2)\cdots\sigma(i_k)$ is ordered as π. We write $\pi \prec \sigma$ to denote that π is a pattern of σ. A permutation σ that does not contain π as a pattern is said to *avoid* π. Let B be a set of permutations. We denote by $\mathrm{Av}_n(B)$ (resp. $\mathrm{Av}(B)$) the set (resp. the class) of permutations in S_n (resp. S) avoiding all patterns $\pi \in B$. For example, $25314 \notin \mathrm{Av}_5(123)$ but $43152 \in \mathrm{Av}_5(123)$. See for instance [2,3]. In the following, a subsequence of a permutation σ order-isomorphic to $D_k = k(k-1)\cdots 21$ will be called a D_k-pattern of σ. A class $\mathcal{C}$ of permutations is *closed* (or *stable*) for the involvement relation $\prec$ if, for any $\sigma \in \mathcal{C}$, for any $\pi \prec \sigma$, then we also have $\pi \in \mathcal{C}$. Such a class can always be defined by a class $\mathrm{Av}(B)$ of pattern-avoiding permutations. In the case where B is minimal (relatively to the relation $\prec$), B is called a *basis*.

Let $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ be a permutation in S_n. We define two bijections r (*cyclic rotation*) and ℓ (*lift rotation*) on S_n as: $r(\sigma) = \sigma_n\sigma_1\sigma_2\cdots\sigma_{n-1}$ and $\ell(\sigma) = (\sigma_1 \bmod (n)+1)(\sigma_2 \bmod (n)+1)\cdots(\sigma_n \bmod (n)+1)$. For the sake of simplicity, we will say that a permutation $\sigma' \in S_n$ is a *cyclic rotation* (resp. *lift rotation*) of a permutation $\sigma \in S_n$ if $\sigma' = r^k(\sigma)$ (resp. $\sigma' = \ell^k(\sigma)$) for some $k \in [1\cdots n]$. For instance, 231 and 312 are both cyclic rotations of $123 \in S_3$; 213 and 132 are both lift rotations of $321 \in S_3$. The *cyclic closure* $\mathrm{cc}(\mathcal{C})$ (resp. *lift closure* $\mathrm{cl}(\mathcal{C})$) of a class $\mathcal{C}$ is the class of permutations that can be obtained by a cyclic rotation (lift rotation) from a permutation in $\mathcal{C}$. Finally, the *toric closure* $\mathrm{tr}(\mathcal{C})$ of $\mathcal{C}$ is defined by $\mathrm{cc}(\mathrm{cl}(\mathcal{C}))$ which is also equal to $\mathrm{cl}(\mathrm{cc}(\mathcal{C}))$.

Given a closed class of permutations, two well-studied problems are that of finding its enumeration and its basis. We are interested in these questions for cyclically closed classes, introduced by Albert et al. (See [1]).

Let $\mathcal{C}$ be the class $\mathrm{Av}(B)$ where B is a set of patterns. We know that $\mathrm{cc}(\mathcal{C})$ is a pattern-avoiding class where its basis is the set of permutations that are minimal with respect to not lying in $\mathrm{cc}(\mathcal{C})$. Let σ be a permutation in $\mathcal{C}$ and θ be a pattern of σ, i.e, a subsequence of σ order-isomorphic to a member of B. Modulo a cyclic rotation of σ, we assume that $\sigma = s\alpha t\beta$

E-mail address: barjl@u-bourgogne.fr.

0012-365X/$ – see front matter © 2010 Elsevier B.V. All rights reserved.
doi:10.1016/j.disc.2010.09.021

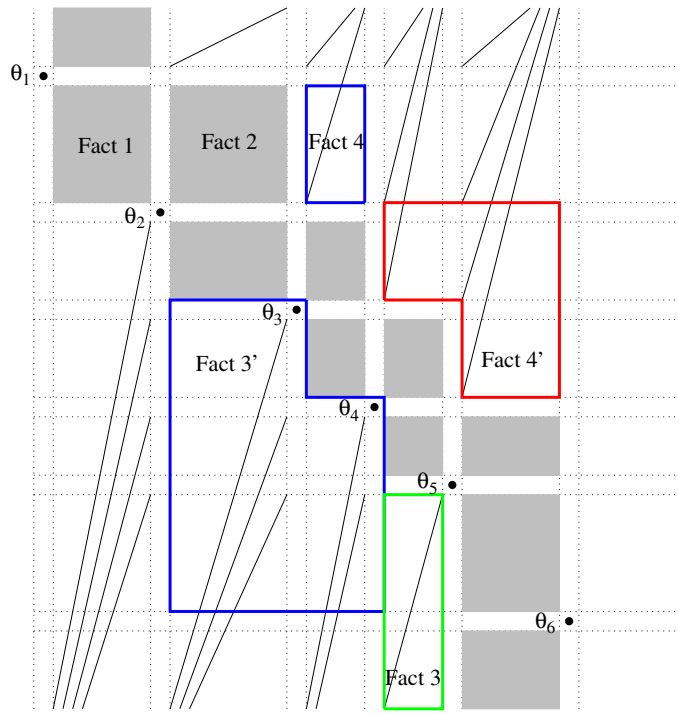


Fig. 1. The special structure of a permutation $\sigma \notin \mathrm{Av}(D_k)$. A filled area does not contain any point of the form (i, σ_i), $1 \le i \le n$. A sequence of j increasing diagonals in an area means that it does not contain a pattern $(j+1)j\cdots 21$.

where α (resp. β) is the first (resp. last) value of θ. The *witnessed segment* $W(\theta)$ of θ is the segment $s\alpha$. Albert et al. give a necessary and sufficient condition for σ to be (or not) in the cyclic closure of $\mathcal{C}$.

Lemma 1 ([1]). *$\sigma \notin \mathrm{cc}(\mathcal{C})$ if and only if the witnessed segments cover σ.*

Moreover, they obtain a sufficient condition for the class $\mathrm{cc}(\mathcal{C})$ to be finitely based.

Proposition 1 ([1]). *Let $\mathcal{C} = \mathrm{Av}(B)$, where B is finite and suppose there is a bound Δ depending on B alone such that, for all $\sigma \notin \mathrm{cc}(\mathcal{C})$, there is a collection of at most Δ witnessed segments that cover σ. Then $\mathrm{cc}(\mathcal{C})$ is finitely based.*

Albert et al. deduce from Proposition 1 that $\mathrm{cc}(\mathrm{Av}(321))$, $\mathrm{cc}(\mathrm{Av}(231))$ and $\mathrm{cc}(\mathrm{Av}(4321))$ are finitely based. They also show that the cyclic closure of a finitely based class can fail to be finitely based (it is the case for the class $\mathrm{Av}(265143)$). They provide several results about enumeration.

This paper is organized as follows. In Section 2, we prove that the cyclic closure of $\mathrm{Av}(k(k-1)\cdots 21)$ has a finite basis B. Moreover, we prove that the smallest length of a minimal permutation of B is $2k-1$, and that these basis permutations are enumerated by $(2k-1).c_k$ where c_k is the kth Catalan number. We also characterize these permutations. In Section 3, we investigate the lift closure of closed classes. We show a duality with the cyclic closure which allows to prove that the lift closure of $\mathrm{Av}(k(k-1)\cdots 21)$ is finitely based. In Section 4, we study the toric closure. We give a sufficient condition for the class $\mathrm{tr}(\mathcal{C})$ to be finitely based. We have obtained a proof that $\mathrm{tr}(\mathrm{Av}(321))$ is finitely based, but we do not give it. Indeed, the proof requires many cases to check and it does not contain any interesting concept. We finish the paper by presenting several open problems.

2. Cyclic closure

Let $\sigma = \sigma_1\cdots\sigma_n$ be a permutation of length n such that $\sigma \notin \mathrm{cc}(\mathrm{Av}(D_k))$ for $k \ge 3$. For $x \in \{1, \ldots, n\}$, we define $\mathcal{W}_\sigma(x)$ to be the set of subsequences $\theta = \theta_1\theta_2\cdots\theta_{k-1}\theta_k$ of σ (considered cyclically), order-isomorphic to $D_k = k(k-1)\cdots 21$ and such that $x \in W(\theta)$. Let $\mathcal{W}_\sigma$ be the union of all $\mathcal{W}_\sigma(x)$ for $1 \le x \le n$. In the following, we will omit the subscript σ for $\mathcal{W}_\sigma(x)$ and $\mathcal{W}_\sigma$ since it should be clear from context. For $\theta, \theta' \in \mathcal{W}$, we define an order relation $\trianglelefteq$ by: $\theta' \trianglelefteq \theta$ if and only if (a) $|W(\theta')| < |W(\theta)|$; or (b) $|W(\theta')| = |W(\theta)|$ and the associated word $\theta'_1\theta'_2\cdots\theta'_k$ is smaller than $\theta_1\theta_2\cdots\theta_k$ in lexicographic order. Thus $\mathcal{W}$ is a non-empty finite totally-ordered set. Let θ be its maximal element. Then, σ has a special structure illustrated in Fig. 1 and described below.

Indeed, σ can be written (modulo rotation):

$$\sigma = \theta_1\alpha_1\theta_2\alpha_2\cdots\theta_{k-1}\alpha_{k-1}\theta_k\alpha_k$$

where $W(\theta) = \alpha_k\theta_1, \theta_i \in \{1, 2, \ldots, n\}$ and α_i are segments of σ for $i \in [1, k]$. By convenience, we set $\theta_{k+1} = 0$ and $\theta_0 = n + 1$. For $i \in [1, k-1]$, we define the subsequence α_i^- (resp. α_i^+) of elements in α_i that are smaller than θ_{i+1} (resp. greater than θ_{i-1}). On the other hand, $\alpha_{i,j}^-$, $i \in [1, k-2]$, $j \in [i+2, k+1]$, (resp. $\alpha_{i,j}^+$, $i \in [2, k-1]$, $j \in [0, i-2]$,) is the subsequence of α_i^- (resp. α_i^+) constituted by all values greater than θ_j (resp. smaller than θ_j).

With these hypotheses, σ verifies the following structural properties.

Fact 1. For $i \in [1, k-1]$, α_i does not contain any value in the interval $[\theta_{i+1}, \theta_i]$.
If there exists $y \in \alpha_i$ such that $y \in [\theta_{i+1}, \theta_i]$, then $\theta' = \theta_1 \cdots \theta_i y \theta_{i+1} \cdots \theta_{k-1}$ is a subsequence of σ belonging to $\mathcal{W}$ such that $|W(\theta')| > |W(\theta)|$ which is a contradiction with the maximality of θ.

Fact 2. For $i \in [1, k-1]$, α_i does not contain any value in $[\theta_i, \theta_{i-1}]$.
If there exists $y \in \alpha_i$ such that $y \in [\theta_i, \theta_{i-1}]$, then $\theta' = \theta_1 \cdots \theta_{i-1} y \theta_{i+1} \cdots \theta_k$ is a subsequence of σ belonging to $\mathcal{W}$ such that $\theta \trianglelefteq \theta'$ which is a contradiction.

Fact 3. For $i \in [1, k-2]$, $j \in [i+2, k]$, $\alpha_{i,j}^-$ lies in $\mathrm{Av}(D_{j-i})$ and α_i^- lies in $\mathrm{Av}(D_{k-i})$.
If $\alpha_{i,j}^-$ contains a subsequence $\rho_1\rho_2 \cdots \rho_{j-i}$ order-isomorphic to D_{j-i}, then $\theta' = \theta_1 \cdots \theta_i \rho_1 \cdots \rho_{j-i} \theta_j \cdots \theta_{k-1}$ belongs to $\mathcal{W}$ and verifies $|W(\theta')| > |W(\theta)|$ which is a contradiction.

Fact 4. For $i \in [2, k-1]$, $j \in [0, i-2]$, $\alpha_{i,j}^+$ lies in $\mathrm{Av}(D_{i-j})$.
If $\alpha_{i,j}^+$ contains a subsequence $\rho_1\rho_2 \cdots \rho_{j-i}$ order-isomorphic to D_{i-j}, then $\theta' = \theta_1 \cdots \theta_j \rho_1 \cdots \rho_{i-j} \theta_{i+1} \cdots \theta_{k-1}$ belongs to $\mathcal{W}$ and verifies $\theta \trianglelefteq \theta'$ which is a contradiction.

The following facts are simple generalizations of Facts 3 and 4. The proofs are obtained *mutatis mutandis*.

Fact 3′. For $i \in [1, k-2]$, $j \in [i+2, k]$, $\ell \in [i+1, k-1]$, $\alpha_{i,j}^-\theta_{i+1}\alpha_{i+1,j}^-\theta_{i+2} \cdots \alpha_{\ell-1,j}^-\theta_\ell$ lies in $\mathrm{Av}(D_{j-i})$ and $\alpha_i^-\theta_{i+1}\alpha_{i+1}^-\theta_{i+2} \cdots \alpha_{\ell-1}^-\theta_\ell$ lies in $\mathrm{Av}(D_{k-i})$.

Fact 4′. For $i \in [2, k-1]$, $\ell \in [i+1, k]$, $j \in [0, \ell-3]$, $\alpha_{i,j}^+\theta_{i+1}\alpha_{i+1,j}^+\theta_{i+2}\alpha_{i+2,j}^+ \cdots \alpha_{\ell-1,j}^+\theta_\ell$ lies in $\mathrm{Av}(D_{\ell-j-1})$.

Theorem 1. *Let $k \geq 3$ be an integer. Then* $\mathrm{cc}(\mathrm{Av}(D_k))$ *is finitely based.*

Proof. Let $\sigma = \sigma_1 \cdots \sigma_n$ be a permutation of length n which does not belong to $\mathrm{cc}(\mathrm{Av}(D_k))$. The crucial point of the proof is to find a bounded number (independently of n) of witnessed segments that cover σ. Finally, we will conclude through Proposition 1.

Let $\theta = \theta_1 \cdots \theta_k$ be the maximal element of $\mathcal{W}$. Modulo cyclic rotation, σ can be decomposed as follows:

$$\sigma = \theta_1\alpha_1\theta_2\alpha_2 \cdots \alpha_{k-1}\theta_k\alpha_k.$$

Notice that α_k and $\theta_1 = \sigma_1$ are obviously covered by $W(\theta)$ since $W(\theta) = \alpha_k\theta_1$. Thus, it suffices to prove that $\alpha_i\theta_{i+1}$ for $1 \leq i \leq k-1$ are covered by a bounded number of witnessed segments.

Let i such that $1 \leq i \leq k-1$ and σ^i be the rotation of σ defined by $\sigma^i = \alpha_i\theta_{i+1} \cdots \theta_k\alpha_k\theta_1\alpha_1 \cdots \theta_{i-1}\alpha_{i-1}\theta_i$. By hypothesis, there is a subsequence ρ of σ^i (considered cyclically) such that $\theta_{i+1} \in W(\rho)$. If α_i is empty, there is nothing to do since $\alpha_i\theta_{i+1}$ is covered by $W(\rho)$. Now, let us suppose that α_i is not empty, and let us consider the first value x_1 of α_i. Then, there is a subsequence $\rho = \rho_1\rho_2 \cdots \rho_k$ of σ^i order-isomorphic to D_k such that $x_1 \in W(\rho)$. We take for ρ the maximal element of $\mathcal{W}(x_1)$.

We discuss on the first value ρ_1 of ρ: (i) $\rho_1 \notin \alpha_i$; (ii) $\rho_1 \in \alpha_i^-$ and (iii) $\rho_1 \in \alpha_i^+$. The case (i) is straightforward since this means that $\alpha_i\theta_{i+1}$ is included into $W(\rho)$.

Let us examine (ii): $\rho_1 \in \alpha_i^-$. Notice that ρ_k appears on the right of θ_k (in σ^i), since otherwise θ would not be maximal in $\mathcal{W}$ (because $\theta_1 \in W(\rho)$ and $|W(\rho)| > |W(\theta)|$). So, let j be the smallest integer, $1 \leq j \leq k-1$, such that ρ_j belongs to $\alpha_i^-\alpha_{i+1}^- \cdots \alpha_{k-1}^-$ but not ρ_{j+1}. Then, the special structure (particularly Fact 3′) of σ necessarily induces that the subsequence $\rho' = \theta_{i+1}\theta_{i+2} \cdots \theta_{i+j}\rho_{j+1}\rho_{j+2} \cdots \rho_k$ is order-isomorphic to D_k and $x \in W(\rho')$. This gives a contradiction with the fact that ρ is the maximal element of $\mathcal{W}(x_1)$. Therefore, the case (ii) never occurs.

Let us examine (iii): $\rho_1 \in \alpha_i^+$. If there exists j, $1 \leq j \leq k-1$ such that $\rho_j \in \alpha_i^+$ and $\rho_{j+1} \in \alpha_i^-$, then we consider ℓ being the smallest integer $j+1 \leq \ell \leq k-1$, such that ρ_ℓ belongs to $\alpha_i^- \cdots \alpha_{k-1}^-$ but not $\rho_{\ell+1}$ (ℓ exists since ρ_k is on the right of θ_k). So, with the same argument as for (ii), Fact 3′ necessarily induces that $\rho' = \rho_1 \cdots \rho_j\theta_{i+1}\theta_{i+2} \cdots \theta_{i+\ell-j}\rho_{\ell+1} \cdots \rho_k$ is a subsequence in $\mathcal{W}(x_1)$ such that $\rho \trianglelefteq \rho'$. This is a contradiction. Therefore, the subsequence $\rho^1 = \rho$ verifies the property that any value of ρ^1 is necessarily in $\alpha_i^+\theta_{i+1}\alpha_{i+1}^+ \cdots \alpha_{k-1}^+\theta_k\alpha_k\theta_1\alpha_1^-\theta_2\alpha_2^- \cdots \alpha_{i-1}^-\theta_i$. Less formally, we say that ρ_1 lies over θ in σ^i. Since this property is crucial for the following of the proof, we call this property the *dominance property* of ρ^1 over θ.

Now we consider the smallest j_1, $1 \leq j_1 \leq k-1$, such that $\rho_{j_1}^1$ belongs to α_i but not $\rho_{j_1+1}^1$. By considering Fact 4 (or 4′) with the dominance property of ρ^1 over θ, we deduce $j_1 < i-1$.

Now, we replace $x_1 = x$ by the value x_2 of α_i just after ρ_1^1 (if it exists; otherwise, we take $x_2 = \theta_{i+1}$). By hypothesis, x_2 is also covered by a witnessed segment $W(\rho^2)$ where ρ^2 is another pattern order-isomorphic to D_k, and such that ρ_k^2 belongs in σ^i on the right of ρ_k^1 or on the left of x_2. With the same argument as above, ρ^2 has the dominance property over ρ^1, i.e, less formally, the sequence ρ^2 lies over ρ^1. Therefore, ρ^2 necessarily contains at most $j_2 \leq j_1 - 1$ values in α_i^+.

By iterating the reasoning for the value just after the first value of ρ^2, we construct a subsequence ρ^3 such that the number j_3 of its values in α_i^+ verifies $j_3 \leq j_2 - 1$.

The process finishes after at most $i-1$ steps. This means that α_i can be covered by the witnessed segments $W(\rho^\ell)$ for some ℓ, such that $1 \le \ell \le i-2 \le k-2$, i.e. with at most $i-2$ witnessed segments, which is a bound depending on k alone. Therefore, $\alpha_i\theta_{i+1}$ can be covered by at most $i-1$ witnessed segments. Thus, σ is covered by at most $1+\sum_{\ell=1}^{k-1}\ell = 1+\frac{k(k-1)}{2}$ witnessed segments. □

Theorem 2. *For* $2 \le \ell \le 2k-1$, $\mathrm{cc}(\mathrm{Av}_{2k-\ell}(D_k)) = S_{2k-\ell}$.

Proof. The proof is straightforward for $2k-\ell \le k$, i.e., $\ell \ge k$. In the following of the proof, we take ℓ such that $2 \le \ell \le k-1$ and we establish the result by contradiction. So, let us assume that there exists σ of length $2k-\ell$ such that $\sigma \notin \mathrm{cc}(\mathrm{Av}_{2k-\ell}(D_k))$ with $2 \le \ell \le k-1$. Modulo cyclic rotation, we set $\sigma_1 = 2k-\ell$. Let $\theta = \theta_1 \cdots \theta_k$ be the leftmost subsequence of σ order-isomorphic to D_k. We have $\theta_1 = 2k-\ell$ and we discuss the value of θ_k: (i) $\theta_k = k-\ell+1$ and (ii) $\theta_k \le k-\ell$.

Case (i). We deduce $\theta_1 = 2k-\ell$, $\theta_2 = 2k-\ell-1, \ldots,$ and $\theta_k = k-\ell+1$. The cyclic rotation σ^1 of σ beginning with θ_k also contains a subsequence θ^1 order-isomorphic to D_k. As θ is the leftmost pattern in σ, θ^1 does not begin after θ_1 in σ^1. This means that θ^1_1 is at most $k-\ell+1$, and thus $\theta^1_k \le k-\ell+1-k+1 = 2-\ell \le 0$ which induces a contradiction.

Case (ii). We have $\theta_k \le k-\ell$. The cyclic rotation σ^1 of σ beginning with θ_k also contains a subsequence θ^1 order-isomorphic to D_k. We consider the leftmost subsequence in σ^1. We necessarily have θ^1_1 on the left of θ_1 and θ^1_k on the right of θ_1 (in σ^1). Moreover θ^1 and θ have a nonempty intersection (since $2k-\ell < 2k$). Let σ^2 be the cyclic rotation of σ beginning with θ^1_k and θ^2 be the leftmost subsequence in σ^2 order-isomorphic to D_k. We necessarily have: θ^2_1 is on the left of θ_k and after θ^1_k (in σ^2). θ^2 necessarily contains at least one element x of $\theta^1_1 \cdots \theta^1_{k-1}$. We discuss the position of x relatively to θ_1.

If x is after θ_1 in σ^2, then we also have θ^2_k after θ_1.

If x is before θ_1 in σ^2. As θ^2 also contains an element y of θ such that y appears after x, we deduce that θ^2_k is on the right of θ_1 in σ^2. With the same reasoning, we construct an infinite sequence of D_k-patterns θ^i different verifying the property that θ_1 is between θ^i_1 and θ^i_k which provides a contradiction. □

Theorem 3. *The set* $\mathrm{cc}(\mathrm{Av}_{2k-1}(D_k))$ *is enumerated by* $(2k-1)!-(2k-1)c_{k-1}$ *where* $(c_k)_{k\ge1}$ *is the well-known Catalan sequence.*

Proof. It suffices to prove that permutations of length $2k-1$ such that $\sigma \notin \mathrm{cc}(\mathrm{Av}_{2k-1}(D_k))$ and $\sigma_1 = 2k-1$ are enumerated by c_{k-1}. Let $\sigma = \sigma_1\sigma_2\cdots\sigma_{2k-1}$ be a permutation that does not belong to $\mathrm{cc}(\mathrm{Av}_{2k-1}(D_k))$ and such that $\sigma_1 = 2k-1$. Obviously σ contains the pattern D_k. We consider the leftmost pattern $\theta_{i_1}\theta_{i_2}\cdots\theta_{i_k}$ in σ (we necessarily have $\theta_{i_1} = 2k-1$).

Now we discuss the value of θ_{i_k}.

Case 1: $\theta_{i_k} = k$. This means that $\theta_{i_j} = 2k-j$ for $1 \le j \le k$, $\sigma_i \le k-1$ for $i > i_k$ and σ contains a decreasing subsequence order-isomorphic to D_k. Let σ^1 be the cyclic rotation of σ beginning with k. By hypothesis, σ^1 contains a subsequence order-isomorphic to D_k. The only one possibility is that this subsequence is exactly $k(k-1)\cdots 21$.

Moreover, remark that the value just after $k+1$ is necessarily at least k, i.e, $k+1$ and k are consecutive in σ. This is due to the hypothesis that θ is the leftmost pattern in σ. Moreover, for the same reason, two successive decreasing values cannot appear between $k+2$ and $k+1$. More generally, there does not exist i decreasing values after $k+i$ for $0 \le i \le k-1$ which characterizes the Catalan numbers. Conversely, such a permutation does not belong to $\mathrm{cc}(\mathrm{Av}_{2k-1}(D_k))$. See Fig. 2 for $k = 4$.

Case 2: $\theta_{i_k} \le k-1$. We repeat *mutatis mutandis* the reasoning of the case (ii) in the proof of Theorem 2. □

Notice that a $(2k-1)$-length permutation $\sigma \notin \mathrm{cc}(\mathrm{Av}(D_k))$ beginning with $2k-1$ does not contain the pattern 123. Moreover, the previous proof induces a constructive bijection between the set of well-formed parentheses of size $2k-2$ and the set of permutations of length $2k-1$, beginning with $2k-1$, and containing a pattern D_k in each of their cyclic rotations. Indeed, we consider the binary representation $b = b_1\cdots b_{2k-2}$ of a well-formed parentheses, i.e. $b_i = 0$ if the ith parenthesis is a closed parenthesis, $b_i = 1$ otherwise). For convenience we add $b_{2k-1} = 0$ on the right of b. Now, let j, $1 \le j \le 2k-1$, be the rank of the rightmost zero on the left of the rightmost one in b (if j does not exist we set $j = 2k-1$). Then, we traverse b from right to left (from indices j to 1), and we label each zero in increasing order from 1. Finally, we continue to label (in increasing order) the unlabeled elements of b from right to left. For instance, the parenthesis ()((()())(()))) has a binary representation 10111010**0**11100000 where $j = 9$ (in boldface). Its corresponding permutation 17 4 16 15 14 3 13 21 12 11 10 9 8 7 6 5 contains the pattern D_9 in each cyclic rotation. See Fig. 2 when $k = 4$.

Theorem 1 shows that the basis B of $\mathrm{cc}(\mathrm{Av}(D_k))$ is finite. Theorems 2 and 3 imply that the smallest length of a minimal permutation is $2k-1$, and that these permutations are enumerated by $(2k-1)\cdot c_{k-1}$. It remains to characterize the other elements of B. Notice that Albert et al. have experimentally obtained the basis elements of $\mathrm{cc}(\mathrm{Av}(321))$ which are 15432, 14325, 164253, 163254 and 1472536. We have experimentally checked that the basis of $\mathrm{cc}(\mathrm{Av}(4321))$ contains 5 minimal permutations of length 7, 32 of length 8, 54 of length 9, and 136 of length 10.

Theorem 4. *The inversions of length* $2k-1$ *which do not lie in* $\mathrm{cc}(\mathrm{Av}(D_k))$ *are enumerated by* $\frac{k(k+1)}{2}$.

Proof. Let σ be an involution of length $2k-1$ that does not belong to $\mathrm{cc}(\mathrm{Av}(D_k))$. We distinguish two cases: (i) there exists i, $k \le i \le 2k-2$, such that $\sigma_{2k-i} = 2k-1$; and (ii) there exists i, $1 \le i \le k-1$, such that $\sigma_{2k-i} = 2k-1$.

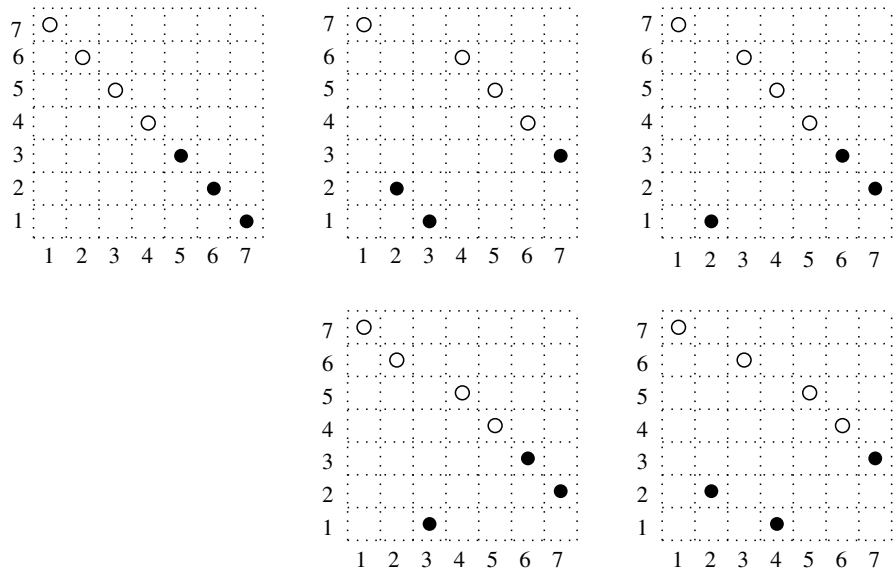


Fig. 2. The five permutations σ beginning with 7 such that $\sigma \notin \mathrm{cc}(\mathrm{Av}_7(D_4))$: **7654**321, 7**21654**3, 7**16543**2, **7615432**, 7**2615**43. The leftmost pattern θ is illustrated in boldface for the one-line notation of σ and with empty points in the representations below. The corresponding well-formed parentheses are respectively $((()))$, $(())()$, $(()())$, $()(())$ and $()()()$.

The first case (i) implies that there is, $k \le i \le 2k-2$, such that $\sigma_{2k-i} = 2k-1$ and $\sigma_{2k-1} = 2k-i$. Moreover, if we take j, $1 \le j < 2k-i$, such that $\sigma_{2k-j} = 1$ then this induces $\sigma_1 = 2k-j$. This means that the cyclic rotation of σ beginning with $2k-1$, contains from right to left the subsequence $1, 2k-i, 2k-j$ which contradicts the remark just after Theorem 3. Thus, there exists j, $1 \le j < 2k-i$, such that $\sigma_j = 1$ and $\sigma_1 = j$. A similar argument as above allows us to conclude that σ is necessarily $2k-i-1, \ldots, 1, 2k-1, \ldots, 2k-i$.

For the second case, there exists i, $1 \le i \le k-1$, such that $\sigma_{2k-i} = 2k-1$ and $\sigma_{2k-1} = 2k-i$. In the same way as above, we deduce that there does not exist j, $j > 2k-i$, such that $\sigma_j = 1$. Thus, let us consider j, $j < 2k-i$, such that $\sigma_j = 1$ and $\sigma_1 = j$. If $j = 2k-i-1$ then it is straightforward to see that we necessarily have $\sigma = (2k-2)\cdots 1(2k-1)\cdots(2k-i)$. If $j \in [k-i, 2k-i-2]$ then we easily see that there does not exist some involutions verifying this hypothesis. If $j \in [1\cdots k-i-1]$ then the only one involution is $\sigma = j\cdots 1(2k-i-1)\cdots(j+1)(2k-1)\cdots(2k-i)$. To summarize, there are $\sum_{i\in[1\cdots k-1]}(k-i-1)+k = \frac{k(k+1)}{2}$ such involutions. □

For $k = 3$ there are 6 involutions of length 5 that do not lie in $\mathrm{cc}(\mathrm{Av}(D_3))$: 54321, 15432, 21543, 32154, 43215 and 14325.

3. Lift closure

In this section, we study the lift closure $\mathrm{cl}(\mathcal{C})$ of a closed class $\mathcal{C}$, i.e., the class of permutations that can be obtained by a lift of a permutation in $\mathcal{C}$. We provide several general results which make links between cyclic and lift closures.

Lemma 2. *If X is a set of permutations then $\pi \in \mathrm{cc}(X) \iff \pi^{-1} \in \mathrm{cl}(X^{-1})$. Consequently, if $X = \mathrm{Av}(B)$ then $\mathrm{cc}(\mathrm{Av}(B)) = \mathrm{cl}(\mathrm{Av}(B^{-1}))^{-1}$. Moreover, if the basis B is stable by inversion, i.e. $B = B^{-1}$, then $\mathrm{cc}(\mathrm{Av}(B)) = \mathrm{cl}(\mathrm{Av}(B))^{-1}$.*

Proof. Indeed, $\pi \in \mathrm{cc}(X)$ means there exists $\sigma = \sigma_1\cdots\sigma_n \in X$ such that $\pi = r^k(\sigma) = \sigma_k\cdots\sigma_n\sigma_1\cdots\sigma_{k-1}$. With $r(\sigma)^{-1} = \ell(\sigma^{-1})$ we deduce $\pi^{-1} = \ell^k(\sigma^{-1})$. Thus $\pi^{-1} \in \mathrm{cl}(\mathrm{Av}(X^{-1}))$. By symmetry, we conclude $\pi^{-1} \in \mathrm{cl}(\mathrm{Av}(X^{-1}))$ implies $\pi \in \mathrm{cc}(X)$. The straightforward relation $\mathrm{Av}(B^{-1}) = \mathrm{Av}(B)^{-1}$ induces the consequences. □

Now we set $\mathcal{C} = \mathrm{Av}(B)$ where B is a basis. As for the case of the cyclic rotation, $\mathrm{cl}(\mathcal{C})$ also is a class of avoiding permutations. Let σ be a permutation in $\mathcal{C}$ and θ be a pattern (in B) of σ. We define by *witnessed interval* of σ the cyclic interval $W'(\theta) = [n]\backslash]\min(\theta)\cdots\max(\theta)]$. Notice that $W'(\theta)$ is an interval in $[n]$ considered cyclically, but it is not necessarily a segment in σ. We can easily remark that $W'(\theta)$ is also the witnessed segments of σ^{-1} (considered cyclically) relatively to the pattern θ^{-1}. Two similar results of Lemma 1 and Proposition 1 can be deduced below.

Lemma 3. *$\sigma \notin \mathrm{cl}(\mathcal{C})$ if and only if the witnessed intervals $W'(\theta)$ cover $[n]$ (or equivalently σ).*

Proof. By applying Lemmas 1 and 2, we obtain: $\sigma \notin \mathrm{cl}(\mathcal{C})$ if and only if $\sigma^{-1} \notin \mathrm{cc}(\mathcal{C}^{-1})$, i.e. iff the witnessed segments (relatively to θ^{-1} and σ^{-1}) cover σ^{-1} (or equivalently $[n]$). With the remark above, this is equivalent to the covering of $[n]$ by the witnessed intervals $W'(\theta)$ of σ. □

Proposition 2. *Let $\mathcal{C} = \mathrm{Av}(B)$, where B is finite and suppose there is a bound Δ depending on B alone such that, for all $\sigma \notin \mathrm{cl}(X)$, there is a collection of at most Δ witnessed intervals that cover σ (or equivalently $[n]$). Then $\mathrm{cl}(X)$ is finitely based.*

Proof. The proof is a direct consequence of Proposition 1 and Lemma 3. □

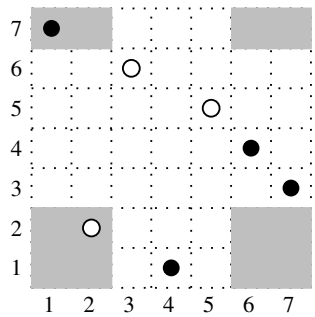


Fig. 3. The witnessed area $W''(265)$ for the permutation 7261543.

Theorem 5. *Let $k \geq 2$ be an integer. Then* $\mathrm{cl}(\mathrm{Av}(D_k))$ *is finitely based.*

Proof. This theorem is a consequence of Theorem 1 combined with Lemma 2. Indeed, we have $\mathrm{cl}(\mathrm{Av}(D_k)) = \mathrm{cc}(\mathrm{Av}(D_k))^{-1} = \mathrm{Av}(B)^{-1} = \mathrm{Av}(B^{-1})$ where B is a finite basis. □

All enumeration results of [1] for the cyclic closure are also valid for the lift closure.

Theorem 6. $\mathrm{cl}(\mathrm{Av}_{2k-1}(D_k))$ *is enumerated by* $(2k-1)! - (2k-1)c_{k-1}$ *where* $(c_k)_{k\geq 1}$ *is the well-known Catalan sequence.*

Proof. This corollary is deduced from Theorem 3. □

4. Going further and conclusion

In this section, we give some general results about the toric closure $\mathrm{tr}(\mathcal{C})$ for a class $\mathcal{C}$ of permutations, i.e. $\mathrm{tr}(\mathcal{C}) = \mathrm{cl}(\mathrm{cc}(\mathcal{C})) = \mathrm{cc}(\mathrm{cl}(\mathcal{C}))$. Let $f_{u,v}$, $1 \leq u, v \leq n$, be the function defined on S_n by $f_{u,v}(\sigma) = \sigma'$ where $\sigma'_j = (\sigma_{(u+j-2) \bmod (n)+1} - v) \bmod (n)+1$, for $j \in [1 \cdots n]$. The set $\{f_{u,v}(\sigma), 1 \leq u, v \leq n\}$ contains exactly all toric rotations of σ since the permutation $f_{u,v}(\sigma)$ equals the permutation $r^{n-u+1}(\ell^{n-v+1}(\sigma))$.

Now assume that $\mathcal{C} = \mathrm{Av}(B)$ where B is a finite set of permutations. Let θ be a pattern (relatively to B) in $\sigma \in \mathcal{C}$. The *witnessed area* $W''(\theta)$ of θ in σ is the direct product of $\sigma^{-1}(W(\theta))$ of θ with the witnessed interval $W'(\theta)$: $W''(\theta) = \sigma^{-1}(W(\theta)) \times W'(\theta)$. For instance, the permutation 7**2**6**1**5**4**3 contains the pattern 132 ($\theta = 265$), $W(\theta)$ is the segment 4372, thus $\sigma^{-1}(W(\theta)) = \{6, 7, 1, 2\}$, $W'(\theta)$ is the interval 712, and $W''(\theta)$ is the area $\{6, 7, 1, 2\} \times \{7, 1, 2\}$. See Fig. 3 for an illustration.

We also obtain similar general results as for the cyclic (or lift) closure.

Lemma 4. $\sigma \notin \mathrm{tr}(\mathcal{C})$ *if and only if the witnessed areas cover the area* $[n] \times [n]$ *where n is the length of* σ.

Proof. If the witnessed areas of σ cover $[n] \times [n]$, then no toric rotation of σ lies in $\mathcal{C}$. Indeed, the toric rotations σ' of σ defined by $\sigma'_j = (\sigma_{(u+j-2) \bmod (n)+1} - v) \bmod (n)+1$, for $j \in [1 \cdots n]$, where $(u, v) \in W''(\theta)$, all contain θ as a pattern without wrap-around and none of these permutations lie in $\mathcal{C}$. Conversely, if the witnessed areas do not cover a point (u, v) then the toric rotation σ' of σ defined as above, contains no pattern (without wrap-around) and thus lies in $\mathrm{Av}(\mathcal{C})$. Consequently, σ lies in $\mathrm{tr}(\mathrm{Av}(\mathcal{C}))$. □

Proposition 3. *Let* $\mathcal{C} = \mathrm{Av}(B)$, *where B is finite and suppose there is a bound* Δ *depending on B alone such that, for all* $\sigma \notin \mathrm{tr}(\mathcal{C})$, *there is a collection of at most* Δ *witnessed areas that cover* σ. *Then* $\mathrm{tr}(\mathcal{C})$ *is finitely based.*

Proof. The proof are obtained mutatis mutandis as for Proposition 1. □

We have obtained a proof that $\mathrm{tr}(\mathrm{Av}(321))$ is finitely based, but we do not present it because it is very technical and requires a lot of cases. Experimental calculations suggest that this basis contains one element of length 5, 2 of length 7, 2 of length 8, 39 of length 9 and 2 of length 10. Thus the problem of finding a nice proof that $\mathrm{tr}(\mathrm{Av}(321))$ is finitely based remains open. We also have the following open questions:

Problem 1. Is $\mathrm{tr}(\mathrm{Av}(231))$ finitely based?

Problem 2. Is $\mathrm{tr}(\mathrm{Av}(k \cdots 21))$ finitely based for $k \geq 4$?

Problem 3. Given a finite basis B, is it decidable if $\mathrm{cc}(\mathrm{Av}(B))$ is finitely based?

Acknowledgements

I would like thank the anonymous referees for their constructive comments and V. Vajnovszki who introduced me to this problem.

References

[1] M.H. Albert, R.E.L. Aldred, M.D. Atkinson, H.P. van Ditmarsch, C.C. Handley, D.A. Holton, D.J. McCaughan, C.W. Monteith, Cyclically closed pattern classes of permutations, Australasian Journal of Combinatorics 38 (2007) 87–100.
[2] M. Bona, Combinatorics of Permutations, in: Discrete Mathematics and its Applications, Chapman & Hall/CRC, 2004.
[3] R. Simion, F.W. Schmidt, Restricted permutations, European Journal of Combinatorics 6 (1985) 383–406.

Classical sequences revisited with permutations avoiding dotted pattern

Jean-Luc Baril

LE2I UMR-CNRS 5158, Université de Bourgogne
B.P. 47 870, 21078 DIJON-Cedex France

barjl@u-bourgogne.fr

Submitted: Mar 13, 2011; Accepted: Jul 21, 2011; Published: Sep 9, 2011
Mathematics Subject Classification: 05A05

Abstract

Inspired by the definition of the barred pattern-avoiding permutation, we introduce the new concept of dotted pattern for permutations. We investigate permutations classes avoiding dotted patterns of length at most 3, possibly along with other classical patterns. We deduce some enumerating results which allow us to exhibit new families of permutations counted by the classical sequences: 2^n, Catalan, Motzkin, Pell, Fibonacci, Fine, Riordan, Padovan, Eulerian.
Keywords: pattern avoiding permutation class; dotted pattern

1 Introduction and notation

Let S_n be the set of permutations on $[n] = \{1, 2, \ldots, n\}$, *i.e.*, all one-to-one correspondences from $[n]$ into itself. We represent a permutation $\pi \in S_n$ in one-line notation $\pi = \pi(1)\pi(2)\ldots\pi(n)$. For instance the identity permutation id_n of length n will be written $12\ldots n$. A permutation $\pi \in S_n$ *avoids* the pattern $\tau \in S_k$ if and only if there does not exist a sequence of indices $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ such that $\pi(i_1)\pi(i_2)\ldots\pi(i_k)$ is order-isomorphic to τ (see [13, 15]). We denote by $S_n(\tau)$ the set of permutations of S_n avoiding the pattern τ. For example, if $\tau = 321$ then $34125 \in S_5(\tau)$ while $\mathbf{43}5\mathbf{1}2 \notin S_5(\tau)$. Many classical sequences in combinatorics appear as the cardinality of pattern-avoiding permutation classes. A large number of these results were firstly obtained by West and Knuth [9, 13, 15, 17, 18] (see the surveys of Kitaev and Mansour [8, 10]).

In the literature we can also find the concept of *barred pattern* (see [7, 16], [11] for a systematic study of these patterns and [2, 5, 15, 19] for some applications). Let τ be a permutation in S_k and $b = b_1 \ldots b_k \in \{0, 1\}^k$. The barred pattern $\bar{\tau}$ is a permutation in

S_k obtained from τ by copying the entries of τ and by putting a bar over τ_i if and only if $b_i = 1$. For example, if $\tau = 312$ and $b = 001$ then $\bar{\tau} = 31\bar{2}$. Let r be the number of barred entries in $\bar{\tau}$ (or equivalently the number of ones in b). We denote the permutation on $[k-r]$ made up of the $(k-r)$ unbarred elements of $\bar{\tau}$ by $\hat{\tau}$, rewritten to be a permutation on $[k-r]$. Obviously τ contains the pattern $\hat{\tau}$ and we will say that τ extends the pattern $\hat{\tau}$ according to the binary string b (the positions of the extended entries are the same as the positions of 1s in b). The permutation $\pi \in S_n$ avoids the pattern $\bar{\tau}$ if and only if each occurrence of the pattern $\hat{\tau}$ in π can be extended into an occurrence of the pattern τ in π according to b. For example, if $\bar{\tau} = 21\overline{34}65$ then $b = 001100$, $\mathbf{31}245\mathbf{76} \in S_7(\bar{\tau})$ and $31\mathbf{42}5\mathbf{76} \notin S_7(\bar{\tau})$. Moreover we will say that $\pi \in S_n$ *weakly avoids* the pattern $\bar{\tau}$ if and only if each occurrence of the pattern $\hat{\tau}$ in π can be extended into an occurrence of the pattern τ in π (without considering the string b). Let $S_n^w(\bar{\tau})$ be the set of n-length permutations that weakly avoid $\bar{\tau}$. For example, if $\bar{\tau} = 12\bar{3}$ then $1243 \in S_4^w(\bar{\tau})$ when $1243 \notin S_4(\bar{\tau})$ since the sequence 24 cannot be extended into a pattern 123 according to $b = 001$. By definition the weak avoidance is less restrictive than the classical avoidance: $S_n(\bar{\tau}) \subset S_n^w(\bar{\tau})$. Surprisingly and to my knowledge, the weak avoidance has not been studied in the literature. One reason could be that the weak avoidance on permutation classes does not provide the classical integer sequences (2^n, Fibonacci, Motzkin, Catalan, ...). However using a variation of this concept we retrieve most of these sequences.

Now we define a *dotted pattern* $\dot{\tau}$ which adds dots over some entries τ_i, $i \in I \subset [k]$, of $\tau \in S_k$. For each $j \notin I$, let $\bar{\tau}^j$ be the barred permutation of length k obtained from τ by adding a bar over the entry τ_j. *We will say that a permutation π avoids $\dot{\tau}$ if and only if π weakly avoids $\bar{\tau}^j$ for all $j \notin I$.* For example, if $\dot{\tau} = 13\dot{2}$ then $\bar{\tau}^1 = \bar{1}32$, $\bar{\tau}^2 = 1\bar{3}2$ and $S_5(\dot{\tau}) = \{15243, 15432, 13254, 13542\}$; if $\dot{\tau} = 23\dot{1}$ then $\bar{\tau}^1 = \bar{2}31$, $\bar{\tau}^2 = 2\bar{3}1$ and $S_3(\dot{\tau}) = \{123, 231\}$. Less formally, if τ^j is the pattern $\bar{\tau}^j$ without its barred entry then π avoids $\dot{\tau}$ if and only if each pattern τ^j ($j \notin I$) can be extended into the pattern τ in π (without restriction for the position of the extended entry). A particular case of this definition was studied in [3].

By convention we set $S_n(\dot{\tau}) = S_n$ in the case where all entries of $\dot{\tau}$ are dotted. In the sequel we consider that the pattern $\dot{\tau}$ contains at least one dotted entry and one undotted entry. More generally we always have

$$S_n(\tau) \subseteq S_n(\tau^j) \subseteq S_n(\dot{\tau}) \subseteq S_n^w(\bar{\tau}^j) \text{ for all } j \notin I, \text{ and}$$

$$S_n(\dot{\tau}) = \bigcap_{j \notin I} S_n^w(\bar{\tau}^j).$$

We say that π *contains* $\dot{\tau}$ if π does not avoid $\dot{\tau}$.

Moreover the Wilf equivalences are also valid: *i.e.*, if $\delta = \dot{\tau}$ then we have

$$|S_n(\delta)| = |S_n(\delta^r)| = |S_n(\delta^c)| = |S_n(\delta^{-1})|$$

where δ^r denotes the reverse of δ where the positions of dots are also reversed (*i.e.*, for $i \in I$, the position i is replaced with $n - i + 1$), δ^c denotes the complement of δ where the

positions of dots are preserved, and δ^{-1} denotes the inverse of δ where the dots are over the entries i for $i \in I$.

In this paper, we present some enumeration results for some classes of permutations avoiding dotted patterns. More precisely, Section 2 presents some basic general lemmas concerning sets of permutations avoiding dotted patterns. Section 3 investigates exhaustively the sets of permutations avoiding one pattern $2\dot{1}$ possibly along with one classical pattern of length 3. Section 4 deals with dotted patterns of length 3. Several open questions are given about the enumeration of one dotted patterns of length 3. However we present others enumeration results for the case where the dotted pattern is associated with one or more other classical patterns. This study allows us to exhibit several new families of avoiding permutations that are enumerated by the well known sequences: Catalan, Motzkin, Fibonacci, Riordan, binary, Fine, Padovan, Eulerian.

2 Some general results

In this section we provide some general lemmas about sets of permutations avoiding dotted patterns. As defined in Section 1, we consider a dotted pattern $\dot{\tau}$ and the permutation $\tau \in S_k$ obtained from $\dot{\tau}$ by deleting its dots. If $I \subset [k]$ is the set of dotted positions in $\dot{\tau}$ then we also need patterns $\bar{\tau}^j$ and τ^j for $j \notin I$ (see Section 1 for these definitions).

Lemma 1 *Let $\dot{\tau}$ be a dotted pattern with only one undotted entry. Then we have*

$$S_n(\dot{\tau}) = S_n^w(\bar{\tau})$$

where the pattern $\bar{\tau}$ is obtained from τ by adding a bar over the entry corresponding to the only one undotted entry of $\dot{\tau}$.

Proof. Here only one position j is undotted in $\dot{\tau}$. So we have $I = [k]\backslash\{j\}$ which induces only one barred pattern $\bar{\tau}^j$. By definition, π avoids $\dot{\tau}$ if and only if π weakly avoids $\bar{\tau}^j = \bar{\tau}$ which achieves the proof. □

Lemma 2 *Let $\dot{\tau}$ be a dotted pattern such that all patterns τ^j, $j \notin I$, are identical. Then $S_n(\dot{\tau}) = S_n^w(\bar{\tau}^j)$ for each $j \notin I$.*

Proof. Let j and k be two integers that do not lie in I. Recall that τ^j (resp. τ^k) is the permutation obtained from $\bar{\tau}^j$ (resp. $\bar{\tau}^k$) by deleting its barred entries. By definition, a permutation π avoids $\dot{\tau}$ if and only if each pattern τ^j ($j \notin I$) can be extended into the pattern τ in π. Therefore we directly deduce $S_n(\dot{\tau}) = S_n^w(\bar{\tau}^j)$ for each $j \notin I$. □

Now Lemma 2 induces the following result:

Lemma 3 *Let $\dot{\tau} = k(k+1)\dots n\dot{\delta}$ be a dotted pattern such that $\dot{\delta}$ is obtained from $\delta \in S_{k-1}$ by dotting all its entries. Then $S_n(\dot{\tau}) = S_n^w(\overline{k}(k+1)\dots n\delta)$.*

In the remainder of this study, we focus on the enumeration of permutation classes avoiding dotted patterns.

3 Avoiding a dotted pattern of length 2

In this section, we present some enumeration results for permutations (and involutions) avoiding a dotted pattern of length 2, possibly along with one or more classical patterns. Considering the Wilf equivalences, we obviously obtain

$$|S_n(2\dot{1})| = |S_n(1\dot{2})| = |S_n(\dot{1}2)| = |S_n(\dot{2}1)|.$$

Moreover Lemma 1 induces $S_n(2\dot{1}) = S_n^w(\bar{2}1)$ and with the definition of the weak avoidance we also obtain $S_n(2\dot{1}) = S_n^w(2\bar{1})$.

Now let us give some useful definitions and notations. A fixed point of a permutation $\pi \in S_n$ is an integer $i \in [n]$ such that $\pi_i = i$. A permutation without fixed point will be called derangement, and the set of n-length derangements is denoted D_n. A *strong fixed point* of a permutation $\pi \in S_n$ is an integer $i \in [n]$ such that $\pi_i = i$ and $\pi_j < \pi_i < \pi_k$ for all j, k verifying $j < i < k$. An involution of length n is a permutation $\pi \in S_n$ such that $\pi^2 = id_n$. Let I_n be the set of n-length involutions.

Theorem 1 *The set $S_n(2\dot{1})$ is the set of the permutations with no strong fixed point. Its generating function is known (see A052186, [14]) and given by $\frac{F(x)}{1+xF(x)}$ where $F(x) = \sum_{n>=0} n!x^n$.*

Proof. Let π be an n-length permutation having a strong fixed point i. We have $\pi_i = i$ and $\pi_j < \pi_i < \pi_k$ for $j < i < k$. Consequently, the entry i cannot be extended into a pattern 21 in π. Thus $\pi \notin S_n(2\dot{1})$. Conversely, if π contains an entry π_i that cannot be extended into a pattern 21 then we necessarily have $\pi_j < \pi_i < \pi_k$ for $j < i < k$ which implies that $\pi_i = i$. Thus π contains a strong fixed point i which completes the proof. □

Theorem 2 *The set $S_n(2\dot{1}, 1\dot{2})$ is the set of the permutations π with no strong fixed point and without i such that $\pi_i = n - i + 1$ and $\pi_j > \pi_i > \pi_k$ for $j < i < k$. We have $|S_n(2\dot{1}, 1\dot{2})| = 2|S_n(2\dot{1})| - n!$.*

Proof. By symmetry a permutation avoiding $1\dot{2}$ has no i such that $\pi_i = n - i + 1$ and $\pi_j > \pi_i > \pi_k$ for $j < i < k$. Thus there does not exist a permutation $\pi \in S_n$, $n \geq 2$, that contains both the patterns $2\dot{1}$ and $1\dot{2}$. So we have $|S_n(2\dot{1}, 1\dot{2})| = |S_n| - 2(|S_n| - |S_n(2\dot{1})|) = 2|S_n(2\dot{1})| - n!$. □

Pattern	Sequence	Sloane [14]
$\{2\dot{1}\}, \{1\dot{2}\}, \{\dot{1}2\}, \{\dot{2}1\}$	no strong fixed point	$A052186$
$\{2\dot{1}, 1\dot{2}\}$	$2\|S_n(2\dot{1})\| - n!$	new

Table 1: Enumeration for the sets of permutations avoiding dotted patterns of length 2

3.1 $2\dot{1}$ and one classical pattern of length 3

In this part we give exhaustive results about the cardinalities of the sets of permutations avoiding $2\dot{1}$ and one classical pattern of length 3. We also consider the sets of involutions avoiding these patterns.

Theorem 3 *For $n \geq 3$, the set $S_n(2\dot{1}, 123)$ is enumerated by $c_n - 2$ (see A120304, [14]) where $c_n = \frac{1}{n+1}\binom{2n}{n}$ is the nth Catalan number.*

Proof. It is well known that $S_n(123)$ is enumerated by the nth Catalan number (see for instance [4]). Now if $\pi \in S_n(123)$ contains $2\dot{1}$ then Theorem 1 implies that there exists a strong fixed point i in π. There are only two possibilities: either $i = 1$ and $\pi = 1n(n-1)\ldots 2$ or $i = n$ and $\pi = (n-1)\ldots 21n$. Thus $|S_n(2\dot{1}, 123)| = c_n - 2$ for all $n \geq 3$. □

Theorem 4 *The set $S_n(2\dot{1}, 321)$ is enumerated by the Fine numbers f_n (see A000957, [14]) defined by $c_n = 2f_n + f_{n-1}$ where $f_1 = 0$, $f_2 = 1$ and c_n is the nth Catalan number.*

Proof. In [12], it is proved that the number of permutations $\pi \in S_n(321)$ with no fixed point is enumerated by the Fine numbers. Thus it suffices to verify that a permutation $\pi \in S_n(321)$ with no strong fixed point does not contain any fixed point. For a contradiction, let us assume that π contains a fixed point $\pi_i = i$. Theorem 1 shows that there exists $j \neq i$ such that at least one of the following conditions is true: (a) $j < i$ and $\pi_j > \pi_i$; (b) $j > i$ and $\pi_j < \pi_i$. For the case (a), this implies that there does not exist π_k, $k > i$ such that $\pi_k < \pi_i = i$. Therefore each value of the set $\{1, 2, \ldots, i-1, j\}$ appears (in π) on the left of π_i which contradicts the hypothesis $\pi_i = i$. The case (b) is obtained similarly. Thus $|S_n(2\dot{1}, 321)| = |S_n(321) \cap D_n|$ where D_n is the set of n-length permutations with no fixed point (derangements). By [12], we conclude that $S_n(2\dot{1}, 321)$ is enumerated by the nth Fine number f_n. □

Theorem 5 *The set $S_n(2\dot{1}, 231)$ is enumerated by the Fine numbers f_n defined by $c_n = 2f_n + f_{n-1}$ where $f_1 = 0$, $f_2 = 1$ (see A000957, [14]) and c_n is the nth Catalan number.*

Proof. We use the well known bijection φ between $S_n(321)$ and $S_n(231)$ found by Simion and Schmidt [13] (see also Bóna [4], Lemma 4.3, p. 130). They consider the right-to-left minima of $\pi \in S_n(231)$; *i.e.*, the entries π_i of π such that $\pi_j \geq \pi_i$ for all $j > i$. The bijection φ keeps the right-to-left minima of $\pi \in S_n(231)$ fixed and to place all other entries in increasing order. For example, if $\pi = 852\mathbf{1}4\mathbf{37}\mathbf{6} \in S_8(231)$ then the right-to-left minima are $1, 3$ and 6 (in boldface) and $\varphi(\pi) = 245\mathbf{1}7\mathbf{3}8\mathbf{6} \in S_8(321)$. As a strong fixed point also is a right-to-left minimum the bijection φ preserves the strong fixed points. Thus φ induces a bijection from $S_n(2\dot{1}, 231)$ to $S_n(2\dot{1}, 321)$. We conclude using Theorem 4. □

Theorem 6 *The set $S_n(2\dot{1}, 213)$ is enumerated by the sequence $c_n - c_{n-1} = \frac{3(2n-2)!}{(n+1)!(n-2)!}$ (see A000245, [14]) where c_n is the nth Catalan number.*

Proof. Let π be a permutation in $S_n(2\dot{1}, 213)$. As π avoids 213, π can be decomposed as follows: $\pi = k\delta\gamma$ where δ (resp. γ) is a sequence made up of the values in $\{(k+1), \ldots, n\}$ (resp. values in $\{1, 2, \ldots, (k-1)\}$) such that δ and γ also avoid the pattern 213 (see Bóna [4], p. 132). Now since π avoids $2\dot{1}$ the first entry k of π must be different from 1. It is straightforward to see that this condition is also sufficient. Thus the cardinality of $S_n(2\dot{1}, 213)$ is exactly $c_n - c_{n-1}$ where c_n is the nth Catalan number. Thus we deduce $|S_n(2\dot{1}, 213)| = \frac{3(2n-2)!}{(n+1)!(n-2)!}$. □

The Wilf equivalences applied to the above results allow us to obtain the enumeration for all sets of permutations avoiding $2\dot{1}$ with a classic pattern of length 3:

Pattern	Sequence	Sloane
$\{2\dot{1}, 321\}, \{2\dot{1}, 312\}, \{2\dot{1}, 231\}$	Fine	$A000957$
$\{2\dot{1}, 132\}, \{2\dot{1}, 213\}$	$c_n - c_{n-1} = \frac{3(2n-2)!}{(n+1)!(n-2)!}$	$A000245$
$\{2\dot{1}, 123\}$	$c_n - 2$	$A120304$

Table 2: Enumeration for the sets of permutations avoiding $2\dot{1}$ and one classical pattern of length 3

Now we present a similar study for the sets of involutions avoiding $2\dot{1}$ and a classical pattern of length 3. Recall that I_n denotes the set of involutions of length n, *i.e.*, the set of n-length permutations π such that $\pi^2 = id_n$.

Theorem 7 *The following statements hold:*

i) *For $n \geq 1$, $|I_n(2\dot{1})| = d_n$ where d_n is recursively defined by $d_1 = 0$ and for $n \geq 2$, $d_n = |I_n| - \sum_{r=1}^{n-1} d_r \cdot |I_{n-1-r}|$.*

For instance $d_n = 0, 1, 1, 4, 9, 31, 94, 337, 1185, 4540$ for $n \leq 10$.

ii) *For $n \geq 3$, $|I_n(2\dot{1}, 123)| = \binom{n}{\lfloor \frac{n}{2} \rfloor} - 2$.*

iii) *For $n \geq 3$, $|I_n(2\dot{1}, 132)| = |I_n(2\dot{1}, 213)| = \binom{n-1}{\lfloor \frac{n-2}{2} \rfloor}$,*

iv) *For $n \geq 3$, $|I_n(2\dot{1}, 231)| = |I_n(2\dot{1}, 312)| = fib_n$ where fib_n is the nth Fibonacci number defined by $fib_1 = 0$, $fib_2 = 1$ and $fib_n = fib_{n-1} + fib_{n-2}$ for $n \geq 3$.*

v) *For $n \geq 1$, $|I_{2n}(2\dot{1}, 321)| = c_n$ and $|I_{2n+1}(2\dot{1}, 321)| = 0$ where c_n is the nth Catalan number.*

Proof. *i)* Let π be an n-length involution that contains $2\dot{1}$. By Theorem 1, π can be written $\pi = \delta k \gamma$ where $k \geq 2$, $\delta \in S_{k-1}(2\dot{1})$ and γ is a subsequence of elements in $\{k+1, \ldots, n\}$. As δ belongs to S_{k-1} and π is an involution, $\delta \in I_{k-1}(2\dot{1})$. On the other hand γ considered as a permutation in S_{n-k} is necessarily an involution. Therefore the

set of involutions that contain $2\dot{1}$ is enumerated by $\sum_{k=2}^{n} |I_{k-1}(2\dot{1})| \cdot |I_{n-k}|$ which induces that for $n \geq 2$,

$$|I_n(2\dot{1})| = |I_n| - \sum_{k=2}^{n} |I_{k-1}(2\dot{1})| \cdot |I_{n-k}|.$$

ii) It is straightforward to see that only two involutions containing $2\dot{1}$ avoid 123: $1n(n-1)\dots 2$ and $(n-1)\dots 21n$. Thus for $n \geq 3$ we have

$$|I_n(2\dot{1}, 123)| = |I_n(123)| - 2 = \binom{n}{\lfloor \frac{n}{2} \rfloor} - 2.$$

iii) An n-length permutation π that contains $2\dot{1}$ and avoids 132 can be written $\pi = \delta k(k+1)\dots n$ where $k \geq 2$, $\delta \in I_{k-1}(2\dot{1})$. So if $a_n = |I_n(2\dot{1}, 132)|$ then $a_n = |I_n(132)| - 1 - \sum_{r=2}^{n-1} a_r = \binom{n}{\lfloor \frac{n}{2} \rfloor} - 1 - \sum_{r=2}^{n-1} a_r$ for $n \geq 3$ anchored by $a_2 = 1$. In order to prove the result we proceed by induction on $n \geq 3$. For $n = 3$ then $a_3 = \binom{3}{\lfloor \frac{3}{2} \rfloor} - 1 - 1 = 1$ thus the result holds. For $n \geq 3$ we have,

$$a_n = \binom{n}{\lfloor \frac{n}{2} \rfloor} - 1 - \sum_{r=2}^{n-1} a_r = \binom{n-1}{\lfloor \frac{n}{2} \rfloor} + \binom{n-1}{\lfloor \frac{n}{2} \rfloor - 1} - 1 - \binom{n-2}{\lfloor \frac{n-3}{2} \rfloor} - \sum_{r=2}^{n-2} a_r.$$

With the recurrence hypothesis and after considering the two cases n odd/even, we deduce $a_n = \binom{n-1}{\lfloor \frac{n-2}{2} \rfloor}$.

iv) We set $a_n = |I_n(2\dot{1}, 231)|$. Let π be an involution of $I_n(2\dot{1}, 231)$. As π avoids $2\dot{1}$, $\pi_n = n$ does not occur. Thus we distinguish two cases: (a) $\pi_n = n-1$ and $\pi_{n-1} = n$; and (b) $\pi_n < n-1$.

For the case (a), the permutation $\pi' \in S_{n-2}$ obtained by deleting the two last entries of π also is an involution in $I_{n-2}(2\dot{1}, 231)$. Conversely, each permutation $\pi \in I_n(2\dot{1}, 231)$ (verifying the condition (a)) can be obtained from a permutation π' as above.

For the case (b), let us assume $\pi_k = n$ with $k < n-1$. We consider $\pi' \in S_{n-1}$ obtained by deleting the entry $\pi_k = n$ in π. As π avoids 231, $n-1$ is on the right of n in π. Moreover, we necessarily have $\pi_{k+1} = n-1$ because otherwise the subsequence $(k+1)\pi_{n-1}\pi_n$ is a pattern 231. Iterating this process, π and π' can be written $\pi = \delta n(n-1)\dots k$ and $\pi' = \delta(n-1)\dots k$ where $\delta \in I_{k-1}(2\dot{1}, 231)$. This induces that π' is a permutation in $I_{n-1}(2\dot{1}, 231)$. Conversely each permutation $\pi \in I_n(2\dot{1}, 231)$ can be obtained from a permutation $\pi' \in I_{n-1}(2\dot{1}, 231)$ by inserting n just after the entry π'_{k-1}.

Therefore we deduce, for $n \geq 3$ that $a_n = a_{n-1} + a_{n-2}$ anchored by $a_1 = 0$ and $a_2 = 1$. This defines the Fibonacci sequence.

v) Let $\pi \in I_{2n+1}(2\dot{1}, 321)$. As the length of π is odd, there exists a fixed point $\pi_k = k$. As π avoids $2\dot{1}$, there exists $j < k$ (or $j > k$) such that $\pi_j > \pi_k$ (or $\pi_j < \pi_k$). In these two cases this means that there exists a pattern 321 that is $\pi_j k \pi_{\pi_j}$ (or $\pi_{\pi_j} k \pi_j$) which is a contradiction. Thus $|I_{2n+1}(2\dot{1}, 321)| = 0$.

Now let $\pi \in I_{2n}(2\dot{1}, 321)$. In this case, we refer to a result of Deutsch, Robertson, and Saracino [6], which states that the number of 321-avoiding involutions on $[2n]$ with no fixed points is c_n. Indeed, π is $2\dot{1}$-avoiding iff it has no fixed point (which follows from the 321-avoiding property and the fact that π is an involution). $\square$

Pattern	Sequence	Sloane
$2\dot{1}$	$d_1 = 0$ and $d_n = \lvert I_n \rvert - \sum_{r=1}^{n-1} d_r \cdot \lvert I_{n-1-r} \rvert$	new
$\{2\dot{1}, 123\}$	$\binom{n}{\lfloor \frac{n}{2} \rfloor} - 2$	new
$\{2\dot{1}, 132\}, \{2\dot{1}, 213\}$	$\binom{n-1}{\lfloor \frac{n-2}{2} \rfloor}$	$A037952$
$\{2\dot{1}, 231\}, \{2\dot{1}, 312\}$	Fibonacci	$A000045$
$\{2\dot{1}, 321\}$	$c_{\frac{n}{2}}$ if n is even; and 0 otherwise	$A126120$

Table 3: Enumeration for the sets of involutions avoiding $2\dot{1}$ and one classical pattern of length 3

3.2 $2\dot{1}$ and two classical patterns of length 3

In this part we give exhaustive results about the cardinalities of the sets of permutations avoiding $2\dot{1}$ and two classical patterns of length 3. For $n \geq 5$, we have $|S_n(2\dot{1}, 123, 321)| = |S_n(123, 321)| = 0$.

Theorem 8 *The set $S_n(2\dot{1}, 123, 213)$ is enumerated by $2^{n-1} - 1$ for $n \geq 2$ (see A000225).*

Proof. It is well known that $S_n(123, 213)$ is enumerated by 2^{n-1}. Theorem 3 proves that $S_n(2\dot{1}, 123)$ is the set $S_n(123)$ except the two permutations $\pi = 1n(n-1)\ldots 2$ and $\pi' = (n-1)\ldots 21n$. However only π belongs to $S_n(213)$. Therefore we deduce $|S_n(2\dot{1}, 123, 213)| = |S_n(123, 213)| - 1 = 2^{n-1} - 1$. $\square$

Theorem 9 *The set $S_n(2\dot{1}, 132, 213)$ is enumerated by $2^{n-1} - 1$ for $n \geq 2$ (see A000225).*

Proof. Let π be a permutation in $S_n(2\dot{1}, 132, 213)$. Using the proof of Theorem 6, π can be decomposed as follows: $\pi = k\delta\gamma$ where $k \neq 1$, δ (resp. γ) is a sequence made up of the values in $\{(k+1), \ldots, n\}$ (resp. $\{1, 2, \ldots, (k-1)\}$) such that δ and γ avoid the pattern 213. The fact that $\pi \in S_n(132)$ implies that $\delta \in S_n(132, 213)$ and γ does not contain any pattern 21. As $|S_n(132, 213)| = 2^{n-1}$, we deduce that the cardinality of $S_n(2\dot{1}, 132, 213)$ is exactly $\sum_{i=1}^{n-1} 2^{i-1} = 2^{n-1} - 1$. $\square$

Theorem 10 *The set $S_n(2\dot{1}, 123, 231)$ is enumerated by $\frac{(n-1)n}{2} - 1$ for $n \geq 3$ (see A000096).*

Proof. It is well known that $S_n(123, 231)$ is enumerated by the polygonal number $\frac{(n-1)n}{2} + 1$. Theorem 3 shows that $S_n(2\dot{1}, 123)$ is the set $S_n(123)$ except the two permutations $\pi = 1n(n-1)\dots 2$ and $\pi' = (n-1)\dots 21n$. Therefore we deduce $|S_n(2\dot{1}, 123, 231)| = |S_n(123, 231)| - 2 = \frac{(n-1)n}{2} - 1$. □

Theorem 11 *The set $S_n(2\dot{1}, 321, 132)$ is enumerated by $n-1$ for $n \geq 2$ (see A007953).*

Proof. Let π be a permutation in $S_n(2\dot{1}, 321, 132)$. As π avoids $2\dot{1}$, $\pi_1 \neq 1$ and $\pi \neq 12\dots n$. Let us assume that $\pi_i = 1$ for some $i \geq 2$. We have $\pi_{i-1} > \pi_i = 1$. As π avoids 321, $1 < \pi_1 < \pi_2 < \dots < \pi_{i-1}$ and $\pi_{i+1} < \pi_{i+2} < \dots < \pi_n$. As π avoids 132, the values $\pi_1, \dots, \pi_{i-1}$ are necessarily consecutive. As π avoids $2\dot{1}$, π does not contain any strong fixed point, which means that there does not exist any value greater than π_{i-1} to the right of 1 in π. Thus $\pi_{i-1} = n$ and $\pi = k(k+1)\dots n12\dots(k-1)$ with $k \neq 1$. Therefore $|S_n(2\dot{1}, 321, 132)| = n-1$ for $n \geq 2$. □

Theorem 12 *The set $S_n(2\dot{1}, 132, 231)$ is enumerated by 2^{n-2} for $n \geq 2$ (see A034008).*

Proof. Let π be a permutation in $S_n(2\dot{1}, 132, 231)$. The case $\pi_1 \neq n$ does not occur. To see this, assume for a contradiction that $\pi_1 = k \leq n-1$. As π avoids 231 and 132 we necessarily have $\pi_n = n$ which is a contradiction with the avoidance of $2\dot{1}$. Thus we have $\pi_1 = n$. Moreover if π' is the permutation in S_{n-1} obtained from π by deleting its first value n then π' also belongs to $S_{n-1}(132, 231)$. Conversely, if we add n on the left from a permutation in $S_{n-1}(132, 231)$, the obtained permutation obviously belongs to $S_n(2\dot{1}, 132, 231)$. Therefore we have $|S_n(2\dot{1}, 132, 231)| = |S_{n-1}(132, 231)| = 2^{n-2}$. □

Theorem 13 *The set $S_n(2\dot{1}, 132, 312)$ is enumerated by 2^{n-2} for $n \geq 2$ (see A034008).*

Proof. This proof is similar to that of Theorem 12 since a permutation $\pi \in S_n(2\dot{1}, 132, 312)$ can be written $\pi = \pi'1$ where π' avoids 132 and 312. As $S_n(132, 312)$ is enumerated by 2^{n-1}, the set $S_n(2\dot{1}, 132, 312)$ also is. □

Theorem 14 *The set $S_n(2\dot{1}, 321, 312)$ is enumerated by the Fibonacci sequence defined by $fib_2 = fib_3 = 1$ and $fib_n = fib_{n-1} + fib_{n-2}$ for $n \geq 4$ (see A000045).*

Proof. Let π be a permutation in $S_n(2\dot{1}, 321, 312)$. As π avoids $2\dot{1}$ we have $\pi_n \neq n$. Moreover, π avoids 312 and 321 implies that π_{n-1} necessarily equals to n and we also have either $\pi_{n-2} = n-1$ or $\pi_n = n-1$. If $\pi_n = n-1$ then the permutation π' obtained from π by deleting the two last values also belongs to $S_{n-2}(2\dot{1}, 321, 312)$. On the other hand, if $\pi_{n-2} = n-1$ then the permutation π' obtained from π by deleting $\pi_{n-1} = n$ belongs to $S_{n-1}(2\dot{1}, 321, 312)$. Conversely, each permutation $\pi \in S_n(2\dot{1}, 321, 312)$ can be obtained from a permutation π' as previous. Therefore, $|S_n(2\dot{1}, 321, 312)| = |S_{n-1}(2\dot{1}, 321, 312)| + |S_{n-2}(2\dot{1}, 321, 312)|$ which defines the Fibonacci sequence. □

Theorem 15 *The set $S_n(2\dot{1}, 312, 231)$ is enumerated by the Fibonacci sequence defined by $fib_2 = fib_3 = 1$ and $fib_n = fib_{n-1} + fib_{n-2}$ for $n \geq 4$ (see A000045).*

Proof. Let π be a permutation in $S_n(2\dot{1}, 312, 231)$. As π avoids $2\dot{1}$ we have $\pi_n \neq n$. Moreover, π avoids 312 and 231 implies that there exists $i \geq 1$ such that $\pi_i = n$ and $\pi_{i+1} = n-1$. If $i = n-1$ then the permutation π' obtained from π by deleting the two last value $n(n-1)$ also belongs to $S_{n-2}(2\dot{1}, 312, 231)$. On the other hand, if $i < n-1$ then the permutation π' obtained from π by deleting $\pi_{n-1} = n$ belongs to $S_{n-1}(2\dot{1}, 312, 231)$. Conversely, each permutation $\pi \in S_n(2\dot{1}, 321, 312)$ can be obtained from a permutation π' as previous. Therefore, $|S_n(2\dot{1}, 312, 231)| = |S_{n-1}(2\dot{1}, 312, 231)| + |S_{n-2}(2\dot{1}, 312, 231)|$ which defines the Fibonacci sequence. □

Pattern	Sequence	Sloane
$\{2\dot{1}, 123, 321\}$	$0, 1, 2, 4, 0, 0, 0 \ldots$	
$\{2\dot{1}, 123, 132\}, \{2\dot{1}, 123, 213\}, \{2\dot{1}, 132, 213\}$	$2^{n-1} - 1$	$A000225$
$\{2\dot{1}, 123, 231\}, \{2\dot{1}, 123, 312\}$	$\frac{(n-1)n}{2} - 1$	$A000096$
$\{2\dot{1}, 132, 231\}, \{2\dot{1}, 132, 312\}$, $\{2\dot{1}, 213, 231\}, \{2\dot{1}, 213, 312\}$	2^{n-2}	$A034008$
$\{2\dot{1}, 132, 321\}, \{2\dot{1}, 213, 321\}$	$n-1$	$A007953$
$\{2\dot{1}, 231, 312\}, \{2\dot{1}, 231, 321\}, \{2\dot{1}, 312, 321\}$	Fibonacci	$A000045$

Table 4: Exhaustive results for $2\dot{1}$ and two classical patterns of length 3.

3.3 $2\dot{1}$ and two classical patterns of length 3 or 4

Below we present three results for permutations avoiding $2\dot{1}$ and two patterns of length greater than two. We also present an open question for permutations avoiding the patterns $\{2\dot{1}, 231, 4123\}$.

Theorem 16 *The set $S_n(2\dot{1}, 231, 4321)$ is enumerated by the Pell numbers (see A000129).*

Proof. Let π be a permutation in $S_n(2\dot{1}, 231, 4321)$. We necessarily have $n-2 \leq \pi_n \leq n-1$. Indeed, Theorem 1 implies that $\pi_n \neq n$; moreover if $\pi_n \leq n-3$ then (as π avoids 231) all values in the interval $I = [\pi_n + 1, n-1]$ appear in decreasing order to the right of n in π. As π avoids 4321, the set I contains at most one element which produces a contradiction with $\pi_n \leq n-3$. Thus we have either $\pi_n = n-1$ or $\pi_n = n-2$.

Let π' be a permutation in $S_{n-1}(2\dot{1}, 231, 4321)$ and π be the permutation in S_n obtained from π' by increasing by one the entries $n-1$ and $n-2$ (resp. the entry $n-1$) and by appending $n-2$ (resp. $n-1$) on the right. Then $\pi \in S_n(2\dot{1}, 231, 4321)$. Conversely, a permutation $\pi \in S_n(2\dot{1}, 231, 4321)$ such that $\pi_n = n-2$ (resp. $\pi_n = n-1$ and $\pi_{n-1} \neq n$) can be obtained by the previous construction from a permutation $\pi' \in S_{n-1}(2\dot{1}, 231, 4321)$. On the other hand, let π' be a permutation in $S_{n-2}(2\dot{1}, 231, 4321)$ and π be the permutation in S_n obtained from π' by appending the sequence $n(n-1)$ on the right.

Then $\pi \in S_n(2\dot{1}, 231, 4321)$. Conversely, a permutation $\pi \in S_n(2\dot{1}, 231, 4321)$ such that $\pi_n = n-1$ and $\pi_{n-1} = n$ can be obtained from $\pi' \in S_{n-2}(2\dot{1}, 231, 4321)$ by this construction. Therefore we have $|S_n(2\dot{1}, 231, 4321)| = 2|S_{n-1}(2\dot{1}, 231, 4321)| + |S_{n-2}(2\dot{1}, 231, 4321)|$ anchored by $|S_2(2\dot{1}, 231, 4321)| = 1$ and $|S_3(2\dot{1}, 231, 4321)| = 2$. We obtain the Pell numbers (see A000129). □

Theorem 17 *The set $S_n(2\dot{1}, 321, 2143)$ is enumerated by $2^{n-1} - n + 1$ (see A000325), i.e., an instance of the Eulerian numbers plus one.*

Proof. Let π be a permutation of $S_n(2\dot{1}, 321, 2143)$. We will say that π has a descent in position $i \in [n]$ if $\pi_i > \pi_{i+1}$. We distinguish two cases: (a) π contains only one descent; and (b) π contains at least two descents (notice that π cannot be the identity permutation since id_n does not avoid $2\dot{1}$). First, we will enumerate n-length permutations verifying the condition (a). Let us assume that the only descent occurs at position $i \geq 1$. We necessarily have $\pi_{i+1} = 1$. To see this, assume to the contrary that $\pi_{i+1} \geq 2$. As π avoids 321, the value 1 is on the left of π_i; as π contains only one descent $\pi_1 = 1$ holds which contradicts that π avoids $2\dot{1}$. Moreover, we have $\pi_i = n$. Otherwise, if $\pi_i \neq n$ then n either is the last entry or just before a value smaller than it. This is a contradiction. The permutation π can be written $\pi = \delta n 1 \gamma$. where δ (resp. γ) is an increasing subsequences of $[\pi_1, n-1]$ (resp. $[2, n-1]$). The cardinality of such permutations is $\sum_{i=1}^{n-1} \binom{n-2}{k-1} = 2^{n-2}$.

Now let us examine the case (b). The permutation π contains at least two descents. Let i be the position of the leftmost descent ($\pi_i > \pi_{i+1}$). As π avoids $2\dot{1}$ we have $\pi_1 \neq 1$ and $\pi_n \neq n$; as π avoids 321 we have $\pi_{i+1} = 1$; as π avoids 2143, if $\pi_j > \pi_{j+1}$ for $j > i+1$ then $\pi_{j+1} < \pi_1$ and $\pi_n = \pi_1 - 1$. These conditions define a particular structure for π: $\pi = k \ldots (k+i-1) 1 \delta n \ell \ldots k-1$ for some ℓ, $2 \leq \ell \leq k-1$. Therefore these permutations are enumerated by: $\sum_{k=3}^{n-1} \sum_{i=1}^{n-k} \binom{n-i-2}{k-3} = 2^{n-2} - n + 1$. Thus $|S_n(2\dot{1}, 321, 2143)| = 2^{n-1} - n + 1$. □

Theorem 18 *The set $I_n(2\dot{1}, 231, 4321)$ is enumerated by the Padovan sequence (see A000931).*

Proof. The proof of Theorem 16 shows the form of a permutation $\pi \in S_n(2\dot{1}, 231, 4321)$. Indeed, we have either $\pi = \delta n(n-1)$ or $\pi = \delta n(n-1)(n-2)$. We deduce $|I_n(2\dot{1}, 231, 4321)| = |I_{n-2}(2\dot{1}, 231, 4321)| + |I_{n-3}(2\dot{1}, 231, 4321)|$ anchored by $|I_1(2\dot{1}, 231, 4321)| = 0$ and $|I_2(2\dot{1}, 231, 4321)| = 1$ which gives the Padovan sequence (see A000931). □

Now we present an open question about the Pisot sequence $a(n)$ defined by $a(4) = 5$, $a(5) = 11$ and $a(n+1)$ is the nearest integer to $\frac{a(n)^2}{a(n-1)}$.

Problem 1 *The set $S_n(2\dot{1}, 231, 4123)$ is enumerated by a sequence $b(n)$ that takes values $5, 11, 24, 52, 113, 246, 536, 1168$ for $4 \leq n \leq 11$. Is it true that $b(n)$ is the Pisot sequence for $n \geq 4$ (see A021008)?*

Pattern	Sequence	Sloane
$\{2\dot{1}, 231, 4321\}$	Pell	$A000129$
Involutions avoiding $\{2\dot{1}, 231, 4321\}$	Padovan	$A000931$
$\{2\dot{1}, 321, 2143\}$	$2^{n-1} - n + 1$	$A000325$

Table 5: Permutations avoiding $2\dot{1}$ and two classical patterns of length 3 or 4

4 Avoiding a dotted pattern of length 3

In this section we investigate the avoidance of one dotted pattern of length 3, possibly along with other classical patterns. We obtain several results when we avoid one dotted pattern of length 3 and one or more classical patterns. Unfortunately, we could not determine the cardinality of $S_n(\dot{\tau})$ where $\dot{\tau}$ is a 3-length dotted pattern of the form $\tau_1\tau_2\dot{\tau}_3$. Thus we present three open problems:

Problem 2 *The set $S_n(23\dot{1})$ is enumerated by the sequence $a(n,0)$ that takes values $1, 1, 2, 5, 17, 71, 357, 2101$ for $n \geq 1$ (see A101897). Is it possible to prove that this sequence is recursively defined by: $a(n,0) = \sum_{k=1}^{n} (-1)^{k-1} \cdot a(n,k)$ where $a(n,k) = \sum_{j=0}^{n-k} a(n-k,j) \cdot a(j+k-1,k-1)$ for $0 < k \leq n$ anchored by $a(0,0) = 0$.*

Problem 3 *Is it possible to obtain the cardinalities of the sets $S_n(12\dot{3})$? The first values are $1, 1, 2, 7, 35, 218, 1598$ for $n \geq 1$. This sequence does not appear in the Sloane Encyclopedia [14].*

Problem 4 *The set $S_n(13\dot{2})$ is enumerated by the sequence $b(n)$ that takes values $1, 1, 2, 4, 13, 58, 299, 180$ for $n \geq 1$. Is it possible to prove that this sequence is recursively defined by: $b(n+1) = a(n,0) - b(n)$ where $a(n,0)$ is defined in Problem 2?*

4.1 Avoiding $23\dot{1}$ and 321 - Catalan

Before presenting the main results we provide two basic technical lemmas.

Lemma 4 *Let $n \geq 1$ be an integer. A permutation π belongs to $S_n(321)$ if and only if for every $i \geq 1$, we have $\pi_i = \min\{\pi_i, \pi_{i+1}, \ldots, \pi_n\}$ or $\pi_i = \max\{\pi_1, \pi_2, \ldots, \pi_i\}$.*

Proof. Let π be an n-length permutation avoiding 321 and i be an integer such that $\pi_i > \pi_{i+1}$. Since π avoids 321, it does not contain any value greater than π_i before π_i. Thus $\pi_i = \max\{\pi_1, \pi_2, \ldots, \pi_i\}$. A similar argument gives $\pi_i = \min\{\pi_i, \pi_{i+1}, \ldots, \pi_n\}$ if $\pi_i < \pi_{i+1}$. Conversely, if π contains the pattern 321 then there are $i < j < k$ such that $\pi_i > \pi_j > \pi_k$. Thus π_j is neither the maximum of elements to its left and nor the minimum of elements to its right which gives the result. $\square$

Lemma 5 *Let $n \geq 1$ be an integer and π be a permutation in $S_n(321)$. Then π avoids $23\dot{1}$ if and only if $\pi_i \neq i-1$ for $i \geq 2$.*

Proof. Let $\pi \in S_n(23\dot{1}, 321)$. Let us assume that $\pi \neq id_n$ avoids $23\dot{1}$ and there exists $i \geq 2$ such that $\pi_i = i-1$. Lemma 4 states that either $\pi_i = i-1 = \min\{\pi_i, \pi_{i+1}, \ldots, \pi_n\}$ or $\pi_i = i-1 = \max\{\pi_1, \pi_2, \ldots, \pi_i\}$. The first case implies that there exists $j < i$ such that $\pi_j > i-1$. Therefore the subsequence $\pi_j\pi_i$ cannot be extended into the pattern 231 which is a contradiction. The second case does not occur since $i-1$ cannot be the maximum of i positive integers. Conversely, if π contains $23\dot{1}$, then there exists $i < j$ such that $\pi_i > \pi_j$ where the subsequence $\pi_i\pi_j$ cannot be extended in π to a 231 pattern. As π avoids 321, all entries before π_j are smaller than π_j. Thus $\pi_j \geq j$ and there exists $j \geq 2$ such that $\pi_j \neq j-1$. □

Theorem 19 *The set $S_n(23\dot{1}, 321)$ is enumerated by the $(n-1)$th Catalan number $c_{n-1} = \frac{1}{n}\binom{2(n-1)}{n-1}$ (see A000108).*

Proof. Let π be a permutation in $S_n(23\dot{1}, 321)$. For $k \in [n+1]\backslash\{n\}$, we define the permutation $\delta \in S_{n+1}$ by: $\delta_{n+1} = k$, and $\delta_i = \pi_i + 1$ if $\pi_i \geq k$, and $\delta_i = \pi_i$ otherwise.

It is straightforward to verify that if $k > \pi_n$ then $\delta \in S_{n+1}(23\dot{1}, 321)$ (Lemma 5 excludes the case $k = n$). We have the same remark when $\pi_n = n$ and $k \leq n-1$. Conversely, each permutation $\delta \in S_{n+1}(23\dot{1}, 321)$ can uniquely be obtained by this construction for some k and $\pi \in S_n(23\dot{1}, 321)$ with $k \neq n$. Indeed, Lemma 5 implies $\delta_{n+1} \neq n$. Now let us assume that there exists $i \leq n$ such that $\delta_i = i > \delta_{n+1}$. As δ avoids 321, we have $\delta_j < i$ for $j < i$. Thus the $(n+1-i)$ values $(i+1), \ldots, (n+1)$ are necessarily on the right of $\delta_i = i$, and by considering the value $\delta_{n+1} < i$, there are $(n+2-i)$ values on the right of δ_i which cannot be possible. Consequently, this case does not occur. The permutation π verifies $\pi_j \neq j-1$ for $j \geq 2$. Lemma 5 implies that $\pi \in S_n(23\dot{1}, 321)$. Thus each permutation π such that $\pi_n = k$, $k \leq n-2$ (resp. $k = n$), produces $(n-k)$ (resp. n) permutations of size $(n+1)$ verifying $\pi_{n+1} = k+1, \ldots, n-1, n+1$ (resp. $\pi_{n+1} = 1, 2, \ldots, n-1, n+1$). If we label by (k) a permutation π such that $\pi_n = n-k$ then such a permutation produces k permutations of size $n+1$ labeled $(2), (3), \ldots, (k+1)$ and 12 is labeled (2). This is the definition of succession rules for Catalan numbers (see the method ECO [1, 2]). □

Theorem 20 *The set $S_n(23\dot{1}, 321, 213)$ is enumerated by the $(n-1)$th central polygonal sequence $\binom{n-1}{2} + 1$ (or Lazy Caterer's sequence, see A000124).*

Proof. Let $\pi \in S_n(23\dot{1}, 321, 213)$ and $\pi \neq id_n$. We consider the smallest i such that there exists $j > i$ verifying $\pi_i > \pi_j$. As π avoids 321 and 213, we immediately have $\pi_k < \pi_j$ for $k < i$; $\pi_k > \pi_i$ for $i < k < j$; and $\pi_j < \pi_k < \pi_i$ for $k > j$. As π avoids $23\dot{1}$, there exists at least one value between the entries π_i and π_j, *i.e.*, $i < j-1$. Conversely, each permutation verifying these properties belongs to $S_n(23\dot{1}, 321, 213)$. Therefore, if we also consider id_n then the cardinality of $S_n(23\dot{1}, 321, 213)$ is $\sum_{j=3}^{n}\sum_{i=1}^{j-2} 1 = \binom{n-1}{2} + 1$. □

4.2 Avoiding $23\dot{1}$ and 132 - Motzkin

Theorem 21 *The set $S_n(23\dot{1}, 132)$ is enumerated by the $(n-1)$th Motzkin number (see A001006).*

Proof. Let M_n be the set $S_n(23\dot{1}, 132)$ and m_n its cardinality. Let π be a permutation in M_n. We write π as follows: $\pi = \delta n \gamma$ where δn and γ are two subsequences of $[n]$ (γ is not empty). See Figure 1 for an illustration of the decomposition. As π avoids 132, the set of values in δn necessarily constitutes an interval $[s, n]$ for some $s \leq n$. Consequently, γ is a permutation in S_{s-1}. As π avoids $23\dot{1}$ and 132, $\gamma \in M_{s-1}$ and δn can be viewed as a permutation in M_{n-s+1}. Conversely, such a permutation $\pi = \delta n \gamma$ verifying the above conditions on δn and γ belongs to M_n. We deduce that $m_n = \sum_{s=1}^{n-1} m_{n-s} m_{s-1}$ anchored by $m_0 = m_1 = 1$. □

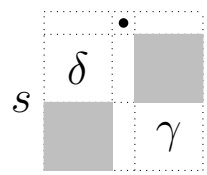


Figure 1: The special structure of a permutation in $S_n(23\dot{1}, 132)$

4.3 Avoiding $23\dot{1}$, 312 and 213 - Fibonacci

Theorem 22 *The set $S_n(23\dot{1}, 312, 213)$ is enumerated by the Fibonacci sequence (see A000045).*

Proof. Let π be a permutation in $S_n(23\dot{1}, 312, 213)$. As π avoids 312 and 213, we have either $\pi_1 = 1$ or $\pi_n = 1$. If $\pi_n = 1$ then the permutation π' obtained from π by deleting $\pi_n = 1$ and by decreasing by one all other values also belongs to $S_{n-1}(23\dot{1}, 312, 213)$. Moreover, as π avoids $23\dot{1}$ we cannot have $\pi'_{n-1} = 1$ (otherwise $\pi_{n-1} = 2$ and $\pi_n = 1$ contradicts that π avoids $23\dot{1}$). Thus we have $\pi'_1 = 1$ and thus $\pi_1 = 2$. Let π'' be the permutation obtained from π' by deleting the first entry 1 and by decreasing by one all other values, then $\pi'' \in S_{n-2}(23\dot{1}, 312, 213)$. Conversely, if $\pi'' \in S_{n-2}(23\dot{1}, 312, 213)$ then the permutation π obtained from π'' by appending 2 (resp. 1) at the left (resp. at the right) then $\pi \in S_n(23\dot{1}, 312, 213)$. On the other hand, if $\pi_1 = 1$ then the permutation π' obtained from π by deleting $\pi_1 = 1$ and by decreasing by one all other values also belongs to $S_{n-1}(23\dot{1}, 312, 213)$. Conversely, each permutation $\pi \in S_n(23\dot{1}, 312, 213)$ can be obtained from a permutation $\pi' \in S_{n-1}(23\dot{1}, 312, 213)$ by adding the entry one on the left and by increasing by one all other values. Finally, if $a(n) = |S_n(23\dot{1}, 312, 213)|$ then $a(n) = a(n-1) + a(n-2)$ anchored by $a(1) = 1$, $a(2) = 1$. □

4.4 Avoiding $12\dot{3}$, 213 and 231 - Binary

Theorem 23 *The set $S_n(12\dot{3}, 213, 231)$ is enumerated by the binary sequence 2^{n-2} (see A034008).*

Proof. Let π be a permutation in $S_n(12\dot{3}, 213, 231)$. As π avoids 213 and 231, such a permutation necessarily has: either $\pi_i = \max\{\pi_1, \ldots, \pi_i\}$ or $\pi_i = \max\{\pi_1, \ldots, \pi_i\}$. Thus we have $\pi_1 = 1$ or $\pi_1 = n$. If $\pi_1 = 1$ then $\pi_2 = 2$ since π avoids $12\dot{3}$. Consequently each permutation π can be written as either $\pi = n\pi'$ or $\pi = 12\pi''$ where π' (resp. π'') belongs to $S_{n-1}(12\dot{3}, 213, 231)$ (resp. $S_{n-2}(213, 231)$). Finally, by induction we obtain $|S_n(12\dot{3}, 213, 231)| = 2^{n-3} + 2^{n-3} = 2^{n-2}$. □

4.5 Avoiding $\dot{2}13$, 231 - Riordan

Theorem 24 *The cardinality of the set $S_n(\dot{2}13, 231)$ is given by the generating function $\frac{\left(1-z-\sqrt{1-2\,z-3\,z^2}\right)z}{1+z+\sqrt{1-2\,z-3\,z^2}}$ (see A005043 for the Riordan sequence).*

Proof. It is well known that a permutation $\pi \in S_n(231)$ can be written $k\delta\gamma$ where $k \geq 1$, $\delta \in S_{k-1}(231)$ and γ is a sequence of values in $\{k+1, \ldots, n\}$ that also avoids 231. Such a permutation can be illustrated by a rooted binary tree T with n nodes as follows: T is the rooted binary tree with its left subtree T_ℓ (resp. right subtree T_r) represents δ (resp. γ considered as a permutation in $S_{n-k}(231)$). Using the classical pre-order bijection between binary trees and Dyck paths the permutation π can be viewed as a Dyck path. See Figure 2 for an example of the correspondence between $\pi \in S_n(231)$, its binary tree with n nodes, and its associated Dyck path.

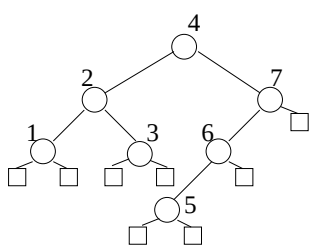


Figure 2: A permutation $\pi = 4213765 \in S_7(\dot{2}13, 231)$ with its corresponding binary tree and the associated Dyck path $UUUDDUDDUUUDDD$ where $U = (1, 1)$ and $D = (1, -1)$.

As π avoids $\dot{2}13$, the sequence γ also avoids $\dot{2}13$; $k \neq 1$ and δ and γ are not empty. Moreover the avoidance of $\dot{2}13$ implies that: (a) each subsequence $\pi_i\pi_j$ of π with $\pi_i < \pi_j$ can be expanded into a 213 pattern; and (b) each subsequence $\pi_i\pi_j$ of π with $\pi_i > \pi_j$ can be expanded into a 213 pattern. These two conditions also hold for the subsequence γ. Moreover the sequence $k\delta$ always verifies (b) since γ contains the value n. Thus the

permutation π belongs to $S_n(\dot{2}13, 231)$ if and only if γ avoids $\dot{2}13$ and 231, and δ verifies (a).

Now if we consider these conditions in the context of Dyck path (via the above bijection), the condition δ *verifies* (a) means that the Dyck path corresponding to the left subtree T_ℓ (of size at least 2) does not contain any occurrence of UDU. To see this, assume for a contradiction that T_ℓ contains an occurrence UDU. Then π verifies $\pi_i = k$ and $\pi_{i+1} = k+1$ for some i and k such that $\pi_j > k$ if $j < i$. Thus the subsequence $k(k+1)$ cannot be expanded into a 213-pattern in π which contradicts the condition δ. So, these Dyck paths are enumerated by the Motzkin numbers (see [14], A001006). Therefore, the cardinalities of $S_n(\dot{2}13, 231)$ is given by the generating function: $A(z) = z^2 M(z)(z + A(z))$ where $M(z)$ is the generating function of the Motzkin numbers. Thus we obtain

$$A(z) = \frac{\left(1 - z - \sqrt{1 - 2\,z - 3\,z^2}\right) z}{1 + z + \sqrt{1 - 2\,z - 3\,z^2}},$$

which is equal to $z(R(z) - 1)$ where

$$R(z) = \frac{2}{\left(1 + z + \sqrt{1 - 2\,z - 3\,z^2}\right)}$$

is the well-known Riordan generating function. □

From this previous proof, we directly deduce the following corollary.

Corollary 1 *The cardinality of the set $S_n^w(\bar{2}13) \cap S_n(231)$ is given by the Motzkin numbers (see A001006).*

4.6 Avoiding $23\dot{1}$, 2413 and 3142 - Catalan

Theorem 25 *The set $S_n(23\dot{1}, 2413, 3142)$ is enumerated by the $(n-1)$th Catalan number.*

Proof. Let π be a permutation in $S_n(23\dot{1}, 2413, 3142)$. As π avoids the two patterns 2413 and 3142, π is separable; *i.e.*, π can be written either (a) $\pi = \delta\gamma$ where $\delta \in S_k$ for some $k \leq n-1$, or (b) $\pi = \delta\gamma$ where $\gamma \in S_k$ for some $k \leq n-1$. For each case, we assume that δ is minimal in the sense that δ does not belong to the considered case. Moreover, π avoids $23\dot{1}$ implies that δ and γ also avoid $23\dot{1}$; and for the case (b), δ is not reduced to one element. Thus the structure of permutations in the cases (a) or (b) induces the two following equalities.

Let $A(z)$ (resp. $B(z)$) be the generating function for permutations avoiding $23\dot{1}$ and verifying (a) (resp. verifying (b)); let $F(z) = A(z) + B(z) + 1$ the generating function for the set $S_n(23\dot{1}, 2413, 3142)$.

$$\begin{cases} A(z) &= zF(z) + B(z)(F(z) - 1) \text{ and} \\ B(z) &= (A(z) - z)(F(z) - 1). \end{cases}$$

If we sum these two equalities we obtain:

$$F(z)^2 - 3F(z) + z + 2 = 0.$$

Thus we conclude $F(z) = \frac{3-\sqrt{1-4z}}{2}$ which is exactly $1+zC(z)$ where $C(z)$ is the well-known Catalan sequence. $\square$

Pattern	Sequence	Sloane
$\{23\dot{1}, 321\}$,	c_{n-1}	$A000108$
$\{23\dot{1}, 321, 213\}$	$\binom{n-1}{2} + 1$	$A000124$
$\{23\dot{1}, 132\}$	Motzkin	$A001006$
$\{23\dot{1}, 312, 213\}, \{23\dot{1}, 312, 132\}$	Fibonacci	$A000045$
$\{31\dot{2}, 132\}$	Riordan	$A005043$
$\{13\dot{2}, 213, 231\}$	2^{n-2}	$A034008$
$\{23\dot{1}, 2413, 3142\}$,	c_{n-1}	$A000108$

Table 6: Some results for a dotted pattern of length 3 and other classical patterns.

Acknowledgements

I would like thank the anonymous referee for their constructive comments.

References

[1] E. Barcucci, A. Del Lungo, E. Pergola and R. Pinzani. ECO: a methodology for the Enumeration of Combinatorial Objects. *Journal of Difference Equations and Applications*, 5:435-490, 1999.

[2] E. Barcucci, A. Del Lungo, E. Pergola and R. Pinzani. From Motzkin to Catalan permutations. *Discrete Mathematics*, 217:33-49, 2000.

[3] J.L. Baril and P.T. Do. ECO-generation for some restricted classes of compositions. *Submitted*, 2011.

[4] M. Bóna. Combinatorics of permutations. Chapman&Hall, 2004, Chap. 4. 129-173.

[5] S. Butler and M. Bousquet-Melou. Forest-like permutations. *Ann. Combin.*, 11:335-354, 2007.

[6] E. Deutsch, A. Robertson and D. Saracino. Refined restricted involutions. *European Journal of Combinatorics*, 28(1):481-498, 2007.

[7] O. Guibert. Combinatoires des permutations à motifs exclus en liaison avec mots, cartes planaires et tableaux de young. Thèse de l'université de Bordeaux I, 1996.

[8] S. Kitaev and T. Mansour. A survey on certain pattern problems. University of Kentucky research report 2003-09, available at **http://www.ms.uky.edu/∼math/MAreport/survey.ps**.

[9] D.E. Knuth. The art of computer programming. Vol. 1, Addison-Wesley, Reading, MA, 1973.

[10] T. Mansour. Permutations avoiding a pattern from S_k and at least two patterns from S_3. *Ars. Combin.*, 62:227-239, 2002.

[11] L. Pudwell. Enumeration schemes for permutations avoiding barred patterns. *The electronic journal of combinatorics*, 17, 2010.

[12] A. Robertson, D. Saracino and D. Zeilberger. Refined restricted permutations. *Ann. Comb.*, 6:427-444, 2003.

[13] R. Simion and F.W. Schmidt. Restricted permutations. *European Journal of Combinatorics*, 6:383–406, 1985.

[14] N.J.A Sloane. On-Line Encyclopedia of Integer Sequences. Published electronically at **http://www.research.att.com/∼njas/sequences/**.

[15] J. West. Permutations with forbidden subsequences and stack-sortable permutations. *Ph.D. Thesis*, MIT, 1990.

[16] J. West. Sorting twice through a stack. *Theoretical Computer Sciences*, 117:303-313, 1993.

[17] J. West. Generating trees and forbidden subsequences. *Proceeding of sixth FPSAC*, 441-450, 1994.

[18] J. West. Generating trees and the Catalan and Schröder numbers. *Discrete Math.*, 146:247-262, 1995.

[19] A. Woo and A. Yong. "When is a Schubert variety Gorenstein?". *Adv. Math.*, 207(1):205-220, 2006.

Vertex Distinguishing Edge- and Total-Colorings of Cartesian and other Product Graphs

Jean-Luc Baril*, Hamamache Kheddouci† and Olivier Togni*

Abstract

This paper studies edge- and total-colorings of graphs in which (all or only adjacent) vertices are distinguished by their sets of colors. We provide bounds for the minimum number of colors needed for such colorings for the Cartesian product of graphs along with exact results for generalized hypercubes. We also present general bounds for the direct, strong and lexicographic products.

Keywords : graph, edge-coloring, vertex-distinguishing, adjacent vertex-distinguishing, total coloring, total adjacent vertex-distinguishing, graph products.

1 Introduction

All the graphs we deal with are simple, finite and with no component K_2, where K_n stands for the complete graph of order n. Let $G = (V, E)$, be a graph with vertex set V and edge set E. An edge between vertex x and vertex y is denoted by xy. Let $\Delta(G)$ be the maximum degree of the graph.

A *proper edge-coloring* c is a mapping from E to $\mathbb{N}$ such that edges incident with the same vertex receive distinct values (or colors). For any vertex x of G, let $S(x)$ denote the set of the colors of all edges incident to x (if necessary, we write $S_c(x)$ to indicate which coloring is used). A proper edge coloring is said to be

- *vertex distinguishing (VD)* if $S(x) \neq S(y), \forall x, y \in V, x \neq y$;
- *adjacent vertex distinguishing (AVD)* if $S(x) \neq S(y), \forall xy \in E$.

*LE2I, UMR 5158 CNRS, Université de Bourgogne, BP 47870, 21078 Dijon cedex, France, {barjl, otogni}@u-bourgogne.fr

†LIESP, Université Claude Bernard Lyon1, 843, Bd. du 11 novembre 1918, 69622 Villeurbanne Cedex France, hkheddou@bat710.univ-lyon1.fr

A total coloring of a graph G is a mapping from $V \cup E$ to $\mathbb{N}$ such that neighboring elements receive distinct colors. For a total coloring, let $S(x)$ be the set of the colors of all edges incident to x plus the color of x: $S(x) = \{c(e)|e = xy\} \cup \{c(x)\}$. A *total adjacent vertex distinguishing (TAVD) coloring* is a total coloring satisfying $S(x) \neq S(y), \forall xy \in E$.

The minimum number of colors among all VD-colorings, AVD-colorings and TAVD-colorings respectively of a graph G will be called the *VD-chromatic index*, *AVD-chromatic index* and *TAVD-chromatic index* denoted by $\chi'_s(G)$, $\chi'_a(G)$ and $\chi''_a(G)$ respectively.

The notation $\chi'(G)$, $\chi(G)$ and $\chi''(G)$ is used to represent respectively the chromatic index, the chromatic number and the total chromatic number of G, as usual. A coloring using the least number of colors with respect to the given constraints will be called a *minimal* coloring.

The notion of VD-coloring was introduced in [BS97] and independently in [ČHS96] where $\chi'_s(G)$ is called the *observability*.

A lower bound for the VD-chromatic index is given by $\pi(G) = \min\{k : \binom{k}{d} \geq n_d \text{ for } 1 \leq d \leq \Delta(G)\}$ where n_d is the number of vertices of degree d. Moreover, it is conjectured in [BS97] that $\chi'_s(G) \leq \pi(G) + 1$ for any graph $G \neq K_2$. This conjecture has been solved for graphs of maximum degree two [BBS02] and for graphs verifying $\Delta(G) \geq \sqrt{2|V(G)|} + 4$ and $\delta(G) \geq 5$ (where $\delta(G)$ is the minimum degree of G) [BKLS04]. It was proved in [BHBLW99] that $\chi'_s(G) \leq |V(G)| + 1$ and in [BHBLW01] the authors obtained $\chi'_s(G) \leq \Delta(G) + 5$ if $\delta(G) \geq \frac{n}{3}$.

The study of AVD-colorings is more recent. In [BGLS07], the authors proved that $\chi'_a(G) \leq 5$ for graphs of maximum degree 3 and $\chi'_a(G) \leq \Delta(G) + 2$ for bipartite graphs. In [BKT06], the AVD-chromatic index of multidimensional meshes was determined. AVD-colorings are also studied in [EHW06, GR06] under the name of neighbour-distinguishing edge colorings. The bound $\chi'_a(G) \geq \Delta(G)$ is trivial. Moreover if G contains two adjacent vertices of degree $\Delta(G)$ then $\chi'_a(G) \geq \Delta(G) + 1$. The following conjecture was made in [ZLW02]:

Conjecture 1 ([ZLW02]) *Let $G \neq C_5$ be a graph of maximum degree Δ, then*

$$\Delta \leq \chi'_a(G) \leq \Delta + 2.$$

In relation with this conjecture and using a probabilistic argument, Hatami proved recently that $\chi'_a(G) \leq \Delta(G) + 300$ provided that $\Delta(G) > 10^{20}$.

As remarked in [EHW06, ZCL+06], the AVD-chromatic index of some regular graphs is in relation with their total chromatic number. More precisely, if G is a regular graph with $\chi'_a(G) = \Delta(G)+1$ then $\chi''(G) = \Delta(G)+1$ and the converse also holds. Therefore, some of the results of the present

paper about the AVD-chromatic index are also new results about the total chromatic number while some other were already known earlier, see [KM03, ZŽ04].

Total adjacent vertex distinguishing colorings were considered in [ZCL+05, LWZW06] in which the authors conjecture that $\chi''_a(G) \leq \Delta + 3$.

In this paper we shall consider VD, AVD and TAVD-colorings of products of graphs (see definitions below), trying to derive bounds for χ'_s, χ'_a and χ''_a of the product of two graphs in term of the value of the same parameter on the factors. In Section 2, we present general bounds for the Cartesian product. As an application, we determine in Sections 3 and 4, the AVD and TAVD-chromatic index of the generalized hypercube and present in Section 5 tight lower and upper bounds for its VD-chromatic index. Section 6 provides bounds for VD, AVD and TAVD-chromatic indices of direct, strong and lexicographic products.

We use the following notation from [IK00] for the standard graph products. Let $G \Box H$, $G \times H$, $G \boxtimes H$ and $G \circ H$ be the *Cartesian*, *direct* (also called Krönecker or categorical), *strong* and *lexicographic* product of G and H respectively. The vertex set of any of these products is $V(G) \times V(H)$ and the edge sets are defined below:
$E(G \Box H) = \{(a,x)(b,y), ab \in E(G) \text{ and } x = y \text{ or } xy \in E(H) \text{ and } a = b\}$.
$E(G \times H) = \{(a,x)(b,y), ab \in E(G) \text{ and } xy \in E(H)\}$.
$E(G \boxtimes H) = E(G \Box H) \cup E(G \times H)$.
$E(G \circ H) = \{(a,x)(b,y), ab \in E(G) \text{ or } a = b \text{ and } xy \in E(H)\}$.

The *d-dimensional generalized hypercube* (also known as Hamming graph) K_n^d is the Cartesian product of the complete graph K_n by itself d times: $K_n^d = K_n \Box K_n \Box \ldots \Box K_n$. The hypercube Q_d is the graph K_2^d.

2 General results for Cartesian products

We first present general results for the VD, AVD and TAVD-coloring of the Cartesian product of graphs.

For AVD-colorings of the Cartesian product of a graph and a path or a cycle, the following results were proved in [BKT06]:

Theorem 1 ([BKT06]) *Let $d \geq 2$ be an integer and let G be a graph of maximum degree $\Delta \leq d-1$. If there exists an AVD-coloring of G with d colors, then*

$$\chi'_a(G \Box P_2) \leq d + 1,$$

$$\chi'_a(G \Box P_k) \leq d + 2, \text{ for } k \geq 3.$$

Theorem 2 ([BKT06]) *Let $d \geq 2$ be an integer and let G be a graph of maximum degree $\Delta \leq d-1$ and of minimum degree $\delta \geq 2$. If there exists an AVD-coloring of G with d colors, then*

$$\chi'_a(G \square C_k) \leq d+2.$$

The following theorem gives an upper bound on χ'_s and χ'_a for the Cartesian product of general graphs.

Theorem 3 *For any two graphs G and H different from K_2, the following hold*

$$\chi'_s(G \square H) \leq \chi'_s(G) + \chi'_s(H),$$
$$\chi'_a(G \square H) \leq \chi'_a(G) + \chi'_a(H).$$

Proof : Note that the product $G' = G \square H$ consists of $|V(G)|$ copies of H; and there is a perfect matching between any two copies of H if the corresponding vertices of G are adjacent. By symmetry, G' also contains $|V(H)|$ copies of G. Let c_G and c_H be two minimal VD- (AVD-)colorings of G and H respectively such that the colors of c_G are different from those used by c_H. A VD-coloring (AVD-coloring) c' of G' is obtained as follows: each copy of G in G' is colored by c_G and each copy of H in G' is colored by c_H.

Indeed, let (a, x) be a vertex of G'. We have $S_{c'}((a, x)) = S_{c_G}(a) \cup S_{c_H}(x)$ and so, (a, x) is distinguished from another (adjacent for AVD-coloring) vertex (b, y) because a is distinguished from b in G or x is distinguished from y in H. $\square$

Notice that, despite the proof of the above theorem is quite simple, it allows to find the exact value of the AVD-chromatic index for graphs G and H verifying $\chi'_a(G) = \Delta(G)$ and $\chi'_a(H) = \Delta(H)$ (for instance, this is the case for trees with no two adjacent vertices of maximum degree [ZLW02]).

For the TAVD-chromatic index of the Cartesian product of two graphs, we have the following.

Theorem 4 *Let G and H be two graphs different from K_2 such that $\chi(H) \leq \chi''_a(G)$, then*

$$\chi''_a(G \square H) \leq \chi''_a(G) + \chi'_a(H).$$

Proof : Let $G' = G \square H$, c_G be a minimal TAVD-coloring of G and c_H be a minimal AVD-coloring of H such that the colors of c_G are different from those used by c_H. We also color the vertices of H with the colors $0, 1, \ldots, \chi(H) - 1$. Let $\alpha = \chi''_a(G)$ and denote by σ_i the permutation on $0, 1, \ldots, \alpha - 1$ defined by $\sigma_i(k) = (k + i) \bmod \alpha$, for $0 \leq k \leq \alpha - 1$ and

$0 \leq i \leq \chi(H) - 1$. By extension, the total coloring $c'_G = \sigma_i(c_G)$ is defined by $c'_G(x) = \sigma_i(c_G(x))\ \forall x \in V(G) \cup E(G)$.

In order to obtain an AVD-coloring c' of G', we first use $\sigma_i(c_G)$ to color each copy G_j of G in G', where i is the color (given by the proper vertex coloring of H) of the vertex corresponding to the copy G_j. Since $\chi(H) \leq \alpha$, all permutations σ_i are pairwise different;

A TAVD-coloring c' of $G' = G \square H$ is obtained as follows: each copy G_j of G in G' is totally colored by $\sigma_i(c_G)$, where i is the color (given by the proper vertex coloring of H) of the vertex of H corresponding to copy G_j and each copy of H in G' is colored by c_H.

Notice that since $\chi(H) \leq \alpha$, all permutations σ_i are pairwise different thus the colors assigned to vertices of G' form a proper coloring. Moreover, as for the previous theorem, the fact that c' is a TAVD-coloring is easily shown since the colors on the vertices induce a proper vertex coloring and adjacent vertices are distinguished either by their sets of colors from c_G or by their sets of colors from c_H. $\square$

We now propose two better results in more specific cases for the AVD and TAVD-chromatic indices.

Theorem 5 *Let G be a graph such that the degree of each vertex is relatively prime with $\chi'_a(G)$, and let H be a graph verifying $\chi(H) \leq \chi'_a(G)$ then*

$$\chi'_a(G \square H) \leq \chi'_a(G) + \Delta(H).$$

Proof : Let $G' = G \square H$, c_G be an AVD-coloring of G in $\chi'_a(G)$ colors and γ_H be a proper edge coloring of H in $\chi'(H)$ colors distinct from those used by c_G. We also color the vertices of H with the colors $0, 1, \ldots, \chi(H) - 1$.

Similarly with the proof of the previous theorem, let $\alpha = \chi'_a(G)$ and denote by σ_i the permutation on $0, 1, \ldots, \alpha - 1$ defined by $\sigma_i(k) = (k + i) \bmod \alpha$, for $0 \leq k \leq \alpha - 1$ and $0 \leq i \leq \chi(H) - 1$. Let also $c'_G = \sigma_i(c_G)$ be the coloring defined by $c'_G(e) = \sigma_i(c_G(e))\ \forall e \in E(G)$.

In order to obtain an AVD-coloring c' of G', we first use $\sigma_i(c_G)$ to color each copy G_j of G in G', where i is the color (given by the proper vertex coloring of H) of the vertex corresponding to the copy G_j.

Since $\chi(H) \leq \alpha$, all permutations σ_i are pairwise different; moreover, if x is a vertex of G, and x_i (resp. x_j) its corresponding vertex in G_i (resp. G_j), then the color set of x_i in G_i is different from the color set of x_j in G_j. Then, we have two cases to consider:

Case 1: $\chi'(H) = \Delta(H)$

In this case, we use the proper coloring γ_H of H to color each copy of H in G'. Let (a, x) and (b, y) be two adjacent vertices of G'. Without

loss of generality, we have $S_{c'}((a,x)) = S_{c_G}(a) \cup S_{\gamma_H}(x)$ and $S_{c'}((b,y)) = \sigma_i(S_{c_G}(b)) \cup S_{\gamma_H}(y)$, for some i.

If $x = y$ (i.e. (a,x) and (b,y) are in the same copy of G), then $i = 0$ and $S_{c'}((a,x)) \neq S_{c'}((b,y))$ since $S_{c_G}(a) \neq S_{c_G}(b)$.

If $x \neq y$, as (a,x) and (b,y) are adjacent then so are x and y in H, thus x and y have different colors and so $i \neq 0$. If $S_{\gamma_H}(y) \neq S_{\gamma_H}(x)$, we have immediately $S_{c'}((a,x)) \neq S_{c'}((b,y))$. If $S_{\gamma_H}(y) = S_{\gamma_H}(x)$, we prove that $S_{c_G}(a) \neq \sigma_i(S_{c_G}(a))$. Indeed, by contradiction, let d be the degree of a in G; if $S = S_{c_G}(a) = \sigma_i(S_{c_G}(a)) = \{s_1, \ldots, s_d\}$, then we have modulo $\alpha = \chi'_a(G)$: $s_j + i = s_k \in S$ for each j, $1 \leq j \leq d$. If we sum all equalities, we obtain $d.i = 0 \bmod \alpha$. So, the hypothesis that d and α are relatively prime gives $\sigma_i = \sigma_0 = Id$ which is a contradiction.

Therefore, when $\chi'(H) = \Delta(H)$ we have $\chi'_a(G\Box H) \leq \chi'_a(G) + \Delta(H)$.

Case 2: $\chi'(H) = \Delta(H) + 1$

Remark that in this case, for each vertex x of H, there exists (at least) one color j such that $j \notin S_{\gamma_H}(x)$ (the *missing color*). In order to complete the coloring c' of G', we use the proper edge-coloring γ_H to color each copy of H. Then, from the above remark, for each copy G_i of G in G', there is a color that is not used by any of the edges incident with any vertex of G_i. So we modify the coloring $\sigma_i(c_G)$ of each copy G_i of G' by changing the color zero by this missing color. With a similar proof as for the first case, we can show that $\chi'_a(G\Box H) \leq \chi'_a(G) + \Delta(H)$. $\Box$

Theorem 6 *Let G be a graph such that the degree of each vertex plus one is relatively prime with $\chi''_a(G)$, and let H be a graph verifying $\chi(H) \leq \chi''_a(G)$ then*

$$\chi''_a(G\Box H) \leq \chi''_a(G) + \Delta(H).$$

Proof : We modify the previous proof as follow: each AVD-coloring is replaced by a TAVD-coloring; thus $\chi'_a(G)$ is changed in $\chi''_a(G)$; each set of d colors $\{s_1, s_2, \ldots, s_d\}$ is changed in a set of $d+1$ colors $\{s_1, s_2, \ldots, s_d, s_{d+1}\}$ and the equality $d \cdot i = 0 \bmod \alpha$ becomes $(d+1) \cdot i = 0 \bmod \alpha$. $\Box$

Now, if α is taken to be the smallest prime number greater than $\chi'_a(G)$ (respectively $\chi''_a(G)$) then we obtain the two following corollaries.

Corollary 1 *Let G be a graph and let p be the smallest prime number greater than or equal to $\chi'_a(G)$. If H is a graph verifying $\chi(H) \leq p$ then*

$$\chi'_a(G\Box H) \leq p + \Delta(H).$$

Corollary 2 *Let G be a graph and let p be the smallest prime number greater than or equal to $\chi''_a(G)$. If H is a graph verifying $\chi(H) \leq p$ then*

$$\chi''_a(G\Box H) \leq p + \Delta(H).$$

3 AVD-coloring of the generalized hypercube

In this section, we determine the AVD-chromatic index of the generalized hypercube K_n^d. We first need to compute the AVD-chromatic index of $K_{2p}\Box K_2$.

Lemma 1 *For $p \geq 2$*

$$\chi'_a(K_{2p}\Box K_2) = 2p + 1.$$

Proof : In order to construct the graph $K_{2p}\Box K_2$, we consider two copies K and K' of K_{2p}. Let $V(K) = \{x_0, x_1, \ldots, x_{2p-1}\}$ and let $V(K') = \{x'_0, x'_1, \ldots, x'_{2p-1}\}$. Let c be the coloring of K_{2p} defined by:

$$c(x_i x_j) = i + j \mod (2p+1) \text{ with } 0 \leq i < j \leq 2p-1.$$

In the following, each integer is considered modulo $(2p+1)$.

Remark that the equality $2i = i - 1$ has no solution for $i \in [0..2p-1]$, and it is easy to see that the color set $S(x_i)$ of each vertex x_i is exactly $S(x_i) = \{0,\ 1,\ \ldots,\ 2p\}\backslash\{2i,\ i-1\}$.

Moreover, for $i \in [0..2p-1]$, $i - 1 \in [0..2p]/\{2p-1\}$ and $2i \in [0..2p]/\{2p-1\}$. So, the color $2p-1$ is the only one which appears in each set $S(x_i)$. Moreover, if we suppose $i \neq j$ then the two equalities $2j = i - 1$ and $2i = j - 1$ induce that $3(i+1) = 0$ which is impossible when 3 does not divide $2p+1$. This proves that c is an AVD-coloring when 3 does not divide $2p+1$.

Now we define another coloring c' for the second copy K' of K_{2p} by setting:

$$c'(x'_i x'_j) = \sigma(c(x_i x_j)) \text{ with } 0 \leq i, j \leq 2p-1,$$

where x'_i is the corresponding vertex of x_i in the second copy K' of K_{2p} and the permutation σ is defined by:

$$\sigma(i-1) = 2i, \text{ with } 1 \leq i \leq 2p.$$

An illustration of the colorings c and c' is given in Appendix for $p = 3$.

As above, remark that if 3 does not divide $2p+1$ and if $i \neq 2p-1$ then $\sigma^2(i) = i$ has no solution. This proves that σ has no cycle of length two in its decomposition into a product of disjoint cycles and c' is also an AVD-coloring if 3 does not divide $2p+1$.

In this case (3 does not divide $2p+1$), we give the color $2i$ to each edge $x_i x'_i$ in $K_{2p}\Box K_2$. So, in $K_{2p}\Box K_2$, the vertex x_i has no incident edge of color $i-1$, and the vertex x'_i has no incident edge of color $\sigma(2i)$. Since σ has no cycle of length 2 in its decomposition then $\sigma(2i) \neq i - 1$. Consequently, $S(x_i) \neq S(x'_i)$ in $K_{2p}\Box K_2$.

So, we have obtained an AVD-coloring of $K_{2p}\Box K_2$ with $2p+1$ colors when 3 does not divide $2p+1$ (see Figure 1 for an AVD-coloring of $K_4\Box K_2$).

In the case where 3 divides $2p+1$, we modify the coloring c into an AVD-coloring d as follows:

Recall that σ has a unique cycle of length two in its disjoint cycles decomposition. Let (a,b) be this cycle where $a>b$. We have necessarily $a = 2\frac{2p+1}{3}-2$ and $b=\frac{2p+1}{3}-2$ since $\sigma^2(i)=i$ has only one solution. Remark also that $a=2(b+1)=\sigma(b)$.

We define the coloring d by:

$$\begin{cases} d(x_{a+1+k}x_{2p-1-k}) & = a \qquad \text{with } 0\le k\le 2p-2-a, \\ d(x_{a+1+k}x_{2p-k}) & = a-1 \qquad \text{with } 1\le k\le 2p-2-a, \end{cases}$$

and in the other cases

$$d(x_ix_j)=c(x_ix_j).$$

This coloring is also an AVD-coloring for K_{2p} with $2p+1$ colors. It is easy to see that the color set $S(x_i)$ of each vertex x_i is exactly $S(x_i)=\{0,\ 1,\ \ldots,\ 2p\}\backslash\{2i,\ i-1\}$ for $i\neq a+1$, $i\neq a/2+p$, and $S(x_{a+1})=\{0,\ 1,\ \ldots,\ 2p\}\backslash\{b,\ a-1\}$, $S(x_{a/2+p})=\{0,\ 1,\ \ldots,\ 2p\}\backslash\{a,\ a/2+p-1\}$

Recall that we had previously with the coloring c

$S(x_{a+1})=\{0,\ 1,\ \ldots,\ 2p\}\backslash\{b,\ a\}$ and $S(x_{a/2+p})=\{0,\ 1,\ \ldots,\ 2p\}\backslash\{a-1,\ a/2+p-1\}$.

As above, excepted the color $2p-1$, all colors appear in at least one color set.

Now, we define another AVD-coloring d' for the second copy of K_{2p} by setting:

$$d'(x'_ix'_j)=\sigma'(d(x_ix_j)) \quad \forall\ 0\le i,j\le 2p-1 \ \text{ with}$$

$$\begin{cases} \sigma'(i) & = \sigma(i) \qquad \forall i\neq b \ \text{ and } i\neq a/2+p-1 \\ \sigma'(b) & = a-1 \\ \sigma'(a/2+p-1) & = a \end{cases}$$

An illustration of the colorings d and d' is given in Appendix for $p=4$.

By construction, σ' has no cycle of length 2 in its decomposition into a product of disjoint cycles. Indeed, we have 'broken' the cycle (a,b) of length 2 in σ.

In $K_{2p}\Box K_2$, for each i, the color sets $S(x_i)$ of x_i and $S(x'_i)$ of x'_i are distinct and verify $S(x'_i)=\sigma'(S(x_i))$. Moreover, as for the previous case,

for each i there exists a color which appears neither in $S(x'_i)$ nor in $S(x_i)$. We give this color to the edge $x_ix'_i$ for each vertex x_i of K_{2p}. Thus, we obtain an AVD-coloring for $K_{2p}\Box K_2$ in $2p+1$ colors when 3 divides $2p+1$.

Finally, as $K_{2p}\Box K_2$ is regular of degree $2p$, we obtain $\chi'_a(K_{2p}\Box K_2) \geq 2p+1$ and thus $\chi'_a(K_{2p}\Box K_2) = 2p+1$.

□

Theorem 7 *For any integers $n \geq 2$ and $d \geq 2$,*

$$\chi'_a(K_n^d) = d(n-1)+1.$$

Proof : It is known that $\chi'_a(K_n) = \chi'_s(K_n) = n+1-\epsilon(n)$, where $\epsilon(n) = 1$ for odd n and 0 for even n.

When n is odd, $n = 2p+1$ for some p, we proceed by induction on d. The result is true for $d = 1$. Assume that $\chi'_a(K_{2p+1}^{d-1}) = 2p(d-1)+1$, thus $\chi'_a(K_{2p+1}^{d-1})$ and $\Delta(K_{2p+1}^{d-1}) = 2p(d-1)$ are relatively prime and using Theorem 5 with $G = K_{2p+1}^{d-1}$ and $H = K_{2p+1}$, we have that $\chi'_a(K_{2p+1}^d) = 2pd+1$.

When $n = 2p$ is even, we obtain $d(2p-1)+1 \leq \chi'_a(K_{2p}^d) \leq d(2p-1)+2$. We show that $\chi'_a(K_{2p}^d) = 2p+1$ by induction on d.

When $d = 2$, by Lemma 1, there axists an AVD-coloring of $K_{2p}\Box K_2$ with $2p+1$ colors, and applying Theorem 5 with $G = K_{2p}\Box K_2$ and $H = K_p$, we conclude that $\chi'_a((K_{2p}\Box K_2)\Box K_p) = 3p$. Now, we add $p-1$ perfect matchings to $(K_{2p}\Box K_2)\Box K_p$ in order to obtain K_{2p}^2, all the edges of each perfect matching being colored with a new color. We therefore obtain $\chi'_a(K_{2p}^2) = 4p-1$.

We suppose by induction that $\chi'_a(K_{2p}^d) = d(2p-1)+1$ for $d \geq 2$ and prove that $\chi'_a(K_{2p}^{d+1}) = (d+1)(2p-1)+1$. As $K_{2p}^{d+1} = K_{2p}^d\Box K_{2p}$, K_{2p}^{d+1} contains $2p$ copies of K_{2p}^d. Let $\{x_i^j,\ 0 \leq i \leq (2p)^d-1\}$ be the set of vertices of the jth copy $(1 \leq j \leq 2p)$. So, we color each copy with the different AVD-colorings c_j, $1 \leq j \leq 2p$ such that: c_1 is an AVD-coloring of K_{2p}^d in $d(2p-1)+1$ colors, and if $j \geq 2$, we define $c_j(x_i^jx_k^j) = c_1(x_{i+j-1}^jx_{k+j-1}^j)$ where the subscripts are modulo $(2p)^d$. In order to obtain K_{2p}^{d+1}, we add $2p-1$ perfect matchings between the $2p$ copies of K_{2p}^d, all the edges of each perfect matching being colored with a new color. So, $\chi'_a(K_{2p}^{d+1}) = d(2p-1)+1+(2p-1) = (d+1)(2p-1)+1$.

□

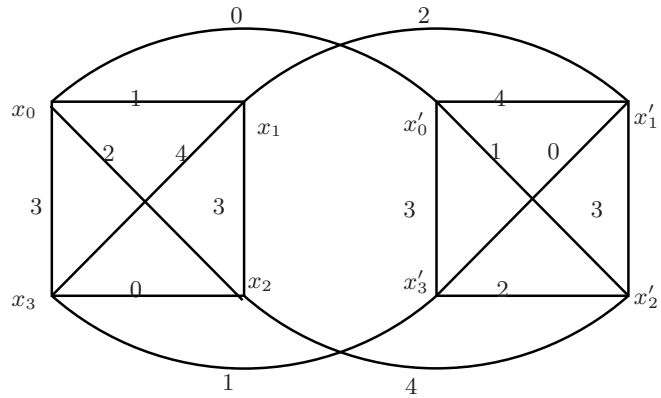


Figure 1: An AVD-coloring for $K_4 \square K_2$ with 5 colors

4 TAVD-coloring of the generalized hypercube

In this section, we determine the TAVD-chromatic index of the generalized hypercube K_n^d. In order to do that, we first need to compute the TAVD-chromatic index of K_{2p+1}^2.

Lemma 2 *For $p \geq 2$*

$$\chi''_a(K_{2p+1}^2) = 4p + 2.$$

Proof : Let $V(K_{2p+1}^2) = \{x_i^j | 0 \leq i, j \leq 2p\}$ and $E(K_{2p+1}^2) = \{x_i^j x_i^{j'} | 0 \leq i, j, j' \leq 2p, j \neq j'\} \cup \{x_i^j x_{i'}^j | 0 \leq i, i', j \leq 2p, i \neq i'\}$. Define a total coloring c of K_{2p+1}^2 by

$$\begin{cases} c(x_i^j) & = & 2i + j \bmod (4p+2), \\ c(x_i^j x_{i'}^j) & = & i + i' + j \bmod (4p+2), \\ c(x_i^j x_i^{j'}) & = & i + j + j' + 2p + 1 \bmod (4p+2). \end{cases}$$

We now show that c is a TAVD-coloring. By the above definition, $S(x_i^j) = \{0, \ldots 4p+1\} \setminus \{i + j + 2p + 1 \bmod (4p+2)\}$. Hence, for $i' \neq i$ and $j' \neq j$, we have $S(x_i^j) \neq S(x_{i'}^j)$ and $S(x_i^j) \neq S(x_i^{j'})$. $\square$

Theorem 8 *For any integers $n \geq 2$ and $d \geq 2$,*

$$\chi''_a(K_n^d) = (n-1)d + 2.$$

Proof : It is known [ZCL+05] that $\chi''_a(K_n) = \chi''_s(K_n) = n+2-\epsilon(n)$, where $\epsilon(n) = 0$ for odd n and 1 for even n.

We proceed by induction on d, by considering two cases depending on the parity of n.

When n is even, $n = 2p$ for some p, the result is true for $d = 1$. Assume that $\chi''_a(K_{2p}^{d-1}) = (2p-1)(d-1)+2$, thus $\chi''_a(K_{2p}^{d-1})$ and $\Delta(K_{2p}^{d-1})+1 = (2p-1)(d-1)+1$ are relatively prime and using Theorem 6 with $G = K_{2p}^{d-1}$ and $H = K_{2p}$, we have that $\chi''_a(K_{2p}^{d}) = (2p-1)d+2$.

When $n = 2p+1$ is odd, the result is true for $d = 2$ by Lemma 2. Assume that $\chi''_a(K_{2p+1}^{d}) = 2pd+2$. Thus $\chi''_a(K_{2p+1}^{d-1})$ and $\Delta(K_{2p+1}^{d-1})+1 = 2pd+1$ are relatively prime and using Theorem 6 with $G = K_{2p+1}^{d-1}$ and $H = K_{2p+1}$, we have that $\chi''_a(K_{2p+1}^{d}) = 2pd+2$.

Therefore, we have proved that $\chi''_a(K_n^d) = (n-1)d+2$.

□

5 VD-coloring of the generalized hypercube

Finding a minimal VD-coloring for the product of a graph by K_2 seems to be difficult, nevertheless we present two simple upper bounds.

Theorem 9 *For any graph G, $\chi'_s(G\Box K_2) \leq 2\chi'_s(G)+1$.*

Proof : A VD-coloring of $G\Box K_2$ can be simply obtained by coloring the edges of each of the two copies of G with two VD-colorings using $\chi'_s(G)$ colors distinct from each other and by coloring the edges between the two copies by a new color. □

Given a graph G with an edge-coloring c, we say that a color j *touches* a vertex x if $j \in S_c(x)$.

Theorem 10 *If there exists a VD-coloring of G with d colors such that one color touches each vertex, then there exists a VD-coloring of $G\Box K_2$ with $d+2$ colors such that one of them touches each vertex.*

Proof : Assume that c is a VD-coloring of G with colors $\{0, 1, \ldots, d-1\}$ such that the color 0 touches each vertex of G. Color each of the two copies of G in $G\Box K_2$ with the coloring c and replace the color 0 by the color d in the first copy and by the color $d+1$ in the second copy. Now, give the color 0 to the edges of the perfect matching between the two copies. The coloring obtained is clearly VD since the coloring in each copy is VD and a vertex of a copy is distinguished with a vertex of the other copy since different colors touch them. Moreover, the color 0 touches each vertex of $G\Box K_2$. □

This theorem allows to obtain the known fact that $\chi'_s(Q_n) \leq 2n$ (see [ČHS96]). Notice that the authors of [ČHS96] have given the asymptotic value of $\chi'_s(Q_n)$ but finding the exact value still remains an open problem.

Theorem 11 *For any integers $n \geq 3$ and d,*

$$d(n-1) \leq \chi'_s(K_n^d) \leq d(2\lfloor \frac{n}{2} \rfloor + 1).$$

Proof : Obviously, $\chi'_s(K_n^d) \geq \Delta(K_n^d) + 1 = d(n-1) + 1$. The relation is true for $d = 1$. By recurrence on d and with Theorem 3, when $n = 2p+1$, $\chi'_s(K_{2p+1}^d) \leq \chi'_s(K_{2p+1}^{d-1}) + \chi'_s(K_{2p+1}) \leq d(2p+1)$.

The case $n = 2p$ is similar.

□

Theorem 12 *For any integers $p \geq 1$ and $d \geq 1$,*

- *if $d \leq \ln(2p+1) + 1$ then $\chi'_s(K_{2p+1}^d) = d(2p+1)$,*
- *if $d \leq \ln(2p)$ then $\chi'_s(K_{2p}^d) \geq 2dp - 1$.*

Proof :

For the first assertion, in order to see if $d(2p+1)$ colors are sufficient to obtain a VD-coloring, we compare $\binom{(2p+1)d-1}{2dp} = \binom{(2p+1)d-1}{d-1}$ with $(2p+1)^d$. Let us define:

$$A = \frac{\binom{(2p+1)d-1}{d-1}}{(2p+1)^{d-1}} = \frac{((2p+1)d-1)((2p+1)d-2)\dots(2dp+1)}{(d-1)(2p+1)(d-2)(2p+1)\dots 1(2p+1)}.$$

Thus $\ln(A) = \sum_{\ell=1}^{d-1} \ln(\frac{2dp+\ell}{\ell(2p+1)}) = \sum_{\ell=1}^{d-1} \ln(\frac{2dp+\ell}{2p+1}) - \sum_{\ell=1}^{d-1} \ln(\ell)$.

With the well-known Darboux sums inequalities, we obtain

$$\begin{aligned} \ln(A) &\leq \int_{\ell=1}^{d} \ln(\tfrac{2dp+\ell}{2p+1}) d\ell - \int_{\ell=1}^{d-1} \ln(\ell) d\ell \text{ and (with Maple)}, \\ &\leq (1+2dp)\ln(1+\tfrac{d-1}{2dp+1}) + (d-1)\ln(\tfrac{d}{d-1}) - 1, \\ &< d-1 \text{ (since } ln(1+x) < x \text{ for } x \neq 0). \end{aligned}$$

So, if $d \leq \ln(2p+1) + 1$ then $\ln(A) < \ln(2p+1)$ and $A < 2p+1$. Thus when $d \leq \ln(2p+1) + 1$, $\chi'_s(K_{2p+1}^d) \geq d(2p+1)$. With the previous theorem, we obtain the result.

The proof of the second point is similar. □

6 Other products

We first present a simple fact about TAVD-colorings that will be useful.

Fact 1 *For any graph G,*

$$\chi''_a(G) \leq \chi'(G) + \chi(G).$$

Actually, coloring separately the edges and the vertices of the graph gives a proper total coloring and the colors assigned to vertices of G distinguish adjacent vertices.

As a consequence, the TAVD-chromatic index of a bipartite graph B with two adjacent vertices of maximum degree is $\Delta(B) + 2$ since $\Delta(B) + 2$ colors are necessary to color and distinguish two adjacent vertices of degree $\Delta(B)$.

6.1 Direct product

Theorem 13 *For any two graphs G and H different from K_2, the following hold*

$$\chi'_s(G \times H) \leq \chi'_s(G)\chi'_s(H),$$
$$\chi'_a(G \times H) \leq \min\{\chi'(G)\chi'_a(H), \chi'_a(G)\chi'(H)\},$$
$$\chi''_a(G \times H) \leq \min\{\chi'(G)\chi''_a(H), \chi''_a(G)\chi'(H)\}.$$

Proof : Let $G' = G \times H$. For the first inequality, given two minimal VD-colorings c_G and c_H of G and H respectively, color each edge $(a, x)(b, y)$ of G' with the color $(c_G(ab), c_H(xy))$. The color set of a vertex (a, x) of G' is then $S((a, x)) = S_{c_G}(a) \times S_{c_H}(x)$. Hence, for any two distinct vertices (a, x) and (b, y) of G', we have $S((a, x)) \neq S((b, y))$ since either $S_{c_G}(a) \neq S_{c_G}(b)$ or $S_{c_H}(x) \neq S_{c_H}(y)$, or both.

For the second inequality, we can suppose without loss of generality that $\min\{\chi'(G)\chi'_a(H), \chi'_a(G)\chi'(H)\} = \chi'_a(G)\chi'(H)$. Let c_G be an AVD-coloring of G with colors $0, 1, \ldots, \alpha - 1$, where $\alpha = \chi'_a(G)$. Let γ_H be a proper edge coloring of H in $\chi'(H)$ colors.

An AVD-coloring c' of G' is obtained by setting for each $ab \in E(G),\ xy \in E(H)$:

$$c'((a, x)(b, y)) = (c_G(ab), \gamma_H(xy)).$$

The fact that c' is AVD is easily seen: we have $S_{c'}((a, x)) = S_{c_G}(a) \times S_{\gamma_H}(x)$. Hence, for any two adjacent vertices (a, x) and (b, y) of G', we have $S((a, x)) \neq S((b, y))$ since $S_{c_G}(a) \neq S_{c_G}(b)$ (a and b are adjacent vertices of G).

The third inequality can be shown in a same way, replacing χ'_a by χ''_a, AVD by TAVD and for each vertex (a, x) of G', setting $c'((a, x)) = (c_G(a), 0)$. □

Theorem 14 *For any graph G, the following holds*

$$\chi'_a(G \times K_2) \leq \chi'_a(G).$$

Moreover, equality holds if G is bipartite, or if G is regular and $\chi'_a(G) = \Delta(G) + 1$, or if $\chi'_a(G) = \Delta(G)$.

Proof : An AVD-coloring c' of $G \times K_2$ is obtained by setting for $ab \in E(G),\ xy \in E(K_2)$:

$$c'((a,x)(b,y)) = c_G(ab).$$

In other words, we give the color of the edge ab to the edge $(a,x)(b,y)$. So, if $S_{c_G}(a) = \{a_1, a_2, \ldots, a_k\}$ then we have also $S_{c'}((a,x)) = \{a_1, a_2, \ldots, a_k\}$ and it is clear that c' is an AVD-coloring of $G \times K_2$. In the case where G is bipartite, then $G \times K_2$ consists in two disconnected copies of G and so $\chi'_a(G \times K_2) = \chi'_a(G)$. In the case where G is regular of degree Δ and $\chi'_a(G) = \Delta+1$, then $G \times K_2$ is also regular and we have $\chi'_a(G \times K_2) \geq \Delta+1$ and $\chi'_a(G \times K_2) \leq \chi'_a(G) = \Delta + 1$. So, $\chi'_a(G \times K_2) = \Delta + 1$. In the case where $\chi'_a(G) = \Delta$, then clearly, $\chi'_a(G \times K_2) = \Delta$.

□

Theorem 15 *For any $m \geq 3$, $n \geq 3$,*

$$\chi'_a(P_m \times P_n) = \begin{cases} 4 & \text{if } m = n = 3 \\ 5 & \text{otherwise.} \end{cases}$$

Proof : If $m = n = 3$, then $P_m \times P_n$ is the disjoint union of a 4-cycle and a 4-star. Hence four colors suffice to AVD-color $P_3 \times P_3$.

If $m \geq 3$, $n \geq 3$ and $m + n > 6$ then, in [BKT06], we proved that $\chi'_a(P_m \square P_n) = 5$ and the coloring used is such that the four edges of each 4-length cycle in $P_m \square P_n$ have pairwise different colors. Now, it is easily seen that $P_m \times P_n$ consists in two connected isomorphic components with two adjacent vertices of degree 4 (thus $\chi'_a(P_m \times P_n) \geq 5$) that are induced subgraphs of some grid $P_{m'} \square P_{n'}$ (see Figure 2). Thus, two adjacent vertices of degree 4 in $P_m \times P_n$ have obviously different sets of colors. Moreover, two adjacent vertices of degree 2 have different sets of colors since they belong to a cycle of length 4 in $P_{m'} \square P_{n'}$.

□

6.2 Strong product

Theorem 16 *For any two graphs G and H different from K_2, the following hold*

$$\chi'_s(G \boxtimes H) \leq \min\{\chi'_s(G \square H) + \chi'(G \times H), \chi'(G \square H) + \chi'_s(G \times H)\},$$

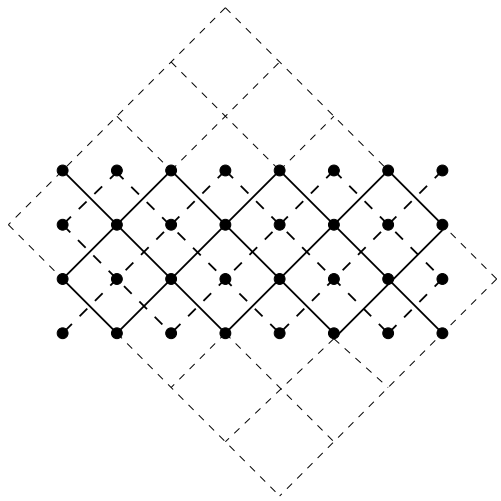


Figure 2: The product $P_8 \times P_4$ with a component (in bold) included in $P_6 \square P_5$.

$$\chi'_a(G \boxtimes H) \leq \chi'_a(G \square H) + \chi'_a(G \times H),$$
$$\chi''_a(G \boxtimes H) \leq \chi'(G \boxtimes H) + \chi(G)\chi(H).$$

Proof : Remember that the edge set of $G \boxtimes H$ is the union of the edge set of $G \square H$ and of $G \times H$. To obtain a VD-coloring of $G \boxtimes H$, VD-color the edges of $G \square H$ in $\chi'_s(G \square H)$ colors and properly color the edges of $G \times H$ in $\chi'(G \times H)$ new colors. Then the coloring is clearly proper and the vertices are distinguished by the VD-coloring of $G \square H$. The same goes when exchanging the roles of G and H. We then obtain $\chi'_s(G \boxtimes H) \leq \min\{\chi'_s(G \square H) + \chi'(G \times H), \chi'(G \square H) + \chi'_s(G \times H)\}$.

For the AVD-coloring of $G \boxtimes H$, AVD-color the edges of $G \square H$ in $\chi'_a(G \square H)$ colors and AVD-color the edges of $G \times H$ in $\chi'_a(G \times H)$ new colors. This coloring is clearly proper and two adjacent vertices are distinguished either by the colors of the edges of $G \square H$ or by the colors of the edges of $G \times H$.

The third inequality is a direct consequence of Claim 1 since $\chi(G \boxtimes H) \leq \chi(G)\chi(H)$ (see [IK00], page 246). $\square$

6.3 Lexicographic product

Theorem 17 *For any two graphs G and H different from K_2, the following hold*

$$\chi'_s(G \circ H) \leq \chi'_s(G) + \chi'_s(H) + (|V(H)| - 1)\chi'(G),$$

$$\chi'_a(G \circ H) \leq \chi'_a(G) + \chi'_a(H) + (|V(H)| - 1)\chi'(G),$$
$$\chi''_a(G \circ H) \leq \chi'(G \circ H) + \chi(G)\chi(H).$$

Proof : Assume G is of order $n \geq 3$ and H is of order $m \geq 3$. The graph $G \circ H$ consists in n copies of H, two copies being linked by a complete bipartite graph $K_{m,m}$ if the corresponding vertices of G are adjacent. Thus, the edges between two copies of H can be decomposed into m perfect matchings. To obtain a VD-coloring of $G \circ H$:

- VD-color the edges of each copy of H with $\chi'_s(H)$ colors,
- for any edge e of G, color one of the m perfect matchings between the two copies of H corresponding to e with the color of e in a VD-coloring of G in $\chi'_s(G)$ new colors,
- properly color the edges of the remaining perfect matchings with $(m-1)\chi'(G)$ new colors.

For the AVD-coloring, the proof is the same as above, replacing VD by AVD and χ'_s by χ'_a.

For the TAVD-coloring, as for the strong product, the third inequality follows from Claim 1 since $\chi(G \circ H) \leq \chi(G)\chi(H)$ ([IK00], page 246). □

7 Concluding remarks

We have obtained bounds for the VD, AVD and TAVD-chromatic indices of the Cartesian product of two graphs and we have shown that some of these bounds are optimal since they allow to determine the AVD and TAVD-chromatic indices of generalized hypercubes. General bounds for the direct, strong and lexicographic products have also been determined. Notice that, despite Theorems 3, 13, 16, 17 have quite simple proofs, they give the exact value of the adjacent vertex distinguishing chromatic index if the factors G and H are such that $\chi'_a(G) = \Delta(G)$ and $\chi'_a(H) = \Delta(H)$ (for instance, this is the case for trees with no two adjacent vertices of maximum degree [ZLW02]). However, the bounds we obtained for the TAVD-chromatic index of the direct, strong and lexicographic products do not seem to be very tight. It could be interesting to investigate this more in details.

References

[BBS02] P. N. Balister, B. Bollobás, and R. H. Schelp. Vertex distinguishing colorings of graphs with $\Delta(G) = 2$. *Discrete Math.*, 252(1-3):17–29, 2002.

[BGLS07] P. N. Balister, E. Győri, J. Lehel, and R. H. Schelp. Adjacent vertex distinguishing edge-colorings. *SIAM J. Discrete Math.*, 21(1):237–250 (electronic), 2007.

[BHBLW99] C. Bazgan, A. Harkat-Benhamdine, H. Li, and M. Woźniak. On the vertex-distinguishing proper edge-colorings of graphs. *J. Combin. Theory Ser. B*, 75(2):288–301, 1999.

[BHBLW01] C. Bazgan, A. Harkat-Benhamdine, H. Li, and M. Woźniak. A note on the vertex-distinguishing proper coloring of graphs with large minimum degree. *Discrete Math.*, 236(1-3):37–42, 2001. Graph theory (Kazimierz Dolny, 1997).

[BKLS04] P. N. Balister, A. Kostochka, Hao Li, and R. H. Schelp. Balanced edge colorings. *J. Combin. Theory Ser. B*, 90(1):3–20, 2004. Dedicated to Adrian Bondy and U. S. R. Murty.

[BKT06] J-L. Baril, H. Kheddouci, and O. Togni. Adjacent vertex distinguishing edge-colorings of meshes. *Australasian Journal of Combinatorics*, 35:89–102, 2006.

[BS97] A. C. Burris and R. H. Schelp. Vertex-distinguishing proper edge-colorings. *J. Graph Theory*, 26(2):73–82, 1997.

[ČHS96] J. Černý, M. Horňák, and R. Soták. Observability of a graph. *Math. Slovaca*, 46(1):21–31, 1996.

[EHW06] K. Edwards, M. Horňák, and M. Woźniak. On the neighbour-distinguishing index of a graph. *Graphs Combin.*, 22(3):341–350, 2006.

[GR06] C. Greenhill and A. Ruciński. Neighbour-distinguishing edge colorings of random regular graphs. *Electron. J. Combin.*, 13(1):Research Paper 77, 12 pp. (electronic), 2006.

[IK00] W. Imrich and S. Klavžar. *Product graphs*. Wiley-Interscience Series in Discrete Mathematics and Optimization. Wiley-Interscience, New York, 2000. Structure and recognition, With a foreword by Peter Winkler.

[KM03] A. Kemnitz and M. Marangio. Total colorings of Cartesian products of graphs. In *Proceedings of the Thirty-Fourth Southeastern International Conference on Combinatorics, Graph Theory and Computing*, volume 165, pages 99–109, 2003.

[LWZW06] J. Li, W. Wang, Z. Zhang, and Z. Wang. Adjacent vertex-distinguishing total coloring on square, cube, biquadratic of pathes. *Int. J. Pure Appl. Math.*, 29(1):7–13, 2006.

[ZCL+05] Z. Zhang, X. Chen, J. Li, B. Yao, X. Lu, and J. Wang. On adjacent-vertex-distinguishing total coloring of graphs. *Sci. China Ser. A*, 48(3):289–299, 2005.

[ZCL+06] Z. Zhang, X. Chen, J. Li, B. Liang, and P. Qiu. A note on the relation of adjacent strong edge coloring and total coloring of graphs. Submitted, 2006.

[ZLW02] Z. Zhang, L. Liu, and J. Wang. Adjacent strong edge coloring of graphs. *Appl. Math. Lett.*, 15(5):623–626, 2002.

[ZŽ04] B. Zmazek and J. Žerovnik. Behzad-Vizing conjecture and Cartesian product graphs. In *Workshop on Graphs and Combinatorial Optimization*, volume 17 of *Electron. Notes Discrete Math.*, pages 297–300 (electronic). Elsevier, Amsterdam, 2004.

Appendix

$$c(K_6) = \left(\begin{array}{cccccccc}
 & x_0 & x_1 & x_2 & x_3 & x_4 & x_5 & \text{Missing colors} \\
x_0 & & 1 & 2 & 3 & 4 & 5 & 0,6 \\
x_1 & 1 & & 3 & 4 & 5 & 6 & 0,2 \\
x_2 & 2 & 3 & & 5 & 6 & 0 & 1,4 \\
x_3 & 3 & 4 & 5 & & 0 & 1 & 2,6 \\
x_4 & 4 & 5 & 6 & 0 & & 2 & 1,3 \\
x_5 & 5 & 6 & 0 & 1 & 2 & & 3,4
\end{array}\right)$$

$$c'(K'_6) = \left(\begin{array}{cccccccc}
 & x'_0 & x'_1 & x'_2 & x'_3 & x'_4 & x'_5 & \text{Missing colors} \\
x'_0 & & 4 & 6 & 1 & 3 & 5 & 0,2 \\
x'_1 & 4 & & 1 & 3 & 5 & 0 & 2,6 \\
x'_2 & 6 & 1 & & 5 & 0 & 2 & 3,4 \\
x'_3 & 1 & 3 & 5 & & 2 & 4 & 0,6 \\
x'_4 & 3 & 5 & 0 & 2 & & 6 & 1,4 \\
x'_5 & 5 & 0 & 2 & 4 & 6 & & 1,3
\end{array}\right)$$

$$d(K_8) = \left(\begin{array}{ccccccccc}
 & x_0 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & \text{Missing colors} \\
x_0 & & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 0,8 \\
x_1 & 1 & & 3 & 4 & 5 & 6 & 7 & 8 & 0,2 \\
x_2 & 2 & 3 & & 5 & 6 & 7 & 8 & 0 & 1,4 \\
x_3 & 3 & 4 & 5 & & 7 & 8 & 0 & 1 & 2,6 \\
x_4 & 4 & 5 & 6 & 7 & & 0 & 1 & 2 & 3,8 \\
x_5 & 5 & 6 & 7 & 8 & 0 & & 2 & 4 & 1,3 \\
x_6 & 6 & 7 & 8 & 0 & 1 & 2 & & 3 & 4,5 \\
x_7 & 7 & 8 & 0 & 1 & 2 & 4 & 3 & & 5,6
\end{array}\right)$$

$$d'(K'_8) = \left(\begin{array}{cccccccccc} & x'_0 & x'_1 & x'_2 & x'_3 & x'_4 & x'_5 & x'_6 & x'_7 & \text{Missing colors} \\ x'_0 & & 3 & 6 & 8 & 1 & 4 & 5 & 7 & 0,2 \\ x'_1 & 3 & & 8 & 1 & 4 & 5 & 7 & 0 & 2,6 \\ x'_2 & 6 & 8 & & 4 & 5 & 7 & 0 & 2 & 1,3 \\ x'_3 & 8 & 1 & 4 & & 7 & 0 & 2 & 3 & 5,6 \\ x'_4 & 1 & 4 & 5 & 7 & & 2 & 3 & 6 & 0,8 \\ x'_5 & 4 & 5 & 7 & 0 & 2 & & 6 & 1 & 3,8 \\ x'_6 & 5 & 7 & 0 & 2 & 3 & 6 & & 8 & 1,4 \\ x'_7 & 7 & 0 & 2 & 3 & 6 & 1 & 8 & & 4,5 \end{array} \right)$$

Computer Science Journal of Moldova, vol.**21**, no.1(61), 2013

ECO-generation for some restricted classes of compositions

Jean-Luc Baril, Phan-Thuan Do

Abstract

We study several restricted classes of compositions by giving one-to-one maps between them and different classes of restricted binary strings or pattern avoiding permutations. Inspired by the ECO method [8], new succession rules for these classes are presented. Finally, we obtain generating algorithms in Constant Amortized Time (CAT) for theses classes.

Keywords : Composition of an integer, ECO method, succession rule, generating tree, pattern avoiding permutation.

1 Introduction

A *composition* of an integer n is an ordered collection of one or more positive integers whose sum is n. So, a composition c of n can be written $c = (c_1, c_2, \ldots, c_k)$ with $c_1 + c_2 + \cdots + c_k = n$ and $c_i \geq 1$ for all $i \leq k$. The integer k corresponds to the number of *parts* of the composition. Let $C(n)$ be the set of compositions of n. It is well known that the cardinality of $C(n)$ is 2^{n-1} and there is a one-to-one correspondence between $C(n)$ and binary strings of length $n-1$ (see Definition 1). There are many studies about enumeration of compositions and their restrictions: $(1, p)$-compositions, *i.e.,* compositions whose parts are 1 or p have been introduced in [12, 15, 16]; compositions with no occurrence of part p have been studied in [17]; see also [1, 14, 24, 25, 27, 28, 30, 32]. However, a very few articles deal with their exhaustive generations. Some Gray codes are given for compositions of a positive integer n in [31, 37]; for compositions with parts of

©2013 by Jean-Luc Baril, Phan-Thuan Do

size smaller than p in [36]; or for $(1,p)$-compositions in [12, 15]. These papers mostly study the classes of compositions in terms of binary strings. On the other hand, in [30], some results are provided using restricted permutations for a few classes of compositions, but they cannot be considered as avoidance patterns. More recently, a generalization of the Simion-Schmidt injection [35] gave a bijection between binary strings and pattern avoiding permutations [29] which creates a natural link between compositions and pattern avoiding permutations. For example, the class of compositions is in one-to-one correspondence with the class of permutations avoiding 321 and 312 [4, 26]; the set of compositions of n with all parts of sizes smaller than $(p+1)$ is enumerated by the p-generalized Fibonacci numbers, see [4, 11, 27, 28] and there is a bijection between this set and permutations avoiding the patterns 321, 312 and $234\cdots(p+1)1$.

In this paper, we use the ECO method [8] (Enumeration Combinatorial Object method) in order to generate some restricted classes of compositions represented as binary strings or pattern avoiding permutations. The *ECO method* is a recursive description of a combinatorial object class which explains how an object of size n can be reached from one and only one object of smaller size (see for example [2, 3, 5, 6, 7, 13, 18, 19, 21, 22, 23]). It consists to define a system of *succession rules* for a combinatorial object class which induces a generating tree such that each node is labeled by the number of its successors. In fact, the set of successions rules describes for each node the label of its successors. More formally, the root of the generating tree is labeled (b), $b \in \mathbb{N}^+$, and we define the rules Ω:

$$\{(k) \rightsquigarrow (e_1(k))(e_2(k))\cdots(e_k(k)), k \in \mathbb{N}\},$$

where $e_i : \mathbb{N}^+ \longrightarrow \mathbb{N}^+$. This means that each node labeled (k) has k successors labeled $(e_1(k))$, $(e_2(k))$,..., $(e_k(k))$. For $\ell \geq 1$, the symbol $\overset{\ell}{\rightsquigarrow}$ means that the succession rule transforms an element of size n into another one of size $n+\ell$. For $\ell = 1$ we frequently omit the superscript ℓ over $\rightsquigarrow$.

By coding each node of the generating tree with either a binary string or a permutation, we deduce new bijections between classes of

restricted compositions, pattern avoiding permutations and restricted binary strings.

This paper is organized as follows. Section 2 recalls the definition of pattern avoiding permutations, and gives existing links between compositions, binary strings and pattern avoiding permutations. Sections from 3 to 7 present succession rules for compositions with a given number of parts, compositions with at most p parts, $(1,p)$-compositions and compositions without parts of a given size. Moreover, each induced generating tree will be encoded by binary strings and pattern avoiding permutations. Finally, we deduce efficient algorithms (Constant Amortized Time algorithms) for generating all these classes (Constant Amortized Time means that the total amount of computation divided by the number of objects is bounded by a constant independent of the size of objects).

2 Definitions and notations

Let $\mathfrak{S}_n$ be the set of permutations on $[n] = \{1, 2, \ldots, n\}$. We represent a permutation $\pi \in \mathfrak{S}_n$ in one line notation: *i.e.,* $\pi = \pi_1\pi_2 \cdots \pi_n$, where $\pi_i = \pi(i)$ for all $i \leq n$. A permutation $\pi \in \mathfrak{S}_n$ contains the pattern $\tau \in \mathfrak{S}_k$ if and only if a sequence of indices $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ exists such that $\pi(i_1)\pi(i_2)\cdots\pi(i_k)$ is order-isomorphic to τ. We denote by $\mathfrak{S}_n(\tau)$ the set of n-length permutations avoiding the pattern τ, *i.e.*, permutations that do not contain τ. For instance, the permutation **52**3**1**64 contains the pattern 321 while $314265 \in \mathfrak{S}_6(321)$. Moreover, we consider a *barred pattern* $\bar{\tau}$, *i.e.,* a permutation in $\mathfrak{S}_k$ having a bar over one or several consecutive entries (see [34]). Let r, $1 \leq r \leq k-1$, be the number of barred elements in $\bar{\tau}$; τ be the permutation on $[k]$ identical to $\bar{\tau}$ but unbarred; and $\hat{\tau}$ be the permutation on $[k-r]$ made up of the $k-r$ unbarred elements of $\bar{\tau}$ rewritten to be a permutation on $[k-r]$. Let $b = b_1 \cdots b_k \in \{0,1\}^k$ such that $b_i = 1$ if and only if the i-th entry of $\bar{\tau}$ is barred. Then $\pi \in \mathfrak{S}_n$ avoids the pattern $\bar{\tau}$ if and only if each pattern $\hat{\tau}$ in π can be expanded into a pattern τ in π such that the positions of the extended entries correspond to the positions of 1s in b. For example, if $\bar{\tau} = 21\overline{34}$, then $b = 0011$ and

$2134 \in \mathfrak{S}_4(\bar{\tau})$ and $21435 \notin \mathfrak{S}_5(\bar{\tau})$ since 43 can not be expanded into a pattern $43ab$, where $4 < a < b$. Now we define a special pattern denoted $\dot{\tau} = 23\cdots(p-1)p\dot{1}$ (see [10]). A permutation π avoids $\dot{\tau}$ if and only if each pattern $23\cdots(p-1)1$ can be extended to a pattern $23\cdots(p-1)p1$ such that the positions of the extended values do not matter, *i.e.*, each pattern $23\cdots(p-1)1$ is contained in a pattern $23\cdots(p-1)p1$ of π. For instance, if $\dot{\tau} = 23\dot{1}$, then $2341 \in \mathfrak{S}_4(\dot{\tau})$ and $21 \notin \mathfrak{S}_2(\dot{\tau})$ while $2341 \notin \mathfrak{S}_4(23\bar{4}1)$.

It is well-known that the set $\mathcal{C}(n+1)$ of compositions of $n+1$ is in one-to-one correspondence with the set $\mathcal{B}(n)$ of binary strings of length n. The following bijection φ shows this correspondence.

Definition 1 *Let $c = (c_1, c_2, \ldots, c_k)$ be a composition of $n+1$. The bijection φ between $\mathcal{C}(n+1)$ and $\mathcal{B}(n)$ is defined by: $\varphi(c) = 1^{c_1-1}01^{c_2-1}0\cdots 1^{c_{k-1}-1}01^{c_k-1}$.*

For instance, if $c = (1, 3, 2, 3)$, then $\varphi(c) = 0\mathbf{11}0\mathbf{1}0\mathbf{11}$.

On the other hand, Juarna and Vajnovszki in [29] gave a bijection ϕ between the binary strings in $\mathcal{B}(n)$ and the permutations in $\mathfrak{S}_{n+1}(321, 312)$. This bijection is considered as a generalization of the Simion-Schmidt injection [35].

Definition 2 *Let $b = b_1b_2\cdots b_n \in \mathcal{B}(n)$. The bijection ϕ between $\mathcal{B}(n)$ and $\mathfrak{S}_{n+1}(321, 312)$ is defined by: $\pi = \phi(b) \in \mathfrak{S}_{n+1}$ which has its i-th value π_i given by the following rule: if $X_i = [n+1] \setminus \{\pi_1, \pi_2, \ldots, \pi_{i-1}\}$, then*

$$\pi_i = \begin{cases} \textit{the minimum value in } X_i, \ \textit{if } b_i = 0, \ \textit{or } i = n+1 \\ \textit{the second minimum value in } X_i, \ \textit{if } b_i = 1. \end{cases}$$

For instance, if $b = \mathbf{0}11\mathbf{0}1\mathbf{0}11$, then $\phi(b) = \mathbf{1}34\mathbf{2}6\mathbf{5}89\mathbf{7}$.

3 Compositions of n with parts of size at most p

The set $\mathcal{C}_{\leq p}(n)$ of compositions of n with parts of size at most p is enumerated by the $(n+1)$-th p-generalized Fibonacci number (see [11]).

The map φ (Section 2) induces a bijection between $\mathcal{C}_{\leq p}(n)$ and the set $\mathcal{B}_{<p}(n-1)$ of binary strings of size $n-1$ without p consecutive ones. It is proved [4, 11] that there are also one-to-one correspondences with the two classes of permutations $\mathfrak{S}_n(321, 231, (p+1)12\cdots p)$ and $\mathfrak{S}_n(321, 312, 23\cdots(p+1)1)$. These permutation classes admit known succession rules (see [4, 26] and Table 1) and they can be generated in constant amortized time (see also [11, 9, 33, 36] for Gray code listing).

4 Compositions of n with exactly p parts

The set $\mathcal{C}_p(n)$ of compositions of n with exactly p parts is enumerated by the binomial coefficient $\binom{n-1}{p-1}$. Also, $\mathcal{C}_p(n)$ is in one-to-one correspondence with the set $\mathcal{B}_{p-1}(n-1)$ of binary strings of length $n-1$ and having exactly $p-1$ zeros. The function φ (see Section 2) shows such a bijection. The following theorem gives a system of succession rules in order to generate the sets $\mathcal{B}_{p-1}(n-1)$ and $\mathfrak{S}_n(132, 312, (p+1)\cdots 21, 12\cdots(n-p+1)(n-p+2))$. In this part, we say that the level of a node in the generating tree is the length of the unique path between the root and this node, plus p (thus the root is on the level p).

Theorem 1 *For $p \geq 1$, a system (Ω_p) of succession rules for the set $\mathcal{C}_p(n)$ is:*

$$(\Omega_p)\begin{cases} (p) \\ (k) \rightsquigarrow (1)(2)\cdots(k-1)(k). \end{cases}$$

Each level $n \geq p$ of the generating tree induced by (Ω_p) can be coded by the binary strings of $\mathcal{B}_{p-1}(n-1)$ or by the permutations in $\mathfrak{S}_n(132, 312, (p+1)\cdots 21, 12\cdots(n-p+1)(n-p+2))$. A node other than the root and labeled (k) is coded by a binary string of the form $b = b'10^{k-1}$ (resp. a permutation $\pi = \pi' n(k-1)(k-2)\cdots 21$) and its successors are obtained from b (resp. π) either by inserting 1 (resp. $n+1$) between two entries of the suffix 10^{k-1} (resp. $n(k-1)(k-2)\cdots 21$) or by appending 1 (resp. $n+1$) on the right (see Figure 1).

Proof. We attach the binary string 0^{p-1} to the root of the generating tree obtained by (Ω_p) and we proceed by induction on the level n of

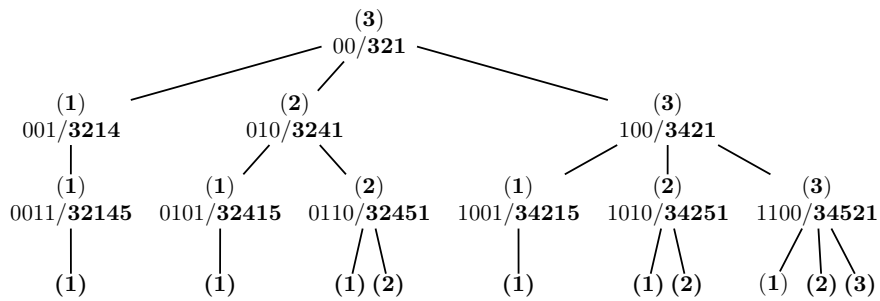


Figure 1. The first levels of the generating tree (Ω_3) (the level of the root is 3). Each node on the level n is coded by one binary string in $\mathcal{B}_2(n-1)$ or by one permutation in $\mathfrak{S}_n(132, 312, 4321, 12\cdots(n-2)(n-1))$.

the tree (the root being on the level p by convenience). So we assume that the level $(n-1)$ generates once each binary string of $\mathcal{B}_{p-1}(n-2)$. Let $b \in \mathcal{B}_{p-1}(n-2)$ such that $b = b'0^k$, $k \leq p-1$, where b' is either empty or has 1 on its right. Therefore, by inserting 1 on the right of b, or between two entries of the suffix 0^k, or on the left of 0^k, we produce $k+1$ binary strings of $\mathcal{B}_{p-1}(n-1)$ and each binary string $c = b'10^{k-\ell+1}10^{\ell-1}$ obtained by this process has ℓ successors labeled (1), (2), $\ldots$, (ℓ). Conversely, each binary string of $\mathcal{B}_{p-1}(n-1)$ can be uniquely obtained from an element of $\mathcal{B}_{p-1}(n-2)$ by this construction.

Now, we define a map ϕ' from $\mathcal{B}_{p-1}(n-1)$ to $\mathfrak{S}_n(132, 312)$.

Let $b = b_1b_2\cdots b_{n-1} \in \mathcal{B}_{p-1}(n-1)$. If $\pi = \phi'(b_1b_2\cdots b_{n-1})$, then $\pi_1 = p$ and for $i \geq 2$,

$$\pi_i = \begin{cases} p-\ell & \text{if } b_{i-1} \text{ is the } \ell\text{-th 0 from the left} \\ p+\ell & \text{if } b_{i-1} \text{ is the } \ell\text{-th 1 from the left.} \end{cases}$$

For instance, if $p = 5$ and $n = 9$, then $\phi'(1\mathbf{00}11\mathbf{0}1\mathbf{0}) = \mathbf{5}6\mathbf{43}78\mathbf{2}9\mathbf{1}$. In fact, the image by ϕ' of an element in $\mathcal{B}_{p-1}(n-1)$ is a permutation of $\mathfrak{S}_n(132, 312)$ verifying $\pi_1 = p$, or equivalently a permutation of $\mathfrak{S}_n(132, 312)$ that avoids the two patterns $(p+1)\cdots 21$ and $12\cdots(n-$

$p+1)(n-p+2)$. Now let us prove that ϕ' is a bijection from $\mathcal{B}_{p-1}(n-1)$ to $\mathfrak{S}_n(132, 312, (p+1)\cdots 21, 12\cdots(n-p+1)(n-p+2))$. Indeed, a permutation π in $\mathfrak{S}_n(132, 312, (p+1)\cdots 21, 12\cdots(n-p+1)(n-p+2))$ verifies that π_i is either $\max\{\pi_j, j \leq i\}$ or $\min\{\pi_j, j \leq i\}$ which are respectively represented by 1 and 0 in order to obtain a binary string b of length $n-1$ (we do not consider the bit corresponding to π_1). Obviously, if π avoids $(p+1)p\cdots 21$ (resp. $12\cdots(n-p+1)(n-p+2)$) then b does not contain p zeros (resp. $n-p+1$ ones) which means that b contains exactly $p-1$ zeros. Moreover, if $c = b'10^{\ell}10^{k-\ell}$ is obtained from $b = b'10^k \in \mathcal{B}_{p-1}(n-2)$ by inserting 1, then $\phi'(c)$ is obtained from $\phi'(b)$ by inserting n on the same position from the right. $\square$

Notice that the set $\mathfrak{S}_n(132, 312, (p+1)\cdots 21, 12\cdots(n-p+1)(n-p+2))$ depends on a pattern of length $n-p+2$. However, it remains the open question: is it possible to find a finite basis B (independent of n) such that $\mathfrak{S}_n(B)$ is enumerated by $\binom{n-1}{p-1}$?

5 Compositions of n with at most p parts

The set $\mathcal{C}_{\#p}(n)$ of compositions of n with at most p parts is enumerated by $\sum_{k=1}^{p}\binom{n-1}{k-1}$. Moreover, $\mathcal{C}_{\#p}(n)$ is in one-to-one correspondence with the set $\mathcal{B}_{\#(p-1)}(n-1)$ of binary strings of length $n-1$ and having at most $p-1$ zeros. The function φ (see Section 2) shows such a bijection. The following theorem gives succession rules in order to generate $\mathcal{B}_{\#(p-1)}(n-1)$. Since $\mathcal{B}_{\#(p-1)}(n-1) = \bigcup_{i=0}^{p-1}\mathcal{B}_i(n-1)$, these rules are obtained by a simple adaptation of the rules described in the previous section. Here we say that the level of a node in the generating tree is the length of the unique path between the root and this node (the root is on the level 0).

Theorem 2 *For $p \geq 1$, a system $(\Omega_{\#p})$ of succession rules for the set*

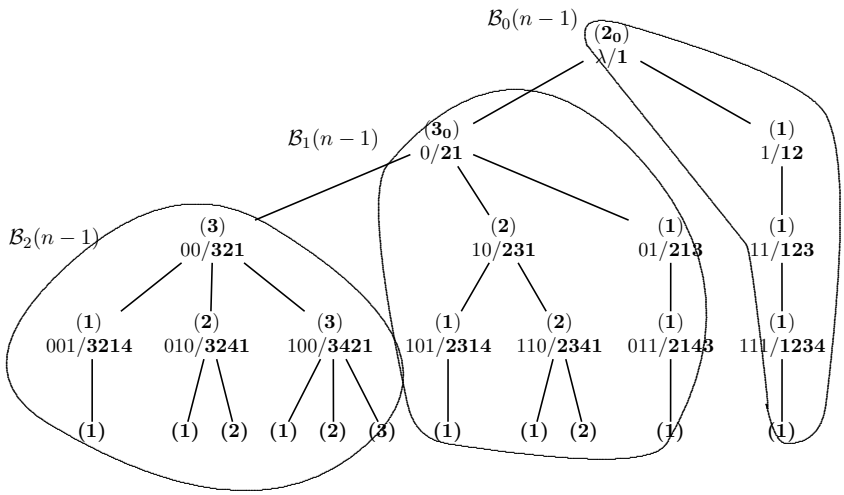


Figure 2. The first levels of the generating tree $(\Omega_{\#3})$. Each node on the level n is coded by one permutation in $\mathfrak{S}_{n+1}(132, 312, 4321)$ or by one binary string in $\mathcal{B}_{\#2}(n)$. Encircled subtrees correspond to the subsets $\mathcal{B}_i(n-1)$ for $0 \leq i \leq 2$.

$\mathcal{C}_{\#p}(n)$ *is:*

$$(\Omega_{\#p}) \begin{cases} (2_0) & \\ (k_0) & \rightsquigarrow ((k+1)_0)(k-1)\cdots(2)(1), \quad \textit{if } 2 \leq k < p \\ (p_0) & \rightsquigarrow (p)(p-1)\cdots(2)(1) \\ (k) & \rightsquigarrow (1)(2)\cdots(k), \quad \textit{if } 1 \leq k \leq p. \end{cases}$$

Each level $n \geq 0$ of the generating tree induced by $(\Omega_{\#p})$ can be coded by the binary strings of $\mathcal{B}_{\#(p-1)}(n)$ or by the permutations in $\mathfrak{S}_{n+1}(132, 312, (p+1)\cdots 21)$.

- *A node labeled (k_0), $2 \leq k \leq p$, is coded by the binary string 0^{k-2} (resp. the permutation $(k-1)(k-2)\cdots 21$) and its successors are obtained either by appending 0 (resp. k) on the left or by inserting 1 (resp. k) between two zeros, on the right or on the left (resp. on the same position from the right as for binary strings).*
- *All other nodes obey to the rules described in Theorem 1.*

Proof. The proof is directly deduced from Theorem 1. Indeed, a node labeled (k_0), $2 \leq k \leq p$, produces $k-1$ nodes labeled $(1), (2), \ldots, (k-1)$ which have the same succession rules as those of Theorem 1, and either one node labeled $((k+1)_0)$ if $k \neq p$ or one node labeled (k) otherwise. This means that the subtree T rooted by a node labeled (k_0), $k \neq p$, has one subtree T_1 rooted by a node labeled $((k+1)_0)$ that generates the sets $\mathcal{B}_k(n-1)$ for $n-1 \geq k$ (see Theorem 1). Now let $T_2 = T \backslash T_1$ be the subtree of T obtained by deleting all nodes of T_1. So, T_2 generates the set $\mathcal{B}_{k-1}(n-1)$ for $n-1 \geq k-1$. Finally, the complete generating tree of $(\Omega_{\#p})$ is exactly the union of subtrees T_i for $0 \leq i \leq p-1$, where T_i generates the set $\mathcal{B}_i(n-1)$, where $n-1 \geq i$. This proves that $(\Omega_{\#p})$ generates $\mathcal{B}_{\#(p-1)}(n)$. By duality and with the same argument, it also generates all permutations in $\mathfrak{S}_{n+1}(132, 312, (p+1)\cdots 21)$ (see Figure 2). $\square$

6 Compositions of n with parts 1 and p

Let $\mathcal{C}_{1,p}(n)$ be the set of compositions of n with parts 1 and p. The following bijection φ' (see for example [12]) gives a bijection between $\mathcal{C}_{1,p}(n)$ and the set $\mathcal{B}_{\geq p-1}(n-p+1)$ of binary strings of length $n-p+1$ with at least $p-1$ zeros between two ones.

Definition 3 *Let $c = (c_1, c_2, \ldots, c_\ell)$ be a composition of n such that $c_i \in \{1, p\}$ for all $i \leq \ell$. We define the bijection φ' between $\mathcal{C}_{1,p}(n)$ and $\mathcal{B}_{\geq p-1}(n-p+1)$ by the following algorithmical process. We initialize $b = \lambda$ (the empty string). For each i from 1 to ℓ, if $c_i = 1$, then we modify b by appending 0 on its right; otherwise (i.e., $c_i = p$), we modify b by appending 10^{p-1} on its right. Finally, we delete $p-1$ zeros on the right of b which defines a binary string b of length $n-p+1$ with at least $p-1$ zeros between two ones.*

For instance, if $n = 12$, $p = 3$ and $c = (1, 3, 1, 3, 3, 1)$, then $\varphi'(c) = 0\mathbf{100}0\mathbf{100}10$.

Theorem 3 *For $p \geq 2$, a system $(\Omega_{1,p})$ of succession rules for the $(1,p)$-compositions is given by:*

$$(\Omega_{1,p})\begin{cases}(2)\\ (2)\rightsquigarrow(2)(1_0)\\ (1_i)\rightsquigarrow(1_{i+1}), & \textit{for } 0\le i<p-2\\ (1_{p-2})\rightsquigarrow(2).\end{cases}$$

Each level n of the generating tree induced by $(\Omega_{1,p})$ can be coded by the binary strings in $\mathcal{B}_{\geq p-1}(n)$ (the root is on the level 0). A binary string of length n can be obtained from a string of length $n-1$ by inserting 0 or 1 on the last position (see Figure 3).

Proof. We will prove by induction that the nodes on the level k can be coded by the binary strings of the set $\mathcal{B}_{\geq p-1}(k)$ for all $k \geq 0$. Remark that this is true for the root which is coded by the empty string λ (by convenience the level of the root will be 0). So let us assume that each level $j \leq k$ is coded by the elements of $\mathcal{B}_{\geq p-1}(j)$.

Let α be a binary string of length $k+1$ on the level $k+1$ and let $\beta \in \mathcal{B}_{\geq p-1}(k)$ be its predecessor on the level k.

If α is obtained from β by inserting 0 on its right, then α also belongs to $\mathcal{B}_{\geq p-1}(k+1)$. If α is obtained by inserting 1 on the right of β, then β has two sons, so its label is (2), and its predecessor γ is labeled by (2) or (1_{p-2}), then $\beta = \gamma 0$.

(i) If γ is labeled (1_{p-2}), its predecessor γ_1 is labeled (1_{p-3}). We repeat this process until γ_{p-2}, *i.e.,* until we reach the label (1_0). Then γ_{p-2} is obtained from a binary string β' labeled (2) by inserting 1 on the right of β'.

Thus α is of the form $\alpha = \beta 1 = \beta' 1 0^{p-1} 1$. Moreover, $\beta' \in \mathcal{B}_{\geq p-1}(k-p)$ by the recurrence hypothesis. We conclude that $\alpha \in \mathcal{B}_{\geq p-1}(k+1)$.

(ii) If γ is labeled (2), its predecessor γ_1 is labeled (1_{p-2}) or (2). If γ_1 is labeled (1_{p-2}), we return to the case (i) just above, so α has at least $p-1$ consecutive zeros between two ones. If γ_1 is labeled (2), we repeat the process by replacing γ_1 with γ and it will finish when: either we reach the label (1_{p-2}) which corresponds to the

case (i), or we reach the root labeled (2). In any case, α contains $p-1$ consecutive zeros between two ones. Then $\alpha \in \mathcal{B}_{\geq p-1}(k+1)$.

Conversely, we consider $\alpha \in \mathcal{B}_{\geq p-1}(k+1)$ and we construct a path on the generating tree $(\Omega_{1,p})$ which generates this string. We distinguish two cases:

- $\alpha = \alpha' 10^j$, where $j \leq p-1$. So $\alpha' 1 \in \mathcal{B}_{\geq p-1}(k+1-j)$. Therefore, α' is labeled (2) and α is obtained from α' with either the path $(2)/\alpha' \rightsquigarrow (1_0)/\alpha' 1 \rightsquigarrow (1_1)/\alpha' 10 \rightsquigarrow \cdots \rightsquigarrow (1_j)/\alpha' 0^j$ or $(2)/\alpha' \rightsquigarrow (1_0)/\alpha' 1 \rightsquigarrow (1_1)/\alpha' 10 \rightsquigarrow \cdots \rightsquigarrow (1_{p-2})/\alpha' 10^{p-2} \rightsquigarrow (2)/\alpha' 10^{p-1}$;
- $\alpha = \alpha'' 10^j$, where $j \geq p$. Then $\alpha' = \alpha'' 10^{p-1} \in \mathcal{B}_{\geq p-1}(k+p-j)$. So, α is obtained from α' with a path of nodes all labeled (2) in the generating tree.

We repeat the same process by replacing α with α' and we will find the path from the root of the generating tree $(\Omega_{1,p})$ to reach α. For instance, $\alpha = 010010001$, the path to reach α from the root in the generating tree $(\Omega_{1,3})$ is:

$(2)/\lambda \rightsquigarrow (2)/0 \rightsquigarrow (1_0)/01 \rightsquigarrow (1_1)/010 \rightsquigarrow (2)/0100 \rightsquigarrow (1_0)/01001 \rightsquigarrow (1_1)/010010 \rightsquigarrow (2)/0100100 \rightsquigarrow (2)/01001000 \rightsquigarrow (1_0)/010010001$.

We finally conclude that the generating tree induced by $(\Omega_{1,p})$ is coded by the set $\mathcal{B}_{\geq p-1}(n)$ of binary strings of length n with at least $p-1$ zeros between two ones. □

Theorem 4 *Each level $n \geq 0$ of the generating tree of $(\Omega_{1,p})$ can be coded by the permutations in $\mathfrak{S}_{n+1}(231, 312, 321, 2\overline{134\cdots(p+1)}(p+3)(p+2))$. A permutation of length n is obtained from a permutation π of length $(n-1)$ by inserting n either on the right of π or just before its last entry (see Figure 3).*

Proof. In [12], Baril and Moreira showed there is a bijection f between $\mathcal{B}_{\geq p-1}(n)$ and $\mathfrak{S}_{n+1}(231, 312, 321, 2\overline{134\cdots(p+1)}(p+3)(p+2))$. Moreover, the insertion of 0 (resp. 1) on the right of $b \in \mathcal{B}_{\geq p-1}(n)$ is equivalent to the insertion of $(n+1)$ on the right of $\pi = f(b)$ (resp. just before the last entry). This means that $\mathfrak{S}_{n+1}(231, 312, 321, 2\overline{134\cdots(p+1)}(p+3)(p+2))$ also codes the generating tree $(\Omega_{1,p})$. □

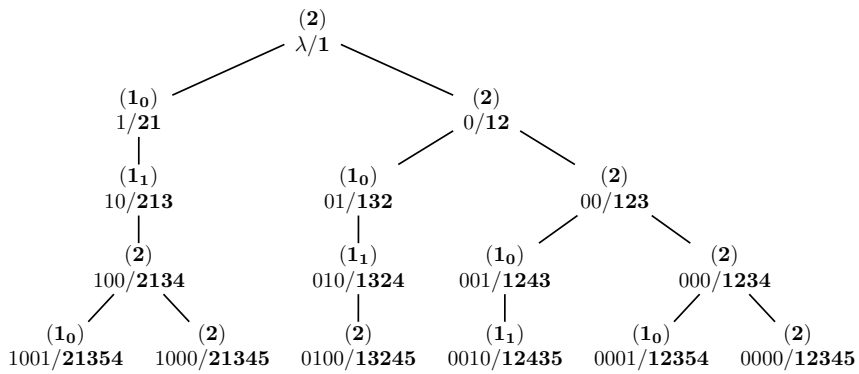


Figure 3. The first five levels of the generating tree $(\Omega_{1,3})$. Each node on the level n is coded by one permutation in $\mathfrak{S}_n(231, 312, 321, 21\overline{34}65)$ or by one binary string in $\mathcal{B}_{\geq 2}(n-1)$.

7 Compositions of n without part of size p

Let $\mathcal{C}_{\widehat{p}}(n)$ be the set of compositions of n without part of size p. The bijection φ (see Section 2) transforms $\mathcal{C}_{\widehat{p}}(n)$ into the set $\mathcal{B}_{\widehat{p-1}}(n-1)$ of the binary strings of length $(n-1)$ without run of ones of length $p-1$ knowing that a run of ones is a maximal substring of consecutive ones. For instance, the binary string $b = 0\mathbf{11}00\mathbf{111}0\mathbf{1}$ contains three runs of ones illustrated in boldface. In this section, we consider the concept of jumping succession rules introduced in [23]. This allows the construction of an element of size greater than $n+1$ from an element of size n (see Section 1).

Theorem 5 *For $p \geq 2$, a system of jumping succession rules $(\Omega_{\widehat{p}})$ for*

the compositions without part of size p is given by:

$$(\Omega_{\widehat{p}})\begin{cases}(2_0)\\ (2_i)\rightsquigarrow(2_0)(2_{i+1}), & \text{for } 0\le i\le p-3\\ (2_{p-2})\overset{1}{\rightsquigarrow}(2_0)\\ \qquad\overset{2}{\rightsquigarrow}(2_{p-1})\\ (2_{p-1})\rightsquigarrow(2_0)(2_{p-1}).\end{cases}$$

Each level n of the generating tree of $(\Omega_{\widehat{p}})$ is coded by the set $\mathcal{B}_{\widehat{p-1}}(n)$ (the root is on the level 0). Let b be a binary string in $\mathcal{B}_{\widehat{p-1}}(n)$ corresponding to a node of level n. Then b has two successors:
- if the two successors of b are on the level $n+1$, they are obtained by inserting 0 or 1 on the right of b.
- if one successor of b is on the level $n+1$ and the other on the level $n+2$, we insert 0 on the right of b in order to obtain the successor on the level $n+1$ and we insert 11 on the right of b in order to obtain that on the level $n+2$ (see Figure 4).

Proof. We proceed by induction. The root of the tree is coded by the empty string λ (by convenience the level of the root is 0). We assume that each level $k\le n$ is coded by the elements of $\mathcal{B}_{\widehat{p-1}}(k)$. Let α be the binary string of length $(n+1)$ corresponding to a node on the level $n+1$ and let β be its predecessor on the level n or $n-1$, then β belongs to $\mathcal{B}_{\widehat{p-1}}(n)$ or $\mathcal{B}_{\widehat{p-1}}(n-1)$.

(i) If β is on the level $n-1$, then β is labeled (2_{p-2}), α is labeled (2_{p-1}) and α is obtained from β by inserting 11 on its right. Thus β is obtained from its predecessor β_1 labeled (2_{p-3}) (if $p-3>0$) by inserting 1 on its right. We repeat this process until we create β_{p-2}, *i.e.,* until we reach a node labeled (2_0). Necessarily β_{p-2} is obtained from its predecessor β' by inserting 0 on its right. Thus $\alpha=\beta 11=\beta' 01^p$, and α does not contain any run of ones of length $p-1$, which implies that $\alpha\in\mathcal{B}_{\widehat{p-1}}(n+1)$.

(ii) If β is on the level n, then α is obtained from β by inserting 0 or 1 on its right.

- if α is obtained from β by inserting 0 on its right, then α obviously belongs to $\mathcal{B}_{\widehat{p-1}}(n+1)$.

- if α is obtained from β by inserting 1 on its right, then β is labeled (2) or (2_i), with $i < p-2$:

(a) if β is labeled (2), it is obtained from its predecessor (also labeled (2)) by inserting 1 on its right. We repeat this process until we reach the label (2_{p-2}). So we retrieve the case (i) above. Then, $\alpha = \alpha' 1^\ell = \beta' 0 1^p 1^\ell$, with $\ell > 0$, and $\beta' \in \mathcal{B}_{\widehat{p-1}}(n-p-l)$. Therefore $\alpha \in \mathcal{B}_{\widehat{p-1}}(n+1)$.

(b) if β is labeled (2_i), with the same process of the case (i), we have $\alpha = \beta' 1^{i+1}$, with $\beta' \in \mathcal{B}_{\widehat{p-1}}(n-i)$ with $i+1 < p-1$. Thus $\alpha \in \mathcal{B}_{\widehat{p-1}}(n+1)$.

Conversely, each string α in $\mathcal{B}_{\widehat{p-1}}(n+1)$ can be constructed on the level $n+1$ of the generating tree (Ω_p). Indeed, if $\beta = \alpha 0$, then $\beta \in \mathcal{B}_{\widehat{p-1}}(n+2)$. So β can be decomposed as $1^{c_1-1}01^{c_2-1}0\cdots 1^{c_\ell-1}0$ such that $c = (c_1, c_2, \ldots, c_\ell) \in \mathcal{C}_{\widehat{p-1}}(n+3)$ (see the bijection $\varphi : \mathcal{C}_{\widehat{p}}(n+1) \rightsquigarrow \mathcal{B}_{\widehat{p-1}}(n)$). Let $\beta = \beta' 1^{c_\ell - 1} 0$, then $\beta' \in \mathcal{B}_{\widehat{p-1}}(n+2-c_\ell)$. We distinguish two cases:

- if $c_\ell < p$, then β is obtained from β' on the generating tree (Ω_p) by the path
  $(2_1) \rightsquigarrow (2_2) \rightsquigarrow \cdots \rightsquigarrow (2_{c_\ell - 1}) \rightsquigarrow (2_0)$,

- if $c_\ell > p$, then β is obtained from β' on the generating tree (Ω_p) by the path
  $(2_1) \rightsquigarrow (2_2) \rightsquigarrow \cdots \rightsquigarrow (2_{p-2}) \overset{2}{\rightsquigarrow} (2_{p-1}) \rightsquigarrow (2_{p-1})^{c_\ell - p - 1} \rightsquigarrow (2_0)$.

We repeat this process by replacing β with β' and we obtain a path from the root to β on the generating tree $(\Omega_{\widehat{p}})$. For instance, on the generating tree $(\Omega_{\widehat{3}})$, if $\alpha = 1110100$, then the path for reaching α from the root is:
$(2_0)/\lambda \rightsquigarrow (2_1)/1 \overset{2}{\rightsquigarrow} (2_2)/111 \rightsquigarrow (2_0)/1110 \rightsquigarrow (2_1)/11101 \rightsquigarrow (2_0)/111010 \rightsquigarrow (2_0)/1110100$. $\square$

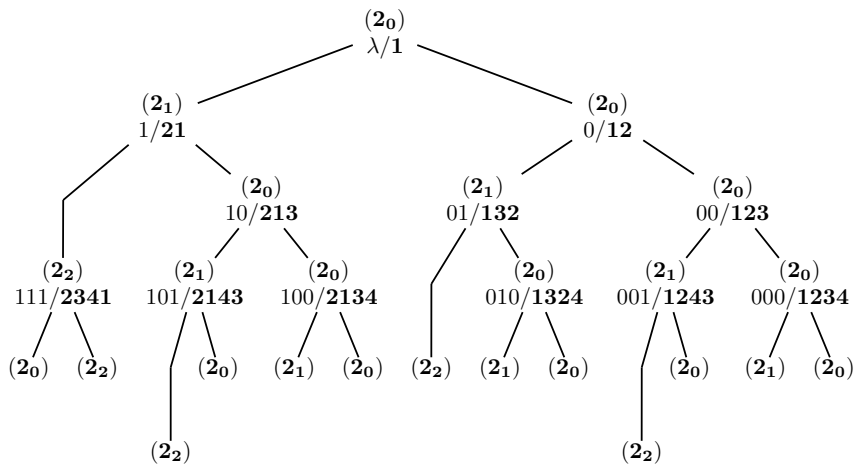


Figure 4. The first levels of the generating tree $(\Omega_{\widehat{3}})$. Each node on the level $n \geq 0$ is coded by a permutation in $\mathfrak{S}_{n+1}(312, 321, 234\dot{1})$ or by a binary string in $\mathcal{B}_{\widehat{2}}(n)$.

In order to code the generating tree by permutations avoiding patterns, we use the new pattern $23\cdots n\dot{1}$ presented in Section 2.

Theorem 6 *Each level n of the generating tree induced by $(\Omega_{\widehat{p}})$ can be coded by the permutations $\pi \in \mathfrak{S}_n(312, 321, 23\cdots(p+1)\dot{1})$ as follows:*
• *If the two successors of π belong to the level $(n+1)$, they are obtained by inserting $n+1$ on the right or just before the last entry of π.*
• *If a successor of π is on the level $(n+1)$ and the other on the level $(n+2)$, we insert $n+1$ on the right of π in order to obtain the successor on the level $n+1$, and we insert $(n+1)(n+2)$ just before the last entry of π in order to obtain that of the level $(n+2)$ (see Figure 4).*

In order to prove this theorem, we present the following proposition.

Proposition 1 *The map ϕ defined in Section 2 is a bijection from $\mathcal{B}_{\widehat{p}}(n)$ to $\mathfrak{S}_{n+1}(312, 321, 23\cdots(p+2)\dot{1})$.*

Proof. Recall that ϕ is a bijection from the set of binary strings of length n to the set of $(n+1)$-length permutations avoiding 321 and 312. So we will prove that $\phi(\mathcal{B}_{\widehat{p}}(n)) = \mathfrak{S}_{n+1}(321, 312, 23\cdots(p+2)\dot{1})$.

Let b be a binary string in $\mathcal{B}_{\widehat{p}}(n)$ and π be its image by ϕ in $\mathfrak{S}_{n+1}(321, 312)$. As $\phi(b) \in \mathfrak{S}_{n+1}(321, 312)$, there exist (see [36]) some indices $0 = k_0 < k_1 < \cdots < k_r < \cdots < k_m = n$ such that π is divided into m blocks

$$\pi = \boxed{\pi_1\pi_2\cdots\pi_{k_1}} \cdots \boxed{\pi_{k_{r-1}+1}\pi_{k_{r-1}+2}\cdots\pi_{k_r}} \cdots$$
$$\cdots \boxed{\pi_{k_{m-1}+1}\pi_{k_{m-1}+2}\cdots\pi_{k_m}}$$

satisfying the two following conditions:

(i) the rightmost elements of each block are in increasing order (*i.e.*, $1 = \pi_{k_1} < \pi_{k_2} < \cdots < \pi_{k_m}$);

(ii) each block $\pi_{k_{r-1}+1}\pi_{k_{r-1}+2}\cdots\pi_{k_r}$ having at least two elements is of the form $(a+1)(a+2)\cdots(a+k_r-k_{r-1}-1)a$ with $a = \pi_{k_r}$.

For instance, if $b = 110111$, then $\pi = \mathbf{231}5674$. It is remarkable from the definition of ϕ that if we add on the last position of b one occurrence of 0 and then divide the obtained binary string into separated blocks where each block contains exactly one occurrence of 0 at its end, then we obtain m blocks corresponding to the m blocks of $\phi(b)$. Furthermore, the number of consecutive 1s in each block of b is also the number, minus one, of elements of the respective block in $\phi(b)$ (see Definition 2). This leads us to the following claim: for $q \geq 2$, if one subsequence of $\phi(b)$ is a pattern $23\ldots q1$ (of length q), then this sequence must belong to only one block of $\phi(b)$ (since the smallest element of each block of $\phi(b)$ is greater than the largest element of the previous blocks of $\phi(b)$). Moreover, since b does not contain exactly p consecutive 1s, $\phi(b)$ does not contain any block of length $p+1$ exactly. Hence, if the subsequence $23\ldots p(p+1)1$ appears in $\phi(b)$, then we can extend it (without considering positions of extended entries) into a sequence $23\ldots(p+1)(p+2)1$, which means that $\phi(b)$ avoids $23\ldots(p+1)(p+2)\dot{1}$.

Conversely, we take a permutation π of $\mathfrak{S}_{n+1}(321, 312, 23\ldots(p+1)(p+2)\dot{1})$. We also have the decomposition of π into blocks as above. It needs to show that $\phi^{-1}(\pi)$ belongs to $B_{\widehat{p}}(n)$. This is induced by the fact

that all blocks of π considered as subsequences of π are either of length less than $p+1$ (if they do not contain the reduced pattern $23\dots p(p+1)1$) or more than $p+1$ (if they contain the extended pattern $23\dots(p+2)1$). Therefore, $\phi^{-1}(\pi)$ is a binary string without p consecutive 1s. $\square$

Now, the proof of Theorem 6 is obtained using the following remark. The insertion of 0 (resp. 1) on the right of $b \in \mathcal{B}_{\widehat{p}}(n)$ is equivalent to the insertion of $n+1$ on the right of $\pi = \phi(b)$ (resp. just before the last entry). And also the insertion of 11 on the right of $b \in \mathcal{B}_{\widehat{p}}(n)$ is equivalent to the insertion of $(n+1)(n+2)$ just before the last entry of $\pi = \phi(b)$. This means that $\mathfrak{S}_n(312, 321, 23\cdots(p+1)\dot{1})$ also codes the generating tree $(\Omega_{\widehat{p}})$.

8 Algorithmic considerations and conclusion

In this section, we explain how all studied classes can be generated efficiently.

Let $\pi \in \mathfrak{S}_n$; the *sites* of π are the positions between two consecutive entries, before the first and after the last entry. We suppose that the sites are numbered from 1 to $n+1$ and from right to left. For example, the third site of the permutation $\pi = 463512$ is between the entries 5 and 1. Moreover, let τ be an n-length permutation (or equivalently a binary string) on a generating tree defined by a succession rule (Ω). Then, the i-th site of τ is said *active* if the element obtained from τ by the insertion of a value into this site also belongs to the generating tree. The active sites of τ are *right-justified* (see [20]) if all sites to the right of the leftmost active site are also active. If each element on the generating tree is right-justified, we will say that the generated class is *regular*. It is crucial to notice that all classes defined in this paper are *regular*.

An algorithm runs in Constant Amortized Time (CAT) if the amount of computations, after a small amount of preprocessing, is proportional to the number of generated objects. Many CAT algorithms exist in the literature, but we will take that presented in [20, 33]. This recursive algorithm acts on regular classes and enables us to ensure

that we can generate all successors of a given node in constant amortized time. Thus the total amount of computations is proportional to the number of recursive calls. Moreover, the number *obj* of generated objects is at least $\frac{c-c_1}{2}$, where c is the total number of recursive calls, and c_1 is the number of recursive calls of degree one. So, for each generating tree of this study, we will calculate c_1.

- We immediately have $c_1 = 0$ for the generating tree defined in Section 7 and $\frac{c}{obj}$ evolves as $\mathcal{O}(1)$.

- In Section 6, a simple observation gets $c_1 \leq p(c - c_1)$ and $\frac{c}{obj}$ evolves as $\mathcal{O}(p)$.

- The generating tree of Section 5 is constituted of several generating trees of Section 4. So it suffices to compute c_1 for Section 4 (see below).

- In Section 4, each node produces exactly once a node of degree one. Thus the number of nodes of degree one and on levels at most $n-1$ in the generating tree (Ω_p) is equal to

$$c_1 = \sum_{i=p-1}^{n-1} \binom{i}{p-1}.$$

A simple calculation proves that if $p \geq \lfloor \frac{n}{2} \rfloor$ and $n \geq 4$, then $c_1 \leq 2\binom{n}{p-1}$, which means that the number of nodes of degree one divided by 2 is smaller than the number of generated objects $obj = \binom{n}{p-1}$ for $p \geq \lfloor \frac{n}{2} \rfloor$. In this case, the complexity is $\mathcal{O}(1)$. The case $p \leq \lfloor \frac{n}{2} \rfloor$ is obtained *mutatis mutandis* by interchanging 0 with 1 in the binary strings of the generating tree.

Finally, this means that the total amount of computation divided by the number of objects is bounded by a constant independently to the size of the objects. Therefore all studied classes in this paper can be generated in Constant Amortized Time (CAT) using an algorithm similar to that of [20, 33]. We summarize our results in Table 1 that contains the succession rules of each studied class and their corresponding pattern avoiding permutation classes.

Table 1. Classes of compositions, corresponding succession rules and corresponding classes of pattern avoiding permutations.

Classes	Succession rules	Avoided patterns
$C(n)$	(2) $(2) \rightsquigarrow (2)(2)$	$\{321, 312\}$ [4, 26]
	(2) $(k) \rightsquigarrow (k+1)(1)^{k-1}$	$\{321, 231\}$ [4, 26]
$C_p(n)$	(p) $(k) \rightsquigarrow (1)(2)\cdots(k)$	$\{132, 312, (p+1)p\cdots 21,$ $12\cdots(n-p)(n-p+1)\}$
$C_{\#p}(n)$	(2_0) $(k_0) \rightsquigarrow ((k+1)_0)(k-1)\cdots(2)(1)$ for $2 \le k < p$ $(p_0) \rightsquigarrow (p)(p-1)\cdots(2)(1)$ $(k) \rightsquigarrow (1)(2)\cdots(k-1)(k)$ for $1 \le k \le p$	$\{132, 312, (p+1)p\cdots 21\}$
$C_{\le p}(n)$	(2_0) $(2_0) \rightsquigarrow (2_0)(2_1)$ $(2_i) \rightsquigarrow (2_0)(2_{i+1})$, for $1 \le i < p-2$ $(2_{p-2}) \rightsquigarrow (2_0)(1)$ $(1) \rightsquigarrow (2_0)$	$\{321, 312, 234\cdots(p+1)1\}$ [4, 11]
	(2) $(k) \rightsquigarrow (k+1)(1)^{k-1}$ $(p) \rightsquigarrow (p)(1)^{k-1}$	$\{312, 231, (p+1)p\cdots 321\}$ [4, 11]
$C_{1,p}(n)$	(2) $(2) \rightsquigarrow (1_0)(2)$ $(1_i) \rightsquigarrow (1_{i+1})$, for $0 \le i < p-2$ $(1_{p-2}) \rightsquigarrow (2)$	$\{231, 312, 321,$ $2\overline{134\cdots(p+1)}(p+3)(p+2)\}$
$C_{\widehat{p}}(n)$	(2_0) $(2_i) \rightsquigarrow (2_0)(2_{i+1})$, for $0 \le i \le p-3$ $(2_{p-2}) \overset{1}{\rightsquigarrow} (2_0)$ $\overset{2}{\rightsquigarrow} (2_{p-1})$ $(2_{p-1}) \rightsquigarrow (2_0)(2_{p-1})$	$\{312, 321, 23\cdots(p+1)\dot{1}\}$

References

[1] K. Alladi and V.E. Hoggatt. Compositions with ones and twos. *Fibonacci Quarterly*, 13(3):233–239, 1975.

[2] S. Bacchelli, E. Barcucci, E. Grazzini, and E. Pergola. Exhaustive generation of combinatorial objects by ECO. *Acta Informatica*, 40:585–602, 2004.

[3] C. Banderier, P. Flajolet, D. Gardy, M. Bousquet-Melou, A. Denise, and D. Gouyou-Beauchamps. Generating functions for generating trees. *Discrete Mathematics*, 246:29–55, 2002.

[4] E. Barcucci, A. Bernini, and M. Poneti. From Fibonacci to Catalan permutations. *PU.M.A.*, 2007(1-2):1–18, 2006.

[5] E. Barcucci, A. Del Lungo, and E. Pergola. Random generation of trees and other combinatorial objects. *Theoretical Computer Science*, 218(2):218–232, 1999.

[6] E. Barcucci, A. Del Lungo, E. Pergola, and R. Pinzani. A methodology for plane tree enumeration. *Discrete Mathematics*, 180:45–64, 1998.

[7] E. Barcucci, A. Del Lungo, E. Pergola, and R. Pinzani. Directed animals, forests of trees and permutations. *Discrete Mathematics*, 204:41–71, 1999.

[8] E. Barcucci, A. Del Lungo, E. Pergola, and R. Pinzani. ECO: a methodology for the enumeration of combinatorial objects. *Journal of Difference Equations and Applications*, 5:435–490, 1999.

[9] J.-L. Baril. More restrictive Gray codes for some classes of pattern avoiding permutations. *Information Processing Letters*, 109:799–804, 2009.

[10] J.-L. Baril. Classical sequences revisited with permutations avoiding dotted pattern. *Electronic Journal of Combinatorics*, 18:p#178, 2011.

[11] J.L. Baril and P.T. Do. ECO-generation for p-generalized Fibonacci and Lucas permutations. *Pu.M.A.*, 17(1-2):19–37, 2006.

[12] J.L. Baril and C. Moreira Dos Santos. Gray code for compositions of n with parts 1 and p. *Advances and Applications in Discrete Mathematics*, 3(1):67–84, 2009.

[13] S. Brlek, E. Duchi, E. Pergola, and S. Rinaldi. On the equivalence problem for succession rules. *Discrete Math.*, 298:142–154, 2005.

[14] L. Carlitz. Restricted compositions. *The Fibonacci Quart.*, 14:254–264, 1976.

[15] T. Chinburg, C.D. Savage, and H.S. Wilf. Combinatorial families that exponentially far from being listable in Gray code sequence. *Transactions of the AMS*, 351:379–402, 1999.

[16] P. Chinn and S. Heubach. $(1, k)$-Compositions. *Congressus Numerantium*, 164:183–194, 2003.

[17] P. Chinn and S. Heubach. Compositions of n with no occurrence of k. *Congressus Numerantium*, 164:33–51, 2003.

[18] A. Del Lungo, A. Frosini, and S. Rinaldi. ECO method and the exhaustive generation of convex polyominoes. *DMTCS*, pages 103–116, 2003.

[19] E. Deutsch, L. Ferrari, and S. Rinaldi. Production matrices. *Advances in Applied Mathematics*, 34:101–122, 2005.

[20] P.T. Do and V. Vajnovszki. Exhaustive generation of some classes of pattern avoiding permutations using succession functions. *Conference in honor of Donald E. Knuth*, 2007.

[21] E. Duchi, A. Frosini, R. Pinzani, and S. Rinaldi. A note on rational succession rules. *Journal of Integer Sequences, Article 03.1.7*, 6, 2003.

[22] L. Ferrari, E. Pergola, R. Pinzani, and S. Rinaldi. An algebraic characterization of the set of succession rules. *Theoretical Computer Science*, 281:351–367, 2002.

[23] L. Ferrari, E. Pergola, R. Pinzani, and S. Rinaldi. Jumping succession rules and their generating functions. *Discrete Math*, 271:29–50, 2003.

[24] R.P. Grimaldi. Compositions with odd summands. *Congressus Numerantium*, 142:113–127, 2000.

[25] R.P. Grimaldi and S. Heubach. Binary strings without odd runs of zeros. *Ars Combinatoria*, 75:241–255, 2005.

[26] O. Guibert. Combinatoire des permutations motifs exclus en liaison avec mots, cartes planaires et tableaux de Young. *Ph.D Thesis, Université Bordeaux 1*, 1995.

[27] S. Heubach and T. Mansour. Combinatorics of compositions and words. *Discrete Mathematics and its Applications (Boca Raton). CRC Press, Boca Raton, FL.*

[28] S. Heubach and T. Mansour. Compositions of n with parts in a set. *Congressus Numerantium*, 168:127–143, 2004.

[29] A. Juarna and V. Vajnovszki. Combinatorial isomorphism between Fibonacci classes. *Journal of Discrete Mathematical Science and Cryptography*, 11(2):147–158, 2008.

[30] S. Kitaev, T. McAllister, and K. Petersen. Enumerating segmented patterns in compositions and encoding with restricted permutations. *Integers: Electronic Journal of Combinatorial Number Theory*, 6:16pp, 2006.

[31] P. Klingsberg. A Gray code for compositions. *J. Algorithms*, 3:41–44, 1982.

[32] A. Knopfmacher and H. Prodinger. On Carlitz compositions. *European Journal of Combinatorics*, 19(5):579–589, 1998.

[33] T. Mansour, W.M. Dukes, M.F. Flanagan, and V. Vajnovszki. Combinatorial Gray codes for classes of pattern avoiding permutations. *Theoretical Computer Science*, 396:35–49, 2008.

[34] L. Pudwell. Enumeration schemes for permutations avoiding barred patterns. *The electronic Journal of Combinatorics*, 17,R29, 2010.

[35] R. Simion and F.W. Schmidt. Restricted permutations. *European J. Combin.*, 6:383–406, 1985.

[36] V. Vajnovszki. A loopless generation of bitstrings without p consecutive ones. *Discrete Mathematics and Theoretical Computer Science*, 1:227–240, Springer 2001.

[37] T.R. Walsh. Loop-free sequencing of bounded integer compositions. *J. Combin. Math. Combin. Comput.*, 33:323–345, 2000.

Jean-Luc Baril, Phan-Thuan Do Received July 16, 2012

Jean-Luc Baril,
LE2I UMR-CNRS 5158, Université de Bourgogne
B.P. 47 870, 21078 DIJON-Cedex France
E–mail: *barjl@u-bourgogne.fr*

Phan-Thuan Do,
Hanoi University of Science and Technology, Vietnam
Department of Computer Science
E–mail: *thuandp@soict.hut.edu.vn*

Information Processing Letters 113 (2013) 17–22

Contents lists available at SciVerse ScienceDirect

Information Processing Letters

www.elsevier.com/locate/ipl

Statistics-preserving bijections between classical and cyclic permutations

Jean-Luc Baril

LE2I UMR-CNRS 5158, Université de Bourgogne, B.P. 47 870, 21078 Dijon Cedex, France

ARTICLE INFO

Article history:
Received 3 January 2012
Received in revised form 28 September 2012
Accepted 15 October 2012
Available online 16 October 2012
Communicated by A. Muscholl

Keywords:
Combinatorial problems
Bijections
Statistics on permutations
Excedance
Left-to-right maximum
Descent
Quasi-fixed point
Cyclic permutation
Derangement

ABSTRACT

Recently, Elizalde (2011) [2] has presented a bijection between the set C_{n+1} of cyclic permutations on $\{1, 2, \ldots, n+1\}$ and the set of permutations on $\{1, 2, \ldots, n\}$ that preserves the descent set of the first n entries and the set of weak excedances. In this paper, we construct a bijection from C_{n+1} to S_n that preserves the weak excedance set and that transfers quasi-fixed points into fixed points and left-to-right maxima into themselves. This induces a bijection from the set D_n of derangements to the set C^q_{n+1} of cycles without quasi-fixed points that preserves the weak excedance set. Moreover, we exhibit a kind of discrete continuity between C_{n+1} and S_n that preserves at each step the set of weak excedances. Finally, some consequences and open problems are presented.

© 2012 Elsevier B.V. All rights reserved.

1. Introduction and notation

Let S_n be the set of permutations of length n, *i.e.*, all one-to-one correspondences from $[n] = \{1, 2, \ldots, n\}$ into itself. We represent a permutation $\sigma \in S_n$ in one-line notation, $\sigma = \sigma_1\sigma_2 \ldots \sigma_n$ where $\sigma_i = \sigma(i)$, $1 \leqslant i \leqslant n$. Moreover, if $\gamma = \gamma(1)\gamma(2)\ldots\gamma(n)$ is a length n permutation then the *product* $\gamma \cdot \sigma$ is the permutation $\gamma(\sigma_1)\gamma(\sigma_2)\ldots\gamma(\sigma_n)$. In S_n, a *k-cycle* $\sigma = \langle i_1, i_2, \ldots, i_k\rangle$ is a length n permutation verifying $\sigma(i_1) = i_2$, $\sigma(i_2) = i_3, \ldots, \sigma(i_{k-1}) = i_k$, $\sigma(i_k) = i_1$ and $\sigma(j) = j$ for $j \in [n]\backslash\{i_1, \ldots, i_k\}$. In particular, a 2-cycle is called a *transposition*. Let $C_n \subset S_n$ be the set of n-cycles. The elements of C_n will be called cyclic permutations (or cycles for short). Obviously C_{n+1} and S_n have the same cardinality.

Any permutation $\sigma \in S_n$ is uniquely decomposed as a product of transpositions of the following form:

$$\sigma = \langle p_1, 1\rangle \cdot \langle p_2, 2\rangle \cdot \langle p_3, 3\rangle \cdots \langle p_n, n\rangle = \prod_{i=1}^{n} \langle p_i, i\rangle, \qquad (1)$$

where p_i are some integers such that $1 \leqslant p_i \leqslant i \leqslant n$. Conversely, any such decomposition provides a permutation in S_n. Therefore, (1) yields a bijection from S_n to the product set $T_n = [1] \times [2] \times \cdots \times [n]$. Then we have another way to represent a permutation:

Definition 1. The transposition array of a permutation $\sigma = \prod_{i=1}^{n} \langle p_i, i\rangle \in S_n$ is defined by $p_1 p_2 \ldots p_n \in T_n$.

For example, if $\sigma = 1\ 4\ 5\ 6\ 3\ 2$ then its decomposition into transpositions is $\langle \mathbf{1}, 1\rangle \cdot \langle \mathbf{2}, 2\rangle \cdot \langle \mathbf{3}, 3\rangle \cdot \langle \mathbf{2}, 4\rangle \cdot \langle \mathbf{3}, 5\rangle \cdot \langle \mathbf{4}, 6\rangle$, and its corresponding transposition array is 1 2 3 2 3 4. Notice that this decomposition is used in [1] in order to obtain Gray codes for restricted classes of length n permutations.

Let σ be a permutation in S_n. A *descent* of σ is a position i, $1 \leqslant i \leqslant n-1$, such that $\sigma(i) > \sigma(i+1)$. Let $D(\sigma)$ be the set of descents of σ. An *excedance* (resp.

E-mail address: barjl@u-bourgogne.fr.

0020-0190/$ – see front matter © 2012 Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.ipl.2012.10.003

weak excedance) of σ is a position i, $1 \leqslant i \leqslant n$, such that $\sigma(i) > i$ (resp. $\sigma(i) \geqslant i$). The set of excedances (resp. weak excedances) of σ will be denoted $E(\sigma)$ (resp. $WE(\sigma)$). A *left-to-right maximum* is a position i, $1 \leqslant i \leqslant n$, such that $\sigma(i) > \sigma(j)$ for all $j < i$. The set of left-to-right maxima of σ will be denoted $L(\sigma)$. A *fixed point* of σ is a position i such that $\sigma(i) = i$. Let $F(\sigma)$ be the set of fixed points of σ. A *quasi-fixed point* of σ is a position i such that $\sigma(i) = i + 1$. Let $QF(\sigma)$ be the set of quasi-fixed points. For instance, if $\sigma = 1\ 3\ 5\ 2\ 4\ 6$ then $D(\sigma) = \{3\}$, $E(\sigma) = \{2, 3\}$, $WE(\sigma) = \{1, 2, 3, 6\}$, $L(\sigma) = \{1, 2, 3, 6\}$, $F(\sigma) = \{1, 6\}$ and $QF(\sigma) = \{2\}$. Let D_n be the set of length n *derangements* (permutations without fixed points).

A combinatorial statistics on S_n is a map $f : S_n \to \mathbb{N}$. The distribution of f is the sequence $(a_i)_{i\in\mathbb{N}}$ where a_i is the cardinality of $f^{-1}(i)$. Many statistics on S_n have been widely studied (number of descents, inversions and excedances, major index, ...), see [4,10,7,8,11] for instance. Descent and excedance statistics were first studied by MacMahon [10] and can be considered as mirror images of each other. Indeed, the numbers of descents and excedances have the same distribution on the length n permutations. They are called Eulerian statistics. Foata [6] presents a bijection from S_n to itself which exchanges excedances and descents. In the literature, descents of a permutation are mostly studied for their many applications. For instance, descents appears in theory of lattice path enumeration (see Gessel and Viennot [9]).

More recently, Elizalde [2] constructs a bijection from S_n to C_{n+1} that preserves the descent set on the first $(n-1)$ positions.

Theorem 1. *(See [2].) For every n, there is a bijection $\varphi : C_{n+1} \to S_n$ such that if $\pi \in C_{n+1}$ and $\sigma = \varphi(\pi)$, then*

$$D(\pi) \cap [n-1] = D(\sigma).$$

Moreover, this bijection also preserves the weak excedance set:

$$E(\pi) = WE(\pi) = WE(\sigma).$$

Inspired by this theorem, we present similar results for other statistics. Section 2 shows how one can characterize n-cycles, left-to-right maxima, (quasi-)fixed points, and (weak) excedances using the transposition array representation. In Section 3, we give a constructive bijection from S_n to C_{n+1} that preserves the weak excedance set and that transfers fixed points into quasi-fixed points and left-to-right maxima into themselves. As a consequence, we deduce a bijection from the set D_n of derangements (permutations without fixed points) to the set C^q_{n+1} of length $n+1$ cycles without quasi-fixed points that preserves the set of excedances. In Section 4, we exhibit a kind of discrete continuity between S_n and C_{n+1} that preserves at each step the set of weak excedances. Finally, we deduce a bijection (preserving the excedance set) between S_n and the set $\mathcal{T}^0_n$ of elements in C_n in which one entry has been replaced with 0.

2. Preliminaries

In this section, we give several elementary lemmas in order to characterize special points of a permutation (left-to-right maximum, fixed point, excedance, ...) using the transposition array representation. They will be used throughout the paper. Before this, we state the two straightforward claims:

Claim 1. *For every $j \geqslant 1$, the permutation $\prod_{i=1}^{j} \langle p_i, i\rangle$ fixes all values strictly larger than j.*

Claim 2. *If $k < \ell$ and if for all $i \geqslant \ell$ we have $p_i \neq k$, then the permutation $\prod_{i=\ell}^{n} \langle p_i, i\rangle$ fixes k.*

In [1], it is shown how an n-cycle can be characterized with its transposition array representation.

Lemma 1. *(See [1, Remark 16].) Let σ be a permutation in S_n and $p = p_1 p_2 \dots p_n$ be its transposition array. Then σ is an n-cycle if and only if its transposition array contains only one fixed point, i.e., $p_1 = 1$ and $p_i \neq i$ for $i \geqslant 2$. More generally, the number of cycles of σ is $n - \ell$ where ℓ is the number of indices i, $1 \leqslant i \leqslant n$, such that $p_i \neq i$.*

Proof. A simple induction on j starting at $j = 1$ shows that the number of cycles (relatively to the classical decomposition into disjoint cycles) in $\prod_{i=1}^{j} \langle p_i, i\rangle$ is $(n - \ell)$, where ℓ is the number of i, $1 \leqslant i \leqslant j$, such that $p_i \neq i$. □

Lemma 2. *Let σ be a permutation in S_n and $p_1 p_2 \dots p_n$ be its transposition array. Then k is a left-to-right maximum of σ if and only if there exists $i \geqslant k$ such that* (a) *$p_i = k$, and* (b) *$p_j > k$ for $j > i$.*

Proof. Assume that k is a left-to-right maximum of σ. We have $\sigma(j) \leqslant \sigma(k)$ for $j \leqslant k$ and in particular $\sigma(k) \geqslant k$. Since $\sigma = \langle p_1, 1\rangle \cdot \langle p_2, 2\rangle \cdots \langle p_n, n\rangle$ with $1 \leqslant p_i \leqslant i \leqslant n$, and using Claims 1 and 2, the fact $\sigma(k) \geqslant k$ induces the existence of $i \geqslant k$ such that $p_i = k$. We choose the rightmost $i \geqslant k$ verifying this property. So, Claims 1 and 2 give $\sigma(k) = i$. Assume for a contradiction that there is j, $j > i$, such that $p_j < p_i = k$; we choose the rightmost j. This implies $\sigma(p_j) = j > i = \sigma(k)$ with $p_j < k$, which contradicts the fact that k is a left-to-right maximum. So (b) is verified. Using Claims 1 and 2, the converse becomes straightforward. □

Lemma 3. *Let σ be a permutation in S_n and $p_1 p_2 \dots p_n$ be its transposition array. Then k is a fixed* (*resp. quasi-fixed*) *point of σ if and only if* (a) *$p_k = k$* (*resp. $p_{k+1} = k$*), *and* (b) *there does not exist $i > k$ such that $p_i = k$* (*resp. $i > k + 1$ such that $p_i = k$*).

Proof. Assume that k is a fixed point of $\sigma = \langle p_1, 1\rangle \cdot \langle p_2, 2\rangle \cdots \langle p_n, n\rangle$ with $1 \leqslant p_i \leqslant i \leqslant n$, *i.e.*, $\sigma(k) = k$. Suppose for a contradiction there exists $i > k$ such that $p_i = k$; we take the rightmost i verifying this property. Claims 1 and 2 give $\sigma(k) = i > k$ which contradicts our hypothesis.

Thus σ verifies (b). Claim 2 induces $\sigma(k) = \sigma'(k)$ where $\sigma' = \langle p_1, 1\rangle \cdot \langle p_2, 2\rangle \cdots \langle p_k, k\rangle$. Since $\sigma(k) = k$, Claim 1 induces $p_k = k$. The converse is straightforward. The proof is obtained *mutatis mutandis* whenever we replace fixed points with quasi-fixed points. □

Lemma 4. *Let σ be a permutation in S_n and $p_1 p_2 \dots p_n$ be its transposition array. Then k is an excedance (resp. a weak excedance) of σ if and only if there exists $i > k$ (resp. $i \geqslant k$) such that $p_i = k$.*

Proof. Assume that k is an excedance of $\sigma = \langle p_1, 1\rangle \cdot \langle p_2, 2\rangle \cdots \langle p_n, n\rangle$ with $1 \leqslant p_i \leqslant i \leqslant n$, *i.e.*, $\sigma(k) > k$. Suppose for a contradiction that there is no $i > k$ such that $p_i = k$. Claim 2 induces $\sigma(k) = \sigma'(k)$ with $\sigma' = \langle p_1, 1\rangle \cdot \langle p_2, 2\rangle \cdots \langle p_k, k\rangle$. We deduce $\sigma(k) \leqslant k$ which contradicts the fact that k is an excedance. The converse is straightforward using Claims 1 and 2, and by taking i to be the rightmost i such that $p_i = k$. The reasoning is similar for the case where k is a weak excedance. □

In order to illustrate these lemmas, we present an example for each of them. For instance, if we set $\sigma = 2\mathbf{4}13$ then σ is a 4-cycle and its transposition array $p_1 p_2 p_3 p_4 = 1122$ contains only one fixed point ($p_1 = 1$). Moreover, $k = 2$ is a left-to-right maximum of σ and there is $i = 4 \geqslant 2$ such that $p_i = 2$ and $p_j > 2$ for $j > 4$. $k = 2$ is an excedance and 2 appears on the right of p_2 in the transposition array 1122 of σ. For $\sigma = 2431$, $k = 3$ is a fixed point, its transposition array is 1132 and we have $p_3 = 3$ and there does not exist $i > 3$ such that $p_i = 3$. Finally, $k = 1$ is a quasi-fixed point, we have $p_2 = 1$ and there does not exist $i > 2$ such that $p_i = 1$.

3. Fixed points, weak excedances and left-to-right maxima

Let ϕ be the map from S_n to S_{n+1} defined, for every $\sigma \in S_n$, by

$$\phi(\sigma) = \langle 1, 1\rangle \cdot \langle p_1, 2\rangle \cdot \langle p_2, 3\rangle \cdots \langle p_n, n+1\rangle,$$

where the transposition array of σ is $p_1 p_2 \dots p_n$.

For example, $\phi(321) = \phi(\langle 1, 1\rangle \cdot \langle 2, 2\rangle \cdot \langle 1, 3\rangle) = \langle 1, 1\rangle \cdot \langle 1, 2\rangle \cdot \langle 2, 3\rangle \cdot \langle 1, 4\rangle = 4312$. By construction the transposition array $1 p_1 p_2 \dots p_n$ of $\phi(\sigma)$ has only one fixed point, that is on the first position. With Lemma 1, the permutation $\phi(\sigma)$ is a cyclic permutation and then belongs to C_{n+1}. Therefore ϕ is a bijection from S_n to C_{n+1} (see Table 1 for $n = 3$). Obviously, this construction allows to go back from $\phi(\sigma)$ to σ.

Now we prove that the bijection ϕ transforms the set of weak excedances of $\sigma \in S_n$ into the excedance set of $\phi(\sigma) \in C_{n+1}$.

Remark 1. If $\sigma \in S_n$ has no fixed points, then $E(\sigma) = WE(\sigma)$. This holds in particular when σ is a cycle of length at least 2.

Theorem 2. *The bijection $\phi : S_n \to C_{n+1}$ satisfies for any $\sigma \in S_n$,*

Table 1
The bijection ϕ from S_3 to C_4. Column $T(\sigma)$ (resp. $T(\phi(\sigma))$) gives the transposition array of σ (resp. $\phi(\sigma)$). The weak excedances are illustrated in **boldface**. The last three columns give respectively the sets of left-to-right maxima, fixed points and weak excedances of σ.

σ	$T(\sigma)$	$\phi(\sigma)$	$T(\phi(\sigma))$	$L(\sigma)$	$F(\sigma)$	$WE(\sigma)$
123	123	**234**1	1123	$\{1, 2, 3\}$	$\{1, 2, 3\}$	$\{1, 2, 3\}$
132	122	**24**13	1122	$\{1, 2\}$	$\{1\}$	$\{1, 2\}$
21**3**	113	**314**2	1113	$\{1, 3\}$	$\{3\}$	$\{1, 3\}$
231	112	**34**21	1112	$\{1, 2\}$	$\emptyset$	$\{1, 2\}$
312	111	**4**123	1111	$\{1\}$	$\emptyset$	$\{1\}$
321	121	**43**12	1121	$\{1\}$	$\{2\}$	$\{1, 2\}$

$$WE(\sigma) = E\big(\phi(\sigma)\big) = WE\big(\phi(\sigma)\big).$$

Moreover if k is a weak excedance of σ then $\sigma'(k) = \sigma(k) + 1$ where $\sigma' = \phi(\sigma)$, and we have

$$F(\sigma) = QF\big(\phi(\sigma)\big).$$

Proof. Let σ be a permutation in S_n and $p_1 p_2 \dots p_n$ its transposition array, *i.e.*, $\sigma = \langle p_1, 1\rangle \cdot \langle p_2, 2\rangle \cdots \langle p_n, n\rangle$. The transposition array of $\phi(\sigma)$ is $q_1 q_2 \dots q_{n+1} = 1 p_1 p_2 \dots p_n$. By Lemma 4, k is a weak excedance of σ if and only if there exists $i \geqslant k$ such that $p_i = k$ which is equivalent to the existence of $j \geqslant k + 1$ such that $q_j = k$. Thus, by Lemma 4 again, k is a weak excedance of σ if and only if k is an excedance of $\phi(\sigma)$, and $WE(\sigma) = E(\phi(\sigma))$. Moreover, for $k \in WE(\sigma)$, let i_0 be the rightmost $i \geqslant k$ such that $p_i = k$. Claims 1 and 2 give $\sigma(k) = i_0$, and since $q_{i_0+1} = p_{i_0} = k$, we obtain $\sigma'(k) = i_0 + 1$ where $\sigma' = \phi(\sigma)$. Thus we have $F(\sigma) \subseteq QF(\phi(\sigma))$. For $k \in WE(\phi(\sigma))$, Remark 1 induces that $k \in E(\phi(\sigma))$, and Lemma 4 ensures the existence of $i > k$ such that $q_i = k$. Let i_0 be the rightmost i. Claims 1 and 2 give $\sigma'(k) = i_0$, and since $q_{i_0} = p_{i_0-1} = k$, we obtain $\sigma(k) = i_0 - 1$ which allows to conclude. □

As a consequence of Theorem 2 we deduce:

Corollary 1. *The bijection $\phi : S_n \to C_{n+1}$ satisfies for any $\sigma \in S_n$,*

$$L(\sigma) = L\big(\phi(\sigma)\big) \quad \textit{and,}$$

if k is a left-to-right maximum of σ then $\sigma'(k) = \sigma(k) + 1$ where $\sigma' = \phi(\sigma)$.

Proof. Let k be a left-to-right maximum of σ, *i.e.*, $\sigma(j) \leqslant \sigma(k)$ for $j \leqslant k$. Since it also is a weak excedance, Theorem 2 induces $\sigma'(k) = \sigma(k) + 1$ where $\sigma' = \phi(\sigma)$. Consider $j < k$. If j is a weak excedance of σ, then we have $\sigma(j) < \sigma(k)$ and with Theorem 2, $\sigma'(j) = \sigma(j) + 1 < \sigma(k) + 1 = \sigma'(k)$. Otherwise $\sigma(j) < j$ and then Theorem 2 ensures that j is not a weak excedance of σ' and $\sigma'(j) < j < k < \sigma'(k) = \sigma(k) + 1$. Thus we have $k \in L(\phi(\sigma))$. The converse is obtained similarly. □

See Table 1 for an illustration of Theorem 2 and Corollary 1 for $n = 3$.

As the bijection ϕ moves fixed points into quasi-fixed points, Theorem 2 and Remark 1 induce the following result:

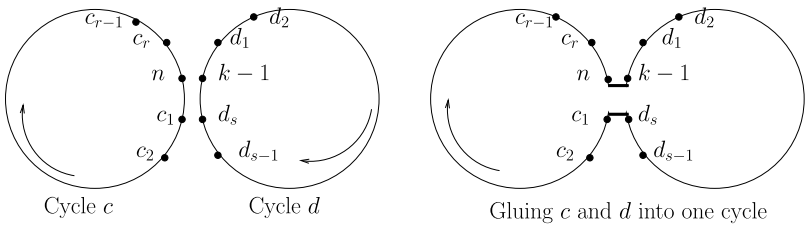


Fig. 1. Gluing of $c = \langle n, c_1, c_2, \ldots, c_r\rangle$ and $d = \langle k-1, d_1, d_2, \ldots, d_s\rangle$.

Table 2
The sets $S_4(k)$ for $k = 1, 2, 3, 4$. The weak excedances on positions $\{1, 2, 3\}$ are illustrated in **boldface**.

$S_4(1)$	$S_4(2)$	$S_4(3)$	$\sigma \in S_4(4) = S_3 \cdot 4$	$WE(\sigma) \cap \{1, 2, 3\}$
4123	**4**123	**4**123	**3**124	$\{1\}$
31**4**2	**3**1**4**2	**2**1**4**3	**2**1**3**4	$\{1, 3\}$
43**1**2	**4**3**1**2	**4**2**1**3	**3**2**1**4	$\{1, 2\}$
24**1**3	**2**4**1**3	**2**4**1**3	**2**3**1**4	$\{1, 2\}$
23**4**1	**1**3**4**2	**1**2**4**3	**1**2**3**4	$\{1, 2, 3\}$
34**2**1	**1**4**2**3	**1**4**2**3	**1**3**2**4	$\{1, 2\}$

Corollary 2. *The bijection ϕ from the set D_n of derangements* (i.e., *length n permutations without fixed points*) *to the set C^q_{n+1} of length $(n+1)$ cycles without quasi-fixed points is a bijection such that: for any $\sigma \in D_n$,*

$$E(\sigma) = E\big(\phi(\sigma)\big).$$

Moreover, if k is an excedance of σ then $\sigma'(k) = \sigma(k) + 1$ where $\sigma' = \phi(\sigma)$.

4. A discrete continuity preserving the set of weak excedances

In this part, we give a kind of discrete continuity between S_n and C_{n+1} that preserves at each step the set of weak excedances. For $1 \leqslant k \leqslant n$, we denote by $S_n(k)$ the set of permutations $\sigma \in S_n$ such that: (a) $\sigma(n) \geqslant k$; and (b) all integers of the interval $[k, n]$ lie in a same cycle of σ (in the decomposition of σ into disjoint cycles). Obviously, $S_{n+1}(1) = C_{n+1}$ and $S_{n+1}(n+1)$ is obtained from S_n by adding $n+1$ to the right of each permutation of S_n. Table 2 shows the different sets $S_4(k)$ for $k = 1, 2, 3, 4$.

Definition 2. For $k \in \{2, \ldots, n\}$, we define $h_k : S_n(k) \to S_n(k-1)$ by:

$$h_k(\sigma) = \begin{cases} \sigma & \text{if } \sigma \in S_n(k-1), \\ \sigma \cdot \langle \sigma^{-1}(k-1), n\rangle & \text{otherwise.} \end{cases}$$

For Definition 2 to be valid, we need to prove that $h_k(\sigma)$ is indeed in $S_n(k-1)$ for any $\sigma \in S_n(k)$. Let σ be a permutation in $S_n(k)$. Its decomposition into disjoint cycles contains a cycle $c = \langle n, c_1, c_2, \ldots, c_r\rangle$ with $r \geqslant 0$, $c_1 \geqslant k$ and such that $[k, n] \subseteq \{n, c_1, c_2, \ldots, c_r\}$. The case $r = 0$ means that c is reduced to the cycle $c = \langle n\rangle$. In the case where $\sigma \notin S_n(k-1)$, $k-1$ does not appear in c and thus, it lies in another cycle $d = \langle k-1, d_1, d_2, \ldots, d_s\rangle$, $s \geqslant 0$. Therefore, the decomposition of $h_k(\sigma) = \sigma \cdot \langle \sigma^{-1}(k-1), n\rangle = \sigma \cdot \langle d_s, n\rangle$ into disjoint cycles is obtained from that of σ by gluing c and d into the cycle $\langle n, k-1, d_1, \ldots, d_s, c_1, c_2, \ldots, c_r\rangle$. See Fig. 1 for an illustration of the gluing. So, this implies that $h_k(\sigma) \in S_n(k-1)$.

Theorem 3. *For $k \in \{2, \ldots, n\}$, h_k is a bijection.*

Proof. Let us prove that h_k is a bijection. In order to show the injectivity we take σ and π in $S_n(k)$ such that $h_k(\pi) = h_k(\sigma)$. We distinguish three cases.

(1) If π and σ belong to $S_n(k-1)$ then $h_k(\pi) = \pi$ and $h_k(\sigma) = \sigma$, and $\pi = \sigma$.

(2) If π and σ do not belong to $S_n(k-1)$ then we have $\sigma \cdot \langle j, n\rangle = \pi \cdot \langle \ell, n\rangle$ where $j = \sigma^{-1}(k-1)$ and $\ell = \pi^{-1}(k-1)$. Let $c = \langle n, c_1, c_2, \ldots, c_r\rangle$ (resp. $c' = \langle n, c'_1, c'_2, \ldots, c'_{r'}\rangle$) be the cycle containing n in π (resp. σ), and $d = \langle k-1, d_1, d_2, \ldots, d_s\rangle$ (resp. $d' = \langle k-1, d'_1, d'_2, \ldots, d'_{s'}\rangle$) be the cycle of π (resp. σ) containing ℓ (resp. j). We have $c \neq d$, $c' \neq d'$, $d_s = \ell$, $d'_{s'} = j$. $h_k(\pi)$ (resp. $h_k(\sigma)$) is obtained by gluing c and d (resp. c' and d') as explained just after Definition 2. Because $h_k(\pi) = h_k(\sigma)$, we deduce that

$$\begin{aligned} &\langle n, k-1, d_1, \ldots, d_s, c_1, c_2, \ldots, c_r\rangle \\ &\quad = \big\langle n, k-1, d'_1, \ldots, d'_{s'}, c'_1, c'_2, \ldots, c'_{r'}\big\rangle. \end{aligned}$$

Since $c_1 \geqslant k$, $c'_1 \geqslant k$, $[k, n] \subseteq \{c_1, c_2, \ldots, c_r\}$ and $[k, n] \subseteq \{c'_1, c'_2, \ldots, c'_{r'}\}$, we necessarily have $s = s'$, $r = r'$, $c_i = c'_i$ for $i \leqslant r$, and $d_i = d'_i$ for $i \leqslant s$. Thus we obtain $\sigma = \pi$.

(3) The case $\pi \in S_n(k) \backslash S_n(k-1)$ and $\sigma \in S_n(k-1)$ does not occur since the last entry of $h_k(\sigma)$ is at least k while the last value of $h_k(\pi)$ is $k-1$.

Therefore h_k is injective.

Now let us consider $\pi \in S_n(k-1)$. If π also belongs to $S_n(k)$ then π is the image of π by h_k. If π does not belong to $S_n(k)$ then we necessarily have $\pi(n) = k-1$. Let $c = \langle n, k-1, c_1, \ldots, c_r\rangle$ be the cycle of π containing n and i_0 be the smallest i such that $c_i \geqslant k$. Then the permutation σ obtained from π by splitting c into the two cycles $c' = \langle n, c_{i_0}, c_{i_0+1}, \ldots, c_r\rangle$ and $d = \langle k-1, c_1, \ldots, c_{i_0-1}\rangle$, belongs to $S_n(k)$ and satisfies $h_k(\sigma) = \pi$. Thus h_k is surjective. □

A consequence of Theorem 3 is that the cardinality of the sets $S_{n+1}(k)$ is $n!$ for each $k \in \{1, 2, \ldots, n+1\}$. A simple combinatorial argument proves the following remarkable equality.

Remark 2. For all $k \in \{1, 2, \ldots, n\}$, we have

$$n! = (n-k+1) \cdot \sum_{i=0}^{k-1} \binom{k-1}{i} (n-k+i)!(k-i-1)!.$$

Table 3
The bijection f^{-1} from S_3 to $\mathcal{T}_3^0$ and the bijection ψ from S_3 to C_4. Excedances are illustrated in **boldface**.

σ	$f^{-1}(\sigma)$	$\psi(\sigma)$	$E(f^{-1}(\sigma)) = E(\sigma)$	$E^+(\sigma) \cup \{1\} = E(\psi(\sigma))$
123	012	**4**123	$\emptyset$	$\{1\}$
1**3**2	0**3**1	**3**1**4**2	$\{2\}$	$\{1, 3\}$
213	**2**01	**24**13	$\{1\}$	$\{1, 2\}$
231	**23**0	**234**1	$\{1, 2\}$	$\{1, 2, 3\}$
312	**3**02	**43**12	$\{1\}$	$\{1, 2\}$
321	**3**10	**34**21	$\{1\}$	$\{1, 2\}$

Both sides of the above equality count the cardinality of $S_{n+1}(k)$. Indeed, a permutation σ belongs to $S_{n+1}(k)$ if and only if $\sigma(n+1) \geqslant k$ and each integer in $[k, n+1]$ lies into the same cycle $c = \langle n+1, c_1, c_2, c_3, \ldots \rangle$ of σ. So, there are $(n-k+1)$ choices for $c_1 = \sigma(n+1) \in [k, n]$. In order to complete c we choose a set I of i values among $\{1, 2, \ldots, k-1\}$, $0 \leqslant i \leqslant k-1$ and we consider all arrangements of elements of the set $I \cup [k, n] \backslash \{c_1\}$. Thus, for a given i, $0 \leqslant i \leqslant k-1$, there are $(n-k+1) \cdot \binom{k-1}{i} \cdot (n-k+i)!$ possible cycles c and $(k-i-1)!$ choices for the remaining values. Moving i from 0 to $k-1$, we obtain the above formula.

Theorem 4. *For $k \in \{2, \ldots, n\}$, the bijection $h_k : S_n(k) \to S_n(k-1)$ satisfies for any $\sigma \in S_n(k)$,*

$$WE(\sigma) \cap \{1, 2, \ldots, n-1\} = WE\big(h_k(\sigma)\big).$$

Proof. Let $\sigma \in S_n(k)$. Notice that n is never a weak excedance of $h_k(\sigma)$.

The case $h_k(\sigma) = \sigma$ is trivial. Now let us assume that $\sigma' = h_k(\sigma) = \sigma \cdot \langle j, n \rangle$ where $j = \sigma^{-1}(k-1)$. We necessarily have $j \leqslant k-1$ and $\sigma(n) \geqslant k$.

Let us take $i \in \{1, 2, \ldots, n-1\}$. In the case where $i \neq j$ and $i \neq n$, we obtain $\sigma'(i) = \sigma(i)$; then i is a weak excedance of σ if and only if it is also one for $\sigma' = h_k(\sigma)$.

If $i = j$ then $\sigma'(i) = \sigma(n)$. As $\sigma \in S_n(k)$, we have $\sigma(n) \geqslant k$ and then $\sigma(n) > k-1$. We obtain $j = i \leqslant k-1$ and $\sigma(i) = \sigma(j) = k-1$, and hence i is a weak excedance of σ. Moreover, we have $\sigma'(i) = \sigma(n) > k-1$ so that i is a weak excedance of σ'. Finally, we have $WE(\sigma) \cap \{1, 2, \ldots, n-1\} = WE(h_k(\sigma))$. □

Corollary 3. *There is a bijection h from S_n to C_{n+1} that preserves the set of all weak excedances.*

Proof. We set $h = h_2 \circ h_3 \circ \cdots \circ h_{n-1} \circ h_n \circ h_{n+1}$ from $S_{n+1}(n+1)$ to $S_{n+1}(1) = C_{n+1}$. Since $S_{n+1}(n+1)$ is the set of permutations in S_n after adding $n+1$ on the right, we have $WE(\sigma) = WE(\sigma \cdot (n+1)) \cap \{1, 2, \ldots, n\}$. Theorem 4 allows to conclude that $WE(\sigma) = WE(h(\sigma))$. □

Notice that the bijection h does not transform fixed points into quasi-fixed points (see for instance $h(132) = 3421$), unlike the bijection ϕ in Section 3.

5. Consequences and open problems

In this section we give some direct consequences of our study and we propose two open problems concerning the descent sets on cyclic permutations and derangements.

Following the notation from [2,3], let $\mathcal{T}_n^0$ be the set whose elements are n-cycles in one-line notation in which one entry has been replaced with 0. For example, $\mathcal{T}_3^0 = \{031, 201, 230, 012, 302, 310\}$. Obviously the cardinality of $\mathcal{T}_n^0$ is $n!$. Let σ be an element of $\mathcal{T}_n^0$, we say that k, $1 \leqslant k \leqslant n$, is an excedance of σ whenever $\sigma(k) > k$ and $E(\sigma)$ will be denote the set of excedances of σ. Theorem 5 is a counterpart of Elizalde's result [2] for the set of excedances on $\mathcal{T}_n^0$.

If E is a subset of $[n]$ then we define the set $E^+ \subseteq \{2, 3, \ldots, n+1\}$ by:

$$E^+ = \{e+1 \mid e \in E\}.$$

Moreover, we define the involution χ_n from S_n into itself as follows: for any $n \geqslant 1$ and $\sigma \in S_n$, $\chi_n(\sigma)(i) = n+1-\sigma(n+1-i)$ for $i \leqslant n$. For instance, we have $\chi_4(4132) = 3241$. Less formally, $\chi_n(\sigma)$ is obtained from $\sigma \in S_n$ by reading the complement of σ from right to left. In the following we will omit the subscript n for χ; it should be clear from the context.

Using the map $\phi : S_n \to C_{n+1}$ presented in Section 3, we consider the bijection ψ from S_n to C_{n+1} defined by:

$$\psi(\sigma) = \chi\big(\phi\big(\chi(\sigma)\big)\big).$$

For example, $\psi(312) = \chi(\phi(231)) = \chi(3421) = 4312$; see Table 3 for an illustration of this map when $n = 3$.

The following corollary appears as a direct consequence of Theorem 2.

Corollary 4. *The bijection $\psi : S_n \to C_{n+1}$ satisfies for any $\sigma \in S_n$,*

$$E^+(\sigma) \cup \{1\} = E\big(\psi(\sigma)\big).$$

Proof. It is obvious that an n-cycle has 1 as excedance; thus $1 \in E(\psi(\sigma))$. Now if $k \notin E^+(\sigma) \cup \{1\}$ then we have $k > 1$, $\sigma(k-1) \leqslant k-1$ and $n+1-(k-1) = n+2-k$ is a weak excedance of $\chi(\sigma)$, and we have $\chi(\sigma)(n+2-k) = n+1-\sigma(k-1)$. With Theorem 2, $n+2-k$ is an excedance of $\phi(\chi(\sigma))$, and we have $\phi(\chi(\sigma))(n+2-k) = n+1-\sigma(k-1)+1$. Thus, $\chi(\phi(\chi(\sigma)))(n+2-(n+2-k)) = n+2-(n+1-\sigma(k-1)+1)$, *i.e.*, $\chi(\phi(\chi(\sigma)))(k) = \sigma(k-1) \leqslant k-1$ which induces that k is not an excedance of $\psi(\sigma)$. Each implication of this reasoning can be also viewed as an equivalence which achieves the proof. □

Theorem 5. *Let n be a positive integer. There is a bijection f between $\mathcal{T}_n^0$ and S_n such that if $\sigma \in \mathcal{T}_n^0$, then*

$$E(\sigma) = E\big(f(\sigma)\big).$$

Proof. Let σ be an element of $\mathcal{T}_n^0$; σ is obtained from a cycle π by replacing $\pi(k)$ with 0 for some k, $1 \leqslant k \leqslant n$. Assume that the cyclic representation of π is $\langle a_1, a_2, \ldots, a_{n-1}, k\rangle$ where $a_i \in [n]\backslash\{k\}$ for all $i \leqslant n-1$. Let us consider π' defined by the cyclic representation $\pi' = \langle a_1+1, a_2+1, \ldots, a_{n-1}+1, k+1, 1\rangle$. Obviously, we have $E(\sigma)^+ \cup \{1\} = E(\pi')$. Now, we set $f(\sigma) = \psi^{-1}(\pi')$ where ψ is the bijection defined above. We immediately have:

$$\begin{aligned} E(\sigma)^+ \cup \{1\} &= E\big(\pi'\big) \\ &= E\big(f(\sigma)\big)^+ \cup \{1\}, \end{aligned}$$

which means $E(\sigma) = E(f(\sigma))$. $\square$

Thus we can directly deduce:

Corollary 5. *For any n and any $I \subseteq [n-1]$,*

$$\big|\big\{\tau \in \mathcal{T}_n^0,\ E(\tau) = I\big\}\big| = \big|\big\{\sigma \in S_n,\ E(\sigma) = I\big\}\big|.$$

This last result appears as a counterpart of the following Elizalde's result for descents:

Theorem 6. *(See [2].) For any n and any $I \subseteq [n-1]$,*

$$\big|\big\{\tau \in \mathcal{T}_n^0, D(\tau) = I\big\}\big| = \big|\big\{\sigma \in S_n, D(\sigma) = I\big\}\big|.$$

We conclude this paper by giving two open problems about descent statistics (see [5] for some other open problems about descent statistics). Experimental investigations allow us to think that the answer to these two problems is positive. Problem 1 appears as a counterpart of Corollary 2 for descent statistics. Problem 2 would be a generalization of the Elizalde's result (Theorem 1, p. 2) and appears as a counterpart of Theorem 4 for descent statistics.

Problem 1. Is it true that there exists a bijection from the set of derangements D_n to the set C_{n+1}^q of $(n+1)$-cycles without quasi-fixed points that preserves the set of descents in $\{1, 2, \ldots, n-1\}$?

For $1 \leqslant k \leqslant n$, we denote $S'_n(k)$ the set of permutations $\sigma \in S_n$ such that the cycle of σ that contains n is of length k.

Problem 2. For $k \in \{2, \ldots, n\}$, is it true that there is a bijection h'_k from $S'_n(k)$ to $S'_n(k-1)$ that preserves the set of descents in $\{1, 2, \ldots, n-2\}$, *i.e.,*

$$D(\sigma) \cap \{1, 2, \ldots, n-2\} = D\big(h'_k(\sigma)\big) \cap \{1, 2, \ldots, n-2\}?$$

Acknowledgements

I would like to thank the anonymous referees for their very careful reading of this paper and their helpful comments and suggestions.

References

[1] J.-L. Baril, Gray code for permutations with a fixed number of cycles, Discrete Math. 307 (2007) 1559–1571.
[2] S. Elizalde, Descent sets of cyclic permutations, Adv. in Appl. Math. 47 (4) (2011) 688–709.
[3] S. Elizalde, The number of permutations realized by a shift, SIAM J. Discrete Math. 23 (2009) 765–786.
[4] R. Ehrenborg, E. Steingrímsson, The excedance set of a permutation, Adv. in Appl. Math. 24 (2000) 284–299.
[5] N. Eriksen, R. Freij, J. Wastlund, Enumeration of derangements with descents in prescribed positions, Electron. J. Combin. 16 (2009) R32.
[6] D.C. Foata, M.P. Schützengerger, Théorie Géométrique des Polynômes Eulériens, Lecture Notes in Math., vol. 138, Springer-Verlag, Berlin, 1970.
[7] D. Foata, On the Netto inversion number of a sequence, Proc. Amer. Math. Soc. 19 (1968) 236–240.
[8] D. Foata, M.P. Schützenberger, Major index and inversion number of permutations, Math. Nachr. 83 (1978) 143–159.
[9] I. Gessel, G. Viennot, Binomial determinants, paths, and hook length formulae, Adv. Math. 58 (1985) 300–321.
[10] P.A. MacMahon, Combinatory Analysis, vols. 1–2, Cambridge University Press, Cambridge, UK, 1915, reprinted by Chelsea, New York, 1955.
[11] R.P. Stanley, Enumerative Combinatorics, vol. I, Cambridge University Press, Cambridge, 1997.

Gray code for permutations with a fixed number of left-to-right minima

Jean-Luc Baril
LE2I UMR-CNRS 5158, Université de Bourgogne
B.P. 47 870, 21078 DIJON-Cedex France
e-mail: barjl@u-bourgogne.fr

January 12, 2009

Abstract

In [1], the author provided a Gray code for the set of n-length permutations with a given number of left-to-right minima in *inversion array representation*. In this paper, we give the first Gray code for the set of n-length permutations with a given number of left-to-right minima in *one-line representation*. In this code, each permutation is transformed into its successor by a product with a transposition or a cycle of length three. Also a generating algorithm for this code is given.

Keywords: Stirling numbers, permutations, left-to-right minima, Gray codes, generating algorithms.

1 Introduction

Let S_n be the set of all permutations of length n ($n \geq 1$). We represent permutations in one-line notation, i.e. if $i_1,\ i_2, \ldots,\ i_n$ are n distinct values in $[n] = \{1,\ 2, \ldots,\ n\}$, we denote the permutation $\sigma \in S_n$ by the sequence $(i_1, i_2, \ldots, i_n)$ if $\sigma(k) = i_k$ for $1 \leq k \leq n$. Moreover, if $\gamma = (\gamma(1), \gamma(2), \ldots, \gamma(n))$ is an n-length permutation then the *composition* (*or product*) $\gamma \cdot \sigma$ is the permutation $(\gamma(\sigma(1)), \gamma(\sigma(2)), \ldots, \gamma(\sigma(n)))$. In S_n, a *k-cycle* $\sigma = \langle i_1,\ i_2, \ldots,\ i_k \rangle$ is an n-length permutation verifying $\sigma(i_1) = i_2,\ \sigma(i_2) = i_3, \ldots,\ \sigma(i_{k-1}) = i_k,\ \sigma(i_k) = i_1$ and $\sigma(j) = j$ for $j \in [n] \backslash \{i_1, \ldots, i_k\}$; in particular, a *transposition* is a 2-cycle ($k = 2$). Now, let $\sigma \in S_n$. We say that $\sigma(i)$ is a *left-to-right minimum* of σ if $\sigma(i) < \sigma(j)$ for $j < i$. For instance, permutation 231546 has two left-to-right minima (2 and 1) and 654321 has six left-to-right minima. For $1 \leq k \leq n$, we denote

by $S_{n,k}$ the set of all n-length permutations with exactly k left-to-right minima. For instance, we have $S_{3,1} = \{123, 132\}$, $S_{3,2} = \{213, 231, 312\}$ and $S_{3,3} = \{321\}$. Obviously, $\{S_{n,k}\}_{1\leq k\leq n}$ forms a partition for S_n and it is well known (see for instance [14], p. 20) that the cardinality of $S_{n,k}$ is given by the signless Stirling numbers of the first kind $s(n,k)$ satisfying:

$$s(n,k) = (n-1)\cdot s(n-1,k) + s(n-1,k-1) \tag{1}$$

with the initial conditions $s(n,k) = 0$ if $n \leq 0$ or $k \leq 0$, except $s(0,0) = 1$. Notice that $s(n,k)$ also enumerates the n-length permutations with exactly k cycles. [1, 14, 18].

A *Gray code* is a family $\{\mathcal{L}_n\}_{n\geq 0}$ of lists of n-length sequences such that in each list the Hamming distance between any two consecutive sequences (i.e. the number of positions in which they differ) is bounded by a constant independently of the length of the sequences. If this constant is minimal then the code is called *optimal*. A Gray code is *cyclic* if the Hamming distance between the first and the last sequence is also bounded by this constant. There are many studies on generating permutations: in any order (or lex order) [5, 13], in Gray code order [4, 15]. Some other results are published for restricted permutations [3, 6, 8, 10, 12], derangements [2, 7], with a fixed number of cycles [1], involutions and fixed-point free involutions [17] or their generalizations (multiset permutations [16]). At [9] is given an implementation of U. Taylor and F. Ruskey [11] generating algorithm for n-length permutations with k left-to-right minima. However the generating order is neither lexicographic nor Gray code order.

In this paper, we provide the first Gray code for permutations (in one-line notation) with a fixed number of left-to-right minima. In this code two successive permutations differ in at most three positions (or equivalently, by a product with a 2 or 3-cycle). This code is optimal; indeed, it is easy to check that the set $S_{4,3} = \{3214, 3241, 3421, 4231, 4312, 4213\}$ can not be list such that two consecutive elements differ in at most two positions. This means there does not exist any Gray code for the complete family $\{S_{n,k}\}_{n\geq k\geq 1}$ such that two consecutive permutations in a list $S_{n,k}$ differ in at most two positions. However by considering the set $I = \mathbb{N}\times\{1\}$, we can obtain a Gray code for $\{S_{n,k}\}_{(n,k)\in I}$ such that the Hamming distance between any two consecutive permutations is two (this case will not be presented here).

Notice that in [1], the author gave a Gray code for n-length permutations with exactly k cycles and deduces a Gray code for n-length permutations with exactly k left-to-right minima in *inversion array representation* but not in *one-line notation*. Since there is not known gray code-preserving bijection between these representations it is of interest to construct directly a Gray code for permutations with a fixed number of left-to-right minima in the natural representation, *i.e.* in one-line notation. This is the main

motivation of our work. In order to obtain this result, we first give two combinatorial interpretations of the recursive relation (1) which allows us to obtain in Section 3 two recursive definition-(2) and (3)- for the set $S_{n,k}$; then, by considering them in terms of lists, we show how one can recursively construct a Gray code for $S_{n,k}$. Finally we give a constant amortized time algorithm for generating our code.

2 Preliminaries

In this section we recursively construct $S_{n,k}$ from $S_{n-1,k}$ and $S_{n-1,k-1}$ which allows us to obtain a constructive proof of the enumerating relation (1). We also provide two lemmas, crucial in our construction of the code.

Let $\gamma \in S_{n-1,k}$ be an $(n-1)$-length permutation with k left-to-right minima, $n-1 \geq k \geq 1$; let also i be an integer, $1 \leq i \leq n-1$. If we denote by σ the permutation in S_n obtained from γ by inserting the entry n in the i-th position (i.e. between $\gamma(i)$ and $\gamma(i+1)$), then σ is an n-length permutation with k left-to-right minima.

Similarly, if $\gamma \in S_{n-1,k-1}$ is an $(n-1)$-length permutation with $(k-1)$ left-to-right minima, $n \geq k \geq 2$, and if σ denotes the permutation in S_n obtained from γ by appending n before the first position, then σ is an n-length permutation with k left-to-right minima. Moreover, each permutation in $S_{n,k}$, $n \geq 2$, can uniquely be obtained by one of these two constructions.

The functions ϕ_n and ψ_n defined below induce a bijection between $S_{n-1,k-1} \cup [n-1] \times S_{n-1,k}$ and $S_{n,k}$ such that its restriction to $S_{n-1,k-1}$ is ϕ_n and its restriction to $[n-1] \times S_{n-1,k}$ is ψ_n.

Definition 1 *1. For $1 \leq k < n$, an integer $i \in [n-1]$ and a permutation $\gamma \in S_{n-1,k}$, we define an n-length permutation $\sigma = \psi_n(i, \gamma)$ by*

$$\sigma(j) = \begin{cases} \gamma(j) & \text{if } j \leq i \\ n & \text{if } j = i+1 \\ \gamma(j-1) & \text{otherwise.} \end{cases}$$

2. For $n \geq k \geq 2$ and a permutation $\gamma \in S_{n-1,k-1}$, we define an n-length permutation $\sigma = \phi_n(\gamma)$ by

$$\sigma(j) = \begin{cases} n & \text{if } j = 1 \\ \gamma(j-1) & \text{otherwise.} \end{cases}$$

On the other hand, we have another bijection between $S_{n-1,k-1} \cup [n] \setminus \{1\} \times S_{n-1,k}$ and $S_{n,k}$ by replacing ϕ_n and ψ_n with the two functions ϕ'_n and ψ'_n given in Definition 2 below. Indeed, let us consider $\sigma \in S_{n,k}$, $n \geq 2$;

if $\sigma(n) = 1$ then $\gamma = (\sigma(1)-1)(\sigma(2)-1)\dots(\sigma(n-1)-1)$ belongs to $S_{n-1,k-1}$ and $\phi'(\gamma) = \sigma$; if $\sigma(n) = i$, $i \neq 1$, then we define γ in $S_{n-1,k}$ as the permutation obtained from σ by deleting the last entry i and by decreasing by one each entry greater than i, *i.e.*, $\sigma = \psi'(i,\gamma)$. Also, each permutation σ in $S_{n,k}$ can be uniquely obtained from one of these two constructions.

Definition 2 *1. For $1 \leq k < n$, an integer $i \in [n]\backslash\{1\}$ and a permutation $\gamma \in S_{n-1,k}$, we define an n-length permutation $\sigma = \psi'_n(i,\gamma)$ by*

$$\sigma(j) = \begin{cases} \gamma(j) & \text{if } \gamma(j) \leq i-1 \\ i & \text{if } j = n \\ \gamma(j)+1 & \text{otherwise.} \end{cases}$$

2. For $n \geq k \geq 2$ and a permutation $\gamma \in S_{n-1,k-1}$, we define an n-length permutation $\sigma = \phi'_n(\gamma)$ by

$$\sigma(j) = \begin{cases} 1 & \text{if } j = n \\ \gamma(j)+1 & \text{otherwise.} \end{cases}$$

In the following, we will omit the subscript n for the functions ϕ_n, ψ_n, ϕ'_n and ψ'_n, and it should be clear from context. Also, we will extend the functions ϕ, ψ, ϕ' and ψ' in a natural way to sets and lists of permutations. Moreover, if $\mathcal{S}$ is a list then $\overline{\mathcal{S}}$ is the list obtained by reversing $\mathcal{S}$, and $first(\mathcal{S})$ ($last(\mathcal{S})$ respectively) is the first (last respectively) element of the list, and obviously $first(\mathcal{S}) = last\left(\overline{\mathcal{S}}\right)$ and $first\left(\overline{\mathcal{S}}\right) = last(\mathcal{S})$; $\mathcal{S}^{(i)}$ is the list $\mathcal{S}$ if i is even, and $\overline{\mathcal{S}}$ if i is odd; if $\mathcal{S}_1$ and $\mathcal{S}_2$ are two lists, then $\mathcal{S}_1 \circ \mathcal{S}_2$ is their concatenation, and generally $\bigcirc_{i=1}^{m} \mathcal{S}_i$ is the list $\mathcal{S}_1 \circ \mathcal{S}_2 \circ \dots \circ \mathcal{S}_m$. Similarly, $\bigcirc_{i=m}^{1} \mathcal{S}_i$ is the list $\mathcal{S}_m \circ \mathcal{S}_{m-1} \circ \dots \circ \mathcal{S}_1$. Finally, for $i \in [n-1]$ and $\mathcal{S}$ a list of $(n-1)$-length permutations we have $\psi(i,\overline{\mathcal{S}}) = \overline{\psi(i,\mathcal{S})}$, $\psi(i, first(\mathcal{S})) = first(\psi(i,\mathcal{S}))$, and $\psi(i, last(\mathcal{S})) = last(\psi(i,\mathcal{S}))$. We obtain similar results for the function ψ' and for ϕ, ϕ' if we do not consider the parameter i.

In the sequel, $\mathcal{S}_{n,k}$ will denote our Gray code for the set $S_{n,k}$.

Here we give some elementary results which are crucial in the construction of our Gray code.

Lemma 1 *Let γ be an $(n-1)$-length permutation, if $n \geq 3$ and $1 \leq i \leq n-2$, then $\psi(i,\gamma) = \psi(i+1,\gamma) \cdot \langle i+1, i+2 \rangle$.*

Lemma 2 *Let γ be an $(n-1)$-length permutation, if $n \geq 3$ and $1 \leq i \leq n-3$, then $\psi(i,\gamma) = \psi(i+2,\gamma) \cdot \langle i+1, i+3, i+2 \rangle$.*

These two Lemmas can also be written for ψ' as follows:

Lemma 3 *Let γ be an $(n-1)$-length permutation, if $n \geq 3$ and $2 \leq i \leq n-1$, then $\psi'(i,\gamma) = \langle i, i+1\rangle \cdot \psi'(i+1,\gamma)$.*

Lemma 4 *Let γ be an $(n-1)$-length permutation, if $n \geq 3$ and $2 \leq i \leq n-2$, then $\psi'(i,\gamma) = \langle i, i+1, i+2\rangle \cdot \psi'(i+2,\gamma)$.*

3 The Gray code

From the remarks before Definition 1 results that the set $S_{n,k}$ can be written as:

$$S_{n,k} = \bigcup_{i=1}^{n-1} \psi(i, S_{n-1,k}) \cup \phi(S_{n-1,k-1}) \tag{2}$$

with $\phi(S_{n,0})$ and $\psi(i, S_{n,n+1})$ empty. Notice that if we consider ϕ' and ψ' the previous relation becomes

$$S_{n,k} = \bigcup_{i=2}^{n} \psi'(i, S_{n-1,k}) \cup \phi'(S_{n-1,k-1}). \tag{3}$$

If $\mathcal{S}$ is a list of permutations where any two consecutive permutations differ in p positions ($p \geq 1$) then so is the image of $\mathcal{S}$ by ψ, ψ', ϕ or ϕ'. Therefore, it is natural to look for a Gray code for the set $S_{n,k}$ of the form

$$\mathcal{S}_1 \circ \mathcal{S}_2 \circ \ldots \circ \mathcal{S}_\ell \circ \mathcal{T} \circ \mathcal{S}_{\ell+1} \circ \ldots \circ \mathcal{S}_{n-1} \tag{4}$$

where $\mathcal{T}$ is the list $\phi(\mathcal{S}_{n-1,k-1})$ or its reverse, and $\mathcal{S}_i$ is $\psi(j, \mathcal{S}_{n-1,k})$ or its reverse, for some j. Notice that (4) is an ordered counterpart of (2) and we have a similar result by considering relation (3).

In order to construct our Gray code, we distinguish five cases (i) $k = 1 \leq n$, (ii) $2 \leq k = n$, (iii) $2 \leq k = n-1$, (iv) $2 \leq k = n-2$ and (v) the other cases. For each case we give a recursive definition for an ordered list $\mathcal{S}_{n,k}$ of the set $S_{n,k}$, and we provide its first element $f_{n,k}$ and last element $\ell_{n,k}$. $\mathcal{S}_{n,k}$ is the concatenation of n lists as in (4) and we prove that it is a Gray code by showing that there is a 'smooth' transition between successive sublists. That is, the last permutation of a sublist and the first one of the next sublist differ in at most three positions. By the remark in introduction, the Gray code will be optimal.

3.1 The case $k = n$, $n \geq 1$

Obviously, $\mathcal{S}_{n,n}$, $n \geq 1$, contains only one element $(n, n-1, \ldots, 2, 1)$, (a)
and in this case, there is nothing to do.

3.2 The case $k = 1$, $n \geq 2$

For $n \geq 2$ we define the relations (b):

• For $n = 2$, $\mathcal{S}_{n,1} = (1, 2)$.

• For $n = 2m \geq 2$,

$$\mathcal{S}_{n,1} = \psi(n-1, \mathcal{S}_{n-1,1}) \circ \bigcirc_{i=m-1}^{1} \psi(2i-1, \mathcal{S}_{n-1,1})^{(m+i-1)} \circ \bigcirc_{i=1}^{m-1} \psi(2i, \mathcal{S}_{n-1,1})^{(m+i)}.$$

• For $n = 2m + 1 \geq 3$,

$$\mathcal{S}_{n,1} = \psi(n-1, \mathcal{S}_{n-1,1}) \circ \bigcirc_{i=m-1}^{1} \psi(2i, \mathcal{S}_{n-1,1})^{(m+i)} \circ \bigcirc_{i=1}^{m} \psi(2i-1, \mathcal{S}_{n-1,1})^{(m-1+i)}.$$

Remark that the function ϕ does not appear in these relations. See Figure 1 for an illustration of this code.

Figure 1: The relations (b) for $n = 8$ and $n = 9$. Each point i corresponds to the list $\psi(i, \mathcal{S}_{n-1,1})$ and each encircled point i represents the list $\overline{\psi(i, \mathcal{S}_{n-1,1})}$, *i.e.*, the reverse list of $\psi(i, \mathcal{S}_{n-1,1})$. For instance, if $n = 8$ then $\mathcal{S}_{8,1} = \psi(7, \mathcal{S}_{7,1}) \circ \psi(5, \mathcal{S}_{7,1}) \circ \overline{\psi(3, \mathcal{S}_{7,1})} \circ \psi(1, \mathcal{S}_{7,1}) \circ \overline{\psi(2, \mathcal{S}_{7,1})} \circ \psi(4, \mathcal{S}_{7,1}) \circ \overline{\psi(6, \mathcal{S}_{7,1})}$.

The lemma below gives the first and the last permutations of the list $\mathcal{S}_{n,1}$.

Lemma 5 *If $n \geq 3$ then*

1. $f_{n,1} = (1, 2, 3, \ldots, n-2, n-1, n)$
2. $\ell_{n,1} = (1, 2, 3, \ldots, n-2, n, n-1)$.

Proof. This holds if $n = 2$. For $n \geq 2$, relations (b) give $f_{n,1} = \psi(n-1, f_{n-1,1})$ and $\ell_{n,1} = \overline{\psi(n-2, \ell_{n-1,1})} = \psi(n-2, f_{n-1,1})$. We obtain the results by induction.

Proposition 1 *In the list $\mathcal{S}_{n,1}$ defined by (b), $n \geq 2$, two consecutive permutation differ in at most three positions.*

Proof. The transitions between $\psi(1, \mathcal{S}_{n-1,1})$ and $\overline{\psi(2, \mathcal{S}_{n-1,1})}$ (or between $\overline{\psi(1, \mathcal{S}_{n-1,1})}$ and $\psi(2, \mathcal{S}_{n-1,1})$) are given by Lemma 3. The transitions between $\psi(i, \mathcal{S}_{n-1,1})$ and $\overline{\psi(i+2, \mathcal{S}_{n-1,1})}$ (or $\overline{\psi(i, \mathcal{S}_{n-1,1})}$ and $\psi(i+2, \mathcal{S}_{n-1,1})$) are given by Lemma 4. Finally, for n even $n \geq 3$, we obtain

$$\begin{aligned} last(\psi(n-1, \mathcal{S}_{n-1,1})) &= \psi(n-1, last(\mathcal{S}_{n-1,1})) \\ &= (1, 2, 3, \ldots, n-3, n-1, n-2, n) \\ &= (1, 2, 3, \ldots, n-3, n, n-2, n-1) \cdot \langle n-2, n \rangle \\ &= \psi(n-3, first(\mathcal{S}_{n-1,1})) \cdot \langle n-2, n \rangle \\ &= first(\psi(n-3, \mathcal{S}_{n-1,1})) \cdot \langle n-2, n \rangle \end{aligned}$$

which gives a smooth transition (product by a transposition) between the two first sublists when n is even. □

3.3 The case $k = n-1$, $n \geq 3$

We define the relations (c) (see Figure 2),

• For $n = 2m \geq 4$,

$\mathcal{S}_{n,n-1} = \psi'(n, \mathcal{S}_{n-1,n-1}) \circ \phi'(\mathcal{S}_{n-1,n-2}) \circ \bigcirc_{i=m-1}^{1} \psi'(2i, \mathcal{S}_{n-1,n-1}) \circ \bigcirc_{i=2}^{m} \psi'(2i-1, \mathcal{S}_{n-1,n-1})$.

• For $n = 2m+1 \geq 3$,

$\mathcal{S}_{n,n-1} = \psi'(n, \mathcal{S}_{n-1,n-1}) \circ \phi'(\mathcal{S}_{n-1,n-2}) \circ \bigcirc_{i=m}^{2} \psi'(2i-1, \mathcal{S}_{n-1,n-1}) \circ \bigcirc_{i=1}^{m} \psi'(2i, \mathcal{S}_{n-1,n-1})$.

Notice that these relations consider ψ' and ϕ' instead of ψ and ϕ.

Figure 2: The relations (c) for $n = 8$ and $n = 9$. Each point $i \neq 0$ corresponds to the list $\psi'(i, \mathcal{S}_{n-1,n-1})$ and the point 0 represents the list $\phi'(\mathcal{S}_{n-1,n-2})$. For instance, if $n = 8$ then $\mathcal{S}_{8,7} = \psi'(8, \mathcal{S}_{7,7}) \circ \phi'(\mathcal{S}_{7,6}) \circ \psi'(6, \mathcal{S}_{7,7}) \circ \psi'(4, \mathcal{S}_{7,7}) \circ \psi'(2, \mathcal{S}_{7,7}) \circ \psi'(3, \mathcal{S}_{7,7}) \circ \psi'(5, \mathcal{S}_{7,7}) \circ \psi'(7, \mathcal{S}_{7,7})$.

Lemma 6 *If $n \geq 3$ then*

1. $f_{n,n-1} = (n-1, n-2, n-3, \ldots, 2, 1, n)$
2. $\ell_{n,n-1} = (n, n-2, n-3, \ldots, 2, 1, n-1)$.

Proof. By relations (c) we obtain by induction $f_{n,n-1} = \psi'(n, \mathcal{S}_{n-1,n-1}) = (n-1, n-2, \ldots, 2, 1, n)$ and $\ell_{n,n-1} = \overline{\psi'(n-1, \mathcal{S}_{n-1,n-1})} = (n, n-2, n-3, \ldots, 2, 1, n-1)$.

Proposition 2 *In the list $\mathcal{S}_{n,n-1}$ defined by (c), $n \geq 3$, two consecutive permutations differ in at most three positions.*

Proof. Here, we just consider the transitions between sublists which are not considered by Lemma 3 and 4. It remains the transitions (1) $\psi'(n, \mathcal{S}_{n-1,n-1}) \circ \phi'(\mathcal{S}_{n-1,n-2})$ and (2) $\phi'(\mathcal{S}_{n-1,n-2}) \circ \psi'(n-2, \mathcal{S}_{n-1,n-1})$.
For the case (1),

$$\begin{aligned} last(\psi'(n, \mathcal{S}_{n-1,n-1})) &= (n-1, n-2, \ldots, 3, 2, 1, n) \\ &= (n-1, n-2, \ldots, 3, 2, n, 1) \cdot \langle n-1, n \rangle \\ &= \phi'(f_{n-1,n-2}) \cdot \langle n-1, n \rangle \\ &= first(\phi'(\mathcal{S}_{n-1,n-2})) \cdot \langle n-1, n \rangle. \end{aligned}$$

For the case (2),

$$\begin{aligned} last(\phi'(\mathcal{S}_{n-1,n-2})) &= (n, n-2, n-3, \ldots, 3, 2, n-1, 1) \\ &= (n, n-1, n-3, \ldots, 3, 2, 1, n-2) \cdot \langle 2, n, n-1 \rangle \\ &= first(\psi'(n-2, \mathcal{S}_{n-1,n-1})) \cdot \langle 2, n, n-1 \rangle. \end{aligned}$$

□

3.4 The case $k = n-2$, $n \geq 4$

We define the relations (d) (see Figure 3),
• For $n = 2m \geq 4$,

$$\mathcal{S}_{n,n-2} = \psi(n-1, \mathcal{S}_{n-1,n-2}) \circ \overline{\phi(\mathcal{S}_{n-1,n-3})} \circ \bigcirc_{i=m-1}^{1} \psi(2i-1, \mathcal{S}_{n-1,n-2})^{(m+i-1)} \circ \bigcirc_{i=1}^{m-1} \psi(2i, \mathcal{S}_{n-1,n-2})^{(m+i)}.$$

• For $n = 2m+1 \geq 5$,

$$\mathcal{S}_{n,n-2} = \psi(n-1, \mathcal{S}_{n-1,n-2}) \circ \phi(\mathcal{S}_{n-1,n-3}) \circ \bigcirc_{i=m-1}^{1} \psi(2i, \mathcal{S}_{n-1,n-2})^{(m+i)} \circ \bigcirc_{i=1}^{m} \psi(2i-1, \mathcal{S}_{n-1,n-2})^{(m+1+i)}.$$

Lemma 7 *If $n \geq 4$ then*

1. $f_{n,n-2} = (n-2, n-3, \ldots, 2, 1, n-1, n)$
2. $\ell_{n,n-2} = (n-2, n-3, \ldots, 2, 1, n, n-1)$.

Proof. By induction and with relations (d) we obtain
$f_{n,n-2} = \psi(n-1, first(\mathcal{S}_{n-1,n-2})) = \psi(n-1, (n-2, n-3, \ldots, 2, 1, n-1)) = (n-2, n-3, \ldots, 2, 1, n-1, n)$ and
$\ell_{n,n-2} = \psi(n-2, first(\mathcal{S}_{n-1,n-2})) = \psi(n-2, (n-2, n-3, \ldots, 2, 1, n-1)) = (n-2, n-3, \ldots, 2, 1, n, n-1)$. □

Figure 3: The relations (d) for $n = 8$ and $n = 9$. Each point $i \neq 0$ corresponds to the list $\psi(i, \mathcal{S}_{n-1,n-2})$ and the point 0 is the list $\phi(\mathcal{S}_{n-1,n-3})$. When the point is encircled, we consider the inverse list. For instance, if $n = 8$ then $\mathcal{S}_{8,6} = \psi(7, \mathcal{S}_{7,6}) \circ \overline{\phi(\mathcal{S}_{7,5})} \circ \psi(5, \mathcal{S}_{7,6}) \circ \overline{\psi(3, \mathcal{S}_{7,6})} \circ \psi(1, \mathcal{S}_{7,6}) \circ \overline{\psi(2, \mathcal{S}_{7,6})} \circ \psi(4, \mathcal{S}_{7,6}) \circ \overline{\psi(6, \mathcal{S}_{7,6})}$.

Proposition 3 *In the list $\mathcal{S}_{n,n-2}$ defined by (d), $n \geq 4$, two consecutive permutations differ in at most three positions.*

Proof. After applying Lemma 1 and 2, it remains the transitions (1) $\psi(n-1, \mathcal{S}_{n-1,n-2}) \circ \phi(\mathcal{S}_{n-1,n-3})$, (2) $\psi(n-1, \mathcal{S}_{n-1,n-2}) \circ \overline{\phi(\mathcal{S}_{n-1,n-3})}$, (3) $\phi(\mathcal{S}_{n-1,n-3}) \circ \overline{\psi(n-3, \mathcal{S}_{n-1,n-2})}$ and (4) $\overline{\phi(\mathcal{S}_{n-1,n-3})} \circ \psi(n-3, \mathcal{S}_{n-1,n-2})$. For the cases (1) and (2),

$$\begin{aligned} last(\psi(n-1, \mathcal{S}_{n-1,n-2})) &= (n-1, n-3, n-4, \ldots, 3, 2, 1, n-2, n) \\ &= (n, n-3, n-4, \ldots, 3, 2, 1, n-2, n-1) \cdot \langle 1, n \rangle \\ &= \phi(f_{n-1,n-3}) \cdot \langle 1, n \rangle \\ &= first(\phi(\mathcal{S}_{n-1,n-3})) \cdot \langle 1, n \rangle \\ &= last(\phi(\mathcal{S}_{n-1,n-3})) \cdot \langle 1, n-1, n \rangle. \end{aligned}$$

For the case (3),

$$\begin{aligned} last(\phi(\mathcal{S}_{n-1,n-3})) &= (n, n-3, n-4, \ldots, 2, 1, n-1, n-2) \\ &= (n-1, n-3, n-4, \ldots, 2, n, 1, n-2) \cdot \langle 1, n-2, n-1 \rangle \\ &= last(\psi(n-3, \mathcal{S}_{n-1,n-2})) \cdot \langle 1, n-2, n-1 \rangle. \end{aligned}$$

For the case (4),

$$\begin{aligned} first(\phi(\mathcal{S}_{n-1,n-3})) &= (n, n-3, n-4, \ldots, 2, 1, n-2, n-1) \\ &= (n-2, n-3, n-4, \ldots, 2, n, 1, n-1) \cdot \langle 1, n-2, n-1 \rangle \\ &= first(\psi(n-3, \mathcal{S}_{n-1,n-2})) \cdot \langle 1, n-2, n-1 \rangle. \end{aligned}$$

□

3.5 The case $2 \leq k \leq n-3$

If $2 \leq k \leq n-3$, we define relations (e) (see Figure 4)

• For $n = 2m$, k odd,

$$\mathcal{S}_{n,k} = \psi(n-1, \mathcal{S}_{n-1,k}) \circ \bigcirc_{i=m-1}^{1} \psi(2i-1, \mathcal{S}_{n-1,k})^{(m+i)} \circ \bigcirc_{i=1}^{(k-1)/2} \psi(2i, \mathcal{S}_{n-1,k})^{(m+i+1)} \circ \phi(\mathcal{S}_{n-1,k-1})^{m+(k-1)/2} \circ \bigcirc_{i=(k+1)/2}^{m-1} \psi(2i, \mathcal{S}_{n-1,k-1})^{(m+i)}.$$

• For $n = 2m$, k even,

$$\mathcal{S}_{n,k} = \psi(n-1, \mathcal{S}_{n-1,k}) \circ \bigcirc_{i=m-1}^{(k+2)/2} \psi(2i-1, \mathcal{S}_{n-1,k})^{(m+i)} \circ \phi(\mathcal{S}_{n-1,k-1})^{m+k/2} \circ$$
$$\bigcirc_{i=k/2}^{1} \psi(2i-1, \mathcal{S}_{n-1,k})^{(m+i-1)} \circ \bigcirc_{i=1}^{m-1} \psi(2i, \mathcal{S}_{n-1,k-1})^{(m+i)}.$$

• For $n = 2m+1$, k odd,

$$\mathcal{S}_{n,k} = \psi(n-1, \mathcal{S}_{n-1,k}) \circ \bigcirc_{i=m-1}^{(k+1)/2} \psi(2i, \mathcal{S}_{n-1,k})^{(m+i-1)} \circ \phi(\mathcal{S}_{n-1,k-1})^{m+(k+1)/2} \circ$$
$$\bigcirc_{i=(k-1)/2}^{1} \psi(2i, \mathcal{S}_{n-1,k})^{(m+i)} \circ \bigcirc_{i=1}^{m} \psi(2i-1, \mathcal{S}_{n-1,k-1})^{(m+i+1)}.$$

• For $n = 2m+1$, k even,

$$\mathcal{S}_{n,k} = \psi(n-1, \mathcal{S}_{n-1,k}) \circ \bigcirc_{i=m-1}^{1} \psi(2i, \mathcal{S}_{n-1,k})^{(m+i-1)} \circ \bigcirc_{i=1}^{(k-1)/2} \psi(2i-1, \mathcal{S}_{n-1,k})^{(m+i)} \circ$$
$$\phi(\mathcal{S}_{n-1,k-1})^{m+(k+1)/2} \circ \bigcirc_{i=(k+1)/2}^{m} \psi(2i-1, \mathcal{S}_{n-1,k-1})^{(m+i+1)}.$$

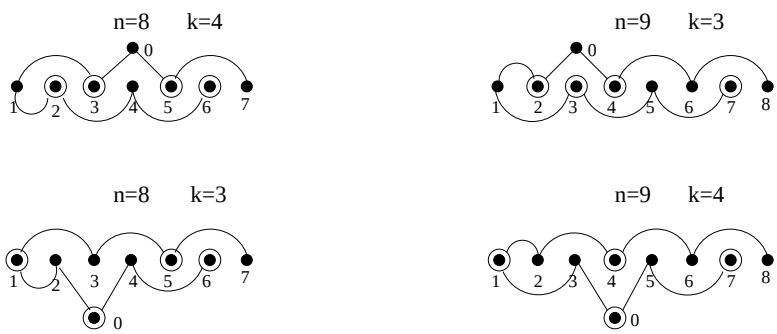


Figure 4: The relations (e) for $n = 8, 9$ and $k = 3, 4$. Each point $i \neq 0$ corresponds to the list $\psi(i, \mathcal{S}_{n-1,k})$ and the point 0 is the list $\phi(\mathcal{S}_{n-1,k-1})$. When the point is encircled, we consider the inverse list. For instance, if $n = 8$ and $k = 4$ then $\mathcal{S}_{8,4} = \psi(7, \mathcal{S}_{7,4}) \circ \overline{\psi(5, \mathcal{S}_{7,4})} \circ \phi(\mathcal{S}_{7,3}) \circ \overline{\psi(3, \mathcal{S}_{7,4})} \circ \psi(1, \mathcal{S}_{7,4}) \circ \overline{\psi(2, \mathcal{S}_{7,4})} \circ \psi(4, \mathcal{S}_{7,4}) \circ \overline{\psi(6, \mathcal{S}_{7,4})}$.

Lemma 8 *If $2 \leq k \leq n-3$ then*

1. $f_{n,k} = (k, k-1, k-2, \ldots, 2, 1, k+1, k+2, \ldots, n-2, n-1, n)$
2. $\ell_{n,k} = (k, k-1, k-2, \ldots, 2, 1, k+1, k+2, \ldots, n-2, n, n-1)$

Proof. We obtain the result with $f_{n,k} = first(\psi(n-1, \mathcal{S}_{n-1,k})) = \psi(n-1, f_{n-1,k})$ anchored by $f_{k,k} = (k, k-1, k-2, \ldots, 3, 2, 1)$. Similarly, $\ell_{n,k} =$

$last(\psi(n-2,\overline{\mathcal{S}_{n-1,k}})) = \psi(n-2, f_{n-1,k})$ also anchored by $f_{k,k} = (k, k-1, k-2, \ldots, 3, 2, 1)$. $\square$

Proposition 4 *In the list $\mathcal{S}_{n,k}$ defined by (e), $2 \leq k \leq n-3$, two consecutive permutations differ by at most three positions.*

Proof. After applying Lemma 1 and 2, it remains the transitions (1) $\psi(k-1, \mathcal{S}_{n-1,k}) \circ \overline{\phi(\mathcal{S}_{n-1,k-1})}$, (2) $\overline{\phi(\mathcal{S}_{n-1,k-1})} \circ \psi(k+1, \mathcal{S}_{n-1,k})$, (3) $\overline{\psi(k-1, \mathcal{S}_{n-1,k})} \circ \phi(\mathcal{S}_{n-1,k-1})$, (4) $\phi(\mathcal{S}_{n-1,k-1}) \circ \overline{\psi(k+1, \mathcal{S}_{n-1,k})}$, (5) $\psi(n-1, \mathcal{S}_{n-1,k}) \circ \psi(n-3, \mathcal{S}_{n-1,k})$.

Case 1:

$$\begin{aligned} last(\psi(k-1, \mathcal{S}_{n-1,k})) &= (k, k-1, \ldots, 2, n, 1, k+1, k+2, \ldots, n-1, n-2) \\ &= (n, k-1, \ldots, 2, 1, k, k+1, k+2, \ldots, n-1, n-2) \cdot \langle 1, k+1, k \rangle \\ &= first(\overline{\phi(\mathcal{S}_{n-1,k-1})}) \cdot \langle 1, k+1, k \rangle. \end{aligned}$$

Case 2:

$$\begin{aligned} first(\psi(k+1, \mathcal{S}_{n-1,k})) &= (k, k-1, \ldots, 2, 1, k+1, n, k+2, \ldots, n-2, n-1) \\ &= (n, k-1, \ldots, 2, 1, k, k+1, k+2, \ldots, n-2, n-1) \cdot \langle 1, k+1, k+2 \rangle \\ &= last(\overline{\phi(\mathcal{S}_{n-1,k-1})}) \cdot \langle 1, k+1, k+2 \rangle. \end{aligned}$$

Case 3:

$$\begin{aligned} last(\overline{\psi(k-1, \mathcal{S}_{n-1,k})}) &= (k, k-1, \ldots, 2, n, 1, k+1, k+2, \ldots, n-2, n-1) \\ &= (n, k-1, \ldots, 2, 1, k, k+1, k+2, \ldots, n-2, n-1) \cdot \langle 1, k+1, k \rangle \\ &= last(\phi(\mathcal{S}_{n-1,k-1})) \cdot \langle 1, k+1, k \rangle. \end{aligned}$$

Case 4:

$$\begin{aligned} first(\overline{\psi(k+1, \mathcal{S}_{n-1,k})}) &= (k, k-1, \ldots, 2, 1, k+1, n, k+2, \ldots, n-1, n-2) \\ &= (n, k-1, \ldots, 2, 1, k, k+1, k+2, \ldots, n-1, n-2) \cdot \langle 1, k+1, k+2 \rangle \\ &= last(\phi(\mathcal{S}_{n-1,k-1})) \cdot \langle 1, k+1, k+2 \rangle. \end{aligned}$$

Case 5:

$$\begin{aligned} first(\psi(n-3, \mathcal{S}_{n-1,k})) &= (k, k-1, \ldots, 2, 1, k+1, \ldots, n-3, n, n-2, n-1) \\ &= (k, k-1, \ldots, 2, 1, k+1, \ldots, n-3, n-1, n, n-2) \cdot \langle n-2, n-1, n \rangle \\ &= last(\psi(n-1, \mathcal{S}_{n-1,k})) \cdot \langle n-2, n-1, n \rangle. \end{aligned}$$

$\square$

The family of lists $\{\mathcal{S}_{n,k}\}_{n \geq k \geq 1}$ is an optimal cyclic Gray code. See Table 1 for some examples. Moreover, let denote $\mathcal{S}_{n,k}^{-1}$ the list obtained from $\mathcal{S}_{n,k}$ by replacing each permutation in $\mathcal{S}_{n,k}$ by its group theoretical inverse. Thus, by a simple calculation, the family of lists $\{\mathcal{S}_{n,k}^{-1}\}_{n \geq k \geq 1}$ is also an optimal cyclic Gray code for the n-length permutations with k left-to-right minima; i.e. two successive permutations in a list $\mathcal{S}_{n,k}^{-1}$ differ by a product with a transposition or a 3-cycle.

4 Algorithmic considerations

In this part, we explain how the recursive definitions (a), (b), (c), (d) and (e) can be implemented into an efficient algorithm, i.e. in a constant

Table 1: The lists $\mathcal{S}_{3,k}$, $1 \leq k \leq 3$, and $\mathcal{S}_{4,k}$ for $1 \leq k \leq 4$. For instance, in $\mathcal{S}_{4,2}$ the sublists of relation (d), $\psi(3, \mathcal{S}_{3,2})$, $\overline{\phi(\mathcal{S}_{3,1})}$, $\psi(1, \mathcal{S}_{3,2})$, and $\overline{\psi(2, \mathcal{S}_{3,2})}$, are alternatively in bold-face and italic.

$\mathcal{S}_{3,1}$		$\mathcal{S}_{3,2}$		$\mathcal{S}_{3,3}$		$\mathcal{S}_{4,1}$		$\mathcal{S}_{4,2}$		$\mathcal{S}_{4,3}$		$\mathcal{S}_{4,4}$	
1	**123**	1	**213**	1	**321**	1	**1234**	1	**2134**	1	**3214**	1	**4321**
2	*132*	2	*231*			2	**1324**	2	**2314**	2	*3241*		
		3	**312**			3	*1423*	3	**3124**	3	*3421*		
						4	*1432*	4	*4132*	4	*4231*		
						5	**1342**	5	*4123*	5	**4312**		
						6	**1243**	6	**2413**	6	*4213*		
								7	**2431**				
								8	**3412**				
								9	*3142*				
								10	*2341*				
								11	*2143*				

amortized time (CAT) algorithm. Such algorithms already exist for derangements, permutations with k cycles or involutions [1, 2, 10, 12], so we will just give here the main difficulties to implement our one.

According to the relation (4) and the cases (a), (b), (c), (d) and (e), the procedure $gen\_up(n, k)$ given in Appendix produces iteratively the sublists $\mathcal{S}_i$ and $\mathcal{T}$ that are recursively generated. So, each call of this procedure fills up entries with indices in an active set $T \subset [n]$ associated with it, and for each recursive call $T = \{i_1, i_2, \ldots, i_k\}$ is replaced by an active subset T' such that $T = T' \cup \{i_j\}$ where $j \in [k]$. As we did it for the generating algorithm the Gray code for derangements [2], the current active set T is represented by four global variables: the integers *head*, *tail*, and the two arrays *succ* and *pred* defined as follows. If at a computational step $T = \{i_1, i_2, \ldots, i_k\}$, then we let $head = i_1$, $tail = i_k$, $succ[i_j] = i_{j+1}$ and $pred[i_j] = i_{j-1}$. So, before each recursive call of the procedure corresponding to the active set T', (1) the procedure $remove(i_j)$ delete the index i_j in T, (2) we perform the recursive call relatively to $T' = T \backslash \{i_j\}$, (3) the procedure $append(i_j)$ add the index i_j in T.

In our algorithm, we consider initially the active set $T = [n+1]$, and the procedure $gen\_up(n, k)$ generates permutations of $\mathcal{S}_{n,k}$ in an array p anchored by the first element of the list. The procedure $type()$ prints out the current permutation of p.

Between any successive calls at least one update statement is performed (according to the relations (a), (b), (c), (d) and (e)), and after each update

statement a new permutation is produced and printed out. The procedure generates at least two recursive calls or produces a permutation. Clearly, the time complexity of *gen_up* is proportional to the total number of recursive calls. Since each call produces at least one new element the time complexity is $\mathcal{O}(s_{n,k})$. A java implementation of our algorithm is avalaible at $http://www.u\text{-}bourgogne.fr/LE2I/jl.baril/leftapplet.html$.

5 Appendix

The call of *gen_up(n,k)* generates the list $\mathcal{S}_{n,k}$. In order to produce $\overline{\mathcal{S}_{n,k}}$ we consider also the procedure *gen_down(n,k)* which has the same instructions of *gen_up(n,k)* in the reverse order. The notation *gen_up/down*$(n-1,k)$ means that we use *gen_up(n,k)* or *gen_down(n,k)* according to the sense of each sublist in the relations (a), (b), (c), (d) and (e). The notation $\langle i,j\rangle$ means that the current permutation p is composed (on the right) by the transposition $\langle i,j\rangle$.

```
public static void gen_up(int n, int k)
 { if(k==n) type();
   if(k==1)
      if(n==3) {type();<succ[head],succ[succ[head]]>;type();}
      else
      { run=pred[tail];remove(run);gen_up(n-1,k);append(run);
        <run,pred[run]>;<pred[run],pred[pred[run]]>;
        if(n mod 2==0)
         <pred[tail],pred[pred[tail]]>;
        for(i=n-3;i>=1;i=i-2)
        {run=pred[pred[run]];remove(run);gen_up/down(n-1,k);append(run);
         if(i>=3)
          {<run,pred[run]>;<pred[run],pred[pred[run]]>;}
         else <succ[head],succ[succ[head]]>;
        }
        if(i==-1) {i=2;run=succ[succ[head]];} else {i=1;run=succ[head];}
        for(int j=i;j<=n-2;j=j+2)
         {remove(run);gen_up/down(n-1,k);append(run);
          if(j+2<=n-2){<run,succ[run]>;<succ[run],succ[succ[run]]>;run=succ[succ[run]];}
         }
      }
   if(k==n-1)
        if(n==3){type();<pred[tail],pred[pred[tail]]>;type();
                        <pred[pred[pred[tail]]],pred[pred[tail]]>;
                        <pred[tail],pred[pred[tail]]>;type();}
        else
         { type();<pred[tail],pred[pred[tail]]>;run=pred[tail];
           remove(run);gen_up(n-1,k-1);append(run);
           run=succ[head];<pred[tail],pred[pred[tail]]>;<pred[tail],run>;type();
           run=head;
           for(i=n-4;i>=2;i=i-2)
            {run=succ[succ[run]];<run,succ[run]>;<run,pred[tail]>;type();}
           <pred[tail],pred[pred[pred[tail]]]>;type();
           if(i==0) {i=3;run=pred[pred[tail]];} else {i=2;run=pred[tail];}
           for(int j=i+2;j<=n-1;j=j+2)
             {run=pred[pred[run]];<run,pred[run]>;<run,pred[tail]>;type();}
         }
   if(k==n-2)
    {   run=pred[tail];
        remove(run);gen_up(n-1,k);append(run);
        if(n mod 2==0)
          {<head,pred[tail]>;<pred[tail],pred[pred[tail]]>;}
        else {<head,pred[tail]>;}
        run=head;
        remove(run);gen_up/down(n-1,k-1);append(run);
        <pred[pred[pred[tail]]],pred[pred[tail]]>;<head,pred[pred[pred[tail]]]>;
        run=pred[tail];
        for(i=n-3;i>=1;i=i-2)
         {run=pred[pred[run]];
          remove(run);gen_up/down(n-1,k);append(run);
          if(i>=3) {<run,pred[run]>;<pred[run],pred[pred[run]]>;}
```

```
        else {<succ[head],succ[succ[head]]>;}
       }
      if(i==-1) {i=2;run=succ[succ[head]];} else {i=1;run=succ[head];}
      for(int j=i;j<=n-2;j=j+2)
       {remove(run);gen_up/down(n-1,k);append(run);
        if(j+2<=n-2){<run,succ[run]>;<succ[run],succ[succ[run]]>;run=succ[succ[run]];}
       }
   }
  if(k!= n && k!= 1 && k!= n-1 && k!= n-2)
   {   run=pred[tail];
       remove(run);gen_up(n-1,k);append(run);
       if(n mod 2==0){<run,pred[run]>;<pred[run],pred[pred[run]]>;}
       else {<run,pred[pred[run]]>;}
       run=pred[tail];
       for(i=n-3;i>=1;i=i-2)
        { run=pred[pred[run]];
          remove(run);gen_up/down(n-1,k);append(run);
          if(i==k+1)
           {<pred[run],run>;<head,pred[run]>;run1=head;
            remove(run1);gen_up/down(n-1,k-1,run1,sens);append(run1);
            <head,pred[run]>;<pred[run],run>;
           }
          if(i>=3) {<run,pred[run]>;<pred[run],pred[pred[run]]>;}
          else {<succ[head],succ[succ[head]]>;}
        }
       if(i==-1) {i=2;run=succ[succ[head]];} else {i=1;run=succ[head];}
       for(int j=i;j<=n-2;j=j+2)
        { remove(run);gen_up/down(n-1,k);append(run);
          if(j==k-1)
           { <succ[run],run>;<head,succ[run]>;
             run1=head;
             remove(run1);gen_up/down(n-1,k-1);append(run1);
             <head,succ[run]>;<succ[run],run>;
           }
          if(j+2<=n-2){<run,succ[run]>;<succ[run],succ[succ[run]]>;run=succ[succ[run]];}
        }
   }
}
```

References

[1] J.L. Baril. Gray code for permutations with a fixed number of cycles. *Discrete Math.*, 307, 13(6): 1559–1571, 2007.

[2] J.L. Baril and V. Vajnovszki. Gray code for derangements. *Discrete Appl. Math.*, 140(1-3): 207–221, 2004.

[3] W.M.B. Dukes, M. F. Flanagan, T. Mansour and V. Vajnovszki. Combinatorial Gray codes for classes of pattern avoiding permutations. *Theoretical Computer Science*, 396: 35–49, 2008.

[4] S.M. Johnson. Generation of permutations by adjacent transpositions. *Mathematics of computation*, 17: 282–285, September 1963.

[5] D.E. Knuth. *The art of computer programming, combinatorial algorithms*, volume 4, fascicle 2, Generating All Tuples and Permutations, Addison Wesley, 2005.

[6] J.F. Korsh. Loopless generation of up-down permutations. *Discrete Math.*, 240(1-3): 97–122, 2001.

[7] J.F. Korsh. Constant time generation of derangements. *Information Process. Letters*, 90(4): 181–186, 2004.

[8] D. Roelants van Baronaigien. Constant time generation of involutions. *Congressus Numerantium*, 90: 87–96, 1992.

[9] F. Ruskey. http://www.theory.cs.uvic.ca/~cos/gen/perm.html.

[10] F. Ruskey and S. Effler. A CAT algorithm for generating permutations with a fixed number of inversions. *Inform. Process. Lett.*, 86(2): 107–112, 2003.

[11] F. Ruskey and U. Taylor. Fast generation of restricted classes of permutations. *Manuscript*, 1995.

[12] F. Ruskey and D. Roelants van Baronaigien. Generating permutations with given ups and downs. *Discrete Appl. Math.*, 36(1): 57–67, 1992.

[13] R. Sedgewick. Permutation generation methods. *Comput. Surveys*, 9(2): 137–164, 1977.

[14] R. Stanley. *Enumerative combinatorics*, volume 1. Cambridge University Press, Cambridge, England, 1997.

[15] H.F. Trotter. *PERM (Algorithm 115)*. Communications of ACM, 5(8), 434–435, 1962.

[16] V. Vajnovszki. A loopless algorithm for generating the permutations of a multiset. *Theoretical Computer Science*, 307: 415–431, 2003.

[17] T. Walsh. Gray code for involutions. *J. Combin. Math. Combin. Comput.*, 36: 95–118, 2001.

[18] R.M. Wilson and J.H. van Lint. *A course in Combinatorics*. Cambridge University Press, 2002.

Information Processing Letters 114 (2014) 31–37

Contents lists available at ScienceDirect

Information Processing Letters

www.elsevier.com/locate/ipl

Motzkin subposets and Motzkin geodesics in Tamari lattices

J.-L. Baril *, J.-M. Pallo

LE2I, UMR 5158, Université de Bourgogne, B.P. 47870, F21078 Dijon Cedex, France

ARTICLE INFO

Article history:
Received 8 May 2013
Received in revised form 26 August 2013
Accepted 3 October 2013
Available online 10 October 2013
Communicated by M. Yamashita

Keywords:
Combinatorial problems
Lattices
Dyck words
Motzkin words
Tamari lattice
Metric
Geodesic

ABSTRACT

The Tamari lattice of order n can be defined by the set $\mathcal{D}_n$ of Dyck words endowed with the partial order relation induced by the well-known rotation transformation. In this paper, we study this rotation on the restricted set of Motzkin words. An upper semimodular join semilattice is obtained and a shortest path metric can be defined. We compute the corresponding distance between two Motzkin words in this structure. This distance can also be interpreted as the length of a geodesic between these Motzkin words in a Tamari lattice. So, a new upper bound is obtained for the classical rotation distance between two Motzkin words in a Tamari lattice. For some specific pairs of Motzkin words, this bound is exactly the value of the rotation distance in a Tamari lattice. Finally, enumerating results are given for join and meet irreducible elements, minimal elements and coverings.

© 2013 Elsevier B.V. All rights reserved.

1. Introduction and notations

The set $\mathcal{D}$ of Dyck words over $\{(,)\}$ is the language defined by the grammar $S \rightsquigarrow \lambda|(S)|SS$ where λ is the empty word, *i.e.* the set of well-formed parentheses strings. Let $\mathcal{D}_n$ be the set of Dyck words of length $2n$, *i.e.* with n open and n close parentheses. The cardinality of $\mathcal{D}_n$ is the nth Catalan number $c_n = (2n)!/(n!(n+1)!)$ (see A000108 in [23]). For instance, $\mathcal{D}_3$ consists of the five words ()()(), (())(), ()(()), (()()) and ((())). A large number of various classes of combinatorial objects are enumerated by the Catalan sequence. This is the case, among others, for ballot sequences, planar trees, binary rooted trees, nonassociative products, stack sortable permutations, triangulations of polygons, and Dyck paths. See [24] for a compilation of such Catalan sets. Some of them are endowed with a partial ordering relation [1–3,10,20,21]. For instance, the coverings of the so-called Tamari lattices [11,13,14,16,18,22,25] can be defined by different elementary transformations depending on the Catalan set considered. The most known is the semi-associative law $x(yz) \longrightarrow (xy)z$ for well-formed parenthesized expressions involving n variables. Also, the Tamari lattice of order n can be defined on the set $\mathcal{T}_n$ of binary rooted trees with $n+1$ leaves. Indeed, from a well-formed parenthesized expression on n variables, we consider the bijection that recursively constructs the binary rooted tree where the left (resp. right) subtree is defined by the left (resp. right) part of the expression. For example, the binary rooted trees associated to the two expressions $x(yz)$ and $(xy)z$ are illustrated in Fig. 1. Moreover, the semi-associative law on parenthesized expressions is equivalent to the well-known left-rotation on binary trees showed in Fig. 1.

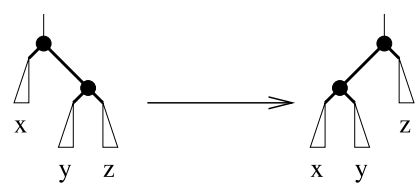


Fig. 1. The left-rotation transformation on binary trees.

* Corresponding author.
E-mail addresses: barjl@u-bourgogne.fr (J.-L. Baril), pallo@u-bourgogne.fr (J.-M. Pallo).

0020-0190/$ – see front matter © 2013 Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.ipl.2013.10.001

Now, let $T \in \mathcal{T}_n$ be a binary tree with $n+1$ leaves. Reading T in prefix order and replacing each internal node (resp. each leaf except the last) with an open (resp. a close) parenthesis, we obtain a bijection from $\mathcal{T}_n$ to $\mathcal{D}_n$ that translates the left-rotation transformation into the elementary transformation $(u)(\longrightarrow ((u)$ where u is a Dyck word. This transformation will be also called left-rotation. More precisely, we say that $d' \in \mathcal{D}_n$ is obtained from $d \in \mathcal{D}_n$ by a left-rotation if $d = \alpha(u)(\beta$ and $d' = \alpha((u)\beta$ where u is a Dyck word and α (resp. β) is some prefix (resp. suffix) of some Dyck word. The inverse transformation is called right-rotation. For instance, $(())((()(())())())$ is obtained from $(())(()(())()\,)(())$ by a left-rotation.

We define the rotation distance between two Dyck words as the minimum number of left- and right-rotations necessary to transform one word into the other. There remains today an open problem whether the rotation distance can be computed in polynomial time. Previous works on rotation distance have focused on approximation algorithms [4,7,18,19].

In this paper, we study this rotation on the restricted set $\mathcal{M}$ of Motzkin words defined by the grammar $S \rightsquigarrow \lambda|(SS)$. In Section 2, an upper semimodular join semilattice is constructed. In this structure, we compute the length of a shortest path between two Motzkin words. This distance can also be interpreted as the length of a geodesic between these two Motzkin words in a Tamari lattice, *i.e.* the length of a shortest path between them by browsing through Motzkin words only. So, a new upper bound is obtained for the classical rotation distance between two Motzkin words in a Tamari lattice. For some specific pairs of Motzkin words, this bound is exactly the value of the well-known rotation distance in a Tamari lattice. In Section 3, enumerating results are given for join and meet irreducible elements, minimal elements and coverings.

2. Motzkin geodesics

Let $\mathcal{M}$ be the set of Motzkin words, *i.e.* the language over $\{(,)\}$ defined by the grammar $S \rightsquigarrow \lambda|(SS)$. Let $\mathcal{M}_n$ be the set of Motzkin words of length $2n$, *i.e.* with n open and n close parentheses. The cardinality of $\mathcal{M}_n$ is the nth term of the Motzkin sequence A001006 in [23]. For example, $\mathcal{M}_4 = \{((((()))), ((()())), ((())()), (((())))\}$. Obviously we have $\mathcal{M}_n \subseteq \mathcal{D}_n$ for $n \geqslant 1$. We refer to [8] and [24] for several combinatorial classes enumerated by the Motzkin numbers.

The following lemma shows how the left-rotation between two Motzkin words can be expressed by a natural transformation $\longrightarrow$ in $\mathcal{M}_n$.

Lemma 1. *Let $m, m' \in \mathcal{M}_n$. Then, the word m' is obtained from m by a left-rotation if and only if m' is obtained from m by a left-transformation $u(v) \longrightarrow (uv)$ where u and v are two Motzkin words, i.e. $m = \alpha u(v)\beta$ and $m' = \alpha(uv)\beta$ where $u, v \in \mathcal{M}$ and α (resp. β) is some prefix (resp. suffix) of some Motzkin word.*

Proof. Let m and m' be two Motzkin words such that m' is obtained from m by a left-rotation. Thus, there is a Dyck word u such that $m = \alpha(u)(\beta$ and $m' = \alpha((u)\beta$ where α and β are some prefix and some suffix of m and m'. Since m' is a Motzkin word, it necessarily is of the form $m' = \alpha((u)v)\beta'$ where (u) and v are Motzkin words with $\beta = v)\beta'$. We deduce that $m = \alpha(u)(v)\beta'$. Furthermore, the fact that (v) and (u) are Motzkin words necessarily implies that m is of the form $m = \alpha'((u)(v))\beta''$ and thus, $m' = \alpha'(((u)v))\beta''$ for some α' and β''. Thus the left-rotation between two Motzkin words is equivalent to the transformation $w(v) \longrightarrow (wv)$ where $w = (u) \neq \lambda$ and v are two Motzkin words. Conversely, let us assume that m' is obtained from m by a transformation $u(v) \longrightarrow (uv)$ where u and v are two Motzkin words. Since, a Motzkin word is obtained by the grammar $S \rightsquigarrow \lambda|(SS)$, we necessarily have $m = \alpha(u(v))\beta$ and $m' = \alpha((uv))\beta$ for some prefix and some suffix α and β which directly induces that m' is obtained from m by a left-rotation. □

In the remainder of the paper, given $m, m' \in \mathcal{M}_n$, we write $m \longrightarrow m'$ if m' is obtained from m by the left-rotation defined in the previous lemma. The right-rotation will be the inverse of $\longrightarrow$. Let $\overset{*}{\longrightarrow}$ denote the reflexive and transitive closure of the rotation transformation $\longrightarrow$ in $\mathcal{M}_n$.

In order to characterize this left-rotation $\longrightarrow$, we exhibit a bijection between Motzkin words and Motzkin paths. A Motzkin path of length n is a lattice path starting at $(0,0)$, ending at $(n,0)$, and never going below the x-axis, consisting of up steps $U = (1,1)$, horizontal steps $H = (1,0)$, and down steps $D = (1,-1)$. Let $\mathcal{P}_n$ be the set of Motzkin path of length $n-1$. It is well known that Motzkin paths are enumerated by the Motzkin numbers (A001006 in [23]).

Let ϕ be the bijection between $\mathcal{M}_n$ and the set $\mathcal{P}_n$ of Motzkin paths of length $n-1$ defined as follows:

- if $m = ()$ then $\phi(m) = \lambda$;
- if $m = (uv)$ where u, v are two non-empty Motzkin words, then $\phi(m) = U\phi(v)D\phi(u)$;
- if $m = (u)$ where u is a non-empty Motzkin word, then $\phi(m) = H\phi(u)$.

For instance, if $m = (()((()()))))$ then $\phi(m) = UHUDD$.

Proposition 1. *Let m and m' be two Motzkin words in $\mathcal{M}_n$. Then $m \longrightarrow m'$ if and only if $\phi(m')$ is obtained from $\phi(m)$ by applying one of the two following transformations: $UH \longrightarrow HU$ and $UD \longrightarrow HH$.*

Proof. Let m and m' be two Motzkin words where m' is obtained from m by a left-rotation in the Tamari lattice of order n. By Lemma 1, we deduce that $m = \alpha(u(v))\beta$ and $m' = \alpha((uv))\beta$ where α and β are some prefix and some suffix of m and m'. Therefore $\phi(m')$ is obtained from $\phi(m)$ by replacing the factor $\phi((u(v)))$ with $\phi(((uv)))$. If v is empty, then we have $\phi((u(v))) = UD\phi(u)$, $\phi(((uv))) = HH\phi(u)$ and $\phi(m')$ is obtained from $\phi(m)$ by a transformation $UD \longrightarrow HH$. If v is not empty, then we have $\phi((u(v))) = UH\phi(v)D\phi(u)$, $\phi(((uv))) = HU\phi(v)D\phi(u)$ and $\phi(m')$ is obtained from $\phi(m)$ by a transformation $UH \longrightarrow HU$. This reasoning can also be considered for the converse. □

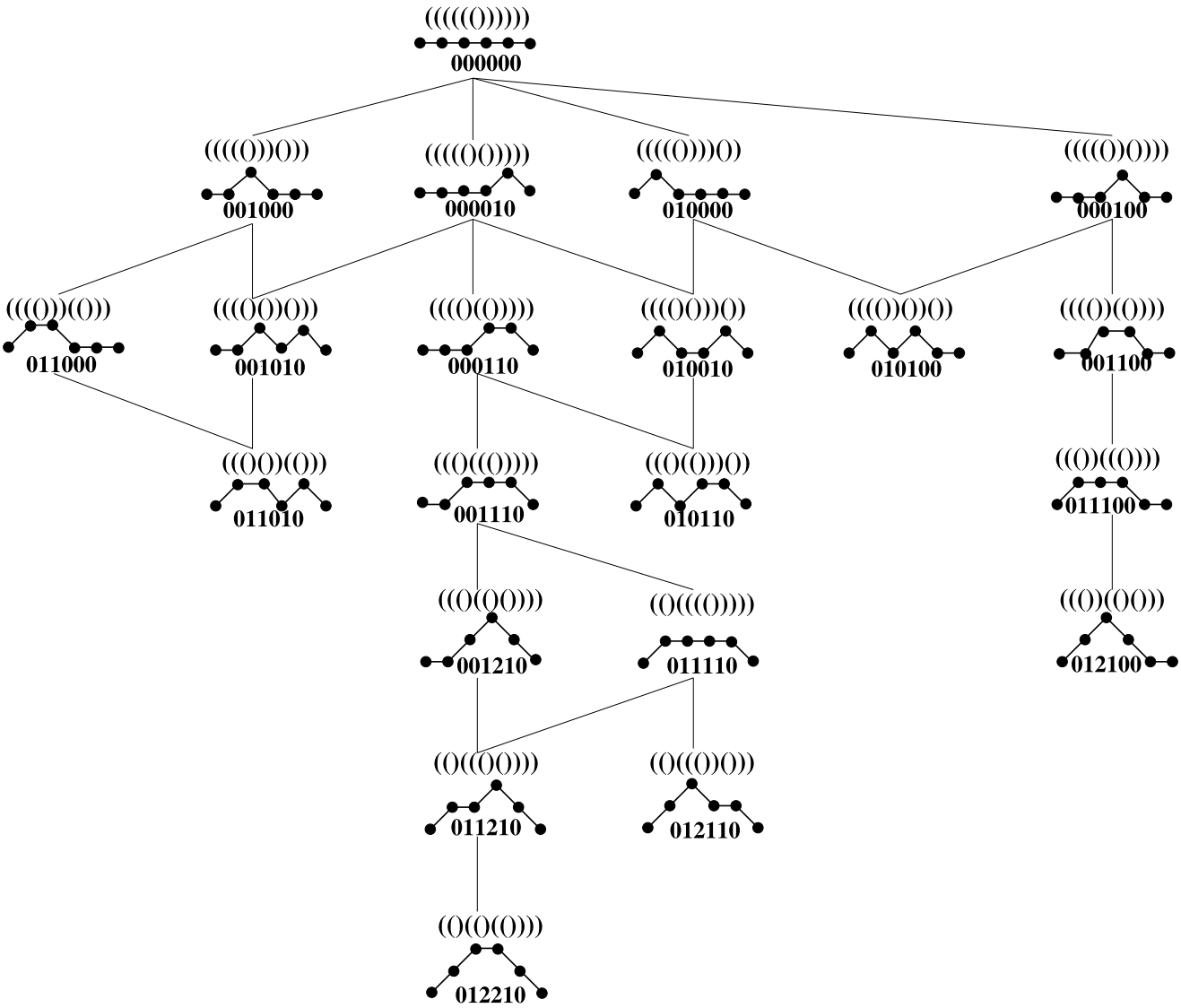


Fig. 2. The Motzkin semilattice $\mathcal{M}_6$. Each Motzkin word m is associated with its Motzkin path $\phi(m)$ and its height sequence χ_m.

A sequence of non-negative integers $\chi = \chi(0)\chi(1)\ldots\chi(n-1)$ will be called *height sequence* of length n if $\chi(0) = \chi(n-1) = 0$ and $|\chi(i) - \chi(i-1)| \leqslant 1$ for $1 \leqslant i \leqslant n-1$. Obviously, there is a one-to-one correspondence between Motzkin words in $\mathcal{M}_n$ and height sequences of length n: if $m \in \mathcal{M}_n$, then we associate the height sequence χ_m defined by $\chi_m(0) = 0$ and for $i \geqslant 1$, $\chi_m(i) - \chi_m(i-1) = 1$ (resp. -1, 0) if the ith step is U (resp. D and H) in the Motzkin path $\phi(m)$. For instance, if $m = (()(()(()))) \in \mathcal{M}_6$, then $\phi(m) = UUHDD$ and $\chi_m = 012210$. Notice that the height sequence of a Motzkin word m corresponds to the ordinates of the different steps in the path $\phi(m)$.

Proposition 2. *Let m and m' be two Motzkin words in $\mathcal{M}_n$. Then $m \longrightarrow m'$ if and only if there exists i, $1 \leqslant i \leqslant n-1$, such that $\chi_{m'}(i-1) = \chi_{m'}(i) = \chi_m(i) - 1$ and for all $j \neq i$, $\chi_{m'}(j) = \chi_m(j)$.*

Proof. Using Proposition 1, $m \longrightarrow m'$ if and only if $\phi(m')$ is obtained from $\phi(m)$ by one of the two transformations $UH \longrightarrow HU$ and $UD \longrightarrow HH$. Considering the height sequences χ_m and $\chi_{m'}$, this implies the existence of some i such that $\chi_m(j) = \chi_{m'}(j)$ for $j \neq i$, and $\chi_{m'}(i-1) = \chi_{m'}(i) = \chi_m(i) - 1$. For the converse, let us assume that the height sequences χ_m and $\chi_{m'}$ satisfy for some i, $\chi_{m'}(j) = \chi_m(j)$ for $j \neq i$, and $\chi_{m'}(i-1) = \chi_{m'}(i) = \chi_m(i) - 1$. Since χ_m is a height sequence, we have $|\chi_m(i+1) - \chi_m(i)| \leqslant 1$. This implies that $|\chi_{m'}(i+1) - \chi_{m'}(i) - 1| \leqslant 1$ and thus $\chi_{m'}(i+1) - \chi_{m'}(i) \geqslant 0$. Finally, if $\chi_{m'}(i+1) = \chi_{m'}(i)$ (resp. $\chi_{m'}(i+1) = \chi_{m'}(i) + 1$) then $\phi(m')$ is obtained from $\phi(m)$ by the transformation $UD \longrightarrow HH$ (resp. $UH \longrightarrow HU$). $\square$

Proposition 3. *The poset $(\mathcal{M}_n, \xrightarrow{*})$ is graded by the rank function $r(m) = \rho - \sum_{i=0}^{n-1} \chi_m(i)$ where $\rho = \lfloor \frac{(n-1)^2}{4} \rfloor$.*

Proof. The poset $(\mathcal{M}_n, \xrightarrow{*})$ is graded by the rank function r whenever $m \xrightarrow{*} m'$ and $r(m') = r(m) + 1$ if and only if $m \longrightarrow m'$. So, Proposition 2 induces that $(\mathcal{M}_n, \xrightarrow{*})$ is graded by the rank function $r(m) = \rho - \sum_{i=0}^{n-1} \chi_m(i)$

where the parameter $\rho = \lfloor \frac{(n-1)^2}{4} \rfloor$ is the maximal value of $\sum_{i=0}^{n-1} \chi_m(i)$ among all Motzkin words m. That is, we have $\rho = \sum_{i=0}^{n-1} \chi_m(i)$ with $m = 0123\ldots\frac{n-1}{2}\ldots3210$ if n is odd, and $m = 0123\ldots\frac{n}{2}-1\frac{n}{2}-1\ldots3210$ otherwise. For the two cases, a simple calculation provides $\rho = \lfloor \frac{(n-1)^2}{4} \rfloor$. $\square$

Remark 1. In the previous proposition, we use the definition of a graded poset given by Grätzer (see [12], p. 233). Notice that other authors like Stanley do not use the same definition (see [24], p. 99). Indeed they require that the minimal elements need to have the same rank (see Fig. 2).

Proposition 4. *For $m, m' \in \mathcal{M}_n$, we have $m \xrightarrow{*} m'$ if and only if the sequences χ_m and $\chi_{m'}$ satisfy the two conditions*:

(a) $\chi_{m'}(i) \leqslant \chi_m(i)$ *for all* $0 \leqslant i \leqslant n-1$, *and*
(b) *there does not exist* i, $1 \leqslant i \leqslant n-1$, *such that* $\chi_{m'}(i-1) > \chi_{m'}(i) < \chi_m(i)$.

Proof. Let m and m', $m \neq m'$, such that their corresponding height sequences χ_m and $\chi_{m'}$ satisfy the conditions (a) and (b). Since $m \neq m'$, there exists some i such that $\chi_{m'}(i) < \chi_m(i)$. We choose the rightmost i with this property. We necessarily have $\chi_{m'}(i+1) - \chi_{m'}(i) = \chi_m(i+1) - \chi_{m'}(i) > \chi_m(i+1) - \chi_m(i) \geqslant -1$ and thus $\chi_{m'}(i+1) \geqslant \chi_{m'}(i)$. Now, there exists some j, $j \leqslant i$, such that $\chi_{m'}(j) \leqslant \chi_{m'}(j-1)$. By contradiction, if there does not exist such a j then $\chi_{m'}(j) > \chi_{m'}(j-1)$ for $j \leqslant i$. With $\chi_{m'}(0) = 0$, we obtain $\chi_{m'}(i) = i$ and $\chi_m(i) > \chi_{m'}(i) = i$ gives a contradiction. So, we have $\chi_{m'}(k) + 1 = \chi_{m'}(k+1)$ for $j \leqslant k \leqslant i-1$, which implies $\chi_{m'}(j) = \chi_{m'}(i) - (i-j)$. On the other hand, we necessarily have $\chi_m(j) \geqslant \chi_m(i) - (i-j)$. Using $\chi_{m'}(i) < \chi_m(i)$, we deduce that $\chi_{m'}(j) < \chi_m(j)$ with the condition $\chi_{m'}(j) \leqslant \chi_{m'}(j-1)$. Using Proposition 2, there exists a Motzkin word m_1 satisfying $m_1 \longrightarrow m'$ and such that $\chi_{m_1}(k) = \chi_{m'}(k)$ for $k \neq j$ and $\chi_{m_1}(j) = \chi_{m'}(j) + 1$. By construction, the two sequences χ_{m_1} and $\chi_{m'}$ also satisfy (a) and (b). Repeating iteratively this process for m and m_i, $i \geqslant 1$, there is a positive integer r so that $m \longrightarrow m_r \longrightarrow \cdots \longrightarrow m_2 \longrightarrow m_1 \longrightarrow m'$.

Conversely, let us assume that $m \xrightarrow{*} m'$. Using Proposition 2, it is straightforward to see that the height sequences χ_m and $\chi_{m'}$ satisfy (a). Moreover, if we have the path $m \longrightarrow m_1 \longrightarrow \cdots \longrightarrow m_r = m'$ for some integer $r \geqslant 1$, Proposition 2 ensures that the height sequence of m_1 is obtained from χ_m by decreasing by one an entry $\chi_m(i)$ such that $\chi_m(i+1) \leqslant \chi_m(i) > \chi_m(i-1)$. Therefore, m and m_1 satisfy also (b), and a simple induction proves that m and $m_r = m'$ satisfy (b). $\square$

By construction, the poset $(\mathcal{M}_n, \xrightarrow{*})$ is included in the Tamari lattice of order n (see Fig. 3). Moreover the previous result proves that it is contained into the Motzkin lattice defined by Ferrari and Munarini in [9]. More precisely, the elements are the same, the partial order is dual but our poset has less covering relations.

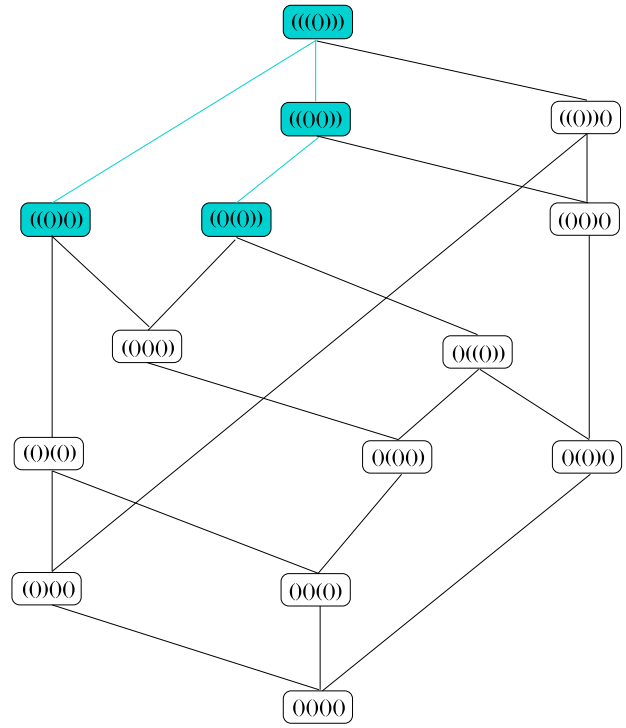


Fig. 3. The Motzkin semilattice $\mathcal{M}_4$ included into the Tamari lattice of order 4.

Theorem 1. *The poset $(\mathcal{M}_n, \xrightarrow{*})$ is a join semilattice with $\mathbf{1} = ((((\ldots)))$ as maximum element.*

Proof. Obviously, Proposition 4 induces that $\mathbf{1} = ((((\ldots)))$ is the maximum element (its height sequence is $0\ldots0$). In order to prove that the poset is a join semilattice, we show that any two elements of $\mathcal{M}_n$ have a least upper bound. Let m and m' be two different Motzkin words and χ_m and $\chi_{m'}$ their height sequences. Now, we construct algorithmically a Motzkin word $m'' \in \mathcal{M}_n$ (or equivalently its height sequence $\chi_{m''}$) that is a candidate to be the join element of m and m', and we will show that this element really is the join.

Join algorithm. (See Algorithm 1.) The inputs are the height sequences of m and m' and the output is the height sequence of the join $m \vee m'$.

For example, if we perform Algorithm 1 for $\chi_m = 0121112110$ and $\chi_{m'} = 0101111010$, Part I gives $\chi_{m''} = 0101111010$, and Part II modifies $\chi_{m''}$ into $\chi_{m''} = 0000000010$.

Part I of Algorithm 1 computes for all i, $0 \leqslant i \leqslant n-1$, $\chi_{m''}(i) = \min\{\chi_m(i), \chi_{m'}(i)\}$. Since the statements of Part II do not increase any value $\chi_{m''}(i)$, the two following conditions $\chi_{m''}(i) \leqslant \chi_m(i)$ and $\chi_{m''}(i) \leqslant \chi_{m'}(i)$, $0 \leqslant i \leqslant n-1$, remain true throughout Algorithm 1.

Statements of Part II modifies $\chi_{m''}$ so that there does not exist i such that $\chi_{m''}(i-1) > \chi_{m''}(i)$ and $(\chi_{m''}(i) < \chi_m(i)$ or $\chi_{m''}(i) < \chi_{m'}(i))$. For this, we traverse $\chi_{m''}$ from right to left and for each i such that $\chi_{m''}(i-1) > \chi_{m''}(i)$ and $(\chi_{m''}(i) < \chi_m(i)$ or $\chi_{m''}(i) < \chi_{m'}(i))$, we replace $\chi_{m''}(j)$ with $\chi_{m''}(i)$ from $j = i-1$ down to $j_0 + 1$ where j_0 is the rightmost index $j \leqslant i-1$ satisfying $\chi_{m''}(j) = \chi_{m''}(i)$.

Algorithm 1. Join algorithm.

```
procedure Join(χ_m, χ_m')
// Part I
for i ← 0 to n − 1 do
    χ_m''(i) ← min{χ_m(i), χ_m'(i)}
end for
// Part II
i ← n − 1
while i > 0 do
    if χ_m''(i) < χ_m''(i − 1) and (χ_m''(i) < χ_m(i) or χ_m''(i) < χ_m'(i)) then
        x ← χ_m''(i)
        i ← i − 1
        while χ_m''(i) > x do
            χ_m''(i) ← x
            i ← i − 1
        end while
    else
        i ← i − 1
    end if
end while
return χ_m''
```

At the end of Algorithm 1, Proposition 4 ensures that $\chi_{m''}$ is a height sequence of a Motzkin word m'' so that $m \xrightarrow{*} m''$ and $m' \xrightarrow{*} m''$.

Now, we will prove that m'' really is the least upper bound of m and m'. Let s be a Motzkin word in $\mathcal{M}_n$ such that $m \xrightarrow{*} s$ and $m' \xrightarrow{*} s$ and let us prove that $m'' \xrightarrow{*} s$. Proposition 4 implies that we have (a) $\chi_s(i) \leqslant \chi_m(i)$ and $\chi_s(i) \leqslant \chi_{m'}(i)$ for $i \geqslant 0$, and (b) there does not exist i such that $\chi_s(i-1) > \chi_s(i)$ and $(\chi_s(i) < \chi_m(i)$ or $\chi_s(i) < \chi_{m'}(i))$.

We distinguish two cases,

- $\chi_s(i) \leqslant \chi_{m''}(i)$ for all i, $0 \leqslant i \leqslant n-1$; then for all j satisfying $\chi_s(j) < \chi_{m''}(j) \leqslant \min\{\chi_m(j), \chi_{m'}(j)\}$, there is a Motzkin word w ($w = m$ or $w = m'$) such that $\chi_s(j) < \chi_w(j)$. Since we have $w \xrightarrow{*} s$, Proposition 4 implies that $\chi_s(j-1) \leqslant \chi_s(j) < \chi_{m''}(j)$. Proposition 4 allows to conclude that $m'' \xrightarrow{*} s$.
- There exists i such that $\chi_s(i) > \chi_{m''}(i)$. Let us recall that $\chi_s(i) \leqslant \min\{\chi_m(i), \chi_{m'}(i)\}$. Then we have $\chi_{m''}(i) < \min\{\chi_m(i), \chi_{m'}(i)\}$ which means that $\chi_{m''}(i)$ was obtained from $\min\{\chi_m(i), \chi_{m'}(i)\}$ using Part II of Algorithm 1.. More precisely, there is $i_1 > i$ such that $\chi_{m''}(i_1) = \chi_{m''}(j) < \min\{\chi_m(j), \chi_{m'}(j)\}$ for $i \leqslant j \leqslant i_1 - 1$, and $\chi_{m''}(i_1) = \min\{\chi_m(i_1), \chi_{m'}(i_1)\}$, and $\chi_{m''}(i_1 - 1) > \chi_{m''}(i_1)$, and $(\chi_{m''}(i_1) < \chi_m(i_1)$ or $\chi_{m''}(i_1) < \chi_{m'}(i_1))$.
  Thus, since $\chi_s(i) > \chi_{m''}(i)$, $\chi_s(j) \leqslant \min\{\chi_m(j), \chi_{m'}(j)\}$ for all j, $\chi_{m''}(i_1) = \min\{\chi_m(i_1), \chi_{m'}(i_1)\}$ and $\chi_{m''}(i_1) = \chi_{m''}(j)$ for $i \leqslant j \leqslant i_1 - 1$, there exists i_2, $i < i_2 \leqslant i_1$ such that $\chi_s(i_2) = \chi_{m''}(i)$ with $\chi_s(i_2 - 1) > \chi_s(i_2)$ and $(\chi_s(i_2) < \chi_m(i_2)$ or $\chi_s(i_2) < \chi_{m'}(i_2))$ which is a contradiction with the fact that $m \xrightarrow{*} s$ and $m' \xrightarrow{*} s$. Therefore, this case does not occur.

Finally, we obtain $m'' \xrightarrow{*} s$ and m'' is really the least upper bound of m and m'. □

Proposition 5. *The semilattice $(\mathcal{M}_n, \xrightarrow{*})$ is upper semimodular, i.e. for all $m_1, m_2, m_3 \in \mathcal{M}_n$ with $m_1 \neq m_2$, $m_3 \longrightarrow m_1$ and $m_3 \longrightarrow m_2$, there exists $m_4 \in \mathcal{M}_n$ such that $m_1 \longrightarrow m_4$ and $m_2 \longrightarrow m_4$.*

Proof. Let χ_{m_i} be the height sequences of m_i, $1 \leqslant i \leqslant 4$. With Proposition 2, the sequence χ_{m_1} (resp. χ_{m_2}) is obtained from χ_{m_3} by decreasing by one the value $\chi_{m_3}(j)$ (resp. $\chi_{m_3}(k)$) with $j < k$, $\chi_{m_3}(j) - 1 \geqslant \chi_{m_3}(j-1)$ and $\chi_{m_3}(k) - 1 \geqslant \chi_{m_3}(k-1)$. In the case where $k \geqslant j+2$, the sequence χ_{m_4} obtained from m_1 by decreasing by one $\chi_{m_1}(k)$ is clearly a height sequence of a Motzkin word m_4 satisfying $m_1 \longrightarrow m_4$ and $m_2 \longrightarrow m_4$. The case $k = j+1$ does not occur. Indeed, we necessarily have the two conditions $\chi_{m_3}(j) - 1 \geqslant \chi_{m_3}(j-1)$ and $\chi_{m_3}(j+1) - 1 \geqslant \chi_{m_3}(j)$ that imply $\chi_{m_1}(j) = \chi_{m_3}(j) - 1 \leqslant \chi_{m_3}(j+1) - 2 = \chi_{m_1}(j+1) - 2$ which contradicts the fact that χ_{m_1} is an height sequence of the Motzkin word m_1. □

Let m and m' be two Motzkin words in $\mathcal{M}_n$. A *geodesic* between m and m' in the Tamari lattice of order n is a shortest path between them browsing through only some Motzkin words, *i.e.* lying in $\mathcal{M}_n$ only. Let $d(m, m')$ be the length of a geodesic between m and m'. Equivalently, $d(m, m')$ is the minimum of left- and right-rotations needed to transform m into m' in $(\mathcal{M}_n, \xrightarrow{*})$. Obviously, $d(m, m')$ is an upper bound of the classical rotation distance between m and m' in the Tamari lattice of order n.

Theorem 2. *Let m and m' be two Motzkin words in $\mathcal{M}_n$. Then, we have $d(m, m') = \sum_{i=0}^{n-1} (\chi_m(i) + \chi_{m'}(i) - 2\chi_{m \vee m'}(i))$.*

Proof. $(\mathcal{M}_n, \xrightarrow{*})$ is an upper semimodular join-semilattice, with a maximal element and graded by the rank function $r(m) = \lfloor \frac{(n-1)^2}{4} \rfloor - \sum_{i=0}^{n-1} \chi_m(i)$ (see Proposition 3, Theorem 1 and Proposition 5). Then, from [5,6,15] we have: $d(m, m') = 2r(m \vee m') - r(m) - r(m')$ and thus $d(m, m') = \sum_{i=0}^{n-1} (\chi_m(i) + \chi_{m'}(i) - 2\chi_{m \vee m'}(i))$. □

Remark 2. A consequence of Theorem 2 is that the classical rotation distance (in a Tamari lattice) between two Motzkin words m and m' is less than or equal to $d(m, m') = \sum_{i=0}^{n-1} (\chi_m(i) + \chi_{m'}(i) - 2\chi_{m \vee m'}(i))$ (see [4,7,17,19] for bounds of the rotation distance). Moreover, this bound can give the exact value of the classic rotation distance for some particular pairs of Motzkin words. Let us define $m = ((()(()(()\ldots(())\ldots))) = \alpha^n \beta^n$ and $m' = ((()(()\ldots(()())\ldots))) = (\alpha^{n-1}()\beta^n$ where $\alpha = (()$ and $\beta =)$. A simple calculation proves that $d(m, m') = n - 1$ which also is the classic rotation distance between m and m' in a Tamari lattice. For instance, if $n = 3$ then we have $m = ((()(()(()))))$, $m' = ((()(()()))))$, $\chi_m = 012210$, $\chi_{m'} = 001210$ and $d(m, m') = 2$; which is exactly the rotation distance in a Tamari lattice between m and m' (see [4]).

3. Some properties of $(\mathcal{M}_n, \xrightarrow{*})$

In this part, we present several enumerating results for some characteristic elements of the semilattice $(\mathcal{M}_n, \xrightarrow{*})$.

Proposition 6. *The generating function for the number of minimal elements in $(\mathcal{M}_n, \xrightarrow{*})$ is given by*

$$\frac{1 - \sqrt{1 - 4x^2 - 4x^3}}{2x^2}$$

(*see* A007477, [23]).

Proof. A minimal element in $(\mathcal{M}_n, \xrightarrow{*})$ is a Motzkin word m such that its associated path $\phi(m)$ satisfies the property that each horizontal step H is either followed by a down step D or ends the path. We distinguish three cases: (i) $\phi(m)$ is empty; (ii) $\phi(m) = H$; and (iii) $\phi(m) = U\phi(m_1)D\phi(m_2)$ where m_1 and m_2 are two minimal Motzkin words. Thus, the generating function for the number of minimal Motzkin paths is $A(x) = 1 + x + x^2A(x)^2$ which gives $A(x) = \frac{1-\sqrt{1-4x^2-4x^3}}{2x^2}$. □

Recall that $m \in \mathcal{M}_n$ is a join (resp. meet) irreducible element if $m = a \vee b$ (resp. $m = a \wedge b$) implies $m = a$ or $m = b$. Since the set $\mathcal{M}_n$ is finite, join (resp. meet) irreducible elements are elements that have a unique lower (resp. upper) cover.

Proposition 7. *The generating function for the number of meet-irreducible elements in* $(\mathcal{M}_n, \xrightarrow{*})$ *is given by* $\frac{x}{(1-x-x^2)(1-x)^2}$ (*see* A001924, [23]).

Proof. A meet-irreducible element in $(\mathcal{M}_n, \xrightarrow{*})$ is a Motzkin word $m \in \mathcal{M}_n$ such that its associated Motzkin path $\phi(m)$ contains only one occurrence of UH or UD. Let a_n (resp. b_n) be the cardinality of the set of meet-irreducible elements (resp. starting with U) in $\mathcal{M}_n$. Moreover, if $\phi(m)$ starts with U then it can be written $\phi(m) = U^kM$ where $1 \leqslant k \leqslant \lfloor \frac{n}{2} \rfloor$ and M is a word of length $n-k$ consisting of k down steps and $n-2k$ horizontal steps. Thus, we obtain $b_n = \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k}$ that is the sequence $A000071$ in [23]. Notice that this sequence are Fibonacci numbers minus 1. Finally, since $a_n = a_{n-1} + b_n$ we have a recurrence which gives precisely sequence A001924 in [23]. □

Proposition 8. *The generating function for the number of join-irreducible elements in* $(\mathcal{M}_n, \xrightarrow{*})$ *is given by*

$$\frac{1 - 2x^2 - \sqrt{1 - 4x^2 - 4x^3}}{2x\sqrt{1 - 4x^2 - 4x^3}}.$$

Proof. A join-irreducible element in $(\mathcal{M}_n, \xrightarrow{*})$ is a Motzkin word $m \in \mathcal{M}_n$ such that its associated Motzkin path $\phi(m)$ contains only one occurrence of HU or HH. Let $A(x)$ (resp. $B(x)$) be the generating function for the join-irreducible elements in $\mathcal{M}_n$ (resp. for the minimal elements in $\mathcal{M}_n$). An element $m \in \mathcal{M}_n$ is join-irreducible if and only if $\phi(m)$ can be written in one of the three following forms: (i) $U\phi(m')D\phi(m'')$ where m' is join-irreducible and m'' is a minimal element; (ii) $U\phi(m')D\phi(m'')$ where m' is a minimal element and m'' is join-irreducible; and (iii) $H\phi(m')$ where m' is a non-empty minimal element. We deduce $A(x) = 2x^2A(x)B(x) + x(B(x) - 1)$ and since $B(x) = \frac{1-\sqrt{1-4x^2-4x^3}}{2x^2}$, we obtain $A(x) = \frac{1-2x^2-\sqrt{1-4x^2-4x^3}}{2x\sqrt{1-4x^2-4x^3}}$. The first values of this sequence are $0, 0, 1, 1, 4, 7, 18, 39, 90, 206, 470, 1085, 2492, 5762$ for $1 \leqslant n \leqslant 14$. □

Proposition 9. *The generating function for the number of coverings in* $(\mathcal{M}_n, \xrightarrow{*})$ *is given by*

$$\frac{(1-x)(1-x-2x^2-\sqrt{1-2x-3x^2})}{2x(1-2x-x^2)}.$$

Proof. Let $C(x) = \sum_{n \geqslant 0} c_n x^n$ be the generating function for the number of coverings of $(\mathcal{M}_n, \xrightarrow{*})$. In order to enumerate the coverings $m \longrightarrow m'$ in $(\mathcal{M}_n, \xrightarrow{*})$, we count the possible elementary transformations $UH \longrightarrow HU$ and $UD \longrightarrow HH$ for Motzkin paths $M = \phi(m)$. So, we distinguish two cases: (i) $M = HM_1$ where M_1 is a Motzkin path; and (ii) $M = UM_1DM_2$ where M_1 and M_2 are two Motzkin paths.

For the case (i), the generating function for coverings whose lower path M has the form $M = HM_1$ is clearly $xC(x)$.

Now let us consider the case (ii) $M = UM_1DM_2$ where M_1 and M_2 are two Motzkin paths. A covering derived from M can be of three different forms:

- it can be derived from a covering of M_1 or M_2; thus there are $\sum_{i=0}^{n-2}(c_i + c_{n-2-i})$ possible coverings of this form, and the corresponding generating function is $2\frac{x^2C(x)}{1-x}$;
- it can be of the form $M = UHM_1'DM_2 \longrightarrow M' = HUM_1'DM_2$ where M_1' and M_2 are two Motzkin paths; thus the generating function is $x^3M(x)^2$ where $M(x) = \frac{1-x-\sqrt{1-2x-3x^2}}{2x^2}$ is the generating function for Motzkin numbers;
- it can be of the form $M = UDM_2 \longrightarrow M' = HHM_2$ where M_2 is a Motzkin path. Thus the generating function is $x^2M(x)$.

Putting this together, the generating function $C(x)$ satisfies the equation:

$$C(x) = xC(x) + x^3M(x)^2 + 2\frac{x^2C(x)}{1-x} + x^2M(x)$$

and we obtain

$$C(x) = \frac{(1-x)(1-x-2x^2-\sqrt{1-2x-3x^2})}{2x(1-2x-x^2)}.$$

The first values of this sequence are $0, 0, 1, 3, 9, 26, 73, 202, 553, 1504, 4073, 11\,003, 29\,689, 80\,094$ for $1 \leqslant n \leqslant 14$. □

Acknowledgements

We would like to thank the anonymous referees for their constructive remarks and suggestions which have greatly improved this paper.

References

[1] E. Barcucci, A. Bernini, L. Ferrari, M. Poneti, A distributive lattice structure connecting Dyck paths, noncrossing partitions and 312-avoiding permutations, Order 22 (4) (2005) 311–328.

[2] J.L. Baril, J.M. Pallo, The phagocyte lattice of Dyck words, Order 23 (2–3) (2006) 97–107.

[3] J.L. Baril, J.M. Pallo, The pruning-grafting lattice of binary trees, Theor. Comput. Sci. 409 (3) (2008) 382–393.

[4] J.L. Baril, J.M. Pallo, Efficient lower and upper bounds of the diagonal-flip distance between triangulations, Inf. Process. Lett. 100 (2006) 131–136.

[5] J.P. Barthélemy, B. Leclerc, B. Monjardet, On the use of ordered sets on problems of comparison and consensus of classifications, J. Classif. 3 (1986) 187–224.

[6] S.A. Boorman, D.C. Olivier, Metrics on spaces of finite trees, J. Math. Psychol. 10 (1973) 26–59.

[7] A. Bonnin, J.-M. Pallo, A shortest path metric on unlabeled binary trees, Pattern Recognit. Lett. 13 (1992) 411–415.

[8] R. Donaghey, L.W. Shapiro, Motzkin numbers, J. Comb. Theory, Ser. A 23 (1977) 291–301.

[9] L. Ferrari, E. Munarini, Lattices of paths: representation theory and valuations, J. Comb. 2 (2) (2011) 265–291.

[10] L. Ferrari, R. Pinzani, Lattices of lattice paths, J. Stat. Plan. Inference 135 (2005) 77–92.

[11] H. Friedman, D. Tamari, Problèmes d'associativité : une structure de treillis fini induite par une loi de demi-associative, J. Comb. Theory 2 (1967) 215–242.

[12] G. Grätzer, General Lattice Theory, second ed., Birkhäuser, 1998.

[13] S. Hanke, T. Ottmann, S.J. Schuierer, The edge-flipping distance of triangulations, J. Univers. Comput. Sci. 2 (1996) 570–579.

[14] S. Huang, D. Tamari, Problems of associativity: A simple proof for the lattice property of systems ordered by a semi-associative law, J. Combin. Theory, Ser. A 13 (1972) 7–13.

[15] B. Monjardet, Metrics on partially ordered sets – A survey, Discrete Math. 35 (1981) 173–184.

[16] F. Müller-Hoissen, J.M. Pallo, J. Stasheff (Eds.), Associahedra, Tamari Lattices and Related Structures, Prog. in Math., vol. 299, Birkhäuser, 2012.

[17] J.M. Pallo, On the rotation distance in the lattice of binary trees, Inf. Process. Lett. 25 (1987) 369–373.

[18] J.M. Pallo, An efficient upper bound of the rotation distance of binary trees, Inf. Process. Lett. 73 (2000) 87–92.

[19] J.M. Pallo, Right-arm rotation distance between binary trees, Inf. Process. Lett. 87 (2003) 173–177.

[20] R. Simion, D. Ullman, On the structure of lattice of noncrossing partitions, Discrete Math. 98 (1991) 193–206.

[21] R. Simion, Noncrossing partitions, Discrete Math. 217 (2000) 367–409.

[22] D.D. Sleator, R.E. Tarjan, W.P. Thurston, Rotation distance, triangulations and hyperbolic geometry, J. Am. Math. Soc. 1 (1988) 647–681.

[23] N.J.A. Sloane, The On-line Encyclopedia of Integer Sequences, available electronically at http://oeis.org.

[24] R.P. Stanley, Enumerative Combinatorics, vol. 2, Cambridge University Press, 1999.

[25] D. Tamari, The algebra of bracketings and their enumeration, Nieuw Arch. Wisk. 10 (1962) 131–146.

Journal of Combinatorics
Volume 0, Number 0, 1–8, 0000

Equivalence classes of permutations modulo excedances

Jean-Luc Baril, Toufik Mansour and Armen Petrossian

We introduce a new equivalence relation on permutations where two permutations are equivalent if and only if they coincide on their excedance sets. This paper studies equivalence classes for several subsets of permutations. Enumerating results are presented for permutations, cycles and permutations avoiding one or two patterns of length three. Also, an open question are proposed.

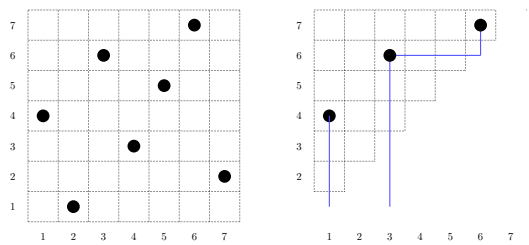
Keywords: permutations; equivalence class; excedance; pattern; Bell, Motzkin, Catalan, Fibonacci numbers.

1. Introduction and notations

Let S_n be the set of permutations of length n, *i.e.*, all one-to-one correspondences from $[n] = \{1, 2, \ldots, n\}$ into itself. The *one-line notation* of a permutation $\pi \in S_n$ is $\pi_1\pi_2\cdots\pi_n$ where $\pi_i = \pi(i)$ for $i \in [n]$. A *cycle* in S_n is a n-length permutation π such that there exist some indices $i_1, i_2, \ldots, i_n$ with $\pi(i_1) = i_2, \pi(i_2) = i_3, \ldots, \pi(i_{n-1}) = i_n$ and $\pi(i_n) = i_1$. According to the context, a cycle will be also denoted by its *cyclic notation* $\pi = \langle i_1, i_2, \ldots, i_n\rangle$. Let $C_n \subset S_n$ be the set of all cycles of length n.

Let π be a permutation in S_n. An *excedance* of π is a position i, $1 \leq i \leq n$, such that $\pi(i) > i$. The set of excedances of π will be denoted $E(\pi)$. For instance, if $\pi = 4\ 1\ 6\ 3\ 5\ 7\ 2$ then $E(\pi) = \{1, 3, 6\}$. The *graphical representation* of π is the set of points in the plane at coordinates (i, π_i) for $i \in [n]$. We define its associated rook placement $R(\pi)$ on the triangular board consisting of the cells located atop the diagonal $y = x$ in the graphical representation of the permutation. For short, $R(\pi)$ will be called *excedance board* of π. Obviously, a point (i, π_i) appears in the excedance board of π if and only if i is an excedance of π. See Figure 1 for the graphical representation of a permutation with its associated excedance board.

We consider the equivalence relation $\sim$ on S_n in which two permutations π and σ are considered to be equivalent if they coincide on their excedance sets, *i.e.*, $E(\pi) = E(\sigma)$ and $\pi(i) = \sigma(i)$ for $i \in E(\pi)$, or again π and σ have the same excedance board. Let $\mathrm{Eq}(\pi, S_n)$ denote the set of permutations being equivalent to π in S_n. For instance, we have $\mathrm{Eq}(43215, S_5) = \{43215, 43125\}$. The set (resp. the cardinality of the set) of distinct equivalence classes in S_n is denoted $S_n^{\sim}$ (resp. $\#S_n^{\sim}$). All these definitions will remain available by replacing the set S_n with a subset of S_n.



14/2/367/5

Figure 1: Graphical representation of $\sigma = 4\ 1\ 6\ 3\ 5\ 7\ 2$ with its excedance board and its associated partition.

The theory of permutation statistics has a long history and has grown at a rapid place in the last few decades. Two classical statistics have been received lot of attention, namely number of descents and number of excedances in a permutation (see [5] and references therein). Recently, Elizalde [6] studied number of permutations that avoid a fixed set T of patterns according to the number of fixed points and excedances. In this paper, we study the number of distinct equivalence classes in either $S_n^{\sim}$ or $S_n^{\sim}(T)$ with a fixed set T of patterns.

A permutation $\pi \in S_n$ *avoids* the pattern $\tau \in S_k$ if and only if there does not exist a sequence of indices $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ such that $\pi(i_1)\pi(i_2)\ldots\pi(i_k)$ is order-isomorphic to τ (see [13, 14]). We denote by $S_n(\tau)$ the set of permutations of S_n avoiding the pattern τ. For example, if $\tau = 123$ then $52143 \in S_5(\tau)$ while $21534 \notin S_5(\tau)$. Many classical sequences in combinatorics appear as the cardinality of pattern-avoiding permutation classes. A large number of these results were firstly obtained by West and Knuth [9, 12, 13, 14, 15, 16] (see books of Kitaev [8] and Mansour [11]).

In Section 2, we prove that equivalence classes for S_n are enumerated by the nth Bell number. We also prove that the number of classes in C_n is given by the sequence the sequence of the number of irreducible set partitions of $[n]$ (see A074664 in the on-line Encyclopedia of Integer Sequences [17]). In Section 3, we give enumerating results for permutations avoiding one pattern of length three (see Table 1). When the pattern is 132, the number of classes is given by the Motzkin numbers which provides a kind of discrete continuity between Catalan and Motzkin numbers. The case where permutations avoid the pattern 123 remains an open question. In Section 4, we investigate permutations avoiding two patterns of length three (see Table 3).

2. Equivalence classes for S_n and C_n

A *partition* of $[n]$ is any collection of non-empty pairwise disjoint subsets, called *blocks*, whose union is $[n]$. Let $\mathcal{P}_n$ be the set of partitions of $[n]$. A partition Π is said to be in *standard form* if it is written $\Pi = B_1/B_2/\cdots$, where the blocks B_i are arranged so that their smallest

Pattern	Sequence	Sloane	$a_n, n \geq 1$
$\{\}$	B_n	$A000110$	$1, 2, 5, 15, 52, 203, 877, 4140, 21147$
$\{132\}, \{213\}$	M_n	$A001006$	$1, 2, 4, 9, 21, 51, 127, 323, 835$
$\{123\}$	Open question	New	$1, 2, 4, 8, 19, 41, 98, 221, 526$
$\{312\}, \{321\}$	$\frac{1}{n+1}\binom{2n}{n}$	$A000108$	$1, 2, 5, 14, 42, 132, 429, 1430, 4862$
$\{231\}$		New	$1, 2, 4, 9, 21, 50, 121, 296, 729$

Table 1: Number of equivalence classes for permutations avoiding at most one pattern of $\mathcal{S}_3$, where B_n (resp. M_n) are the Bell numbers (resp. Motzkin numbers) defined by the generating function $\sum_{k\geq 0} \frac{x^k}{\prod_{j=1}^{k}(1-jx)}$ (resp. $\frac{1-x-\sqrt{1-2x-3x^2}}{2x^2}$).

elements are in increasing order. For convenience, we assume also that elements in a same block are arranged in increasing order. An *atomic partition* is a partition that does not have a proper subset of blocks with a union equal to a subset $[j]$ with $1 \leq j \leq n-1$.

From a permutation $\pi \in S_n$, we associate the partition $\Pi \in \mathcal{P}_n$ defined as follows. Two elements $x < y$ belong to the same block in Π if and only if there exist $k \geq 1$ and $i_1 = x < i_2 < \cdots < i_k = y$ such that $\pi_{i_j} = i_{j+1}$ for $1 \leq j \leq k-1$. For instance, the partition associated to $\pi = 4163572$ is $\Pi = 14/2/367/5$ (see Figure 1). Indeed, if $i_1 = 1$ then we have $i_1 = 1 < i_2 = \pi_{i_1} = 4$ and $\pi_{i_2} = 3 < 4$ which induces that 14 is a block; if $i_1 = 2$ then we have $\pi_{i_1} = 1 < 2$, so 2 is a block; if $i_1 = 3$ then we have $i_1 = 3 < i_2 = \pi_{i_1} = 6 < i_3 = \pi_{i_2} = 7$ and $\pi_{i_3} = 2 < 7$, so 367 forms a block, and so on. Notice that this definition appears as a counterpart for permutations of the partition associated to a rook placement on a *Ferrers board* (see Stanley [18], p. 75, or Mansour [10], p. 99).

2.1. For S_n - Bell numbers

Theorem 2.1. *The sets $S_n^{\sim}$, $n \geq 1$, are enumerated by the Bell numbers.*

Proof. Two permutations lie in a same equivalence class $\mathrm{Eq}(\pi, S_n)$ if and only if they have the same excedance board $R(\pi)$. Then they have the same associated partition. Moreover, the associated partitions of two non-equivalent permutations are different.

Conversely, let $\Pi = B_1/B_2/\cdots/B_k$ be a partition of $[n]$ in the standard form and such that each block B_i, $1 \leq i \leq k$ is arranged in increasing order. We define the permutation $\pi \in S_n$ such that $\pi_i = j$ if j is just after i into a same block (j is the smallest element greater than i in the block) and the image by π of the largest element of a block is the smallest element of this block. For instance, if $\Pi = 14/2/367/5$ we obtain the permutation $\pi = 4261573$. By construction, this process defines a permutation in S_n such that Π is its associated partition. So, there is a bijection between $S_n^{\sim}$ and the set $\mathcal{P}_n$ of partitions of $[n]$. Therefore, $S_n^{\sim}$ is enumerated by the n-th Bell number. $\square$

2.2. For C_n - A074664

Lemma 1. *The associated partition of a cyclic permutation $\pi \in C_n$ is atomic.*

Proof. For $\pi \in C_n$, let $\Pi = B_1/B_2/\cdots/B_k$ be its associated partition in standard form. For a contradiction, we assume that Π is not atomic, *i.e.*, there exists a proper subset of blocks $B_1, B_2, \ldots, B_\ell$, $\ell < k$ with a union equal to a subset $[j]$ for some $j \in [n-1]$. Then for any $i \in [j]$, we have $\pi_i \in [j]$. Indeed, if there was $i \leq j$ such that $\pi_i > j$, then i would be an excedance and π_i would be in a same block. This case does not occur since the hypothesis induces that the block containing i and π_i is included in $[j]$. Therefore, the image by π of the interval $[j]$, $j < n$, still remains $[j]$, which contradicts the fact that π is a cycle in C_n. □

Theorem 2.2. *The sets $C_n^{\sim}$, $n \geq 1$, are enumerated by the sequence A074664 in [17].*

Proof. The previous lemma proves that the associated partition of a cycle in C_n is necessarily atomic. Moreover, the associated partitions of two non-equivalent cycles are different. Now, we will prove that for any atomic partition $\Pi \in \mathcal{P}_n$, there exists a cycle $\pi \in S_n$ having Π as associated partition (see the definition on the previous page). Let $\Pi = B_1/B_2/\cdots/B_k$ be an atomic partition (not necessarily in standard form). Let us assume that $1 \in B_1$ and put $i_1 = 1$. For $1 \leq i \leq k$, we denote by $M(B_i)$ (resp. $m(B_i)$) the maximum (resp. minimum) element in the block B_i. In the case where $M(B_1) \neq n$, the atomicity of Π induces that there is a block B_{i_2} such that $1 < m(B_{i_2}) < M(B_{i_1}) < M(B_{i_2})$. We choose the block B_{i_2} having its smallest element the lowest possible. Iterating this process by replacing B_{i_1} with B_{i_2}, we exhibit a sequence of blocks $B_{i_1}, B_{i_2}, \ldots, B_{i_r}$ such that $m(B_{i_j}) < m(B_{i_{j+1}}) < M(B_{i_j}) < M(B_{i_{j+1}})$ for $1 \leq j \leq r-1$ and such that B_{i_r} contains the value n. Let π be the cycle in C_n defined as follows: $\pi = \langle B_{i_1} B_{i_2} \cdots B_{i_r} A_1 A_2 \cdots A_s \rangle$ where each block B_{i_j}, $1 \leq j \leq r$ has its elements in increasing order, and the blocks A_i, $1 \leq i \leq s$, are the remaining blocks of the partition Π ordered by their decreasing minima and each of them ordered in increasing order. By construction π is a cycle in C_n, and it is straightforward to see that Π is its associated partition. Using Lemma 1, we have exhibited a constructive bijection between $C_n^{\sim}$, $n \geq 1$, and the subset of atomic partitions in $\mathcal{P}_n$ which is enumerated by the sequence A074664 in [17] (see Bergeron and Zabrocki [4]). □

A permutation $\pi = \pi_1\pi_2\cdots\pi_n \in S_n$ is called *indecomposable* if there does not exist $i < n$ such that $\pi_1\pi_2\cdots\pi_i$ is a permutation of $[i]$. Let $\mathrm{I}nd_n$, $n \geq 1$, be the sets of indecomposable permutations of length n. Lemma 1 remains valid if we replace cyclic permutation with indecomposable permutation. Moreover, since a cyclic permutation is necessarily indecomposable, the proof of the following theorem is obtained, *mutatis mutandis*, from the proof of Theorem 2.2.

Theorem 2.3. *The sets $Ind_n^{\sim}$, $n \geq 1$, are enumerated by the sequence A074664 in [17].*

3. Equivalence classes for $S_n(\alpha)$ with $\alpha \in S_3$

In this section, we study enumerating results about the equivalence classes of sets of permutations avoiding one pattern of length three.

For the pattern 321, the excedances of $\pi \in S_n(321)$ determine π completely which means that every permutation $\pi \in S_n(321)$ is the unique element in its equivalence class. Therefore, the cardinality of the set $S_n^{\sim}(321)$ is given the n-th Catalan number (see sequence A000108 in [17]).

For the pattern 312, we use the Simion-Schmidt bijection modulo symmetry from $S_n(321)$ to $S_n(312)$ (see [13]). Since this bijection preserves the position of points (i, π_i) whenever $\pi_i > i$, we directly conclude that the cardinality of the set $S_n^{\sim}(312)$ is given by the cardinality of $S_n(312)$, that also is the n-th Catalan number. We have not succeeded to find the cardinality of $S_n^{\sim}(123)$ for any $n \geq 1$ (first values are $1, 2, 4, 8, 19, 41, 98, 221, 526$). So, we leave open this problem.

In order to study other patterns, we define some other equivalence relations between two permutations by extending the definition of an excedance. Let π be a permutation in S_n. A *k-excedance* of π, $1 - n \leq k \leq n$, is a position i, $1 \leq i \leq n$, such that $\pi(i) > i - k$. The set of k-excedances of π will be denoted $E^k(\pi)$. For instance, if $\pi = 312$ then $E^{-2}(\pi) = \emptyset$, $E^{-1}(\pi) = E^0(\pi) = E^1(\pi) = \{1\}$ and $E^2(\pi) = [3]$. Obviously, for any $\pi \in S_n$, we have $E^0(\pi) = E(\pi)$, $E^{1-n}(\pi) = \emptyset$ and $E^n(\pi) = [n]$. For any k, $1 - n \leq k \leq n$, we define the *k-equivalence relation* $\sim^k$ on S_n in which two permutations π and σ are equivalent if and only if they coincide on their k-excedance sets, *i.e.*, $\pi(i) = \sigma(i)$ for $i \in E^k(\pi) = E^k(\sigma)$. Clearly, when $k = 0$ we retrieve the equivalence relation defined in the introduction. Let $\mathrm{Eq}^k(\pi, S_n)$ denote the set of permutations in S_n k-equivalent to π; we call it the *k-class* of π. For instance, we have $\mathrm{Eq}^0(43215, S_5) = \mathrm{Eq}(43215, S_5) = \{43215, 43125\}$, $\mathrm{Eq}^2(43215, S_5) = \{43215\}$ and $\mathrm{Eq}^{-4}(43215, S_5) = S_5$. The set of distinct k-classes in S_n is denoted $S_n^{\sim^k}$.

3.1. The sets $S_n^{\sim}(132)$ and $S_n^{\sim}(213)$- Motzkin

We focus our study on the pattern 132 and the symmetry *reverse* $\rightarrow$ *complement* $\rightarrow$ *inverse* will provide the result for 213.

Theorem 3.1. *The sets $S_n^{\sim}(132)$, $n \geq 1$, are enumerated by the Motzkin numbers.*

Proof. For $k \geq 0$, let M_n^k be the number of k-classes in $S_n(132)$. Let π be a permutation in $S_n(132)$. Since π avoids the pattern 132, π can be written $\pi = \sigma\gamma i$, $1 \leq i \leq n$, where γi is a subsequence of $[i]$ avoiding the pattern 132 and σ is obtained from a permutation $\sigma' \in S_{n-i}(132)$ by adding i on all its entries, *i.e.*, $\sigma(j) = i + \sigma'(j)$ for $1 \leq j \leq n - 1 - i$.

We distinguish two cases: (1) $i \geq n - k$, and (2) $i \leq n - k - 1$.

Case 1. In this case, n is either a k-excedance (*i.e.*, $\pi_n = i \geq n - k + 1$) or $\pi_n = i = n - k$. The decomposition of $\pi = \sigma\gamma i$ is illustrated by the left part of Figure 2. The blue (resp. green) square corresponds to σ (resp. γ), and the filled gray area does not contain any point (j, π_j).

It is straightforward and crucial to see that for a given i, $i \geq n-k$, the above decomposition of π induces that:

an integer $j > n-i$ is a k-excedance of π if and only if $j-n+i$ is a $(k+i-n)$-excedance of γi where γi is considered as a permutation in $S_i(132)$.

Using the decomposition of $\pi = \sigma\gamma i$, the number of k-classes having a representative ending by $i \geq n-k$ is equal to the product between the number of $(k+i-n)$-classes of $S_{i-1}(132)$ and the number of subsequences σ of length $n-i$ avoiding 132, that is $M_{i-1}^{k+i-n} \cdot c_{n-i}$ where $c_{n-i} = \frac{1}{n-i+1}\binom{2(n-i)}{n-i}$ is the $(n-i)$th Catalan number. Varying i from $n-k$ to n, the number of k-classes having a representative with a last element greater or equal than $n-k$ is exactly $M_{n-1}^{k}c_0 + M_{n-2}^{k-1}c_1 + \ldots + M_{n-k-1}^{0}c_k$.

Now, let us examine the second case.

Case 2. We assume $i \leq n-k-1$ and we will prove that each k-class contains a permutation having its last element equal to one. Let π be a permutation such that $\pi_n = i \leq n-k-1$ and let us assume that $\pi_n = i \neq 1$. Using the decomposition $\pi = \sigma\gamma i$, the value 1 appears in γ, *i.e.*, $\pi_j = 1$ implies $j \geq n-i+1$. Since π avoids 132, all values after $\pi_j = 1$ appear in increasing order in the one-line representation of π and are less or equal than i. This implies that for $\ell \geq j$, π_ℓ are less or equal than $i-(n-\ell)$.

Now, let us define the permutation $\pi' \in S_n(132)$ such that $\pi'_n = 1$, $\pi'_\ell = \pi_\ell$ if $\ell \in [j-1]$, and $\pi'_\ell = \pi_{\ell+1}$ for $\ell \in [j, n-1]$. By construction, $\pi' \in S_n(132)$ and π belongs to the same k-class. Thus, the number of k-classes (in this case) is equal to the number of $(k+1)$-classes in $S_{n-1}(132)$ (we omit the last value 1 of π'), which is given by M_{n-1}^{k+1} (see the right part of Figure 2).

In accordance with the two previous cases, we obtain the following recursive formula for $k \geq 0$, $n \geq 1$ and $n > k$:

$$M_n^k = M_{n-1}^{k+1} + M_{n-1}^{k}c_0 + M_{n-2}^{k-1}c_1 + \ldots + M_{n-k-1}^{0}c_k$$

anchored with $M_n^n = c_n$ for $n \geq 0$, where c_n is the nth Catalan number (see A000108 in [17]).

Setting $D^n(x) = \sum_{k\geq 0} M_{k+n}^k x^k$ for $n \geq 1$, the bivariate generating function $F(x,u)$ where the coefficient of $x^k u^n$ is the number of k-classes in $S_{n+k}(132)$ satisfies $F(x,u) = \sum_{n\geq 1} D^n(x)u^n$. A simple calculation from the previous recurrence relation provides the functional equation for $n \geq 3$:

$$D^n(x) = C(x)D^{n-1}(x) + \frac{D^{n-2}(x) - D^{n-2}(0)}{x}$$

with $D^1(x) = \frac{C(x)-1}{x}$ and $D^2(x) = \frac{D^1(x)-1}{x}$ where $C(x) = \frac{1-\sqrt{1-4x}}{2x}$ is the generating function for the Catalan numbers.

With this last relation, we obtain:

$$F(x,u)(x - C(x)xu - u^2) = D^1(x)xu + D^2(x)xu^2 - C(x)xD^1(x)u^2 - u^2F(0,u).$$

Using the kernel method (see [1, 9]), we compute

$$x = \frac{u}{2}(1 + u - 2u^2 - \sqrt{1 - 2u - 3u^2}).$$

This induces $F(0, u) = \frac{1-u-\sqrt{1-2u-3u^2}}{2u^2}$ which is the generating function for the Motzkin numbers. Therefore, M_n^0 is the nth term of the Motzkin sequence.

Finally, a simple calculation provides

$$F(x, u) = \frac{u - u\sqrt{1-4x} - ux + u^2 - u^2\sqrt{1-4x} - x + x\sqrt{1-2u-3u^2}}{x(2x - u + u\sqrt{1-4x} - 2u^2)}.$$

□

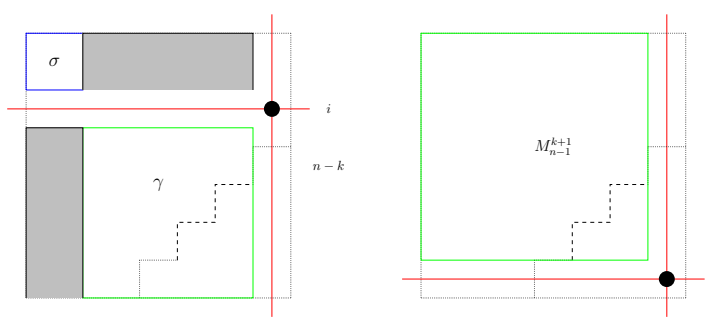


Figure 2: Illustration of the two cases studied in the proof of Theorem 4.

The bivariate generating function obtained in the proof of Theorem 3.1 provides a kind of *discrete continuity* between the well-known Catalan and Motzkin sequences (see Table 2). See [3] for another discrete continuity between these sequences and [2] for a discrete continuity between Fibonacci and Catalan sequences. Using Maple, the first terms of the Taylor expansion of $F(x, u)$ are $u + 2u^2 + 4u^3 + 9u^4 + 21u^5 + 51u^6 + 127u^7 + 2xu + 5xu^2 + 12xu^3 + 30xu^4 + 76xu^5 + 196xu^6 + 5x^2u + 14x^2u^2 + 37x^2u^3 + \ldots$.

Table 2: Number M_n^k of k-classes in $S_n(132)$ for some n and k, $n > k \geq 0$.

$k \setminus n$	1	2	3	4	5	6	7
0	1	2	4	9	21	51	127
1		2	5	12	30	76	196
2			5	14	37	99	265
3				14	42	118	331
4					42	132	387
5						132	429

3.2. The set $S_n^{\sim}(231)$

Theorem 3.2. *The generating function for the number of classes in $S_n(231)$, $n \geq 1$, is given by the continued fraction:*

$$\cfrac{1}{1+[a_1]-\cfrac{x}{1+[a_2]-\cfrac{x}{1+[a_3]-\cfrac{x}{1+[a_4]+\cdots}}}}$$

where $[a_\ell] = \frac{x^2(1-x^{\ell-1})}{1-x}$ for $\ell \geq 1$.

Proof. For $k \leq 0$, let M_n^k be the number of k-classes in $S_n(231)$. Recall that two permutations σ and π lie in a same k-class if and only if they coincide on positions i satisfying $\pi_i > i-k$ or $\sigma_i > i-k$. Let π be a permutation in $S_n(231)$ and denote by i, $1 \leq i \leq n$, the index such that $\pi_i = n$. Since π avoids the pattern 231, π can be written $\pi = \sigma n \gamma$, where σ is a permutation of $[i-1]$ avoiding the pattern 231 and γ is obtained from a permutation $\gamma' \in S_{n-i}(231)$ by adding $i-1$ on all its entries, *i.e.*, $\gamma(j) = i-1+\gamma'(j)$ for $i+1 \leq j \leq n$.

We distinguish two cases: (1) $i \leq n-1+k$, and (2) $i > n-1+k$.

Case 1. In this case, i is a k-excedance since $\pi_i = n > i-k+1 > i-k$. Notice that for a given i, $i \leq n-1+k$, the above decomposition of π induces that:

an integer $j \neq i$ is a k-excedance of π if and only either j is a k-excedance of σ, or $j-i+1$ is a $(k-1)$-excedance of γ' considered as a permutation in $S_{n-i}(231)$.

Thus, the number of k-classes having a representative with n in position i is equal to the product $M_{i-1}^k \cdot M_{n-i}^{k-1}$. Varying i from 1 to $n-1+k$, the number of k-classes in this case is exactly $\sum_{i=0}^{n-2+k} M_{n-1-i}^{k-1} M_i^k$.

Now, let us examine the second case.

Case 2. We have $i > n-1+k$ and i is not a k-excedance. We will prove that the k-class of π contains a permutation having its last element equal to n.

Let us consider the permutation π' defined by $\pi' = \sigma\gamma n$. It is straightforward to check that π' avoids 231 and that π' belongs to the same k-class of π. Thus, the number of k-classes (in this case) is equal to the number of k-classes having a representative ending with n, which is given by M^k_{n-1}. In accordance with the two previous cases, we obtain the following recursive formula for $k \leq 0$ and $n \geq 1$:

$$\begin{cases} M^k_n = M^k_{n-1} + \sum\limits_{\ell=0}^{n-2+k} M^k_\ell M^{k-1}_{n-1-\ell} & \text{for } n-1 > -k, \\ M^k_n = 1 & \text{for } n-1 \leq -k. \end{cases}$$

Let $D^k(x) = \sum_{n\geq 0} M^k_n x^n$ be the generating function for the number of k-classes in $S_n(231)$. Some elementary calculations from the previous recurrence relation provide the following functional equation for $k \leq 0$:

$$D^k(x)\left(1 + x(\frac{1-x^{-k}}{1-x} - 1) - xD^{k-1}(x)\right) = 1.$$

Finally, we obtain $D^k(x)$ expressed as a continued fraction

$$\cfrac{1}{1+[a_{-k+1}] - \cfrac{x}{1+[a_{-k+2}] - \cfrac{x}{1+[a_{-k+3}] - \cfrac{x}{1+[a_{-k+4}]+\cdots}}}}$$

where $[a_\ell] = \frac{x^2(1-x^{\ell-1})}{1-x}$ for $\ell \geq 1$. When $k = 0$, we obtain the expected result. □

By the equation of $D^k(x)$ (see line 4), wee see that

$$\lim_{k\to-\infty} D^k(x)\left(1 - x + x\frac{1-x^{-k}}{1-x} - xD^{k-1}(x)\right) = 1.$$

Let $D(x) = \lim_{k\to-\infty} D^k(x)$ and assume that $|x| < 1$. Then $D(x)\left(1 - x + \frac{x}{1-x} - xD(x)\right) = 1$, which implies that either $D(x) = 1/(1-x)$ or $D(x) = (1-x)/x$. Since $D^k(0) = 1$ for all $k < 0$, we have that $D(0) = 1$. Hence, $D(x) = \lim_{k\to-\infty} D^k(x) = \frac{1}{1-x}$

Finally, the first terms of the Taylor expansion of the continued fraction $D^0(x)$ can be obtained by replacing the rest of the continued fraction by its limit $\frac{1}{1-x}$. More precisely, if we

need the first five terms then we expand the rational fraction

$$\cfrac{1}{1+[a_1]-\cfrac{x}{1+[a_2]-\cfrac{x}{1+[a_3]-\cfrac{x}{1+[a_4]+\frac{x}{1-x}}}}}$$

which provides $1+x+2x^2+4x^3+9x^4+\mathcal{O}(x^5)$.

4. Equivalence classes for $S_n(\alpha,\beta)$ with α and β in S_3

In this part, we give enumerating results for classes of permutations avoiding two patterns of length three. Below, we present proofs for the most interesting cases which appear in the last three rows of Table 3, *i.e.*, for the cases where the equivalence classes are not all reduced to a single permutation. All other cases can be easily proved using similar arguments or by applying classical symmetries on permutations (inverse, reverse, and complement).

4.1. The set $S_n^{\sim}(123,231)$ - A096777

Theorem 4.1. *The number of classes in $S_n^{\sim}(123,231)$, $n\geq 1$, is given by*

$$1+\frac{1}{2}\cdot\lceil\frac{n-2}{3}\rceil\cdot(1-2n+3\lceil\frac{n-2}{3}\rceil)+\lfloor\frac{n}{2}\rfloor\cdot(2n-1-2\lfloor\frac{n}{2}\rfloor)$$

(see sequence A096777 in [17]).

Proof. Let π be a permutation in $S_n(123,231)$. For $k\in[n+1]$, we define the subsequence $\sigma(k)=n(n-1)\dots(n-k+1)$ if $k\leq n$, and $\sigma(k)$ is empty if $k=n+1$. Thus, a permutation $\pi\in S_n(123,231)$ has a unique decomposition of the form $\pi=\sigma(k)\gamma(\ell)\delta$ where $\delta=(n-k)\cdots(\ell+1)$ and $\gamma(\ell)=\ell(\ell-1)\cdots 21$ with $k\in[n+1]$ and $\ell\in[n-k-1]$ (see Figure 3 for an illustration of this decomposition).

Let a_n (respectively b_n^k) be the number of classes in $S_n(123,231)$ (respectively starting with $\sigma(k)$ for $k\leq\lfloor\frac{n}{2}\rfloor$). The above decomposition induces that

$$a_n=1+\sum_{k=1}^{\lfloor\frac{n}{2}\rfloor-1}b_n^k+(n-1)$$

for $n\geq 1$. Indeed, if $k\geq\lfloor\frac{n}{2}\rfloor$ then $\sigma(k)$ fills the semi-diagonal $y=n-x+1$ in the excedance board, and π and $n(n-1)\cdots 21$ belong to the same class. There is only one class whenever

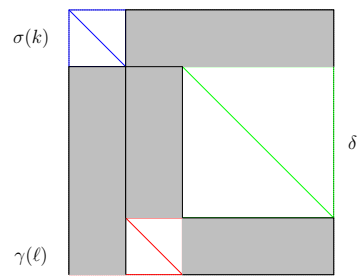


Figure 3: Illustration of the decomposition of a permutation in $S_n(123, 231)$.

$k \geq \lfloor \frac{n}{2} \rfloor$. The second term $\sum_{k=1}^{\lfloor \frac{n}{2} \rfloor - 1} b_n^k$ corresponds to permutations having a non-empty $\sigma(k)$ that does not fill the semi-diagonal, *i.e.*, $1 \leq k \leq \lfloor \frac{n}{2} \rfloor - 1$; and the last term $n-1$ enumerates classes of permutations having an empty $\sigma(k)$.

Now, we will calculate b_n^k.

For convenience, we will say that γ (resp. δ) intersects the excedance board of π if the associated block of γ (resp. δ) illustrated in red (resp. green) in Figure 3 has a non-empty intersection with the excedance board of π.

Using Figure 3, a simple observation provides that γ and δ does not intersect the excedance board if and only if $\ell \leq k+1$ and $\ell + n - \ell - k \leq \ell + k + 1$ which is equivalent to $n - 2k - 1 \leq \ell \leq k + 1$. This case can occur whenever $k + 1 \geq n - 2k - 1$, that is $k \geq \lceil \frac{n-2}{3} \rceil$.

We distinguish two cases.

Case 1. If $k \in [1, \lceil \frac{n-2}{3} \rceil - 1]$ then there is at least one block issue from γ or δ that intersects the excedance board of π. This means that γ and δ are entirely determined by ℓ that describes the interval $[n-k-1]$. Therefore, in this case $b_n^k = n - k - 1$ and this case contains exactly $\sum_{k=1}^{\lceil \frac{n-2}{3} \rceil - 1} (n - k - 1)$ classes.

Case 2. If $k \geq \lceil \frac{n-2}{3} \rceil$ then there exists ℓ, $n - 2k - 1 \leq \ell \leq k + 1$, such that the two blocks associated to γ and δ do not intersect the excedance board of π. Moreover, all permutations obtained by varying ℓ from $n - 2k - 1$ to $k + 1$ lie in a unique class for a given k. Now, let us assume that either $\ell \in [n - 2k - 2]$ or $\ell \geq k + 2$. If $\ell \in [n - 2k - 2]$ then the block associated to γ intersects the excedance board of π that also determines δ. The number of classes in this subcase is exactly $n - 2k - 2$ for a given k, $k \geq \lceil \frac{n-2}{3} \rceil$. If $\ell \geq k + 2$ then the block associated to δ intersects the excedance board of π that also determines γ. The number of classes in this subcase is exactly $n - k - 1 - (k + 2) + 1 = n - 2k - 2$. So, we have $b_n^k = 2(n - 2k - 2) + 1$ and the total number of classes satisfying Case 2 is exactly $\sum_{k=\lceil \frac{n-2}{3} \rceil}^{\lfloor \frac{n}{2} \rfloor - 1} (2(n - 2k - 2) + 1)$.

In accordance with these two cases, we finally obtain

$$a_n = 1 + n - 1 + \sum_{k=1}^{\lceil \frac{n-2}{3} \rceil - 1} (n-k-1) + \sum_{k=\lceil \frac{n-2}{3} \rceil}^{\lfloor \frac{n}{2} \rfloor - 1} (2(n-2k-2)+1).$$

With a simple calculation on this formula, we obtain the expected result and the sequence A096777 in [17]. □

4.2. The set $S_n^{\sim}(132, 231)$ - Fibonacci

Theorem 4.2. *The sets $S_n^{\sim}(132, 231)$, $n \geq 1$, are enumerated by the Fibonacci sequence (the n-th Fibonacci number F_n is defined by $F_n = F_{n-1} + F_{n-2}$ with $F_0 = F_1 = 1$, see sequence A000045 in [17]).*

Proof. For $n \geq 1$, we denote by a_n the cardinality of $S_n^{\sim}(132, 231)$. Let π be a permutation avoiding 132 and 231. Thus, π can be written either (1) $\pi = \pi_1 \cdots \pi_{n-1} n$ or (2) $\pi = n\pi_2 \cdots \pi_n$. The number of classes in $S_n(132, 231)$ satisfying case (1) is also the number a_{n-1} (we omit the last value n). Now, let us study the second case. There exists $j \geq 2$ such that $\pi_j = 1$ and $\pi_1 = n > \pi_2 > \cdots > \pi_j = 1 < \pi_{j+1} < \cdots \pi_n$. It is straightforward to see that $\pi_i \leq i - 1$ for $i \geq j$. So, we set $\pi' = (\pi_2 - 1) \cdots (\pi_{j-1} - 1)(\pi_{j+1} - 1) \cdots (\pi_n - 1)$. Obviously, $\pi' \in S_{n-2}(132, 231)$ and there is a one-to-one correspondence between classes of $S_n(132, 231)$ having n in first position, and classes of $S_{n-2}(132, 231)$. Finally the number of classes in $S_n(132, 231)$ (in this case) is equal to a_{n-2} (we omit n and 1 in π). According to the two previous cases, we have $a_n = a_{n-1} + a_{n-2}$ anchored with $a_1 = 1$ and $a_2 = 2$ which defines the Fibonacci sequence. □

4.3. The set $S_n^{\sim}(123, 132)$ - A052955

Theorem 4.3. *The sets $S_n^{\sim}(123, 132)$, $n \geq 1$, are enumerated by $2^{\frac{n+1}{2}} - 1$ if n odd, and by $3 \cdot 2^{\frac{n}{2}-1} - 1$ otherwise (see sequence A052955 in [17]).*

Proof. Let π be a permutation avoiding 123 and 132. Let $i_1 < i_2 < \cdots < i_k$ be the indices of the right-to-left maxima of π, *i.e.*, the indices i_j, $j \in [k]$, such that $\pi(i_j) > \pi(\ell)$ for $\ell > i_j$. Then π can be obtained by gluing from left to right the blocks $(\pi(i_j)-1)(\pi(i_j)-2) \cdots (\pi(i_{j+1})+1)\pi(i_j)$ for $1 \leq j \leq k$ by setting $\pi(i_1) = n$ and for convenience $i_{k+1} = n + 1$ and $\pi(i_{k+1}) = 0$.

We distinguish two cases: (1) $i_1 > \lfloor \frac{n}{2} \rfloor$, and (2) $i_1 \leq \lfloor \frac{n}{2} \rfloor$.

Case 1. We have $i_1 > \lfloor \frac{n}{2} \rfloor$. So, π can be written $\pi = (n-1)(n-2) \cdots (\pi(i_2)+1)n\gamma$ where γ has all its values less or equal than $\pi(i_2)$ and so that these values do not lie in the excedance board of π. We set $\pi' = (n-1)(n-2) \cdots (\pi(i_2)+1)n\pi(i_2)(\pi(i_2)-1) \cdots 3\ 2\ 1$. Obviously, π' avoids 123 and 132, and π' and π belong to the same class. Varying i_1 from $\lfloor \frac{n}{2} \rfloor$ to n, the number of classes (in this case) is exactly $\sum_{k=\lfloor \frac{n}{2} \rfloor + 1}^{n} 1 = n - \lfloor \frac{n}{2} \rfloor$.

Case 2. We have $i_1 \leq \lfloor \frac{n}{2} \rfloor$. We will prove that there is a permutation π' in the class of π such that $\pi' = (n-1)(n-2)\cdots(\pi(i_2)+1)n\delta i_1(i_1-1)\cdots 21$.

Let r be the smallest index j such that $\pi(i_j) \leq i_j$. Less formally, the point $(i_r, \pi(i_r))$ is the highest point that does not lie in the excedance board of π.

Since $i_1 \leq \lfloor \frac{n}{2} \rfloor$ and $\pi(i_1) = n$, we necessarily have $r \geq 2$. The permutation π can be written $\pi = \gamma\ (\pi(i_r)-1)\cdots(\pi(i_{r+1})+1)\pi(i_r)\sigma$ where all values in γ appear in the excedance board of π, and all values in σ do not lie in the excedance board of π. So, there exists an index $t = \pi_{i_r}$, $i_{r-1} < t \leq i_r$, such that $\pi(t) = t = \pi_{i_r}$.

Now let us consider the permutation π' obtained from π by replacing the block $(\pi(i_r) - 1)\cdots(\pi(i_{r+1})+1)\pi(i_r)$ with the block

$$(\pi(i_{r-1}+1))\cdots\pi(t-1)\pi(i_r)\pi(t)\cdots(\pi(i_{r+1})+1).$$

Moreover, we replace σ by the decreasing sequence $(\pi(i_r)-1)\cdots 21$. It is straightforward to see that π and π' lie in the same class. Moreover, π' is in the expected form $\pi' = (n-1)(n-2)\cdots(\pi(i_2)+1)n\delta i_1(i_1-1)\cdots 21$ where δ is a subsequence of length $n-2i_1$ that avoids 123 and 132.

Thus, the number of classes in $S_n(123, 132)$ such that the point (i_1, n) lies in the excedance board is exactly the number a_{n-2i_1} of classes in $S_{n-2i_1}(123, 132)$. Varying i_1 from 1 to $\lfloor \frac{n}{2} \rfloor$, the number b_n of classes (in this case) is given by $b_n = \sum_{i_1=1}^{\lfloor \frac{n}{2} \rfloor} a_{n-2i_1}$ with $a_1 = 1$ and $a_2 = 2$.

Considering the two cases, we obtain

$$a_n = n - \lfloor \frac{n}{2} \rfloor + \sum_{i_1=1}^{\lfloor \frac{n}{2} \rfloor} a_{n-2i_1}$$

and finally a simple calculation provides $a_n = 2a_{n-2} + 1$ with $a_1 = 1$ and $a_2 = 2$ which is the sequence A052955 in [17]. □

5. Acknowledgement

I would like to thank the anonymous referees for their very careful reading of this paper and their helpful comments and suggestions.

References

[1] C. Banderier, M. Bousquet-Mélou, A. Denise, P. Flajolet, D. Gardy, and D. Gouyou-Beauchamps. Generating functions of generating trees. *Discrete Mathematics*, 246(1-3), 29-55, 2002.

Pattern	Sequence	Sloane	$a_n, n \geq 1$
$\{123, 321\}$			$1, 2, 4, 4, 0, 0, 0, \ldots$
$\{132, 312\}, \{213, 312\}, \{231, 312\}, \{231, 321\}, \{321, 312\}$	2^{n-1}	$A000079$	$1, 2, 4, 8, 16, 32, 64, 128, 256$
$\{123, 312\}, \{132, 321\}, \{213, 321\}$	$(n-1)n/2+1$	$A000124$	$1, 2, 4, 7, 11, 16, 22, 29, 37$
$\{132, 231\}, \{213, 231\}$	F_n	$A000045$	$1, 2, 3, 5, 8, 13, 21, 34, 55$
$\{123, 231\}$	$\lfloor (n-2)^2/3 \rfloor + n$	$A096777$	$1, 2, 3, 5, 8, 11, 15, 20, 25$
$\{123, 132\}, \{123, 213\}, \{132, 213\}$	$(5+(-1)^n)2^{\lfloor (n-1)/2 \rfloor - 1} - 1$	$A052955$	$1, 2, 3, 5, 7, 11, 15, 23, 31$

Table 3: Number of equivalence classes for permutations avoiding two patterns in $\mathcal{S}_3$, where F_n is the n-th Fibonacci number defined as $F_n = F_{n-1} + F_{n-2}$ with $F_0 = F_1 = 1$.

[2] E. Barcucci, A. Bernini, M. Poneti. From Fibonacci to Catalan permutations. *Pure Mathematics And Applications*, 17, 1-17, 2006.

[3] E. Barcucci, A. Del Lungo, E. Pergola and R. Pinzani. From Motzkin to Catalan permutations. *Discrete Mathematics*, 217, 33-49, 2000.

[4] N. Bergeron and M. Zabrocki. The Hopf algebras of symmetric functions and quasisymmetric functions in non-commutative variables are free and cofree. *J. Algebra Appl.*, 08(4), 581-600, 2009.

[5] R. Ehrenborg and E. Steingrimsson. The Excedance set of a permutation. *Advances in Applied Mathematics*, 24, 284-299, 2000.

[6] S. Elizalde. Statistics on Pattern-avoiding Permutations. Ph.D thesis, M.I.T. 2004, https://www.math.dartmouth.edu/ sergi/papers/thesis.pdf.

[7] S. Elizalde and E. Deutsch. A simple and unusual bijection for Dyck paths and its consequences. *Ann. Comb.*, 7(3), 281-297, 2003.

[8] S. Kitaev. Patterns in permutations and words, Springer-Verlag, 2011.

[9] D.E. Knuth. The Art of Computer Programming. Vol. 1, Addison-Wesley, Reading MA, 1973, Third edition, 1997.

[10] T. Mansour. Combinatorics of Set Partitions. *Discrete Mathematics and its Applications*, Chapman & Hall/CRC, 2013.

[11] T. Mansour. Permutations avoiding a pattern from S_k and at least two patterns from S_3 *Ars. Combin.*, 62, 227-239, 2002.

[12] T. Mansour and A. Vainshtein Restricted 132-avoiding permutations *Adv. Appl. Math.*, 26, 258-269, 2001.

[13] R. Simion and F.W. Schmidt. Restricted permutations. *European Journal of Combinatorics*, 6, 383406, 1985.

[14] J. West. Permutations with forbidden subsequences and stack-sortable permutations. Ph.D. Thesis, MIT, 1990.

[15] J. West. Generating trees and forbidden subsequences. *Proceeding of sixth FPSAC*, 441-450, 1994.

[16] J. West. Generating trees and the Catalan and Schröder numbers. *Discrete Math.*, 146, 247-262, 1995.

[17] N.J.A. Sloane. On-Line Encyclopedia of Integer Sequences. Published electronically at http://oeis.org/.

[18] R. Stanley. Enumerative Combinatorics. *Cambridge University Press*, Vol. 1, 1997.

Jean-Luc Baril

LE2I UMR-CNRS 6306, Université de Bourgogne, B.P. 47 870, 21078 Dijon, France

E-mail address: barjl@u-bourgogne.fr

Toufik Mansour

Department of Mathematics, University of Haifa, 3498838 Haifa, Israel

E-mail address: tmansour@univ.haifa.ac.il

Armen Petrossian

LE2I UMR-CNRS 6306, Université de Bourgogne, B.P. 47 870, 21078 Dijon, France

E-mail address: armen.petrossian@u-bourgogne.fr

Received January 1, 0000

Discrete Mathematics 338 (2015) 655–660

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Equivalence classes of Dyck paths modulo some statistics

Jean-Luc Baril *, Armen Petrossian

LE2I UMR-CNRS 6306, Université de Bourgogne, B.P. 47 870, 21078 DIJON-Cedex, France

ARTICLE INFO

Article history:
Received 25 July 2014
Received in revised form 3 December 2014
Accepted 4 December 2014
Available online 29 December 2014

Keywords:
Dyck path
Equivalence relation
Statistics
Rise
Peak
Valley
Catalan
Motzkin and Fibonacci sequences

ABSTRACT

We investigate new equivalence relations on the set $\mathcal{D}$ of Dyck paths relatively to the three statistics of double rises, peaks and valleys. Two Dyck paths are r-equivalent (respectively p-equivalent and v-equivalent) whenever the positions of their double rises (respectively peaks and valleys) are the same. Then, we provide generating functions for the numbers of r-, p- and v-equivalence classes of $\mathcal{D}$.

© 2014 Elsevier B.V. All rights reserved.

1. Introduction and notations

A large number of various classes of combinatorial objects are enumerated by the Catalan numbers (A000108 in the online encyclopedia of integer sequences [10]). It is the case, among others, for planar trees, Young tableaux, stack sortable permutations, Dyck paths, and so on (see [11]).

A *Dyck path* of semilength n, $n \geq 0$, is a lattice path starting at $(0, 0)$, ending at $(2n, 0)$, and never going below the x-axis, consisting of up steps $U = (1, 1)$ and down steps $D = (1, -1)$. Let $\mathcal{D}$ be the set of all Dyck paths. Any non-empty Dyck path $P \in \mathcal{D}$ has a unique first return decomposition of the form $P = U\alpha D\beta$ where α and β are two Dyck paths (see [1]).

A *double rise* of a Dyck path is an occurrence UU of two consecutive up steps. A *peak* (resp. *valley*, resp. *zigzag*) is an occurrence of UD (resp. DU, resp. DUD). More generally, a *pattern* consists of consecutive steps of a Dyck path. We will say that a pattern is at position i, $i \geq 1$, in a Dyck path whenever the first step of the pattern appears at the ith step of the Dyck path. The *height* of a pattern is the minimal ordinate reached by this pattern. For instance, the path $P = UUDUDUUDDD$ contains two double rises at positions 1 and 6, three peaks at positions 2, 4, 7, and one zigzag of height one at position 3.

In the literature, many statistics on Dyck paths have been studied. Almost always, it is shown how we can enumerate Dyck paths according to several parameters, such as length, number of peaks or valleys, number of double rises, number of returns to the x-axis (see for instance [1–9,12]). Here, we take a new approach to study these statistics. We define three equivalence relations on $\mathcal{D}$.

Two Dyck paths of the same semilength are r-equivalent (resp. p-equivalent and v-equivalent) whenever the positions of their double rises (resp. peaks and valleys) are the same.

* Corresponding author.
E-mail addresses: barjl@u-bourgogne.fr (J.-L. Baril), armen.petrossian@u-bourgogne.fr (A. Petrossian).

http://dx.doi.org/10.1016/j.disc.2014.12.003
0012-365X/© 2014 Elsevier B.V. All rights reserved.

Table 1
Number of equivalence classes for Dyck paths.

Pattern	Sequence	Sloane	a_n, $1 \le n \le 9$
$\{UU\}$, $\{DD\}$	$\frac{1-x+\sqrt{1-2x-3x^2}}{1-3x+x^2+x^3+(1-x^2)\sqrt{1-2x-3x^2}}$	New	1, 2, 4, 9, 22, 56, 147, 393, 1065
$\{UD\}$	$\frac{1-6x+12x^2-8x^3+x^4}{(1-2x)^2(1-3x+x^2)}$	New	1, 2, 5, 14, 41, 121, 354, 1021, 2901
$\{DU\}$	$\frac{1-2x}{1-3x+x^2}$	A001519	1, 2, 5, 13, 34, 89, 233, 610, 1597

For instance, the path $UD\mathbf{UU}DDUD$ is r-equivalent to $UD\mathbf{UU}DUDD$ since they coincide on the unique double rise at position 3. The paths $U\mathbf{UD}U\mathbf{UD}DD\mathbf{UD}U\mathbf{UD}D$ and $U\mathbf{UD}D\mathbf{UD}UU\mathbf{UD}D\mathbf{UD}D$ are p-equivalent since they coincide on their four peaks at positions 2, 5, 9 and 12.

In this paper, we provide generating functions for the numbers of r-, p- and v-equivalence classes in $\mathcal{D}$, with respect to the semilength (see Table 1). The general method used in Sections 2–4 consists of exhibiting one-to-one correspondences between some subsets of Dyck paths and the different sets of equivalence classes by using combinatorial reasonings, and then, evaluating algebraically the generating functions for these subsets.

2. Equivalence classes modulo double rises

Throughout this section, we study the r-equivalence in $\mathcal{D}$.

Let $\mathcal{A}$ be the set of Dyck paths where all occurrences of DUD are at height 0 or 1, and where the pattern $DUDD$ does not appear. For instance, the Dyck path $UU\mathbf{DUDD}UDUD$ does not belong to $\mathcal{A}$, while $UUDUDUUDDD \in \mathcal{A}$, and the Dyck paths of semilength three in $\mathcal{A}$ are $UDUDUD$, $UDUUDD$, $UUUDDD$ and $UUDDUD$.

Lemma 1. *There is a bijection between $\mathcal{A}$ and the set of r-equivalence classes of $\mathcal{D}$.*

Proof. Let P be a Dyck path in $\mathcal{D}$. Let us prove that there exists a Dyck path $P' \in \mathcal{A}$ (with the same semilength as P) such that P and P' belong to the same class.

Before describing the construction of P', it is worth to notice the following fact.

If a Dyck path P avoids $DUDD$ then for any zigzag DUD at height $h \ge 2$ there is an occurrence of DDD at height $h-1$ on its right.

Indeed, a zigzag DUD at height $h \ge 2$ is followed by a subpath $Q \in \mathcal{D}$, which is followed by the first D reaching height $h-1$ after this zigzag. Obviously, Q is neither empty nor it ends with UD, since otherwise an occurrence of $DUDD$ would appear. Hence, Q ends with DD, thus forming an occurrence of DDD at height $h-1$. This occurrence DDD will be called *the right abutment* of the zigzag DUD.

Now, we define a sequence of Dyck paths $P_0 = P, P_1, P_2, \ldots, P_{k-1}, P_k = P'$, $k \ge 1$.

For any i, $1 \le i \le k$, the Dyck path P_i is obtained from P_{i-1} by performing successively the two following processes (1) and (2):

(1) If $P_{i-1} = a_0 \prod_{j=1}^{k} U^{r_j} a_j$, where $a_0 = (UD)^{\lambda}$, $\lambda \ge 0$, a_j avoids UU, $j \in [k]$ and each r_j is taken to be maximal, then set $P'_{i-1} = a_0 \prod_{j=1}^{k} U^{r_j} D^{\mu_j} (UD)^{\nu_j}$, where ν_j and $\nu_j + \mu_j$ are the number of U's and the number of D's in a_j respectively.
(2) Swap the leftmost zigzag of height at least two in P'_{i-1} with its right abutment, to obtain P_i.

The process finishes because performing (1) and (2) necessarily shifts to the right the position of the leftmost zigzag of height at least two. At the end of the process, the Dyck path P' belongs to $\mathcal{A}$ since it contains neither an occurrence of $DUDD$ nor a zigzag at height of at least two. Moreover, all Dyck paths $P_0 = P, P_1, \ldots, P_k = P'$ belong to the same equivalence class.

For instance, let us define $P = P_0 = UUUUDUDDUUUDDUDDDD$. Performing (1), we obtain $UUUUDDUDUUUDDDDDUD$ and after (2) we have $P_1 = UUUUDDDDUUUDDUDDUD$. The process continues since there is an occurrence of $DUDD$ in P_1. After (1), we have $UUUUDDDDUUUDDDUDUD$ that does not contain any occurrence of zigzag at height at least two. The process finishes with $P' = P_2 = UUUUDDDDUUUDDDUDUD$. See Fig. 1 for an illustration of this example.

Now, it suffices to prove that two different Dyck paths P and P' with the same semilength in $\mathcal{A}$ cannot belong to the same class. Let us assume that P and P' belong to the same class. In the case where P does not contain any double rise, we necessarily have $P = P' = (UD)^n$. Whenever P contains at least one double rise, we decompose P and P' as follows:

$$P = \alpha_0 U^{r_1} \alpha_1 U^{r_2} \alpha_2 \ldots U^{r_k} \alpha_k \quad \text{and} \quad P' = \alpha'_0 U^{r_1} \alpha'_1 U^{r_2} \alpha'_2 \ldots U^{r_k} \alpha'_k$$

where $k \ge 1$, α_i, α'_i, $0 \le i \le k$, do not contain any double rise and where each $r_i \ge 2$, $1 \le i \le k$, is maximal.

Obviously, we necessarily have $\alpha_0 = \alpha'_0 = (UD)^{s_0}$ for some $s_0 \ge 0$. Moreover, for $i \ge 0$, α_i (resp. α'_i) contains neither a double rise nor an occurrence of $DUDD$ which means that α_i (resp. α'_i) is necessarily of the form $\alpha_i = D^{t_i}(UD)^{s_i}$ for some $t_i \ge 1$ and $s_i \ge 0$ (resp. $\alpha'_i = D^{t'_i}(UD)^{s'_i}$ for some $t'_i \ge 1$ and $s'_i \ge 0$).

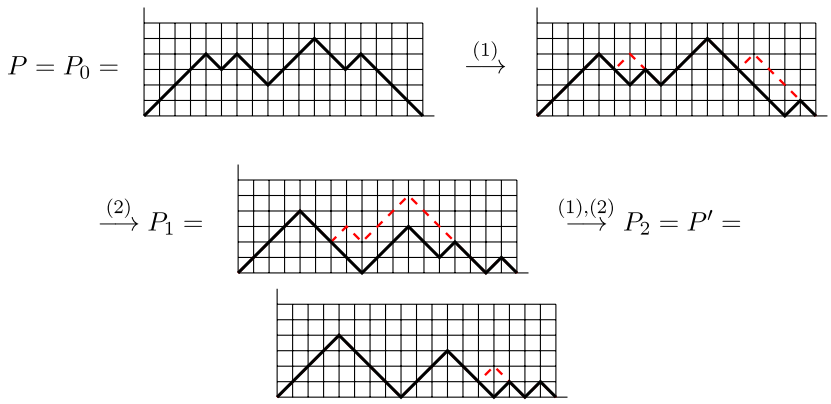


Fig. 1. Illustration of the example described in the proof of Lemma 1.

Since P and P' belong to the same class, we have $t_i + 2s_i = t'_i + 2s'_i$ for all $i \geq 0$. For a contradiction, let us assume that there is $j \geq 0$ such that $s_j \neq s'_j$ (we choose the smallest $j \geq 0$). Without loss of generality, we set $s_j < s'_j$.

So, the difference between the height h of the first point of α_j (which also is the height of the first point of α'_j) and the height of the last point of α_j (resp. α'_j) is t_j (resp. t'_j). The equality $t_j + 2s_j = t'_j + 2s'_j$ induces that $t_j - t'_j \geq 2$ which implies that α'_j has the height of its last point out of the interval [0, 1]. Since P and P' do not have any zigzag at height at least two, we deduce that α'_j does not contain any zigzag, that is $s'_j = 0$. A contradiction is obtained, since $s_j < s'_j$. Finally we necessarily have $\alpha_i = \alpha'_i$ for $1 \leq i \leq k$, and then $P = P'$ which completes the proof. □

Before proving Theorem 1, we give a preliminary result in Lemma 2. Let $\mathcal{B}$ be the set of Dyck paths where all zigzags are at height 0, and without zigzag at the end. For instance, the Dyck paths of semilength three in $\mathcal{B}$ are *UDUUDD* and *UUUDDD*.

Lemma 2. *The generating function of the set $\mathcal{B}$ with respect to the semilength is given by*

$$\frac{2 - x^2 - x + x\sqrt{1 - 2x - 3x^2}}{1 - x + \sqrt{1 - 2x - 3x^2}}.$$

Proof. Let $B(x)$ be the generating function of the set $\mathcal{B}$ with respect to the semilength. Let P be a non-empty Dyck path of $\mathcal{D}$ and $P = U\alpha D\beta$ its first return decomposition where α and β are two Dyck paths. Then, a non-empty Dyck path P belongs to $\mathcal{B}$ if and only if α avoids *DUD*, and β belongs to $\mathcal{B}$. It is well known (see for instance [12]) that the generating function for the set of Dyck paths avoiding *DUD* is given by $1 + xM(x)$, where

$$M(x) = \frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x^2}$$

is the generating function for the classical Motzkin sequence (A001006 in [10]). Therefore, we deduce the functional equation

$$B(x) = 1 + x \cdot (1 + x \cdot M(x)) \cdot (B(x) - x)$$

where $M(x)$ is the generating function for the Motzkin sequence defined above. A simple calculation provides the expected result. □

Theorem 1. *The generating function for the set of r-equivalence classes in $\mathcal{D}$ (i.e., modulo the positions of the double rises) with respect to the semilength is given by*

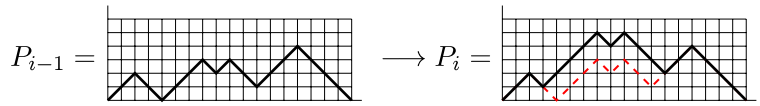
$$\frac{1 - x + \sqrt{1 - 2x - 3x^2}}{1 - 3x + x^2 + x^3 + (1 - x^2)\sqrt{1 - 2x - 3x^2}}.$$

Proof. By Lemma 1, it suffices to provide the generating function $A(x)$ for the the set $\mathcal{A}$, with respect to the semilength. Let P be a non-empty Dyck path of $\mathcal{A}$. It has a unique first return decomposition of the form $P = U\alpha D\beta$ where α belongs to $\mathcal{B}$ and β belongs to $\mathcal{A}$. This induces the functional equation $A(x) = 1 + x \cdot B(x) \cdot A(x)$ where $B(x)$ is the generating function for the set $\mathcal{B}$ (see Lemma 2 for the calculation of $B(x)$), giving the required result. □

3. Equivalence classes modulo peaks

Throughout this section, we study the p-equivalence in $\mathcal{D}$.

Let $\mathcal{E}$ be the set of Dyck paths such that there is no peak *UD* both on the right of an occurrence of *DD* and on the left of an occurrence of *UU*. For instance, *UU***DD***U***UD***DU***UU***DDD* does not belong to $\mathcal{E}$, while *UUDUUUDDUUDDDD* belongs to $\mathcal{E}$.



$$UUDDUUUDUDDUUUDDDD \to UUDUUUUDUDDDUUDDDD$$

Fig. 2. Illustration of the example described in the proof of Lemma 3.

Lemma 3. *There is a bijection between $\mathcal{E}$ and the set of p-equivalence classes of $\mathcal{D}$.*

Proof. Let P be a Dyck path in $\mathcal{D}\backslash\mathcal{E}$. Let us prove that there exists a Dyck path $P' \in \mathcal{E}$ of the same semilength as P such that P and P' belong to the same class.

We define a sequence of Dyck paths $P_0 = P, P_1, P_2, \ldots, P_{k-1}, P_k = P', k \geq 1$.

For any i, $1 \leq i \leq k$, the Dyck path P_i is obtained from P_{i-1} by performing the following process.

We write

$$P_{i-1} = \alpha UD^r \beta U^s D\gamma$$

where $r, s \geq 2$, α (resp. γ) avoids DD (resp. UU) and $\beta = U\delta D$ for some δ. Notice that this decomposition is unique.

We set

$$P_i = \alpha UD^{r-t} U^t \beta D^t U^{s-t} D\gamma$$

where $t = \min\{r-1, s-1\}$.

Now, let us consider the decomposition of P_i (as above)

$$P_i = \alpha' UD^{r'} \beta' U^{s'} D\gamma'$$

where α', β', γ', r' and s' satisfy the same properties as α, β, γ, r and s. Since at least one of the two values $r-t$ and $s-t$ is equal to one, we can assume $r-t=1$ without loss of generality. So, we deduce that α' begins with αUDU^tU and γ' ends with γ which implies that the length of $D^{r'}\beta' U^{s'}$ is smaller than the length of $D^r\beta U^s$. Thus, the process finishes whenever either β' becomes empty or there is no occurrence of DD lying on the left of an occurrence of UU, *i.e.*, $P_k \in \mathcal{E}$. Moreover, this construction preserves the positions of peaks which implies that P_{i-1} and P_i belong to the same class. So, at the end of the process P' belongs to $\mathcal{E}$ and to the same p-equivalence class of P. See Fig. 2 for an illustration whenever $P_{i-1} = \underline{U}UDD\mathbf{UUUDUDD}UUUD\overline{DDD}$ and $P_i = UUDUUUUDUDDDUUDDDD$ (α, β and γ appear respectively in underlined type, bold face and overlined type in P_{i-1}).

Now, it suffices to prove that two different Dyck paths P and P' in $\mathcal{E}$ cannot belong to the same class. Let us assume that P and P' belong to the same class. Whenever P and P' contain k peaks, $k \geq 1$, we can decompose uniquely P and P' as follows:

$$P = U^{r_1}D^{s_1}U^{r_2}D^{s_2}\ldots U^{r_k}D^{s_k} \quad \text{and} \quad P' = U^{r'_1}D^{s'_1}U^{r'_2}D^{s'_2}\ldots U^{r'_k}D^{s'_k}$$

where $r_i, s_i \geq 1$ and $r'_i, s'_i \geq 1$ for $1 \leq i \leq k$.

Since the positions of peaks in P and P' are the same, we have $r_1 = r'_1$, $s_k = s'_k$ and $s_i + r_{i+1} = s'_i + r'_{i+1}$ for $1 \leq i \leq k-1$.

For a contradiction, let us assume that there is $j \geq 1$ such that $s_j \neq s'_j$ (we choose the smallest $j \geq 1$). Without loss of generality, we consider $s_j < s'_j$ and thus $s'_j \geq 2$. Since P' belongs to $\mathcal{E}$, we necessarily have $r'_\ell = 1$ for all $\ell \geq j+2$. As $s_j < s'_j$ the height of the $(j+1)$th peak of P is greater than the height of the $(j+1)$th peak of P'. Since $r'_\ell (= 1)$ is minimal for any $\ell \geq j+2$, the height of the ℓth peak of P is greater than the height of the ℓth peak of P' for all $\ell \geq j+1$. We obtain a contradiction with $s_k = s'_k$ which means that the kth peak of P and P' are located at the same height. Thus, we necessarily have $P = P'$ which completes the proof. □

Theorem 2. *The generating function for the set of p-equivalence classes of $\mathcal{D}$ (i.e., modulo the positions of peaks) with respect to the semilength is given by*

$$\frac{1-6x+12x^2-8x^3+x^4}{(1-2x)^2(1-3x+x^2)}.$$

Proof. By Lemma 3, it suffices to provide the generating function $E(x)$ of the set $\mathcal{E}$, with respect to the semilength. Every nonempty path $P \in \mathcal{E}$ is decomposed as $P = U\alpha D\beta$, where either α is empty and $\beta \in \mathcal{E}$, or α is a nonempty element of $\mathcal{E}$ that contains DDU and $\beta = (UD)^\lambda$, $\lambda \geq 0$, or α avoids DDU and β avoids DUU. It is known that the generating function of Dyck paths avoiding DDU (equivalently DUU) is equal to $\frac{1-x}{1-2x}$ (see [1]). The above decomposition gives that

$$E(x) = 1 + xE(x) + x\left(E(x) - 1 - \frac{1-x}{1-2x}\right)\frac{1}{1-x} + x\left(\frac{1-x}{1-2x} - 1\right)\frac{1-x}{1-2x},$$

which implies the results. □

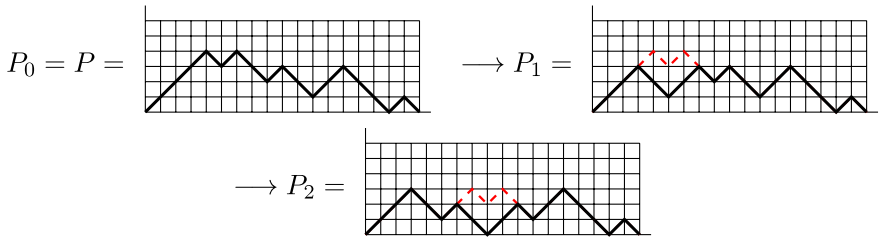


Fig. 3. Illustration of the example described in the proof of Lemma 4.

4. Equivalence classes modulo valleys

Throughout this section, we study the v-equivalence in $\mathcal{D}$.

Let $\mathcal{H}$ be the set of Dyck paths such that the height of any valley DU is at most one. For instance, $UUDDUUDUDD$ belongs to $\mathcal{H}$, while $UUU\mathbf{DU}DDUDD$ does not belong to $\mathcal{H}$ since its first valley DU is at height 2.

Lemma 4. *There is a bijection between $\mathcal{H}$ and the set of v-equivalence classes of $\mathcal{D}$.*

Proof. Let P be a Dyck path in $\mathcal{D}$. Let us prove that there exists a Dyck path $P' \in \mathcal{H}$ of the same semilength as P such that P and P' belong to the same class.

Before describing the construction of P', it is easy but worth to notice the following fact.

If a Dyck path P contains a valley DU of height at least one, then there exists an occurrence of UU on its left and an occurrence of DD on its right.

Let DU be a valley of height $h \geq 2$. We call *left-abutment* (resp. *right-abutment*) of this valley the rightmost occurrence of UU located on the left of the valley (resp. the leftmost occurrence of DD located on the right of the valley). Of course, the right- and left-abutments are at height at least $h-1$.

For instance, if $P = UU\underline{UU}DUD\mathbf{DU}UU\overline{DD}UDDDD$ then the second valley DU (bold face) has its left-abutment at position 3 (underlined) and its right-abutment at position 12 (overlined).

Now, we define a sequence of Dyck paths $P_0 = P, P_1, P_2, \ldots, P_k = P', k \geq 1$.

For any i, $1 \leq i \leq k$, the Dyck path P_i is obtained from P_{i-1} by performing the following process.

We consider the leftmost valley of height at least two and we replace its left-abutment UU with UD and its right-abutment DD with UD.

The process finishes because at each step the leftmost valley of P_{i-1} at height $h \geq 2$ is moved into a valley in P_i at height $h-2$. At the end of the process, the Dyck path P' belongs to $\mathcal{H}$ since it does not contain any valley of height at least two. Moreover, all Dyck paths $P_0, P_1, \ldots, P_k$ belong to the same equivalence class.

For instance, if we perform the above process on $P = UUUUDUDDUDDUUDDDUD$, then we obtain $P_1 = UUUDDUUDUDDUUDDDUD$ and $P_2 = P'$ is given by $UUUDDUDDUUDUUDDDUD$ (see Fig. 3 for an illustration of this example).

Now, it suffices to prove that two different Dyck paths P and P' of the same semilength in $\mathcal{H}$ cannot belong to the same class. Let us assume that P and P' belong to the same class. Whenever P and P' contain k valleys ($k \geq 0$), we can uniquely decompose P and P' as follows:

$$P = U^{r_1}D^{s_1}U^{r_2}D^{s_2}\ldots U^{r_{k+1}}D^{s_{k+1}} \quad \text{and} \quad P' = U^{r'_1}D^{s'_1}U^{r'_2}D^{s'_2}\ldots U^{r'_{k+1}}D^{s'_{k+1}}$$

where $r_i, s_i \geq 1$ and $r'_i, s'_i \geq 1$ for $1 \leq i \leq k+1$.

Since the positions of valleys are the same in P and P', we have $s_i + r_i = s'_i + r'_i$ for $1 \leq i \leq k+1$. For a contradiction, let us assume that there is $j \geq 1$ such that $s_j \neq s'_j$ (we choose the smallest $j \geq 1$). Without loss of generality, we assume $s'_j < s_j$. Considering $s_j + r_j = s'_j + r'_j$ we deduce from $s'_j \leq s_j - 1$ the inequality $r'_j - s'_j \geq r_j - s_j + 2$.

Notice that the height of the jth valley in P (resp. P') is given by $\sum_{i=1}^{j}(r_i - s_i)$ (resp. $\sum_{i=1}^{j}(r'_i - s'_i)$). Therefore, as P and P' belong to $\mathcal{H}$, we necessarily have $0 \leq \sum_{i=1}^{j}(r_i - s_i) \leq 1$ and $0 \leq \sum_{i=1}^{j}(r'_i - s'_i) \leq 1$.

Thus, we have:

$$\begin{aligned}
\sum_{i=1}^{j}(r'_i - s'_i) &= \sum_{i=1}^{j-1}(r'_i - s'_i) + r'_j - s'_j \\
&= \sum_{i=1}^{j-1}(r_i - s_i) + r'_j - s'_j \\
&\geq \sum_{i=1}^{j-1}(r_i - s_i) + r_j - s_j + 2 = 2 + \sum_{i=1}^{j}(r_i - s_i) \geq 2.
\end{aligned}$$

This means that P' has its jth valley of height greater than or equal two, which is a contradiction. Thus, we necessarily have $P = P'$, which completes the proof. $\square$

Theorem 3. *The generating function for the set of v-equivalence classes of $\mathcal{D}$ (i.e., modulo the positions of valleys) with respect to the semilength is given by the generating function for the Fibonacci sequence restricted to the odd ranks (see A001519 in* [10]*)*

$$\frac{1-2x}{1-3x+x^2}.$$

Proof. By Lemma 4, it suffices to provide the generating function $H(x)$ for the set $\mathcal{H}$ with respect to the semilength. Let P be a non-empty Dyck path in $\mathcal{H}$ and $P = U\alpha D\beta$ its first return decomposition where α and β are some Dyck paths.

So, $P = U\alpha D\beta$ does not contain any valley at height at least two if and only if α has all its valleys at height 0, and β belongs to $\mathcal{H}$. This means that $\alpha = U^jD^j\alpha'$ for some j, $1 \le j \le n$, and where α' has all its valleys at height 0. The generating function for the number of such α is given by $\frac{1-x}{1-2x}$. Finally we have the functional equation $H(x) = 1 + x \cdot \frac{1-x}{1-2x} \cdot H(x)$ which gives the result. □

Acknowledgments

We would like to thank the anonymous referees for their very careful reading of this paper and their helpful comments and suggestions.

References

[1] E. Deutsch, Dyck path enumeration, Discrete Math. 204 (1999) 167–202.
[2] T. Mansour, Counting peaks at height k in a dyck path, J. Integer Seq. 5 (2002) 02.1.1.
[3] T. Mansour, Statistics on dyck paths, J. Integer Seq. 9 (2006) 06.1.5.
[4] T.Mansour M, Shattuck. counting dyck paths according to the maximum distance between peaks and valleys, J. Integer Seq. 15 (2012) 12.1.1.
[5] D. Merlini, R. Sprugnoli, M.C. Verri, Some statistics on dyck paths, J. Statist. Plann. Inference 101 (2002) 211–227.
[6] A. Panayotopoulos, A. Sapounakis, On the prime decomposition of dyck paths, J. Combin. Math. Combin. Comput. 40 (2002) 33–39.
[7] P. Peart, W.J. Woan, Dyck paths with no peaks at height k, J. Integer Seq. 4 (2001) 01.1.3.
[8] A. Sapounakis, I. Tasoulas, P. Tsikouras, Counting strings in dyck paths, Discrete Math. 307 (23) (2007) 2909–2924.
[9] A. Sapounakis, I. Tasoulas, P. Tsikouras, Some strings in dyck paths, Australas. J. Combin. 39 (2007) 49–72.
[10] N.J.A. Sloane, The On-line Encyclopedia of Integer Sequences, available electronically at http://oeis.org.
[11] R.P. Stanley, Enumerative Combinatorics, Vol. 2, Cambridge University Press, 1999.
[12] Y. Sun, The statistic "number of udu's" in dyck paths, Discrete Math. 287 (2004) 177–186.

Discrete Mathematics 338 (2015) 1370–1378

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

A Motzkin filter in the Tamari lattice

Jean-Luc Baril *, Jean-Marcel Pallo

LE2I, UMR 6306, Université de Bourgogne B.P. 47870, F21078 DIJON-Cedex, France

ARTICLE INFO

Article history:
Received 16 October 2014
Received in revised form 26 February 2015
Accepted 27 February 2015
Available online 20 March 2015

Keywords:
Lattice
Ideal
Filter
Binary tree
Motzkin
Tamari
Diameter

ABSTRACT

The Tamari lattice of order n can be defined on the set $\mathcal{T}_n$ of binary trees endowed with the partial order relation induced by the well-known rotation transformation. In this paper, we restrict our attention to the subset $\mathcal{M}_n$ of Motzkin trees. This set appears as a filter of the Tamari lattice. We prove that its diameter is $2n-5$ and that its radius is $n-2$. Enumeration results are given for join and meet irreducible elements, minimal elements and coverings. The set $\mathcal{M}_n$ endowed with an order relation based on a restricted rotation is then isomorphic to a ranked join-semilattice recently defined in Baril and Pallo (2014). As a consequence, we deduce an upper bound for the rotation distance between two Motzkin trees in $\mathcal{T}_n$ which gives the exact value for some specific pairs of Motzkin trees.

© 2015 Elsevier B.V. All rights reserved.

1. Introduction

Interpreting associativity as a leftward substitution rule on parenthesizations leads to what is known as a Tamari lattice [9,13,14]. This partial order on a Catalan set first appeared in 1951 in Dov Tamari's thesis at the Sorbonne in Paris [27]. The Tamari order was originally defined as a partial order on parenthesizations, but it can also be understood as an order on binary trees endowed with the well-known rotation operation occurring among other in computer science. Quite a number of important papers have been published on the topic in many areas such as algebra, combinatorics, and physics. However, among this plentiful literature, there are only a few studies related to specific subsets of the Tamari lattice. For instance, the paper [17] highlights a Boolean sublattice of the Tamari lattice. More recently, it has been proved that the subset of balanced binary trees is closed by interval in the Tamari lattice [10]. The subset of Motzkin words has also been studied whenever this subset is endowed with the Tamari partial order on parenthesizations [2].

In this paper, we tackle the problem by studying how the rotation transformation acts on the subset $\mathcal{M}_n$ of Motzkin trees of order n which are binary trees such that the internal nodes whose left subtree is a leaf also have a leaf as their right subtree. These trees are in bijection with Motzkin paths, which explains their name.

In Section 3, we show that $\mathcal{M}_n$ is a filter in the Tamari lattice $\mathcal{T}_n$ of binary trees of order n. We compute the diameter and the radius of $\mathcal{M}_n$. In Section 4, enumeration results are given for join and meet irreducible elements, minimal elements and coverings. In Section 5, we endow the set $\mathcal{M}_n$ with a partial order based on a restricted rotation transformation, and we prove that this poset is isomorphic to a ranked join-semilattice presented in a recent paper of the authors [2]. As a consequence, we deduce an upper bound for the rotation distance between two Motzkin trees in $\mathcal{T}_n$ which gives the exact value of the classical distance rotation for some specific pairs of Motzkin trees. This result suggests that $\mathcal{M}_n$ is better behaved than the Tamari lattice regarding the rotation distance and the diameter.

* Corresponding author.
E-mail addresses: barjl@u-bourgogne.fr (J.-L. Baril), pallo@u-bourgogne.fr (J.-M. Pallo).

http://dx.doi.org/10.1016/j.disc.2015.02.024
0012-365X/© 2015 Elsevier B.V. All rights reserved.

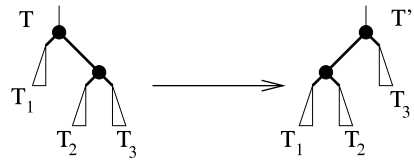


Fig. 1. The left-rotation transformation on binary trees.

2. Definition and notations

The Tamari lattice $\mathcal{T}_n$ of order n is defined on the set of binary rooted ordered trees with n internal nodes and thus $n+1$ leaves (see [9,13,14,28]). In this lattice, a tree T' covers a tree T when it is obtained from it by a left-rotation (see Fig. 1).

Now, we introduce the Polish notation of binary trees that will be convenient later for the proofs. An internal node of a binary ordered tree admits a left and a right subtree. The *prefix order* on a binary tree is defined recursively by visiting the root and then the left subtree and the right subtree. The *infix order* is defined recursively by visiting the left subtree, the root and the right subtree. The Polish (or linear) notation of T is obtained by reading T in prefix order and replacing each internal node (resp. each leaf) with $\bigcirc$ (resp. with $\square$). The left-rotation transformation $\longrightarrow$ on a tree T can be viewed on the Polish notation of trees as the elementary transformation $\bigcirc T_1 \bigcirc T_2 T_3 \longrightarrow \bigcirc \bigcirc T_1 T_2 T_3$ where T_1, T_2 and T_3 are the Polish notations of three subtrees of T. For instance, $\bigcirc \bigcirc \square \bigcirc \square \square \bigcirc \square \square$ is obtained from $\bigcirc \square \bigcirc \bigcirc \square \square \bigcirc \square \square$ by a left-rotation. The inverse transformation $\longleftarrow$ will be called a right-rotation and the transitive closure of the left-rotation will be denoted $\xrightarrow{*}$.

The rotation transformation has been widely studied using weight sequences of binary trees introduced in [15]. Some of our proofs consist in switching from one of the three representations to the other (tree, Polish notation and weight sequence). So we provide the definition of the *weight sequence* of $T \in \mathcal{T}_n$ (w-sequence for short).

Given $T \in \mathcal{T}_n$, the *weight* of T is the number of its leaves, *i.e.* $n+1$. The w-sequence of $T \in \mathcal{T}_n$ is $w_T = w_T(1)w_T(2)\ldots w_T(n)w_T(n+1)$, where $w_T(i)$ is the weight of the largest subtree of T whose last leaf is the ith leaf of T in prefix order. For convenience, we do not use the last value $w_T(n+1)$ which is always equal to $n+1$. Two distinct trees cannot have the same w-sequence (see [15]).

Proposition 1 (*Theorem 1 in [15]*)**.** *A necessary and sufficient condition for an integer sequence w of length n to be the w-sequence of a tree in $\mathcal{T}_n$ is $1 \le w(i) \le i$ for all $i \in [n]$, and if $j \in [i - w(i) + 1, i]$ then $i - w(i) \le j - w(j)$.*

Proposition 2 (*Lemma 2 in [21]*)**.** *Given $T \in \mathcal{T}_n$ with w-sequence $w_T = w_T(1)w_T(2)\ldots w_T(n)$, then the tree obtained by performing a left-rotation on the kth internal node in infix order (if it possible) has the w-sequence $w_T(1)w_T(2)\ldots w_T(k-1)(w_T(k) + w_T(k - w_T(k)))w_T(k+1)\ldots w_T(n)$.*

For example, the left-rotation that transforms $\bigcirc \square \bigcirc \bigcirc \square \square \bigcirc \square \square$ into $\bigcirc \square \bigcirc \bigcirc \bigcirc \square \square \square \square$ corresponds to changing the w-sequence 1121 into the w-sequence 1123.

Theorem 1 (*Theorem 2 in [15]*)**.** *Given T and T' in $\mathcal{T}_n$, we have:*

$$T \xrightarrow{*} T' \iff w_T(i) \le w_{T'}(i) \quad \text{for all } i \in [n].$$

We define the rotation distance $d(T, T')$ between two binary trees $T, T' \in \mathcal{T}_n$ as the minimum number of left- and right-rotations needed to transform T into T' (see [12,22]). Previous works on rotation distance have focused on approximation algorithms [1,4,18]. However, there remains today an open problem whether the rotation distance can be computed in polynomial time.

3. The Motzkin filter $\mathcal{M}_n$

Let $\mathcal{M}_n$ be the set of Motzkin trees with n internal nodes, *i.e.* binary trees where all internal nodes that have a leaf $\square$ as their left subtree also have a leaf $\square$ as their right subtree. Equivalently, Motzkin trees are the ones whose Polish notation does not contain any occurrence of $\bigcirc \square \bigcirc$. It is well known that this set is enumerated by the nth term of the Motzkin sequence A001006 in [23] (see for instance [5,26]). For example, $\mathcal{M}_4 = \{\bigcirc \bigcirc \bigcirc \bigcirc \square \square \square \square \square, \bigcirc \bigcirc \bigcirc \square \bigcirc \square \square \square \square, \bigcirc \bigcirc \bigcirc \square \square \bigcirc \square \square \square, \bigcirc \bigcirc \square \square \bigcirc \bigcirc \square \square \square\}$. We refer to [8,24] for other combinatorial classes enumerated by the Motzkin numbers. See Fig. 2 for an illustration of a Motzkin tree. For readability, binary trees undermentioned will be sometimes illustrated without leaves (see Fig. 3).

Due to the definition of the left-rotation (see Fig. 1), the set $\mathcal{M}_n$ is closed under the left-rotation transformation, *i.e.* any left-rotation on a Motzkin tree creates a tree that belongs to $\mathcal{M}_n$. Notice that this property means that $\mathcal{M}_n$ is a *filter* in $\mathcal{T}_n$ which is the notion dual to that of an *ideal* [6,11]. Hence the following proposition holds.

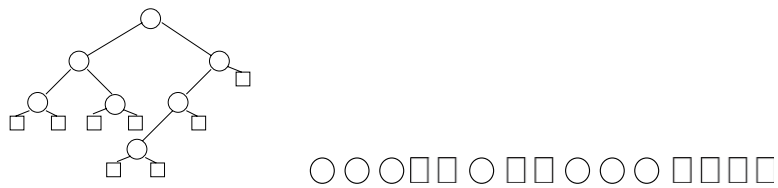


Fig. 2. A Motzkin tree, with w-sequence 1214123, and its Polish notation.

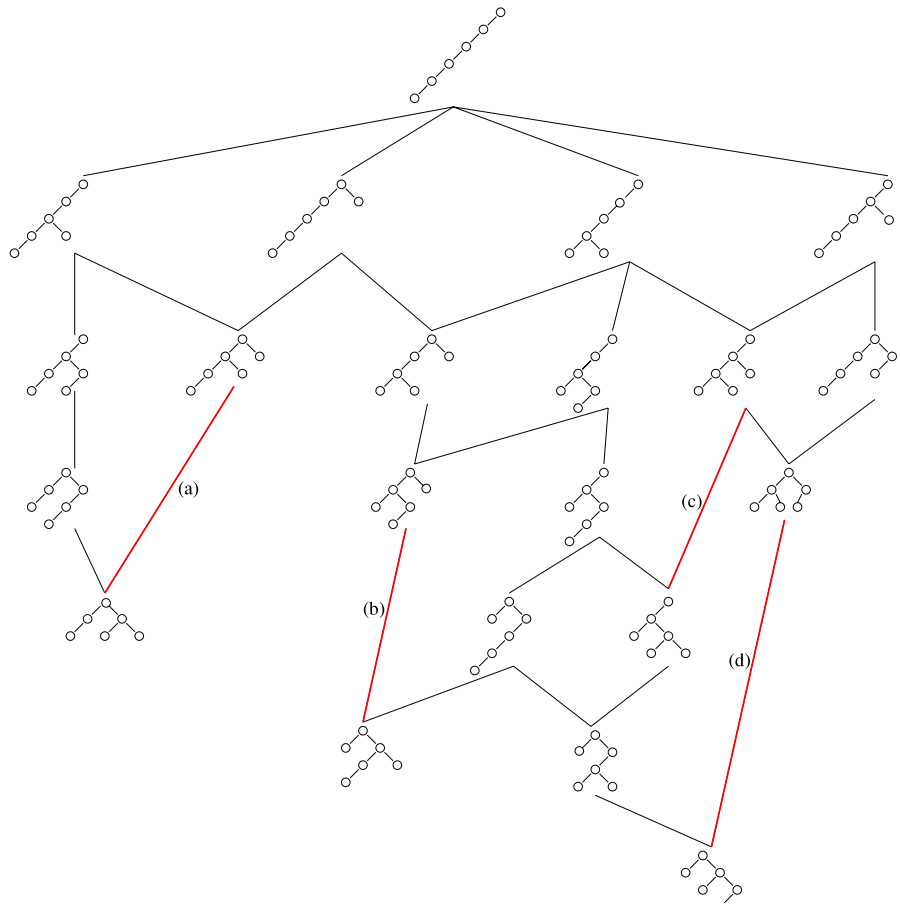


Fig. 3. The Motzkin semilattice $\mathcal{M}_6$.

Proposition 3. *The poset $\mathcal{M}_n$ is a filter of $\mathcal{T}_n$ whose maximum element* **1** *has the w-sequence $w_{\mathbf{1}} = 123\ldots n$. Thus $(\mathcal{M}_n, \xrightarrow{*})$ is a join-semilattice.*

Let $\delta(T, T')$ be the rotation distance in $\mathcal{M}_n$ between two Motzkin trees T and T', *i.e.* the minimum number of left- and right-rotations needed to transform T into T' by passing through Motzkin trees in $\mathcal{M}_n$. Obviously, we have $d(T, T') \leq \delta(T, T')$ where d is the classical rotation distance in $\mathcal{T}_n$. The *diameter* $\delta(\mathcal{M}_n)$ of $\mathcal{M}_n$ is the maximum distance in $\mathcal{M}_n$ among all pairs of Motzkin trees. The *radius* $rad(\mathcal{M}_n)$ of $\mathcal{M}_n$ is the minimum (on $T \in \mathcal{M}_n$) of the greatest distance of T from any other Motzkin tree (see [7]). The following theorems give the diameter and the radius of $\mathcal{M}_n$. Notice that computing the diameter $d(\mathcal{T}_n)$ of the Tamari lattice was for many years an open problem. Recently, Pournin has proved that $d(\mathcal{T}_n) = 2n - 6$ for $n > 10$ using only combinatorial arguments [20].

Lemma 1. *For all $n \geq 3$, the diameter of $\mathcal{M}_n$ satisfies*

$$\delta(\mathcal{M}_n) \leq 2n - 5.$$

Proof. The proof uses the same general idea as that of Lemma 2 in [22]. We build a path between two arbitrary objects by transforming them into a canonical object. The proof is also similar in the way this transformation is done, since we cluster the leaves of the trees to the right of the Polish notation, while in [22], they increase the incidence of a given vertex in two triangulations. We proceed by induction on $n \geq 3$. For $n = 3$, it is clear that $\delta(\mathcal{M}_3) = 1 \leq 2 \cdot 3 - 5$. Now let us assume that

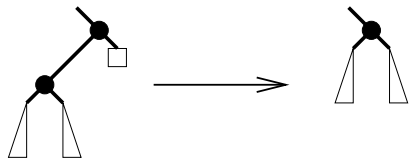


Fig. 4. The transformation ϕ.

$\delta(\mathcal{M}_k) \leq 2k-5$ for all $k < n$, and let us prove that $\delta(\mathcal{M}_n) \leq 2n-5$. Let T be a Motzkin tree in $\mathcal{M}_n$ and $\ell(T) \leq n$ be the number of internal nodes before its first leaf in its Polish notation. If $\ell(T) < n$ then there necessarily exists a rotation on T that increases by one the number $\ell(T)$. Thus, we can produce the maximum tree $\mathbf{1}$ with $\ell(\mathbf{1}) = n$ by performing $n - \ell(T)$ rotations. Then, given any two trees T and T' we can convert T into T' in $2n - \ell(T) - \ell(T')$ rotations via $\mathbf{1}$. Thus, we have $\delta(T, T') \leq 2n - \ell(T) - \ell(T')$.

Due to the fact that T and T' belong to $\mathcal{M}_n$, we have $\ell(T) \geq 2$ and $\ell(T') \geq 2$. In the case where $\ell(T) = 2$ and $\ell(T') = 2$, T and T' have the same left subtree ◯□□. Therefore, the distance between T and T' is the distance between the two right subtrees of T and T' that lie in $\mathcal{M}_{n-2}$. The induction hypothesis implies $\delta(T, T') \leq 2(n-2) - 5 = 2n - 9 \leq 2n - 5$.

In the case where $\ell(T) > 2$ or $\ell(T') > 2$, we have $\ell(T) + \ell(T') \geq 5$ and we have $\delta(T, T') \leq 2n - 5$.

According to the two previous cases, we deduce $\delta(\mathcal{M}_n) \leq 2n - 5$ which completes the induction. □

We obtain the lower bounds using the same general argument and a similar construction as in [20]. In particular, we exhibit a pair of Motzkin trees with n internal nodes and show they are at distance $2n - 5$ using a map ϕ (which works as the deletions from [20]) that removes an internal node from a Motzkin tree, and that removes the rotations involving this node from any path within the graph of the semilattice $\mathcal{M}_n$. Lemma 4 of this paper corresponds to Corollary 1 from [20].

Now let us define the transformation ϕ from $\mathcal{T}_n$ to $\mathcal{T}_{n-1}$ such that $\phi(T)$ is obtained from T by replacing the last internal node in infix order with its left subtree (its right subtree being necessarily a leaf). Notice that whenever the left subtree of the last node of T is a leaf, then $\phi(T)$ is obtained from T by replacing the last node with a leaf. For instance, if $T =$ ◯◯□□◯◯□□◯□□ then $\phi(T) =$ ◯◯□□◯◯□□□, and if $T' =$ ◯◯□□◯◯□□◯◯□□□ then $\phi(T') =$ ◯◯□□◯◯□□◯□□. In terms of w-sequences, we have $w_T = 12121$, $w_{\phi(T)} = 1212$ and $w_{T'} = 121212$, $w_{\phi(T')} = 12121$. See Fig. 4 for an illustration of ϕ.

Lemma 2. *Let $T \in \mathcal{T}_n$ and $w(1)w(2)\ldots w(n)$ be its w-sequence. Then, the w-sequence of $\phi(T)$ is $w(1)w(2)\ldots w(n-1)$. Therefore, if $T \in \mathcal{M}_n$ then $\phi(T) \in \mathcal{M}_{n-1}$.*

Proof. For $1 \leq i \leq n-1$, the transformation ϕ does not modify the largest subtree of T whose last leaf is the ith leaf of T in prefix order. Thus, the w-sequence of $\phi(T)$ is $w(1)w(2)\ldots w(n-1)$ and $\phi(T)$ belongs to $\mathcal{M}_{n-1}$. □

Lemma 3. *Let T and T' be two Motzkin trees in $\mathcal{M}_n$ such that T' is obtained from T by a rotation involving the last internal node of T in infix order. Then, we have $\phi(T) = \phi(T')$.*

Proof. Let us assume that T' is obtained from T by a left-rotation (resp. right-rotation) involving the last internal node of T in infix order. Then, the w-sequence of T' is obtained from that of T by increasing (resp. decreasing) the last value $w_T(n)$. With Lemma 2, we deduce that the w-sequences of $\phi(T)$ and $\phi(T')$ are the same, and thus $\phi(T) = \phi(T')$. □

Lemma 4. *Let $T = T_0, T_1, \ldots, T_k = T'$ be a shortest path in $\mathcal{M}_n$ between T and T'. Let $p \geq 0$ be the number of (left or right) rotations involving the last internal node in infix order. Then, we have*

$$\delta(\phi(T), \phi(T')) \leq \delta(T, T') - p.$$

Proof. According to Lemma 2, $\phi(T) = \phi(T_0), \phi(T_1), \ldots, \phi(T_k) = \phi(T')$ is a path in $\mathcal{M}_{n-1}$ between $\phi(T)$ and $\phi(T')$, provided one removes duplicates from this sequence. Two consecutive trees in the sequence are then indeed related by a rotation, which follows from Lemma 2 and from Proposition 2. With Lemma 3, there are p pairs (T_i, T_{i+1}) such that $\phi(T_i) = \phi(T_{i+1})$. Thus, the length of the previous path between $\phi(T)$ and $\phi(T')$ is $\delta(T, T') - p$, which implies $\delta(\phi(T), \phi(T')) \leq \delta(T, T') - p$. □

Theorem 2. *For $n \geq 3$, we have $\delta(\mathcal{M}_n) = 2n - 5$.*

Proof. Considering Lemma 1, it suffices to exhibit a family of pairs of Motzkin trees $T, T' \in \mathcal{M}_n$, $n \geq 3$, satisfying $\delta(T, T') = 2n - 5$. For n even, $n \geq 4$, we define T and T' by their weight sequences $w_T = 121212\ldots12$ and $w_{T'} = 1231212\ldots121$. For n odd, $n \geq 3$, T and T' are defined by $w_T = 121212\ldots121$ and $w_{T'} = 1231212\ldots12$.

We proceed by induction on $n \geq 3$. It is straightforward to verify that $\delta(121, 123) = 1 = 2\cdot3-5$ and $\delta(1212, 1231) = 3$. Therefore the cases $n = 3$ and $n = 4$ hold. Let us assume that $\delta(T, T') = 2k - 5$ for all k, $3 \leq k < n$, and let us prove that $\delta(T, T') = 2n - 5$ whenever T and T' belong to $\mathcal{M}_n$.

Exchanging T and T' according to the parity of n (if needed), we assume that $w_T = \cdots 12$ and $w_{T'} = \cdots 121$. Let $T = T_0, T_1, \ldots, T_k = T'$ be a shortest path in $\mathcal{M}_n$ between T and T'. Let p be the number of rotations in this path that involve

the last internal node in infix order. Lemma 4 induces that $\delta(\phi(T), \phi(T')) \leq \delta(T, T') - p$. By Lemma 2, we use the induction hypothesis and we deduce $\delta(T, T') \geq 2(n-1) - 5 + p = 2n - 7 + p$. Now, let us prove that $p \geq 2$. Indeed, a path in $\mathcal{M}_n$ between $w_T = \cdots 12$ and $w_{T'} = \cdots 121$ necessarily moves the last value $w_T(n) = 2$ of w_T. We distinguish two cases: (i) the first rotation r involving the last node of a tree in the path increases the value $w_T(n)$, and (ii) the first rotation r involving the last node of a tree in the path moves the last value $w_T(n) = 2$ into one. We will prove that case (i) is the only possibility.

In the case (i), it is clear that we need at least one more rotation in order to decrease to one the last value. Thus, we necessarily have $p \geq 2$.

In the case (ii), whenever we decrease $w_T(n) = 2$ to one, it is necessary to have $w_T(n-1) \neq 1$ (otherwise the obtained tree would not be a Motzkin tree). Thus, the path contains a rotation before r that moves the value $w_T(n-1) = 1$. However the only possibility to move it, is that $w_T(n) \neq 2$ (see the characterization of a w-sequence in Proposition 1), which means that $w_T(n) = 2$ must be changed before. This case does not occur since r was the first rotation moving the last value.

Hence, we have $p \geq 2$ and we deduce $\delta(T, T') \geq 2(n-1) - 5 + p \geq 2n - 5$ which completes the induction. □

Theorem 3. *For $n \geq 3$, we have $rad(\mathcal{M}_n) = n - 2$.*

Proof. It is clear that (see for instance [7]), we have the inequality

$$rad(\mathcal{M}_n) \leq \delta(\mathcal{M}_n) \leq 2 \cdot rad(\mathcal{M}_n).$$

Using Theorem 2, we deduce that the radius of $\mathcal{M}_n$ is at least $n - 2$. On the other hand, we consider the Motzkin tree defined in the proof of Theorem 2 with the weight sequence $w_T = 121212\ldots$ Since the distance between a Motzkin tree and the tree $\mathbf{1} = 123\ldots n$ is the Hamming distance of their w-sequences (see Section 3 in [16] for instance), we have $\delta(1212\ldots, 1234\ldots n) = n - 2$ and $n - 2$ is the maximum distance between the tree $\mathbf{1} = 1234\ldots n$ and any Motzkin tree. Thus, the radius of $\mathcal{M}_n$ is at most $n - 2$. □

4. Enumeration results for $(\mathcal{M}_n, \xrightarrow{*})$

In this part, we present several enumeration results for some specific elements of the semilattice $(\mathcal{M}_n, \xrightarrow{*})$. Given $T \in \mathcal{M}_n$, we denote T_L (resp. T_R) its left (resp. right) subtree, *i.e.* $T = \bigcirc T_L T_R$.

Proposition 4. *The generating function for the number of minimal elements in $(\mathcal{M}_n, \xrightarrow{*})$ is given by*

$$\frac{1 + x - x^3}{1 - x^2 - x^3}.$$

For $0 \leq n \leq 12$, the first values are 1, 1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16 (see Padovan sequence A000931 in [23]).

Proof. A minimal element T in $(\mathcal{M}_n, \xrightarrow{*})$ is a Motzkin tree where any right-rotation creates a tree that does not belong in $\mathcal{M}_n$. Given $T = \bigcirc T_L T_R \in \mathcal{M}_n$, then T_L and T_R are necessarily minimal elements. Moreover, the right-rotation involving the root of T necessarily creates a tree that does not lie in $\mathcal{M}_n$. This means that the right subtree of T_L is necessarily a leaf. By induction, T_L does not contain any right subtree not reduced to a leaf. In the case where T_L contains at least three internal nodes, then the right-rotation involving its root creates a Motzkin tree. Therefore, the only two possibilities are either $T_L = \bigcirc \square \square$ or $T_L = \bigcirc \bigcirc \square \square \square$.

Let $A(x)$ be the generating function for the number of minimal elements in $\mathcal{M}_n$ for $n \geq 0$. Then, we have the functional equation $A(x) = x^2 A(x) + x^3(A(x) - 1) + 1 + x$ which gives $A(x) = \frac{1+x-x^3}{1-x^2-x^3}$. □

Recall that $T \in \mathcal{M}_n$ is a join (resp. meet) irreducible element if $T = T_1 \vee T_2$ (resp. $T = T_1 \wedge T_2$) implies $T = T_1$ or $T = T_2$. Since the set $\mathcal{M}_n$ is finite, join (resp. meet) irreducible elements are elements that have a unique lower (resp. upper) cover.

Proposition 5. *For $n \geq 1$, the meet irreducible elements in $(\mathcal{M}_n, \xrightarrow{*})$ are enumerated by the triangular numbers $\frac{(n-2)(n-1)}{2}$.*

For $1 \leq n \leq 12$, the first values are 0, 0, 1, 3, 6, 10, 15, 21, 28, 36, 45, 55 (see A000217 in [23]).

Proof. A meet irreducible element in $(\mathcal{M}_n, \xrightarrow{*})$ is a Motzkin tree T where only a single left-rotation can be performed. Thus, only one internal node can be a right child which means that the weight sequence of T is necessarily of the form $w_T = 12\ldots\ell 12\ldots k(k+\ell+1)\ldots n$ where $k \geq 1$ is the number of internal nodes of the unique subtree T' of T whose root is a right child, and $\ell \geq 2$ is the number of internal nodes of the left subtree of the unique node having an internal node as right child. Finally, the number of such trees is given by $\sum_{k=1}^{n-2} \sum_{\ell=2}^{n-k} 1 = \frac{(n-2)(n-1)}{2}$. □

Proposition 6. *The generating function for the number of join-irreducible elements in $(\mathcal{M}_n, \xrightarrow{*})$ is given by*

$$\frac{x^3(1 + x + 2x^2 + 3x^3 + 3x^4 + x^5)}{(1 - x^2 - x^3)^3}.$$

For $1 \leq n \leq 12$, the first values are 0, 0, 1, 1, 5, 9, 18, 34, 58, 100, 164, 265.

Proof. A join-irreducible element in $(\mathcal{M}_n, \xrightarrow{*})$ is a Motzkin tree T on which only one right-rotation is possible. Let $B(x)$ be the generating function for the number of join irreducible elements in $\mathcal{M}_n$. The Polish notation of the only one join irreducible in $\mathcal{M}_3$ is $\bigcirc\bigcirc\bigcirc\square\square\square\square$. Now we assume $n \geq 4$. Given $T = \bigcirc T_L T_R$ and $A(x)$ be the generating function for the number of minimal Motzkin trees (see Proposition 4).

Case 1: if T_R is a leaf then the right-rotation involving the root provides a Motzkin tree, which implies that T_L is necessarily a minimal element of weight at least four. Thus, the corresponding generating function is $x(A(x) - 1 - x - x^2)$.

Case 2: if T_R is a minimal Motzkin tree of weight at least two, then T_L is either (i) a minimal tree of weight at least four, or (ii) a join irreducible element whose right subtree is a leaf. Indeed, in sub-case (i), the unique possible right-rotation is the one at the root of T. The generating function for the case (i) is $x(A(x) - 1)(A(x) - 1 - x - x^2)$. For the case (ii), the unique right-rotation is the one that can be performed in T_L. So, a join irreducible element with a leaf as right subtree has necessarily a minimal left subtree. Thus, the number of Motzkin trees satisfying (ii) is given by the generating function $x^2(A(x) - 1)(A(x) - 1 - x)$.

Case 3: if T_R is a join irreducible Motzkin tree of weight at least two, then T_L is either $\bigcirc\square\square$ or $\bigcirc\bigcirc\square\square\square$. Indeed, if T_L is minimal then the right subtree of T_L must be a leaf (otherwise the right-rotation at the root of T would transform T into a Motzkin tree). So, the corresponding generating function is $x^2B(x) + x^3B(x)$. Finally we have the following functional equation that gives the result:

$$\begin{aligned} B(x) &= x^3 + x(A(x) - 1 - x - x^2) + x(A(x) - 1)(A(x) - 1 - x - x^2) \\ &\quad + x^2(A(x) - 1)(A(x) - 1 - x) + x^2B(x) + x^3B(x), \end{aligned}$$

where $A(x)$ is given in Proposition 4. $\square$

Proposition 7. *The generating function for the number of coverings in $(\mathcal{M}_n, \xrightarrow{*})$ is given by*

$$\frac{(1-x)(1-2x-x^2-(1-x)\sqrt{1-2x-3x^2})}{2x(1-2x-x^2)}.$$

For $1 \leq n \leq 12$, the first values are 0, 0, 1, 3, 10, 30, 88, 252, 712, 1992, 5537, 15 323.

Proof. Let c_n be the number of coverings in $\mathcal{M}_n$, $C(x)$ be the associated generating function and $M(x) = \sum_{i\geq 0} M_i x^i$ be the generating function for the number of Motzkin trees.

Since we have $c_0 = c_1 = c_2 = 0$, we assume $n \geq 3$. Given $T = \bigcirc T_L T_R$, we distinguish three cases.

Case 1: T_R is a leaf. Thus, there are c_{n-1} possible left-rotations in T_L. The corresponding generating function is $xC(x)$.

Case 2: if $T_R = \bigcirc\square\square$. There are M_{n-2} left-rotations involving the root of T and c_{n-2} possible rotations in T_L. The corresponding generating function is $x^2M(x) + x^2C(x)$.

Case 3: if the weight of T_R is at least three. Let i, $1 \leq i \leq n-3$, be the number of internal nodes of T_L. There are $c_i + c_{n-i-1}$ left-rotations in the two subtrees T_L and T_R, and M_iM_{n-i-1} left-rotations involving the root of T. Varying i from 1 to $n-3$, there are $\sum_{i=1}^{n-3}(c_i + c_{n-i-1}) + \sum_{i=1}^{n-3} M_iM_{n-i-1}$ possible left-rotations. Since $\sum_{i=1}^{n-3} M_iM_{n-i-1} = M_n - M_{n-1} - M_{n-2}$ (see [25] for instance), the corresponding generating function is $\frac{x^2C(x)}{1-x} + \frac{x^3C(x)}{1-x} + M(x) - xM(x) - x^2M(x) - x$.

Considering all cases, we have the following functional equation and the result is deduced:

$$C(x) = xC(x) + x^2C(x) + x^2\frac{C(x)}{1-x} + x^3\frac{C(x)}{1-x} + M(x) - xM(x) - x,$$

where $M(x) = \frac{1-x-\sqrt{1-2x-3x^2}}{2x}$. $\square$

5. A ranked Motzkin poset included in $(\mathcal{M}_n, \xrightarrow{*})$

Let $\mathcal{W}$ be the set of Motzkin words, *i.e.* the language over $\{(,)\}$ defined by the grammar $S \rightsquigarrow \lambda|(SS)$, and $\mathcal{W}_n$ be the set of Motzkin words of length $2n$ (with n open and n close parentheses). From a Motzkin word in $\mathcal{W}_n$ we can associate a binary tree in $\mathcal{T}_n$ where its Polish notation is obtained by replacing each open (resp. close) parenthesis with $\bigcirc$ (resp. $\square$), and by adding $\square$ at the end. For instance, the Polish notation of the associated tree of the Motzkin word $(()(()))$ is $\bigcirc\bigcirc\square\bigcirc\bigcirc\square\square\square\square$.

Let $\mathcal{MW}_n$ be the set of binary trees in $\mathcal{T}_n$ associated to the Motzkin words belonging to $\mathcal{W}_n$. Since a Motzkin word is obtained from the rule $S \rightsquigarrow \lambda|(SS)$, the Polish notation of its associated tree is either of the form (i) $\bigcirc T_L\square$ or (ii) $T = \bigcirc T_L T_R$ where T_L and T_R lie in some sets $\mathcal{MW}_k$ for $k < n$, and such that T_R satisfies (i). Actually, the set $\mathcal{MW}_n$ consists of the mirrors of binary trees whose Polish notation has no three consecutive internal nodes.

In [2], we investigate the rotation transformation $\rightarrow$ on the set $\mathcal{MW}_n$. We have proved that $(\mathcal{MW}_n, \xrightarrow{*})$ is a ranked join-semilattice.

In this part, we construct an isomorphism between $(\mathcal{MW}_{n-1}, \xrightarrow{*})$ and $(\mathcal{M}_n, \overset{*}{\rightarrowtail})$ where $\rightarrowtail$ is the restricted left-rotation defined by

$$\bigcirc T_1 \bigcirc T_2 \square \rightarrowtail \bigcirc\bigcirc T_1 T_2 \square$$

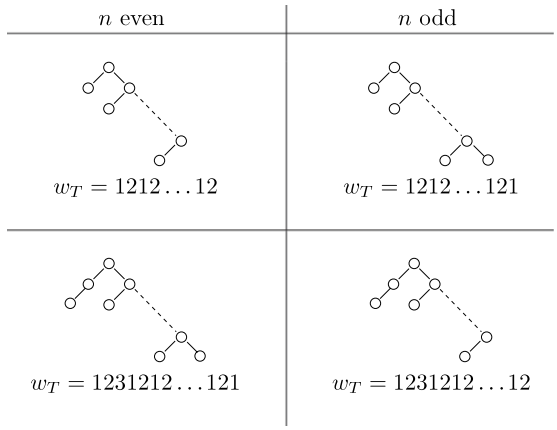


Fig. 5. The Motzkin trees T and T' in the proof of Theorem 2.

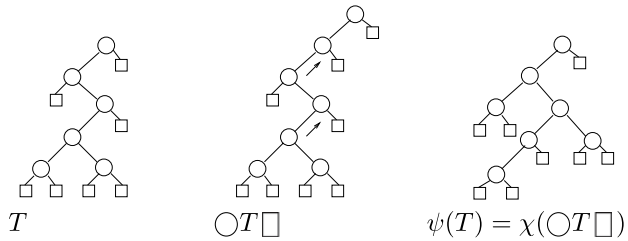


Fig. 6. The bijection ψ.

where T_1, T_2 are the Polish notations of some subtrees. Notice that in [4], the authors study on binary trees an analogous restricted rotation defined by $\bigcirc\,\square\,\bigcirc\,T_2T_3 \rightarrowtail \bigcirc\,\bigcirc\,\square\,T_2T_3$.

Let ψ be the map from $\mathcal{MW}_{n-1}$ to $\mathcal{T}_n$ defined by the following recursive rule. For $T = \bigcirc T_L T_R \in \mathcal{MW}_{n-1}$, we define

$$\psi(T) = \chi(\bigcirc T\,\square),$$

where $\chi(\bigcirc T\,\square)$ is recursively defined by

$$\chi(\bigcirc\,\bigcirc\,T_L T_R\,\square) = \bigcirc\chi(\bigcirc T_L\,\square)\chi(T_R),$$

anchored with $\chi(\square) = \square$ and $\chi(\bigcirc\,\square\,\square) = \bigcirc\,\square\,\square$.

Less formally, $\psi(T)$ is obtained from $\bigcirc T\,\square$ by performing the following process: for all nodes x and y such that y is the left child of x, the right subtree of y is moved into the right subtree of x.

For example, if $T = \bigcirc\bigcirc\,\square\,\bigcirc\bigcirc\bigcirc\,\square\,\square\,\bigcirc\,\square\,\square\,\square\,\square$ then $\psi(T) = \chi(\bigcirc T\,\square) = \bigcirc\bigcirc\bigcirc\,\square\,\square\,\bigcirc\bigcirc\bigcirc\,\square\,\square\,\square\,\bigcirc\,\square\,\square\,\square$ (see Fig. 5).

Lemma 5. *The map ψ is a bijection from $\mathcal{MW}_{n-1}$ to $\mathcal{M}_n$.*

Proof. By the recursive definition of ψ, it is straightforward to see that $\psi(T)$ does not contain any pattern $\bigcirc\,\square\,\bigcirc$. Thus, we have $\psi(\mathcal{MW}_{n-1}) \subseteq \mathcal{M}_n$. Moreover, from the recursive definition of χ, ψ is necessarily injective. Indeed, for any T, T' such that $T = \bigcirc T_L T_R$, $T' = \bigcirc T'_L T'_R$ and $\psi(T) = \psi(T')$, we have $\bigcirc\chi(\bigcirc T_L\,\square)\chi(T_R) = \bigcirc\chi(\bigcirc T'_L\,\square)\chi(T'_R)$. Using the induction hypothesis, we obtain $T'_L = T_L$, $T'_R = T_R$ and thus, $T = T'$. The two sets $\mathcal{MW}_{n-1}$ and $\mathcal{M}_n$ being enumerated by the Motzkin numbers (see [5]), we deduce that ψ is a bijection from $\mathcal{MW}_{n-1}$ to $\mathcal{M}_n$. Notice that the bijections described in [5] induce a different isomorphism between $\mathcal{M}_n$ and the set $\mathcal{MW}_{n-1}$. $\square$

Theorem 4. *The two join-semilattices $(\mathcal{MW}_{n-1}, \overset{*}{\longrightarrow})$ and $(\mathcal{M}_n, \overset{*}{\rightarrowtail})$ are isomorphic.*

Proof. According to Lemma 5, it suffices to prove that the map ψ transports the rotation transformation $\longrightarrow$ between two trees in $\mathcal{MW}_{n-1}$ onto the restricted rotation $\rightarrowtail$ in $\mathcal{M}_n$, and *vice versa*. Let $T, T' \in \mathcal{MW}_{n-1}$ be so that T' is obtained from T by a left-rotation. It is worth noticing that a left-rotation between two trees of $\mathcal{MW}_{n-1}$ is a restricted rotation between these two trees. Since the rotation transformation is a local transformation, we will consider T and T' near the node involved by the rotation. Therefore, we give arguments using $T = \cdots\bigcirc\,\bigcirc A\,\bigcirc\,\bigcirc B\,\square\,\square C\ldots$ and $T' = \cdots\bigcirc\,\bigcirc\,\bigcirc\,A\,\bigcirc\,B\,\square\,\square C\ldots$ (see the trees on the top of Fig. 6).

We have $\psi(T) = \cdots\bigcirc\,\chi(\bigcirc A\,\square)\,\bigcirc\,\chi(\bigcirc B\,\square)\,\square\ldots$ and $\psi(T') = \cdots\bigcirc\,\bigcirc\chi(\bigcirc A\,\square)\chi(\bigcirc B\,\square)\,\square\ldots$. Setting $A' = \chi(\bigcirc A\,\square)$ and $B' = \chi(\bigcirc B\,\square)$, we recognize the restricted rotation $\ldots\bigcirc\,A'\,\bigcirc\,B'\,\square\ldots \rightarrowtail \ldots\bigcirc\,\bigcirc A'B'\,\square\ldots$ (see Fig. 6 for an illustration of this proof). This argument still remains available for the converse *mutatis mutandis*. Finally, the map ψ transports the rotation transformation $T \longrightarrow T'$ where $T, T' \in \mathcal{MW}_{n-1}$ into the restricted rotation $\psi(T) \rightarrowtail \psi(T')$ where $\psi(T)$ and $\psi(T')$ belong to $\mathcal{M}_n$ (see Fig. 7). $\square$

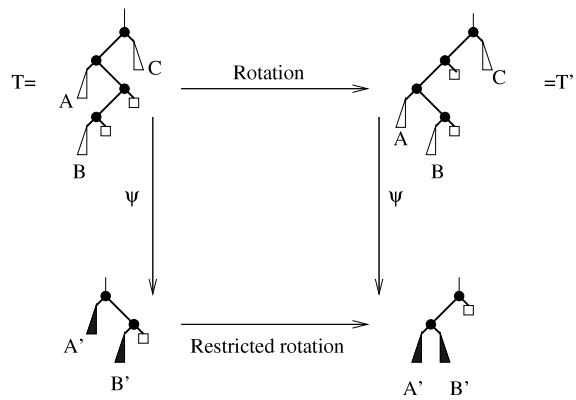


Fig. 7. An illustration for the proof of Theorem 4.

The image of the semilattice $(\mathcal{MW}_5, \xrightarrow{*})$ by the map ψ can be viewed in Fig. 3 by not taking into account the four rotations labeled (a), (b), (c) and (d).

In [2], the authors compute the length $\rho(T, T')$ of a shortest path between T and T' in $\mathcal{MW}_{n-1}$. The following corollary provides an upper bound for the rotation distance d in $\mathcal{T}_n$ (and also in $\mathcal{M}_n$).

Corollary 1. *Given T and T' in $\mathcal{M}_n$, we have*

$$d(T, T') \leq \delta(T, T') \leq \rho(\psi^{-1}(T), \psi^{-1}(T')).$$

Since computing the rotation distance d in $\mathcal{T}_n$ is a difficult problem, our upper bounds are valuable, especially because they are sometimes sharp and because ρ can be computed easily. Indeed, the bounds give the exact value of the classical distance rotation d for some specific pairs of Motzkin trees. For example, if n is even, $n \geq 4$, then we define T and T' by their w-sequences $w_T = 121212\ldots121n$ and $w_{T'} = 121212\ldots12$. If n is odd, $n \geq 3$, then T and T' are defined by $w_T = 121212\ldots12n$ and $w_{T'} = 121212\ldots121$. A simple calculation proves that $d(T, T') = \lfloor \frac{n-1}{2} \rfloor = \delta(T, T') = \rho(\psi^{-1}(T), \psi^{-1}(T'))$.

6. Others research directions

Motzkin trees are in bijection with trees where internal nodes have one or two children. How the rotation operation can be described on these trees?

Motzkin trees can be defined as binary trees whose Polish notation avoids the pattern ◯ □ ◯ (or equivalently, a certain binary tree pattern). Is there a criterion to decide, for a given set of patterns P, if the set $\mathcal{M}_n^P$ of binary trees avoiding P form a subposet (resp. a sublattice, a join-semilattice, a meet-semilattice) of the Tamari lattice of order n?

Recently, some studies have focused on m-Tamari lattices which generalize the classical Tamari lattices for trees where internal nodes are of arity $m + 1$ (see the survey paper of Bergeron [3] for any m, and [19] for ternary trees). Is there a generalization of Motzkin trees and the results of this paper for trees where internal nodes are of arity $m + 1$?

Acknowledgments

We would like to thank the anonymous referees for their very careful reading of this paper and their helpful comments and suggestions.

References

[1] J.L. Baril, J.M. Pallo, Efficient lower and upper bounds of the diagonal-flip distance between triangulations, Inform. Process. Lett. 100 (2006) 131–136.
[2] J.L. Baril, J.M. Pallo, Motzkin subposets and Motzkin geodesics in Tamari lattices, Inform. Process. Lett. 114 (2014) 31–37.
[3] F. Bergeron, Combinatorics of r-Dyck paths, r-Parking functions, and the r-Tamari lattices, arXiv:math/1202.6269v4.
[4] A. Bonnin, J.-M. Pallo, A shortest path metric on unlabeled binary trees, Pattern Recognit. Lett. 13 (1992) 411–415.
[5] D. Callan, Bijections from restricted Dyck paths to Motzkin paths, 2004. arXiv:math/0406381v1.
[6] B.D. Davey, H.A. Priestley, Introduction to Lattices and Order, second ed., Cambridge U. Press, 2002.
[7] R. Diestel, Graph Theory, third ed., Springer, New York, 2005.
[8] R. Donaghey, L.W. Shapiro, Motzkin numbers, J. Combin. Theory Ser. A 23 (1977) 291–301.
[9] H. Friedman, D. Tamari, Problèmes d'associativité: une structure de treillis fini induite par une loi de demi-associative, J. Combin. Theory 2 (1967) 215–242.
[10] S. Giraudo, Intervals of balanced binary trees in the Tamari lattice, Theoret. Comput. Sci. 420 (2012) 1–27.
[11] G. Grätzer, General Lattice Theory, second ed., Birkhäuser, 1998.
[12] S. Hanke, T. Ottmann, S.J. Schuierer, The edge-flipping distance of triangulations, J.UCS 2 (1996) 570–579.

[13] S. Huang, D. Tamari, Problems of associativity: a simple proof for the lattice property of systems ordered by a semi-associative law, J. Combin. Theory Ser. A 13 (1972) 7–13.
[14] F. Müller-Hoissen, J.M. Pallo, J. Stasheff (Eds.), Associahedra, Tamari Lattices and Related Structures, in: Progress in Mathematics, vol. 299, Birkhäuser, 2012.
[15] J.M. Pallo, Enumerating, ranking and unranking binary trees, Comput. J. 29 (1986) 171–175.
[16] J.M. Pallo, On the rotation distance in the lattice of binary trees, Inform. Process. Lett. 25 (1987) 369–373.
[17] J.M. Pallo, Some properties of the rotation lattice of binary trees, Comput. J. 31 (1988) 564–565.
[18] J.M. Pallo, An efficient upper bound of the rotation distance of binary trees, Inform. Process. Lett. 73 (2000) 87–92.
[19] J.M. Pallo, The rotation χ-lattice of ternary trees, Computing 66 (2001) 297–308.
[20] L. Pournin, The diameter of associahedra, Adv. Math. 259 (2014) 13–42.
[21] D. Roelants, van Baronaigien and F. Ruskey: a Hamilton path in the rotation lattice of binary trees, Congr. Numer. 59 (1987) 313–318.
[22] D.D. Sleator, R.E. Tarjan, W.P. Thurston, Rotation distance, triangulations and hyperbolic geometry, J. Amer. Math. Soc. 1 (1988) 647–681.
[23] N.J.A. Sloane, The On-line Encyclopedia of Integer Sequences. Available electronically at http://oeis.org.
[24] R.P. Stanley, Enumerative Combinatorics, Vol. 2, Cambridge University Press, 1999.
[25] P.R. Stein, M.S. Waterman, On some new sequences generalizing the Catalan and Motzkin numbers, Discrete Math. 26 (1978) 261–272.
[26] Y. Sun, Statistic number of *udu*'s in Dyck paths, Discrete Math. 287 (1–3) (2004) 177–186.
[27] D. Tamari, Monoïdes préordonnés et chaînes de Malcev, Doctorat ès Sciences Mathématiques, Paris, 1951.
[28] D. Tamari, The algebra of bracketings and their enumeration, Nieuw Arch. Wiskd. 10 (1962) 131–146.

DE GRUYTER OPEN

PU. M. A. Vol. **25** (2015), No.1, pp. 19 – 29

Equivalence classes of permutations modulo descents and left-to-right maxima

Jean-Luc Baril
Université de Bourgogne, B.P. 47 870
LE2I UMR-CNRS 6306
21078 Dijon, France
e-mail: barjl@u-bourgogne.fr

and
Armen Petrossian
Université de Bourgogne, B.P. 47 870
LE2I UMR-CNRS 6306
21078 Dijon, France
email: armen.petrossian@u-bourgogne.fr

(Received: July 8, 2014, and in revised form March 23, 2015)

Abstract. In a recent paper [2], the authors provide enumerating results for equivalence classes of permutations modulo excedances. In this paper we investigate two other equivalence relations based on descents and left-to-right maxima. Enumerating results are presented for permutations, involutions, derangements, cycles and permutations avoiding one pattern of length three.

Mathematics Subject Classification(2010). 05A05.

Keywords: permutation, equivalence class, descent, left-to-right maximum, pattern, Bell number, Motzkin number, Catalan number, Fine number.

1 Introduction and notations

Let S_n be the set of length-n permutations, *i.e.*, all one-to-one correspondences from $[n] = \{1, 2, \ldots, n\}$ into itself. The *one-line notation* of a permutation $\pi \in S_n$ is $\pi_1\pi_2\cdots\pi_n$ where $\pi_i = \pi(i)$ for $i \in [n]$. The *graphical representation* of $\pi \in S_n$ is the set of points in the plane at coordinates (i, π_i) for $i \in [n]$. A *cycle* in S_n is an n-length permutation π such that there exist some indices $i_1, i_2, \ldots, i_n$ with $\pi(i_1) = i_2, \pi(i_2) = i_3, \ldots, \pi(i_{n-1}) = i_n$ and $\pi(i_n) = i_1$. A cycle will also be denoted by its *cyclic notation* $\pi = \langle i_1, i_2, \ldots, i_n\rangle$. Let $C_n \subset S_n$ be the set of all cycles of length n. We denote by I_n the set of involutions of length n, *i.e.*, permutations π such that $\pi^2 = Id$ where Id is the identity permutation.

Let π be a permutation in S_n. A *fixed point* of π is a position $i \in [n]$ where $\pi(i) = i$. The set of n-length permutations with no fixed points (called *derangements*) will be denoted D_n. An *excedance* of π is a position $i \in [n-1]$, such that $\pi(i) > i$. The set of excedances of π will be denoted $E(\pi)$. A *descent* of π is a position $i \in [n-1]$, such that $\pi(i) > \pi(i+1)$. Let $D(\pi)$ be the set of descents in π, and $DD(\pi)$ be the set of pairs $(\pi(i), \pi(i+1))$ for $i \in D(\pi)$. By abuse of language, we also use the

DOI: 10.1515/puma-2015-0002

term *descent* for such a pair. A *left-to-right maximum* is a position $i \in [n]$, such that $\pi(i) > \pi(j)$ for all $j < i$. The set of left-to-right maxima of π will be denoted $L(\pi)$. For instance, if $\pi = 1\ 4\ 2\ 7\ 5\ 3\ 8\ 6$ then $E(\pi) = \{2, 4, 7\}$, $D(\pi) = \{2, 4, 5, 7\}$, $DD(\pi) = \{(4, 2), (7, 5), (5, 3), (8, 6)\}$ and $L(\pi) = \{1, 2, 4, 7\}$.

In [2], the authors consider the equivalence relation on S_n in which two permutations π and σ are equivalent if they coincide on their excedance sets, *i.e.*, $E(\pi) = E(\sigma)$ and $\pi(i) = \sigma(i)$ for $i \in E(\pi)$. In this paper we investigate the counterpart of this equivalence relation for descents and left-to-right maxima. More precisely, we define the ℓ-equivalence relation $\sim_\ell$ where $\pi \sim_\ell \sigma$ if and only if π and σ coincide on their left-to-right maximum sets, *i.e.*, $L(\pi) = L(\sigma)$ and $\pi(i) = \sigma(i)$ for $i \in L(\pi)$. Also, we define the d-equivalence relation $\sim_d$ where two permutations π and σ are equivalent if $DD(\pi) = DD(\sigma)$. The motivation for studying this d-equivalence relation is that two permutations π and σ are equivalent under excedance ([2]) if and only if $\phi(\pi)$ and $\phi(\sigma)$ are d-equivalent, where ϕ is the Foata's first transformation [5] (see Theorem 6). All these definitions remain available for subsets of S_n. For instance, the permutation $32541 \in S_5$ is ℓ-equivalent to $32514, 31524, 31542$ and d-equivalent to 54132. The set of ℓ-equivalence (resp. d-equivalence) classes in S_n is denoted $S_n^{\sim_\ell}$ (resp. $S_n^{\sim_d}$).

In this paper we propose to compute the number of ℓ- and d-equivalence classes for several subsets of permutations.

A permutation $\pi \in S_n$ *avoids* the pattern $\tau \in S_k$ if and only if there does not exist any sequence of indices $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ such that $\pi(i_1)\pi(i_2)\ldots\pi(i_k)$ is order-isomorphic to τ (see [13, 14]). We denote by $S_n(\tau)$ the set of permutations of S_n avoiding the pattern τ. For example, if $\tau = 123$ then $52143 \in S_5(\tau)$ while $21534 \notin S_5(\tau)$. Many classical sequences in combinatorics appear as the cardinality of pattern-avoiding permutation classes. A large number of these results were firstly obtained by West and Knuth [8, 12, 13, 14, 15, 16] (see books of Kitaev [7] and Mansour [11]).

In Section 2, we investigate the equivalence relation based on the set of left-to-right maxima. We enumerate ℓ-equivalence classes for S_n, C_n, I_n, D_n and several sets of pattern avoiding permutations. In Section 3, we study equivalence relation for descents and also provide enumerating results for some restricted sets of permutations. See Table 1, 2 and 3 for an overview of these results.

2 Enumeration of classes under ℓ-equivalence relation

Throughout this section two permutations π and σ belong to a same class whenever they coincide on their sets of left-to-right maxima, *i.e.*, $L(\pi) = L(\sigma)$ and $\pi(i) = \sigma(i)$ for $i \in L(\pi)$.

A *Dyck path* of semilength n, $n \geq 0$, will be a lattice path starting at $(0, 0)$, ending at $(2n, 0)$, and never going below the x-axis, consisting of up steps $U = (1, 1)$ and down steps $D = (1, -1)$. Let $\mathcal{P}_n$ be the set of all Dyck paths of semilength n. A *peak* of *height* $h \geq 0$ in a Dyck path is a point of ordinate h which is both at the end of an up step and at the beginning of a down step.

From a permutation $\pi \in S_n$, we consider the path on the graphical representation of π with up and right steps along the edges of the squares that goes from the lower-left corner to the upper-right corner and leaving all the points (i, π_i), $i \in [n]$, to the right and remaining always as close to the diagonal $y = x$ as possible (the path can possibly reach the diagonal but never crosses it). Let us define the Dyck path of length $2n$ (called *Dyck path associated with* π) obtained from this lattice path by reading an up-step U every time the path moves up, and a down-step D every time the path moves to the right. It is crucial to notice that only the points (i, π_i) with $i \in L(\pi)$ involve in this construction. See Figure 1 for an illustration of this classical construction.

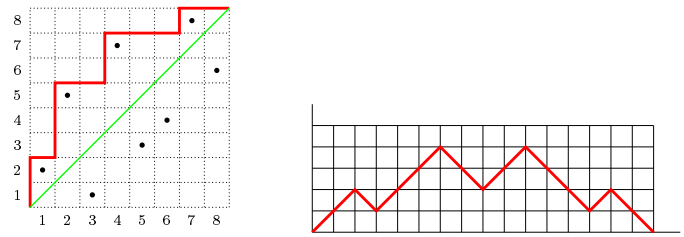


Figure 1: Permutation $\sigma = 2\ 5\ 1\ 7\ 3\ 4\ 8\ 6$.

Using this construction, all permutations of a same class provide the same Dyck path. Moreover, any Dyck path in $\mathcal{P}_n$ can be obtained from a permutation in S_n. Indeed, we define the sequence $\ell = \ell_1\ell_2\dots\ell_r$, (resp. $k = k_1k_2\dots k_r$), $r \geq 1$ where ℓ_i (resp. k_i) is the number (resp. the number plus one) of up steps U (resp. down steps D) before the i-th peak. Since P is a Dyck path, we have $k_i \leq \ell_i$ for $i \leq r$. So, we define the permutation $\pi = \ell_1A_1\ell_2A_2\dots\ell_rA_r$ where each ℓ_i, $i \leq r$, is at position k_i, and such that the concatenated block $A_1A_2\dots A_r$ consists of the increasing sequence of values in $[n]\backslash\{\ell_1, \ell_2, \dots, \ell_r\}$. Therefore, ℓ_{i+1} is greater than all elements in A_i which means that the set of left-to-right maxima of π is $L(\pi) = \{k_1, k_2, \dots, k_r\}$ with $\pi(k_i) = \ell_i$ for $i \leq r$. By construction, π avoids the pattern 321 and P is its associated Dyck path which gives a bijection from $S_n^{\sim\ell}$ to $\mathcal{P}_n$ that induces Theorem 1. In the following, the permutation π will be called the *associated permutation* of P. Notice that a similar construction already exists in the literature (see the Krattenthaler bijection Ψ defined in [9], Section 4).

THEOREM 2.1 *The sets $S_n^{\sim\ell}$ (resp. $S_n(321)^{\sim\ell}$), $n \geq 1$, are enumerated by the Catalan numbers (sequence A000108 in the on-line Encyclopedia of Integer Sequences [18]).*

2.1 Equivalence classes for classical subsets of permutations

In this part we give several enumerating results for classical subsets of S_n (see Table 1).

THEOREM 2.2 *The sets $D_n^{\sim\ell}$, $n \geq 1$, are enumerated by the Fine numbers (A000108 in [18]).*

Proof. Using the above construction, we construct a Dyck path P of length $2n$ from $\pi \in D_n$. Since π does not contain any fixed point, P does not contain any peak of height one. Conversely, let P be a Dyck path with no peak of height one and $\pi \in S_n(321)$ be its associated permutation of P (see the above construction). Since P does not contain any peak of height one, this implies that there does not exist $i \in L(\pi)$ such that $\pi_i = i$. Now, for a contradiction, let us assume that there is $j \notin L(\pi)$ such that $\pi_j = j$. Since $j \notin L(\pi)$, there is $i < j$ such that $\pi_i > j = \pi_j$. So, there are at most $j - 2$ values $\pi_k < \pi_j$ for $k < j$, or equivalently there is at least one value $\pi_k < \pi_j$ for $k > j$ which contradicts the fact that π avoids the pattern 321. Finally, the result follows because the set of Dyck paths with no peak of height one is enumerated by the Fine numbers (see [4]). □

THEOREM 2.3 *Let Irr_n be the set of permutations $\pi \in S_n$ such that $\pi_i \neq \pi_{i+1} - 1$, $1 \leq i \leq n-1$. The sets $Irr_n^{\sim\ell}$, $n \geq 1$, are enumerated by the sequence A078481 in [18].*

Proof. Let π be a permutation in Irr_n. Then the Dyck path associated with π does not contain any consecutive steps of the form $UDUD$. Conversely, let P be a Dyck path which does not contain any occurence of consecutive $UDUD$, and $\pi \in S_n(321)$ be its associated permutation. Since P does not contain any occurrence of $UDUD$, this implies that there does not exist $i \in L(\pi)$ such that $\pi_i = \pi_{i+1}-1$. In the case where there is j, $j \notin L(\pi)$ such that $\pi_j = \pi_{j+1} - 1$, we define the blocks of maximal length $J_1, J_2, \ldots, J_s$ of the form $a, a+1, \ldots, b$ such that $a \leq b$ and $\pi_a = \pi_{a+1} - 1$, $\pi_{a+1} = \pi_{a+2} - 1$, $\ldots$, $\pi_{b-1} = \pi_b - 1$ where $a \notin L(\pi)$. We consider the permutation σ obtained from π by the following process: for each block $J_k = a, a+1, \ldots, b$, $1 \leq k \leq s$, in the one-line notation of π we replace the block $\pi(J_k)$ with its mirror $\pi(J'_k)$ where $J'_k = b, b-1, \ldots, a$. So, σ and π belong to the same class, and $\sigma \in Irr_n$. For instance, the Dyck path $UUUUDDDD$ would produce $\pi = 4123 \notin Irr_4$, then applying the described process, the permutation $\sigma = 4321 \in Irr_4$ is obtained. Finally, the result is obtained since the set of Dyck paths with no occurrence of $UDUD$ is enumerated by the sequence A078481 in [18] (see [17]). □

THEOREM 2.4 *The sets $I_n^{\sim_\ell}$, $n \geq 1$, are enumerated by the Motzkin numbers (A001006 in [18]).*

Proof. We will show that each equivalence class contains a unique involution that avoids the pattern 4321 (see A001006 in [18] and [6] for the enumeration of $I_n(4321)$ by Motzkin numbers). Let π be an involution in I_n. If there exists a position i, $i < \pi_i$, such that i is not a left-to-right maximum, then there is $j \in [n]$ such that $j < i < \pi_i < \pi_j$ which means that π contains the pattern 4321. So, we define the involution σ satisfying $L(\pi) = L(\sigma)$ and verifying the additional conditions $\sigma_i = i$ whenever $i \notin L(\pi)$. By construction, σ avoids the pattern 4321 and belongs to the same class of π. Conversely, let σ be an involution avoiding the pattern 4321. Then, the inequality $j < i < \sigma_i < \sigma_j$, $i, j \in [n]$ does not occur. Therefore, if $i \notin L(\sigma)$, then i is necessarily a fixed point. Therefore, there is a unique involution $\sigma \in I_n(4321)$ having $L(\sigma)$ as set of left-to-right maxima. □

THEOREM 2.5 *The sets $C_n^{\sim_\ell}$, $n \geq 1$, are enumerated by the Catalan numbers (A000108 in [18]).*

Proof. Any permutation $\pi \in S_{n-1}$ can uniquely be decomposed as a product of transpositions

$$\pi = \langle p_1, 1\rangle \cdot \langle p_2, 2\rangle \cdots \langle p_{n-1}, n-1\rangle$$

where p_i are some integers such that $1 \leq p_i \leq i \leq n-1$ (see for instance [1]).

Let ϕ be the map from S_{n-1} to S_n defined, for every $\pi \in S_{n-1}$, by

$$\phi(\pi) = \langle 1, 1\rangle \cdot \langle p_1, 2\rangle \cdots \langle p_{n-1}, n\rangle$$

where $\pi = \langle p_1, 1\rangle \cdot \langle p_2, 2\rangle \cdots \langle p_{n-1}, n-1\rangle$.

Using Corollary 1 in [1], ϕ is a bijection from S_{n-1} to C_n satisfying $L(\pi) = L(\phi(\pi))$ for any $\pi \in S_n$ and such that $\phi(\pi)(k) = \pi(k) + 1$ for $k \in L(\pi)$. Therefore, ϕ induces a bijection from $S_{n-1}^{\sim_\ell}$ to $C_n^{\sim_\ell}$. With Theorem 2.1, the cardinality of $C_n^{\sim_\ell}$ is the $(n-1)$-th Catalan number. □

2.2 Equivalence classes for $S_n(\alpha)^{\sim_\ell}$ with $\alpha \in S_3$

In this part we give several enumerating results for the sets $S_n(\alpha)^{\sim_\ell}$ where the pattern α lies in S_3 (see Table 2).

Set	Sequence	Sloane	$a_n, 1 \leq n \leq 9$
$S_n^{\sim_\ell}$	Catalan	$A000108$	$1, 2, 5, 14, 42, 132, 429, 1430, 4862$
$C_n^{\sim_\ell}$	Catalan	$A000108$	$1, 1, 2, 5, 14, 42, 132, 429, 1430$
$I_n^{\sim_\ell}$	Motzkin	$A001006$	$1, 2, 4, 9, 21, 51, 127, 323, 835$
$D_n^{\sim_\ell}$	Fine	$A000957$	$0, 1, 2, 6, 18, 57, 186, 622, 2120$
$Irr_n^{\sim_\ell}$	Dyck with no UDUD	$A078481$	$1, 1, 3, 7, 19, 53, 153, 453, 1367$

Table 1: Number of equivalence classes for classical subsets of permutations.

Theorem 1 proves that $S_n(321)^{\sim_\ell}$ is enumerated by the nth Catalan number. Let ϕ be the bijection from $S_n(321)$ to $S_n(312)$ described (modulo a basic symmetry) in [3] (Lemma 4.3, page 148). It has the property to leave all left-to-right maxima fixed. Therefore, it induces a bijection from $S_n(321)^{\sim_\ell}$ to $S_n(312)^{\sim_\ell}$.

Now, let us examine the cases where the pattern α belongs to $\{123, 132, 213, 231\}$.

THEOREM 2.6 *The sets $S_n(123)^{\sim_\ell}$, $n \geq 1$, are enumerated by the central polygonal numbers $1 + \frac{n(n-1)}{2}$ (A000124 in [18]).*

Proof. Let π be a permutation in $S_n(123)$. It is straightforward to see that the left-to-right maxima of π are 1 and i where $\pi_i = n$ for some i, $1 \leq i \leq n$. We necessarily have $i - 1 \leq \pi_1$ because the condition $\pi_1 < \pi_j < n$ implies $j < i$. Since the values i and j, $1 \leq i, j \leq n$, characterize a class in $S_n(123)^{\sim_\ell}$, it follows that the cardinality of $S_n(123)^{\sim_\ell}$ is given by $1 + \sum_{i=2}^{n} \sum_{j=i-1}^{n-1} 1 = 1 + \frac{n(n-1)}{2}$. □

THEOREM 2.7 *For $\alpha \in \{132, 213, 231\}$, the sets $S_n(\alpha)^{\sim_\ell}$, $n \geq 1$, are enumerated by the binary numbers 2^{n-1}.*

Proof. Let π be a permutation in $S_n(231)$. It can be written $\pi = \sigma n \gamma$ where $\sigma \in S_k(231)$ for some k, $0 \leq k \leq n-1$, and γ is obtained from a permutation in $S_{n-k-1}(231)$ by adding k on all these entries. Therefore, the set $L(\pi)$ of left-to-right maxima of π is the union of $\{k+1\}$ with the set $L(\sigma)$ of left-to-right maxima of σ. For $n \geq 1$, let a_n be the cardinality of $S_n(231)^{\sim_\ell}$. Varying k from 0 to $n-1$, we have $a_n = 1 + \sum_{k=1}^{n-1} a_k$ anchored with $a_1 = 1$. Thus, we deduce $a_n = 2^{n-1}$ for $n \geq 1$.

Basic symmetries on permutations allow to obtain the result whenever α lies in $\{132, 312\}$. □

3 Enumeration of classes under d-equivalence relation

In this section two permutations π and σ belong to a same class whenever $DD(\pi) = DD(\sigma)$, *i.e.*, if the set of pairs (π_i, π_{i+1}) for $i \in D(\pi)$ is equal to the set of pairs (σ_i, σ_{i+1}) for $i \in D(\sigma)$.

A *partition* Π of $[n]$ is any collection of non-empty pairwise disjoint subsets, called *blocks*, whose union is $[n]$. The *standard form* of Π is $\Pi = B_1/B_2/\ldots$, where the blocks B_i are arranged so that their smallest elements are in increasing order. For convenience, we assume also that elements in a

Pattern	Sequence	Sloane	$a_n, 1 \leq n \leq 9$
$\{123\}$	Central polygonal	$A000124$	$1, 2, 4, 7, 11, 16, 22, 29, 37$
$\{312\}, \{321\}$	Catalan	$A000108$	$1, 2, 5, 14, 42, 132, 429, 1430, 4862$
$\{132\}, \{213\}, \{231\}$	Binary	$A000079$	$1, 2, 4, 8, 16, 32, 64, 128, 256$

Table 2: Number of equivalence classes for permutations avoiding one pattern in $\mathcal{S}_3$.

same block are arranged in decreasing order. From a permutation $\pi \in S_n$, we associate the unique partition Π defined as follows. Two elements $x > y$ belong to the same block in Π if and only if there exist i and j, $i < j$, such that $\pi_i = x > \pi_{i+1} > \cdots > \pi_{j-1} > \pi_j = y$. Conversely, any partition $\Pi = B_1/B_2/\ldots/B_k$, $k \geq 1$, is the associated to the permutation $B_1B_2\ldots B_k$. Theorem 8 becomes a straightforward consequence.

THEOREM 3.1 *The sets $S_n^{\sim_d}$, $n \geq 1$, are enumerated by the Bell numbers (A000110 in [18]).*

THEOREM 3.2 *The sets $S_n(321)^{\sim_d}$, $n \geq 1$, are enumerated by the Motzkin numbers (A001006 in [18]).*

Proof. Let $\pi \in S_n(321)$ and $DD(\pi) = \{(M_1, m_1), (M_2, m_2), \ldots, (M_r, m_r)\}$, $r \geq 0$, be the set of pairs (π_i, π_{i+1}) where i is a descent of π. Since π avoids 321, $DD(\pi)$ does not contain two pairs of the form (π_i, π_{i+1}) and (π_{i+1}, π_{i+2}). Then, we define the involution $\sigma \in I_n$ as follows: $\sigma(M_i) = m_i$, $\sigma(m_i) = M_i$ for $1 \leq i \leq r$ and $\sigma(k) = k$ if k does not appear in any pair of $DD(\pi)$. For a contradiction, let us assume that σ contains a pattern 4321. Then there exist two pairs (M_i, m_i) and (M_j, m_j) in $DD(\pi)$ such that $M_i > M_j > m_j > m_i$. If the descent (M_i, m_i) is on the left (in π) of the descent (M_j, m_j), then the subsequence $M_iM_jm_j$ is a pattern 321 of π; otherwise, the subsequence $M_jm_jm_i$ also is a 321-pattern. In the two cases we obtain a contradiction, which ensures that the involution σ avoids the pattern 4321.

Conversely, let σ be an involution avoiding the pattern 4321. There exists a sequence of pairs $(M_1, m_1), \ldots, (M_r, m_r)$, $r \geq 0$, such that $M_i < M_{i+1}$, $m_i < m_{i+1}$ for $i \leq r-1$ and such that $M_i > m_i$, $\sigma(M_i) = m_i$ and $\sigma(m_i) = M_i$ for $i \leq r$ and $\sigma(k) = k$ for $k \in [n] \backslash \{M_1, \ldots, M_r, m_1, \ldots, m_r\}$. We define the permutation π with the following process. We start with the sequence $M_1m_1M_2m_2\ldots M_rm_r$; we insert in increasing order all other values k satisfying $\sigma(k) = k$ as follows: if $m_{i-1} < k < m_i$ then we insert k between m_{i-1} and M_i; if $k < m_1$ then we insert k before M_1; and if $k \geq m_r$ then we insert k after m_r. Obviously, this construction induces that π avoids 321. Moreover, σ can be obtained from π by the construction of the beginning of this proof. Thus, there is a bijection between $S_n(321)^{\sim_d}$ and $I_n(4321)$ which is enumerated by the Motzkin numbers (see A001006 in [18] and [6]). □

LEMMA 3.3 *Let π and π' be two permutations in $S_n(132)$ belonging to the same d-equivalence class. If we have $\pi_1 = \pi'_1$ then $\pi = \pi'$.*

Proof. We proceed by induction on n. A simple observation gives the result for $n \leq 3$. Now, let us assume that Lemma 1 is true for $k \leq n-1$. Let π and π' be two permutations in $S_n(132)$ such that $\pi_1 = \pi'_1$. We can write $\pi = \alpha n \beta$ (resp. $\pi' = \alpha' n \beta'$) where $\beta \in S_k(132)$ for some k, $0 \leq k \leq n-1$ (resp. $\beta' \in S_{k'}(132)$ for some k', $0 \leq k' \leq n-1$), and α (resp. α') is obtained from a permutation in

$S_{n-k-1}(132)$ (resp. $S_{n-k'-1}(132)$) by adding k (resp. k') on all these entries. Let m (resp. m') be the minimal value of α (resp. α').

Without loss of generality, we assume that $m' \leq m \leq \pi_1 = \pi'_1$. For a contradiction, assume that $m' < m$. So, there is two consecutive entries a and m' in α' such that $a > m'$. As $m' < m$, the descent (a, m') does not appear in α. Thus, (a, m') appears in β. Let α_1 be the first value of α; the subsequence $\alpha_1 a m'$ is necessarily a pattern 132 which is a contradiction. Thus, we have $m = m'$ and then $k = k'$. We deduce α and α' (resp. β and β') are d-equivalent and the recurrence hypothesis gives $\alpha = \alpha'$. Moreover the descent (n, β_1) is equal to the descent (n, β'_1) and then $\beta_1 = \beta'_1$. Using the recurrence hypothesis we conclude $\beta = \beta'$ and then, $\pi = \pi'$. □

THEOREM 3.4 *The sets $S_n(132)^{\sim_d}$, $n \geq 1$, are enumerated by $c_n - c_{n-1} + 1$ where $c_n = \frac{1}{n+1}\binom{2n}{n}$ is the n-th Catalan number.*

Proof. Let a_n be the cardinality of $S_n(132)^{\sim_d}$. We distinguish three kinds of classes: (1) classes with a representative π satisfying $\pi_1 = n$; (2) classes with a representative π satisfying $\pi_n = n$; (3) the remaining classes.

Case (1). Such a class contains a permutation π such that $\pi_1 = n$, *i.e.*, $\pi = n\pi'$ with $\pi' \in S_{n-1}(132)$. Using Lemma 3.3, there is a unique $\sigma = \pi' \in S_{n-1}(132)$ such that $n\sigma$ and π belong to the same class. Thus, the number of classes in this case is also the cardinality of $S_{n-1}(132)$, that is the $(n-1)$-th Catalan number c_{n-1}.

Case (2). Such a class contains a permutation π such that $\pi_n = n$. So, the number of classes in this case is also the number of elements in $S_{n-1}(132)^{\sim_d}$, that is a_{n-1}.

Case (3). Now we consider the classes that do not lie in the two previous cases. Any permutation π of such a class satisfies $\pi_i = n$ for some $i \in [2, n-1]$, and since π avoids 132, π can be written $\pi = \alpha n \beta$ where $\beta \in S_{n-i}(132)$ and α is obtained by adding $(n-i)$ on all entries of a permutation in $S_{i-1}(132)$.

Let us consider j, $1 \leq j \leq i-1$, the position where α reaches its minimum m.

If $j = 1$ then $\pi = (n-i+1)\dots(n-1)n\beta$ and this permutation lies in the same class of $n\beta(n-i+1)\dots(n-1)$ that satisfies Case (1). Then, $j = 1$ does not occur.

Now we assume $j \geq 2$ and let σ be a permutation in $S_n(132)$ lying in the same class of π. Then, σ must contain the two descents (n, π_{i+1}) and (π_{j-1}, m). These two descents necessarily appear in the same order as in π (otherwise, a pattern 132 would be created with $\pi_{i+1}\pi_{j-1}m$). Thus, the minimum m' of values on the left of n in σ is necessarily less or equal to m. For a contradiction, let us assume that $m' < m$. The value m' appears necessarily on the right of the descent (π_{j-1}, m) in σ (otherwise, $m'\pi_{j-1}m$ would be a pattern 132). Therefore, a descent of the form (a, b), $a \geq m$ and $b < m$ would necessarily exists in σ, which is not possible because such a descent cannot belong in π.

Thus, we deduce $m = m'$ and σ has the similar decomposition $\sigma = \alpha' n \beta'$ where $\beta' \in S_{n-i}(132)$ and α' is obtained by adding $(n-i)$ on all entries of some permutation in $S_{i-1}(132)$. So, α (resp. β) is equivalent to α' (resp. β'). Hence, Lemma 3.3 implies that $\beta = \beta'$. Then, for a given $i \in [2, n-1]$, there are exactly $c_{n-i} \cdot (a_{i-1} - 1)$ classes verifying this case (we subtract one to a_{i-1} because we do not consider $\pi = (n-i+1)\dots n\beta$).

So, such classes are enumerated by $\sum_{k=2}^{n-1} (a_{k-1} - 1) \cdot c_{n-k}$.

Considering the three cases, the cardinality a_n of $S_n(132)^{\sim_d}$ satisfies for $n \geq 2$,

$$a_n = c_{n-1} + a_{n-1} + \sum_{k=2}^{n-1} (a_{k-1} - 1) \cdot c_{n-k}.$$

A simple calculation proves that $a_n = c_n - c_{n-1} + 1$ for $n \geq 2$. □

THEOREM 3.5 *The sets $S_n(123)^{\sim_d}$, $n \geq 2$, are enumerated by $c_n + n - (n+2) \cdot 2^{n-3} + \frac{(n-2)(n-1)}{2}$ where $c_n = \frac{1}{n+1}\binom{2n}{n}$ is the n-th Catalan number.*

Proof. Let π be a permutation in $S_n(123)$. Then π has a unique decomposition into blocks of decreasing sequences, *i.e.*, $\pi = A_1 A_2 \dots A_r$, $1 \leq i \leq r$, where blocks A_i consist of sequences of decreasing values (possibly reduced to one value) and such that $\ell_i < f_{i+1}$ for $1 \leq i \leq r-1$, where f_i (resp. ℓ_i) is the first (resp. last) element of A_i.

We distinguish three cases (1) $r = 1$; (2) $r = 2$; and (3) $r \geq 3$.
Case (1). We necessarily have $\pi = n(n-1)\dots 21$ and its equivalence class contains only one element.
Case (2). We have $\pi = A_1 A_2$ with $\ell_1 < f_2$. (i) If $f_1 < \ell_2$ then the class of π contains only one element since the permutation $\sigma = A_2 A_1$ is not there. Therefore, there are $(n-1)$ such classes. (ii) If $f_1 > \ell_2$ then the class of π contains exactly two elements π and $\sigma = A_2 A_1$. Since the number of permutations of length n with $n-2$ descents is $2^n - (n+1)$, the number of classes for the subcase (ii) is $\frac{2^n - (n+1) - (n-1)}{2} = 2^{n-1} - n$. Finally, there are $2^{n-1} - n + n - 1 = 2^{n-1} - 1$ classes for Case (2).
Case (3). π contains at least three blocks. We decompose $\pi = A_1 B A_r$ where $B = A_2 \dots A_{r-1}$, $r \geq 3$, such that $f_2 = n$, $\ell_{r-1} = 1$ with A_1 and A_r possibly empty.

Let σ be a permutation in $S_n(123)$ belonging to the class of π. We will prove that σ is either π or $A_r B A_1$.

For this, we will prove that the block B also appears in σ. It is obvious whenever B is a decreasing sequence. Now, let us assume that B is the concatenation of at least two blocks, that is $B = A_2 \dots A_{r-1}$ with $r \geq 4$.

(i) For a contradiction, we suppose that there exist i and j, $2 \leq i < j \leq r-1$ such that A_i appears on the right of A_j into σ, *i.e.*, $\sigma = \alpha A_j \beta A_i \gamma$ for some α, β, γ possibly empty. Since σ avoids 123 and $f_j < f_i$, α does not contain any value a such that $a < f_j$ (otherwise $a f_j f_i$ would be a pattern 123). Also, α does not contain any value a such that $a > f_j$ (otherwise there would be $b \leq a$ such that (b, f_j) is descent in σ that does not appear in π). Thus α is necessarily empty. By a simple symmetry , γ is also empty. This implies that all other blocks of π appear between the two blocks A_j and A_i in σ. Thus, all other blocks consist of values a such that $a \in [1, \ell_j - 1] \cup [f_i + 1, n]$ (otherwise $\ell_j a f_i$ would be a pattern 123). Since π contains at least three blocks there is at least one block between A_j and A_i in σ.

The case $\ell_j = 1$ does not occur. Indeed, this would mean that all blocks between A_j and A_i contain values x greater than f_i which creates a descent of the form (x, f_i) that does not appear in π. A similar argument proves that $f_i = n$ does not occur.

Let us consider the case where n and 1 do not appear in A_i and A_j. Since B contains at least two blocks, n and 1 do not appear in the same block in B. Let R (resp. S) be the block containing n (resp. 1). In σ, the last element $\ell(R)$ of R is necessarily less than ℓ_j (otherwise σ would contain a pattern 123, that is $\ell_j \ell(R) x$ where x is the value just after $\ell(R)$ in σ). A same argument shows that the first

element $f(S)$ of S is greater than f_i. In π, this would mean that $\ell(R)\ell_i f(S)$ is a pattern 123 which is a contradiction. Finally, all blocks of B appear in σ in the same order as π.

(ii) Now we will prove that A_1 does not appear between A_2 and A_{r-1} in σ. For a contradiction, let us assume that A_1 appears between A_2 and A_{r-1} in σ. Let a be the value of σ just after the block A_1. If $a < \ell_1$ then σ contains the descent (ℓ_1, a) that does not appear in π; otherwise, $\ell_2\ell_1 a$ would be a pattern 123 in σ which gives a contradiction. A same argument proves that A_r does not appear between A_2 and A_{r-1} in σ.

Therefore, we deduce that either $\sigma = \pi$ or $\sigma = A_r B A_1$.

Now we will enumerate permutations $\pi \in S_n(123)$ of Case (3) such that there is $\sigma \in S_n(123)$, $\sigma \neq \pi$, belonging to the same class of π, *i.e.*, $A_1 B A_r$ and $A_r B A_1$ do not contain any pattern 123. This case is characterized by the fact that there is no value a in the block B such that $\min\{\ell_1, \ell_r\} < a < \max\{f_1, f_r\}$ (otherwise, one of the two permutations $A_1 B A_r$, $A_r B A_1$ would contain the pattern 123). See Figure 1 for a graphical representation of such a permutation.

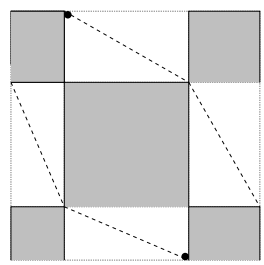


Figure 2: Illustration of $\pi = A_1 B A_r \in S_n(123)$ having two elements in its class.

If B contains only one block, A_1 and A_r are non-empty blocks. Varying the size k of B from 2 to $n-2$, and the size ℓ of A_1 from 1 to $n-k-1$, the number of permutations having two elements in its class is $a_n = \sum_{k=2}^{n-2} (k-1) \cdot \sum_{\ell=1}^{n-k-1} \binom{n-k}{\ell}$.

If B contains at least two blocks, A_1 and A_r are blocks (possibly empty). Varying the size k of B from 4 to $n-1$ and varying the size ℓ of A_1 from 0 to $n-k$, the number of permutations having two elements in its class is $b_n = \sum_{k=4}^{n-1} (2^{k-2} - (k-1)) \cdot \sum_{\ell=0}^{n-k} \binom{n-k}{\ell}$.

Finally, the number of classes in $S_n(123)$ is obtained from c_n by subtracting the number of classes having two elements, that is $c_n - \frac{1}{2}(2^n - 2n + a_n + b_n) = c_n - (n+2) \cdot 2^{n-3} + \frac{n(n-1)}{2} + 1$. □

References

[1] J.-L. Baril, *Statistics-preserving bijections between classical and cyclic permutations*, Inform. Process. Lett., 113 (2003) 17–22.

[2] J.-L. Baril, T. Mansour and A. Petrossian, *Equivalence classes of permutations modulo excedances*, J. Comb., 5 (2014) 453–469.

Pattern	Sequence	Sloane	$a_n, 1 \leq n \leq 9$
$\{\}$	Bell	$A000110$	$1, 2, 5, 15, 52, 203, 877, 4140, 21147$
$\{231\}, \{312\}$	Catalan	$A000108$	$1, 2, 5, 14, 42, 132, 429, 1430, 4862$
$\{321\}$	Motzkin	$A001006$	$1, 2, 4, 9, 21, 51, 127, 323, 835$
$\{132\}, \{213\}$	$c_n - c_{n-1} + 1$	New	$1, 2, 4, 10, 29, 91, 298, 1002, 3433$
$\{123\}$	$c_n - (n+2) \cdot 2^{n-3} + \frac{n(n-1)}{2} + 1$	New	$1, 2, 4, 9, 25, 84, 307, 1139, 4195$

Table 3: Number of equivalence classes for permutations avoiding at most one pattern of $\mathcal{S}_3$.

[3] M. Bona, *Combinatorics of permutations*, Chapman and Hall/CRC, Second Edition, 2012.

[4] E. Deutsch and L. Shapiro, *A survey of the Fine numbers*, Discrete Math., 241 (2001) 241–265.

[5] D. C. Foata and M. P. Schützenberger, *Théorie Géométrique des Polynômes Euleriens*, Lecture Notes in Math., 138, Springer-Verlag, Berlin, 1970.

[6] O. Guibert, E. Pergola and R. Pinzani, *Vexillary involutions are enumerated by Motzkin numbers*, Ann. Comb., 5 (2001) 153–174.

[7] S. Kitaev, *Patterns in permutations and words*, Springer-Verlag, 2011.

[8] D. E. Knuth, *The art of computer programming, Volume I*, Addison-Wesley, Reading MA, Third edition, 1997.

[9] C. Krattenthaler, *Permutations with restricted patterns and Dyck paths*, Adv. in Appl. Math., 27 (2001) 510–530.

[10] T. Mansour, *Combinatorics of Set Partitions*, Discrete Mathematics and its Applications, Chapman & Hall/CRC, 2013.

[11] T. Mansour, *Permutations avoiding a pattern from S_k and at least two patterns from S_3*, Ars. Combin., 62 (2002) 227–239.

[12] T. Mansour and A. Vainshtein, *Restricted 132-avoiding permutations*, Adv. Appl. Math., 26 (2001) 258–269.

[13] R. Simion and F.W. Schmidt, *Restricted permutations*, European J. Combin., 6 (1985) 383–406.

[14] J. West, *Permutations with forbidden subsequences and stack-sortable permutations*, Ph.D. Thesis, MIT, 1990.

[15] J. West, *Generating trees and forbidden subsequences*, Local Proceedings of FPSAC 1994, 441–450.

[16] J. West, *Generating trees and the Catalan and Schröder numbers*, Discrete Math., 146 (1995) 247–262.

[17] A. Sapounakis, I. Tasoulas and P. Tsikouras, *Counting strings in Dyck paths*, Discrete Math., 307 (2007) 2909–2924.

[18] N.J.A. Sloane, *On-Line Encyclopedia of Integer Sequences*, http://oeis.org/.

[19] R. P. Stanley, *Enumerative Combinatorics, Volume I*, Cambridge University Press, 2002.

Journal of Integer Sequences, Vol. 18 (2015),
Article 15.7.1

Equivalence Classes of Motzkin Paths Modulo a Pattern of Length at Most Two

Jean-Luc Baril and Armen Petrossian
LE2I UMR CNRS 6306
University of Bourgogne
21078 Dijon
France
barjl@u-bourgogne.fr
armen.petrossian@u-bourgogne.fr

Abstract

For any pattern α of length at most two, we enumerate equivalence classes of Motzkin paths where two paths of the same length are equivalent whenever they coincide on all occurrences of the pattern α.

1 Introduction and notation

A *Motzkin path* of length n, $n \geq 0$, is a lattice path starting at point $(0,0)$, ending at $(n,0)$, and never going below the x-axis, consisting of up steps $U = (1,1)$, down steps $D = (1,-1)$ and flat steps $F = (1,0)$. See Figure 1 for an illustration of a Motzkin path of length 18. Let $\mathcal{M}$ be the set of all Motzkin paths. A *pattern* of length one (resp., two) in $M \in \mathcal{M}$ consists of one step (resp., two consecutive steps). We will say that an *occurrence* of a pattern is at position i, $i \geq 1$, in M whenever the first step of this occurrence appears at the i-th step of the path. The *height* of this occurrence is the minimal ordinate reached by its points. For instance, the path $M = UUFDDFFUDUFUDDFFUD$ (see Figure 1) contains three occurrences of the pattern FU at positions 7, 11 and 16, respectively at heights 0, 1 and 0. A Motzkin path will be called *Dyck path* when it does not contain any flat step F. Let $\mathcal{D}$ be the set of all Dyck paths.

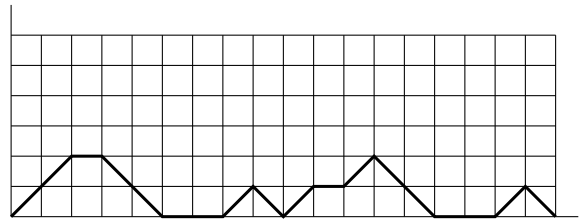


Figure 1: A Motzkin path $M = UUFDDFFUDUFUDDFFUD$.

Many statistics on Motzkin and Dyck paths have been studied. Almost always, it is shown how we can enumerate these paths according to several parameters, such as length, number of occurrences of a pattern, number of returns to the x-axis (see, for instance, [4, 7, 8, 9, 11, 12, 13, 15, 16, 19] for Dyck paths and [2, 3, 5, 6, 10, 14, 17] for Motzkin paths). Recently in [1], the authors investigate equivalence relations on the set $\mathcal{D}$ of Dyck paths where two Dyck paths of the same length are equivalent whenever they coincide on all occurrences of a given pattern.

In this paper, we study these equivalence relations on the set $\mathcal{M}$ of Motzkin paths for any pattern α of length at most two, i.e.,

two Motzkin paths of the same length are α-equivalent whenever all occurrences of the pattern α appear at the same positions in the two paths.

For instance, $UDUFDF\mathbf{FU}UF\mathbf{FU}DDD\mathbf{FU}D$ is FU-equivalent to the path in Figure 1 since all occurrences of the pattern FU (in boldface) appear at the same positions in the two paths.

For any pattern α of length at most two, we provide generating functions for the number of α-equivalence classes in $\mathcal{M}$, with respect to the length (see Table 1). The general method used consists of exhibiting one-to-one correspondences between some subsets of Motzkin paths and the different sets of equivalence classes by using combinatorial reasonings, and then, evaluating algebraically the generating functions for these subsets.

2 Equivalence classes modulo a pattern of length one

Throughout this section, we study the α-equivalence in $\mathcal{M}$ for $\alpha \in \{U, D, F\}$.

2.1 Modulo $\alpha = F$

Let $\mathcal{A}$ be the set of Motzkin paths without any points of ordinate greater than one. For instance, we have $UFDUDF \in \mathcal{A}$ and $UUDDUD \notin \mathcal{A}$.

Lemma 1. *There is a bijection between $\mathcal{A}$ and the set of F-equivalence classes of $\mathcal{M}$.*

Pattern	Sequence	Sloane	$a_n, 1 \leq n \leq 10$
$\{U\}, \{D\}$	$\frac{2}{1-2x+\sqrt{1-4x^2}}$	A001405	$1, 2, 3, 6, 10, 20, 35, 70, 126, 252$
$\{F\}$	2^n	A000079	$1, 2, 4, 8, 16, 32, 64, 128, 256, 512$
$\{UU\}, \{DD\}$	$\frac{(x-1)^2-\sqrt{(1-3x^2)(x^2+1)}}{2x(x^3-(1-x)^2)}$	A191385	$1, 1, 1, 2, 3, 5, 7, 12, 18, 31$
$\{DU\}$	Shift of Fibonacci	A132916	$1, 1, 1, 2, 3, 5, 8, 13, 21, 34$
$\{UD\}$	Fibonacci	A000045	$1, 2, 3, 5, 8, 13, 21, 34, 55, 89$
$\{UF\}, \{DF\}$, $\{FU\}, \{FD\}$	$\frac{2}{1-2x-\sqrt{1-4x^3}}$	A165407	$1, 1, 2, 3, 4, 7, 11, 16, 27, 43$
$\{FF\}$	$\frac{x(1-x-x^2+3x^3-5x^4+3x^5-2x^6-x^7-x^9)}{(1-x-2x^4-x^6-x^7)(1-2x+x^2-x^3)}$	New	$1, 2, 2, 5, 9, 17, 29, 53, 94, 168$

Table 1: Number of α-equivalence classes for Motzkin paths.

Proof. Let M be a Motzkin path in $\mathcal{M}\backslash\mathcal{A}$. Let us prove that there exists $M' \in \mathcal{A}$ (with the same length as M) such that M and M' belong to the same class. Since M does not belong to $\mathcal{A}$, we can write $M = RUQDS$, where R is of maximal length so that it does not contain any point of ordinate two, and Q is a (possibly empty) Motzkin path.

Now, we define a sequence of Motzkin paths $M_0 = M, M_1, M_2, \ldots, M_{k-1}, M_k = M'$, $k \geq 1$, where for any i, $1 \leq i \leq k$, the Motzkin path M_i is obtained from M_{i-1} by performing the following process.

Let $M_{i-1} = RUQDS$ be the above decomposition. Then, $M_i = RDQUS$ is obtained from M_{i-1} by swapping the two steps U and D contiguous to Q.

The process finishes because it decreases the number of points of ordinate greater than one. All Motzkin paths $M_0 = M, M_1, \ldots, M_k = P'$ belong to the same equivalence class. So, at the end of the process M_k belongs to $\mathcal{A}$. For instance, if

$$M = UUUDUFDFUFFDFFDDUD,$$

then we obtain $M_1 = M' = UDUDUFDFUFFDFFUDUD$. See Figure 2 for an illustration of this example.

Now we will prove that any F-equivalence class contains at most one element in $\mathcal{A}$. For a contradiction, let M and M' be two different Motzkin paths in $\mathcal{A}$ belonging to the same class. We write $M = QR$ and $M' = QS$ where R and S start with two different steps. Since M and M' lie in the same class, these steps cannot be F. Without loss of generality, let us assume that the first step of R is an up step U and then, the first step of S is a down step D. This means that the last point of Q has its ordinate equal to zero (otherwise, R would contain a point of ordinate at least two). As S starts with a down step, S has its second point (from the left) with an ordinate equal to -1, which gives a contradiction. $\square$

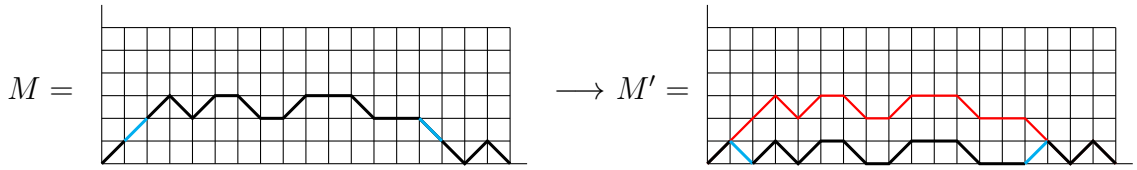


Figure 2: Illustration of the example described in the proof of Lemma 1.

Theorem 2. *The generating function for the set of F-equivalence classes in $\mathcal{M}$ with respect to the length is given by*

$$\frac{1-x}{1-2x}.$$

Proof. Using Lemma 1, it suffices to obtain the generating function $A(x)$ for the set $\mathcal{A}$. A non-empty Motzkin path $M \in \mathcal{A}$ can be written either $M = FQ$ where $Q \in \mathcal{A}$ or $M = UF^rDQ$ where $r \geq 0$ and $Q \in \mathcal{A}$. Then, we deduce the functional equation $A(x) = 1 + xA(x) + \frac{x^2}{1-x}A(x)$ which implies that $A(x) = \frac{1-x}{1-2x}$. □

2.2 Modulo $\alpha = U$ or $\alpha = D$

The result for the case $\alpha = D$ is deduced from $\alpha = U$ using a simple symmetry (the mirror transformation on Motzkin paths). So, we set $\alpha = U$ in this part.

Let $\mathcal{B}$ be the set of Motzkin paths without any flat steps at positive height. For instance, we have $UDFUD \in \mathcal{B}$ and $UFDUD \notin \mathcal{B}$.

Lemma 3. *There is a bijection between $\mathcal{B}$ and the set of U-equivalence classes of $\mathcal{M}$.*

Let M be a Motzkin path in $\mathcal{M} \backslash \mathcal{B}$. Let us prove that there exists a Motzkin path $M' \in \mathcal{B}$ (with the same length as M) such that M and M' belong to the same class. Since M does not belong to $\mathcal{B}$, there exists a Motzkin path Q (possibly empty) such that $M = RFQDS$ where R is maximal so that it does not contain any flat step at positive height, its last point has an ordinate of at least one, and Q is a Motzkin path.

Now, we define a sequence of Motzkin paths $M_0 = M, M_1, M_2, \ldots, M_{k-1}, M_k = M'$, $k \geq 1$, where for any i, $1 \leq i \leq k$, the Motzkin path M_i is obtained from M_{i-1} by performing the following process.

Let $M_{i-1} = RFQDS$ be the above decomposition. Then, we define the Motzkin path $M_i = RDQFS$ obtained from M_{i-1} by swapping the two steps F and D contiguous to Q.

The process finishes because it decreases the height of the leftmost flat step having a positive height. All Motzkin paths $M_0 = M, M_1, \ldots, M_k = M'$ belong to the same equivalence class. So, at the end of the process, M_k belongs to $\mathcal{B}$. For instance, if we perform the above process on $M = UUUDFFFFUUDDFFDDFF$, then we obtain $M_1 = UUUDDFFFUUDDFFFDFF$ and $M_2 = M' = UUUDDDFFUUDDFFFFFF$ (see Figure 3 for an illustration of this example).

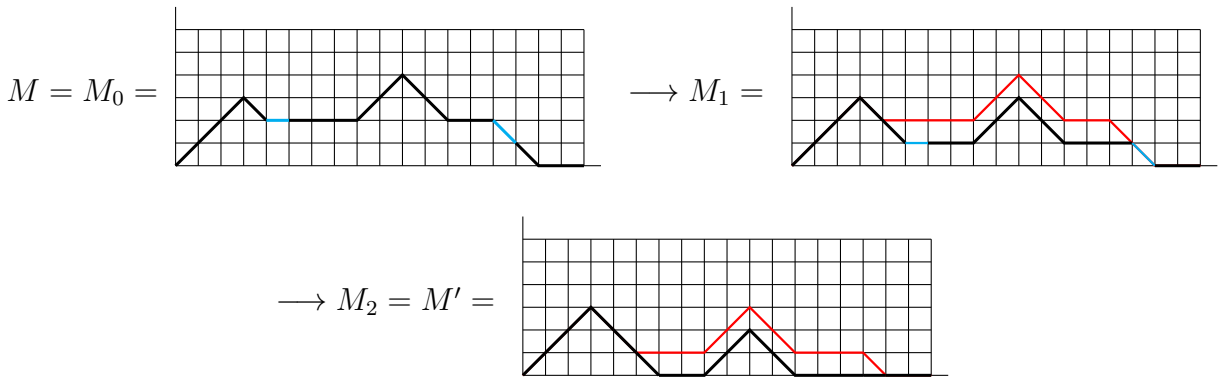


Figure 3: Illustration of the example described in the proof of Lemma 3.

Now we will prove that any U-equivalence class contains at most one element in B. For a contradiction, let M and M' be two different Motzkin paths in $\mathcal{B}$ belonging to the same class. We write $M = QR$ and $M' = QS$ where R and S start with two different steps. Since M and M' lie in the same class, these steps cannot be U. Without loss of generality, let us assume that the first step of R is a down step D and then, the first step of S is a flat step F. This means that the last point of Q has its ordinate equal to zero (otherwise M' could not belong to $\mathcal{B}$). As the first step of R is D, the second point of R has an ordinate equal to -1 which gives a contradiction. □

Theorem 4. *The generating function for the set of U-equivalence (resp., D-equivalence) classes in $\mathcal{M}$ with respect to the length is given by*

$$\frac{2}{1 - 2x + \sqrt{1 - 4x^2}}.$$

Proof. Using Lemma 3, it suffices to obtain the generating function $B(x)$ for the set $\mathcal{B}$. A non-empty Motzkin path $M \in \mathcal{B}$ can be written either $M = FQ$ where $Q \in \mathcal{B}$ or $M = UQDQ'$ where Q is a Dyck path in $\mathcal{D}$ and $Q' \in \mathcal{B}$. Then, we deduce the functional equation $B(x) = 1 + xB(x) + x^2C(x^2)B(x)$ where $C(x) = \frac{1-\sqrt{1-4x}}{2x}$ is the generating function for the set $\mathcal{D}$ of Dyck paths. A simple calculation gives the result. □

3 Equivalence classes modulo a pattern of length two

Throughout this section, we study the α-equivalence in $\mathcal{M}$ for any pattern α of length two. By considering the mirror transformation of Motzkin paths, it suffices to examine the cases where $\alpha \in \{UU, UD, UF, FU, DU, FF\}$.

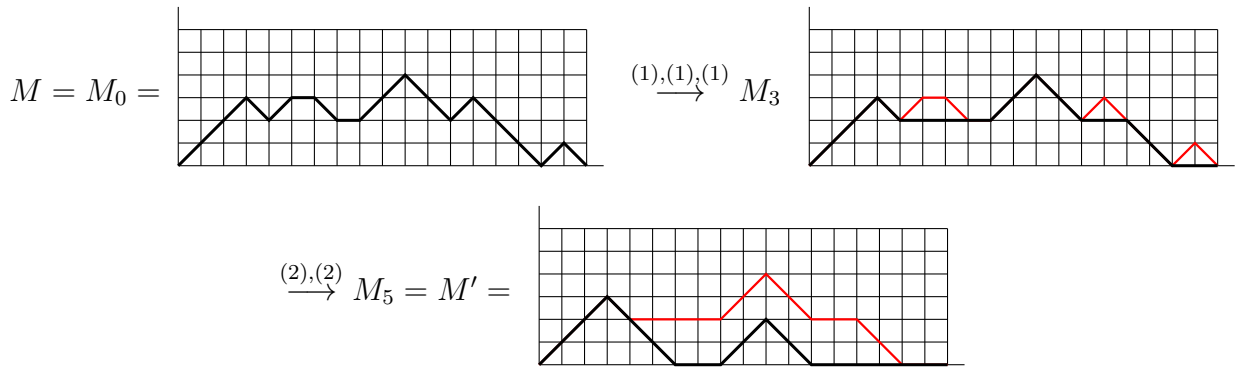


Figure 4: Illustration of the example described in the proof of Lemma 5.

3.1 Modulo $\alpha = UU$

Let $\mathcal{E}$ be the set of Motzkin paths without any flat steps at positive height and having no ascents of length one, i.e., no maximal consecutive up steps of length one. For instance, we have $UUDDF \in \mathcal{E}$ and $FUDFF \notin \mathcal{E}$.

Lemma 5. *There is a bijection between $\mathcal{E}$ and the set of UU-equivalence classes of $\mathcal{M}$.*

Proof. Let M be a Motzkin path in $\mathcal{M}\backslash\mathcal{E}$. Let us prove that there exists a Motzkin path $M' \in \mathcal{E}$ of the same length as M such that M and M' belong to the same class.

We define a sequence of Motzkin paths $M_0 = M, M_1, M_2, \ldots, M_{k-1}, M_k = M'$, $k \geq 1$, where for any i, $1 \leq i \leq k$, the Motzkin path M_i is obtained from M_{i-1} by performing the following process.

(1) If there is an ascent of length one in M_{i-1}, then we write $M_{i-1} = RUQDS$ where R is of maximal length so that it does not contains any ascent of length one, R does not finish with U, and Q is a Motzkin path (possibly empty) that does not start with an up step. So, we set $M_i = RFQFS$;

(2) Otherwise, if there is a flat step at positive height, then we perform the process described in the proof of Lemma 3.

This process finishes because part (1) of the process decreases by one the number of ascents of length one, and part (2) decreases the height of the leftmost flat step having a positive height (see the proof of Lemma 3). All Motzkin paths $M_0 = M, M_1, \ldots, M_k = M'$ belong to the same class. At the end of the process, M_k belongs to $\mathcal{E}$. For instance, if we perform the above process on $M = M_0 = UUUDUFDFUUDDUDDDUD$ then we obtain $M_3 = UUUDFFFFUUDDFFDDFF$ after three runs of (1), and we obtain $M_5 = UUUDDDFFUUDDFFFFFF$ after two runs of (2) (see Figure 4 for an illustration of this example).

For a contradiction, let M and M' be two different Motzkin paths in $\mathcal{E}$ belonging to the same class. We write $M = QR$ and $M' = QS$ where R and S start with two different steps,

and R and S cannot start with UU. We distinguish two cases. If the last step of Q is a down step D or a flat step F, then R and S cannot start with UF and UD (and also with U) because an ascent of length one would be created; if the last step of Q is an up step U, then R and S cannot start with U because a pattern UU would be created. Without loss of generality, we can assume (for the two last cases) that R (resp., S) starts with D (resp., with F); so, the last point of Q has a positive ordinate that implies that the first step of S is a flat step of positive height, which is a contradiction. □

Theorem 6. *The generating function for the set of UU-equivalence classes of $\mathcal{M}$ with respect to the length is given by*

$$\frac{(x-1)^2 - \sqrt{(1-3x^2)(x^2+1)}}{2x(x^3-(1-x)^2)}.$$

Proof. Using Lemma 5, it suffices to give the generating function for the set $\mathcal{E}$, which is already known (see A191385 in [18]). □

3.2 Modulo $\alpha = DU$

Let $\mathcal{F}$ be the set of Motzkin paths without any points of ordinate greater than one, and without any flat steps at height zero. For instance, we have $UFDUFFDUD \in \mathcal{F}$ and $UDFUUDD \notin \mathcal{F}$.

Lemma 7. *There is a bijection between $\mathcal{F} \cup \{F\}$ and the set of DU-equivalence classes of $\mathcal{M}$.*

Proof. Let $M \neq F$ be a Motzkin path in $\mathcal{M} \backslash \mathcal{F}$ (the case where $M = F$ is clear). Let us prove that there exists a Motzkin path $M' \in \mathcal{F}$ of the same length as M such that M and M' belong to the same class.

Any Motzkin path $M \in \mathcal{M}$ can be uniquely written either

$$M = \beta_0 \qquad \text{or} \qquad M = \beta_0(DU\beta_1)\cdots(DU\beta_r),$$

where $r \geq 1$, and β_i, $0 \leq i \leq r$, (possibly empty) does not contain any pattern DU.

If $M = \beta_0$ then we set $M' = UF^{s_0-2}D$ where $s_0 \geq 2$ is the length of M; otherwise, we set $M' = UF^{s_0-1}(DUF^{s_1})\cdots(DUF^{s_{r-1}})(DUF^{s_r-1})D$ where s_i, $0 \leq i \leq r$, is the length of β_i. By construction, M and M' belong to the same equivalence class and M' lies in $\mathcal{F}$. For instance, if $M = FUFDUUUDUDDUUFFDDD$ then we obtain $M' = UFFDUFFDUFFDUFFFFD$ (see Figure 5 for an illustration of this example).

Any Motzkin path $M \in \mathcal{F}$ is characterized by the positions of its pattern DU. So, two different Motzkin paths in $\mathcal{F}$ cannot belong to the same class, which completes the proof. □

Theorem 8. *The generating function for the set of DU-equivalence classes of $\mathcal{M}$ with respect to the length is given by $F(x) = \frac{1-x^2-x^3}{1-x-x^2}$, that corresponds to a shift of the Fibonacci sequence.*

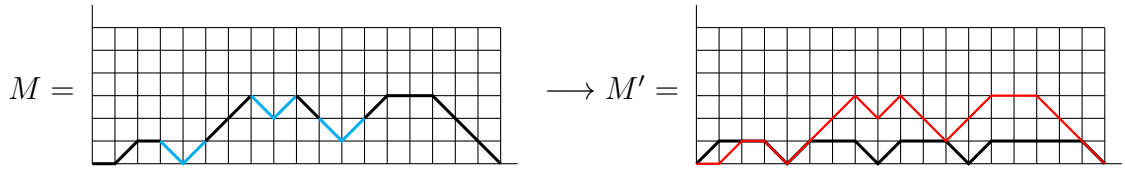


Figure 5: Illustration of the example in the proof of Lemma 9.

Proof. Using Lemma 7, it suffices to provide the generating function $F(x)$ for the set $\mathcal{F} \cup \{F\}$. Let M be a non-empty Motzkin path in $\mathcal{F}$. It can be written $M = UF^rDQ$ where $r \geq 0$ and $Q \in \mathcal{F}$. So, we have the functional equation $F(x) = 1 + x + \frac{x^2}{1-x}(F(x) - x)$ which implies the result. $\square$

3.3 Modulo $\alpha = UD$

Let $\mathcal{G}$ be the set of Motzkin paths without any points of ordinate greater than one, and without any flat steps at positive height. For instance, we have $FUDFUD \in \mathcal{G}$ and $UFDUUDD \notin \mathcal{G}$.

Lemma 9. *There is a bijection between $\mathcal{G}$ and the set of UD-equivalence classes of $\mathcal{M}$.*

Proof. Let M be a Motzkin path in $\mathcal{M} \backslash \mathcal{G}$. Let us prove that there exists a Motzkin path $M' \in \mathcal{G}$ of the same length as M such that M and M' belong to the same class.

Any Motzkin path $M \in \mathcal{M}$ can be uniquely written either

$$M = \beta_0 \qquad \text{or} \qquad M = \beta_0(UD\beta_1)\cdots(UD\beta_r),$$

where $r \geq 1$, and β_i, $0 \leq i \leq r$, (possibly empty) does not contain any pattern UD.

If $M = \beta_0$ then we set $M' = F^{s_0}$ where $s_0 \geq 0$ is the length of M; otherwise, we set $M' = F^{s_0}(UDF^{s_1})\cdots(UDF^{s_r})$ where s_i, $0 \leq i \leq r$, is the length of β_i. Obviously, M and M' belong to the same equivalence class and M' lies in $\mathcal{G}$. For instance, if $M = FUFDUUUDUDDUUUDDDD$ then we obtain $M' = FFFFFFUDUDFFFUDFFF$ (see Figure 6 for an illustration of this example).

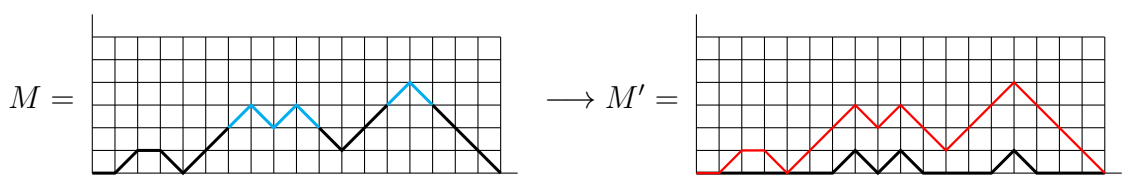


Figure 6: Illustration of the example in the proof of Lemma 9.

Any Motzkin path $M \in \mathcal{G}$ is characterized by the positions of its pattern UD. So, two different Motzkin paths in $\mathcal{G}$ cannot belong to the same class, which completes the proof. $\square$

Theorem 10. *The generating function for the set of UD-equivalence classes of $\mathcal{M}$ with respect to the length is given by $F(x) = \frac{1}{1-x-x^2}$, that corresponds to the Fibonacci sequence.*

Proof. Using Lemma 9, it suffices to provide the generating function $G(x)$ for the set $\mathcal{G}$. Let M be a non-empty Motzkin path in $\mathcal{G}$. It can be written either $M = FQ$ or $M = UDQ$ where $Q \in \mathcal{G}$ which implies that $G(x) = \frac{1}{1-x-x^2}$. □

3.4 Modulo $\alpha = UF$

Let $\mathcal{H}$ be the set of Motzkin paths without any patterns UD and UU, and such that all patterns FF and DF appear at height zero. For instance, we have $FFUFUFDDF \in \mathcal{H}$ and $UDUFUFDDF \notin \mathcal{H}$.

Lemma 11. *There is a bijection between $\mathcal{H}$ and the set of UF-equivalence classes of $\mathcal{M}$.*

Proof. Let M be a Motzkin path in $\mathcal{M}\backslash\mathcal{H}$. Let us prove that there exists a Motzkin path $M' \in \mathcal{H}$ of the same length as M such that M and M' belong to the same class.

Any Motzkin path $M \in \mathcal{M}$ can be uniquely written either

$$M = \beta_0 \qquad \text{or} \qquad M = \beta_0 \Pi_{i=1}^{r} (UF)^{k_i} \beta_i,$$

where $r \geq 1$, $k_i \geq 1$ for $1 \leq i \leq r$ and β_i, $0 \leq i \leq r$, (non-empty) does not contain any pattern UF.

If $M = \beta_0$ then we set $M' = F^{s_0}$ where $s_0 \geq 0$ is the length of M; since an element in $\mathcal{H}$ avoids UU and UD, M' is the only one element in $\mathcal{H}$ lying in the class of M (that avoids UF).

Otherwise, we set $M' = F^{s_0} \Pi_{i=1}^{r} (UF)^{k_i} \beta_i'$ with $\beta_i' = D^{t_i} F^{s_i - t_i}$ for $1 \leq i \leq r$, where s_i, $0 \leq i \leq r$, is the length of β_i and t_i, $i \geq 1$, is recursively defined by the equation

$$t_i = \min(s_i, \sum_{j=1}^{i} k_j - \sum_{j=1}^{i-1} t_j).$$

It is worth noticing that $\sum_{j=1}^{i} k_j$ (resp., $\sum_{j=1}^{i-1} t_j$) is the number of up steps (resp., down steps) before β_i' in M'. So, setting $t_i = \min(s_i, \sum_{j=1}^{i} k_j - \sum_{j=1}^{i-1} t_j)$ means that β_i' consists of the maximum possible of down steps, eventually completed with some flat steps at height zero. Then, M and M' belong to the same equivalence class and M' lies in $\mathcal{G}$. For instance, if we have $M = UUUFDFDUFUFFFDUDDD$ then we obtain $M' = FFUFDFFUFUFDDFFFFF$ (see Figure 7 for an illustration of this example).

Moreover, M' is the unique element in $\mathcal{H}$ in the same class as $M = \beta_0 \Pi_{i=1}^{r} (UF)^{k_i} \beta_i$. Indeed, β_i, $i \geq 1$, does not contains any pattern UU, UF and UD implies that β_i does not contain any up steps; M does not contains FF and DF at positive height implies that $\beta_i = D^{k_i} F^{s_i - k_i}$ where k_i is either the length s_i of β_i or the difference between the numbers of up steps and down steps before β_i, that is precisely $\sum_{j=1}^{i} k_j - \sum_{j=1}^{i-1} t_j$. Thus, we necessarily have $M = M'$ which completes the proof.

□

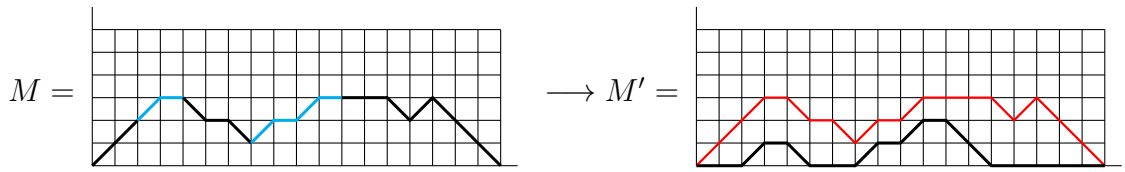


Figure 7: Illustration of the example in the proof of Lemma 11.

Theorem 12. *The generating function for the set of UF-equivalence classes of $\mathcal{M}$ with respect to the length is given by $H(x) = \frac{2}{1-2x-\sqrt{1-4x^3}}$.*

Proof. Using Lemma 11, it suffices to provide the generating function $H(x)$ for the set $\mathcal{H}$. Let M be a non-empty Motzkin path in $\mathcal{H}$. It can be written either $M = FQ$ or $M = UQDR$ where Q and R are two Motzkin paths.

If we have $M = FQ$, then M belongs to $\mathcal{H}$ if and only if $Q \in \mathcal{H}$. Otherwise, $M = UQDR$ belongs to $\mathcal{H}$ if and only if $R \in \mathcal{H}$ and $Q \in \mathcal{H}$ such that $Q = F\Pi_{i=1}^{r}(UF)^{k_i}D^{\ell_i}$ for some k_i and ℓ_i, with $1 \leq i \leq r$. Since Q is a Motzkin path of length $n = 1 + 3\sum_{i=1}^{r} k_i$, this implies that $Q' = \Pi_{i=1}^{r}U^{k_i}D^{\ell_i}$ is a Dyck path of semilength $\sum_{i=1}^{r} k_i = \frac{n-1}{3}$. Conversely, any Dyck path can be associated to a Motzkin path of the form of Q. So we deduce the functional equation $H(x) = 1+xH(x)+x^3C(x^3)H(x)$ where $C(x) = \frac{1-\sqrt{1-4x}}{2x}$ is the generating function for the set $\mathcal{D}$ of Dyck paths, which completes the proof. □

3.5 Modulo $\alpha = FU$

Let $\mathcal{I}$ be the set of Motzkin paths starting with a flat step, without any patterns DU, UU, FD and such that all patterns FF appear at height zero. For instance, we have $FFUFUDFUDD \in \mathcal{I}$ and $FFUFUFDDFUD \notin \mathcal{I}$.

Lemma 13. *There is a bijection between $\mathcal{I}$ and the set of FU-equivalence classes of $\mathcal{M}$.*

Proof. This is a simple adaptation of the proof of Lemma 11 by replacing UF with FU in the decomposition $M = \beta_0\Pi_{i=1}^{r}(UF)^{k_i}\beta_i$. So, we do not present this proof here. □

Theorem 14. *The generating function for the set of FU-equivalence classes of $\mathcal{M}$ with respect to the length is given by $I(x) = \frac{2}{1-2x-\sqrt{1-4x^3}}$.*

Proof. Using Lemma 13, it suffices to provide the generating function $I(x)$ for the set $\mathcal{I}$. Let M be a non-empty Motzkin path in $\mathcal{I}$. It can be uniquely decomposed $M' = F^{s_0}\Pi_{i=1}^{r}(FU)^{k_i}\beta_i'$ with $\beta_i' = D^{t_i}F^{s_i-t_i}$ for $1 \leq i \leq r$, where s_i, $0 \leq i \leq r$, is the length of β_i and t_i is recursively defined by $t_i = \min(s_i, \sum_{j=1}^{i} k_j - \sum_{j=1}^{i-1} t_j)$. The transformation that consists of replacing all occurrence FU into an occurrence UF, induces a one-to-one correspondence between $\mathcal{I}$ and $\mathcal{H}$. Therefore, the generating function $I(x)$ for $\mathcal{I}$ verifies $I(x) = H(x)$. Using Theorem 12, we complete the proof. □

3.6 Modulo $\alpha = FF$

Let $M \in \mathcal{M}$ be a Motzkin path. A *long flat* in M is a maximal sequence of at least two consecutive flat steps in M. An *isolate flat step* is a flat step which is not contiguous to another flat step.

Let $\mathcal{J}$ be the set of Motzkin paths without any points of ordinate greater than one, without any isolate flat steps, such that two long flats are separated by at most two steps, and at most one step precedes (resp., follows) the first (resp., last) long flat. For instance, there are five Motzkin paths in $\mathcal{J}$ of length 6: $FFFFFF$, $FFUFFD$, $UFFDFF$, $UFFFFD$ and $FFUDFF$. For $r, s \in \{0, 1\}$, we define the set $\mathcal{J}_r^s$ of elements of $\mathcal{J}$ such that the height of the leftmost long flat is r and the height of the rightmost long flat is s. Obviously, we have $\mathcal{J} = \cup_{r,s\in\{0,1\}} \mathcal{J}_r^s$.

Let M be a Motzkin path without any points of ordinate greater than one. We denote by $|M|$ its length, i.e., the number of the steps in M.

Fact 1: There is a unique decomposition of the form

$$M = \beta_0 \Pi_{i=1}^k (Q_i \beta_i)$$

with Q_i of maximal length in $\mathcal{J}$ for all $1 \leq i \leq k$ and where $\beta_i \in \mathcal{M}$, $0 \leq i \leq k$, does not contain any long flat, and only β_0 and β_k can possibly be empty.

Fact 2: If $M = \beta_0 \Pi_{i=1}^k (Q_i \beta_i)$ is the decomposition obtained in Fact 1, then there exists M' in the same class as M such that $M' = \beta'_0 \Pi_{i=1}^k (Q_i \beta'_i)$ with $\beta'_i = F$ if $|\beta_i| = 1$, $\beta'_i = (UD)^{\frac{|\beta_i|}{2}}$ if $|\beta_i| \geq 2$ is even and $\beta'_i = UFD(UD)^{\frac{|\beta_i|-3}{2}}$ if $|\beta_i| \geq 3$ is odd. In the following, a Motzkin path β'_i of the previous form will be called a *hinge*. It is worth noticing that any hinge is entirely characterized by its length.

Let $\mathcal{K}$ be the set of Motzkin paths $M = \beta_0 \Pi_{i=1}^k (Q_i \beta_i)$ verifying the conditions described in Fact 2, and such that all Q_i starts with FF whenever β_{i-1} and β_i are non-empty. For instance, the third Motzkin path $UDUDUDFFUFFDUDFFUD$ in Figure 8 belongs to $\mathcal{K}$. Its decomposition provides $\beta_0 = UDUDUD$, $Q_1 = FFUFFD$, $\beta_1 = UD$, $Q_2 = FF$ and $\beta_2 = UD$.

Lemma 15. *There is a bijection between $\mathcal{K}$ and the set of FF-equivalence classes of $\mathcal{M}$.*

Proof. For any Motzkin path in $\mathcal{M} \backslash \mathcal{K}$, the process described in the proof of Lemma 1 ensures that there is a Motzkin path $M \in \mathcal{A}$ in the same F-equivalence class, and thus in the same FF-equivalence class.

Let $M = \beta_0 \Pi_{i=1}^k (Q_i \beta_i)$ be the decomposition described in Fact 2.

We define a sequence of Motzkin path $M_0 = M, M_1, M_2, \ldots, M_{k-1}, M_k = M'$, $k \geq 1$.

For any i, $1 \leq i \leq k$, the Motzkin path M_i is obtained from M_{i-1} by performing the following process.

(a) If β_{i-1} and β_i are non-empty and $Q_i = UWD$, then we replace Q_i with $\bar{W}$ where $\bar{W}$ is obtained from W by exchanging U and D, and we replace β_{i-1} (resp., β_i) with a hinge of length $|\beta_{i-1}| + 1$ (resp., of length $|\beta_i| + 1$);

(b) If β_{i-1} and β_i are non-empty and $Q_i = UWF$, then we replace Q_i with $\bar{W}FD$ where $\bar{W}$ is obtained from W by exchanging U and D, and we replace β_{i-1} (resp., β_i) with a hinge of length $|\beta_{i-1}| + 1$ (resp., of length $|\beta_i| - 1$).

By construction, all Motzkin paths $M_0 = M, M_1, M_2, \ldots, M_{k-1}, M_k = M'$, belong to the same class and M' belongs to $\mathcal{K}$. Moreover, M' is the unique element in $\mathcal{K}$ in the same class as M; indeed, let $M = \beta_0 \Pi_{i=1}^{k}(Q_i\beta_i)$ and $M' = \beta'_0 \Pi_{i=1}^{k'}(Q'_i\beta'_i)$ two different elements in $\mathcal{K}$ lying in the same class.

– From the definition of the sets $\mathcal{K}$ and $\mathcal{J}$, it is clear that $k = k'$.

– If the long steps in M appear in M' with the same height as in M, then we necessarily have $Q_i = Q'_i$ for $1 \leq i \leq k$. Thus, we deduce $|\beta_i| = |\beta'_i|$ for $i \geq 0$, and with the last remark of Fact 1, $\beta_i = \beta'_i$ for all $i \geq 0$, which is a contradiction with $M \neq M'$.

– If there is a long flat in M with height $h \in [0, 1]$ and such that it appears in M' at height $1 - h$. We choose the leftmost long flat with this property. Let $i \geq 1$ be the integer such that this long flat of M appears in Q_i. Since any element in $\mathcal{J}$ has at most two steps between two long flats, and at most one step before (resp., after) the first (resp., the last) long flat step, any long flat step of height $h' \in [0, 1]$ in Q_i appears at height $1 - h'$ in Q'_i. Since $Q_i \in \mathcal{J}$, we have either $Q_i = FFWFF$ and $Q'_i = UFF\bar{W}FFD$, or $Q_i = FFWFFD$ and $Q'_i = UFF\bar{W}FF$, or $Q_i = UFFWFF$ and $Q'_i = FF\bar{W}FFD$, or $Q_i = UFFWFFD$ and $Q'_i = FF\bar{W}FF$.

Observing these four cases, Q_i cannot be a prefix nor a suffix of M (otherwise M' could not have the leftmost FF at the same position as M). This means that β_{i-1} and β_i are non-empty. Thus, there is an element among M and M' where there is i such that Q_i whenever the element is M, Q'_i whenever the element is M', is contiguous with two non-empty hinges and such that the first long flat is of height zero. We obtain a contradiction with the fact that M and M' belong to $\mathcal{K}$ which completes the proof.

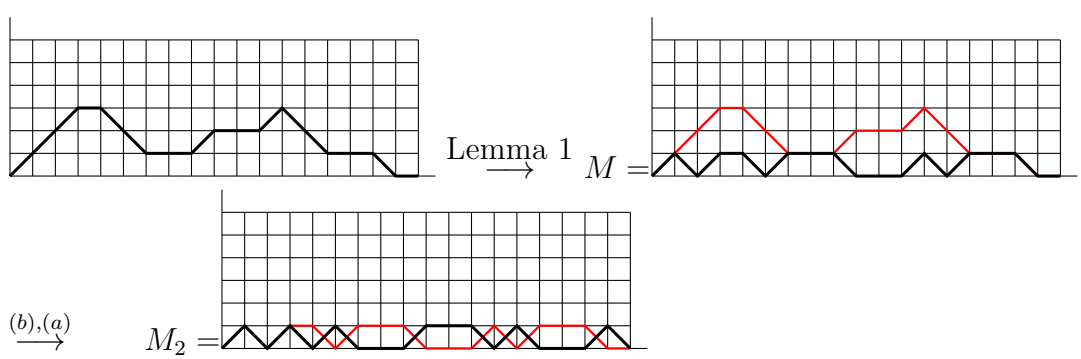


Figure 8: Illustration of the example in the proof of Lemma 15.

For instance, if $M = UFDUDUFFDFFUDUFFDF$ then the decomposition in Fact 2 implies $\beta_0 = UFDUD$, $Q_1 = UFFDFF$, $\beta_1 = UD$, $Q_2 = UFFD$ and $\beta_2 = F$. For

$i = 1$, β_0 and β_1 are non-empty and $Q_1 = UWF$ with $W = FFDF$; the process (b) creates $M_1 = UDUDUDFFUFFDFUFFDF$ where β_0 (resp., β_1 and Q_1) is replaced with $UDUDUD$ (resp., F and $FFUFFD$). For $i = 2$, $\beta_1 = F$ and $\beta_2 = F$ are non-empty and $Q_2 = UFFD$; the process (a) creates $M_2 = UDUDUDFFUFFDUDFFUD$ where β_1 (resp., β_2 and Q_2) is replaced with UD (resp., UD and FF). See Figure 8 for an illustration of this example. □

Lemma 16. *The generating function for the set $\mathcal{J}$, with respect to the length is given by* $J(x) = \frac{x^4-x^2+x-1}{x^7+x^6+2x^4+x-1}$.

Proof. In order to obtain the generating function $J(x)$ for the set $\mathcal{J}$, we calculate the generating functions $J_r^s(x)$ for the set $\mathcal{J}_r^s$ with $r, s \in \{0, 1\}$.

– If M is a Motzkin path in $\mathcal{J}_0^0$. We have either $M = F^k$ or $M = WF^k$ or $M = W'UDF^k$ with $k \geq 2$, $W \in \mathcal{J}_0^1$ and $W' \in \mathcal{J}_0^0$;

– if M is a Motzkin path in $\mathcal{J}_1^0$. We have either $M = WF^k$ or $M = W'UDF^k$ with $k \geq 2$, $W \in \mathcal{J}_1^1$ and $W' \in \mathcal{J}_1^0$;

– If M is a Motzkin path in $\mathcal{J}_0^1$. We have either $M = WUF^kD$ or $M = W'UF^kD$ with $k \geq 2$, $W \in \mathcal{J}_0^0$ and $W' \in \mathcal{J}_0^1$;

– If M is a Motzkin path in $\mathcal{J}_1^1$. We have either $M = UF^kD$ or $M = WUF^kD$ or $M = W'UF^kD$ with $k \geq 2$, $W \in \mathcal{J}_1^0$ and $W' \in \mathcal{J}_1^1$.

We deduce the following system of functional equations:

$$\begin{cases} J_0^0(x) = & \frac{x^2}{1-x}(J_0^1(x) + x^2 J_0^0(x) + 1) \\ J_1^0(x) = & \frac{x^2}{1-x}(J_1^1(x) + x^2 J_1^0(x)) \\ J_1^1(x) = & \frac{x^4}{1-x}(J_0^0(x) + J_0^1(x)) \\ J_1^0(x) = & \frac{x^4}{1-x}(J_1^0(x) + J_1^1(x) + 1). \end{cases}$$

A simple calculation (with Maple for instance) provides the generating functions for the four sets $\mathcal{J}_0^0, \mathcal{J}_0^1, \mathcal{J}_1^1$ and $\mathcal{J}_1^0$, and thus for the set $\mathcal{J} = \cup_{r,s\in\{0,1\}}\mathcal{J}_r^s$ which completes the proof. □

Theorem 17. *The generating function for the set of FF-equivalence classes of $\mathcal{M}$ with respect to the length is given by*

$$\frac{x(1 - x - x^2 + 3x^3 - 5x^4 + 3x^5 - 2x^6 - x^7 - x^9)}{(1 - x - 2x^4 - x^6 - x^7)(1 - 2x + x^2 - x^3)}.$$

Proof. With Lemma 15, it suffices to obtain the generating function $K(x)$ for the set $\mathcal{K}$. First, we calculate the generating functions for the sets $\mathcal{K}_r$ with $r, s \in \{0, 1, 2\}$, where $\mathcal{K}_0$ (resp., $\mathcal{K}_1$, resp., $\mathcal{K}_2$) is the subset of $\mathcal{K}$ of Motzkin path starting by FF (resp., UFF, resp.,

a hinge). Let $J_r^s(x)$, $r, s \in \{0,1\}$, be the generating function obtained in the proof of Lemma 16. We set $J_1(x) = J_1^0(x) + J_1^1(x)$ and $J_0(x) = J_0^0(x) + J_0^1(x)$.

– Let M be a Motzkin path in $\mathcal{K}_0$. We discuss on the number of hinges in M.

If M does not contains any hinge, then this means that $M \in \mathcal{J}_0$; If M has only one hinge, then either $M = QH_kS$, $k \geq 2$, or $M = QH_kT$, $k \geq 3$, or $M = RH_kS$, $k \geq 1$, or $M = RH_kT$, $k \geq 2$ with $Q \in \mathcal{J}_0^0$, $R \in \mathcal{J}_0^1$, $S \in \mathcal{J}_1^0 \cup \mathcal{J}_1^1 \cup \{\epsilon\}$, $T \in \mathcal{J}_0^0 \cup \mathcal{J}_0^1$ and H_k is a hinge of length at least k; If M has at least two hinge, then either $M = QH_kW$, $k \geq 3$, or $M = RH_kW$, $k \geq 2$, with $Q \in \mathcal{J}_0^0$, $R \in \mathcal{J}_0^1$ and $W \in \mathcal{K}_0 \backslash \mathcal{J}_0$.

Thus we have the functional equation:

$$\begin{aligned} K_0(x) = \quad & J_0(x) + J_0^0(x)\tfrac{x^2}{1-x}(J_1(x)+1) + J_0^0(x)\tfrac{x^3}{1-x}J_0(x) + J_0^1(x)\tfrac{x}{1-x}(J_1(x)+1) + \\ & J_0^1(x)\tfrac{x^2}{1-x}J_0(x) + J_0^0(x)\tfrac{x^3}{1-x}(K_0(x) - J_0(x)) + J_0^1(x)\tfrac{x^2}{1-x}(K_0(x) - J_0(x)) \end{aligned}.$$

– Let M be a Motzkin path in $\mathcal{K}_1$. We discuss on the number of hinges in M. If M does not contain any hinge, then this means that $M \in \mathcal{J}_1$; If M has only one hinge, then either $M = Q'H_kS$, $k \geq 2$, or $M = Q'H_kT$, $k \geq 3$, or $M = R'H_kS$, $k \geq 1$, or $M = R'H_kT$, $k \geq 2$ with $Q' \in \mathcal{J}_1^0$, $R' \in \mathcal{J}_1^1$, $S \in \mathcal{J}_1^0 \cup \mathcal{J}_1^1 \cup \{\epsilon\}$, $T \in \mathcal{J}_0^0 \cup \mathcal{J}_0^1$ and H_k is a hinge of length at least k; If M has at least two hinges, then either $M = Q'H_kW$, $k \geq 3$, or $M = R'H_kW$, $k \geq 2$, with $Q' \in \mathcal{J}_1^0$, $R' \in \mathcal{J}_1^1$ and $W \in \mathcal{K}_1 \backslash \mathcal{J}_1$.

Thus we have the functional equation:

$$\begin{aligned} K_1(x) = \quad & J_1(x) + J_1^0(x)\tfrac{x^2}{1-x}(J_1(x)+1) + J_1^0(x)\tfrac{x^3}{1-x}J_0(x) + J_1^1(x)\tfrac{x}{1-x}(J_1(x)+1) + \\ & J_1^1(x)\tfrac{x^2}{1-x}J_0(x) + J_1^0(x)\tfrac{x^3}{1-x}(K_0(x) - J_0(x)) + J_1^1(x)\tfrac{x^2}{1-x}(K_0(x) - J_0(x)) \end{aligned}.$$

– Let M be a Motzkin path in $\mathcal{K}_2$. We have either $M = H_kS$, $k \geq 1$, or $M = H_kT$, $k \geq 2$ with $S \in \mathcal{J}_1^0 \cup \mathcal{J}_1^1$, $T \in \mathcal{J}_0^0 \cup \mathcal{J}_0^1 \cup \{\epsilon\}$ and H_k is a hinge of length at least k.

Thus we have the functional equation:

$$K_2(x) = J_1(x)\frac{x}{1-x} + (K_0(x)+1)\frac{x^2}{1-x}.$$

Using the Lemma 16, a simple calculation with Maple gives the generating function $K(x) = K_0(x) + K_1(x) + K_2(x)$ for the set $\mathcal{K}$. □

References

[1] J.-L. Baril and A. Petrossian, Equivalence of Dyck paths modulo some statistics, *Discrete Math.* **338** (2015), 655–660.

[2] E. Barcucci, R. Pinzani and R. Sprugnoli, The Motzkin family, *Pure Math. Appl. Ser. A* **2** (1991), 249–279.

[3] C. Brennan and S. Mavhungu, Peaks and valleys in Motzkin paths, *Quaest. Math.* **33** (2010), 171–188.

[4] E. Deutsch, Dyck path enumeration, *Discrete Math.* **204** (1999), 167–202.

[5] R. Donaghey and L. W. Shapiro, Motzkin numbers, *J. Combin. Theory Ser. A* **23** (1977), 291–301.

[6] D. Drake and R. Gantner, Generating functions for plateaus in Motzkin paths, *J. Chungcheong Math. Society* **25** (2012), 475–489.

[7] T. Mansour, Statistics on Dyck paths, *J. Integer Sequences* **9** (2006), Article 06.1.5.

[8] T. Mansour, Counting peaks at height k in a Dyck path, *J. Integer Sequences* **5** (2002), Article 02.1.1.

[9] T. Mansour and M. Shattuck, Counting Dyck paths according to the maximum distance between peaks and valleys, *J. Integer Sequences* **15** (2012), Article 12.1.1.

[10] T. Mansour, M. Schork and Y. Sun, Motzkin numbers of higher rank: Generating function and explicit expression, *J. Integer Sequences* **10** (2007), Article 07.7.4.

[11] D. Merlini, R. Sprugnoli and M. C. Verri, Some statistics on Dyck paths, *J. Statis. Plann. Inference* **101** (2002), 211–227.

[12] A. Panayotopoulos and A. Sapounakis, On the prime decomposition of Dyck paths, *J. Combin. Math. Combin. Comput.* **40** (2002), 33–39.

[13] P. Peart and W. J. Woan, Dyck paths with no peaks at height k, *J. Integer Sequences* **4** (2001), Article 01.1.3.

[14] H. Prodinger and S. Wagner, Minimal and maximal plateau lengths in Motzkin paths, 2007 Conference on Analysis of Algorithms, *Discrete Math. Theor. Comput. Sci. Proc.*, AH (2007), 353–364.

[15] A. Sapounakis, I. Tasoula and P. Tsikouras, Counting strings in Dyck paths, *Discrete Math.* **307** (2007), 2909–2924.

[16] A. Sapounakis, I. Tasoulas and P. Tsikouras, Some strings in Dyck paths, *Australasian J. Combin.* **39** (2007), 49–72.

[17] A. Sapounakis and P. Tsikouras, Couting peaks and valleys in k-colored Motzkin words, *Electron. J. Comb.* **12** (2005), Article 12.

[18] N. J. A. Sloane: The On-line Encyclopedia of Integer Sequences, available electronically at `http://oeis.org`.

[19] Y. Sun, The statistic "number of udu's" in Dyck paths, *Discrete Math.* **287** (2004), 177–186.

[20] R. P. Stanley, *Enumerative Combinatorics*, Vol. 2, Cambridge University Press, 1999.

2010 *Mathematics Subject Classification*: Primary 68R15.
Key words: Motzkin path, equivalence relation, statistics.

(Concerned with sequences A001405, A000079, A191385, A132916, A000045, A165407, and A191385.)

Received January 12 2015; revised versions received May 20 2015; June 4 2015. Published in *Journal of Integer Sequences*, June 16 2015.

Return to Journal of Integer Sequences home page.

Discrete Mathematics 339 (2016) 1199–1205

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Rooted planar maps modulo some patterns

Jean-Luc Baril [a,*], Richard Genestier [b], Alain Giorgetti [b], Armen Petrossian [a]

[a] LE2I UMR CNRS 6306, University of Bourgogne, B.P. 47 870, 21078 Dijon Cedex, France

[b] FEMTO-ST, UMR CNRS 6174 (UBFC/UFC/ENSMM/UTBM), University of Franche-Comté, 25030 Besançon Cedex, France

ARTICLE INFO

Article history:
Received 19 June 2015
Received in revised form 13 November 2015
Accepted 16 November 2015
Available online 15 December 2015

Keywords:
Rooted planar map
Shuffle of Dyck words
Equivalence relation
Pattern

ABSTRACT

We provide generating functions for the number of equivalence classes of rooted planar maps where two maps are equivalent whenever their representations in shuffles of Dyck words coincide on all occurrences of a given pattern.

© 2015 Elsevier B.V. All rights reserved.

1. Introduction and notations

A *map* is defined topologically as a 2-cell imbedding of a connected graph, loops and multiple edges allowed, in a 2-dimensional surface. A *rooted planar map* is a map on a sphere with a distinguished edge, called the *root*, assigned with an orientation. Fig. 1(a) shows a rooted planar map with seven edges. We refer to [1,15,16] for the enumeration of rooted planar maps with respect to the number of edges. Planar maps have been widely studied for their combinatorial structure and their links with other domains such as theoretical physics where they appear in some models of 2-dimensional quantum gravity for instance. From a combinatorial point of view, it is proved in [8] that rooted planar maps are in one-to-one correspondence with shuffles of Dyck words that avoid a specific pattern. Then, it becomes natural to extend to maps statistical studies on Dyck words (see [7,9,12,10,11,13] for instance). It is one of the main purpose of this paper.

A *Dyck word* on the alphabet $\{a, \bar{a}\}$ is a word generated by the context-free grammar $S \to \varepsilon \mid aS\bar{a}S$ where ε is the empty word. Let $\mathcal{D}_a$ be the set of all Dyck words on the alphabet $\{a, \bar{a}\}$, and let $\mathcal{D}_b$ be the set of all Dyck words on the alphabet $\{b, \bar{b}\}$. It is well known that Dyck words of semilength n are counted by Catalan numbers (A000108 in the on-line encyclopedia of integer sequences [14]), and that any non-empty Dyck word $w \in \mathcal{D}_a$ has a unique decomposition of the form $w = a\alpha\bar{a}\beta$ where α and β are two Dyck words in $\mathcal{D}_a$ (see [7]). Also, a word w on the alphabet $\{a, \bar{a}\}$ belongs to $\mathcal{D}_a$ if and only if the following two properties hold: (i) the number of letters a is equal to the number of letters $\bar{a}$ in w, and (ii) in any prefix of w, the number of letters a is greater than or equal to the number of letters $\bar{a}$.

We say that an occurrence of the letter a *matches* an occurrence of $\bar{a}$ located to its right in $w \in \mathcal{D}_a$, whenever the subword of w consisting of all letters between these two occurrences also belongs to $\mathcal{D}_a$. In this case, the pair $(a, \bar{a})$ is called a *matching* in w. For instance, if $w = a\bar{a}aa\bar{a}aa\bar{a}\bar{a}\bar{a}$, then the second occurrence of the letter a matches the last occurrence of $\bar{a}$ since $a\bar{a}aa\bar{a}\bar{a}$ is a Dyck word.

* Corresponding author.
E-mail addresses: barjl@u-bourgogne.fr (J.-L. Baril), richard.genestier@femto-st.fr (R. Genestier), alain.giorgetti@femto-st.fr (A. Giorgetti), armen.petrossian@u-bourgogne.fr (A. Petrossian).

http://dx.doi.org/10.1016/j.disc.2015.11.007
0012-365X/© 2015 Elsevier B.V. All rights reserved.

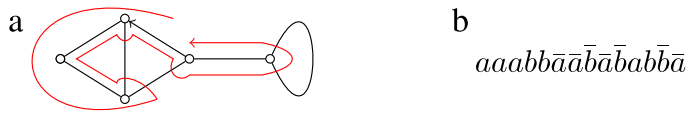
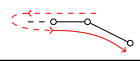


Fig. 1. A rooted planar map with 7 edges and its non-crossing shuffle representation.

Table 1
Correspondence between patterns in a shuffle and patterns in a rooted planar map.

Shuffle	a	aa	$a\bar{a}$	b	$b\bar{b}$	$\bar{a}a$	$a\bar{b}$
Map							Does not appear

A *shuffle* of two Dyck words $v \in \mathcal{D}_a$ and $w \in \mathcal{D}_b$ is a word s on the alphabet $\{a, \bar{a}, b, \bar{b}\}$ where s is constructed by interspersing the letters of v and w. Let $\mathcal{S}$ be the set of all shuffles of two Dyck words of $\mathcal{D}_a$ and $\mathcal{D}_b$. Shuffles of semilength n, $n \geq 0$, in $\mathcal{S}$ are enumerated by the sequence A005568 in [14]. The first values for $n \geq 0$ are 1, 2, 10, 70, 588, 5544, 56628. For instance, $s = aab\bar{a}\bar{a}\bar{b}ab\bar{b}\bar{a}$ is a shuffle of the two Dyck words $aa\bar{a}\bar{a}a\bar{a} \in \mathcal{D}_a$ and $b\bar{b}b\bar{b} \in \mathcal{D}_b$.

For any shuffle s, we denote by $w_a(s)$ (resp. $w_b(s)$) the Dyck word in $\mathcal{D}_a$ (resp. $\mathcal{D}_b$) obtained from s by deleting the letters b and $\bar{b}$ (resp. a and $\bar{a}$). In the following, we will extend the definition of $w_a(s)$ and $w_b(s)$ for any word in $\{a, \bar{a}, b, \bar{b}\}^*$. In particular, if s is a prefix of shuffles, $w_a(s)$ (resp. $w_b(s)$) becomes a prefix of a Dyck word in $\mathcal{D}_a$ (resp. $\mathcal{D}_b$). For instance, if $s = aab\bar{a}\bar{a}\bar{b}ab\bar{b}\bar{a}$, then $w_a(s) = aa\bar{a}\bar{a}a\bar{a}$ and $w_b(s) = b\bar{b}b\bar{b}$; and if $s = aab\bar{a}$, it is a prefix of a shuffle, and $w_a(s) = aa\bar{a}$ and $w_b(s) = b$ are prefixes of Dyck words. Then, *a word s is a shuffle in $\mathcal{S}$ if and only if $w_a(s) \in \mathcal{D}_a$ and $w_b(s) \in \mathcal{D}_b$.*

A shuffle s of two Dyck words $v \in \mathcal{D}_a$ and $w \in \mathcal{D}_b$ will be called *crossing* whenever there exists a matching $(a, \bar{a})$ in v and a matching $(b, \bar{b})$ in w such that s can be written $s = \alpha b \beta a \gamma \bar{b} \delta \bar{a} \eta$ where $\alpha, \beta, \gamma, \delta$ and η belong to $\{a, \bar{a}, b, \bar{b}\}^*$. Then the occurrence $ba\bar{b}\bar{a}$ will be called a *pair of crossing matchings*. Let $\mathcal{NCS} \subset \mathcal{S}$ be the subset of non-crossing shuffles in $\mathcal{S}$, *i.e.*, shuffles with no pair of crossing matchings. For instance, $a\bar{a}b\bar{b}a\bar{a}\bar{b}\bar{b}a\bar{a}$ is in $\mathcal{NCS}$, while $a\bar{a}bb\bar{b}a\bar{b}\bar{a}$ is not in $\mathcal{NCS}$. Notice that non-crossing shuffles are called *canonical parenthesis-bracket systems* in [17]. The shuffles of semilength $n \geq 0$ in $\mathcal{NCS}$ are enumerated by the sequence A000168 in [14] whose first values for $n \geq 0$ are 1, 2, 9, 54, 378, 2916, 24057, 208494. They are in one-to-one correspondence with the rooted planar maps with n edges [5,6,8,17]. See Fig. 1 for an example of rooted planar map with its representation as a non-crossing shuffle in $\mathcal{NCS}$. This one-to-one correspondence is obtained by the following process. Starting with the root edge, we follow or cross all edges of the map by making its tour in counter-clockwise direction. Each edge in the map must be reached twice. If the final vertex of the encountered edge has not yet been considered, then we follow this edge and we write the letter a; otherwise, if the edge is reached for the first time, then we write the letter b and we cross it; in the other cases the edge is reached for the second time, and we write $\bar{a}$ (resp. $\bar{b}$) and we follow the edge (resp. we cross the edge) whenever the first assignment of the edge was the letter a (resp. b). Table 1 gives the correspondence between some patterns of length at most two in a shuffle with their meaning in terms of map. For instance, a pattern $b\bar{b}$ in a shuffle of $\mathcal{NCS}$ corresponds to an empty loop on a vertex of the corresponding rooted planar map. Also, the pattern $a\bar{b}$ cannot occur in the shuffle representation of a rooted planar map because it necessarily creates a pair of crossing matchings.

In a recent paper [3] (for equivalence classes of permutations see [2]), the authors introduced an equivalence relation on the set of Dyck words where *two Dyck words are equivalent whenever the positions of the occurrences of a given pattern are the same in both words.* They provided generating functions for the numbers of equivalence classes whenever the patterns are of length two: aa, $a\bar{a}$, $\bar{a}\bar{a}$ and $\bar{a}a$. See also [4] for a study of this equivalence relation on Motzkin words.

The motivation of this paper is to present a similar study for shuffles of Dyck words and for rooted planar maps considered with their representation by shuffles in $\mathcal{NCS}$. So, we define the equivalence relation on $\mathcal{S}$ and $\mathcal{NCS}$ for any pattern π of length at most two:

two shuffles with the same semilength are π-equivalent whenever they coincide on the positions of occurrences of the pattern π.

For instance, the shuffle $s = aaba\bar{b}\bar{a}b\bar{a}\bar{b}\bar{a}$ is $\bar{b}\bar{a}$-equivalent to $s' = a\bar{a}ab\bar{b}\bar{a}ab\bar{b}\bar{a}$ since the occurrences of $\bar{b}\bar{a}$ appear in positions 5 and 9 in s and s'.

In Section 2, we show that the problem of the enumeration of the π-equivalence classes in $\mathcal{NCS}$ is the same as in $\mathcal{S}$ whenever π is a pattern of length at most two that does not belong to the set $\{\bar{b}a, b\bar{a}, a\bar{b}, \bar{a}b\}$. In Sections 3–6, we present enumerative results by providing generating functions for the number of π-equivalence classes when $\pi \notin \{\bar{b}a, b\bar{a}, a\bar{b}, \bar{a}b\}$. Using the one-to-one correspondence between rooted planar maps and non-crossing shuffles of Dyck words, this induces enumerative results for equivalence classes of rooted planar maps relatively to the positions of some patterns. See Table 2 for an overview of these results. Notice that the pattern $a\bar{b}$ does not appear in any shuffle of a rooted planar map, and that we did not succeed to obtain the number of π-equivalence classes in $\mathcal{S}$ and $\mathcal{NCS}$ for $\pi \in \{\bar{a}b, \bar{b}a, b\bar{a}\}$. So, we leave these cases as open problems.

2. Some preliminary results

In this section, we prove that for some specific patterns π the numbers of π-equivalence classes in $\mathcal{S}$ and $\mathcal{NCS}$ are the same.

Table 2
Enumeration of π-equivalence classes for shuffles and rooted planar maps.

Pattern	Sequence	Sloane	a_n, $1 \le n \le 9$
$\{a\}$, $\{\bar{a}\}$, $\{b\}$, $\{\bar{b}\}$	$\frac{1}{\sqrt{1-4x}}$	Central binomial coeff. A000984	2, 6, 20, 70, 252, 924, 3432,12870,48620
$\{a\bar{a}\}$, $\{b\bar{b}\}$	$\frac{1-x}{1-3x+x^2}$	Shift of A001519	2,5,13,34,89,233,610,1597,4181
$\{\bar{a}a\}$, $\{\bar{b}b\}$	$\frac{1-2x}{1-3x+x^2}$	A001519	1,2,5,13,34,89,233,610,1597
$\{aa\}$, $\{\bar{a}\bar{a}\}$, $\{bb\}$, $\{\bar{b}\bar{b}\}$	$\frac{1+x-\sqrt{1-2x-3x^2}}{2x^2+3x-1+\sqrt{1-2x-3x^2}}$	Bisection of A191385	1,2,5,12,31,81,216,583,1590
$\{ab\}$, $\{ba\}$, $\{\bar{a}\bar{b}\}$, $\{\bar{b}\bar{a}\}$	$\frac{-2\sqrt{3}\sin\left(1/3\ \arcsin\left(3/2\sqrt{3}x\right)\right)}{4\left(\sin\left(1/3\ \arcsin\left(3/2\sqrt{3}x\right)\right)\right)^2-3x}$	A138164	1,2,4,9,20,47,109,262,622

Lemma 1. *Let $w = \alpha\bar{a}\beta\gamma$ be a Dyck word in $\mathcal{D}_a$ with $\beta\gamma$ in $\{a, \bar{a}\}^*$. Then the word $w' = \alpha\beta\bar{a}\gamma$ is also in $\mathcal{D}_a$.*

Proof. The Dyck word w' is obtained from w by a shift on the right of a letter $\bar{a}$. So, the number of a and the number of $\bar{a}$ remains unchanged in w'. Moreover, in any prefix of w the number of a is greater than or equal to the number of $\bar{a}$ and this property remains satisfied in w'. Using the characterization of a Dyck word given in the introduction, w' is in $\mathcal{D}_a$. □

Lemma 2. *Let $w = \alpha a\beta\bar{a}\gamma$ be a Dyck word in $\mathcal{D}_a$. If $\beta = \beta_1\beta_2\ldots\beta_k$ is in $\mathcal{D}_a$ then for any i, $1 \le i \le k$, the word $w' = \alpha a\beta_1\ldots\beta_{i-1}\bar{a}\beta_i\ldots\beta_k\gamma$ is also in $\mathcal{D}_a$.*

Proof. Since w is in $\mathcal{D}_a$, (i) the number of a and the number of $\bar{a}$ are equal in w, and in any prefix (ii) the number of a is greater than or equal to the number of $\bar{a}$. Since β is in $\mathcal{D}_a$, the number of a in any prefix of $\alpha a\beta_1\ldots\beta_{i-1}$ is strictly greater than the number of $\bar{a}$. This means that $\alpha a\beta_1\ldots\beta_{i-1}\bar{a}$ satisfies Condition (ii). Then, w' satisfies (ii) which implies that w' belongs to $\mathcal{D}_a$. □

Of course, the last two lemmas also hold if we replace the letter a with b. So, we use them indifferently for a and b.

Theorem 1. *Let s be a shuffle in $\mathcal{S}$ and π be a pattern of length at most two not in $\{a\bar{b}, \bar{a}b, \bar{b}a, b\bar{a}\}$. Then, there exists a shuffle s' in $\mathcal{NCS}$ with the same semilength as s so that s and s' are π-equivalent.*

Proof. Let s be a shuffle in $\mathcal{S}$ that does not lie in $\mathcal{NCS}$, *i.e.* s contains a pair of crossing matchings. Then, s can be decomposed $s = \alpha b\beta a\gamma\bar{b}\delta\bar{a}\eta$ so that α, β, γ, δ and η are in $\{a, \bar{a}, b, \bar{b}\}^*$ and the crossing matching $ba\bar{b}\bar{a}$ is chosen the leftmost possible in s.

We distinguish three cases: (i) $\pi \in \{a, aa, ab, ba\}$; (ii) $\pi = a\bar{a}$; and (iii) $\pi = \bar{a}a$. The other cases are easily obtained using classical symmetries on shuffles ($a \leftrightarrow b$ and mirror).

Case (i): Let us consider the word $s' = \alpha b\beta a\gamma\bar{a}\delta\bar{b}\eta$ obtained from s by an exchange of $\bar{a}$ and $\bar{b}$. With Lemmas 1 and 2, s' is also a shuffle of two Dyck words. The two shuffles s and s' belong to the same π-equivalence class since this operation cannot create or delete a pattern in $\{a, aa, ab, ba\}$. Moreover the leftmost pair of crossing matchings in s' is shifted on the right compared to s. We repeat the process for as long as required, and we obtain a shuffle in $\mathcal{NCS}$ with the same semilength as s, which is π-equivalent to s.

Case (ii): Let us assume that $s = s_1s_2\ldots s_{2k}$ for some $k \ge 1$ with $s_i \in \{a, \bar{a}, b, \bar{b}\}$, $1 \le i \le 2k$. We obtain s' from s by the following process: we preserve the positions of all occurrences of the pattern $a\bar{a}$; the other letters of s' are chosen so that the restriction of s' to them is the Dyck word of the form $b^\ell\bar{b}^\ell$ for some $\ell \ge 0$. It is straightforward to see that $s' \in \mathcal{NCS}$ and that the two shuffles s and s' belong to the same $a\bar{a}$-equivalence class.

Case (iii): Let us assume that $s = s_1s_2\ldots s_{2k}$ for some $k \ge 1$ with $s_i \in \{a, \bar{a}, b, \bar{b}\}$, $1 \le i \le 2k$. We obtain s' from s by the following process: we preserve the positions of all occurrences of the pattern $\bar{a}a$ and we set $s'_1 = a$ and $s'_{2k} = \bar{a}$; the other letters of s' are chosen so that the restriction to them is the Dyck word of the form $a^\ell\bar{a}^\ell$ for some $\ell \ge 0$. It is straightforward to see that $s' \in \mathcal{D}_a \subset \mathcal{NCS}$ and that the two shuffles s and s' belong to the same $\bar{a}a$-equivalence class. □

Corollary 1. *Let π be a pattern of length at most two not in $\{\bar{b}a, b\bar{a}, a\bar{b}, \bar{a}b\}$. For each word length the number of π-equivalence classes in $\mathcal{S}$ is also the number of π-equivalence classes in $\mathcal{NCS}$.*

This corollary allows us to limit our study of the general case to shuffles in $\mathcal{S}$ and patterns π not in $\{\bar{b}a, b\bar{a}, a\bar{b}, \bar{a}b\}$. All results in the following sections will hold for both sets $\mathcal{S}$ and $\mathcal{NCS}$.

3. Equivalence modulo $\pi \in \{a, b, \bar{a}, \bar{b}\}$

The results for $\pi \in \{b, \bar{a}, \bar{b}\}$ are deduced from the ones for $\pi = a$ by using the classical symmetries (mirror and $a \leftrightarrow b$) on shuffles. So, we set $\pi = a$ in this section.

Let $\mathcal{A}$ be the set of shuffles in $\mathcal{S}$ defined by the grammar $\mathcal{A} \to \mathcal{D}_a \mid \mathcal{D}_a b\mathcal{D}_a\bar{b}\mathcal{A}$ where $\mathcal{D}_a$ is the set of Dyck words on the alphabet $\{a, \bar{a}\}$.

Lemma 3. *The set $\mathcal{A}$ and the set of a-equivalence classes of $\mathcal{S}$ are in length-preserving one-to-one correspondence.*

Proof. For $k \geq 1$ let $s = s_1 s_2 \dots s_{2k}$ be a shuffle in $\mathcal{S} \backslash \mathcal{A}$ with $s_i \in \{a, \bar{a}, b, \bar{b}\}$ for $1 \leq i \leq 2k$. Let us prove that there exists a shuffle $s' \in \mathcal{A}$ (with the same semilength as s) such that s and s' belong to the same a-equivalence class. We define the word s' by performing the following process on s for i from 1 to $2k$:

- If $s_i = a$ then we set $s'_i = a$;
- Otherwise, we distinguish three cases:
  (i) If $w_a(s'_1 \dots s'_{i-1}) \notin \mathcal{D}_a$ then we set $s'_i = \bar{a}$;
  (ii) If $w_a(s'_1 \dots s'_{i-1}) \in \mathcal{D}_a$ and $w_b(s'_1 \dots s'_{i-1}) \notin \mathcal{D}_b$ then we set $s'_i = \bar{b}$;
  (iii) If $w_a(s'_1 \dots s'_{i-1}) \in \mathcal{D}_a$ and $w_b(s'_1 \dots s'_{i-1}) \in \mathcal{D}_b$ then we set $s'_i = b$.

It is worth noticing that $w_a(s'_1 \dots s'_i)$ (resp. $w_b(s'_1 \dots s'_i)$) is a prefix of a Dyck word in $\mathcal{D}_a$ (resp. $\mathcal{D}_b$) for all i, $1 \leq i \leq 2k$. Moreover, at the end of the process we necessarily have $w_a(s'_1 \dots s'_{2k}) \in \mathcal{D}_a$ and $w_b(s'_1 \dots s'_{2k}) \in \mathcal{D}_b$ which means that s' belongs to $\mathcal{S}$.

Since the process preserves the occurrences of the letter a in s and do not introduce other letters a in s', s and s' belong to the same a-equivalence class. Moreover, s' necessarily lies in $\mathcal{A}$. Indeed, the process sets $s'_i = \bar{a}$ whenever $w_a(s'_1 \dots s'_{i-1})$ is not a Dyck word of $\mathcal{D}_a$; in all other cases, it sets $s'_i = b$ or $s'_i = \bar{b}$ alternatively. If s' does not contain any occurrence of b, then $s' \in \mathcal{D}_a \subset \mathcal{A}$; otherwise, the previous construction ensures that s' has a prefix of the form $\alpha b \beta \bar{b}$ with α and β in $\mathcal{D}_a$. This means that s' satisfies the grammar $\mathcal{A} \to \mathcal{D}_a \mid \mathcal{D}_a b \mathcal{D}_a \bar{b} \mathcal{A}$. Thus, we have $s' \in \mathcal{A}$.

Now, it suffices to prove that two distinct shuffles s and s' with the same semilength in $\mathcal{A}$ are not a-equivalent. For a contradiction, let us assume that s and s' belong to the same class.

We distinguish two cases:

- If $s \in \mathcal{D}_a$ or $s' \in \mathcal{D}_a$ with the same positions of the occurrences of a then we have $s = s'$.
- Otherwise, s can be written $s = \alpha b \beta \bar{b} \gamma$ with α and β in $\mathcal{D}_a$ and γ in $\mathcal{A}$. Also, s' can be written $s' = \alpha' b \beta' \bar{b} \gamma'$ with α' and β' in $\mathcal{D}_a$ and γ' in $\mathcal{A}$.

Since α and α' belong to $\mathcal{D}_a$ and have the same positions of occurrences of the letter a, we necessarily have $\alpha = \alpha'$. The same argument for β implies that $\beta = \beta'$. We complete the proof by induction on the length for γ and γ' in $\mathcal{A}$ and we conclude $s = s'$. □

Theorem 2. *The generating function for the set of a-equivalence classes in $\mathcal{S}$ (or in $\mathcal{NCS}$) with respect to the semilength is given by*

$$\frac{1}{\sqrt{1-4x}} = \sum_{n \geq 0} \binom{2n}{n} x^n.$$

Proof. By Lemma 3, it suffices to provide the generating function $A(x)$ of the set $\mathcal{A}$, with respect to the semilength. Every nonempty shuffle $s \in \mathcal{A}$ is obtained by the grammar $\mathcal{A} \to \mathcal{D}_a \mid \mathcal{D}_a b \mathcal{D}_a \bar{b} \mathcal{A}$ where $\mathcal{D}_a$ is the set of Dyck words on the alphabet $\{a, \bar{a}\}$. This induces the functional equation:

$$A(x) = D(x) + xD(x)^2 A(x)$$

where $D(x) = \frac{1-\sqrt{1-4x}}{2x}$ is the Catalan generating function for the set $\mathcal{D}_a$, which implies the required result. □

4. Equivalence modulo $\pi \in \{a\bar{a}, b\bar{b}, \bar{a}a, \bar{b}b\}$

The results for $\pi \in \{b\bar{b}, \bar{b}b\}$ are deduced from the ones for $\pi = a\bar{a}$ and $\pi = \bar{a}a$ by using the classical symmetries. Then, we study in this section the two cases $\pi = a\bar{a}$ and $\pi = \bar{a}a$.

Let $\mathcal{B}$ be the subset of $\mathcal{S}$ defined by the grammar $\mathcal{B} \to a\bar{a}\mathcal{B} \mid b\mathcal{B}'\bar{b}\mathcal{B} \mid \varepsilon$ where $\mathcal{B}' \to a\bar{a}\mathcal{B}' \mid \varepsilon$. It is worth noticing that the set $\mathcal{B}'$ consists of words of the form $(a\bar{a})^k$ for all $k \geq 0$. Moreover, it is straightforward to see that for any $s \in \mathcal{B}$ we have $w_b(s) = (b\bar{b})^i$ for some $i \geq 0$ and $w_a(s) = (a\bar{a})^j$ for some $j \geq 0$. Finally, if $X = a\bar{a}$ then the set $\mathcal{B}$ consists of all shuffles of words X^i, $i \geq 0$, with Dyck words of the form $(b\bar{b})^j$, $j \geq 0$, where the occurrences of $X = a\bar{a}$ are never split. For instance, $ba\bar{a}a\bar{a}\bar{b}a\bar{a}b\bar{b}$ belongs to $\mathcal{B}$, while $ba\bar{a}a\bar{b}\bar{a}a\bar{a}b\bar{b} \notin \mathcal{B}$.

Lemma 4. *The set $\mathcal{B}$ and the set of $a\bar{a}$-equivalence classes of $\mathcal{S}$ are in length-preserving one-to-one correspondence.*

Proof. For $k \geq 1$ let $s = s_1 s_2 \dots s_{2k}$ be a shuffle in $\mathcal{S} \backslash \mathcal{B}$ with $s_i \in \{a, \bar{a}, b, \bar{b}\}$ for $1 \leq i \leq 2k$. Let us prove that there exists a shuffle $s' \in \mathcal{B}$ (with the same semilength as s) such that s and s' belong to the same class. We define the word s' by performing the following process on s:

- all occurrences of the pattern $a\bar{a}$ in s are preserved in s';
- for i from 1 to $2k$ such that s_i does not belong to a pattern $a\bar{a}$ in s, we distinguish two cases:
  (i) if $w_b(s'_1 \dots s'_{i-1}) = (b\bar{b})^j$ for some $j \geq 0$, then we set $s'_i = b$,
  (ii) if $w_b(s'_1 \dots s'_{i-1}) = (b\bar{b})^j b$ for some $j \geq 0$, then we set $s'_i = \bar{b}$.

For all i ($1 \le i \le 2k$) we have either $w_b(s'_1 \dots s'_i) = (b\bar{b})^j$ or $w_b(s'_1 \dots s'_i) = (b\bar{b})^j b$ for some $j \ge 0$. Moreover, we have $w_a(s'_1 \dots s'_i) = (a\bar{a})^\ell$ for some $\ell \ge 0$ so that a and $\bar{a}$ appear necessarily adjacent in s' in an occurrence $a\bar{a}$. This means that s' is a shuffle that lies in $\mathcal{B}$. Since the process preserves the occurrences of the pattern $a\bar{a}$, s and s' belong to the same $a\bar{a}$-equivalence class.

Less formally, s' is obtained from s by fixing the occurrences of $a\bar{a}$ and by replacing all other letters by a Dyck word of the form $(b\bar{b})^j$ for some $j \ge 0$. Therefore, the definition of $\mathcal{B}$ ensures us that s' is the unique element in $\mathcal{B}$ that lies into the same $a\bar{a}$-equivalence class as s. □

Theorem 3. *The generating function for the set of $a\bar{a}$-equivalence classes in $\mathcal{S}$ with respect to the semilength is given by*

$$\frac{1-x}{1-3x+x^2} = \sum_{n\ge 0} F_{2n+1}x^n$$

where F_m is the mth Fibonacci number (defined by $F_m = F_{m-1} + F_{m-2}$ with $F_0 = 0$ and $F_1 = 1$).

Proof. By Lemma 4, it suffices to provide the generating function $B(x)$ of the set $\mathcal{B}$, with respect to the semilength. Every shuffle $s \in \mathcal{B}$ is obtained by the grammar $\mathcal{B} \to a\bar{a}\mathcal{B} \mid b\mathcal{B}'\bar{b}\mathcal{B} \mid \varepsilon$ where $\mathcal{B}' \to a\bar{a}\mathcal{B}' \mid \varepsilon$. This induces the functional equation:

$$B(x) = 1 + xB(x) + xB(x)B'(x)$$

where $B'(x) = 1 + xB'(x)$ is the generating function for the set $\mathcal{B}'$.

A simple calculation provides the result. □

Now we consider the equivalence relation for the pattern $\pi = \bar{a}a$. Let $\mathcal{B}''$ be the set $\mathcal{B}'' = a\mathcal{B}\bar{a} \mid \varepsilon$ where $\mathcal{B}$ is defined at the beginning of this section.

Lemma 5. *The set $\mathcal{B}''$ and the set of $\bar{a}a$-equivalence classes of $\mathcal{S}$ are in length-preserving one-to-one correspondence.*

Proof. Let $s = s_1 s_2 \dots s_{2k}$ ($k \ge 1$) be a nonempty shuffle in $\mathcal{S}\backslash\mathcal{B}''$ with $s_i \in \{a, \bar{a}, b, \bar{b}\}$ for $1 \le i \le 2k$. Let us prove that there exists a shuffle $s' \in \mathcal{B}''$ (with the same semilength as s) such that s and s' belong to the same $\bar{a}a$-equivalence class.

We define the shuffle s' as follows:

- we set $s'_1 = a$ and $s'_{2k} = \bar{a}$;
- all occurrences of the pattern $\bar{a}a$ in s are preserved in s';
- for i from 2 to $2k-1$ such that s_i does not belong to a pattern $\bar{a}a$, we distinguish two cases:
  (i) if $w_b(s'_1 \dots s'_{i-1}) = (b\bar{b})^j$ for some $j \ge 0$, then we set $s'_i = b$,
  (ii) if $w_b(s'_1 \dots s'_{i-1}) = (b\bar{b})^j b$ for some $j \ge 0$, then we set $s'_i = \bar{b}$.

For all i, $1 \le i \le 2k$ we have either $w_b(s'_1 \dots s'_i) = (b\bar{b})^j$ or $w_b(s'_1 \dots s'_i) = (b\bar{b})^j b$ for some $j \ge 0$. Moreover, we have $w_a(s'_1 \dots s'_i) = (a\bar{a})^\ell$ for some $\ell \ge 0$ so that a and $\bar{a}$ appear necessarily adjacent in s' in an occurrence $\bar{a}a$, except for $s'_1 = a$ and $s'_{2k} = \bar{a}$. This means that s' belongs to $\mathcal{B}''$. Since the process preserves the occurrences of the pattern $\bar{a}a$, s and s' belong to the same $\bar{a}a$-equivalence class.

This process ensures us that s' is the unique element in $\mathcal{B}''$ that lies into the same $\bar{a}a$-equivalence class as s. □

Theorem 4. *The generating function for the set of $\bar{a}a$-equivalence classes in $\mathcal{S}$ with respect to the semilength is given by*

$$\frac{1-2x}{1-3x+x^2} = \sum_{n\ge 0} F_{2n-1}x^n$$

where F_m is the mth Fibonacci number (with $F_{-1} = 1$).

Proof. By Lemma 5, it suffices to provide the generating function for $\mathcal{B}'' = a\mathcal{B}\bar{a} \mid \varepsilon$ which is $B''(x) = xB(x) + 1 = \frac{1-2x}{1-3x+x^2}$. □

5. Equivalence modulo $\pi \in \{aa, \bar{a}\bar{a}, bb, \bar{b}\bar{b}\}$

The results for $\pi \in \{\bar{a}\bar{a}, bb, \bar{b}\bar{b}\}$ are deduced from the ones for $\pi = aa$ by using the classical symmetries.

Let $\mathcal{C}$ be the subset of $\mathcal{S}$ defined by the grammar $\mathcal{C} \to \mathcal{C}' \mid \mathcal{C}'b\mathcal{C}'\bar{b}\mathcal{C}$ where $\mathcal{C}' \to aa\bar{a}\mathcal{C}'\bar{a}\mathcal{C}' \mid a(\mathcal{C}'\backslash\varepsilon)\bar{a}\mathcal{C}' \mid \varepsilon$. In fact, the set $\mathcal{C}'$ consists of Dyck words in $\mathcal{D}_a$ such that every occurrence of the letter a is contiguous with another occurrence of a.

Lemma 6. *The set $\mathcal{C}$ and the set of aa-equivalence classes of $\mathcal{S}$ are in length-preserving one-to-one correspondence.*

Proof. Let $s = s_1s_2\dots s_{2k}$ $(k \geq 1)$ be a shuffle in $\mathcal{S}\backslash\mathcal{C}$ with $s_i \in \{a, \bar{a}, b, \bar{b}\}$ for $1 \leq i \leq 2k$. Let us prove that there exists a shuffle $s' \in \mathcal{C}$ (with the same semilength as s) such that s and s' belong to the same class. We define the word s' by performing the following process on s:

- all occurrences of the pattern aa in s are preserved in s';

- for i from 1 to $2k$ such that s_i does not belong to a pattern aa, we distinguish three cases:
  (i) If $w_a(s'_1s'_2\dots s'_{i-1}) \in \mathcal{D}_a$ and $w_b(s'_1s'_2\dots s'_{i-1}) = (b\bar{b})^jb$ for some $j \geq 0$, then we set $s'_i = \bar{b}$;
  (ii) If $w_a(s'_1s'_2\dots s'_{i-1}) \in \mathcal{D}_a$ and $w_b(s'_1s'_2\dots s'_{i-1}) = (b\bar{b})^j$ for some $j \geq 0$, then we set $s'_i = b$;
  (iii) If $w_a(s'_1s'_2\dots s'_{i-1}) \notin \mathcal{D}_a$ then $s'_i = \bar{a}$.

For all i, $1 \leq i \leq 2k$ we have either $w_b(s'_1s'_2\dots s'_i) = (b\bar{b})^j$ or $w_b(s'_1s'_2\dots s'_i) = (b\bar{b})^jb$ for some $j \geq 0$; $w_a(s'_1s'_2\dots s'_i)$ is a prefix of a Dyck word in $\mathcal{D}_a$, $w_a(s'_1s'_2\dots s'_{2k}) \in \mathcal{D}_a$ and $w_b(s'_1s'_2\dots s'_{2k}) = (b\bar{b})^j$ for some $j \geq 0$. So, the word s' is a shuffle in $\mathcal{S}$. Moreover, any occurrence of the letter a in s' is always contiguous with another occurrence of a. Let $i_1 \geq 1$ (resp. $i_2 > i_1$) be the position of the leftmost b (resp. $\bar{b}$) in s'; then the prefix $w_a(s'_1s'_2\dots s'_{i_1-1})$ (resp. $w_a(s'_1s'_2\dots s'_{i_2-1})$) lies into $\mathcal{D}_a$ which implies that the shuffle s' is of the form $\alpha b\beta\bar{b}\gamma$ with α and β in $\mathcal{C}'$ and γ in $\mathcal{C}$; thus we have $s' \in \mathcal{C}$.

The process preserves the positions of the occurrences of aa. So, s and s' belong to the same aa-equivalence class.

The proof for the unicity of s' in $\mathcal{C}$ is obtained *mutatis mutandis* from the proof of Lemma 3. □

Theorem 5. *The generating function for the set of aa-equivalence classes in $\mathcal{S}$ with respect to the semilength is given by*

$$\frac{1 + x - \sqrt{1 - 2x - 3x^2}}{2x^2 + 3x - 1 + \sqrt{1 - 2x - 3x^2}}.$$

This sequence corresponds to the values of even ranks in A191385 [14], and the first values for $n \geq 1$ are 1, 2, 5, 12, 31, 81, 216, 583, 1590.

Proof. By Lemma 6, it suffices to provide the generating function $C(x)$ of the set $\mathcal{C}$, with respect to the semilength. Every nonempty shuffle $s \in \mathcal{C}$ is obtained by the grammar $\mathcal{C} \to \mathcal{C}' \mid \mathcal{C}'b\mathcal{C}'\bar{b}\mathcal{C}$ where $\mathcal{C}' \to aa\bar{a}\mathcal{C}'\bar{a}\mathcal{C}' \mid a(\mathcal{C}'\backslash\varepsilon)\bar{a}\mathcal{C}' \mid \varepsilon$. Let $C'(x)$ be the generating function for the set $\mathcal{C}'$. From the above grammar, we deduce the functional equation:

$$C'(x) = x^2C'(x)^2 + x(C'(x) - 1)C'(x) + 1$$

and

$$C(x) = C'(x) + xC'(x)^2C(x)$$

which provides $C'(x) = \frac{x+1-\sqrt{1-2x-3x^2}}{2x(x+1)}$ and $C(x) = \frac{1+x-\sqrt{1-2x-3x^2}}{2x^2+3x-1+\sqrt{1-2x-3x^2}}$. □

6. Equivalence modulo $\pi \in \{ab, ba, \bar{a}\bar{b}, \bar{b}\bar{a}\}$

The results for $\pi \in \{ba, \bar{a}\bar{b}, \bar{b}\bar{a}\}$ are deduced from the ones for $\pi = ab$ by using the classical symmetries.

Let $\mathcal{E}$ be the set of shuffles in $\mathcal{S}$ defined by the grammar $\mathcal{E} \to \mathcal{E}' \mid \mathcal{E}'b\mathcal{E}'\bar{b}\mathcal{E}$ where $\mathcal{E}' \to ab\mathcal{E}'\bar{b}\mathcal{E}'\bar{a}\mathcal{E}' \mid \varepsilon$. In fact, the set $\mathcal{E}'$ consists of shuffles s in $\mathcal{S}$ such that every occurrence of a letter $x \in \{a, b\}$ appears in a pattern ab, and there is no matchings $(a, \bar{a})$ and $(b, \bar{b})$ such that $ab\bar{a}\bar{b}$ appears in s.

Lemma 7. *The set $\mathcal{E}$ and the set of ab-equivalence classes of $\mathcal{S}$ are in length-preserving one-to-one correspondence.*

Proof. Let $s = s_1s_2\dots s_{2k}$ $(k \geq 1)$ be a shuffle in $\mathcal{S}\backslash\mathcal{E}$ with $s_i \in \{a, \bar{a}, b, \bar{b}\}$ for $1 \leq i \leq 2k$. Let us prove that there exists a shuffle $s' \in \mathcal{E}$ (with the same semilength as s) such that s and s' belong to the same class. We define the shuffle s' by performing the following process on s:

- all occurrences of the pattern ab in s are preserved in s';

- for i from 1 to $2k$ such that s_i does not belong to a pattern ab, we distinguish two cases:
  (i) If $w_b(s'_1s'_2\dots s'_{i-1}) \in \mathcal{D}_b$ and $w_a(s'_1s'_2\dots s'_{i-1}) \in \mathcal{D}_a$, then we set $s'_i = b$;
  (ii) If $w_b(s'_1s'_2\dots s'_{i-1}) \notin \mathcal{D}_b$ or $w_a(s'_1s'_2\dots s'_{i-1}) \notin \mathcal{D}_a$ then there is a letter $x \in \{a, b\}$ which is not matched in the prefix $s'_1s'_2\dots s'_{i-1}$ (we choose x the rightmost possible). If $x = a$, we set $s'_i = \bar{a}$, otherwise we set $s'_i = \bar{b}$.

Now, let us prove that this process produces a shuffle in $\mathcal{E}$. We distinguish two cases.

- Let us assume that any letter $x \in \{a, b\}$ belongs to an occurrence ab in s'. If $s'_i \notin \{a, b\}$ then $s'_i = \bar{a}$ (resp. $s'_i = \bar{b}$) whenever the rightmost non-matched $x \in \{a, b\}$ in $s'_1s'_2\dots s'_{i-1}$ is $x = a$ (resp. $x = b$). So, $s'_1s'_2\dots s'_{i-1}s'_i$ can be written either $ab\alpha\bar{b}\beta\bar{a}$ or $ab\alpha\bar{b}$ according to the value of s'_i ($\bar{a}$ or $\bar{b}$), such that $w_a(\alpha) \in \mathcal{D}_a$, $w_a(\beta) \in \mathcal{D}_a$, $w_b(\alpha) \in \mathcal{D}_b$ and $w_b(\beta) \in \mathcal{D}_b$. Moreover, α and β also satisfy the property that any $x \in \{a, b\}$ belongs to an occurrence ab. Then, s' can be generated by the grammar $\mathcal{E}' \to ab\mathcal{E}'\bar{b}\mathcal{E}'\bar{a}\mathcal{E}' \mid \varepsilon$.

- Now, let us assume that there exists an occurrence b that does not lie in a pattern ab (the process ensures that any occurrence of the letter a lies into a pattern ab in s'). We choose the leftmost b with this property. Then, s' can be written $s' = \alpha b \beta \bar{b} \gamma$ where $\alpha \in \mathcal{E}'$, $\beta \in \mathcal{E}'$ and $\gamma \in \{a, b, \bar{a}, \bar{b}\}^*$. By induction, we have $\gamma \in \mathcal{E}$ and s' can be generated by the grammar $\mathcal{E} \to \mathcal{E}' b \mathcal{E}' \bar{b} \mathcal{E}$.

Considering the two cases, s' can be generated by the grammar $\mathcal{E} \to \mathcal{E}' \mid \mathcal{E}' b \mathcal{E}' \bar{b} \mathcal{E}$ where $\mathcal{E}' \to ab\mathcal{E}'\bar{b}\mathcal{E}'\bar{a}\mathcal{E}' \mid \varepsilon$.

It is clear that two shuffles s and s' in $\mathcal{E}'$ lying into the same ab-equivalence class are necessarily identical. Using a recursive argument, this induces the unicity of a shuffle in $\mathcal{E}$ for a given ab-equivalence class. □

Theorem 6. *The generating function for the set of ab-equivalence classes in $\mathcal{S}$ with respect to the semilength is given by*

$$\frac{-2\sqrt{3}\sin\left(1/3\ \arcsin\left(3/2\sqrt{3}x\right)\right)}{4\left(\sin\left(1/3\ \arcsin\left(3/2\sqrt{3}x\right)\right)\right)^2 - 3x}.$$

This is the sequence A138164 in [14], and the first values for $n \geq 1$ are 1, 2, 4, 9, 20, 47, 109, 262, 622.

Proof. By Lemma 7, it suffices to provide the generating function $E(x)$ of the set $\mathcal{E}$, with respect to the semilength. Every nonempty shuffle $s \in \mathcal{E}$ is obtained by the grammar $\mathcal{E} \to \mathcal{E}' \mid \mathcal{E}' b \mathcal{E}' \bar{b} \mathcal{E}$ where $\mathcal{E}' \to ab\mathcal{E}'\bar{b}\mathcal{E}'\bar{a}\mathcal{E}' \mid \varepsilon$.

Let $E'(x)$ be the generating function for the set $\mathcal{E}'$. From the above grammar, we deduce the functional equation

$$E'(x) = 1 + x^2 E'(x)^3$$

and

$$E(x) = E'(x) + xE'(x)^2 E(x).$$

The generating function $E'(x)$ is known (see A001764 in [14]) and given by

$$E'(x) = \frac{2\sqrt{3}\sin\left(1/3\ \arcsin\left(3/2\sqrt{3}x\right)\right)}{3x}.$$

A simple calculation gives

$$E(x) = \frac{-2\sqrt{3}\sin\left(1/3\ \arcsin\left(3/2\sqrt{3}x\right)\right)}{4\left(\sin\left(1/3\ \arcsin\left(3/2\sqrt{3}x\right)\right)\right)^2 - 3x}. \quad \square$$

7. Going further

Our study focused on the enumeration of the π-equivalence classes of rooted planar maps in shuffle representation for a pattern π of length at most two not in $\{a\bar{b}, \bar{a}b, b\bar{a}, \bar{b}a\}$. Since the pattern $a\bar{b}$ does not occur in any shuffle of a rooted planar map, only three patterns are left as open enumeration problems.

The positions of the occurrences of the pattern π involve our equivalence relation. So, it would be interesting to study the weaker equivalence relation where two maps are equivalent when they have the same number of occurrences of a given pattern in their shuffle representations.

More generally, such a study could be made on maps using other representations and patterns.

References

[1] D. Arquès, Une relation fonctionnelle nouvelle sur les cartes planaires pointées, J. Combin. Theory Ser. B 39 (1) (1985) 27–42.
[2] J.-L. Baril, T. Mansour, A. Petrossian, Equivalence classes of permutations modulo excedances, J. Comb. 5 (4) (2014) 453–469.
[3] J.-L. Baril, A. Petrossian, Equivalence of Dyck paths modulo some statistics, Discrete Math. 338 (2015) 655–660.
[4] J.-L. Baril, A. Petrossian, Equivalence classes of Motzkin paths modulo a pattern of length at most two, J. Integer Seq. 18 (2015) 15.7.1.
[5] O. Bernardi, Bijective counting of tree-rooted maps and shuffles of parenthesis systems, Electron. J. Combin. 14 (2007) #R9.
[6] R. Cori, J. Richard, Enumeration des graphes planaires à l'aide des séries formelles en variables non commutatives, Discrete Math. 2 (1972) 115–162.
[7] E. Deutsch, Dyck path enumeration, Discrete Math. 204 (1999) 167–202.
[8] A.B. Lehman, T. Walsh, Counting rooted maps by genus II, J. Comb. Theory B 13 (1972) 122–141.
[9] T. Mansour, Statistics on Dyck words, J. Integer Seq 9 (2006) 06.1.5.
[10] T. Mansour, Counting peaks at height k in a Dyck path, J. Integer Seq. 5 (2002) 02.1.1.
[11] T. Mansour, M. Shattuck, Counting Dyck paths according to the maximum distance between peaks and valley, J. Integer Seq. 15 (2012) 12.1.1.
[12] D. Merlini, R. Sprugnoli, M.C. Verri, Some statistics on Dyck words, J. Statis. Plann. Inference 101 (2002) 211–227.
[13] A. Sapounakis, I. Tasoulas, P. Tsikouras, Counting strings in Dyck paths, Discrete Math. 307 (2002) 211–227.
[14] N.J.A. Sloane, The On-line Encyclopedia of Integer Sequences, available electronically at http://oeis.org.
[15] W.T. Tutte, A census of planar maps, Canad J. Math. 15 (1963) 249–271.
[16] W.T. Tutte, On the enumeration of planar maps, Bull. Amer. Math. Soc. 74 (1968) 64–74.
[17] T.R.S. Walsh, Combinatorial enumeration of non-planar maps (Ph.D. thesis), University of Toronto, 1971.

Avoiding patterns in irreducible permutations

Jean-Luc Baril

LE2I UMR-CNRS 6306, Université de Bourgogne
B.P. 47 870, 21078 DIJON-Cedex France
e-mail: {barjl}@u-bourgogne.fr

received 18th December 2014,

We explore the classical pattern avoidance question in the case of irreducible permutations, *i.e.*, those in which there is no index i such that $\sigma(i+1)-\sigma(i)=1$. The problem is addressed completely in the case of avoiding one or two patterns of length three, and several well known sequences are encountered in the process, such as Catalan, Motzkin, Fibonacci, Tribonacci, Padovan and Binary numbers. Also, we present constructive bijections between the set of Motzkin paths of length $n-1$ and the sets of irreducible permutations of length n (respectively fixed point free irreducible involutions of length $2n$) avoiding a pattern α for $\alpha \in \{132, 213, 321\}$. This induces two new bijections between the set of Dyck paths and some restricted sets of permutations.

Keywords: Pattern avoiding permutation; irreducible permutation; succession; involution; Motzkin path

1 Introduction and notation

Let $\mathcal{S}_n$ be the set of permutations on $[n]=\{1,2,\ldots,n\}$, *i.e.*, all one-to-one correspondences from $[n]$ into itself. Let $\mathcal{S}$ be the set of all permutations. We represent a permutation $\sigma \in \mathcal{S}_n$ in one-line notation $\sigma=\sigma_1\sigma_2\cdots\sigma_n$ where $\sigma_i=\sigma(i)$ for all $i\in[n]$. We denote by σ^r, σ^c and σ^{-1} the classical symmetries of σ, *i.e.*, the *reverse* $\sigma^r=\sigma_n\cdots\sigma_1$, the *complement* $\sigma^c=(n-\sigma_1+1)\cdots(n-\sigma_n+1)$ and the *inverse*.

Let $q=q_1\cdots q_k$, $k\geq 1$, be a sequence of pairwise different positive integers. The *reduction* red(q) of q is the permutation in S_k obtained from q by replacing the i-th smallest number of q with i for $1\leq i\leq k$. For instance, if $q=53841$ then we have red$(q)=42531$. A permutation $\sigma\in\mathcal{S}_n$ *avoids* the pattern $\pi\in\mathcal{S}_k$, $k\geq 1$, if and only if there does not exist a sequence of indices $1\leq i_1<i_2<\cdots<i_k\leq n$ such that $\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_k}$ is order-isomorphic to π (see [SS85, Wes90]), *i.e.* such that $\pi=$red$(\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_k})$. We denote by $\mathcal{S}_n(\pi)$ the set of permutations in $\mathcal{S}_n$ avoiding the pattern π. For example, if $\pi=123$ then $52143\in\mathcal{S}_5(\pi)$ while $\mathbf{2}15\mathbf{34}\notin\mathcal{S}_5(\pi)$. In the case where σ does not avoid π (or equivalently *contains* π), π is said to be *involved* in σ which is denoted $\pi\preceq\sigma$. A set $\mathcal{F}$ of permutations is called a *permutation class* if it is closed downwards under this involvement relation. Many classical sequences in combinatorics appear as the cardinality of pattern-avoiding permutation classes. A large number of these results were firstly obtained by West and Knuth [Knu73, SS85, Wes90, Wes94, Wes95] (see the surveys of Kitaev and Mansour [KM, Man02]). Also Bóna [Bón04] has written a book that is dedicated to the notion of pattern avoiding permutations. Some generalizations of pattern avoidance can be viewed in [Bar11, Pud10].

subm. to DMTCS © by the authors Discrete Mathematics and Theoretical Computer Science (DMTCS), Nancy, France

A *succession* in a permutation $\sigma \in \mathcal{S}_n$ is a pair (σ_i, σ_{i+1}), $1 \leq i \leq n-1$, with $\sigma_{i+1} - \sigma_i = 1$. For instance, the pair $(3, 4)$ is the only one succession in $\sigma = 53421$. Using the terminology of Atkinson and Stitt (see [AS02]), a permutation with no successions will be also called *irreducible*. They show how the notion of irreducibility plays, in some sense, a dual role compared to indecomposability (a permutation $\sigma = \sigma_1 \cdots \sigma_n$ is *indecomposable* if there does not exist $p \leq n-1$ such that $\sigma_1 \cdots \sigma_p$ is a permutation of $[p]$). Indeed, irreducible permutations are crucial for the construction of a basis of some wreath products $\mathcal{C} \wr \mathcal{I}$ where $\mathcal{C}$ is a class of permutations and $\mathcal{I}$ is the class of identity permutations (see [AS02]). On the other hand, an irreducible permutation can be also viewed as a permutation avoiding the bi-vincular pattern $(12, \{1\}, \{1\})$ (see [Par09]).

Let $\mathcal{S}_n^{irr}$ be the set of irreducible permutations of length n. For example, we have $\mathcal{S}_3^{irr} = \{132, 213, 321\}$. The cardinality of the sets $\mathcal{S}_n^{irr}$ is known (see [Ros68, Tan76, Tie43]) and given by

$$(n-1)! \sum_{k=0}^{n-1} (-1)^k \frac{n-k}{k!}.$$

More generally, permutations with a given number of successions also appeared in combinatorial theory literature (see [Kap44, Rio65]).

In this paper, we explore the classical pattern avoidance question in the case of irreducible permutations. The problem is addressed completely in the case of avoiding one or two patterns of length three, and several well-known sequences are encountered, such as the Catalan, Motzkin, Fibonacci, Binary, Tribonacci and Padovan numbers. In Section 2, we present general enumerative results for sets of permutations that are *expanded* (*i.e., closed under inflation*) and *closed under deflation*. In particular, we give bivariate generating functions according to the length of permutations and the number of successions. Generalizing the two concepts of inflation and deflation for sets of fixed point free involutions, we obtain similar results for involutions. In Section 3, we give exhaustive enumerative results for sets of irreducible permutations avoiding one pattern of length three. For the sets counted by the Motzkin numbers, we exhibit bijections between them and the set of Motzkin paths. In Section 4, we focus on sets of irreducible permutations avoiding two patterns of length three. In Sections 5 and 6, we study irreducible (fixed point free) involutions avoiding one pattern of length three. Exhaustive enumerative results are obtained, and we construct bijections between fixed point free irreducible involutions of length $2n$ avoiding $\alpha \in \{132, 213, 321\}$ and Motzkin paths of length $n-1$.

2 Preliminaries

A set $\mathcal{F}$ of permutations is said to be *closed under inflation* (also called *expanded*, [AS02]) if whenever a permutation $\sigma = \beta\, i\, \gamma \in \mathcal{F}$ so is the permutation obtained by replacing i with $i\ (i+1)$, after increasing by one each other's values greater than i in σ. A set $\mathcal{F}$ of permutations is said to be *closed under deflation* if whenever a permutation $\sigma = \beta\, i\ (i+1)\ \gamma$ so is the permutation obtained by replacing $i\ (i+1)$ with i, after decreasing by one each other's values greater than $(i+1)$ in σ. In the following, these two last elementary transformations will be respectively called *inflation* and *deflation* of σ. Obviously, a set $\mathcal{F}$ of irreducible permutations is closed under deflation since its elements do not contain any successions. A set closed under pattern involvement is closed under deflation. Moreover, a set closed under pattern involvement is expanded if and only if every basis element is irreducible (see Lemma 5, [AS02]).

In this section we provide general lemmas about generating functions for sets of permutations and fixed point free involutions which are expanded and closed under deflation.

Lemma 1 *Let $\mathcal{F}$ be a set of permutations which is expanded and closed under deflation; let $\mathcal{G}$ be the set of its irreducible permutations. We denote by $g(x)$ the generating function of $\mathcal{G}$, and $f(x,y)$ the bivariate generating function of $\mathcal{F}$ where the coefficient of $x^n y^k$ is the number of permutations of length n with exactly k successions. Then we have*

$$f(x,y) = g\left(\frac{x}{1-yx}\right).$$

In particular, the generating function of $\mathcal{F}$ is $g\left(x/(1-x)\right)$.

Proof. Let $\mathcal{F}_n^k$, $n \geq 1$, $0 \leq k \leq n-1$, be the set of permutations of length n in $\mathcal{F}$ containing exactly k successions. Obviously, we have $\mathcal{F}_n^{irr} = \mathcal{G}_n = \mathcal{F}_n^0$ for $n \geq 1$. Let σ be a permutation in $\mathcal{F}_n^k$, *i.e*, σ contains exactly k successions. We define the succession set I_σ of indices $i \in [n-1]$ such that (σ_i, σ_{i+1}) is a succession of σ, and $J_\sigma = \{i \in [n], i-1 \in I_\sigma\}$. Since $\mathcal{F}$ is expanded and closed under deflation, a permutation $\pi \in \mathcal{F}_{n-|I_\sigma|}^{irr}$ can be uniquely obtained from a permutation $\sigma \in \mathcal{F}_n^k$ by deleting all entries σ_i, $i \in J_\sigma$ and reducing the result to a permutation of $\mathcal{F}_{n-|I_\sigma|}^{irr}$. This construction produces a simple bijection from $\mathcal{F}_{n-|I_\sigma|}^{irr}$ to the set $\mathcal{F}_n^{I_\sigma}$ of permutations in $\mathcal{F}_n$ having the succession set equal to I_σ.

Since inflations of terms in an irreducible permutation correspond exactly to the terms in the geometric series expansion $\frac{x}{1-yx} = x + yx^2 + y^2x^3 + \ldots$, the above bijection implies

$$f(x,y) = g(x + yx^2 + y^2x^3 + \ldots) = g(\frac{x}{1-yx}).$$

□

Now we give a similar Lemma for fixed point free involutions. In this case, we slightly modify the two concepts of inflation and deflation. Let $\mathcal{F}$ be a set of involutions with no fixed points. We say that $\mathcal{F}$ is *expanded* (*closed under inflation*) if whenever an involution $\sigma \in \mathcal{F}$ and (i,j), $1 \leq i < j \leq n$, such that $\sigma_i = j$, so is the permutation σ' obtained from σ by replacing σ_i with $(\sigma_i+1)(\sigma_i+2)$, σ_j with $\sigma_j(\sigma_j+1)$, and increasing by inc(k) =card$\{\ell \in \{i+1, j+1\}, \ell \leq k\}$ each other's values k. For instance, if $\sigma = 3\ \mathbf{4}\ 1\ \mathbf{2}\ 7\ 8\ 5\ 6$, $i=2$, $j=4$, then $\sigma' = 4\ \mathbf{5}\ \mathbf{6}\ 1\ \mathbf{2}\ \mathbf{3}\ 9\ (10)\ 7\ 8$. On the other hand, we say that $\mathcal{F}$ is *closed under deflation* if whenever an involution $\sigma \in \mathcal{F}$ and (i,j), $1 \leq i < j \leq n-1$, such that $\sigma_i = j$, $\sigma_{i+1} = j+1$, so is the permutation σ' obtained from σ by deleting $(i+1)$ and $(j+1)$ and reducing the result to a permutation. Obviously the set of all fixed point free involutions is expanded and closed under deflation.

Lemma 2 *Let $\mathcal{F}$ be a set of fixed point free involutions which is expanded and closed under deflation; let $\mathcal{G}$ be the set of its irreducible involutions. We denote by $g(x)$ the generating function of $\mathcal{G}$, and $f(x,y)$ the bivariate generating function of $\mathcal{F}$ where the coefficient of $x^n y^k$ is the number of involutions of length n with exactly k successions. Then we have*

$$f(x,y) = g\left(\sqrt{\frac{x^2}{1-y^2x^2}}\right).$$

In particular, the generating function of $\mathcal{F}$ is $g\left(\sqrt{x^2/(1-x^2)}\right)$.

Proof. This proof is a simple counterpart of the previous one. It suffices to remark that an involution with no fixed points is necessarily of even length. □

Lemma 1 ensures that if we know the generating function of a set $\mathcal{G}$ of irreducible permutations, then we can easily obtain the bivariate generating function (according to the length and the number of successions) of the wreath product $\mathcal{G} \wr \mathcal{I}$ where $\mathcal{I}$ is the class of identity permutations (the wreath product $\mathcal{G} \wr \mathcal{I}$ is the closure under inflation of $\mathcal{G}$). An immediate consequence is: *if $\mathcal{G}$ is a set of irreducible permutations such that the wreath product $\mathcal{G} \wr \mathcal{I}$ is enumerated by the Catalan numbers ([Slo], A00018) then $\mathcal{G}$ is enumerated by the Motzkin numbers ([Slo], A001006).* Therefore, irreducible permutations of $\mathcal{S}_n(\alpha)$, $n \geq 1$, $\alpha \in \{321, 213, 132\}$, are enumerated by the Motzkin numbers (see Theorem 2).

3 Avoiding a pattern of length 3

In this section, we study the avoidance of one pattern of length 3 for irreducible permutations. The first part uses arguments with generating functions while the second part presents bijective proofs for the sets of permutations counted by Motzkin numbers. All enumerative results of this section are listed in Table 1.

3.1 Using generating functions

For $\alpha = 231$, Atkinson and Stitt give a proof of Remark 1 (see [AS02], Section 6.1). The bijection $\chi : \sigma \to (\sigma^c)^r$, allows to conclude for $\alpha = 312$.

Remark 1 *For $\alpha \in \{231, 312\}$, the generating function for the sets $\mathcal{S}_n^{irr}(\alpha)$, $n \geq 0$, of irreducible permutations avoiding α is given by*

$$-1 + \frac{2(1+x)}{1+x+x^2+\sqrt{(1-x+x^2)^2-8x^2}}$$

(see [Slo], A078481).

Theorem 1 *The generating function for the sets $\mathcal{S}_n^{irr}(123)$, $n \geq 0$, of irreducible permutations of length n avoiding the pattern 123 is given by (see [Slo], A114487):*

$$\frac{2}{1+2x^2+\sqrt{1-4x}}.$$

Proof. Let σ be a permutation in $\mathcal{S}_n(123) \backslash \mathcal{S}_n^{irr}(123)$. It necessarily contains a succession $(k, k+1)$, $1 \leq k \leq n-1$; let us take the leftmost succession. Then σ can be written $\sigma = uk(k+1)v$ where $1 \leq k \leq n-1$, such that u and v are two subsequences of $[n]$. Since σ avoids 123, the set of values in u necessarily equals to the interval $[k+2, n]$. So v belongs to $\mathcal{S}_{k-1}(123)$, u avoids the pattern 123 and does not contain any successions. Let $f(x)$ be the generating function for the sets $\mathcal{S}_n^{irr}(123)$, $n \geq 0$, the above decomposition of σ implies

$$c(x) - f(x) = x^2 \cdot f(x) \cdot c(x),$$

where $c(x) = \frac{1-\sqrt{1-4x}}{2x}$ is the generating function for the sets $\mathcal{S}_n(123)$, *i.e.*, the generating function for the well-known Catalan numbers (see [Slo], A00018). We deduce:

$$f(x) = \frac{c(x)}{1+x^2 \cdot c(x)} = \frac{2}{1+2x^2+\sqrt{1-4x}}.$$

□

Theorem 2 *For $\alpha \in \{321, 213, 132\}$, the sets $\mathcal{S}_n^{irr}(\alpha)$, $n \geq 1$, are enumerated by the Motzkin numbers (see [Slo], A001006).*

Proof. For $\alpha \in \{321, 213, 132\}$, the set $\mathcal{S}(\alpha) = \bigcup_{n=1}^{\infty} \mathcal{S}_n(\alpha)$ is expanded and closed under deflation. Let $g(x)$ be the generating function of $\mathcal{S}^{irr}(\alpha)$; Lemma 1 of Section 2 proves that $c(x) = g(x/(1-x))$ and thus:

$$g(x) = c(x/(1+x)).$$

More precisely, the set $\mathcal{S}_n^{irr}(\alpha)$, $n \geq 1$, is enumerated by the $(n-1)$-th term of the Motzkin sequence ([Slo], A001006). □

In fact, Lemma 1 yields the more general result:

Theorem 3 *For $\alpha \in \{321, 213, 132\}$, let $\mathcal{S}(\alpha)$ be the set of permutations avoiding α, and $f(x, y)$ its bivariate generating function, where the coefficient of $x^n y^k$ is the number of permutations of length n with k successions in $\mathcal{S}(\alpha)$. Then we have*

$$f(x, y) = c\left(\frac{x}{1 + (1-y)x}\right),$$

where $c(x)$ is the generating function for the Catalan numbers.

Pattern	Sequence	Sloane	$a_n, n \geq 1$
$\{231\}, \{312\}$ [AS02]	$-1 + \frac{2(1+x)}{1+x+x^2+\sqrt{(1-x+x^2)^2-8x^2}}$	$A078481$	$1, 1, 3, 7, 19, 53, 153$
$\{123\}$	$\frac{2}{1+2x^2+\sqrt{1-4x}}$	$A114487$	$1, 1, 3, 10, 31, 98, 321$
$\{321\}, \{213\}, \{132\}$	$\frac{1-x-\sqrt{1-2x-3x^2}}{2x^2}$	$A001006$	$1, 1, 2, 4, 9, 21, 51$

Tab. 1: Wilf-equivalence classes for patterns of length 3 in irreducible permutations.

3.2 Bijective proofs

A Motzkin path of length n is a lattice path starting at $(0, 0)$, ending at $(n, 0)$, and never going below the x-axis, consisting of up steps $U = (1, 1)$, horizontal steps $H = (1, 0)$, and down steps $D = (1, -1)$. A Dyck path of length $2n$ is a Motzkin path of the same length that does not contain any horizontal steps. Dyck paths of length $2n$ are enumerated by the well-known Catalan numbers ([Slo], A00018) and Motzkin paths by the Motzkin numbers ([Slo], A001006). We refer to Donaghey, Shapiro [DS77] and Stanley [Sta] for several combinatorial classes enumerated by the Motzkin and Catalan numbers.

• *Bijective proof for $\mathcal{S}_n^{irr}(132)$.*

Here we construct a bijection between $\mathcal{S}_n^{irr}(132)$ and the set of Motzkin paths of length $n-1$. We define the map f that transforms a permutation $\sigma = \sigma_1\sigma_2 \cdots \sigma_n \in \mathcal{S}_n^{irr}(132)$ into the Motzkin path M of length $n-1$ obtained by the following process:

For each i from 1 to $n-1$,

(a) if $\sigma_i < \sigma_{i+1}$, then the i-th step of M is a down step D;

(b) if $\sigma_i > \sigma_{i+1}$ and there exists $j > i+1$ such that $\sigma_j = \sigma_i + 1$, then the i-th step of M is an up step U;

(c) otherwise, the i-th step of M is an horizontal step H.

For instance, the permutations 21, 321, 213, 4213, 3214, 3241 and 4321 are respectively transformed by f into the Motzkin paths H, HH, UD, HUD, UHD, UDH, HHH (see Figure 1 for an example with $n = 14$).

Let us prove that f is a one-to-one correspondence between $\mathcal{S}_n^{irr}(132)$ and the set of Motzkin paths of length $(n-1)$. Let σ be a permutation of $\mathcal{S}_n^{irr}(132)$. Let us take i, $1 \leq i \leq n-1$, such that (b) is verified. Then, there exists $j > i+1$ such that $\sigma_j = \sigma_i + 1$. Since σ avoids 132, $\sigma_{j-1} < \sigma_j$ and the index $j-1 > i$ verifies the case (a). Conversely, if i, $1 \leq i \leq n-1$, satisfies (a) then $\sigma_i < \sigma_{i+1}$. Since σ is irreducible, we have $\sigma_{i+1} \neq \sigma_i + 1$. Thus, there exists $j \neq i$ and $j \neq i+1$ such that $\sigma_j = \sigma_{i+1} - 1$. As σ avoids 132, we necessarily have $j < i$ and j verifies (b). Hence, there is a one-to-one correspondence between up steps and down steps such that each up step is associated with a down step on its right. This is precisely the characterization of the Motzkin paths.

Conversely, let M be a Motzkin path of length $(n-1)$ and let us prove that there exists a permutation $\sigma \in \mathcal{S}_n^{irr}(132)$ such that $f(\sigma) = M$. We proceed by induction on n in order to construct the permutation $\sigma \in \mathcal{S}_n^{irr}(132)$.

We distinguish two cases: (1) $M = mHm'$ where m and m' are two Motzkin paths and such that m does not contain any horizontal steps on the x-axis, and (2) $M = mUm'D$ where m and m' are two Motzkin paths such that m does not belong to the case (1).

- (1) If $M = mHm'$, then we have $M = f(\sigma)$ where $\sigma = \beta n \gamma$ such that γ (resp. red(βn)) is recursively obtained from the Motzkin path m' (resp. m). Notice that the position of n in σ creates the horizontal step H between m and m'.

- (2) If $M = mUm'D$, then we have $M = f(\sigma)$ where $\sigma = \beta(n-1)\gamma n$ such that γ (resp. red$(\beta(n-1))$) is recursively obtained from the Motzkin path m' (resp. m). Since m does not belong to (1), its associated permutation necessarily have its greatest element on the last position. We conclude with a simple induction.

Notice that paths of type (1) are mapped to permutations whose last element is not n, and paths of type (2) are mapped to permutations that end with n.

For example, if $M = mUm'D$ with $m = UHD$ and $m' = UDH$ ($n = 9$), then $\beta 8 = 7658$, red$(\beta 8) = 3214$, $\gamma = 3241$ and $\sigma = 765832419$. If $M = mHm'$ with $m = UHD$ and $m' = UDH$ ($n = 8$), then $\beta 8 = 7658$, red$(\beta 8) = 3214$, $\gamma = 3241$ and $\sigma = 76583241$.

Finally, the map f is a one-to-one correspondence between $\mathcal{S}_n^{irr}(132)$ and the set of Motzkin paths of length $(n-1)$. □

• *Bijective proof for* $\mathcal{S}_n^{irr}(321)$.

Here we construct a bijection between $\mathcal{S}_n^{irr}(321)$ and the set of Motzkin paths of length $n-1$. Let $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ be a permutation in $\mathcal{S}_n^{irr}(321)$, and for $i \in [n]$ we denote by s_i the maximum of the set $\{\sigma_1, \sigma_2, \ldots, \sigma_i\}$. We define the map g that transforms a permutation $\sigma \in \mathcal{S}_n^{irr}(321)$ into the Motzkin path M of length $n-1$ obtained by the following process:

For each i from 1 to $n-1$,

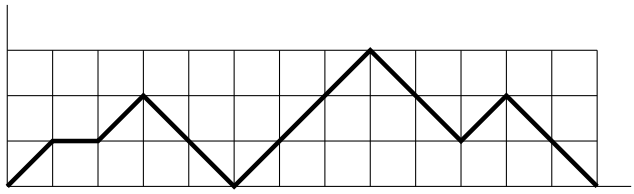


Fig. 1: The Motzkin path $UHUDDUUUDDUDD$ and its corresponding permutations $\sigma = (12)\ (11)\ 9\ 8\ (10)\ (13)\ 5\ 3\ 2\ 4\ 6\ 1\ 7\ (14)$ and $\sigma' = 2\ 4\ 1\ 6\ 3\ 5\ 8\ (10)\ (12)\ 7\ 9\ (14)\ (11)\ (13)$ lying respectively in $\mathcal{S}_{14}^{irr}(132)$ and $\mathcal{S}_{14}^{irr}(321)$.

(a) if $s_i + 1 < \sigma_{i+1}$, then the i-th step of M is an up step U;

(b) if there exists $j < i$ such that $\sigma_{i+1} = s_j + 1$ and $s_j + 1 < \sigma_{j+1}$, then the i-th step of M is a down step D;

(c) otherwise, the i-th step of M is an horizontal step H.

For instance, the permutations 21, 132, 213, 1324, 3142, 2143 and 2413 are respectively transformed by g into the Motzkin paths H, UD, HH, UDH, HUD, HHH, UHD (see Figure 1 for an example of length 14).

Let us prove that g is a one-to-one correspondence between $\mathcal{S}_n^{irr}(321)$ and the set of Motzkin paths of length $(n-1)$. Let σ be a permutation of $\mathcal{S}_n^{irr}(321)$. Let us take i, $1 \leq i \leq n-1$, such that (a) is verified. Since $s_i + 1 < \sigma_{i+1}$, there exists $j > i+1$, such that $\sigma_j = s_i + 1$. A consequence is that the index $j-1$, $i < j-1 < n$, verifies (b). Conversely, let us take i, $1 \leq i \leq n-1$, satisfying (b). So there is $j < i$ such that $\sigma_{i+1} = s_j + 1$ and $s_j + 1 < \sigma_{j+1}$ and the index j verifies (a). Thus, there is a one-to-one correspondence between up steps and down steps such that each up step is associated with a down step on its right. This is precisely the characterization of the Motzkin paths.

Conversely, let M be a Motzkin path of length $(n-1)$ and let us prove that there exists a permutation $\sigma \in \mathcal{S}_n^{irr}(321)$ such that $g(\sigma) = M$. We proceed by induction on n in order to construct the permutation $\sigma \in \mathcal{S}_n^{irr}(321)$.

We distinguish four cases: (1) $M = U^k Dm$; (2) $M = U^k H^\ell m$; (3) $M = H^k Um$; and (4) $M = HH \ldots H$, with $k \geq 1$, $\ell \geq 0$, and where m is a suffix of the Motzkin path M.

- If $M = U^k Dm$, then we have $M = g(\sigma)$ where σ is recursively obtained from the permutation π such that $g(\pi) = U^{k-1}Hm$ by adding 1 on the left of π and after increasing by one all values of π. Notice that π cannot begin with 1 since there would be a down step just after U^{k-1}. Thus σ does not contain any successions and avoids 321.

- If $M = U^k H^\ell m$ where m does not begin with H.

If $\ell = 1$, then we have $M = g(\sigma)$ where σ is recursively obtained from the permutation π such that $g(\pi) = U^k m$ by inserting 1 between the $(k+1)$ and $(k+2)$-th positions of π and after increasing by one all values of π greater or equal than 1. Since $g(\pi) = U^k m$ and m does not start with H, we have $\pi_{k+2} \neq 1$ and the previous insertion of 1 does not create any succession and any pattern 321.

If $M = U^k H^\ell m$ where m does not begin with H and with $\ell \neq 1$, then we have $M = g(\sigma)$ where σ is recursively obtained from the permutation π such that $g(\pi) = U^k H^{\ell-1} m$ by inserting $\pi_{k+\ell-1} + 1$

between the $(k+\ell)$ and $(k+\ell+1)$-th positions of π and after increasing by one all values of π greater or equal than $\pi_{k+\ell-1}+1$.

Since the first step of m is not H, we necessarily have either $\pi_{k+\ell+1} > s_{k+\ell}+1$ or $\pi_{k+\ell+1} < s_{k+\ell}$ which implies that the insertion of $\pi_{k+\ell-1}+1$ does not create a succession at position $(k+\ell+1)$ in σ. Also, the insertion of $\pi_{k+\ell-1}+1$ between the $(k+\ell)$ and $(k+\ell+1)$-th positions cannot create a succession at position $(k+\ell)$. So, the permutation σ does not contain any succession.

Let us assume $s_{k+\ell} = \pi_{k+\ell}$. Since the $(k+\ell-1)$-th step is H, there is $j < k+l-1$ such that $\pi_j + 1 = \pi_{k+\ell}$. Therefore, the insertion of $\pi_{k+\ell-1}+1$ between the $(k+\ell)$ and $(k+\ell+1)$-th positions does not create a pattern 321 (since π does not contain any pattern 321).

Now, let us assume that $s_{k+\ell} \geq \pi_{k+\ell}+1$. If we have $\pi_{k+\ell-1} = s_{k+\ell}$ then the insertion of $s_{k+\ell}+1$ between the $(k+\ell)$ and $(k+\ell+1)$-th positions cannot create a pattern 321. If we have $\pi_{k+\ell-1} \neq s_{k+\ell}$, we necessarily have $\pi_{k+\ell-1} < \pi_{k+\ell}$ and all values lying in $[1, \pi_{k+\ell-1}]$ appear at positions $j < k+\ell$. So, the insertion of $s_{k+\ell}+1$ between the $(k+\ell)$ and $(k+\ell+1)$-th positions cannot create a pattern 321. Finally, the permutation σ belongs to $\mathcal{S}_n^{irr}(321)$.

- If $M = H^k Um$, then we have $M = g(\sigma)$ where $\sigma = \pi u \pi'$ such that $g(\pi u) = H^k$ and $g(\text{red}(u\pi')) = Um$.

- If $M = H^k$, then we have $M = g(\sigma)$ where σ is recursively obtained from the permutation π such that $g(\pi) = H^{k-1}$ by inserting $\pi_{k-1}+1$ in the last position and after increasing by one all values of π greater or equal than $\pi_{k-1}+1$.

Below, we give an example for each previous case.

- If $M = U^k Dm$ with $k=2$, $m = UHDD$, then $\pi = 3517246$, $g(\pi) = UHUHDD$ and $\sigma = \mathbf{1}4628357$;

- if $M = U^k H^\ell m$ with $k=2$, $\ell = 3$ and $m = DD$, then $\pi = 2461735$, $g(\pi) = UUHHDD$ and $\sigma = 35718\mathbf{2}46$;

- if $M = H^k Um$ with $k=3$, $m = D$, then $u = 2$, $\pi u = 3142$, $g(\pi u) = HHH$, $u\pi' = 265$, $g(132) = UD$ and $\sigma = 314265$;

- if $M = H^k$ with $k=5$, then $\pi = 31425$, $g(\pi) = HHHH$ and $\sigma = 41526\mathbf{3}$. □

The bijection f (resp. g) from $\mathcal{S}_n^{irr}(132)$ (resp. $\mathcal{S}_n^{irr}(321)$) to the set of Motzkin paths of length $(n-1)$ induces a new constructive bijection between Dyck paths and some restricted irreducible permutations:

Corollary 1 *Let P_{2n+1} be the set of permutations $\sigma \in \mathcal{S}_{2n+1}^{irr}(132)$ such that for all i, $1 \leq i \leq 2n-1$, with $\sigma_i > \sigma_{i+1}$ there exists $j > i+1$ with $\sigma_j = \sigma_i + 1$. Then the map f induces a constructive bijection between P_{2n+1} and the set of Dyck Paths of length $2n$.*

Corollary 2 *Let P'_{2n+1} be the set of permutations $\sigma \in \mathcal{S}_{2n+1}^{irr}(321)$ such that $\sigma_1 = 1$ and $\sigma_{i+1} - \max\{\sigma_1, \ldots, \sigma_i\} \leq 2$ and $\sigma_{i+1} - \max\{\sigma_1, \ldots, \sigma_i\} \neq 1$ for all $1 \leq i \leq 2n-1$. The map g induces a constructive bijection between P'_{2n+1} and the set of Dyck Paths of length $2n$.*

For example, we have $P_3 = \{213\}$, $P_5 = \{42135, 32415\}$, $P_7 = \{6421357, 5462137, 6324157, 5324617, 4352617\}$, $P'_3 = \{132\}$, $P'_5 = \{13254, 13524\}$ and $P'_7 = \{1325476, 1325746, 1352476, 1352746, 1357246\}$.

4 Avoiding two patterns of length 3

In this section, we explore the avoidance of two patterns of length 3 for irreducible permutations. All enumerative results of this section are listed in Table 2.

Theorem 4 *For $\alpha \in \{231, 312\}$, the sets $\mathcal{S}_n^{irr}(321, \alpha)$, $n \geq 1$, are enumerated by the Padovan's spiral numbers defined by the generating function $\frac{1+x}{1-x^2-x^3}$ (see [Slo], A134816).*

Proof. Let $\sigma = \beta n \gamma$ be a permutation in $\mathcal{S}_n^{irr}(321, \alpha)$ where β and γ are two subsequences of $[n-1]$. Since σ avoids 321, $\gamma = \gamma_1\gamma_2\cdots\gamma_k$ is an increasing subsequence (possibly empty) of $[n-1]$. We distinguish two cases: (1) γ is empty; and (2) γ is not empty.

Case (1). This means that $\beta \in \mathcal{S}_{n-1}^{irr}(321, \alpha)$ and the last entry of β is different from $n-1$. Conversely, if we add n to the right of a permutation π in $\mathcal{S}_{n-1}^{irr}(321, \alpha)$ such that $\pi_{n-1} \neq n-1$, we obtain a permutation of $\mathcal{S}_n^{irr}(321, \alpha)$.

Case (2). Now $\gamma = \gamma_1\gamma_2\cdots\gamma_k$ is not empty. For a contradiction, let us assume $k \geq 2$. Then, σ contains the pattern 312. Since σ is irreducible, there is an element b of β such that $\gamma_{k-1} < b < \gamma_k < n$ which implies that σ contains the pattern 231. Thus, we necessarily have $k = 1$. In this case, σ can be written $\beta n(n-1)$ where $\beta \in \mathcal{S}_{n-2}^{irr}(321, \alpha)$.

Let $f(x)$ (resp. $g(x)$) be the generating function for permutations $\sigma \in \mathcal{S}_n^{irr}(321, \alpha)$ (resp. $\sigma \in \mathcal{S}_n^{irr}(321, \alpha)$ such that $\sigma_n \neq n$). The previous study induces that $f(x) = 1 + x \cdot g(x) + x^2 \cdot f(x)$ with $g(x) = f(x) - x \cdot g(x)$. We obtain $g(x) = \frac{1}{1-x^2-x^3}$ and $f(x) = \frac{1+x}{1-x^2-x^3} = \frac{1}{1-x} \cdot h(x)$ where $h(x)$ is the generating function for the Padovan numbers (see [Slo], A000931). This corresponds to the Padovan's spiral numbers [Slo], A134816). □

Notice that this last proof shows that the three sets $\mathcal{S}_n^{irr}(321, 231)$, $\mathcal{S}_n^{irr}(321, 312)$ and $\mathcal{S}_n^{irr}(321, 231, 312)$ are identical, not just equinumerous. To my knowledge, there is no analogous result in the enumeration of general (as opposed to irreducible) permutations.

Theorem 5 *For $\alpha \in \{132, 213\}$, the set $\mathcal{S}_n^{irr}(321, \alpha)$ is empty for $n \geq 4$. The set $\mathcal{S}_n^{irr}(321, 123)$ is empty for $n \geq 5$.*

Proof. The result is well known for $\alpha = 123$. Now, let us take $\alpha = 213$ (the case $\alpha = 132$ will be obtained with a smple Wilf equivalence). Let σ be a permutation of $\mathcal{S}_n^{irr}(321, 213)$. Writing $\sigma = \beta n \gamma$ with β and γ are two subsequences of $[n-1]$, the avoidance of 213 implies that $\beta = \beta_1 \cdots \beta_k$, where $\beta_1 < \beta_2 < \cdots < \beta_k$; the avoidance of 321 implies that $\gamma = \gamma_1\gamma_2 \ldots \gamma_\ell$, where $\gamma_1 < \gamma_2 < \cdots < \gamma_\ell$. Since σ is irreducible and avoids 213, γ and β contain at most one element which implies that there does not exist any permutations σ in $\mathcal{S}_n^{irr}(321, 213)$ whenever $n \geq 4$. □

Theorem 6 *For $n \geq 1$, the set $\mathcal{S}_n^{irr}(213, 132)$ is reduced to the unique permutation $n(n-1)\cdots 321$.*

Proof. Let σ be a permutation in $\mathcal{S}_n^{irr}(213, 132)$. We can write $\sigma = \beta n \gamma$ where β and γ are two subsequences of $[n-1]$. Since σ avoids 213, we have $\beta = \beta_1\beta_2\cdots\beta_k$, where $\beta_1 < \beta_2 < \cdots < \beta_k$. Since σ is irreducible, we have $\beta_k \neq n-1$, and thus $n-1$ lies in γ. Since γ avoids 132, β is empty (otherwise, $\beta_1 n(n-1)$ would be a pattern 132). Thus we have $\sigma = n\gamma$. A straightforward induction provides $\sigma = n(n-1)\cdots 321$. □

Theorem 7 *The sets $\mathcal{S}_n^{irr}(231, 312)$, $n \geq 0$, are enumerated by the Tribonacci numbers (see [Slo], A000213).*

Proof. Let σ be a permutation in $\mathcal{S}_n^{irr}(231,312)$. Since σ avoids 312, it can be written $\sigma = \beta n \gamma$ where β and γ are two subsequences of $[n-1]$ such that either $\gamma = \gamma_1 \cdots \gamma_k$ with $\gamma_1 > \gamma_2 > \cdots > \gamma_k$ or γ is empty. Since γ avoids 231, β does not contain any values greater than γ_k. Thus we have either γ is empty or $\gamma = \beta n(n-1)(n-2)\cdots(n-k)$ where $\beta \in \mathcal{S}_{n-k-1}^{irr}(231,312)$.

Let $f(x)$ (resp. $g(x)$) be the generating function for $\mathcal{S}_n^{irr}(231,312)$, $n \geq 0$, (resp. for $\mathcal{S}_n^{irr}(231,312)$, $n \geq 1$, with the restriction that permutations do not end with n). Then we have $f(x) = 1 + x + xg(x) + g(x)$. The above structure of a permutation in $\mathcal{S}_n^{irr}(231,312)$ ensures that $g(x) = \left(\frac{1}{1-x} - 1 - x\right) \cdot f(x)$. Thus we have $f(x) = \frac{1-x^2}{1-x-x^2-x^3}$ that corresponds to the Tribonacci numbers [Slo], $A000213$. □

Theorem 8 *For $\alpha \in \{231, 312\}$ and $\beta \in \{132, 213\}$, the sets $\mathcal{S}_n^{irr}(\alpha,\beta)$, $n \geq 1$, are enumerated by the Fibonacci numbers (see [Slo], A000045).*

Proof. Let $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ be a permutation in $\mathcal{S}_n^{irr}(231,213)$. Since σ avoids 231 and 213, then σ_i, $1 \leq i \leq n$, is either the smallest or the greatest element of the set $\{\sigma_i, \sigma_{i+1}, \ldots, \sigma_n\}$. We deduce that σ can be written either $\sigma = 1n\gamma$ or $\sigma = n\gamma$, where γ avoids 231 and 213. If $a_n = \text{card}(\mathcal{S}_n^{irr}(231,213))$ then $a_1 = 1$, $a_2 = 1$, and $a_n = a_{n-1} + a_{n-2}$ which defines the Fibonacci numbers.

The classical symmetries deal with the remaining cases. □

Theorem 9 *For $\alpha \in \{312, 231\}$, the sets $\mathcal{S}_n^{irr}(123,\alpha)$, $n \geq 1$, are enumerated by the Triangular numbers $\frac{n(n-1)}{2}$ (see [Slo], A000217).*

Proof. Let $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ be a permutation in $\mathcal{S}_n^{irr}(123,312)$. Since σ avoids 123 and 312, σ can be written $\sigma = k(k-1)\cdots\ell n\gamma$ where $1 \leq \ell \leq k \leq n$ and γ is a decreasing sequence of integers. For k fixed, $1 \leq k \leq n-2$, there are k possible permutations obtained whenever ℓ describes $[1,k]$. For $k = n-1$, the permutation $(n-1)n(n-2)\cdots 21$ is not irreducible, thus we do not consider it. In this case, there are $(n-2)$ possible permutations. Finally, $\mathcal{S}_n^{irr}(123,312)$ is enumerated by $1+2+3+\cdots+(n-2)+(n-2)+1 = \frac{n(n-1)}{2}$ which is the Sloane's sequence A000217. A simple symmetry gives the result for $\alpha = 231$. □

Theorem 10 *For $\alpha \in \{132, 213\}$, the sets $\mathcal{S}_n^{irr}(123,\alpha)$, $n \geq 1$, are enumerated by the sequence [Slo], A005251.*

Proof. Let $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ be a permutation in $\mathcal{S}_n^{irr}(123,132)$. Since σ avoids 123 and 132, σ_1 is either $(n-1)$ or n. In the case where $\sigma_1 = n$, then σ can be written $\sigma = n\gamma$ where $\gamma \in \mathcal{S}_{n-1}^{irr}(123,132)$. In the case where $\sigma_1 = n-1$, σ can be written: $\sigma = (n-1)(n-2)\cdots(n-k)n\gamma$, $2 \leq k \leq n-1$, where γ belongs to $\mathcal{S}_{n-1-k}^{irr}(123,132)$. If $a_n = \text{card}(\mathcal{S}_n^{irr}(123,132))$ then we deduce $a_n = a_{n-1} + a_{n-3} + a_{n-4} + \cdots + a_1 + 1$ where $a_1 = 1$ and $a_2 = 1$ which corresponds to the Sloane's sequence A005251. The case $\alpha = 213$ is obtained by symmetry. □

5 Fixed point free involutions

In this section, we study the avoidance of one pattern of length 3 for fixed point free irreducible involutions, *i.e.*, for involutions with no fixed points and no successions. All enumerative results of this section are listed in Table 3. Let $\mathcal{DI}$ (resp. $\mathcal{DI}^{irr}$) be the set of fixed point free (resp. fixed point free irreducible) involutions. These sets restricted to their length n elements will be respectively denoted $\mathcal{DI}_n$ and $\mathcal{DI}_n^{irr}$.

Pattern	Sloane	Sequence
$\{321, 231\}, \{321, 312\}$	$A134816$	Padovan's spiral
$\{321, 132\}, \{321, 213\}$		$1, 1, 1, 0, 0, 0, \ldots$
$\{321, 123\}$		$1, 1, 2, 3, 0, 0, \ldots$
$\{213, 132\}$		$1, 1, 1, 1, \ldots$
$\{231, 312\}$	$A000213$	Tribonacci
$\{231, 132\}, \{231, 213\}, \{312, 132\}, \{312, 213\}$	$A000045$	Fibonacci
$\{123, 231\}, \{123, 312\}$	$A000217$	$\frac{n(n-1)}{2}$
$\{123, 132\}, \{123, 213\}$	$A005251$	$1, 1, 2, 4, 7, 12, 21, 37, 65$

Tab. 2: Wilf-equivalence classes for two pattern subsets of S_3 in irreducible permutations.

5.1 Enumerative results

Theorem 11 *The sets $\mathcal{DI}_n^{irr}$, $n \geq 0$, are enumerated by the sequence a_n defined by $a_0 = 0$, $a_2 = 1$, $a_{2n} = (2n-2)a_{2n-2} + (2n-4)a_{2n-4}$ for $n \geq 2$ and $a_{2n+1} = 0$ for $n \geq 0$ (see [Slo], A165968).*

Proof. Obviously, there does not exist any fixed point free involutions of odd length; thus $a_{2n+1} = 0$ for $n \geq 0$. Now, let σ be an irreducible involution of length $2n$, $n \geq 2$, with no fixed points. Let k be the index in $[2n-1]$ such that $\sigma_k = 2n$ (and also $\sigma_{2n} = k$).

Let π be the involution of length $(2n-2)$ obtained from σ by deleting k and $2n$, and reducing the result to an involution of length $(2n-2)$. Two cases occur: (a) π is irreducible; and (b) π has the two successions (π_{k-1}, π_k) and $(k-1, k)$. Conversely, each irreducible involution of length $2n$ can be obtained from an involution π belonging to one of the two previous cases (a) and (b).

Thus, the generating function for the permutations σ of type (a) is $(2n-2)a_{2n-2}$.

Now let us consider involutions of type (b). Let $\mathcal{A}_{2n-2}$ be the set of involutions of length $(2n-2)$ with exactly two successions and with no fixed points. An involution in $\mathcal{A}_{2n-2}$ can be extended into an irreducible involution of length $2n$ of type (b) in two possible ways. So, let π be an involution in A_{2n-2} such that (π_k, π_{k+1}) and $(k, k+1)$ are the two successions. From π, we construct the pair $(k, \pi') \in [2n-4] \times \mathcal{DI}_{2n-4}^{irr}$ where π' is obtained from π by deleting the two entries π_{k+1} and $(k+1)$, and reducing the result to a permutation of length $(2n-4)$. Since we obtain the same permutation in $\pi' \in \mathcal{DI}_{2n-4}^{irr}$ by deleting π_k and k, the cardinality of $\mathcal{A}_{2n-2}$ is half the cardinality of $[2n-4] \times \mathcal{DI}_{2n-4}^{irr}$. Therefore, the generating function for the number of involutions of type (b) is given by $2 \cdot \frac{(2n-4) \cdot a_{2n-4}}{2}$.

Finally, we conclude $a_{2n} = (2n-2)a_{2n-2} + (2n-4)a_{2n-4}$ for $n \geq 2$. $\square$

More generally, the proofs of Lemmas 1 and 2 imply that there are $\frac{n-1}{k} \cdot a_{2n}$ fixed point free involutions of length $2n$ with exactly $2k$ successions, where a_{2n} is the sequence defined in Theorem 11.

Theorem 12 *For $\alpha \in \{132, 213, 321\}$ and $n \geq 0$, the sets $\mathcal{DI}_{2n}^{irr}(\alpha)$ of irreducible involutions of length $2n$ without fixed points and avoiding the pattern α, are enumerated by the Motzkin numbers (see [Slo], A001006). The generating function for $\mathcal{DI}_n^{irr}(\alpha)$ is given by $\frac{1+x^2-\sqrt{(1-3x^2)(1+x^2)}}{2x^2}$.*

Proof. Since the set $\mathcal{DI}(\alpha)$ is expanded and closed under deflation, we apply Lemma 2 of Section 2. As the generating function for $\mathcal{DI}_n(\alpha)$, $n \geq 0$, is $f(x) = c(x^2)$, where $c(x) = \frac{1-\sqrt{1-4x}}{2x}$ is the generating function for the Catalan numbers, the generating function for $\mathcal{DI}_n^{irr}(\alpha)$ is

$$g(x) = f(\frac{x}{\sqrt{1+x^2}}) = c(\frac{x^2}{1+x^2}) = \frac{1+x^2-\sqrt{(1-3x^2)(1+x^2)}}{2x^2}.$$

□

A simple application of Lemma 2 yields the bivariate generating function $f(x,y) = c\left(\frac{x^2}{1+(1-y)x^2}\right)$ of the set $\mathcal{DI}(\alpha)$ of fixed point free involutions avoiding the pattern α for $\alpha \in \{321, 213, 132\}$, where the coefficient of $x^n y^k$ gives the number of elements in $\mathcal{DI}_n(\alpha)$ with exactly k successions.

Theorem 13 *For $\alpha \in \{231, 312\}$ and $n \geq 0$, the sets $\mathcal{DI}_{2n}^{irr}(\alpha)$ of irreducible involutions of length $2n$ without fixed points and avoiding the pattern α, are enumerated by 2^{n-1}.*

Proof. Let $\alpha = 231$ and $\sigma = \sigma_1\sigma_2\cdots\sigma_{2n}$ be an irreducible involution of length $2n$ without fixed points and avoiding α. Since σ avoids 231, we can decomposed $\sigma = \beta(2n)\gamma(2k+1)$ where $2k+1 \leq 2n-1$, $\beta \in \mathcal{DI}_{2k}^{irr}(\alpha)$ and $(2n)\gamma(2k+1)$ is a decreasing sequence of consecutive numbers in the interval $[2k+1, 2n]$. Therefore, if a_{2n} enumerates the set $\mathcal{DI}_{2n}^{irr}(\alpha)$, we deduce the induction $a_{2n} = a_0 + a_2 + a_4 + \cdots + a_{2n-2}$ where $a_0 = 1$. We obtain $a_{2n} = 2^{n-1}$ for $n \geq 1$ with $a_0 = 1$. The case $\alpha = 312$ is handled by the symmetry $\sigma \to \sigma^{-1}$. □

Theorem 14 *The generating function for the sets $\mathcal{DI}_n^{irr}(123)$, $n \geq 0$, of irreducible involutions of length n without fixed points and avoiding the pattern 123 is*

$$\frac{2x^2(x^2\sqrt{1-4x^2}-1)}{\sqrt{1-4x^2}\cdot(1+2x^4+\sqrt{1-4x^2})}.$$

Proof. Let σ be an involution of length $2n$ without fixed points containing at least one succession. We suppose that (σ_k, σ_{k+1}) is the leftmost succession. Since σ avoids 123, it is straightforward to see that $\sigma_k = 2n-k$ and $\sigma_{k+1} = 2n-k+1$.

Then σ can be written: $\sigma = \lambda(2n-k)(2n-k+1)\beta k(k+1)\gamma$ where λ (resp. γ) is a 123-avoiding sequence of elements in $[2n-k+2, 2n]$ (resp. $[1, k-1]$) without successions, such that red$(\lambda) = \gamma^{-1}$ and β is a sequence of elements in $[k+2, 2n-k-1]$ where red(β) is an involution of length $2n-2k-2$ without fixed points. See Figure 2 for an illustration of the structure of σ.

Let $g(x) = \frac{\frac{1}{1-4x^2}-1}{2}$ be the generating function for the set of involutions without fixed points and avoiding 123 (see for instance [DRS07]). Let $f(x)$ be the generating function of irreducible involutions without fixed points and avoiding 123. We have

$$g(x) - f(x) = x^4 \cdot h(x^2) \cdot (g(x)+1)$$

where $h(x)$ is the generating function for irreducible permutations avoiding the pattern 123 (see Theorem 1). Finally, a simple calculation gives the desired results. □

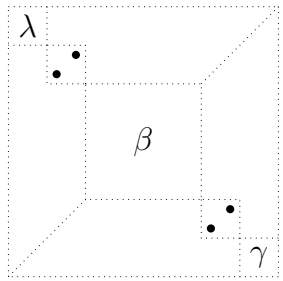


Fig. 2: Proof of Theorem 14. The special structure of a fixed point free involution avoiding 123 and with at least one succession.

Pattern	Sequence	Sloane	$a_{2n}, n \geq 1$
$\{\}$	$\begin{cases} a_{2n+1} = 0 \\ a_{2n} = (2n-2)a_{2n-2} + (2n-4)a_{2n-4} \end{cases}$	$A165968$	$1, 2, 10, 68, 604$
$\{132\}, \{213\}, \{321\}$	Motzkin	$A001006$	$1, 1, 2, 4, 9, 21$
$\{231\}, \{312\}$	2^{n-1}	$A000079$	$1, 2, 4, 8, 16, 32$
$\{123\}$	$\frac{2x^2(x^2\sqrt{1-4x^2}-1)}{\sqrt{1-4x^2}\cdot(1+2x^4+\sqrt{1-4x^2})}$	New	$1, 2, 8, 30, 109, 401$

Tab. 3: Fixed point free irreducible involutions avoiding at most one pattern of $\mathcal{S}_3$.

5.2 Bijective proofs

Recall that a Motzkin path of length n is a lattice path starting at $(0,0)$, ending at $(n,0)$, and never going below the x-axis, consisting of up steps $U = (1,1)$, horizontal steps $H = (1,0)$, and down steps $D = (1,-1)$.

• *Bijective proof for* $\mathcal{DI}_{2n}^{irr}(132)$.

A fixed point free irreducible involution σ of length $2n$ that avoids the pattern 132 is necessarily of the form $\sigma = \beta\beta'$ where β' belongs to $\mathcal{S}_n^{irr}(132)$ and such that red$(\beta) = \beta'^{-1}$ (all values of β are greater than those of β'). Using the bijection f from $\mathcal{S}_n^{irr}(132)$ to the set of Motzkin paths of length $(n-1)$ (see Section 3.2), we deduce immediately a constructive bijection $\bar{f}$ between $\mathcal{DI}_{2n}^{irr}(132)$ and the set of Motzkin paths of length $(n-1)$ as follows: $\bar{f}(\sigma) = f(\beta')$. □

• *Bijective proof for* $\mathcal{DI}_{2n}^{irr}(321)$.

This part presents a constructive bijection between $\mathcal{DI}_{2n}^{irr}(321)$ and the set of Motzkin paths of length $(n-1)$.

Let $\sigma = \sigma_1 \cdots \sigma_{2n}$ be an involution of length $2n$ without fixed points and that avoids 321. Then σ is the product of n transpositions (cycles of length 2): $\sigma = \langle \ell_1, r_1 \rangle \cdot \langle \ell_2, r_2 \rangle \cdots \langle \ell_n, r_n \rangle$ where $\ell_1 < \ell_2 < \cdots < \ell_n$, $r_1 < r_2 < \cdots < r_n$ and $r_i < \ell_i$ for $1 \leq i \leq n$. In fact, the values ℓ_i (resp. r_i) for $1 \leq i \leq n$, are the left-to-right maxima (resp. right-to-left minima) of σ, *i.e.*, ℓ_i (resp. r_i) is greater (resp. less) than all values on its left (resp. right) in σ.

Remark 2 *It is well-known (see [DRS07] for instance) that involutions of length $2n$ without fixed points and avoiding 321 are enumerated by the n-th Catalan number. Such an involution is associated to the Dyck path from $(0,0)$ to $(2n,0)$ by the following process: reading σ from left to right, we replace each left-to-right maxima with $U=(1,1)$ and each right-to-left minima with $D=(1,-1)$. For example, the involution $3\ 5\ 1\ 7\ 2\ 8\ 4\ 6\ 10\ 9 = \langle 3,1\rangle \cdot \langle 5,2\rangle \cdot \langle 7,4\rangle \cdot \langle 8,6\rangle \cdot \langle 10,9\rangle$ is associated to the Dyck path $UUDUDUDDUD$.*

On the other hand, σ can be viewed as a matching by putting $2n$ points labeled from 1 to $2n$ in this order, and then connecting, for $1 \leq j \leq n$, the numbers ℓ_j and r_j by an arc (see Figure 4 for an illustration).

Fig. 3: The two forbidden configurations in the matching of an involution $\sigma \in \mathcal{DI}_{2n}^{irr}(321)$.

Therefore, a fixed point free involution σ avoids 321 means that its matching does not contain nesting arcs, *i.e.* the configuration (1) in Figure 3 does not occur. Moreover, σ does not contain any successions if and only if the configuration (2) in Figure 3 does not occur, *i.e.*, there does not exist two arcs a_1 and a_2 such that a_2 is obtained from a_1 by an horizontal translation of $(1,0)$. These two last conditions can also be expressed using the one-line notation of σ as follows.

A fixed point free irreducible involution σ avoids 321 if and only if, for $1 \leq i \leq n-1$, at least one of the two following statements is verified:

(A_i) there exists a right-to-left minimum in σ between the i-th and $(i+1)$-th left-to-right maxima;

(B_i) there exists a left-to-right maximum between the i-th and $(i+1)$-th right-to-left minima.

Let us define the map h that transforms $\sigma \in \mathcal{DI}_{2n}^{irr}(321)$ into the Motzkin path M from $(0,0)$ to $(n-1,0)$ defined as follows:

For i from 1 to $n-1$,

(a) if B_i is verified but not A_i, then the i-th step of M is an up step U;

(b) if A_i is verified but not B_i, then the i-th step of M is a down step D;

(c) if A_i and B_i are verified, then the i-th step of M is an horizontal step H.

For instance, involutions 2143, 351624, 214365, 35172846, 35162487, 21573846 and 21436587 are respectively transformed by h into H, UD, HH, UHD, UDH, HUD and HHH (see Figure 4 for an example with $2n=20$).

Let us prove that h is a one-to-one correspondence between $\mathcal{DI}_{2n}^{irr}(321)$ and the set of Motzkin paths of length $n-1$. Let σ be an involution in $\mathcal{DI}_{2n}^{irr}(321)$. Using Remark 2, σ is associated with a Dyck path where left-to-right maxima (resp. right-to-left minima) correspond to up steps (resp. down steps). So, there is a one-to-one correspondence between up steps and down steps such that the image of an up step is a down step lying on its right. This induces a one-to-one correspondence c between the set of all occurrences of UU and the set of all occurrences of DD such that the image by c of a UU-occurrence lies

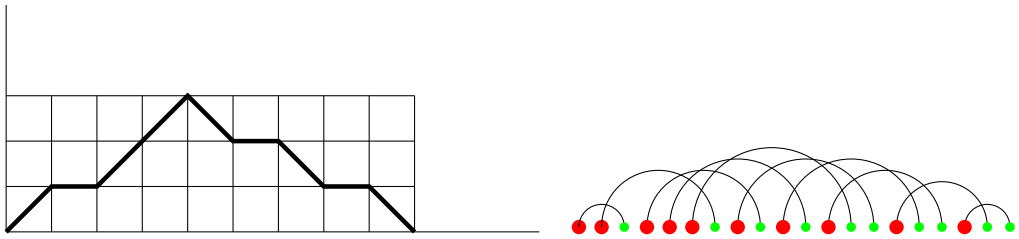


Fig. 4: The Motzkin path $UHUUDHDHD$ and its corresponding involution $\sigma = 3719(11)(13)2(14)4(16)5(17)68(19)(10)(12)(20)(15)(18)$.

on its right. This implies that there is a one-to-one correspondence $\bar{c}$ between the sets $Up = \{i, 1 \leq i \leq n-1, (a) \text{ is verified}\}$ and $Down = \{i, 1 \leq i \leq n-1, (b) \text{ is verified}\}$ such that $\bar{c}(i)$ is greater than i (we have $\bar{c}(i) \neq i$ since the configuration (2) of Figure 3 does not occur). As the function h associates an up step when $i \in Up$, a down step when $i \in Down$ and an horizontal step otherwise, $h(\sigma)$ is a Motzkin path of length $(n-1)$. Moreover this construction ensures that the images by h of two different involutions necessarily yields two different Motzkin paths.

Conversely, from any Motzkin path M of length $n-1$, we construct an involution of length $2n$ without fixed points and that avoids 321, by the following process. More precisely, we will construct a sequence of red and green points where red (resp. green) points correspond to left-to-right maxima (resp. right-to-left minima) of the involution of length $2n$. This sequence will characterize the desired involution.

We start the process with a red point followed by a green point. Through the Motzkin path M from left to right:

- if we meet an up step U, then we add one red point just after the last red point, and we add one red and one green points (in this order) on the right;
- if we meet a down step D, then we add one green point on the right;
- if we meet horizontal step H, then we add one red and one green points (in this order) on the right.

For instance, this process applying to the Motzkin path $M = UHD$ provides the following steps:

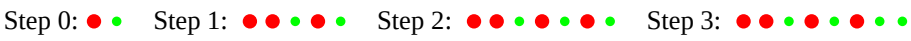


Fig. 5: Construction of the involution $\sigma = 35172846$ from the Motzkin path $M = UHD$.

At each step of this process, we add only one green point on the right. Moreover, the number of red points is, at each step, at most the number of green points; at the end of the process, there is equality. The configuration (2) of Figure 3 does not occur. Therefore, the obtained matching corresponds to that of a fixed point free irreducible involution that avoids 321 and such that its image by h is exactly the Motzkin path M. We conclude that h is a one-to-one correspondence between $\mathcal{DI}_{2n}^{irr}(321)$ and the set of Motzkin paths of length $n-1$. $\square$

Notice that the above construction appears as a generalization of the bijection of P. Manara and C. Perelli Cippo, [MC10], which transforms a restricted set of Motzkin paths into the set of simple involutions avoiding the pattern 321.

6 Pattern avoiding involutions

In this section, we present enumerative results for sets of irreducible involutions avoiding one pattern of length three (see Table 4).

Theorem 15 *For $\alpha \in \{231, 312\}$, the sets $\mathcal{I}_n^{irr}(\alpha)$, $n \geq 0$, of irreducible involutions of length n avoiding α are enumerated by the Tribonacci numbers (see [Slo], A000213).*

Proof. A length n irreducible involution σ avoiding 231 can be written $\beta n \gamma k$, $1 \leq k \leq n$, where $\beta \in \mathcal{I}_{k-1}^{irr}(\alpha)$ such that the last value of β is different from $n-1$ whenever $k = n$, and where $n\gamma k = n(n-1)\cdots(k+1)k$. Let $g(x)$ (resp. $h(x)$) be the generating function for the sets $\mathcal{I}_n^{irr}(\alpha)$, $n \geq 0$, (resp. for the sets of irreducible involutions of length n avoiding α such that the last value is different from n).

According to the above structure of σ, it is straightforward to see that $g(x) = 1 + x^2\frac{g(x)}{1-x} + xh(x)$ with $(1+x)h(x) = g(x)$. Thus we obtain $g(x) = \frac{x^2-1}{x^3+x^2+x-1}$ which is the generating function for the Tribonacci numbers. The case $\alpha = 312$ is handled by the symmetry $\sigma \to \sigma^{-1}$. □

Theorem 16 *The generating function for the sets $\mathcal{I}_n^{irr}(123)$, $n \geq 0$, of irreducible involutions of length n avoiding 123 is given by $\frac{1-2x-2x^2+4x^3+\sqrt{1-4x^2}}{(1-2x)(1+2x^4+\sqrt{1-4x^2})}$.*

Proof. Let σ be an involution of length n containing at least one succession. We suppose that (σ_k, σ_{k+1}) is the leftmost succession. Since σ avoids 123, it is straightforward to see that $\sigma_k = n-k$ and $\sigma_{k+1} = n-k+1$. We distinguish two cases: either (1) $\sigma_k \neq k$ or (2) $\sigma_k = k$.

In the first case, σ can be written $\sigma = \lambda(n-k)(n-k+1)\beta k(k+1)\gamma$ where λ (resp. γ) is a 123-avoiding sequence of elements in $[n-k+2, n]$ (resp. $[1, k-1]$) without successions, such that red(λ)= γ^{-1} and β is a sequence of elements in $[k+2, n-k-1]$ where red(β) is an involution of length $n-2k-2$.

In the second case, σ can be written $\sigma = \lambda k(k+1)\gamma$ where λ (resp. γ) is a 123-avoiding sequence of elements in $[k+2, n]$ (resp. $[1, k-1]$) without successions, such that red(λ)= γ^{-1}.

Let $g(x) = \frac{-1+2x+\sqrt{1-4x^2}}{2x-4x^2}$ be the generating function for the set of involutions avoiding 123 (see [SS85]). Let $f(x)$ be the generating function of irreducible involutions avoiding 123. The above structure of σ implies

$$g(x) - f(x) = x^4 \cdot h(x^2) \cdot g(x) + x^2 \cdot h(x^2)$$

where $h(x)$ is the generating function for irreducible permutations avoiding the pattern 123 (see Theorem 1). Finally, a simple calculation yields the desired results. □

Theorem 17 *For $\alpha \in \{132, 213, 321\}$, the generating function for the sets $\mathcal{I}_n^{irr}(\alpha)$, $n \geq 0$, of irreducible involutions of length n avoiding α is given by (see [Slo], A125189):*

$$\frac{2(1+x)(1+x^2)}{1-x+x^2-x^3+(1+x)\sqrt{1-2x^2-3x^4}}.$$

Proof. For $\alpha = 132$, $\sigma \in \mathcal{I}_n^{irr}(\alpha)$ can be written either $\sigma = \lambda n \beta \gamma k$ where $(\gamma k)^{-1} = \text{red}(\lambda n) \in \mathcal{S}_k^{irr}(132)$ and such that $\beta \in \mathcal{I}_{n-2k}^{irr}(132)$ or $\sigma = \beta n$ where $\beta \in \mathcal{I}_{n-1}^{irr}(\alpha)$ such that its last value is different from $n-1$. If $f(x)$ is the generating function for $\mathcal{I}_n^{irr}(\alpha)$, the above structure of σ implies that $f(x) = g(x^2)f(x) + xg(x^2)f(x) + 1 + x$ where $g(x)$ is the generating function for irreducible permutations avoiding 132 and ending with n. So, we have $g(x) = xh(x)$ where $h(x)$ is the generating function for irreducible permutations avoiding 132 and not ending with n; thus $g(x) = x(M(x) - g(x) + 1)$ where $M(x) = \frac{1-x-\sqrt{1-2x-3x^2}}{2x}$ is the generating function for irreducible permutations avoiding 132 (see Theorem 2). A simple calculation gives the result for $\alpha = 132$.

For $\alpha = 213$, the result is obtained from $\alpha = 132$ with the symmetry $\sigma \to (\sigma^r)^c$.

For $\alpha = 321$, let $f(x)$ be the generating function for $\mathcal{I}_n^{irr}(\alpha)$. Then we have $f(x) = h(x) + g(x)$ where $h(x)$ (resp. $g(x)$) is the generating function for $\mathcal{DI}_n^{irr}(\alpha)$ (resp. for $\mathcal{I}_n^{irr}(\alpha) \backslash \mathcal{DI}_n^{irr}(\alpha)$). On the other hand, we have $f(x) = f_1(x) + f_2(x)$ and $g(x) = h(x)f_1(x)$ where $f_1(x)$ (resp. $f_2(x)$) is the generating function for permutations in $\mathcal{I}_n^{irr}(\alpha)$ with the last value equal to n (resp. different from n). Then we have $f_1(x) = xf_2(x)$, $f_1(x) = \frac{x}{1+x} \cdot f(x)$ and $g(x) = \frac{x}{1+x} \cdot h(x) \cdot f(x)$. We obtain $f(x) = h(x) + \frac{x}{1+x} \cdot h(x) \cdot f(x)$. Since $h(x)$ is known using Theorem 12, a simple calculation gives the desired result. □

Problem: It remains to obtain the generating function for the sets $\mathcal{I}_n^{irr}$, $n \geq 0$, of irreducible involutions of length n.

Pattern	Sequence	Sloane	$a_n, n \geq 1$
$\{\}$	?	?	$1, 3, 5, 13, 37, 107, 341, 1141$
$\{132\}, \{213\}, \{321\}$	$\frac{2(1+x)(1+x^2)}{1-x+x^2-x^3+(1+x)\sqrt{1-2x^2-3x^4}}$	$A125189$	$1, 2, 2, 3, 5, 7, 11, 17, 27, 42$
$\{123\}$	$\frac{1-2x-2x^2+4x^3+\sqrt{1-4x^2}}{(1-2x)(1+2x^4+\sqrt{1-4x^2})}$	New	$1, 3, 4, 9, 16, 31, 58, 112$
$\{231\}, \{312\}$	Tribonacci	$A000213$	$1, 1, 1, 3, 5, 9, 17, 31, 57$

Tab. 4: Irreducible involutions avoiding at most one pattern of $\mathcal{S}_3$.

7 Acknowledgement

I would like to thank the anonymous referees for their very careful reading of this paper and their helpful comments and suggestions.

References

[AS02] M.D. Atkinson and T. Stitt. Restricted permutations and the wreath product. *Discrete Mathematics*, 259(1-3):19–36, 2002.

[Bar11] J.-L. Baril. Classical sequences revisited with permutations avoiding dotted pattern. *Electr. J. Comb.*, 18(1), 2011.

[Bón04] M. Bóna. Combinatorics of permutations. *Chapman&Hall*, Chap. 4:129–173, 2004.

[DRS07] E. Deutsch, A. Robertson, and D. Saracino. Refined restricted involutions. *European Journal of Combinatorics*, 28(1):481–498, 2007.

[DS77] R. Donaghey and L.W. Shapiro. Motzkin numbers. *Journal of Combinatorial Theory, Series A*, 23:291–301, 1977.

[Kap44] I. Kaplansky. Symbolic solutions of certain problems in permutations. *Bull. Amer. Math. Soc.*, 50:906–914, 1944.

[KM] S. Kitaev and T. Mansour. A survey on certain pattern problems. *University of Kentucky research report 2003-09*, **http://www.ms.uky.edu/∼math/MAreport/survey.ps**.

[Knu73] D.E. Knuth. The art of computer programming. *Addison-Wesley, Reading, MA*, Vol. 1, 1973.

[Man02] T. Mansour. Permutations avoiding a pattern from S_k and at least two patterns from S_3. *Ars. Combin.*, 62:227–239, 2002.

[MC10] P. Manara and C. Perelli Cippo. The fine structure of 321 avoiding involutions. *ArXiv:1010.5919v1*, 2010.

[Par09] R. Parviainen. Wilf classification of bi-vincular permutation patterns. *ArXiv:0910.5103v2*, 2009.

[Pud10] L. Pudwell. Enumeration schemes for permutations avoiding barred patterns. *Electr. J. Comb.*, 17(1), 2010.

[Rio65] J. Riordan. A recurrence for permutations without rising or falling successions. *Ann. Math. Statist.*, 36:745–748, 1965.

[Ros68] D.P. Roselle. Permutations by number of rises and successions. *Proc. Amer. Math. Soc.*, 19:8–16, 1968.

[Slo] N.J.A. Sloane. On-line encyclopedia of integer sequences, **http://www.research.att.com/∼njas/sequences/**.

[SS85] R. Simion and F.W. Schmidt. Restricted permutations. *European Journal of Combinatorics*, 6:383–406, 1985.

[Sta] R.P. Stanley. Catalan addendum. *Version of 22 October 2011*, **http://www-math.mit.edu/∼rstan/ec/catadd.pdf**.

[Tan76] S.M. Tanny. Permutations and successions. *Journal of Combinatorial Theory (A)*, 21:196–202, 1976.

[Tie43] H. Tietze. *Ü*ber gewisse unordnungen von permutationen. *S.-B. Math.-Natur. Abt. Bayer. Akad. Wiss.*, pages 281–293, 1943.

[Wes90] J. West. Permutations with forbidden subsequences and stack-sortable permutations. *Ph.D. Thesis, MIT*, 1990.

[Wes94] J. West. Generating trees and forbidden subsequences. *Proceeding of sixth FPSAC*, pages 441–450, 1994.

[Wes95] J. West. Generating trees and the catalan and schröder numbers. *Discrete Math.*, 146:247–262, 1995.

Discrete Mathematics and Theoretical Computer Science DMTCS vol. **18**:2, 2017, #12

Right-jumps & pattern avoiding permutations

Cyril Banderier[1]* Jean-Luc Baril[2]† Céline Moreira Dos Santos[2]‡

[1] *LIPN UMR-CNRS 7030, Université de Paris Nord, France.*
[2] *LE2I UMR-CNRS 6306, Université de Bourgogne, France.*

received 8th Dec. 2015, revised 8th Feb. 2017, accepted 8th Feb. 2017.

We study the iteration of the process of moving values to the right in permutations. We prove that the set of permutations obtained in this model after a given number of iterations from the identity is a class of pattern avoiding permutations. We characterize the elements of the basis of this class and enumerate it by giving their bivariate exponential generating function: we achieve this via a catalytic variable, the number of left-to-right maxima. We show that this generating function is a D-finite function satisfying a differential equation of order 2. We give some congruence properties for the coefficients of this generating function, and show that their asymptotics involves a rather unusual algebraic exponent (the golden ratio $(1+\sqrt{5})/2$) and some closed-form constants. We end by proving a limit law: a forbidden pattern of length n has typically $(\ln n)/\sqrt{5}$ left-to-right maxima, with Gaussian fluctuations.

Keywords: Permutation pattern, left-to-right maximum, insertion sort, generating function, analytic combinatorics, D-finite function, supercongruence

1 Introduction

In computer science, many algorithms related to sorting a permutation have been analysed and shown to have behaviours linked to nice combinatorial properties (see e.g. Knuth (1998)). Their complexity can be analysed in terms of memory needed, or number of key operations (like comparisons or pointer swaps). An important family of algorithms, like the so-called insertion algorithms, or in situ permutations, are quite efficient in terms of the number of pointer swaps (but are not the fastest ones in terms of comparisons). Due to this higher cost, they have been much less studied than the faster stack sorting algorithms. Like for the stack algorithms, instead of seeing them as an input/output pair, we can see them like a *process*: input and set of intermediate steps. This opens a full realm of questions on such processes, and they often lead to nice links with other parts of mathematics (like the link between trees, birth and death processes, random walks in probability theory, or permutations and Young tableaux in algebraic combinatorics). Our article will investigate a link between a sorting algorithm, patterns in permutations, and their asymptotics counterparts.

*http://lipn.fr/~banderier
†http://jl.baril.u-bourgogne.fr/
‡http://le2i.cnrs.fr/-Celine-Moreira-Dos-Santos-

ISSN 1365–8050 © 2017 by the author(s) Distributed under a Creative Commons Attribution 4.0 International License

Another motivation to analyse such processes comes from the field of bioinformatics. Indeed, in genomics, a crucial part of study is to estimate the similarity of two genomes. This consists in finding the length of a shortest path of evolutionary mutations that transforms one genome into another. Usually, the main operations used in the rearrangement of a genome are of three different types: substitutions (one gene is replaced with another), insertions (a gene is added) and deletions (a gene is removed). For instance, we refer to Jones and Pevzner (2004); Meidanis and Setubal (1997) for an explanation of these operations, and the notions of transposons or jumping genes.

As the problem is too hard in full generality, many simpler mathematical models of the genome are used (see Ewens et al. (1982)): one of them is using permutations of $\{1, 2, \ldots, n\}$ where each gene is assigned a number. Following the idea of transposition mutations (see Jones and Pevzner (2004)), our motivation is to find some combinatorial properties in terms of pattern avoiding permutations whenever one element is deleted and inserted in a position to its right. This operation will be called a right-jump.

$$1\ 2\ 3\ 4\ 5\ 6\ 7 \quad \rightsquigarrow \quad 1\ 2\ 4\ 5\ 6\ 3\ 7$$

Figure 1: A right-jump in the permutation $\sigma = 1234567$. In this article, we investigate the structure of permutations obtained after several iterations of such right-jumps.

This operation is a variant of genome duplication, which consists of copying a part of the original genome inserted into itself, followed by the loss of one copy of each of the duplicated genes. In particular, it is comparable to the whole duplication-random loss model studied in Chaudhuri et al. (2006). Although there are many connections between these models, it is surprising that the behaviour of their combinatorial properties depends on different parameters: the right-jump model reveals some links with *left-to-right maximum* statistics (see Baril (2013); Bóna (2012)), while the whole duplication-random loss model reveals links with *descent* statistics (see Baril and Vernay (2010); Bouvel and Ferrari (2013); Bouvel and Pergola (2010); Bouvel and Rossin (2009); Chaudhuri et al. (2006); Mansour and Yan (2010)).

In the literature, such right-jumps in permutations are also found in the domain of sorting theory. Indeed, it corresponds (modulo a mirror symmetry) to the *insertion-sorting algorithms* on permutations (see Knuth (1998)). Since the seminal work of Knuth on this subject, many articles related to sorting with a stack exhibit links with pattern avoiding permutations. In contrast, for insertion sorting algorithms (also the subject of a vivid literature), only one study exhibits links with pattern avoiding permutations. Indeed, in his thesis, Magnússon (2013) proves that the set of permutations that can be sorted with one step of the insertion-sorting operator is the class of permutations avoiding the three patterns 321, 312 and 2143.

Plan of the article. In Section 2, we recall some basic facts on permutations patterns. In Section 3, we generalize the result of Magnússon by studying the iteration of right-jumps in terms of pattern avoiding permutations: we prove that the set of permutations obtained from the identity after a given number of right-jumps is the class of permutations avoiding some patterns, which we characterize. In Section 4, we enumerate these forbidden patterns by giving their bivariate exponential generating function (involving an additional parameter: the number of left-to-right maxima), and we give the corresponding asymptotics and limit law. We also give some modular congruences for our main enumeration sequence. In Section 5, we conclude with several possible extensions of this work.

2 Patterns in permutations

In this section, we give some classical definitions and properties on patterns in permutations. For any permutation $\sigma \in \mathcal{S}_n$ (the set of permutations of length n), the *graphical representation* of $\sigma = \sigma_1\sigma_2\ldots\sigma_n$ is the set of points in the plane at coordinates (i, σ_i) for $i \in [\![n]\!]$[(i)]. For instance, the permutation 53621487 has the graphical representation illustrated in Figure 2. A *left-to-right maximum* of $\sigma \in \mathcal{S}_n$ is a value σ_i, $1 \leq i \leq n$, such that $\sigma_j \leq \sigma_i$ for $j \leq i$. A value σ_i of σ, $1 \leq i \leq n$ which is not a left-to-right maximum will be called a *non-left-to-right-maximum* of σ. For instance, if $\sigma = 53621487$ then the left-to-right maxima are $5, 6, 8$ and the non-left-to-right-maxima are $1, 2, 3, 4, 7$.

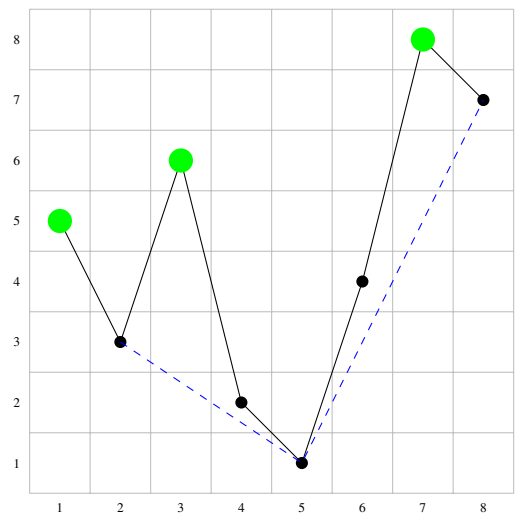


Figure 2: The graphical representation of $\sigma = 53621487$. We show an occurrence of a pattern 213 with a dashed line; big green points are the left-to-right maxima and small black points are non-left-to-right-maxima.

A permutation π of length k, is a *pattern* of a permutation $\sigma \in \mathcal{S}_n$ if there is a subsequence of σ which is order-isomorphic to π, *i.e.*, if there is a subsequence $\sigma_{i_1}\ldots\sigma_{i_k}$ of σ with $1 \leq i_1 < \cdots < i_k \leq n$ and such that $\sigma_{i_j} < \sigma_{i_\ell}$ whenever $\pi_j < \pi_\ell$. We write $\pi \prec \sigma$ to denote that π is a pattern of σ. A permutation σ that does not contain π as a pattern is said to *avoid* π. For example, $\sigma = 2413$ contains the patterns 231, 132, 213 and 312, but σ avoids the patterns 123 and 321. The set of all permutations avoiding the patterns $\pi_1, \ldots, \pi_m$ is denoted by $\mathrm{Avoid}(\pi_1, \ldots, \pi_m)$. We say that $\mathrm{Avoid}(\pi_1, \ldots, \pi_m)$ is a *class* of pattern avoiding permutations with *basis* $\{\pi_1, \ldots, \pi_m\}$. For instance, we refer to the book of Kitaev (2011) and Bóna (2012) to deepen these notions. A set $\mathcal{C}$ of permutations is *stable* for the involvement relation $\prec$ if, for any $\sigma \in \mathcal{C}$, for any $\pi \prec \sigma$, then we also have $\pi \in \mathcal{C}$.

Now, we formulate a definition that is crucial for the present study.

(i) In this article, we write $[\![n]\!]$ for $\{1, 2, \ldots, n\}$.

Definition 1 (Permutation basis and basis permutations)**.** *If a set $\mathcal{C}$ of permutations is stable for the involvement relation $\prec$, then $\mathcal{C}$ is a class of pattern avoiding permutations: $\mathcal{C} = \mathrm{Avoid}(\mathcal{B})$. The basis $\mathcal{B}$ of forbidden patterns is then given by $\mathcal{B} = \{\sigma \notin \mathcal{C}, \forall \pi \prec \sigma \text{ with } \pi \neq \sigma, \pi \in \mathcal{C}\}$. In other words, the basis $\mathcal{B}$ is the set of minimal permutations σ that do not belong to $\mathcal{C}$, where minimal is intended in the sense of the pattern-involvement relation $\prec$ on permutations, that is: if $\pi \prec \sigma$ and $\pi \neq \sigma$ then $\pi \in \mathcal{C}$. Notice that $\mathcal{B}$ might be infinite. We call* basis permutations *the permutations belonging to $\mathcal{B}$.*

Equipped with these definitions, our mission consists now in giving a description of the permutations belonging to the basis $\mathcal{B}_p$ (the permutations which are the minimal forbidden patterns) for the set $\mathcal{C}_p$ of permutations at distance at most p from the identity, i.e., permutations obtained from the identity after at most p right-jumps.

3 Iteration of right-jumps in permutations: a structural description of the forbidden patterns

In this section we study the iteration of right-jumps in terms of pattern avoiding permutations. We establish that the set $\mathcal{C}_p$ of permutations obtained from the identity after at most p right-jumps is a class of permutations avoiding some patterns that we characterize.

Lemma 1 (Characterization of the distance)**.** *A permutation obtained from the identity after p right-jumps contains at most p non-left-to-right-maxima.*

Proof: The result holds for $p = 1$; indeed a right-jump transformation of the identity permutation creates the permutation $1\ 2\ \ldots (i-1)\ (i+1) \ldots (j-1)\ i\ j \ldots n$ for $1 \leq i < j$, where i is the only one non-left-to-right-maximum. Now, let us assume that each permutation π obtained from the identity after $(p-1)$ right-jumps contains at most $p-1$ non-left-to-right-maxima. Let σ be a permutation obtained from the identity after p right-jumps. Using the recurrence hypothesis, σ is obtained from a permutation π with at most $p-1$ non-left-to-right-maxima by moving an element π_i, $1 \leq i < n$, in a position to its right.

We distinguish two cases: (1) π_i is a non-left-to-right-maximum, and (2) π_i is a left-to-right maximum.

Case (1): Since π_i is a non-left-to-right-maximum, there exists $j < i$ such that π_j is a left-to-right maximum satisfying $\pi_j > \pi_i$. Since we move π_i to its right, π_j remains on the left of π_i in σ which implies that π_i is a non-left-to-right-maximum in σ. Using the same argument, any non-left-to-right-maximum π_k in π remains a non-left-to-right-maximum in σ. Moreover, let π_k be a left-to-right maximum in π, *i.e.*, $\pi_j < \pi_k$ for all $j < k$. Since the right-jump transformation moves to the right of a non-left-to-right-maximum, all values on the left of π_k in σ are lower than π_k, which proves that π_k remains a left-to-right maximum in σ. Therefore, σ contains at most $p-1$ non-left-to-right-maxima (as π does).

Case (2): π_i is a left-to-right maximum, *i.e.*, $\pi_j < \pi_i$ for all $j < i$. Since π_i is moved to its right, any left-to-right maximum located on the left of π_i in π remains a left-to-right maximum in σ. On the other hand, any left-to-right maximum located on the right of π_i in π is greater than π_i and thus, it remains a left-to-right maximum in σ. Therefore, the number of left-to-right maxima in σ is at least the number of left-to-right maxima in π minus one (we do not consider π_i). This means that the number of non-left-to-right-maxima in σ is at most p.

Considering the two previous cases allows to complete the proof, by induction. $\square$

We now derive our first enumeration result for the set $\mathcal{D}_p$ of permutations at distance p from the identity, *i.e.*, the set of permutations reachable from the identity with p right-jumps, but that one cannot reach with less than p right-jumps: $\mathcal{C}_p = \cup_{k \in [\![p]\!]} \mathcal{D}_k$, and this union is disjoint.

Theorem 1 (Permutations after p right-jumps)**.** *The set $\mathcal{D}_p$ of permutations at distance p from the identity is the set of permutations with exactly p non-left-to-right-maxima. Accordingly, the number $d_{n,p}$ of permutations of length n in $\mathcal{D}_p$ is counted by the Stirling numbers $s(n, n-p)$:*

$$d_{n,p} = s(n, n-p) = \sum_{0 \le j \le h \le p} (-1)^j \binom{h}{j} \binom{n-1+h}{p+h} \binom{n+p}{p-h} \frac{(j-h)^{p+h}}{h!} .$$

Proof: After considering Lemma 1, it suffices to prove that any permutation σ with at most p non-left-to-right-maxima can be obtained from the identity after p right-jumps. Let σ be a permutation with $p \ge 1$ non-left-to-right-maxima. Let us assume that the leftmost non-left-to-right-maximum is σ_i and let $j < i$ be the position of the smallest left-to-right maximum σ_j such that $\sigma_j > \sigma_i$. Then we set $\sigma' = \sigma_1 \dots \sigma_{j-1}\sigma_i\sigma_j \dots \sigma_{i-1}\sigma_{i+1} \dots \sigma_n$. Since we have $\sigma_j > \sigma_i$ and also $\sigma_i > \sigma_{j-1}$ (if σ_{j-1} exists), σ_i becomes a left-to-right maximum in σ'. Thus, σ' contains exactly $p-1$ non-left-to-right-maxima and by construction, σ can be obtained from σ' by a right-jump. This proves that permutations at distance p from the identity are exactly the permutations with $n-p$ left-to-right-maxima, which are known to be counted by $s(n, n-p)$, the signless Stirling number of the first kind (see Flajolet and Sedgewick (2009) for the closed-form formula due to Schlömilch, and sequence `OEIS A094638` in Sloane and collaborators (2016) for many occurrences of the corresponding triangular array). □

For instance, the values of $d_{n,p}$ for $n = 7$ and $0 \le p < 7$ are 1, 21, 175, 735, 1624, 1764, 720.

The following corollary says a little more on the lattice structure associated to our process "a particle jumps to the right".

Corollary 1 (Changing the starting point and sorting algorithms)**.** *For any permutation p, one denotes by t_p its number of non-left-to-right-maxima. Let σ and π be two permutations, then $t_{\sigma^{-1}.\pi}$ right-jumps are necessary to obtain π from σ. In particular, $t_{\sigma^{-1}}$ right-jumps are necessary and sufficient to sort by insertion the permutation σ into the identity.*

Proof: Firstly, t_σ right-jumps are necessary and sufficient to obtain σ from the identity. Therefore, $t_{\sigma^{-1}.\pi}$ transformations are sufficient and necessary to obtain $\sigma^{-1} \cdot \pi$ from the identity. We set $t = t_{\sigma^{-1}.\pi}$ and let $Id = \chi_0, \chi_1, \dots, \chi_{t-1}, \chi_t = \sigma^{-1} \cdot \pi$ be a shortest path between the identity and $\sigma^{-1} \cdot \pi$. Now, let us prove that if a permutation β is obtained from α by one right-jump, then for any permutation γ, $\gamma \cdot \beta$ is also obtained from $\gamma \cdot \alpha$ by one right-jump. Indeed, if we have $\alpha = \alpha_1\alpha_2 \dots \alpha_n$ then β can be written as $\beta = \alpha_1 \dots \alpha_{i-1}\alpha_{i+1} \dots \alpha_{j-1}\alpha_i\alpha_j \dots \alpha_n$. Composing by a permutation γ, we obtain $\gamma \cdot \alpha = \gamma(\alpha_1)\gamma(\alpha_2) \dots \gamma(\alpha_n)$ and $\gamma \cdot \beta = \gamma(\alpha_1) \dots \gamma(\alpha_{i-1})\gamma(\alpha_{i+1}) \dots \gamma(\alpha_{j-1})\gamma(\alpha_i)\gamma(\alpha_j) \dots \gamma(\alpha_n)$ which proves that $\gamma \cdot \beta$ is also obtained from $\gamma \cdot \alpha$ by one right-jump. So if we compose by σ at each step of the above shortest path, then we obtain a shortest path of $t_{\sigma^{-1}.\pi}$ right-jumps from σ to π, which completes the proof. □

Since the set $\mathcal{C}_p$ of permutations obtained after p right-jumps is stable for the relation $\prec$, $\mathcal{C}_p$ is also a class $\mathrm{Avoid}(\mathcal{B}_p)$ of pattern avoiding permutations where $\mathcal{B}_p$ is the basis consisting of minimal permutations σ that are not in $\mathcal{C}_p$ (see Definition 1). Theorem 2 gives the explicit description of these basis permutations.

Theorem 2 (Structural description of the basis permutations)**.** *A permutation $\sigma \in \mathcal{S}_n$ belongs to the basis $\mathcal{B}_p$ of forbidden patterns, if and only if the following conditions hold:*

(i) σ contains exactly $p+1$ non-left-to-right-maxima.

(ii) $n-1$ is a non-left-to-right-maximum.

(iii) σ_2 is a non-left-to-right-maximum.

(iv) For any three left-to-right maxima, σ_i, σ_j and σ_k (with $i<j<k$) such that there is no left-to-right maximum between them, there exists a non-left-to-right-maximum σ_t (with $j<t<k$) satisfying $\sigma_t > \sigma_i$.

Proof: N.B.: Figure 3 on next page illustrates the different claims and notations of this theorem.

Let $\sigma \in \mathcal{S}_n$ be a permutation belonging to the basis $\mathcal{B}_p$, *i.e.*, $\sigma \notin \mathcal{C}_p$ and $\pi \prec \sigma$ implies $\pi \in \mathcal{C}_p$. Throughout this proof, we refer to Figure 3 for an illustration of the three conditions (ii), (iii) and (iv).

- First, the deletion of a non-left-to-right-maximum in σ decreases the number of non-left-to-right-maxima by one exactly. Therefore, the minimality of σ implies that σ necessarily contains exactly $p+1$ non-left-to-right-maxima, which proves (i).

- For a contradiction, assume that (ii) is not satisfied, *i.e.*, $n-1$ is a left-to-right maximum. Since n is always a left-to-right maximum, n is on the right of $n-1$ in σ. Thus, the permutation π obtained by deleting n from σ also contains $p+1$ non-left-to-right-maxima (a non-left-to-right-maximum on the right of n in σ remains a non-left-to-right-maximum on the right of $n-1$ in π). Therefore, π does not belong to $\mathcal{C}_p$; this gives a contradiction with the minimality of σ.

- For a contradiction, assume that (iii) is not satisfied, *i.e.*, σ_2 is a left-to-right maximum and thus, σ_1 is smaller than σ_2. Thus, the permutation π obtained by deleting σ_1 from σ also contains $p+1$ non-left-to-right-maxima. Indeed, a non-left-to-right-maximum σ_i in σ such that $\sigma_i < \sigma_1$ becomes a non-left-to-right-maximum $\sigma_i < \sigma_2 - 1$ in π. Moreover, a non-left-to-right-maximum σ_i in σ such that $\sigma_i > \sigma_1$ (there is ℓ, $2 \leq \ell < i$, with $\sigma_i < \sigma_\ell$) becomes a non-left-to-right-maximum $\sigma_i - 1$ in π with $\sigma_i - 1 < \sigma_\ell - 1$. Therefore, π does not belong to $\mathcal{C}_p$; this gives a contradiction with the minimality of σ.

- For a contradiction, assume that (iv) is not satisfied; *i.e.*, there are (i,j,k), $1 \leq i < j < k \leq n$, such that σ_i, σ_j and σ_k are three consecutive left-to-right maxima of σ (consecutive means that there is no other left-to-right maximum between σ_i and σ_j and between σ_j and σ_k), and such that there is no non-left-to-right-maximum σ_ℓ, $j < \ell < k$, satisfying $\sigma_i < \sigma_\ell$. Let π be the permutation obtained from σ by deleting σ_j. It is clear that any non-left-to-right-maximum on the left of σ_j in σ remains a non-left-to-right-maximum in π. Let σ_ℓ, $\ell > k$, be a non-left-to-right-maximum on the right of σ_k in σ. If $\sigma_\ell < \sigma_j$ then $\sigma_\ell < \sigma_j \leq \sigma_k - 1$ and σ_ℓ remains a non-left-to-right-maximum in π. If $\sigma_\ell > \sigma_j$ then there is σ_t, $t \geq k$, such that $\sigma_\ell \leq \sigma_t \geq \sigma_k$, and thus, there is $\sigma_t - 1$ on the left of $\sigma_k - 1$ in π with $\sigma_t - 1 > \sigma_k - 1$ which means that $\sigma_\ell - 1$ is a non-left-to-right-maximum in π. Let σ_ℓ, $j < \ell < k$, be a non-left-to-right-maximum between σ_j and σ_k in σ. Assuming that (iv) is not satisfied, we deduce that $\sigma_\ell < \sigma_i$, and σ_ℓ

remains a non-left-to-right-maximum in π. Finally, π also contains $p+1$ non-left-to-right-maxima; this gives a contradiction with the minimality of σ.

Conversely, let σ be a permutation satisfying (i), (ii), (iii) and (iv) and π be a permutation obtained by deleting σ_i, $1 \leq i \leq n$, from σ. Let us prove that π belongs to $\mathcal{C}_p$, that is, π contains at most p non-left-to-right-maxima.

- If σ_i is a non-left-to-right-maximum of σ, then π has p non-left-to-right-maxima and thus, $\pi \in \mathcal{C}_p$.

- Now, let us assume that σ_i is a left-to-right maximum of σ. If $\sigma_i = n$, then (ii) implies that $n-1$ is a non-left-to-right-maximum of σ and a left-to-right maximum in π; this implies that π contains p non-left-to-right-maxima and thus, $\pi \in \mathcal{C}_p$. If $\sigma_i = \sigma_1$, then (iii) implies that σ_2 is a non-left-to-right-maximum of σ and $\sigma_2 - 1$ is a left-to-right maximum in π; this implies that π contains p non-left-to-right-maxima and thus, $\pi \in \mathcal{C}_p$. If there exists (j,k), $1 \leq j < i < k \leq n$, such that σ_j and σ_k are left-to-right maxima (we choose j the greatest possible and k the lowest possible with this property). Then, (iv) implies that there is σ_ℓ, $i < \ell < k$, such that $\sigma_j < \sigma_\ell < \sigma_i$ (we choose the lowest possible $\ell > i$). Thus, σ_ℓ is a non-left-to-right-maximum in σ and becomes a left-to-right maximum in π, which implies that π contains exactly p non-left-to-right-maxima, and thus $\pi \in \mathcal{C}_p$.

Finally, the permutation π necessarily belongs to $\mathcal{C}_p$, which completes the proof. $\square$

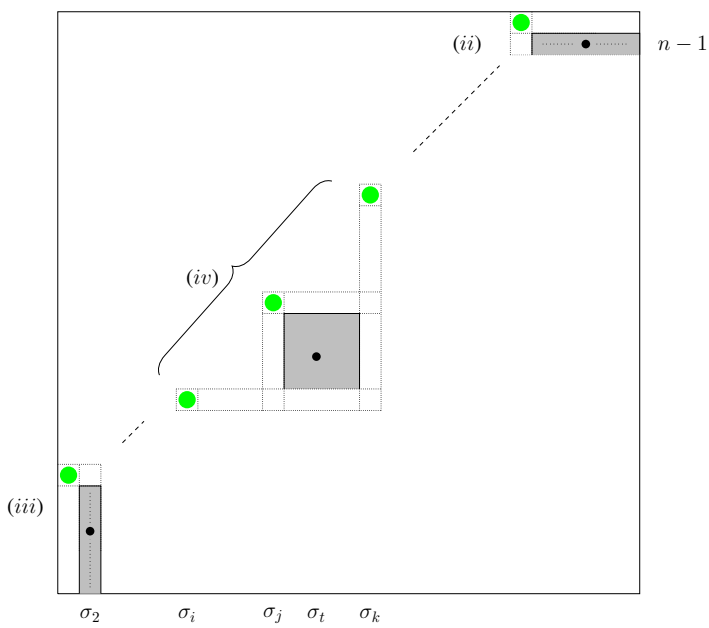


Figure 3: An illustration of Theorem 2 that characterizes the basis permutations of $\mathcal{B}_p$. Condition (i) states such a basis permutation has $p+1$ non-left-to-right maxima (drawn with a small black point, while left-to-right maxima are drawn with a big green point), condition (ii) states that $n-1$ is not a left-to-right maximum, condition (iii) states that σ_2 is not a left-to-right maximum, and condition (iv) states that there is a "higher" non-left-to-right maximum between 3 left-to-right maxima.

Corollary 2 (Length of the forbidden patterns)**.** *Permutations in $\mathcal{B}_p$ have length $\leq 2(p+1)$ and $\geq p+2$. As a consequence, $\mathcal{B}_p$ is a finite set.*

Proof: Theorem 2 implies that the number of left-to-right maxima in a basis permutation is at most the number of non-left-to-right-maxima. Since a basis permutation of $\mathcal{B}_p$ has $p+1$ non-left-to-right-maxima, its length is at most $2(p+1)$. □

For instance, the basis for $p = 0, 1, 2$ are respectively $\mathcal{B}_0 = \{21\}$, $\mathcal{B}_1 = \{312, 321, 2143\}$ (recovering the result of Magnússon), and $\mathcal{B}_2 = \{4123, 4132, 4213, 4231, 4312, 4321, 21534, 21543, 31254, 32154, 31524, 31542, 32514, 32541, 214365\}$.

4 Enumerative results for basis permutations

In order to obtain a recursive formula for the number $b_{n,p}$ of permutations of length n in the basis $\mathcal{B}_p$, we present the following preliminary lemma.

Lemma 2 (A recursive description)**.** *Let $\sigma \in \mathcal{S}_n$ be a basis permutation having $p \geq 1$ non-left-to-right-maxima and such that $\sigma_{k+1} = n$, $k \geq 0$. Let α be the subsequence $\sigma_1\sigma_2 \ldots \sigma_k$ and π be the permutation in $\mathcal{S}_k$ isomorphic to α. Then, π is a basis permutation with $p - n + k + 1$ non-left-to-right-maxima.*

Proof: Any permutation σ can be uniquely written as $\sigma = \alpha n \beta$ where α and β are two subsequences of $[\![n-1]\!]$. Let k be the length of α and let $\pi = \pi_1\pi_2 \ldots \pi_k$ be the permutation of $[\![k]\!]$ that is isomorphic to the subsequence α. Let us prove that π is minimal.

Since σ is minimal, it satisfies the three conditions (ii), (iii) and (iv) of Theorem 2. Since all elements in β are non-left-to-right-maxima in σ, π contains exactly $p - (n - 1 - k)$ non-left-to-right-maxima and thus, $n - p - 1$ left-to-right maxima.

- The condition (iii) of Theorem 2 on σ does not involve the part $n\beta$. Therefore, π satisfies (iii).

- The deletion of $n\beta$ from σ preserves the condition (iv) on π. Thus, π satisfies (iv).

- Let σ_i, σ_j and $\sigma_{k+1} = n$, $1 \leq i < j \leq k$, be the last three left-to-right maxima of σ. After the deletion of $n\beta$, the two left-to-right maxima of σ, σ_i and σ_j, are respectively transported in π into π_i and $\pi_j = k$. Condition (iv) on σ ensures that there is σ_ℓ, between σ_j and n such that $\sigma_\ell > \sigma_i$. The greatest value σ_ℓ satisfying this property is then transported in π into $k-1$, which proves that $k-1$ is on the right of k in π. Thus, π satisfies (ii).

Using Theorem 2, the permutation π is a basis permutation with $p + \ell - n + 1$ non-left-to-right-maxima. □

Theorem 3 (An infinite recursion)**.** *The number $b_{n,p}$ of basis permutations of length n in $\mathcal{B}_p$ (or equivalently having exactly $p+1$ non-left-to-right-maxima) is given by the following recurrence relation (for $p < n-2$):*

$$b_{n,p} = \sum_{\ell=0}^{p-1} (\ell+1)! \cdot \binom{n-2}{\ell} \cdot b_{n-\ell-2,p-\ell-1}$$

anchored with $b_{n,p} = 0$ if $p < (n-2)/2$ or $p > n-2$, and $b_{n,n-2} = (n-1)!$ for $n > 1$.

Proof: Any permutation σ of length $n \geq 1$ contains at least one left-to-right maximum and thus, at most $n-1$ non-left-to-right-maxima which implies that $b_{n,p} = 0$ for $n \leq p+1$. Using the proof of Corollary 2, we also have $b_{n,p} = 0$ for $n > 2(p+1)$. Moreover, the basis permutations of length n with $n-1$ non-left-to-right-maxima are the permutations of the form $n\alpha$ where $\alpha \in \mathcal{S}_{n-1}$. So, we have $b_{n,n-2} = (n-1)!$ for $n > 1$.

Now, let us prove the recursive relation. Let $\sigma \in \mathcal{S}_n$ be a basis permutation with $p+1$ non-left-to-right-maxima. We consider its unique decomposition $\sigma = \alpha n \beta$ where α and β are some subsequences of $[\![n-1]\!]$. Let $\ell+2$, $\ell \geq 0$, be the length of $n\beta$ and let π be the permutation in $\mathcal{S}_{n-\ell-2}$ isomorphic to α. Using Lemma 2 with $k = n-\ell-2$, π is minimal with $p-\ell-1$ non-left-to-right-maxima. So, we can associate to $\sigma = \alpha n \beta$ the pair (π, γ) where $\pi \in \mathcal{S}_{n-\ell-2}$ is minimal with $p-\ell-1$ non-left-to-right-maxima and $\gamma \in \mathcal{S}_{\ell+1}$ is isomorphic to β.

Conversely, let π be a basis permutation of length $n-\ell-2$ with $p-\ell-1$ non-left-to-right-maxima and $\gamma \in \mathcal{S}_{\ell+1}$. We construct a basis permutation σ of length n with $p+1$ non-left-to-right-maxima as follows. From $\gamma \in \mathcal{S}_{\ell+1}$, we construct a subsequence β of $[\![n-1]\!]$ of length $\ell+1$ such that β contains the value $n-1$ and such that β is isomorphic to γ. Since $n-1$ belongs to β, its position in β also is the position of the greatest value of γ. So, β is characterized by the choice of ℓ values among $[\![n-2]\!]$. Now, we define the unique subsequence α of $[\![n-2]\!] \backslash X$ isomorphic to π where X is the set of values used in β. This construction ensures that $\sigma = \alpha n \beta$ is a basis permutation of length n with $p+1$ non-left-to-right-maxima, and so $\sigma \in \mathcal{B}_p$. So, there are $\binom{n-2}{\ell}$ possibilities to choose the values of β and $(\ell+1)!$ possibilities to choose γ and $b_{n-\ell-2,p-\ell-1}$ possibilities to choose a basis permutation $\pi \in \mathcal{S}_{n-\ell-2}$ with $p-\ell-1$ non-left-to-right-maxima. Varying ℓ from 0 to $p-1$, we obtain the recursive formula. $\square$

Theorem 3 allows us to find the bivariate exponential generating function for the number of basis permutations according to the number of non-left-to-right-maxima.

Theorem 4 (Closed-form for the bivariate generating function)**.** *Consider the bivariate exponential generating function $B(x,y) = \sum_{n\geq 0, p\geq 0} b_{n,p} \frac{x^n y^p}{n!}$ where the coefficient of $\frac{x^n y^p}{n!}$ is the number $b_{n,p}$ of basis permutations of length n in $\mathcal{B}_p$. Then, we have*

$$B(x,y) = \frac{1}{2y}\left(1 - \frac{1}{V}\right)(1-xy)^{\frac{1}{2}(1+V)} + \frac{1}{2y}\left(1 + \frac{1}{V}\right)(1-xy)^{\frac{1}{2}(1-V)} - \frac{1}{y}\,,$$

$$\textit{where } V := \sqrt{1+4/y}\,.$$

Proof: Setting $F_p(x) := \sum_{n\geq 0} b_{n,n-p} \frac{x^n}{n!}$ and $F(x,y) := \sum_{p\geq 0} F_p(x) y^p$, we have $B(x,y) = F(xy, 1/y)$. (We work with the generating function $F_p(x)$ of the $(b_{n,n-p})$'s rather than the generating function of the $(b_{n,p})$'s because then the derivation of the proof is simpler to write).

Taking the second derivative of $F(x,y)$ with respect to x gives

$$\partial_x^2 F(x,y) = \partial_x^2 \left(\sum_{p\geq 0} F_p(x) y^p\right) = \partial_x^2 F_0(x) + \partial_x^2 F_1(x) y + \partial_x^2 F_2(x) y^2 + \sum_{p\geq 3} \partial_x^2 F_p(x) y^p\,. \tag{1}$$

Now, the recursive relation of Theorem 3 for $b_{n+2,n-p+2}$ implies for $p \geq 3$:

$$\begin{aligned}\partial_x^2 F_p(x) &= \sum_{n\geq 0} b_{n+2,n-p+2}\frac{x^n}{n!} = \sum_{n\geq 0}\frac{x^n}{n!}\sum_{\ell=0}^{n-p+1}(\ell+1)!\cdot\binom{n}{\ell}\cdot b_{n-\ell,(n-\ell)-(p-1)}\\ &= \sum_{n\geq 0}(n+1)!\frac{x^n}{n!}\cdot\sum_{n\geq 0} b_{n,n-p+1}\frac{x^n}{n!} = \frac{1}{(1-x)^2}F_{p-1}(x).\end{aligned}$$

Plugging this recurrence into the equation (1) (and using $F_0(x) = F_1(x) = 0$) gives:

$$\partial_x^2 F(x,y) = \partial_x^2 F_2(x)y^2 + \sum_{p\geq 2}\frac{y}{(1-x)^2}F_p(x)y^p.$$

It remains to simplify $F_2(x)$; the initial conditions of Theorem 3 ($b_{n,n-2} = (n-1)!$ and $b_{n,p} = 0$ for $n < 2$) imply that

$$F_2(x) = \sum_{n\geq 2} b_{n,n-2}\frac{x^n}{n!} = \sum_{n\geq 2}\frac{(n-1)!}{n!}x^n = -\ln(1-x) - x\,.$$

This leads to the main differential equation:

$$\partial_x^2 F(x,y) = \partial_x^2 F_2(x)y^2 + \frac{y}{(1-x)^2}F(x,y) = \frac{y}{(1-x)^2}(y + F(x,y))\,. \tag{2}$$

First, by plug & prove, the solutions of $\partial_x^2 F(x) = yF(x)/(1-x)^2$ are a linear combination of $(1-x)^\alpha$, with $\alpha = (1+\sqrt{1+4y})/2$, or $\alpha = (1-\sqrt{1+4y})/2$. Now, $F(x,y) = -y$ is a trivial particular solution of the non-homogeneous differential equation (2), so the general solution of this differential equation is of the form:

$$F(x,y) = K(y)\cdot(1-x)^{\frac{1}{2}(1+\sqrt{1+4y})} + L(y)\cdot(1-x)^{\frac{1}{2}(1-\sqrt{1+4y})} - y.$$

Since $F(0,y) = 0$ and $\frac{\partial F(x,y)}{\partial x}\rfloor_{x=0} = 0$, we respectively deduce the two equations

$$K(y) + L(y) - y = 0 \qquad \text{and} \qquad -K(y)\cdot(1+\sqrt{1+4y}) - L(y)\cdot(1-\sqrt{1+4y}) = 0\,,$$

thus we obtain

$$K(y) = \frac{y}{2}\frac{\sqrt{1+4y}-1}{\sqrt{1+4y}} \qquad \text{and} \qquad L(y) = \frac{y}{2}\frac{\sqrt{1+4y}+1}{\sqrt{1+4y}}\,.$$

This gives

$$F(x,y) = \frac{y}{2}\frac{\sqrt{1+4y}-1}{\sqrt{1+4y}}\cdot(1-x)^{\frac{1}{2}(1+\sqrt{1+4y})} + \frac{y}{2}\frac{\sqrt{1+4y}+1}{\sqrt{1+4y}}\cdot(1-x)^{\frac{1}{2}(1-\sqrt{1+4y})} - y\,,$$

and therefore the theorem, as $B(x,y) = F(xy, 1/y)$. □

Theorem 5 (Asymptotics)**.** *The exponential generating function for the number of basis permutations with respect to their length is given by*

$$\begin{aligned} B(x) &= B(x,1) = \sum_{n\geq 0} b_n \frac{x^n}{n!} = \frac{\sqrt{5}-1}{2\sqrt{5}} \cdot (1-x)^{\frac{1+\sqrt{5}}{2}} + \frac{\sqrt{5}+1}{2\sqrt{5}} \cdot (1-x)^{\frac{1-\sqrt{5}}{2}} - 1 \\ &= \frac{x^2}{2!} + 2\frac{x^3}{3!} + 7\frac{x^4}{4!} + 32\frac{x^5}{5!} + 179\frac{x^6}{6!} + 1182\frac{x^7}{7!} + 8993\frac{x^8}{8!} + O(x^9). \end{aligned}$$

It is a D-finite transcendental function satisfying the following differential equation

$$B(x) - (1-x)^2 \partial_x^2 B(x) + 1 = 0, \qquad B(0) = B'(0) = 0\,. \tag{3}$$

Equivalently, its coefficients b_n satisfy the recurrence

$$b_{n+2} = 2nb_{n+1} + (1+n-n^2)b_n, \qquad b_0 = b_1 = 0, b_2 = 1\,, \tag{4}$$

and the asymptotics are given by

$$b_n \sim \frac{\phi}{\sqrt{5}\,\Gamma(\phi-1)} \frac{1}{n^{2-\phi}} (1+o(1))\,,$$

where ϕ is the golden ratio $\phi = (1+\sqrt{5})/2$, and $\Gamma(z) := \int_0^{+\infty} t^{z-1}\exp(-t)dt$ is the Euler gamma function.

Accordingly, a permutation of length n has a probability asymptotically 0 to be an element of the basis of forbidden patterns, however, this probability is not "very small" as it decays only polynomially:

$$\mathrm{Prob}(s \in \mathcal{S}_n \textit{ belongs to } \cup_{p\in\mathbb{N}} \mathcal{B}_p) = \frac{b_n}{n!} \approx 0.499/n^{0.381}(1+o(1))\,.$$

Proof: Setting $y = 1$ in the bivariate exponential generating function given in Theorem 4 gives $B(x)$. If a function $B(z) = \sum b_n z^n$ is D-finite (it is satisfying a linear differential equation, with polynomial coefficients in z), then its coefficients b_n are polynomially recursive (in short, P-recursive): they satisfy a linear recurrence, with polynomial coefficients in n. See e.g. Flajolet and Sedgewick (2009); Stanley (2012) for more on these two equivalent notions. Starting from the building blocks $(1-x)^a$, which are D-finite, and then using the closure properties of D-finite functions (by sum and product) gives the differential equation (3) (this is e.g. implemented in the Gfun Maple package, see Salvy and Zimmermann (1994)). The recurrence is obtained by extracting the coefficient of x^n on both sides of the differential equation. The asymptotics follows from a singularity analysis (see Flajolet and Sedgewick (2009)) on each term of the shape $(1-x)^a$, indeed, for any $a \in \mathbb{R}$ which is not an integer, one has:

$$[x^n](1-x)^a = \frac{1}{\Gamma(-a)n^{1+a}} \left(1 + \frac{1}{2}a(a+1)\,\frac{1}{n} + O(\frac{1}{n^2})\right)\,.$$

□

Note that $B(x,y)$ is D-finite in the variable x:

$$1 + yB(x,y) - (1-xy)^2\partial_x^2 B(x,y) = 0 \qquad \text{with } B(0,y) = (\partial_x B)(0,y) = 0,$$

but it is not D-finite in the variable y. This follows from a saddle point analysis on $B(1,y) = \sum_n \beta_n y^n$, indeed the asymptotics of β_n involve arbitrarily large $(\ln n)^d$, while the asymptotics of a D-finite function can only have a finite sum of such powers of log, see Flajolet and Sedgewick (2009). This argument is thus similar to a proof that $(1-y)^{1-y}$ is not D-finite.

Remark [Irrational critical exponent]: It is very seldom that a combinatorial problem leads to some asympotics involving an irrational number as exponent. In fact, in combinatorics and in statistical physics, most of the asymptotics of integer sequences are of the shape $b_n \sim Cn^\alpha A^n$, and the exponent α which appears there is a key quantity: its value is often the signature of some universal phenomena (in physics, it is called a critical exponent). For D-finite sequences, the theory implies that it is an algebraic number, however, this exponent is very often -3/2, or a dyadic number (for the reasons explained in Banderier and Drmota (2015)), or a rational number (due to a results on G-functions). Indeed, a theorem (resulting from the works of Katz, André, Chudnovsky & Chudnovsky, see Chambert-Loir (2002)) states that G-functions (D-finite functions with integer coefficients and non-zero radius of convergence) have a rational critical exponent. Now, instead of considering the exponential generating function $B(x) = \sum b_n x^n/n!$, we may consider its inverse Borel transform, *i.e.*, the ordinary generating function $\sum b_n x^n$. It is also a D-finite function, because D-finite functions are closed by Hadamard product, and therefore the Borel transform (and the inverse Borel transform) of a D-finite function is D-finite (*i.e.*, if the sequence b_n is P-recursive, so are $n!b_n$ and $b_n/n!$). We have thus a new D-finite function with integer coefficients and irrational critical exponent (involving the golden ratio ϕ), but this is not contradicting the G-function theorem, because, due to the multiplication by $n!$, we now have a 0 radius of convergence. In conclusion, we have the pleasure to have here one of the few examples in combinatorics of a problem leading to an irrational critical exponent. Other examples are given via the KPZ formula in physics, or via quantities related to quadtrees, see Flajolet and Sedgewick (2009).

Pushing further the asymptotics from Theorem 5, we get the following limit law:

Theorem 6 (Limit Law)**.** *In the model where all permutations of length n are equidistributed, a random permutation of length n in $\cup_{p\in\mathbb{N}}\mathcal{B}_p$ is typically a member of $\mathcal{B}_p$, for $p \sim n - (\ln n)/\sqrt{5}$, with Gaussian fluctuations. Equivalently, the average number of left-to-right-maxima in a random basis permutation is $p \sim (\ln n)/\sqrt{5}$ with Gaussian fluctuations.*

Proof: This follows from the closed-expression for $B(x,y)$, or from a singularity analysis of the differential equation. Indeed, the average and standard deviation follow from the computation of $\partial_y B(x,y)$ and $\partial_y^2 B(x,y)$ at $y=1$. The Gaussian limit law follows from the quasi-power theorem applied to a variable exponent perturbation or to our non-confluent differential equation (see Theorem IX.11 and Theorem IX.18 from Flajolet and Sedgewick (2009)). □

As a random permutation of $\mathcal{S}_n$ has $\ln n$ left-to-right maxima on average, the above theorem quantifies to what extent the right-jump process kills the left-to-right maxima when one starts from the identity permutation.

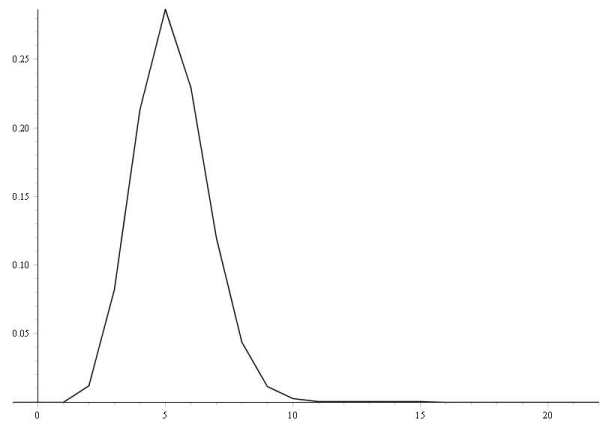


Figure 4: This histogram illustrates Theorem 6: the average number of left-to-right-maxima in a random basis permutation is $p \sim (\ln n)/\sqrt{5}$ with Gaussian fluctuations. However the speed of convergence and the small amplitude of the variance makes that large values of n have to be considered to recognize clearly the bell curve of the Gaussian (above, this is a histogram for $n = 4000$).

For the combinatorial structure $\mathcal{B}_p$, it could be possible that its complementary set has a nicer structure. Those permutations not in the basis are for sure counted by $u_n = n! - b_n$; this sequence satisfies $u_{n+3} = (n+1)(n^2 - n - 1)u_n - (3n^2 + 3n - 1)u_{n+1} + 3(n+1)u_{n+2}$, which is still a nice recurrence but of order one more than the recurrence for b_n, so it is a heuristic confirmation than b_n is a more fundamental sequence than u_n.

The first values of b_n (the number of basis permutations of length n) are $1, 2, 7, 32$, 179, 1182, 8993, 77440 for $2 \leq n \leq 9$. We added this sequence to the On-line Encyclopedia of Integer Sequences (hereafter abbreviated OEIS), see Sloane and collaborators (2016):

$p \backslash n$	2	3	4	5	6	7	8	9	10	11	$\#\mathcal{B}_p$
0	1										1
1		2	1								3
2			6	8	1						15
3				24	58	18	1				101
4					120	444	244	32	1		841
5						720	3708	3104	700	50	8232
6							5040	33984	39708	13400	78732
Σ	1	2	7	32	179	1182	8993	77440	744425	7901410	

Table 1: Number $b_{n,p}$ of basis permutations of length n (the "minimal forbidden patterns" of $\mathcal{B}_p$, or equivalently, with $p+1$ non-left-to-right-maxima) where $2 \leq n \leq 11$ and $0 \leq p \leq 6$ `(OEIS A265163)`. The last column contains $\beta_p := \sum_n b_{n,p}$ `(OEIS A265164)`; the last line contains $b_n := \sum_p b_{n,p}$ `(OEIS A265165)`.

There is a vast literature in number theory analysing the modular congruences of famous sequences

(Pascal triangle, Fibonacci, Catalan, Motzkin, Apéry numbers, see Deutsch and Sagan (2006); Xin and Xu (2011); Rowland and Zeilberger (2014); Kauers et al. (2011)). The properties of $b_n \bmod m$ are sometimes called "supercongruences" when m is the power of a prime number: many articles consider $m = 2^r$, or $m = 3^r$. We now give a result which holds for any m (not necessarily the power of a prime number).

Theorem 7 (Supercongruences for D-finite functions)**.**
Consider any P-recurrence of order r:

$$P_0(n)u_n = \sum_{i=1}^{r} P_i(n)u_{n-i}\,.$$

If the polynomial $P_0(n)$ is ultimately invertible $\bmod m$ (i.e., $\gcd(P_0(n), m) = 1$, *for all n large enough), then the sequence (u_n) is ultimately periodic*[ii] $\bmod m$, *and there is an algorithm to get this period.*

In particular, recurrences such that $P_0(n) = 1$ are periodic $\bmod m$.

Accordingly, our sequence $b_n \bmod m$ (defined by recurrence (4)) is periodic for any m.

Proof: Indeed, as the leading term P_0 is invertible, we can write:

$$u_n \bmod m = \sum_{i=1}^{r} \frac{P_i(n) \bmod m}{P_0(n) \bmod m}(u_{n-i} \bmod m)\,,$$

in which each term has just a finite set of possible values. What is more, for any polynomial $P(n)$ with integer coefficients, $P(n) \bmod m$ is of period p, for some $p|m$. (This follows from the fact that the sum and the product is preserving periodicity $\bmod m$, as we did not require in the definition of "period" that m is the smallest m such that the sequence is m periodic). Therefore, one can then construct a Markov chain (an automaton): the states are all the possible $2r+2$-tuples of values $\bmod m$ for $(P_0(n), \ldots, P_r(n), u_n, \ldots, u_{n-r})$, and the recurrence dictates the transitions in this Markov Chain. The pigeonhole principle implies that there is a loop in this finite graph, and this gives our period.

Besides, starting with the hypothesis that $u_n = u_{n+p} \bmod m$ for $n \in [n_1+1, n_1+p]$ (such a p can be found by brute-force, as n_1 has to be smaller than the number of states in the automaton, and p has to be smaller than m^{2r}), it is enough to check that this property goes on for $r+1$ steps to prove that u_n is p periodic. □

This theorem explains the periodic behaviour of $b_n \bmod m$. By brute-force computation, we can get $b_n \bmod m$, for any given m. For example $b_n \bmod 15$ is periodic of period 12: for $n \geq 9$, one has

$$b_n \bmod 15 = (10, 5, 10, 10, 0, 10, 5, 10, 5, 5, 0, 5)^{\infty}.$$

The period can be quite large, for example $b_n \bmod 3617$ has period 26158144. We say more on this phenomena for P-recursive sequences in Banderier et al. (2017).

[ii] In the sequel, we will omit the word "ultimately": a periodic sequence of period p is thus a sequence for which $u_{n+p} = u_n$ for all large enough n. Some authors use the terminology "eventually periodic" instead.

5 Conclusion

In this article, we analysed the iteration of the process "a particle jumps to the right" in a permutation, and we gave the typical properties of the patterns which are not reached after p moves. We expect our approach (introducing a catalytic variable and getting a D-finite function) to work in many other cases. However, we already know a nice permutation class for which the basis is not D-finite. Indeed, as an extension of this work, an interesting question is to consider a model in which both right-jumps and left-jumps are allowed: this is a very natural process, also related to sorting algorithms and bioinformatics processes. In a forthcoming work, we show that for this new model, the basis of forbidden minimal patterns for permutations obtained by p iterations of the process is related to Young Tableaux with 2 equally long first rows (but it is no longer D-finite, unlike the pure right-jump iteration process that we considered in this article).

Another natural question is: is it the case that using e.g. the correspondence between records and cycles in permutations, there is an elegant process corresponding to a "particle jumps to the right", with permutations at distance p from identity being counted in terms of cycles in the permutation? To get direct "bijective" proofs of our formulae is also an interesting question: as a credo, it cannot be the case that such nice formulae/recurrences are only reached by solving differential equations (like we did in this article). It may be the case that a generating tree approach leads to the simple recurrence (4) we get for b_n (see e.g. Banderier et al. (2002) or Barcucci et al. (1999)). With respect to the asymptotics, it is noteworthy that the process analysed in the present article involves the golden ratio (this is very unusual in combinatorics to have this constant *as critical exponent*): is it the trace of some universality class? (Like it is sometimes the case for problems coming from statistical mechanics, see our remark after Theorem 5.)

Last but not least, we already mentioned a vast literature of publications in number theory analysing the modular congruences of famous sequences (Pascal triangle, Fibonacci, Catalan, Motzkin, Apéry numbers, ..., Deutsch and Sagan (2006); Xin and Xu (2011)). It seems to us that our approach to tackle them at the level of D-finite functions is new (see also Kauers et al. (2011); Rowland and Zeilberger (2014)), and it would be worth analysing these properties in full generality. In this article, we proved by a mixture of Ansatz and brute force proof that $b_n \bmod m$ (where m can be any integer) is a periodic function (of period bounded by a polynomial in m). In fact, we prove in Banderier et al. (2017) that this period and the values of $b_n \bmod m$ for any given m can be made explicit.

Acknowledgments: We thank our laboratories (LE2I and LIPN) for funding this collaboration, and presentation of preliminary results at the conferences AofA'15 and Permutation Patterns'15. We also thank the two referees for their careful reading.

References

Cyril Banderier and Michael Drmota. Formulae and asymptotics for coefficients of algebraic functions. *Combin. Probab. Comput.*, 24(1):1–53, 2015. [online pdf].

Cyril Banderier, Mireille Bousquet-Mélou, Alain Denise, Philippe Flajolet, Danièle Gardy, and Dominique Gouyou-Beauchamps. Generating functions for generating trees. *Discrete Math.*, 246(1-3):29–55, 2002. URL http://dx.doi.org/10.1016/S0012-365X(01)00250-3. Formal power series and algebraic combinatorics (Barcelona, 1999).

Cyril Banderier, Jean-Luc Baril, and Céline Moreira Dos Santos. P-recursive sequences, D-finite functions, supercongruences and mod-k periodicity. *In preparation*, 2017.

Elena Barcucci, Alberto Del Lungo, Elisa Pergola, and Renzo Pinzani. ECO: a methodology for the enumeration of combinatorial objects. *J. Differ. Equations Appl.*, 5(4-5):435–490, 1999. URL http://dx.doi.org/10.1080/10236199908808200.

Jean-Luc Baril. Gray code for permutations with a fixed number of left-to-right minima. *Ars Combin.*, 111:225–239, 2013. [online pdf].

Jean-Luc Baril and Rémi Vernay. Whole mirror duplication-random loss model and pattern avoiding permutations. *Inform. Process. Lett.*, 110(11):474–480, 2010. URL http://dx.doi.org/10.1016/j.ipl.2010.04.016.

Miklós Bóna. *Combinatorics of permutations*. Discrete Mathematics and its Applications. CRC Press, Boca Raton, FL, second (1st ed: 2004) edition, 2012. URL http://dx.doi.org/10.1201/b12210.

Mathilde Bouvel and Luca Ferrari. On the enumeration of d-minimal permutations. *Discrete Math. Theor. Comput. Sci.*, 15(2):33–48, 2013. [online pdf].

Mathilde Bouvel and Elisa Pergola. Posets and permutations in the duplication-loss model: minimal permutations with d descents. *Theoret. Comput. Sci.*, 411(26-28):2487–2501, 2010. URL http://dx.doi.org/10.1016/j.tcs.2010.03.008.

Mathilde Bouvel and Dominique Rossin. A variant of the tandem duplication–random loss model of genome rearrangement. *Theoret. Comput. Sci.*, 410(8-10):847–858, 2009. URL http://dx.doi.org/10.1016/j.tcs.2008.11.017.

Antoine Chambert-Loir. Théorèmes d'algébricité en géométrie diophantienne (d'après J.-B. Bost, Y. André, D. & G. Chudnovsky). *Astérisque*, pages 175–209, 2002. [online pdf].

Kamalika Chaudhuri, Kevin Chen, Radu Mihaescu, and Satish Rao. On the tandem duplication-random loss model of genome rearrangement. In *Proceedings of the Seventeenth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 564–570. ACM, New York, 2006. URL http://dx.doi.org/10.1145/1109557.1109619.

Emeric Deutsch and Bruce E. Sagan. Congruences for Catalan and Motzkin numbers and related sequences. *J. Number Theory*, 117(1):191–215, 2006. [online pdf].

Warren J. Ewens, Geoffrey A. Watterson, Thomas Eric Hall, and A. Morgan. The chromosome inversion problem. *Journal of Theoretical Biology*, 99:1–7, 1982. URL http://dx.doi.org/10.1016/0022-5193(82)90384-8.

Philippe Flajolet and Robert Sedgewick. *Analytic combinatorics*. Cambridge University Press, Cambridge, 2009. [online pdf].

Neil C. Jones and Pavel A. Pevzner. *An Introduction to Bioinformatics Algorithms*. MIT Press, 2004.

Manuel Kauers, Christian Krattenthaler, and Thomas W. Müller. A method for determining the $\mathrm{mod}\text{-}2^k$ behaviour of recursive sequences, with applications to subgroup counting. *Electron. J. Combin.*, 18(2): Paper 37, 83p., 2011. [online pdf].

Sergey Kitaev. *Patterns in permutations and words*. Monographs in Theoretical Computer Science. An EATCS Series. Springer, Heidelberg, 2011. URL http://dx.doi.org/10.1007/978-3-642-17333-2.

Donald E. Knuth. *The art of computer programming. Vol. 3*. Addison-Wesley, Reading, MA, 1998. Sorting and searching, Second edition (1st ed: 1973).

Hjalti Magnússon. *Sorting Operators and Their Preimages*. Thesis, Reykjavik University, 2013.

Toufik Mansour and Sherry H. F. Yan. Minimal permutations with d descents. *European J. Combin.*, 31 (5):1445–1460, 2010. ISSN 0195-6698. URL http://dx.doi.org/10.1016/j.ejc.2010.01.001.

Carlos Meidanis and João Carlos Setubal. *Introduction to Computational Molecular Biology*. PWS Publishing, 1997.

Eric Rowland and Doron Zeilberger. A case study in meta-automation: automatic generation of congruence automata for combinatorial sequences. *J. Difference Equ. Appl.*, 20(7):973–988, 2014. URL http://dx.doi.org/10.1080/10236198.2013.876422.

Bruno Salvy and Paul Zimmermann. Gfun: a Maple package for the manipulation of generating and holonomic functions in one variable. *ACM Transactions on Mathematical Software*, 20(2):163–177, 1994. URL http://dx.doi.org/10.1145/178365.178368.

Neil J.A. Sloane and collaborators. On-Line Encyclopedia of Integer Sequences. Published electronically at https://oeis.org/, 2016.

Richard P. Stanley. *Enumerative combinatorics. Volume 1*, volume 49 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second (1st ed: 1997) edition, 2012.

Guoce Xin and Jing-Feng Xu. A short approach to Catalan numbers modulo 2^r. *Electron. J. Combin.*, 18 (1):Paper 177, 12p., 2011. [online pdf].

Discrete Applied Mathematics 224 (2017) 9–15

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

A permutation code preserving a double Eulerian bistatistic

Jean-Luc Baril, Vincent Vajnovszki *

LE2I, Université de Bourgogne Franche-Comté, BP 47870, 21078 Dijon Cedex, France

ARTICLE INFO

Article history:
Received 19 October 2016
Received in revised form 10 February 2017
Accepted 16 February 2017
Available online 22 March 2017

Keywords:
Set-valued/Eulerian statistic on permutation
Permutation/Lehmer code
Subexcedant sequence
Constructive bijection

ABSTRACT

In 1977 Foata proved bijectively, among other things, that the joint distribution of ascent and distinct nonzero value numbers on the set of subexcedant sequences is the same as that of descent and inverse descent numbers on the set of permutations, and the generating function of the corresponding bistatistics is the double Eulerian polynomial. In 2013 Foata's result was rediscovered by Visontai as a conjecture, and then reproved by Aas in 2014.

In this paper, we define a permutation code (that is, a bijection between permutations and subexcedant sequences) and show the more general result that two 5-tuples of set-valued statistics on the set of permutations and on the set of subexcedant sequences, respectively, are equidistributed. In particular, these results give another bijective proof of Foata's result.

© 2017 Published by Elsevier B.V.

1. Introduction

In enumerative combinatorics, it is a classical result that the descent number des and the inverse descent number ides (defined as $\mathsf{ides}\,\pi = \mathsf{des}\,\pi^{-1}$) on permutations are Eulerian statistics, and their distributions on the set $\mathfrak{S}_n$ of length-n permutations are given by the nth Eulerian polynomial A_n, that is

$$A_n(u) = \sum_{\pi\in\mathfrak{S}_n} u^{\mathsf{des}\,\pi+1} = \sum_{\pi\in\mathfrak{S}_n} u^{\mathsf{ides}\,\pi+1},$$

and the joint distribution of des and ides is given by the nth double Eulerian polynomial,

$$A_n(u,v) = \sum_{\pi\in\mathfrak{S}_n} u^{\mathsf{des}\,\pi+1} v^{\mathsf{ides}\,\pi+1},$$

see for instance [2,8].

An alternative way to represent a permutation is its Lehmer code [5], which is a subexcedant sequence. The ascent number asc on the set S_n of subexcedant sequences is still an Eulerian statistic (see for example [9]), and in [7] the statistic that counts the number of distinct nonzero symbols in a subexcedant sequence (that following [1] we denote by row) is proved to be still Eulerian; this result is credited to Dumont by the authors of [7]. In terms of generating functions, we have

$$A_n(u) = \sum_{s\in S_n} u^{\mathsf{asc}\,s+1} = \sum_{s\in S_n} u^{\mathsf{row}\,s+1}.$$

* Corresponding author.
E-mail addresses: barjl@u-bourgogne.fr (J.-L. Baril), vvajnov@u-bourgogne.fr (V. Vajnovszki).

http://dx.doi.org/10.1016/j.dam.2017.02.014
0166-218X/© 2017 Published by Elsevier B.V.

In 1977 Foata [4] proved, among other things, that the joint distribution of des and ides on the set of permutations is the same as that of asc and row on the set of subexcedant sequences, that is

$$A_n(u, v) = \sum_{\pi \in \mathfrak{S}_n} u^{\mathsf{des}\,\pi+1} v^{\mathsf{ides}\,\pi+1} = \sum_{s \in S_n} u^{\mathsf{asc}\,s+1} v^{\mathsf{row}\,s+1}.$$

In the present paper, we define a bijection between permutations and subexcedant sequences (i.e., a permutation code) and show that the tuple of set-valued statistics

(Des, Ides, Lrmax, Lrmin, Rlmax) on the set of permutations, and
(Asc, Row, Posz, Max, Rlmax) on the set of subexcedant sequences

have the same distribution (each of the occurring statistics is defined below). In particular, our bijection provides set-valued partners for Asc (answering to a question in [1]) and gives a alternative proof of Foata's result.

2. Notation and definitions

A length-n *word* w over the alphabet A is a sequence $w_1 w_2 \dots w_n$ of symbols in A, and we will consider only finite alphabets $A \subset \mathbb{N}$.

Statistics

A *statistic* on a set X of words is simply a function from X to $\mathbb{N}$; a *set-valued statistic* is a function from X to $2^{\mathbb{N}}$; and a *multistatistic* is a tuple of statistics.

Let $w = w_1 w_2 \dots w_n$ be a length-n word. A *descent* in w is a position i in w, $1 \le i < n$, with $w_i > w_{i+1}$, and the *descent set* of w is

$$\mathsf{Des}\, w = \{i : 1 \le i < n \text{ with } w_i > w_{i+1}\}.$$

A *left-to-right maximum* in w is a position i in w, $1 \le i \le n$, with $w_j < w_i$ for all $j < i$, and the *set of left-to-right maxima* is

$$\mathsf{Lrmax}\, w = \{i : 1 \le i \le n \text{ with } w_j < w_i \text{ for all } j < i\}.$$

Clearly, $1 \in \mathsf{Lrmax}\, w$, and Des and Lrmax are classical examples of set-valued statistics on words. We define similarly the sets $\mathsf{Asc}\, w$ of *ascents*, $\mathsf{Lrmin}\, w$ of *left-to-right minima*, $\mathsf{Rlmax}\, w$ of *right-to-left maxima* and $\mathsf{Rlmin}\, w$ of *right-to-left minima* in w.

To each set-valued statistic St corresponds an (integer-valued) statistic st defined as $\mathsf{st}\, w = \mathsf{card}\, \mathsf{St}\, w$, for example $\mathsf{des}\, w$ and $\mathsf{lrmax}\, w$ count, respectively, the number of descents and the number of left-to-right maxima in w.

Let X and X' be two sets of words, and st and st′ be two statistics defined on X and X', respectively. We say that st on X has the *same distribution* as st′ on X' (or equivalently, st and st′ are *equidistributed*) if, for any integer u,

$$\mathsf{card}\{w \in X : \mathsf{st}\, w = u\} = \mathsf{card}\{w \in X' : \mathsf{st}'\, w = u\},$$

and the multistatistic $(\mathsf{st}_1, \mathsf{st}_2, \dots, \mathsf{st}_p)$ defined on X has the same distribution as the multistatistic $(\mathsf{st}'_1, \mathsf{st}'_2, \dots, \mathsf{st}'_p)$ defined on X' (or the two multistatistics are equidistributed) if, for any integer p-tuple $u = (u_1, u_2, \dots, u_p)$,

$$\mathsf{card}\{w \in X : (\mathsf{st}_1, \mathsf{st}_2, \dots, \mathsf{st}_p)\, w = u\} = \mathsf{card}\{w \in X' : (\mathsf{st}'_1, \mathsf{st}'_2, \dots, \mathsf{st}'_p)\, w = u\}.$$

The notion of equidistribution of (multi)statistics can naturally be extended to set-valued (multi)statistics.

Permutations, subexcedant sequences and codes

This paper deals with two particular classes of words: permutations and subexcedant sequences. A permutation is a length-n word over $\{1, 2, \dots, n\}$ with distinct symbols. Alternatively, a permutation is an element of the symmetric group on $\{1, 2, \dots, n\}$ written in one line notation, and $\mathfrak{S}_n$ denotes the set of length-n permutations. If two permutations $\pi = \pi_1 \pi_2 \dots \pi_n$ and $\sigma = \sigma_1 \sigma_2 \dots \sigma_n$ are such that $\sigma_{\pi_1} \sigma_{\pi_2} \cdots \sigma_{\pi_n} = 12 \cdots n$ (i.e., the identity in $\mathfrak{S}_n$), then σ is the *inverse* of π, which is denoted by π^{-1}.

A length-n *subexcedant sequence*[1] is a word $s = s_1 s_2 \dots s_n$ over $\{0, 1, \dots, n-1\}$ with $0 \le s_i \le i - 1$ for $1 \le i \le n$, and S_n denotes the set of length-n subexcedant sequences; and we have $S_n = \{0\} \times \{0, 1\} \times \cdots \times \{0, 1, \dots, n-1\}$.

Some statistics are consistently defined only on particular classes of words, *e.g.* permutations or subexcedant sequences.

For a permutation $\pi \in \mathfrak{S}_n$, an *inverse descent* (*ides* for short) in π is a position i for which $\pi_i + 1$ appears to the left of π_i in π. Equivalently, i is an ides in π if π_i is a descent in π^{-1}. The *ides set* is defined as

$$\mathsf{Ides}\, \pi = \{i : 1 < i \le n \text{ with } \pi_i + 1 \text{ appears in } \pi \text{ to the left of } \pi_i\},$$

and $\mathsf{ides}\, \pi = \mathsf{des}\, \pi^{-1}$, but in general $\mathsf{Ides}\, \pi$ is not equal to $\mathsf{Des}\, \pi^{-1}$.

Let $s = s_1 s_2 \dots s_n$ be a subexcedant sequence in S_n. The Posz statistic gives the positions of 0s in s,

$$\mathsf{Posz}\, s = \{i : 1 \le i \le n, s_i = 0\},$$

[1] Known in the literature also as *inversion sequence*, *inversion table* or *subexceedant function*.

and obviously $1 \in \mathsf{Posz}\, s$. The Max statistic is defined as

$$\mathsf{Max}\, s = \{i : 1 \le i \le n, s_i = i - 1\},$$

and as above, $1 \in \mathsf{Max}\, s$.

A *last-value position* in s is a position i in s such that $s_i \neq 0$ and s_i does not occur in the suffix $s_{i+1}s_{i+2}\dots s_n$ of s. The *last-value position set*, denoted by Row, is defined as

$$\mathsf{Row}\, s = \{i : s_i \neq 0 \text{ and } s_i \text{ does not occur in the suffix } s_{i+1}s_{i+2}\dots s_n\}.$$

Clearly, $1 \notin \mathsf{Row}\, s$, and $\mathsf{row}\, s = \mathsf{card}\, \mathsf{Row}\, s$ counts the number of distinct nonzero symbols in s.

Example 1. If $\pi = 6\,2\,5\,8\,7\,3\,1\,4 \in \mathfrak{S}_8$ and $s = 0\,1\,1\,0\,2\,3\,6\,3 \in S_8$, then

$\mathsf{Des}\, \pi = \mathsf{Asc}\, s = \{1, 4, 5, 6\}$,
$\mathsf{Ides}\, \pi = \mathsf{Row}\, s = \{3, 5, 7, 8\}$,
$\mathsf{Lrmax}\, \pi = \mathsf{Posz}\, s = \{1, 4\}$,
$\mathsf{Lrmin}\, \pi = \mathsf{Max}\, s = \{1, 2, 7\}$,
$\mathsf{Rlmax}\, \pi = \mathsf{Rlmin}\, s = \{4, 5, 8\}$.

An *inversion* in a permutation $\pi = \pi_1\pi_2\dots\pi_n \in \mathfrak{S}_n$ is a pair (i, j) with $i < j$ and $\pi_i > \pi_j$. The set $\mathfrak{S}_n$ is in bijection with S_n, and any such bijection is called *permutation code*. The *Lehmer code* L defined in [5] is a classical example of permutation code; it maps each permutation $\pi = \pi_1\pi_2\dots\pi_n$ to a subexcedant sequence $s_1s_2\dots s_n$ where, for all j, $1 \le j \le n$, s_j is the number of inversions (i, j) in π (or equivalently, the number of entries in π larger than π_j and on its left). For example $L(6\,2\,5\,8\,7\,3\,1\,4) = 0\,1\,1\,0\,1\,4\,6\,4$. See also [11] for a family of permutation codes in the context of Mahonian statistics on permutations.

In [3] is showed that dmc statistic which counts the number of distinct nonzero symbols in the Lehmer code of a permutation π (the statistic $\pi \mapsto \mathsf{row}\, L(\pi)$ with the above notations) is Eulerian, and so has the same distribution as des, asc or ides on $\mathfrak{S}_n$. See also [10] where Dumont's statistic dmc is extended to words.

Although the following properties are folklore, they are easy to check.

Property 1. *If $\pi \in \mathfrak{S}_n$ and $L(\pi) \in S_n$ is its Lehmer code, then* $\mathsf{Des}\, \pi = \mathsf{Asc}\, L(\pi)$, $\mathsf{Lrmax}\, \pi = \mathsf{Posz}\, L(\pi)$, $\mathsf{Lrmin}\, \pi = \mathsf{Max}\, L(\pi)$, *and* $\mathsf{Rlmax}\, \pi = \mathsf{Rlmin}\, L(\pi)$.

3. The permutation code *b*

We define a mapping $b\colon \mathfrak{S}_n \to S_n$ and Theorem 1 shows that b is a bijection, that is, a permutation code, and it is the main tool in proving that (Des, Ides, Lrmax, Lrmin, Rlmax) on $\mathfrak{S}_n$ has the same distribution as (Asc, Row, Posz, Max, Rlmax) on S_n (see Theorem 2).

A position i in $\pi = \pi_1\pi_2\dots\pi_n \in \mathfrak{S}_n$, $1 \le i \le n$, can satisfy the following properties:

P1: $\pi_i + 1$ occurs in π at the right of π_i,
P2: $\pi_i - 1$ occurs in $\pi 0$ at the right of π_i,

where $\pi 0$ is the permutation of $\{0, 1, \dots, n\}$ obtained by adding a 0 at the end of π. And, to each position i in π, we associate an integer $\lambda_i(\pi) \in \{0, 1, 2, 3\}$ according to i satisfies both, one, or none of these properties:

$$\lambda_i(\pi) = \begin{cases} 0, & \text{if } i \text{ satisfies both P1 and P2,} \\ 1, & \text{if } i \text{ satisfies P2 but not P1,} \\ 2, & \text{if } i \text{ satisfies P1 but not P2,} \\ 3, & \text{if } i \text{ satisfies neither P1 nor P2,} \end{cases}$$

and we denote it simply by λ_i when there is no ambiguity.

Alternatively, using the Iverson bracket notation ($[P] = 1$ iff the statement P is true), we have the more concise expression: $\lambda_i = [\pi_i = n \text{ or } \pi_i + 1 \text{ occurs at the left of } \pi_i] + 2 \cdot [\pi_i - 1 \text{ occurs at the left of } \pi_i]$.

For example, for any $\pi \in \mathfrak{S}_n$ we have $\lambda_1 = 0$ except $\lambda_1 = 1$ if $\pi_1 = n$; and when $n > 1$, then $\lambda_n = 3$ except $\lambda_n = 2$ if $\pi_n = 1$. Each λ_i is uniquely determined by π, for instance if $\pi = 6\,2\,5\,8\,7\,3\,1\,4$, then $\lambda_1, \lambda_2, \dots, \lambda_8 = 0, 0, 1, 1, 3, 2, 1, 3$, see Fig. 2.

An *interval* $I = [a, b]$, $a \le b$, is the set of integers $\{x : a \le x \le b\}$; and a *labeled interval* is a pair (I, ℓ) where I is an interval and ℓ is an integer. In order to give the construction of the mapping b, we define below the slices of a permutation, and some of their properties are given in Remark 1.

Definition 1. For a permutation $\pi = \pi_1\pi_2\dots\pi_n \in \mathfrak{S}_n$ and an i, $0 \le i < n$, the ith *slice* of π is the sequence of labeled intervals $U_i(\pi) = (I_1, \ell_1), (I_2, \ell_2), \dots, (I_k, \ell_k)$, defined by the following process (see Fig. 1).

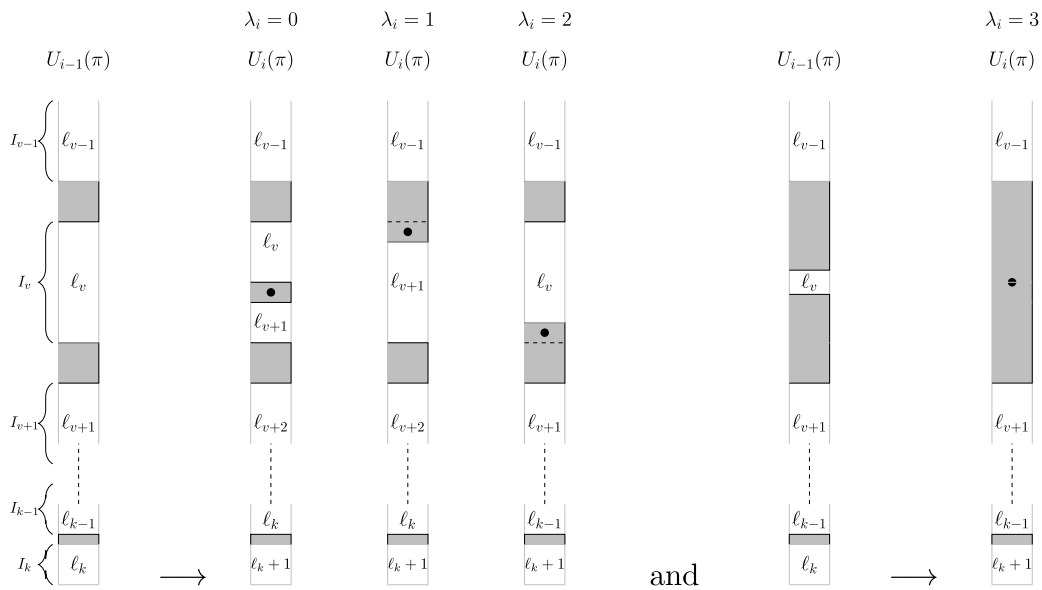
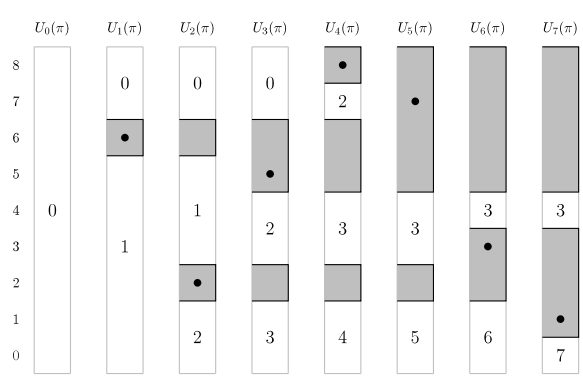


Fig. 1. The four cases in Definition 1.

- $U_0(\pi) = ([0, n], 0)$.
- For $i \geq 1$, let $U_{i-1}(\pi) = (I_1, \ell_1), (I_2, \ell_2), \ldots, (I_k, \ell_k)$ be the $(i-1)$th slice of π and v, $1 \leq v \leq k$, be the integer such that $\pi_i \in I_v$. The ith slice $U_i(\pi)$ of π is defined according to λ_i:
  - If $\lambda_i = 0$ (or equivalently, $\min I_v < \pi_i < \max I_v$), then
  $$U_i(\pi) = (I_1, \ell_1), \ldots, (I_{v-1}, \ell_{v-1}), (H, \ell_v), (J, \ell_{v+1}), (I_{v+1}, \ell_{v+2}), \ldots, (I_{k-1}, \ell_k), (I_k, \ell_k + 1),$$
  where $H = [\pi_i + 1, \max I_v]$ and $J = [\min I_v, \pi_i - 1]$;
  - If $\lambda_i = 1$ (or equivalently, $\min I_v < \max I_v = \pi_i$), then
  $$U_i(\pi) = (I_1, \ell_1), \ldots, (I_{v-1}, \ell_{v-1}), (J, \ell_{v+1}), (I_{v+1}, \ell_{v+2}), \ldots, (I_{k-1}, \ell_k), (I_k, \ell_k + 1),$$
  where $J = [\min I_v, \pi_i - 1]$;
  - If $\lambda_i = 2$ (or equivalently, $\min I_v = \pi_i < \max I_v$), then
  $$U_i(\pi) = (I_1, \ell_1), \ldots, (I_{v-1}, \ell_{v-1}), (J, \ell_v), (I_{v+1}, \ell_{v+1}), \ldots, (I_{k-1}, \ell_{k-1}), (I_k, \ell_k + 1),$$
  where $J = [\pi_i + 1, \max I_v]$;
  - If $\lambda_i = 3$ (or equivalently, $\min I_v = \max I_v = \pi_i$), then
  $$U_i(\pi) = (I_1, \ell_1), \ldots, (I_{v-1}, \ell_{v-1}), (I_{v+1}, \ell_{v+1}), \ldots, (I_{k-1}, \ell_{k-1}), (I_k, \ell_k + 1).$$

Example 2. For the permutation $\pi = 6\,2\,5\,8\,7\,3\,1\,4$ in Fig. 2, $\lambda_1(\pi), \lambda_2(\pi), \ldots, \lambda_8(\pi) = 0, 0, 1, 1, 3, 2, 1, 3$, and the process described in Definition 1 gives the slices below.

$$U_0(\pi) = ([0, 8], 0);$$
$$U_1(\pi) = ([7, 8], 0), ([0, 5], 1);$$
$$U_2(\pi) = ([7, 8], 0), ([3, 5], 1), ([0, 1], 2);$$
$$U_3(\pi) = ([7, 8], 0), ([3, 4], 2), ([0, 1], 3);$$
$$U_4(\pi) = ([7, 7], 2), ([3, 4], 3), ([0, 1], 4);$$
$$U_5(\pi) = ([3, 4], 3), ([0, 1], 5);$$
$$U_6(\pi) = ([4, 4], 3), ([0, 1], 6);$$
$$U_7(\pi) = ([4, 4], 3), ([0, 0], 7).$$



Remark 1. Let $U_i(\pi) = (I_1, \ell_1), (I_2, \ell_2), \ldots, (I_k, \ell_k)$ be the ith slice of π, $0 \leq i < n$. Then, the following properties can be easily checked:

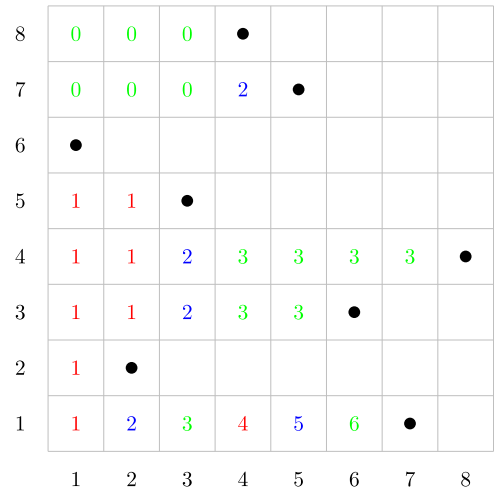


Fig. 2. The permutation $\pi = 6\,2\,5\,8\,7\,3\,1\,4$ with $b(\pi) = 0\,1\,1\,0\,2\,3\,6\,3$ and $\lambda_1(\pi), \lambda_2(\pi), \ldots, \lambda_8(\pi) = 0, 0, 1, 1, 3, 2, 1, 3$.

- the intervals $I_1, I_2, \ldots, I_k$ are in decreasing order, that is $\max I_{j+1} < \min I_j$ for any j, $1 \le j < k$;
- the sequence $\ell_1, \ell_2, \ldots, \ell_k$ is increasing, that is $\ell_j < \ell_{j+1}$ for any j, $1 \le j < k$;
- $\{\ell_1, \ell_2, \ldots, \ell_k\} \subseteq [0, i]$, and $\ell_k = i$;
- $0 \in I_k$;
- $\cup_{j=1}^{k} I_j = \{\pi_{i+1}, \pi_{i+2}, \ldots, \pi_n\} \cup \{0\}$;
- the $(i+1)$th entry of the Lehmer code of π is given by the number of entries $\pi_j > \pi_{i+1}$, with $j < i+1$, that is the cardinality of $[\pi_{i+1}, n] \setminus \cup_{j=1}^{k} I_j$.

A byproduct of Definition 1 is the construction of $b\colon \mathfrak{S}_n \to S_n$ defined below.

Definition 2. Let $\pi = \pi_1\pi_2 \ldots \pi_n \in \mathfrak{S}_n$. For each i, $1 \le i \le n$, we define $b_i = \ell_v$, where v is such that (I_v, ℓ_v) is a labeled interval in the $(i-1)$th slice of π with $\pi_i \in I_v$, and we denote by $b(\pi)$ the sequence $b_1 b_2 \ldots b_n$.

From Remark 1 it follows that $b(\pi)$ is a subexcedant sequence, see for instance Example 2 and Fig. 2.

Proposition 1. *Let $\pi = \pi_1\pi_2 \ldots \pi_n \in \mathfrak{S}_n$, $b(\pi) = b_1 b_2 \ldots b_n$, and let an i, $1 \le i \le n$.*

1. *i is a descent in π iff i is an ascent in $b(\pi)$;*
2. *i is an ides in π iff b_i does not occur in $b_{i+1}b_{i+2} \ldots b_n$;*
3. *i is a left-to-right maximum in π iff $b_i = 0$;*
4. *i is a left-to-right minimum in π iff $b_i = i-1$;*
5. *i is a right-to-left maximum in π iff i is right-to-left minimum in b.*

Proof. Points 1 and 2 obviously follow from the definition of b.

Point 3. Let j be such that $\pi_j = n$; $b_i = 0$ iff $i \le j$ and π_i lies in the first interval of the $(i-1)$th slice of π, which in turn is equivalent to i is a left-to-right maximum in π.

Point 4. Similarly, let j be such that $\pi_j = 1$; $b_i = i-1$ iff $i \le j$ and π_i lies in the last interval of the $(i-1)$th slice of π, which in turn is equivalent to i is a left-to-right minimum in π.

Point 5. By the construction of b, i is a right-to-left maximum in π iff π_i is the largest element of the first interval of the $(i-1)$th slice of π, which in turn is equivalent to b_i is smaller than any of $b_{i+1}, b_{i+2}, \ldots, b_n$. □

See for instance Example 1, where $s = b(\pi)$.

For a length-n subexcedant sequence $b = b_1 b_2 \ldots b_n$, let us consider the following properties that a position i, $1 \le i \le n$, can satisfy:

R1: b_i occurs in the suffix $b_{i+1}b_{i+2} \ldots b_n$ of b,
R2: $i-1$ occurs in b.

The next proposition shows that each $\lambda_i(\pi)$ can be obtained solely from $b(\pi)$.

Proposition 2. *Let $\pi \in \mathfrak{S}_n$ and $b(\pi) = b_1 b_2 \ldots b_n$. Then, for any i, $1 \le i \le n$, we have*

$$\lambda_i(\pi) = \begin{cases} 0, & \text{if } i \text{ satisfies both R1 and R2,} \\ 1, & \text{if } i \text{ satisfies R2 but not R1,} \\ 2, & \text{if } i \text{ satisfies R1 but not R2,} \\ 3, & \text{if } i \text{ satisfies neither R1 nor R2.} \end{cases}$$

Proof. By the construction given in Definition 2 for $b(\pi)$ from the slices of π, it follows that the position i in π satisfies property P1 (resp. P2) if and only if the position i in $b(\pi)$ satisfies property R1 (resp. R2), and the statement holds. □

Proposition 3. *Let $\pi, \sigma \in \mathfrak{S}_n$ with $b(\pi) = b(\sigma)$. Then*

1. *$\lambda_i(\pi) = \lambda_i(\sigma)$ for any i, $1 \le i \le n$.*
2. *If $(I_1, \ell_1), (I_2, \ell_2), \ldots, (I_k, \ell_k)$ is the ith slice of π, and $(J_1, m_1), (J_2, m_2), \ldots, (J_p, m_p)$ that of σ, for some i, $1 \le i < n$, then $k = p$ and $\ell_j = m_j$, for $1 \le j \le k$.*

Proof. The first point is a consequence of Proposition 2.

The second point follows by the next considerations. The ith slice of π, $i \le 1 < n$, has the same number of intervals as its $(i-1)$th slice, except in two cases: $\lambda_i(\pi) = 0$ (when an interval is split into two intervals); and when $\lambda_i(\pi) = 3$ (when a one-element interval is removed). The result follows by considering the first point and by induction on i. □

The sequence $b(\pi) = b_1 b_2 \ldots b_n \in S_n$ was defined by means of the slices of π, but in proving the bijectivity of b we need rather the complement of these slices. Let $\pi \in \mathfrak{S}_n$ and $U_i(\pi) = (I_1, \ell_1), (I_2, \ell_2), \ldots, (I_k, \ell_k)$ be the ith slice of π for an i, $1 \le i < n$. The ith *profile* of π is the sequence $X_1, X_2, \ldots, X_p$ of decreasing nonempty maximal intervals (that is, $\max X_{j+1} < \min X_j$, and none of them has the form $X_j \cup X_{j+1}$) with $\cup_{j=1}^{p} X_j = \{1, 2, \ldots, n\} \setminus \cup_{j=1}^{k} I_j$. And, clearly, $\cup_{j=1}^{p} X_j$ is the set of entries in π to the left of π_{i+1}, and $\sum_{j=1}^{p} \operatorname{card} X_j = i$.

Example 3. The vertical gray regions on the right side of Example 2 correspond to the profiles of $\pi = 6\,2\,5\,8\,7\,3\,1\,4$ in Fig. 2. These profiles are [6, 6]; [6, 6], [2, 2]; [5, 6], [2, 2]; [8, 8], [5, 6], [2, 2]; [5, 8], [2, 2]; [5, 8], [2, 3]; and [5, 8], [1, 3].

In the proof of Theorem 1, we need the next result.

Proposition 4. *Let $\pi, \sigma \in \mathfrak{S}_n$ with $b(\pi) = b(\sigma)$, and let an i, $1 \le i < n$. If $X_1, X_2, \ldots, X_p$ and $Y_1, Y_2, \ldots, Y_m$ are the ith profiles of π and of σ, then*

- *$n \in X_1$ if and only if $n \in Y_1$,*
- *$p = m$, and*
- *$\operatorname{card} X_j = \operatorname{card} Y_j$ for any j, $1 \le j \le p$.*

Proof. It is easy to see that $n \in X_1$ iff 0 does not appear in $b_{i+1}(\pi) \ldots b_n(\pi) = b_{i+1}(\sigma) \ldots b_n(\sigma)$, that is, iff $n \in Y_1$. And, if $i = 1$, then the first profile of π and of σ are one-element intervals, and the statement holds.

From the first point of Proposition 3, we have $\lambda_i(\pi) = \lambda_i(\sigma)$. Suppose that the statement is true for $i-1$, and we will prove it for i.

In passing from the $(i-1)$th profiles of π and of σ to their ith profiles, the following cases can occur (we refer the reader to Definition 1 and Fig. 1).

- If $\lambda_i(\pi) = \lambda_i(\sigma) = 0$, or $\lambda_i(\pi) = \lambda_i(\sigma) = 1$ and $b_i(\pi) = b_i(\sigma) = 0$, then a new one-element interval is added to the ith profile of π and of σ. Moreover, since $b(\pi) = b(\sigma)$, by the second point of Proposition 3, it follows that these intervals are both, for some k, the kth intervals in the ith profile of π and σ.
- If $\lambda_i(\pi) = \lambda_i(\sigma) = 1$ and $b_i(\pi) = b_i(\sigma) \ne 0$, then for some k, a new element is added to the kth interval of both ith profiles of π and σ; this element is the smallest one in the obtained intervals.
- If $\lambda_i(\pi) = \lambda_i(\sigma) = 2$, or $\lambda_i(\pi) = \lambda_i(\sigma) = 3$ and $b_i(\pi) = b_i(\sigma) = 0$, then for some k, a new element is added to the kth interval of both ith profiles of π and σ; this element is the largest one in the obtained intervals.
- If $\lambda_i(\pi) = \lambda_i(\sigma) = 3$ and $b_i(\pi) = b_i(\sigma) \ne 0$, then two consecutive intervals are merged in the ith profiles of π and of σ: the kth and $(k+1)$th ones, for some k. □

Now, we explain how the Lehmer code $c_1 c_2 \ldots c_n$ is linked to the profiles of a permutation. By definition, $c_1 = 0$ and c_i, $i > 1$, is the number of entries in π at the left of π_i and larger than π_i. If $X_1, X_2, \ldots, X_p$ is the $(i-1)$th profile of π, it follows that $c_i = \sum_{j=1}^{u} \operatorname{card} X_j$, where u is such that $\cup_{j=1}^{u} X_j$ is the set of entries in π at the left of π_i and larger than π_i, and so $c_i = \operatorname{card} \cup_{j=1}^{u} X_j$.

Theorem 1. *The mapping $b\colon \mathfrak{S}_n \to S_n$ is a bijection.*

Proof. Let $\pi, \sigma \in \mathfrak{S}_n$ with $b(\pi) = b(\sigma)$, and $c_1 c_2 \ldots c_n$ and $d_1 d_2 \ldots d_n$ be the Lehmer codes of π and σ. Let also i be an integer, $1 < i \le n$, and $(I_1, \ell_1), (I_2, \ell_2), \ldots, (I_k, \ell_k)$ be the $(i-1)$th slice of π, and v such that $\pi_i \in I_v$ (see Definition 2). If $X_1, X_2, \ldots, X_p$ is the $(i-1)$th profile of π, then

if $n \in X_1$, it follows that $c_i = \sum_{j=1}^{v} \operatorname{card} X_j$, and

if $n \notin X_1$, it follows that $c_i = \sum_{j=1}^{v-1} \operatorname{card} X_j$.

Since $b(\pi) = b(\sigma)$, combining Proposition 4 and the second point of Proposition 3, we have that $c_i = d_i$. It follows that the Lehmer code of π and of σ are equal, and so are π and σ, and thus b is injective. And, by cardinality reasons, it follows that b is bijective. □

It is straightforward to see that the 4-tuple of statistics (Des, Lrmax, Lrmin, Rlmax) on $\mathfrak{S}_n$ has the same distribution as (Asc, Posz, Max, Rlmin) on S_n. Indeed, for the Lehmer code $L(\pi)$ of a permutation π, we have (Des, Lrmax, Lrmin, Rlmax) π = (Asc, Posz, Max, Rlmin) $L(\pi)$, see Property 1. But, generally, Ides π is different from Row $L(\pi)$. For example, if $\pi = 6\,2\,5\,8\,7\,3\,1\,4$, then $L(\pi) = 0\,1\,1\,0\,1\,4\,6\,4$, Ides $\pi = \{3, 5, 7, 8\}$ and Row $L(\pi) = \{5, 7, 8\}$.

Combining Theorem 1 and Proposition 1, it follows that b behaves not only as the Lehmer code for the above 4-tuples of statistics, but also it transforms Ides π into Row $b(\pi)$. Formally, we have the next theorem, which subsequently gives Row as a set-valued partner for Asc, thereby answering to an open question stated in [1].

Theorem 2. *For any $\pi \in \mathfrak{S}_n$,*

$$(\mathsf{Des}, \mathsf{Ides}, \mathsf{Lrmax}, \mathsf{Lrmin}, \mathsf{Rlmax})\, \pi = (\mathsf{Asc}, \mathsf{Row}, \mathsf{Posz}, \mathsf{Max}, \mathsf{Rlmin})\, b(\pi),$$

and so the multistatistic (Des, Ides, Lrmax, Lrmin, Rlmax) *on $\mathfrak{S}_n$ has the same distribution as* (Asc, Row, Posz, Max, Rlmin) *on S_n.*

The next corollaries are consequences of Theorem 2. The first of them is Foata's result in [4] saying that (asc, row) on subexcedant sequences is a double Eulerian bistatistic.

Corollary 1. *The bistatistics* (asc, row) *on the set of subexcedant sequences has the same distribution as* (des, ides) *on the set of permutations.*

Corollary 2. *The bistatistics* (asc, row) *and* (row, asc) *are equidistributed on the set of subexcedant sequences.*

Proof. Let $s \in S_n$ and let us define $t = b(\sigma)$ where $\sigma = \pi^{-1}$ with $\pi = b^{-1}(s)$. It is clear that (asc, row) s = (des, ides) π = (ides, des) σ = (row, asc) t. □

Final remarks: Recently, Lin and Kim [6] defined a bijection Ψ between two combinatorial classes counted by the large Schröder numbers, namely the set $S_n(021)$ of subexcedant sequences avoiding the pattern 021, and the set $\mathfrak{S}_n(2413, 4213)$ of permutations avoiding the patterns 2413 and 4213. In [6], it is shown that Ψ proves the equidistribution of two 6-tuples of set-valued statistics, that is

$$(\mathsf{Asc}, \mathsf{Row}, \mathsf{Posz}, \mathsf{Max}, \mathsf{Rlmin}, \mathsf{Expo})\, s = (\mathsf{Des}, \mathsf{Ides}, \mathsf{Lrmax}, \mathsf{Lrmin}, \mathsf{Rlmax}, \mathsf{Rlmin})\, \Psi(s),$$

for any $s \in S_n(021)$. Here, Expo is the *exposed positions statistic* define in [6], and if the last statistic of both 6-tuples is dropped out, then the 5-tuples in our Theorem 2 are obtained. Also in [6] is pointed out that the restriction of our mapping b yields a bijection between $\mathfrak{S}_n(2413, 4213)$ and $S_n(021)$, but unlike Ψ^{-1}, this restriction of b does not transform Rlmin to Expo.

References

[1] E. Aas, The double Eulerian polynomial and inversion tables, arXiv:1401.5653, 2014.
[2] M. Beck, S. Robins, Computing the Continuous Discretely: Integer-Point Enumeration in Polyhedra, in: Undergraduate Texts in Mathematics, Springer, New York, 2007.
[3] D. Dumont, Interprétations combinatoires des nombres de Genocchi, Duke Math. J. 41 (2) (1974) 305–318.
[4] D. Foata, Distributions Eulériennes et Mahoniennes sur le groupe des permutations, in: Higher Combinatorics, in: NATO Adv. Study Inst. Ser., Ser. C: Math. Phys. Sci., vol. 31, Reidel, Dordrecht, Boston, Mass, 1977, pp. 27–49. (in French). With a comment by Richard P. Stanley.
[5] D.H. Lehmer, Teaching combinatorial tricks to a computer, in: Proc. Sympos. Appl. Math., Vol. 10, Amer. Math. Soc., 1960, pp. 179–193.
[6] Z. Lin, D. Kim, A sextuple equidistribution arising in pattern avoidance, arXiv:1612.02964v2, 2016.
[7] R. Mantaci, F. Rakotondrajao, A permutation representation that knows what Eulerian means, Discrete Math. Theor. Comput. Sci. 4 (2) (2001) 101–108 (electronic).
[8] T.K. Petersen, Two-sided Eulerian numbers via balls in boxes, Math. Mag. 86 (2013) 159–176.
[9] C.D. Savage, M.J. Schuster, Ehrhart series of lecture hall polytopes and Eulerian polynomials for inversion sequences, J. Combin. Theory Ser. A 119 (4) (2012) 850–870.
[10] M. Skandera, Dumont's statistic on words, Electron. J. Combin. 8 (1) (2001) #R11.
[11] V. Vajnovszki, Lehmer code transforms and Mahonian statistics on permutations, Discrete Math. 313 (2013) 581–589.

Discrete Mathematics 340 (2017) 2550–2558

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

The pure descent statistic on permutations

Jean-Luc Baril, Sergey Kirgizov

LE2I UMR-CNRS 6306, Université de Bourgogne, B.P. 47 870, 21078 DIJON-Cedex, France

ARTICLE INFO

Article history:
Received 20 June 2016
Received in revised form 6 June 2017
Accepted 7 June 2017
Available online 7 July 2017

Keywords:
Stirling number
Permutation
Descent
Dyck path
Popularity

ABSTRACT

We introduce a new statistic based on permutation descents which has a distribution given by the Stirling numbers of the first kind, *i.e.*, with the same distribution as for the number of cycles in permutations. We study this statistic on the sets of permutations avoiding one pattern of length three by giving bivariate generating functions. As a consequence, new classes of permutations enumerated by the Motzkin numbers are obtained. Finally, we deduce results about the popularity of the pure descents in all these restricted sets.

© 2017 Elsevier B.V. All rights reserved.

1. Introduction and notations

Let S_n be the set of permutations of length n, *i.e.*, all one-to-one correspondences from $[n] = \{1, 2, \ldots, n\}$ into itself. We represent a permutation $\pi \in S_n$ in one-line notation, $\pi = \pi_1\pi_2 \ldots \pi_n$ where $\pi_i = \pi(i)$, $1 \le i \le n$. Moreover, if $\sigma = \sigma(1)\sigma(2)\ldots\sigma(n)$ is a length n permutation then the *product* $\sigma \cdot \pi$ is the permutation $\sigma(\pi_1)\sigma(\pi_2)\ldots\sigma(\pi_n)$. In S_n, a *k-cycle* $\pi = \langle i_1, i_2, \ldots, i_k\rangle$ is a length n permutation verifying $\pi(i_1) = i_2, \pi(i_2) = i_3, \ldots, \pi(i_{k-1}) = i_k, \pi(i_k) = i_1$ and $\pi(j) = j$ for $j \in [n] \setminus \{i_1, \ldots, i_k\}$. For $1 \le k \le n$, we denote by $C_{n,k}$ the set of all n-length permutations which admit a decomposition in a product of k disjoint cycles. The cardinality of $C_{n,k}$ is given by the signless Stirling numbers of the first kind $c(n, k)$ satisfying the relation:

$$c(n, k) = (n-1)c(n-1, k) + c(n-1, k-1) \tag{1}$$

with the initial conditions $c(n, k) = 0$ if $n \le 0$ or $k \le 0$, except $c(0, 0) = 1$. See for instance [12,16] and the sequence A132393 in the Sloane's on-line encyclopedia of integer sequences [15]. These numbers are also usually defined by means of the following combinatorial identity (see for instance [7,8,16])

$$\sum_{k=1}^{n} c(n, k)x^k = x(x+1)(x+2)\cdots(x+n-1). \tag{2}$$

Also, they enumerate n-length permutations π having k left-to-right maxima, *i.e.*, values $i \in [n]$ such that $\pi_j < \pi_i$ for every $j < i$. Indeed, erasing the parentheses from the *standard representation* of a permutation π in a product of disjoint cycles (each cycle is written with its largest element first and the cycles are in increasing order of their largest element), one constructs a bijection on S_n (see [16]) that transports cycles into left-to-right maxima. See [1,2] for efficient generating algorithms of the sets of permutations with a given number of cycles or left-to-right maxima.

Surprisingly and to our knowledge, these two linked statistics on permutations are the only examples studied in the literature that interpolate the Stirling numbers of the first kind. In this paper, we introduce a new statistic on permutations

E-mail addresses: barjl@u-bourgogne.fr (J.-L. Baril), sergey.kirgizov@u-bourgogne.fr (S. Kirgizov).

http://dx.doi.org/10.1016/j.disc.2017.06.005
0012-365X/© 2017 Elsevier B.V. All rights reserved.

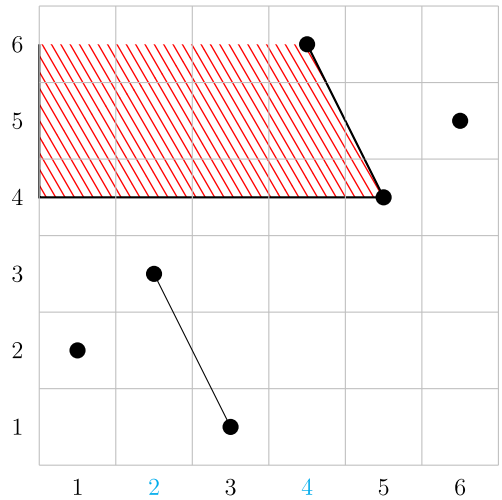


Fig. 1. The value 4 is a pure descent in $\pi = 231645$, while 2 is a descent but not a pure descent.

(called *pure descent statistic*) which has the same distribution as the number of cycles. Its definition is based on the well known Eulerian statistic of descents in permutations (a descent in π is a value $i \in [n-1]$ such that $\pi_i > \pi_{i+1}$). In Section 2, we define the notion of pure descent and we show how their distribution on permutations realizes an interpolation of the Stirling numbers of the first kind. In Section 3, we focus our study on the sets of permutations avoiding one pattern of length three by giving bivariate generating functions with respect to the length of permutation and the number of pure descents. Finally, we deduce the total number of pure descents in all permutations of these restricted sets, which is usually called the *popularity* [6,13].

2. Pure descent statistics

Let π be a permutation in S_n. Let us recall that a *descent* in π is a value $i \in [n-1]$ such that $\pi_i > \pi_{i+1}$. In the case where there is no $j < i$ such that $\pi_j \in [\pi_{i+1}, \pi_i]$, we call it a *pure descent*. We denote by $D_{n,k}$ the set of permutations of length n with k pure descents. For instance, in $\pi = 231645$, the descents are 2 and 4, and 4 is the only one pure descent (see Fig. 1).

Theorem 1. *For $0 \le k \le n-1$, the number of n-length permutations with k pure descents is given by the signless Stirling number of the first kind $c(n, k+1)$.*

Proof. Let us define the insertion operator ψ from $S_{n-1} \times [n]$ to S_n by $\psi(\pi, j) = \pi'$ where

$$\pi'_i = \begin{cases} \pi_i & \text{if } \pi_i < j, \\ \pi_i + 1 & \text{if } \pi_i \ge j, \\ j & \text{if } i = n. \end{cases}$$

For instance, if $\pi = 2413$ then we have $\psi(\pi, 3) = 25143$ and $\psi(\pi, 2) = 35142$.

Let $\pi \in D_{n-1,k}$ be a permutation of length $n-1$ with k pure descents. If $j \ne \pi_{n-1}$ then the insertion operator provides a permutation $\pi' = \psi(\pi, j)$ that lies into $D_{n,k}$. Indeed, the pure descents of π are preserved in π' and no pure descents are created. Setting $d_{n,k} = |D_{n,k}|$, there are $(n-1)d_{n-1,k}$ permutations $\pi' \in S_n$ with k pure descents satisfying $\pi'_n \ne \pi'_{n-1} - 1$.

Now, let $\pi \in D_{n-1,k-1}$ be a permutation of length $n-1$ with $k-1$ pure descents. If $j = \pi_{n-1}$ then the insertion provides $\pi' = \psi(\pi, j)$ that lies into $D_{n,k}$. Indeed, the pure descents of π are preserved in π', and a new pure descent is created between π'_{n-1} and $\pi'_n = \pi'_{n-1} - 1$. Hence, there are $d_{n-1,k-1}$ permutations $\pi' \in S_n$ with k pure descents for which $\pi'_n = \pi'_{n-1} - 1$.

This induces the recurrence relation $d_{n,k} = (n-1)d_{n-1,k} + d_{n-1,k-1}$ for $n, k \ge 0$ with the initial conditions $d_{n,k} = 0$ if $n \le 0$ or $k < 0$, except $d_{0,-1} = 1$. Using relation (1), we obtain $d_{n,k} = c(n, k+1)$. □

Now, we define recursively a bijection ϕ on S_n that transports the number of pure descents into the number of cycles. Let π be a permutation in S_n and σ the permutation in S_{n-1} such that $\psi(\sigma, \pi_n) = \pi$, *i.e.*, σ is obtained from π by deleting the element π_n on the right, and after a normalization in S_{n-1}. Then we set $\phi(1) = 1$ and

$$\phi(\pi) = \begin{cases} \phi(\sigma) \cdot \langle n, n \rangle & \text{if } \pi_n = \pi_{n-1} - 1, \\ \phi(\sigma) \cdot \langle \pi_n, n \rangle & \text{if } \pi_n < \pi_{n-1} - 1, \\ \phi(\sigma) \cdot \langle \pi_n - 1, n \rangle & \text{if } \pi_n > \pi_{n-1}. \end{cases}$$

For instance, if $\pi = 2\,731\,645 \in D_{7,2}$ then we have $\sigma = 263154$, $\pi_7 = 5$ and $\phi(\pi) = \langle 1, 2 \rangle \cdot \langle 3, 3 \rangle \cdot \langle 1, 4 \rangle \cdot \langle 3, 5 \rangle \cdot \langle 6, 6 \rangle \cdot \langle 4, 7 \rangle = \langle 1, 4, 7, 2 \rangle \cdot \langle 3, 5 \rangle \cdot \langle 6, 6 \rangle = 4157362$ has a decomposition in three disjoint cycles.

Table 1
Number of permutations with no pure descents and avoiding one pattern of length three.

Pattern	Sequence	Sloane	a_n, $1 \le n \le 11$
{}	$(n-1)!$	A000142	1, 1, 2, 6, 24, 120, 720, ...
{231}	1	A000124	1, 1, 1, ...
{123}		A007477	1, 1, 1, 2, 3, 6, 11, 22, 44, 90, 187
{132}, {213}, {312}	Motzkin	A001006	1, 1, 2, 4, 9, 21, 51, 127, 323, 835, 2188
{321}		A035929	1, 1, 2, 6, 19, 61, 200, 670, 2286, 7918, 27 770

Theorem 2. *The map ϕ defined above is a bijection on S_n. Moreover, for $k \ge 0$, $\phi(\pi)$ has $k+1$ cycles if and only if π has k pure descents.*

Proof. Due to the recursive definition of ϕ, it is straightforward to see that ϕ is injective and surjective. Furthermore, $\phi(\pi)$ consists of a product of 2-cycles $\Pi_{i=1}^{n}\langle p_i, i\rangle$ where $p_1p_2\dots p_n$ appears in the literature as the transposition array of $\phi(\pi)$ (see for instance [3]). Lemma 1 in [3] proves that the number of cycles of $\phi(\pi)$ is the number of indices i, $1 \le i \le n$, such that $p_i = i$. According to the recursive definition of $\phi(\pi)$, the number of these indices minus one (we do not consider $i = 1$) is exactly the number of pure descents in π; indeed, if $i \ne 1$ then $p_i = i$ means that after the ith step of the induction, the current permutation $\sigma = \sigma_1\sigma_2\dots\sigma_{n-1}\sigma_n$ satisfies $\phi(\sigma) = \langle p_1, 1\rangle \cdot \langle p_2, 2\rangle \cdots \langle p_i, i\rangle$ with $p_i = i$. Using the definition of ϕ, σ is such that $\sigma_i = \sigma_{i-1} - 1$, which implies that $i-1$ is a pure descent in π since π has no value $\pi_j, j < i$, such that $\pi_i < \pi_j < \pi_{i-1}$. All these arguments are reversible, so the converse is also available. □

An immediate consequence of these last two results is the following corollary about the *popularity* [6] of the pure descents among S_n, *i.e.*, the total number of pure descents in all permutations of length n.

Corollary 1. *The popularity of the pure descents among the set S_n is given by the generalized Stirling number (see* A001705 *in* [15]*)*

$$n! \cdot (H_n - 1)$$

where $H_n = \sum_{k=1}^{n} \frac{1}{k}$ is the nth harmonic number.

Proof. Using Theorem 1, the total number of pure descents in all permutations of length n is given by $a(n) = \sum_{k=0}^{n-1} k \cdot d(n, k)$. Since $d(n, k) = c(n, k+1)$, we deduce $a(n) = \sum_{k=1}^{n}(k-1) \cdot c(n, k)$, and $a(n) = \sum_{k=1}^{n} k \cdot c(n, k) - \sum_{k=1}^{n} c(n, k)$. The first sum equals the total number of cycles in all permutations of length n, it is enumerated by $n!H_n$ (see A000254 in [15]); the second sum is $n!$. We obtain the result. □

Notice that the total number of pure descents in all permutations A001705 equals the total number of cycles of length at least two in all permutations, while the distributions of these two statistics are different.

3. Pure descents in permutations avoiding one pattern of length three

A permutation π of length k is a *pattern* of a permutation $\sigma \in S_n$ if there is a subsequence of σ which is order-isomorphic to π, *i.e.*, a subsequence $\sigma_{i_1}\dots\sigma_{i_k}$ of σ with $1 \le i_1 < \cdots < i_k \le n$ and such that $\sigma_{i_\ell} < \sigma_{i_m}$ whenever $\pi_\ell < \pi_m$. A permutation σ that does not contain π as a pattern is said to *avoid* π. For example, $\sigma = 2413$ avoids the pattern 123. We denote by $S_n(\pi)$ the set of permutations of length n avoiding the pattern π. See for instance [5,10,14,17].

In this section, we provide bivariate generating functions for the distributions of the pure descents on the sets of permutations avoiding one pattern of length three. Table 1 focuses on the cases of permutations that do not contain any pure descent. Notice that Proposition 2.1 in [9], which enumerates permutations avoiding the pattern 213 and the generalized pattern $\bar{2}$–31, is a particular case of our result for permutations having no pure descents and avoiding the pattern 213.

3.1. The pattern $\alpha \in \{132, 213, 312\}$

The following theorem proves that the pure descent statistic has the same distribution on the sets $S_n(\alpha)$ of permutations avoiding a pattern $\alpha \in \{132, 213, 312\}$.

Theorem 3. *Let $A(x, y) = \sum_{n\ge0,k\ge0} a_{n,k}x^ny^k$ be the bivariate generating function where the coefficient of x^ny^k is the number $a_{n,k}$ of permutations of length n with k pure descents and avoiding the pattern α, $\alpha \in \{132, 213, 312\}$. Then, we have*

$$A(x, y) = \frac{1 + x - xy - \sqrt{x^2y^2 + 2x^2y - 3x^2 - 2xy - 2x + 1}}{2x}.$$

For instance, the first terms of $A(x, y)$ are $1 + x + x^2 + 2x^3 + 4x^4 + 9x^5 + 21x^6 + 51x^7 + x^2y + 2x^3y + 6x^4y + 16x^5y + 45x^6y + x^3y^2 + 3x^4y^2 + 12x^5y^2 + x^4y^3$ (see Table 2).

Table 2
Coefficients $a_{n,k}$ for $1 \le n \le 8$ and $0 \le k \le 6$.

k	n							
	1	2	3	4	5	6	7	8
0	1	1	2	4	9	21	51	127
1		1	2	6	16	45	126	357
2			1	3	12	40	135	441
3				1	4	20	80	315
4					1	5	30	140
5						1	6	42
6							1	7

Proof. Case $\alpha = 132$: let π be a permutation in $S_n(132)$. If π is not empty, then it can be written $\pi = \sigma n \gamma$ where $\gamma \in S_k(132)$ for some $k, 0 \le k \le n-1$, and σ is obtained from a permutation in $S_{n-k-1}(132)$ by adding k on all these entries. We distinguish two cases: (i) σ is empty, and (ii) σ is not empty.

For the case (i), the permutation π is of the form $\pi = n\gamma$ where $\gamma \in S_{n-1}(132)$; if γ is not empty then γ has one pure descent less than π (π contains the pure descents of γ and the pure descent created by n at the first position). Hence, the generating function for these permutations is given by $x + xy(A(x, y) - 1)$.

For the case (ii), the entry n does not create any pure descent in π. Thus, the generating function for these permutations is given by $x(A(x, y) - 1)A(x, y)$.

Combining the two cases, the following functional equation provides the result

$$A(x, y) = 1 + x + xy(A(x, y) - 1) + x(A(x, y) - 1)A(x, y).$$

Case $\alpha = 312$: let π be a permutation in $S_n(312)$. If π is not empty, then it can be written $\pi = \sigma 1 \gamma$ where $\sigma 1 \in S_k(312)$ for some k, $1 \le k \le n$, and γ is obtained from a permutation in $S_{n-k}(312)$ by adding k on all these entries. We distinguish two cases: (i) γ is empty, and (ii) γ is not empty. Let $A_1(x, y)$ (resp. $A_2(x, y)$) be the generating function for the set of permutations π satisfying (i) (resp. (ii)). Obviously, we have $A(x, y) = 1 + A_1(x, y) + A_2(x, y)$.

For the case (i), the permutation π is of the form $\pi = \sigma 1$, where σ is obtained from a permutation in $S_{n-1}(312)$ by adding 1 to all these entries. If σ belongs to the case (i), then σ has one pure descent less than π (π contains the pure descents of σ and the pure descent created by the last value of σ and 1); otherwise, σ and π have the same number of pure descents. Hence, the generating function for these permutations satisfies the functional equation $A_1(x, y) = xyA_1(x, y) + x(A_2(x, y) + 1)$.

For the case (ii), $\sigma 1$ satisfies the case (i), and γ can satisfy the two cases. Thus, the generating function for these permutations satisfies the functional equation $A_2(x, y) = A_1(x, y)(A(x, y) - 1)$.

Combining the two cases, we obtain the following system of functional equations:

$$\begin{cases} A_1(x, y) = xyA_1(x, y) + x(A_2(x, y) + 1) \\ A_2(x, y) = A_1(x, y)(A_1(x, y) + A_2(x, y)). \end{cases}$$

Thus, we obtain $A_1 = \frac{1+x-xy-\sqrt{x^2y^2+2x^2y-3x^2-2xy-2x+1}}{2(1+x-xy)}$, and
$A_2 = \frac{1-x-2xy-2x^2+x^2y+x^2y^2+(xy-1)\sqrt{x^2y^2+2x^2y-3x^2-2xy-2x+1}}{2x(1+x-xy)}$ which completes the proof.

Case $\alpha = 213$: let π be a permutation in $S_n(213)$. If π is not empty, it can be written $\pi = \sigma 1 \gamma$ where $1\gamma \in S_k(213)$ for some k, $1 \le k \le n$, and σ is obtained from a permutation in $S_{n-k}(213)$ by adding k on all these entries. We distinguish two cases: (i) γ is empty and (ii) otherwise. Let $A_1(x, y)$ (resp. $A_2(x, y)$) be the generating function for the set of permutations π satisfying (i) (resp. (ii)). Obviously, we have $A(x, y) = 1 + A_1(x, y) + A_2(x, y)$.

For the case (i), the permutation π is of the form $\pi = \sigma 1$, where σ is obtained from a permutation in $S_{n-1}(213)$ by adding 1 to all these entries. If σ belongs to the case (i), then σ has one pure descent less than π (π contains the pure descents of σ and the pure descent created by the last value of σ and 1); otherwise, σ and π have the same number of pure descents. Hence, the generating function for these permutations satisfies the functional equation $A_1(x, y) = xyA_1(x, y) + x(A_2(x, y) + 1)$.

For the case (ii), $\sigma 1$ satisfies the case (i), and γ can satisfy the two cases. Thus, the generating function for these permutations satisfies the functional equation $A_2(x, y) = A_1(x, y)(A(x, y) - 1)$. We conclude *mutatis mutandis* as for the pattern 312. □

Corollary 2. *For $\alpha \in \{132, 213, 312\}$ and $n \ge 1$, the permutations in $S_n(\alpha)$ with no pure descent are enumerated by the Motzkin numbers (see* A001006 *in* [15]*).*

Proof. We obtain the result by calculating $A(x, 0) = \frac{1+x-\sqrt{-3x^2-2x+1}}{2x}$, which is the generating function for the Motzkin numbers. □

Corollary 3. *For $\alpha \in \{132, 213, 312\}$ and $n \ge 1$, the popularity of the pure descents among the set $S_n(\alpha)$ is given by the binomial coefficient $\binom{2n-2}{n-2}$ (see* A001791 *in* [15]*).*

Proof. We need to calculate the numbers $\sum_{k=0}^{n-1} k.a_{n,k}$ for $n \geq 0$, which are the coefficients of x^n in $\frac{\partial A(x,y)}{\partial y}\Big|_{y=1}$. A simple calculation provides $\frac{\partial A(x,y)}{\partial y}\Big|_{y=1} = \frac{1-2x-\sqrt{1-4x}}{2\sqrt{1-4x}}$ which induces the result. □

3.2. The pattern 231

Since any descent in a permutation avoiding 231 is also a pure descent, the set of permutations of length n with no pure descents and avoiding the pattern 231 is clearly reduced to the identity permutation $12\ldots n$.

Theorem 4. *Let $B(x, y) = \sum_{n\geq 0, k\geq 0} b_{n,k} x^n y^k$ be the bivariate generating function where the coefficient of $x^n y^k$ is the number $b_{n,k}$ of permutations of length n with k pure descents and avoiding the pattern 231. Then, we have*

$$B(x, y) = \frac{1 - x + xy - \sqrt{1 - 2x - 2xy + x^2 - 2x^2y + x^2y^2}}{2xy}.$$

For instance, the first terms of $B(x, y)$ are $1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^2y + 3x^3y + 6x^4y + 10x^5y + 15x^6y + x^3y^2 + 6x^4y^2 + 20x^5y^2 + x^4y^3$ (see the Narayana numbers A001263 in [15]).

Proof. Let π be a permutation in $S_n(231)$. If π is not empty, then it can be written $\pi = \sigma n \gamma$ where $\sigma \in S_k(231)$ for some k, $0 \leq k \leq n-1$, and γ is obtained from a permutation in $S_{n-k-1}(231)$ by adding k on all these entries.

In the case where γ is empty, the generating function for these permutations is given by $xB(x, y)$ (π and σ have the same number of pure descents); otherwise, the generating function is given by $xyB(x, y)(B(x, y) - 1)$ (π contains the pure descents of σ and the pure descent created by n and the first value of γ).

Hence we deduce the functional equation:

$$B(x, y) = 1 + xB(x, y) + xyB(x, y)(B(x, y) - 1),$$

and a simple calculation gives the result. □

Corollary 4. *The popularity of the pure descents among the set $S_n(231)$, $n \geq 1$, is given by the binomial coefficient $\binom{2n-1}{n-2}$ (see A002054 in [15]).*

Proof. Using Theorem 4, the result is given by the coefficient of x^n in $\frac{\partial B(x,y)}{\partial y}\Big|_{y=1} = \frac{1-3x-(1-x)\sqrt{1-4x}}{2x\sqrt{1-4x}}$. □

3.3. The pattern $\alpha \in \{321, 123\}$

In this section, we investigate the distribution of pure descents on the sets $S_n(\alpha)$ for $\alpha \in \{321, 123\}$, but we proceed differently than we did in the Sections 3.1 and 3.2. Using classical bijections between $S_n(123)$ (or $S_n(321)$) and Dyck paths, the method consists in translating the pure descents on permutations into some specific patterns on Dyck paths on which we study the distributions.

A *Dyck path* of semilength n, $n \geq 0$, is a lattice path starting at $(0, 0)$, ending at $(2n, 0)$, and never going below the x-axis, consisting of up steps $U = (1, 1)$ and down steps $D = (1, -1)$. Let $\mathcal{P}_n$ be the set of all Dyck paths of semilength n. The *first return decomposition* of a Dyck path P is the unique decomposition $P = UQDR$ where Q and R are two (possibly empty) Dyck paths.

Given a permutation $\pi \in S_n(321)$, we consider the path on the graphical representation of π with up and right steps along the edges of the squares that goes from lower-left corner to the upper-right corner and leaving all the points (i, π_i), $i \in [n]$, to the right and remaining always as close to the diagonal $y = x$ as possible (the path can possibly reach the diagonal but never crosses it). See Fig. 2 for an example of this construction. Let us define the Dyck path of length $2n$ (called Dyck path associated with π) obtained from this lattice path by reading an up-step U every time the path moves up, and a down-step D every time the path moves to the right. This already known construction (see for instance [11]) induces a bijection χ from $S_n(321)$ to $\mathcal{P}_n$. The following lemma shows how this bijection translates pure descents of permutations into some particular configurations of Dyck paths.

Lemma 1. *Let π be a permutation in $S_n(321)$. Then, π has a pure descent in position d, $d \geq 1$, if and only if its associated Dyck path $\chi(\pi)$ is of the form $\gamma U^k DD\delta$, $k \geq 2$, where γ is a Dyck path of semilength $d - 1 \geq 0$ and δ is a word in $\{U, D\}^*$.*

Proof. Let π be a permutation in $S_n(321)$ and let d be a pure descent in π. Let $\lambda_1 < \lambda_2 < \cdots < \lambda_\ell$ be the left-to-right maxima of π, *i.e.*, the values π_i such that $\pi_j < \pi_i$ for $j < i$. So, we can write $\pi = \lambda_1 w_1 \ldots \lambda_\ell w_\ell$ where w_j are some words possibly empty so that the word $w_1 \ldots w_\ell$ is increasing.

So, the descents in π occur necessarily between λ_j and the first letter of w_j whenever w_j is not empty. Thus, the descents in π are in one to one correspondence with the occurrences of UDD in $\chi(\pi)$. Since π avoids 321, all integers j such that

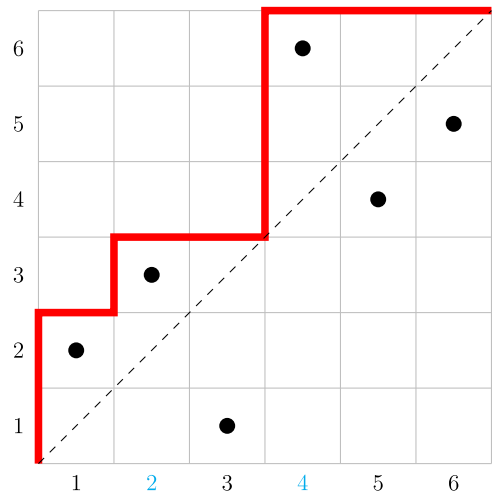


Fig. 2. The Dyck path $\chi(\pi) = UUDUDDUUUDDD$ associated to the permutation $\pi = 231\,645 \in S_6(321)$.

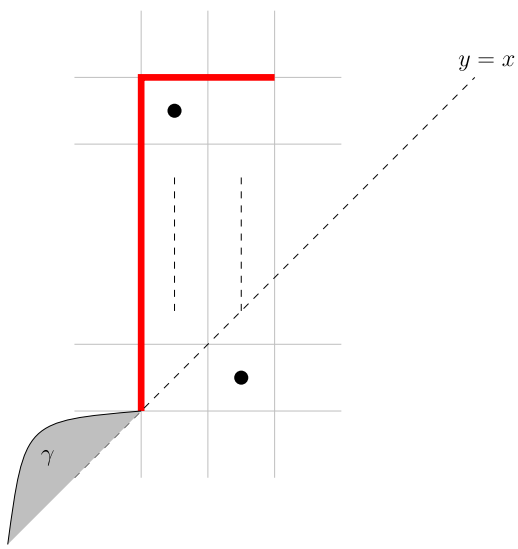


Fig. 3. The configuration $\gamma U^k DD$, $k \geq 2$, for a pure descent in $S_n(321)$.

$j < \pi_{d+1}$ must be placed before π_d, and every other integer must stay on the right of π_{d+1}, since d is a pure descent. This implies that $\gamma = \chi(\pi_1 \dots \pi_{d-1})$ is a Dyck path and $\chi(\pi)$ is of the form $\gamma U^k DD\delta$ for some $k \geq 2$ and δ is a word in $\{U, D\}^*$. See Fig. 3 for an illustration of this proof. □

Theorem 5. *Let $F(x, y) = \sum_{n\geq 0, k\geq 0} f_{n,k} x^n y^k$ be the bivariate generating function where the coefficient of $x^n y^k$ is the number $f_{n,k}$ of permutations of length n with k pure descents and avoiding the pattern 321. Then, we have*

$$F(x, y) = \frac{1 - \sqrt{1 - 4x}}{3x - 2x^2 + 2x^2 y - xy + x(y - 1)\sqrt{1 - 4x}}.$$

For instance, the first terms of $F(x, y)$ are $1 + x + x^2 + 2\,x^3 + 6\,x^4 + 19\,x^5 + 61\,x^6 + 200\,x^7 + x^2 y + 3\,x^3 y + 7\,x^4 y + 18\,x^5 y + 53\,x^6 y + x^4 y^2 + 5\,x^5 y^2$.

Proof. Using Lemma 1, we need to provide the bivariate generating function for Dyck paths of semilength n with respect to the number of different decompositions of the form $\gamma U^k DD\delta$, $k \geq 2$, where γ is a Dyck path and δ is a word in $\{U, D\}^*$. For short, the occurrence of $U^k DD$ in this decomposition will be called a *crochet*.

Now, we consider the first return decomposition of a Dyck path $P = UQDR$ where Q and R are two Dyck paths.

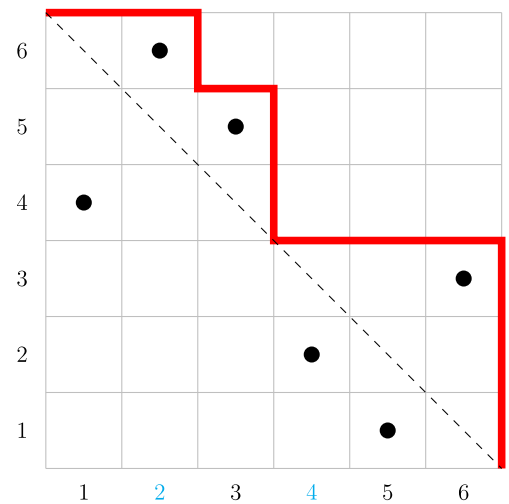


Fig. 4. The Dyck path $\Omega(\pi) = UUDUDDUUUDDD$ associated to the permutation $\pi = 465\,213 \in S_6(123)$.

We distinguish two cases: (i) P starts with an occurrence of U^kDU for some $k \geq 1$, or $P = UD$; and (ii) otherwise. Let $F_1(x, y)$ (resp. $F_2(x, y)$) be the bivariate generating function for the set of permutations π satisfying (i) (resp. (ii)). Obviously, we have $F(x, y) = F_1(x, y) + F_2(x, y)$.

In the first case, the Dyck paths $P = UQDR$ and R have the same number of crochets (the crochets of Q do not appear in P). Hence, the generating function $F_1(x, y)$ for these paths satisfies the functional equation $F_1(x, y) = x(F_1(x, 1) - x + 1)F(x, y)$.

In the second case, P is either empty, or P is a Dyck path starting with a crochet U^kDD for some $k \geq 2$. Then, for the non trivial case, $P = UQDR$ has one pure descent more than R (a crochet is created at the beginning of the path). Hence, the generating function $F_2(x, y)$ for these paths satisfies the functional equation $F_2(x, y) = 1 + xy(F_2(x, 1) - 1 + x)F(x, y)$.

So, we obtain the functional equation:

$$F(x, y) = 1 + x(F_1(x, 1) - x + 1)F(x, y) + xy(F_2(x, 1) - 1 + x)F(x, y), \tag{3}$$

where $F_1(x, 1) + F_2(x, 1) = C(x) = \frac{1-\sqrt{1-4x}}{2x}$ is the Catalan generating function for the sets $S_n(321)$, $n \geq 0$.

By fixing $y = 1$ in (3), we obtain $F_1(x, 1) = \frac{(1-x)(1-2x-\sqrt{1-4x})}{2x}$ and $F_2(x, 1) = \frac{2-3x+(x-2)\sqrt{1-4x}}{1-\sqrt{1-4x}}$ which allows to entirely determine the generating function $F(x, y)$. □

Corollary 5. *The popularity of the pure descents among the set $S_n(321)$, $n \geq 1$, is given by the generating function*

$$\frac{1 - 3x + (x - 1)\sqrt{1 - 4x}}{2x}$$

(see A000245 in [15]).

Proof. Using Theorem 5, the result is given by $\left.\frac{\partial F(x,y)}{\partial y}\right|_{y=1} = \frac{1-3x+(x-1)\sqrt{1-4x}}{2x}$. □

Finally, we study the distribution of the pure descents on the sets $S_n(123)$. From a permutation $\pi \in S_n(123)$, we consider the path on the graphical representation of π with right and down steps along the edges of the squares that goes from upper-left corner to the lower-right corner and leaving all the points (i, π_i), $i \in [n]$, to the left and remaining always as close to the diagonal $y = n - x + 1$ as possible (the path can possibly reach the diagonal but never crosses it). See Fig. 4 for an example of this construction. Let us define the Dyck path of length $2n$ (called Dyck path associated with π) obtained from this lattice path by reading an up-step U every time the path moves right, and a down-step D every time the path moves down. This construction induces a bijection Ω from $S_n(123)$ to $\mathcal{P}_n$. The following lemma shows how this bijection translates pure descents of permutations into some particular configurations of Dyck paths. Notice that Lemma 2 is a refinement for pure descents of the Proposition 1 in [4].

Lemma 2. *Let π be a permutation in $S_n(123)$. Then, π has a pure descent in position d, $d \geq 1$, if and only if its associated Dyck path $\Omega(\pi)$ has one of the three following forms:*

(a) *$\gamma UDUD\delta$, where γ and δ are two words in $\{U, D\}^*$ and γ contains $d - 1$ steps U,*
(b) *$\gamma UUU\delta$, where γ and δ are two words in $\{U, D\}^*$ and γ contains $d - 1$ steps U,*
(c) *$\gamma UDU\delta$, where γ is a Dyck path of semilength $d - 1 \geq 0$, and $U\delta$ is a Dyck path of semilength at least two.*

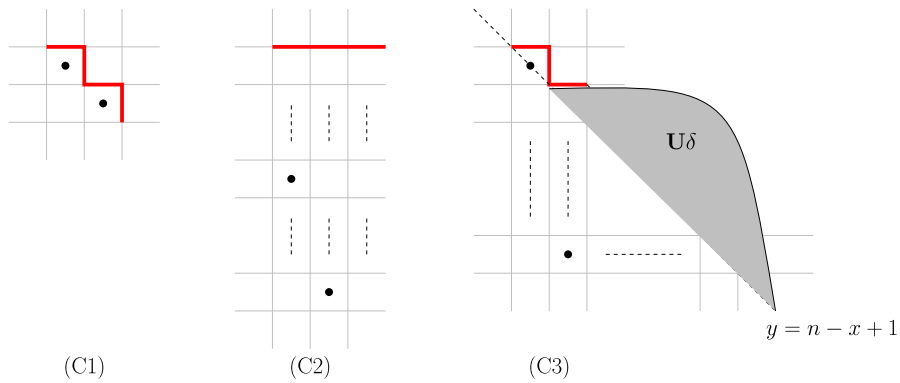


Fig. 5. The three configurations (C1)–(C3) for a pure descent in $S_n(123)$.

Proof. Using Proposition 1 in [4], Barnabei et al. prove that the descents of a permutation $\pi \in S_n(123)$ are translated by Ω, modulo a symmetry, into the occurrences of UUU and DU in a Dyck path and vice versa.

Now let us study what happens whenever the descent is pure. Let $\lambda_1 > \lambda_2 > \cdots > \lambda_\ell$ be the right-to-left maxima of π, *i.e.*, the values π_i such that $\pi_i > \pi_j$ for $i < j$. So, we can write $\pi = w_1\lambda_1 \dots w_\ell\lambda_\ell$ where w_j are some words possibly empty so that the word $w_1 \dots w_\ell$ is decreasing.

The pure descents appear in the three following configurations (see Fig. 5):

(C1) between two consecutive right-to-left maxima with consecutive values,

(C2) between two consecutive letters of a word w_i,

(C3) between the right-to-left maximum λ_i and the first letter x of w_{i+1} (whenever w_{i+1} is non-empty), such that there does not exist any value π_j lying in $[x, \lambda_i]$ on the left of λ_i.

Translating configurations (C1) and (C2) in terms of Dyck paths using the map Ω, we obtain easily the following correspondences:

- a pure descent satisfying (C1) is mapped into an occurrence of $UDUD$,
- a pure descent satisfying (C2) is mapped into an occurrence of UUU.

For the case (C3), since a pure descent d is obviously a descent, it is mapped with an occurrence of $UDU\delta$ where $U\delta$ is a Dyck path.

Moreover, if d is a pure descent, then there is no $j < d$ such that $\pi_{d+1} < \pi_j < \pi_d$. As π_d is a right-to-left maximum, we have $\pi_d \geq d$; since π_{d+1} is not right-to-left maximum there is $\pi_k > \pi_{d+1}$ for $k > d+1$, so the avoidance of 123 implies that all values on the left of π_{d+1} are greater than π_{d+1}. Finally, for all $j < d$, we have $\pi_j > \pi_d \geq d$, which implies that $\pi_d = d$. Hence, $\pi_1 \dots \pi_{d-1}$ is obtained from a permutation of $S_{d-1}(123)$ by adding d to all its values. This implies that $\Omega(\pi_1 \dots \pi_{d-1})$ is a Dyck path and $\Omega(\pi)$ is of the form $\gamma UDU\delta$ where γ is a Dyck path of semilength $d-1$ and $U\delta$ is also a Dyck path. Finally, the fact that π_{d+1} is not a right-to-left maximum ensures that this case is not the same as (C1) and thus, we necessarily have $U\delta$ is of semilength at least two. □

Theorem 6. *Let $G(x, y) = \sum_{n\geq 0, k\geq 0} g_{n,k} x^n y^k$ be the bivariate generating function where the coefficient of $x^n y^k$ is the number $g_{n,k}$ of permutations of length n with k pure descents and avoiding the pattern 123. Then, we have*

$$G(x, y) = \frac{x^3y^3 - 2x^3y^2 + x^3y - 3x^2y^2 + 3x^2y + 3xy - x - 1 + (xy - x - 1)R(x, y)}{x^3y^3 - x^3y^2 - 3x^2y^2 + x^2y + 2x^2 + 3xy - 1 + (xy - 1)R(x, y)}$$

where

$$R(x, y) = \sqrt{x^4y^4 - 2x^4y^3 + x^4y^2 - 4x^3y^3 + 4x^3y^2 + 4x^3y + 6x^2y^2 - 4x^3 - 2x^2y - 4x^2 - 4xy + 1}.$$

For instance, the first terms of $G(x, y)$ are $1 + x + x^2 + x^3 + 2x^4 + 3x^5 + 6x^6 + 11x^7 + x^2y + 3x^3y + 5x^4y + 13x^5y + 26x^6y + x^3y^2 + 6x^4y^2 + 15x^5y^2 + x^4y^3$.

Proof. Using Lemma 2, we need to provide the bivariate generating function for the number of Dyck paths of semilength n with respect to the number of configurations (C1)–(C3).

We distinguish two cases: (i) the Dyck path starts with UD, and (ii) otherwise.

Let $H(x, y, z)$ (resp. $H^{UD}(x, y, z)$, $H^{UU}(x, y, z)$) be the trivariate generating function where the coefficient of $x^n y^k z^\ell$ is the number of Dyck paths of semilength n (resp. starting with UD, resp. starting with UU), having k configurations (C1) or (C2) and ℓ configurations (C3). Obviously, we have $H(x, y, z) = 1 + H^{UD}(x, y, z) + H^{UU}(x, y, z)$.

Using the first return decomposition, we obtain the system of functional equations (4):

$$\begin{cases} H(x, y, z) &= 1 + H^{UD}(x, y, z) + H^{UU}(x, y, z) \\ H^{UD}(x, y, z) &= x + xzH^{UU}(x, y, z) + xyH^{UD}(x, y, z) \\ H^{UU}(x, y, z) &= xyH^{UU}(x, y, 1)H(x, y, z) + xH^{UD}(x, y, 1)H(x, y, z). \end{cases} \tag{4}$$

Indeed, let P be a non empty Dyck path having its first return decomposition $P = UQDR$ where Q and R are two Dyck paths. When Q is empty (P starts with UD): if R starts with UD, then P and R have the same number of configurations (C2) and (C3), and P has one configuration (C1) more than R. Hence the generating function for this subcase is $xyH^{UD}(x, y, z)$; if R starts with UU, then P and R have the same number of configurations (C1) and (C2), and P has one configuration (C3) more than R. Hence the generating function for this subcase is $xzH^{UU}(x, y, z)$.

When Q is not empty, the juxtaposition with R does not create any new configuration (C1), (C2) or (C3): if Q starts with UU, then P has one configuration (C2) more than the total number of configurations (C2) in Q and R; the configurations (C3) of Q do not appear in P. Hence the generating function for this subcase is $xyH^{UU}(x, y, 1)H(x, y, z)$. In the same way, we obtain the generating function $xH^{UD}(x, y, 1)H(x, y, z)$ whenever Q starts with UD.

Resolving this system of Eqs. (4), after fixing $z = y$, we obtain the expected bivariate generating function $G(x, y) = H(x, y, y)$. □

Corollary 6. *The popularity of the pure descents among the set $S_n(123)$, $n \geq 1$, is given by the generating function*

$$\frac{4x^2 - 9x + 2 + (5x - 2)\sqrt{1 - 4x}}{8x - 2}$$

(see A129869 in [15]).

Proof. Using Theorem 6, the result is given by $\left.\frac{\partial G(x,y)}{\partial y}\right|_{y=1} = \frac{4x^2-9x+2+(5x-2)\sqrt{1-4x}}{8x-2}$. □

Acknowledgements

We would like to thank the anonymous referees for their very careful reading of this paper and their helpful comments and suggestions.

References

[1] Jean-Luc Baril, Gray code for permutations with a fixed number of cycles, Discrete Math. 307 (13) (2007) 1559–1571.
[2] Jean-Luc Baril, Gray code for permutations with a fixed number of left-to-right minima, Ars Combin. 111 (2013) 225–239.
[3] Jean-Luc Baril, Statistics-preserving bijections between classical and cyclic permutations, Inform. Process. Lett. 113 (1) (2013) 17–22.
[4] Marilena Barnabei, Flavio Bonetti, Matteo Silimbani, The descent statistic on 123-avoiding permutations, Sém. Lothar. Combin. 63 (2010) B63a.
[5] Miklós Bóna, Combinatorics of Permutations, CRC Press, 2012.
[6] Miklós Bóna, Bruce E. Sagan, Vincent R. Vatter, Pattern frequency sequences and internal zeros, Adv. Appl. Math. 28 (3–4) (2002) 395–420.
[7] D. Callan, Notes on stirling cycle numbers. Preprint at http://www.stat.wisc.edu/~callan/notes.
[8] Ricky X.F. Chen, A note on the generating function for the stirling numbers of the first kind, J. Integer Seq. 18 (2) (2015) 3.
[9] Sergi Elizalde, Generating trees for permutations avoiding generalized patterns, Ann. Comb. 11 (3) (2007) 435–458.
[10] Sergey Kitaev, Patterns in Permutations and Words, Springer Science & Business Media, 2011.
[11] Christian Krattenthaler, Permutations with restricted patterns and dyck paths, Adv. Appl. Math. 27 (2–3) (2001) 510–530.
[12] Jacobus Hendricus van Lint, Richard Michael Wilson, A Course in Combinatorics, Cambridge university press, 2002.
[13] Kate Rudolph, Pattern popularity in 132-avoiding permutations, Electron. J. Combin. 20 (1) (2013) #P8.
[14] Rodica Simion, Frank W. Schmidt, Restricted permutations, European J. Combin. 6 (4) (1985) 383–406.
[15] Neil J.A. Sloane, et al., The on-line encyclopedia of integer sequences. Available electronically at http://oeis.org.
[16] Richard P. Stanley, Enumerative Combinatorics, Cambridge Univ Press, 1999.
[17] Julian West, Permutations with Forbidden Subsequences, and, Stack-Sortable Permutations (Ph.D. thesis), Massachusetts Institute of Technology, 1990.

Discrete Mathematics 340 (2017) 2946–2954

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Patterns in treeshelves

Jean-Luc Baril, Sergey Kirgizov, Vincent Vajnovszki *

LE2I CNRS, Université Bourgogne Franche-Comté, 21078 Dijon, France

ARTICLE INFO

Article history:
Received 23 November 2016
Received in revised form 17 July 2017
Accepted 24 July 2017
Available online 16 August 2017

Keywords:
Binary increasing tree
Pattern
Statistic
Popularity
Bell/Euler(ian)/Lah number

ABSTRACT

We study the distribution and the popularity of left children on sets of treeshelves avoiding a pattern of size three. (Treeshelves are ordered binary increasing trees where every child is connected to its parent by a left or a right link.) The considered patterns are sub-treeshelves, and for each such a pattern we provide exponential generating function for the corresponding distribution and popularity. Finally, we present constructive bijections between treeshelves avoiding a pattern of size three and some classes of simpler combinatorial objects.

© 2017 Elsevier B.V. All rights reserved.

1. Introduction and notation

The study of patterns in permutations was first introduced by Knuth [13], and continues to be an active area of research today. Recently, patterns have been studied in contexts other than permutations, see for instance [5,16] where the combinatorial class under consideration are inversion sequences, which can be seen as an alternative representation for permutations. The present paper deals with treeshelves (formally defined below) which are still another class in bijection with permutations, and patterns are sub-treeshelves contained or avoided in a similar way as consecutive patterns in permutations or in inversion sequences. More precisely, we consider the class of unrestricted treeshelves and of those avoiding a pattern of size 3 (treeshelves avoiding a pattern of size 2 collapse trivially to a singleton set). We not only enumerate these classes for any avoider of size 3, but also give bivariate generating functions with respect to the size and to the number of occurrences of a second pattern of size 2. As a byproduct we obtain the popularity (*i.e.*, the cumulative number of occurrences) among these classes of the pattern of size 2, obtaining counting sequences which are not yet recorded in Sloane's Encyclopedia of Integer Sequences [20].

Treeshelves are particular classes of binary increasing trees, considered for example in Françon's work [10] in the context of data structures for binary search methods. An *increasing tree* of size n, is an ordered rooted tree with n nodes labeled by distinct integers in $\{1, 2, \ldots, n\}$, so that the sequences of labels are increasing along all branches starting at the root (and thus, the root is labeled by 1). A *binary increasing tree* (sometimes called 0–1–2 increasing tree) is an increasing tree where every node has at most two children. Many studies (e.g., [1,2,3,6,15]) investigate binary increasing trees, but very few deal with such trees endowed with the additional property that every child (including those with no siblings) is connected to its parent by either a left or a right link. We call such a binary increasing tree *treeshelf* (or *t-shelf* for short), and its size is the number of its nodes, see Fig. 1 for a size 7 t-shelf. We denote by $\mathcal{B}_n$ the set of size n t-shelves, and $\mathcal{B}_1$ consists of the single one-node t-shelf. Often it is more convenient to represent graphically t-shelves by trees where edges are elongated so that

* Corresponding author.
E-mail addresses: jean-luc.baril@u-bourgogne.fr (J.-L. Baril), sergey.kirgizov@u-bourgogne.fr (S. Kirgizov), vvajnov@u-bourgogne.fr (V. Vajnovszki).

http://dx.doi.org/10.1016/j.disc.2017.07.021
0012-365X/© 2017 Elsevier B.V. All rights reserved.

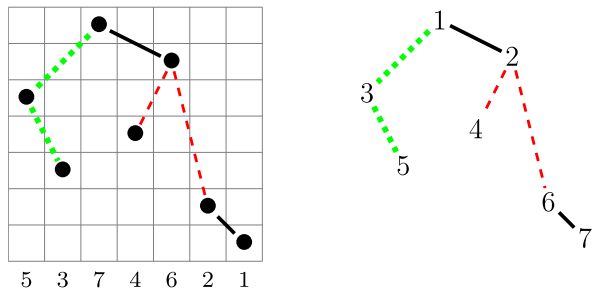


Fig. 1. The t-shelf corresponding to the permutation 5 3 7 4 6 2 1; dotted (dashed) lines correspond to an occurrence of the third (fourth) pattern in the set $\mathcal{B}_3$ listed above.

vertex i is at level i, for each i. For example, the size 3 t-shelf

is represented by

, and

is represented by

,

see also Fig. 1. In this representation, $\mathcal{B}_2 = \{$, $\}$, and $\mathcal{B}_3 = \{$, , , , , , $\}$.

We denote $\cup_{n\geq 0}\mathcal{B}_n$ by $\mathcal{B}$, and $\cup_{n\geq 1}\mathcal{B}_n$ by $\mathcal{B}^\bullet$. The labeled tree rooted at the left child of the root of a t-shelf T becomes a t-shelf after appropriately relabeling its nodes, and in the following we refer to it as the *left t-shelf of* T, and similarly for the *right t-shelf of* T .

There is a bijection between $\mathcal{B}_n$ and the set of permutations of size n, and so the cardinality of $\mathcal{B}_n$ is $n!$. Indeed, to any t-shelf T in $\mathcal{B}_n$ we can uniquely associate the length n permutation $\pi = \alpha(n - r(T) + 1)\beta$, where $r(T)$ is the label of the root of T, and α (resp. β) is recursively defined from the left (resp. right) t-shelf of T (see again Fig. 1). As mentioned by Bergeron, Flajolet, and Salvy [1], this construction appears in [10] and thereafter recalled in Stanley's book [21]. Additional information (including historical notes) about binary and other families of increasing trees can be found for example in [1,4,11].

In this paper we are interested in the sets of t-shelves avoiding a pattern $P \in \mathcal{B}_3$, *i.e.*, the sets of those that do not contain any occurrence of P. The containment/avoidance of a pattern in a t-shelf can most easily be explained with examples. The avoidance of in a t-shelf T means that T does not contain any node where the label of its left child is less than that of its right child. The t-shelf in Fig. 1 contains only one pattern (illustrated by dashed lines), one pattern (dotted) and avoids the pattern .

Since the number of patterns in a t-shelf is equal to the size of the t-shelf minus the number of patterns, minus one, in the following we will consider only patterns. Moreover, an occurrence of the pattern is equivalent to that of a left child in the underlying tree of the t-shelf, we will refer to this pattern as a left child (similarly the pattern corresponds to a right child). Also, since the patterns and are equivalent by symmetry, and so are the patterns and , and the patterns and , we will consider only avoiders P in $\{$, , $\}$.

T-shelves are labeled combinatorial objects, and so it is appropriate to use exponential generating functions (e.g.f.) for the enumerative analysis of them. In Section 2, for each of the avoiders P above mentioned, we consider the set $\mathcal{B}(P)$ of t-shelves avoiding P, or $\mathcal{B}^\bullet(P)$ when we restrict to non-empty t-shelves. We provide a bivariate exponential generating function for each $\mathcal{B}(P)$ with respect to the size and the number of left children, that is, function where the coefficient of $\frac{z^n y^k}{n!}$ in its series expansion is the number of t-shelves of size n having exactly k left children, and deduce the e.g.f. for $\mathcal{B}(P)$ with respect to the size. We also give the e.g.f. for the popularity of the left children among $\mathcal{B}(P)$, function where the coefficient of $\frac{z^n}{n!}$ in its series expansion is the total number of left children appearing in all size n t-shelves in $\mathcal{B}(P)$. These results are summarized in Tables 1 and 2.

Our method consists in constructing recursively the combinatorial class in question from two smaller classes, $\mathcal{A}_1$ and $\mathcal{A}_2$, using the usual labeled product $\mathcal{A}_1 \star \mathcal{A}_2$ and the boxed product $\mathcal{A}_1^{\square} \star \mathcal{A}_2$. The boxed product $\mathcal{A}_1^{\square} \star \mathcal{A}_2$ is a subset of $\mathcal{A}_1 \star \mathcal{A}_2$ where the smallest label appears in the $\mathcal{A}_1$ component. See [8] for more information about the boxed product and its application on labeled combinatorial structures.

Theorems 4–6 in Section 3 give constructive proofs of some results in Section 2, namely constructive bijections between: (i) t-shelves avoiding and set partitions, (ii) unordered binary increasing trees where every node of degree one has either a left or a right child and t-shelves avoiding the pattern , and (iii) unordered binary increasing trees and t-shelves avoiding the pattern .

2. T-shelves avoiding a size 3 pattern

We begin this section by considering unrestricted t-shelves, then we extend our approach to those avoiding a pattern in $\{$, , $\} \subset \mathcal{B}_3$.

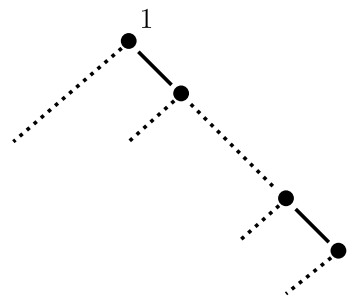


Fig. 2. The shape of a t-shelf avoiding pattern .

A t-shelf is either empty or consists of a root with two (possibly empty) children. Thus, the set $\mathcal{B}$ of unrestricted t-shelves can be expressed as

$$\mathcal{B} = \epsilon + \mathcal{Z}^{\square} \star \mathcal{B}^2,$$

where $\mathcal{Z}$ corresponds to the atom, *i.e.*, the singleton formed by the unique object of size one.

As the boxed product $\mathcal{A}_1^{\square} \star \mathcal{A}_2$ has its exponential generating function given by $\int_0^z \partial_t A_1(t) \cdot A_2(t)\,\mathrm{d}t$, where $A_1(t)$ and $A_2(t)$ are the exponential generating functions of $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively (see [8, Theorem II.5]), we obtain the differential equation

$$B(z) = 1 + \int_0^z B^2(t)\,\mathrm{d}t,$$

which, with the initial condition $B(0) = 1$, gives as expected $B(z) = \frac{1}{1-z}$, the e.g.f. for the sequence $n!$.

If we are interested in the bivariate exponential generating function $B(z, y)$ where the coefficient of $\frac{z^n y^k}{n!}$ is the number of t-shelves of size n having exactly k left children (or, equivalently by symmetry, k right children), then it is more convenient to consider the set $\mathcal{B}^\bullet$ of non-empty t-shelves. A t-shelf $T \in \mathcal{B}^\bullet$ can be in one of the following cases: the root of T either

- has no children (T is reduced to one root node), in this case the set of such T is $\mathcal{Z}$; or
- has only a left or only a right child, in both cases the set of such T is $\mathcal{Z}^{\square} \star \mathcal{B}^\bullet$; or
- has both left and right children, the set of such T is $\mathcal{Z}^{\square} \star \mathcal{B}^\bullet \star \mathcal{B}^\bullet$.

Thus, $\mathcal{B}^\bullet$ can be expressed as

$$\mathcal{B}^\bullet = \mathcal{Z} + \mathcal{Z}^{\square} \star \mathcal{B}^\bullet + \mathcal{Z}^{\square} \star \mathcal{B}^\bullet + \mathcal{Z}^{\square} \star (\mathcal{B}^\bullet)^2,$$

and after multiplying by y whenever a new left child is created, we obtain the differential equation

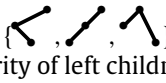
$$B^\bullet(z, y) = z + \int_0^z B^\bullet(t, y)\,\mathrm{d}t + y \int_0^z B^\bullet(t, y)\,\mathrm{d}t + y \int_0^z (B^\bullet(t, y))^2\,\mathrm{d}t,$$

where $B(0, y)^\bullet = 0$, and its solution is $B^\bullet(z, y) = \frac{1-\mathrm{e}^{z(y-1)}}{\mathrm{e}^{z(y-1)}-y}$. Finally,

$$B(z, y) = 1 + B^\bullet(z, y) = \frac{1-y}{\mathrm{e}^{z(y-1)} - y},$$

and we retrieve two well known results, see [18, Exercise 1.9]:

- $B(z, y)$ is given by a shift of the Eulerian numbers (sequence A008292 in OEIS [20]); and
- the popularity of the left (or right) children among $\mathcal{B}$, which is the coefficient of $\frac{z^n}{n!}$ in $\partial_y B(z, y)|_{y=1} = \frac{z^2}{2z^2-4z+2}$, is given by the Lah numbers (sequence A001286 in OEIS [20]).

In the following, for each t-shelf $P \in \{$  $\}$ we will count the class $\mathcal{B}(P)$ (or $\mathcal{B}^\bullet(P)$) of t-shelves avoiding P, and explore the distribution and the popularity of left children (*i.e*, of the pattern) within each class.

2.1. Pattern

Here we consider $\mathcal{B}(P)$ with $P =$, that is, t-shelves where each left child has no right child, we refer to Fig. 2 for an illustration of the shape of such a t-shelf.

Theorem 1. *Let P be the pattern* *. The bivariate e.g.f. for $\mathcal{B}(P)$ with respect to the size of t-shelves and the number of left children is given by*

$$C(z, y) = \mathrm{e}^{\frac{\mathrm{e}^{zy}-1}{y}}.$$

Proof. Let $\mathcal{C} = \mathcal{B}(P)$ and $T \in \mathcal{C}$. According to the shape of the t-shelves in $\mathcal{C}$ (see Fig. 2), if T is non-empty, then it is obtained from a pair of t-shelves, namely a non-empty t-shelf with no right children containing the smallest label of T, and a second unrestricted t-shelf in $\mathcal{C}$. The set of such non-empty t-shelves is $\mathcal{D}^{\square} \star \mathcal{C}$, where $\mathcal{D}$ is the set of non-empty t-shelves with no right children. Thus we have

$$\mathcal{C} = \epsilon + \mathcal{D}^{\square} \star \mathcal{C}.$$

Since the bivariate exponential generating function for $\mathcal{D}$ is $D(z, y) = \frac{\mathrm{e}^{zy}-1}{y}$, we obtain the differential equation

$$C(z, y) = 1 + \int_0^z \mathrm{e}^{ty} \cdot C(t, y) \cdot \mathrm{d}t$$

where $C(0, y) = 1$, with the solution $C(z, y) = \mathrm{e}^{\frac{\mathrm{e}^{zy}-1}{y}}$. $\square$

By calculating $C(z, 1)$ we have the following corollary.

Corollary 1. *The exponential generating function for the set $\mathcal{B}(P)$ with respect to the size of t-shelves is $Bell(z) = \mathrm{e}^{\mathrm{e}^z-1}$, which generates the Bell numbers (sequence A000110 in OEIS [20]).*

Corollary 2. *The popularity of the left children among the set $\mathcal{B}(P)$ is given by the exponential generating function*

$$PC(z) = \left(z\mathrm{e}^z - \mathrm{e}^z + 1\right) \mathrm{e}^{\mathrm{e}^z-1}.$$

Moreover, the coefficient pc_n of $\frac{z^n}{n!}$ in $PC(z)$ satisfies $pc_n = (n+1)b_n - b_{n+1}$ where b_n is the nth Bell number. The asymptotic value of pc_n is given by

$$\sqrt{n}\left(\frac{n}{W(n)}\right)^{n+\frac{1}{2}} \mathrm{e}^{\frac{n}{W(n)}-n-1},$$

where W is the Lambert function [7,19], that is, $W(n)$ is the unique real solution of $W(n) \cdot \mathrm{e}^{W(n)} = n$.

(The first terms of pc_n, $n \geq 2$, are 1, 5, 23, 109, 544, 2876, 16113, 95495.)

Proof. The popularity is given by $\partial_y C(z, y)|_{y=1} = (z\mathrm{e}^z - \mathrm{e}^z + 1)\, \mathrm{e}^{\mathrm{e}^z-1}$. The recurrence for pc_n is directly obtained from the relation $(z-1)\partial_z \mathrm{Bell}(z) + \mathrm{Bell}(z) = PC(z)$.

Finally, the asymptotic follows from the asymptotic formula due to M. Klazar [12, Proposition 2.6] and D.E. Knuth [14, eq. (30), p. 69]:

$$\frac{b_{n+1}}{b_n} \sim \frac{n}{\ln(n)},$$

and from the well known asymptotic for the Bell numbers (see A.M. Odlyzko [17]):

$$\frac{1}{\sqrt{n}}\left(\frac{n}{W(n)}\right)^{n+\frac{1}{2}} \mathrm{e}^{\frac{n}{W(n)}-n-1},$$

where $W(n)$ is the unique real solution of $W(n) \cdot \mathrm{e}^{W(n)} = n$. $\square$

2.2. Pattern

Here we consider the set $\mathcal{B}(P)$ of t-shelves avoiding the pattern $P =$.

Theorem 2. *Let P be the pattern* *. Then the bivariate e.g.f. for $\mathcal{B}(P)$ with respect to the size of t-shelves and the number of left children is given by*

$$E(z, y) = \frac{2y-1}{y \cosh\left(z\sqrt{-2y+1} + \ln\left(\frac{1}{y}\left(y + \sqrt{-2y+1} - 1\right)\right)\right) + y}.$$

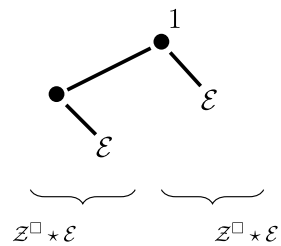


Fig. 3. Illustration of a t-shelf satisfying the third case in the proof of Theorem 2.

Proof.
Let $\mathcal{E} = \mathcal{B}(P)$ and $T \in \mathcal{E}$. One of the following cases can occur.

- T is empty.
- T is not empty, and its root does not have a left child. In this case, the right t-shelf of T belongs to $\mathcal{E}$ and the set of such t-shelves T is $\mathcal{Z}^{\square} \star \mathcal{E}$.
- The root of T has a left child. In this case T is obtained from a pair of t-shelves satisfying the second point above, namely one formed by the root of T together with its right t-shelf, and the other one being the left t-shelf of T. See Fig. 3 for an illustration of this case. So, T is the product of two t-shelves satisfying the second point above and, with the smallest label belonging to the first t-shelf. Thus, the set of such t-shelves T is $\mathcal{F}^{\square} \star \mathcal{F}$ where $\mathcal{F} = \mathcal{Z}^{\square} \star \mathcal{E}$.

Combining these cases we have

$$\mathcal{E} = \bullet + \mathcal{Z}^{\square} \star \mathcal{E} + \left(\mathcal{Z}^{\square} \star \mathcal{E}\right)^{\square} \star \left(\mathcal{Z}^{\square} \star \mathcal{E}\right),$$

which yields the differential equation

$$E(z, y) = 1 + \int_0^z E(t, y)\mathrm{d}t + y \cdot \int_0^z \left(E(u, y) \cdot \int_0^u E(t, y)\,\mathrm{d}t\right) \mathrm{d}u,$$

with the initial conditions $E(0, y) = 1$ and $\partial_z E(z, y)|_{z=0} = 1$. A simple calculation (using Maple for instance) gives the desired result. $\square$

The next corollary is obtained by calculating $E(z, 1)$.

Corollary 3. *The exponential generating function for the set $\mathcal{B}(P)$ with respect to the size of trees is given by*

$$Eul(z) = \frac{1}{1 - \sin z},$$

which yields a shift of the Euler numbers (sequence A000111 in OEIS [20]–not to be confused with Eulerian numbers).

Corollary 4. *The popularity pe_n of the left children among the set $\mathcal{B}(P)$ is given by the exponential generating function*

$$PE(z) = \frac{-\sin z + 1 + (z - 1)\cos z}{(1 - \sin z)^2}.$$

Moreover, the coefficient pe_n of $\frac{z^n}{n!}$ satisfies $pe_n = (n + 1)e_n - e_{n+1}$ where e_n is the shifted Euler number defined by the e.g.f. $Eul(z) = \frac{1}{1-\sin(z)}$. The asymptotic value of pe_n is given by

$$\frac{8(\pi - 2)}{\pi^3} n^2 \left(\frac{2}{\pi}\right)^n.$$

(The first terms of pe_n, $n \geq 2$, are 1, 4, 19, 94, 519, 3144, 20903, 151418).

Proof. Using Theorem 2, $PE(z)$ is obtained by calculating $\partial_y E(z, y)|_{y=1}$. The recurrence relation is directly obtained with the relation $PE(z) = (z - 1)\partial_z \mathrm{Eul}(z) + \mathrm{Eul}(z)$, and the asymptotic follows from the classical singularity analysis (see for instance [8]). $\square$

2.3. Pattern

We conclude this section by considering the pattern $P =$ and the set $\mathcal{B}^{\bullet}(P)$ of non-empty t-shelves avoiding P.

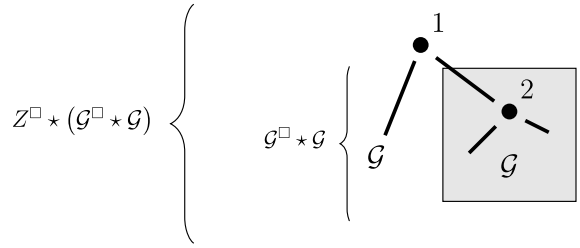


Fig. 4. Illustration of a t-shelf satisfying fourth case in the proof of Theorem 3.

Theorem 3. *Let P be the pattern* . *Then the bivariate e.g.f. for $\mathcal{B}^{\bullet}(P)$ with respect to the size of t-shelves and the number of left children is given by*

$$G(z,y) = \frac{-2}{1+y-\sqrt{y^2+1}\coth\left(\frac{z\sqrt{y^2+1}}{2}\right)}.$$

Proof. For $P =$, a non-empty t-shelf T in $\mathcal{G} = \mathcal{B}^{\bullet}(P)$ is in one of the following cases:

- T is reduced to one (root) node.
- T has at least two nodes and its root does not have a left child. In this case the set of such t-shelves T is $\mathcal{Z}^{\square} \star \mathcal{G}$.
- T has at least two nodes and its root does not have a right child. As above, the set of such t-shelves T is $\mathcal{Z}^{\square} \star \mathcal{G}$.
- The root of T has both left and right children (see Fig. 4). In this case T is obtained from a pair of t-shelves in $\mathcal{G}$ connected by a common root, with the smallest label of T in its right t-shelf. The set of such T is $\mathcal{Z}^{\square} \star (\mathcal{G}^{\square} \star \mathcal{G})$.

Combining these four cases we obtain

$$\mathcal{G} = \mathcal{Z} + \mathcal{Z}^{\square} \star \mathcal{G} + \mathcal{Z}^{\square} \star \mathcal{G} + \mathcal{Z}^{\square} \star \left(\mathcal{G}^{\square} \star \mathcal{G}\right)$$

which induces the differential equation

$$G(z,y) = z + \int_0^z G(t,y)\,\mathrm{d}t + y \cdot \int_0^z G(t,y)\,\mathrm{d}t + y \cdot \int_0^z \int_0^u \partial_t G(t,y) \cdot G(t,y)\,\mathrm{d}t\,\mathrm{d}u,$$

with the initial conditions $G(0,y) = 0$ and $\partial_z G(z,y)|_{z=0} = 1$. A simple calculation (using Maple for instance) gives the desired result. □

Corollary 5. *The exponential generating function for the set $\mathcal{B}(P)$ with respect to the size of t-shelves is given by*

$$1 + \frac{-2}{-\sqrt{2}\coth\left(\frac{z}{\sqrt{2}}\right) + 2}$$

which generates the sequence A131178 *in OEIS* [20].

Corollary 6. *The popularity pg_n of the left children among the set $\mathcal{B}(P)$ is given by the exponential generating function*

$$PG(z) = \frac{\mathrm{e}^{\sqrt{2}z}(4z-4) - \left(\sqrt{2}-2\right)\mathrm{e}^{2\sqrt{2}z} + \sqrt{2} + 2}{\left(\left(\sqrt{2}-2\right)\mathrm{e}^{\sqrt{2}z} + 2 + \sqrt{2}\right)^2}.$$

Moreover, the asymptotic value of the coefficient pg_n of $\frac{z^n}{n!}$ is given by

$$n\left(\frac{\sqrt{2}}{\ln\left(2\sqrt{2}+3\right)}\right)^{n+1}.$$

(The first terms of pg_n, $n \geq 2$, are 1, 5, 24, 128, 770, 5190, 38864, 320704.)

Proof. Using Theorem 3, $PG(z)$ is obtained by calculating $\partial_y G(z,y)|_{y=1}$, and the asymptotic follows from the classical singularity analysis. □

Table 1
Number of t-shelves avoiding the pattern P.

Pattern P	Sequence counting $\mathcal{B}(P)$	OEIS [20]	Comments
	1, 1, 2, 5, 15, 52, 203, 877, 4140, 21147, …	A000110 (Bell)	Corollary 1 and Theorem 4
	1, 1, 2, 5, 16, 61, 272, 1385, 7936, 50521, …	A000111 (Euler)	Corollary 3 and Theorem 6
	1, 1, 2, 5, 16, 64, 308, 1730, 11104, 80176, …	A131178	Corollary 5 and Theorem 5

Table 2
Popularity of left children among t-shelves avoiding the pattern P. None of these sequences is yet recorded in OEIS [20].

Pattern P	Popularity of left children in $\mathcal{B}(P)$	Comments
	1, 5, 23, 109, 544, 2876, 16113, …	Corollary 2
	1, 4, 19, 94, 519, 3144, 20903, 151418, …	Corollary 4
	1,5,24,128,770,5190,38864,320704, …	Corollary 6

3. Constructive bijections

The counting sequences for t-shelves avoiding a pattern of length 3 given in Corollaries 1, 3 and 5 are known (see Table 1), and these results deserve bijective proofs. Here, for each pattern $P \in \{$ $\}$, we give an explicit bijection between $\mathcal{B}(P)$ and a simpler combinatorial class. These results are stated in the next three theorems, the first two of them are straightforward.

Theorem 4. *There is a bijection between the set of partitions of* $\{1, 2, \ldots, n\}$ *and the set* $\mathcal{B}_n(P)$ *of t-shelves of size n avoiding the pattern* $P =$.

Proof. For a partition $S_1, S_2, \ldots, S_k$ of a set $S \subseteq \{1, 2, \ldots, n\}$ with $\min S_1 < \min S_2 < \cdots < \min S_k$ we define a t-shelf T with nodes labeled by integers from S. If $k = 1$, then T is simply the t-shelf with no right children (and with labels in $S = S_1$). Otherwise, T is defined recursively as:

- the root of T is labeled by $\min S_1$,
- the left t-shelf of T has size equal to $\operatorname{card} S_1 - 1$ and does not have a right child; its nodes are labeled by integers in $S_1 \setminus \{\min S_1\}$,
- the right t-shelf of T is obtained recursively from the partition $S_2, \ldots, S_k$ of $S \setminus S_1$.

Clearly, the t-shelf T corresponding to a set partition of $\{1, 2, \ldots, n\}$ is a size n t-shelf avoiding P. See the recursive definition of $\mathcal{B}(P)$ in the proof of Theorem 1 and the shape of T given in Fig. 2. This construction is reversible, and the statement holds □

Unordered binary increasing trees are the non-ordered counterpart of t-shelves: in an unordered binary increasing tree the sibling nodes are not longer ordered among themselves, and nodes with no sibling are not 'labeled' left/right. Thus each unordered binary increasing tree T can be expanded into 2^k different t-shelves of the same size, where k is the number of nodes of T having at least one child. Theorem 6 below establishes a bijection between size $n+1$ unordered binary increasing trees and size n t-shelves avoiding . An interesting intermediate ordered/unordered combinatorial class is that of binary increasing trees where, as above, the sibling nodes are not ordered, but nodes with no sibling are still 'labeled' left/right. We denote by $\mathcal{J}$ the set of these trees.

We define a transformation ϕ acting on unordered binary increasing trees and on trees in $\mathcal{J}$ by ordering the nodes having a sibling: if a node of a tree has two children, then we consider the child with the smaller label as the right one (and thus, that with the larger label as the left one). This configuration is depicted below.

, when $z < y$.

Clearly, applying ϕ on a tree in $\mathcal{J}$ a t-shelf avoiding is obtained. Moreover, this transformation is reversible, and since $\mathcal{J}$ is counted by the sequence A131178 in [20] (see the references therein), the next theorem gives a constructive proof for the counting sequence of t-shelves avoiding .

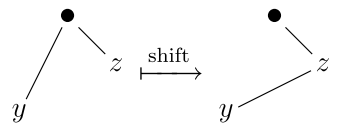


Fig. 5. The shift operation. The label of z is smaller than that of y.

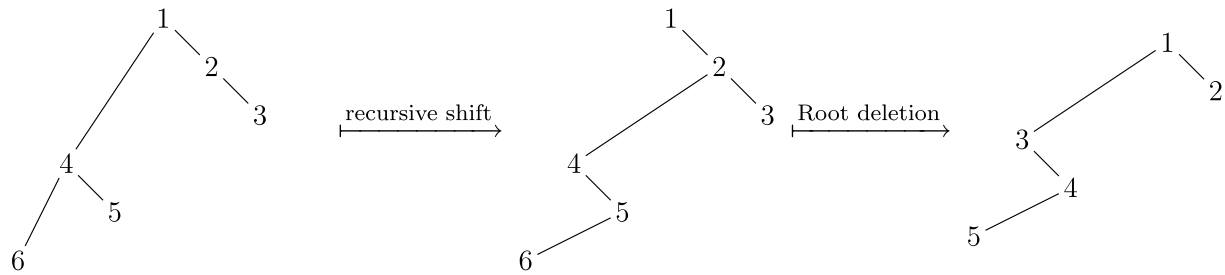


Fig. 6. An unordered binary increasing tree in standard representation and its image after the recursive shift process, and after the deletion of the root.

Theorem 5. *There is a bijection between the set $\mathcal{J}$ and the set $\mathcal{B}(P)$ of t-shelves avoiding the pattern $P =$* .

In order to obtain a bijection between unordered binary increasing trees and t-shelves that avoid (next theorem), we apply the shift (defined below) on unordered binary increasing trees in standard representation. The *standard representation* of such a tree is the t-shelf obtained after ordering sibling nodes, which is obtained by performing the above transformation ϕ, together with considering as right child each node with no sibling, as depicted below.

The *shift* of a node y of a t-shelf has effect only if the following conditions are satisfied: (i) y is a left child and it has a right sibling, say z; and (ii) z in turn does not have a left child and its label is smaller than that of y. Otherwise the shift has no effect. With this notation, the shift of a node y satisfying the two conditions above consists of pruning y from its parent and grafting it as the left child of z, see Fig. 5.

Finally, the shift of a t-shelf T is defined recursively by shifting, in order, the right t-shelf of T, the root of T, and then the left t-shelf of T. See the first part of Fig. 6 for an illustration. Obviously, the shift of a t-shelf T is still a t-shelf, and if T is the standard representation of some unordered binary increasing tree, then the shift of T avoids , and its root does not have a left child.

Theorem 6. *There is a bijection between unordered binary increasing trees with $n+1$ nodes and the set $\mathcal{B}_n(P)$ of t-shelves of size n avoiding the pattern $P =$* .

Proof. Let S be an unordered binary increasing tree with $n+1$ nodes and T be the shift of its standard representation. As mentioned above, T is a t-shelf and its root does not have a left child. We define the mapping $S \mapsto U$, where U is the t-shelf obtained after deleting the root of T and decreasing by one each label of the obtained t-shelf, see Fig. 6. This mapping is injective, and any t-shelf with n nodes that avoids can be obtained by this mapping from an unordered binary increasing tree. □

Let us remark that unordered binary increasing trees are equinumerous with alternating permutations that start with a descent, as proved by Foata and Schützenberger [9]. The corresponding bijection is given by Donaghey [6]. Using Donaghey's bijection together with the bijection in Theorem 6, we obtain a one-to-one correspondence between alternating permutations starting with a descent and t-shelves avoiding .

Acknowledgments

The authors wish to express their thanks to the referees for a very careful reading of our manuscript and many valuable suggestions.

References

[1] F. Bergeron, P. Flajolet, B. Salvy, Varieties of increasing tree, in: Proceedings of CAAP'92, Lecture Notes in Comput. Sci. 581 (1992) 24–48.
[2] O. Bodini, A. Genitrini, Cuts in increasing trees, in: Proceedings of the Meeting on Analytic Algorithmics and Combinatorics, 2015, pp. 66–77.
[3] D. Callan, A note on downup permutations and increasing 0-1-2 trees. Preprint at http://www.stat.wisc.edu/callan/notes/.
[4] L. Comtet, Advanced Combinatorics, Reidel, Dordrecht, 1974.
[5] S. Corteel, M.A. Martinez, C.D. Savage, M. Weselcouch, Patterns in inversion sequences I, Discrete Math. Theor. Comput. Sci. 18 (2) (2016) R2.
[6] R. Donaghey, Alternating permutations and binary increasing trees, J. Combin. Theory Ser. A 18 (1975) 141–148.
[7] L. Euler, De serie Lambertina Plurimisque eius insignibus proprietatibus, Acta Acad. Scient. Petropol. 2 (1783) 29–51. Also in Opera Omnia, Series Prima, vol. 6: Commentationes Algebraicae, Leipzig, Teubner, 1921, 350-369.
[8] P. Flajolet, R. Sedgewick, Analytic Combinatorics, Cambridge University Press, 2000.
[9] D. Foata, M.P. Schützenberger, Nombres d'Euler et permutations alternantes, in: A Survey of Combinatorial Theory, North Holland Publishing Company, 1973.
[10] J. Françon, Arbres binaires de recherche : propriétés combinatoires et applications, Rev. Fr. Autom. Inform. Rech. Oper. 10 (1976) 35–50.
[11] I.P. Goulden, D.M. Jackson, Combinatorial Enumeration, John Wiley, 1983.
[12] M. Klazar, Counting set systems by weight, Electron. J. Combin. 12 (2005) R11.
[13] D.E. Knuth, The Art of Computer Programming, Volume 3, Sorting and Searching, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1973.
[14] D.E. Knuth, The Art of Computer Programming, Volume 4, Fascicle 3, Pearson Education, 2005.
[15] A.G. Kutznetsov, I.M. Pak, A.E. Postnikov, Increasing trees and alternating permutations, Russian Math. Surveys 49 (6) (1994) 157–176.
[16] T. Mansour, M. Shattuck, Pattern avoidance in inversion sequences, Pure Math. Appl. 25 (2) (2015) 157–176.
[17] A.M. Odlyzko, Asymptotic Enumeration Methods, in: Handbook of Combinatorics, vol. 2, MIT Press, Cambridge, MA, 1995, pp. 1063–1229.
[18] T.K. Petersen, Eulerian Numbers, Birkhäuser Advanced Texts, Springer, 2015.
[19] G. Pólya, G. Szegő, Aufgaben und Lehrsätze der Analysis, Springer-Verlag, Berlin, 1925.
[20] N.J.A. Sloane, The on-line encyclopedia of integer sequences. Available electronically at http://oeis.org.
[21] R.P. Stanley, Enumerative Combinatorics, vol. 2, Cambridge University Press, 1999.

DE GRUYTER OPEN

PU. M. A. Vol. **27** (2018), No.1, pp. 18 – 31

Forests and pattern-avoiding permutations modulo pure descents

Jean-Luc Baril
LE2I, Université de Bourgogne
B.P. 47 870, 21078 Dijon, France
email: barjl@u-bourgogne.fr

and
Sergey Kirgizov
LE2I, Université de Bourgogne
B.P. 47 870, 21078 Dijon, France
email: sergey.kirgizov@u-bourgogne.fr

and
Armen Petrossian
LE2I, Université de Bourgogne
B.P. 47 870, 21078 Dijon, France
email: armen.petrossian@u-bourgogne.fr

(Received: June 16, 2017, and in revised form September 4, 2017.)

Abstract. We investigate an equivalence relation on permutations based on the pure descent statistic. Generating functions are given for the number of equivalence classes for the set of all permutations, and the sets of permutations avoiding exactly one pattern of length three. As a byproduct, we exhibit a permutation set in one-to-one correspondence with forests of ordered binary trees, which provides a new combinatorial class enumerated by the single-source directed animals on the square lattice. Furthermore, bivariate generating functions for these sets are given according to various statistics.

Mathematics Subject Classification(2010). 05A05, 05A15, 05A19.

Keywords: permutation, equivalence class, pure descent, pattern, Catalan and Motzkin numbers, forest, directed animal.

1 Introduction, definitions and notations

Many statistics on permutations have been studied for many years, but two of them appear more frequently in the literature: the number of descents and the number of excedances. These two statistics were introduced by MacMahon [15] and are closely related since they have the same distribution. However, many more articles deal with the descents which have links with other fields such as Coxeter groups [6] or the theory of lattice paths [11].

Recently [3, 4], two equivalence relations on permutations based on the excedance and descent statistics were introduced. The main results of these works consist of giving generating functions for the number of equivalence classes for several restricted sets of permutations such as involutions, cycles,

DOI: 10.1515/puma-2015-0025

derangements, and permutations avoiding at most one pattern of length three. So, it becomes natural to conduct a similar study for equivalence relations based on other statistics. This paper investigates an equivalence relation based on the pure descent statistic, first introduced in [5] and formally defined below. Moreover, we show how these equivalence classes are in one-to-one correspondence with certain forests of ordered trees, providing some links between several statistics on these sets. As a consequence, we exhibit a new set of pattern-avoiding permutations with the same cardinality as the set of single-source directed animals on the square lattice (see Barcucci and al. [2], Bousquet-Mélou [9] for two studies concerning directed animals, forests and pattern-avoiding permutations).

Now, we present some basic definitions and notation. Let S_n be the set of permutations of length n, *i.e.*, all one-to-one correspondences from $[n] = \{1, 2, \ldots, n\}$ into itself. The *one-line notation* of a permutation $\pi \in S_n$ is $\pi_1\pi_2\cdots\pi_n$ where $\pi_i = \pi(i)$ for $i \in [n]$. The *graphical representation* of $\pi \in S_n$ is the set of points in the plane at coordinates (i, π_i) for $i \in [n]$ (see Figure 1).

Let π be a permutation in S_n. A *descent* of π is an integer $i \in [n-1]$ such that $\pi_i > \pi_{i+1}$. Whenever there does not exist $j < i$ such that $\pi_{i+1} < \pi_j < \pi_i$, we call it a *pure descent*. Let $D(\pi)$ be the set of pure descents in π, and $DD(\pi)$ be the set of pairs (π_i, π_{i+1}) for $i \in D(\pi)$. By abuse of language, such a pair will be also called a *pure descent*. For instance, if $\pi = 1\ 4\ 2\ 7\ 5\ 3\ 8\ 6$ then $D(\pi) = \{2, 4\}$ and $DD(\pi) = \{(4, 2), (7, 5)\}$. In [5, Theorem 1], the authors prove that the number of length n permutations with k pure descents is given by the signless Stirling number of the first kind $c(n, k+1)$ where $c(n, k)$ satisfies

$$c(n, k) = (n-1) \cdot c(n-1, k) + c(n-1, k-1)$$

with the initial conditions $c(n, k) = 0$ if $n \leq 0$ or $k \leq 0$, except $c(0, 0) = 1$ ([20, 22] and the sequence A132393 in the Sloane's On-line Encyclopedia of Integer Sequences [19]).

We define the following equivalence relation on permutations of length n:

$$\pi \sim \sigma \iff DD(\pi) = DD(\sigma).$$

The set of equivalence classes in S_n (resp. in a restricted set $R \subset S_n$) is denoted $S_n^{\sim}$ (resp. $R^{\sim}$). For instance, the permutations $\pi = 1\ 4\ 2\ 7\ 5\ 3\ 8\ 6$ and $\sigma = 1\ 7\ 5\ 6\ 4\ 2\ 3\ 8$ belong to the same equivalence class (see Figure 1) because $DD(\pi) = DD(\sigma) = \{(4, 2), (7, 5)\}$, and $S_3^{\sim}$ is constituted of the 5 classes $\{123, 231\}$, $\{132\}$, $\{213\}$, $\{321\}$ and $\{312\}$.

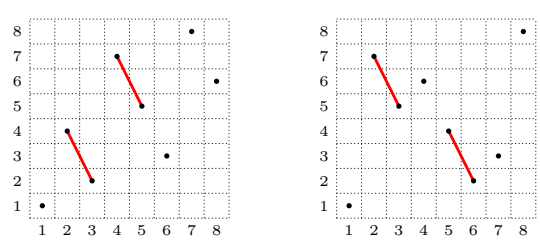


Figure 1: Two permutations $\pi = 1\ 4\ 2\ 7\ 5\ 3\ 8\ 6$ and $\sigma = 1\ 7\ 5\ 6\ 4\ 2\ 3\ 8$ in the same equivalence class of $S_8^{\sim}$ with $DD(\pi) = DD(\sigma) = \{(4, 2), (7, 5)\}$.

A permutation $\pi \in S_n$ *avoids* the pattern $\tau \in S_k$ if and only if there is no sequence of indices $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ such that $\pi(i_1)\pi(i_2)\cdots\pi(i_k)$ is order-isomorphic to τ (see [14, 18]). We denote by $S_n(\tau)$ the set of permutations of S_n avoiding the pattern τ. For example, if $\tau = 123$ then $52143 \in S_5(\tau)$ while $21534 \notin S_5(\tau)$. Many classical sequences in combinatorics appear as the cardinality of pattern-avoiding permutation sets. A large number of these results were firstly obtained in [7, 12, 14, 15, 18, 21]. Also, we refer to the books of Kitaev [13] and Bóna [8].

Later, Babson and Steingrímsson [1] defined *generalized patterns* (also called *vincular patterns*) where any pair of two adjacent values in the pattern may be underlined, which means that the corresponding values in the permutation must be adjacent. For instance, the pattern $2\underline{31}$ occurs in the permutation 316452 two times as the subsequences 352 and 452, while the pattern 231 occurs four times.

Moreover, we will consider a *barred pattern* $\bar{\tau}$, *i.e.*, a permutation in S_k having a bar over one value (see [17] and [21] for instance). Let τ be the permutation on $[k]$ identical to $\bar{\tau}$ but unbarred, and $\hat{\tau}$ be the permutation on $[k-1]$ made up of the $k-1$ unbarred values of $\bar{\tau}$ rewritten to be a permutation on $[k-1]$. Then $\pi \in S_n$ avoids the pattern $\bar{\tau}$ if and only if each pattern $\hat{\tau}$ in π can be expanded into a pattern τ in π where the expanded value corresponds to the barred value in $\bar{\tau}$. For instance, the permutation 3241 does not avoid $21\bar{3}$ since 41 cannot be expanded into a 213 pattern, while 3124 avoids it.

In the following, we will consider permutations avoiding the generalized and barred pattern $\underline{51}\bar{4}23$, consisting in permutations where any pattern $\underline{41}23$ can be expanded into a pattern $\underline{51}423$.

The main goal of this paper is to calculate the number of equivalence classes (modulo pure descents) for some subsets of permutations avoiding at most one pattern of length three. See Table 1 for an overview of our results.

The paper is organized as follows. In Section 2, we give a one-to-one correspondence between $S_n^{\sim}$ and the set of noncrossing partitions of $[n]$, proving that the cardinalities of $S_n^{\sim}$ for $n \geq 1$ are given by the Catalan numbers (see A000108 in the On-line Encyclopedia of Integer Sequences [19]). For the case of permutations avoiding the pattern 231, we prove that any equivalence class contains only one permutation on which we construct a forest of ordered trees. Also, we prove that $S_n(312)^{\sim}$ and $S_n(321)^{\sim}$ are enumerated by 2^{n-1} (A011782 in [19]).

In Section 3, we describe a bijection between forests of ordered binary trees with n nodes and the set $S_n(231, \underline{51}\bar{4}23)$, giving a new set of pattern-avoiding permutations in bijection with the single-source directed animals on the square lattice (see Barcucci et al. [2], and Bousquet-Mélou [9]). Bivariate generating functions are given for these sets according to various statistics.

In Section 4, we investigate the equivalence relation on the set $S_n(123)$ of permutations avoiding the pattern 123. We give a constructive bijection between forests of ordered binary trees and the classes in $S_n(123)^{\sim}$, proving that the cardinality of $S_n(123)^{\sim}$ is also given by the sequence A005773 that counts the single-source directed animals as above.

2 Enumeration of $S_n^{\sim}$, $S_n(231)^{\sim}$, $S_n(312)^{\sim}$ and $S_n(321)^{\sim}$

In this section, we provide the cardinality of $S_n^{\sim}$, $S_n(231)^{\sim}$, $S_n(312)^{\sim}$ and $S_n(321)^{\sim}$. Note that if a permutation π avoids the pattern 231 then any descent of π is a pure descent.

A *partition* Π of $[n]$ is a collection of non-empty pairwise disjoint subsets, called *blocks*, whose union

Pattern	Sequence	OEIS	$a_n, 1 \leq n \leq 9$
$\{\}, \{231\}$	Catalan	A000108	$1, 2, 5, 14, 42, 132, 429, 1430, 4862$
$\{312\}, \{321\}$	2^{n-1}	A011782	$1, 2, 4, 8, 16, 32, 64, 128, 256$
$\{231, \underline{51}\bar{4}23\}$	Directed animals	A005773	$1, 2, 5, 13, 35, 96, 267, 750, 2123$
$\{123\}$	Directed animals	A005773	$1, 2, 5, 13, 35, 96, 267, 750, 2123$

Table 1: Number of equivalence classes for the set of all permutations, and for some restricted sets of pattern-avoiding permutations.

is $[n]$ (see [16]). The *standard form* of Π is $\Pi_1/\Pi_2/\cdots$, where the blocks Π_i are arranged in increasing order of their smallest elements, and elements in a same block are in decreasing order. Let $\mathcal{P}_n$ be the set of partitions of $[n]$, and $\mathcal{NCP}_n \subset \mathcal{P}_n$ be the set of noncrossing partitions, *i.e.* all partitions Π where there do not exist four integers $p < q < r < s$ such that $p, r \in \Pi_i$ and $q, s \in \Pi_j$ with $i \neq j$.

We associate to a permutation $\pi \in S_n$ the unique partition Π defined as follows. Two elements x and y, $x > y$, belong to the same block in Π if and only if there exist i and j, $i < j$, such that the pairs $(x = \pi_i, \pi_{i+1}), (\pi_{i+1}, \pi_{i+2}), \ldots, (\pi_{j-1}, \pi_j = y)$ are pure descents in π. For instance, the two permutations in Figure 1 are associated to the same partition (in standard form) $\Pi = 1/42/3/75/6/8$. In fact, the associated partitions are always noncrossing partitions. Indeed, let us consider a block $\Pi_i = \pi_a\pi_{a+1}\cdots\pi_b$ with $a < b$, $\pi_a > \pi_{a+1} > \cdots > \pi_b$. Since $\pi_a\pi_{a+1}\cdots\pi_b$ is a subsequence of consecutive pure descents in π, there is no π_c, $c < a$, such that $\pi_c \in [\pi_b, \pi_a]$. So, let us assume that there is $c > b$ such that $\pi_c \in [\pi_b, \pi_a]$; then, for the same argument, all elements in the same block as π_c are greater than π_b and lower than π_a, which implies that there is no π_d in the block of π_c such that $\pi_d < \pi_b < \pi_c < \pi_a$ with $a < b < c < d$. *Mutatis mutandis*, there is no π_d in the block of π_c such that $\pi_b < \pi_c < \pi_a < \pi_d$ with $a < b < c < d$. Thus, the partition Π is noncrossing.

Conversely, any noncrossing partition Π of standard form $\Pi = \Pi_1/\Pi_2/\ldots/\Pi_k$, $k \geq 1$, is associated to the permutation $\pi = \Pi_1\Pi_2\cdots\Pi_k$ that avoids the pattern 231. Indeed, the noncrossing property forces all descents of π to be pure, implying that π does not contain any pattern 231. As the set $\mathcal{NCP}_n$ (and also $S_n(231)$) is enumerated by the nth Catalan number (see A000108, [19]), we obtain Theorem 2.1. As an immediate consequence, equivalence classes in $S_n(231)^\sim$ are singletons, and the set $S_n(231)$ is a set of representatives of $S_n^\sim$.

THEOREM 2.1 *The sets $S_n^\sim$ (resp. $S_n(231)^\sim$), $n \geq 1$, are enumerated by the Catalan numbers.*

As a byproduct of Theorem 2.1, we obtain the cardinalities of $S_n(312)^\sim$ and $S_n(321)^\sim$. Since $S_n(231)$ is a set of representatives of $S_n^\sim$, there is a unique $\pi' \in S_n(231)$ equivalent to $\pi \in S_n(312)$, and π' is obtained from the noncrossing partition associated to π (in standard form) $\Pi = \Pi_1/\Pi_2/\ldots/\Pi_k$ by deleting all '/', *i.e.*, $\pi' = \Pi_1\Pi_2\cdots\Pi_k$. Notice that for any permutation $\pi \in S_n(312)$, a pure descent in π is necessarily an adjacency, *i.e.*, a descent (π_i, π_{i+1}) with $\pi_{i+1} = \pi_i - 1$. Then, any block Π_j, $1 \leq j \leq k$, is an interval, which implies that π' avoids also the pattern 312. So, the set $S_n(312)^\sim$ is in one-to-one correspondence with the set of $S_n(231, 312)$ which induces Theorem 2.2 (see Simion and Schmidt [18]). Theorem 2.3 is obtained *mutatis mutandis*.

THEOREM 2.2 *The sets $S_n(312)^\sim$, $n \geq 1$, are enumerated by 2^{n-1}.*

THEOREM 2.3 *The sets* $S_n(321)^{\sim}$, $n \geq 1$, *are enumerated by* 2^{n-1}.

3 Forests and 231-avoiding permutations

In this section, we establish a constructive bijection between $S_n(231)$ and the set $\mathcal{F}_n$ of forests of ordered trees, *i.e.* collections of rooted trees in which children of each node are ordered and the total number of nodes is n. Taking advantage of the recursive definition of the forests, we exhibit a new set of permutations $S_n(231, \underline{51}\overline{4}23)$ having the same cardinality as the set of single-source directed animals on the square lattice (see [2, 9] and A005773, [19]). Moreover, we show how the bijection transports various statistics (see Table 2). As a byproduct, we provide several bivariate generating functions with respect to the length and these statistics for the two sets $S_n(231)$ and $S_n(231, \underline{51}\overline{4}23)$ (see Theorems 3.1 and 3.2).

Let π be a permutation in $S_n(231)$. We construct a forest $f_\pi \in \mathcal{F}_n$ as follows: we cross the graphical representation of π from left to right; if the point (i, π_i) is a left-to-right maximum (that is $\pi_i > \pi_j$ for all $j < i$), then it corresponds to the root of a new tree in f_π; otherwise we add an edge between (i, π_i) and (j, π_j) where j is the rightmost $j < i$ such that $\pi_j > \pi_i$. See Figure 2 for an example of this construction. Notice that in [10], the authors have a different way of converting a permutation to a graph (not necessary a forest) based on ascents.

By construction, the map $\phi : S_n(231) \to \mathcal{F}_n$ defined by $\pi \mapsto f_\pi$ is injective. Since $\mathcal{F}_n$ is enumerated by the nth Catalan number as $S_n(231)$, we deduce the bijectivity of ϕ.

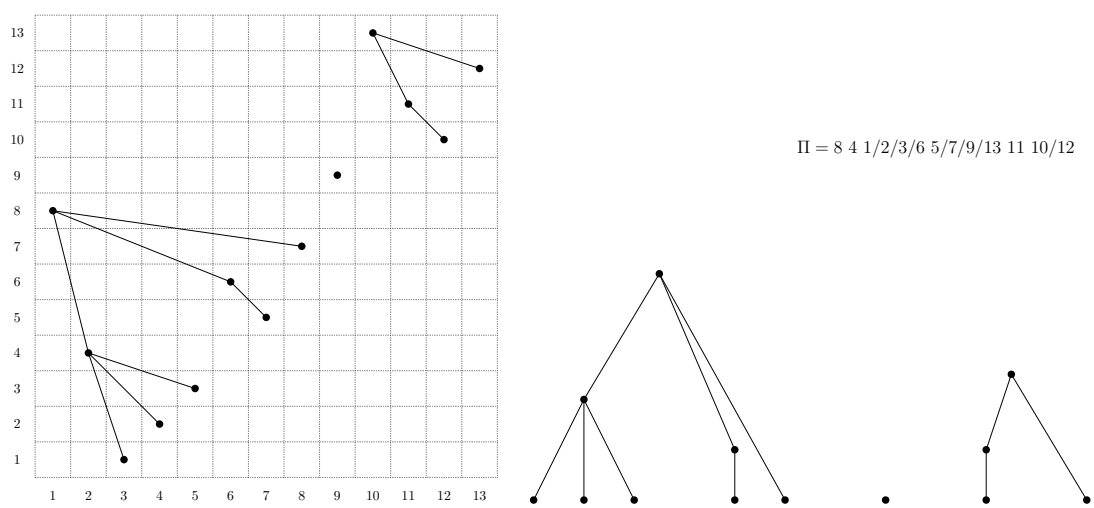


Figure 2: The permutation $\pi = 8\ 4\ 1\ 2\ 3\ 6\ 5\ 7\ 9\ 13\ 11\ 10\ 12$ with its corresponding forest f_π and the noncrossing partition $\Pi = 8\ 4\ 1/2/3/6\ 5/7/9/13\ 11\ 10/12$.

Now we define some statistics on $S_n(231)$ and $\mathcal{F}_n$, and we show how the map ϕ establishes a correspondence between them.

For a permutation $\pi \in S_n(231)$, we define:

• **des**(π) = *number of descents* (which is also the number of pure descents);
• **ides**(π) = *number of descents in* π^{-1} (for $\pi \in S_n(231)$, we have **ides**(π) = **des**(π));
• **adj**(π) = *number of adjacencies*, *i.e.* descent (π_i, π_{i+1}) such that $\pi_{i+1} = \pi_i - 1$;
• **lrM**(π) = *number of left-to-right maxima*, *i.e.* $i \geq 1$ such that $\pi_i > \pi_j$ for all $j < i$;
• **rlm**(π) = *number of right-to-left minima*, *i.e.* $i \geq 1$ such that $\pi_i < \pi_j$ for all $j > i$;
• **inv**(π) = *number of inversions*, *i.e.* pairs (π_i, π_j) with $\pi_i > \pi_j$ and $i < j$,
• **lmax**(π) = *maximum value of the Lehmer code* $\ell_1\ell_2\cdots\ell_n$ of π, *i.e.* $\max_{1\leq i\leq n} \ell_i$ where $\ell_i = |\{\pi_j > \pi_i,\ j < i\}|$;
• **lsum**(π) = *sum of all values of the Lehmer code* of π.

For instance, if $\pi = 8\,4\,1\,2\,3\,6\,5\,7\,9\,13\,11\,10\,12$ is the permutation in Figure 2, then we have **des**$(\pi) = 5$, **ides**$(\pi) = 5$, **adj**$(\pi) = 2$, **lrM**$(\pi) = 3$, **rlm**$(\pi) = 8$, **inv**$(\pi) = 15$, **lmax**$(\pi) = 2$, and **lsum**$(\pi) = 15$. For a forest $f \in \mathcal{F}_n$, we define

• **ledg**(f) = *number of left edges*, *i.e.*, leftmost edges among its siblings;
• **redg**(f) = *number of right edges*, *i.e.*, rightmost edges among its siblings (**ledg**(f)=**redg**(f));
• **nod1**(f) = *number of nodes with only one child*;
• **ordt**(f) = *number of ordered trees*;
• **leav**(f) = *number of leaves*, *i.e.*, nodes without child;
• **vpat**(f) = *number of vertical paths* (a vertical path is a path between a node and one of its ancestors);
• **dept**(f) = *depth*, *i.e.*, the maximal length of a vertical path;
• **inpl**(f) = *internal path length*, *i.e.*, the sum of the lengths of all paths from a node to the root.

For instance, if f is the associated forest of the permutation in Figure 2, then we have **ledg**$(f) = 5$, **redg**$(f) = 5$, **nod1**$(f) = 2$, **ordt**$(f) = 3$, **leav**$(f) = 8$, **vpat**$(f) = 15$, **dept**$(f) = 2$, and **inpl**$(f) = 15$.

In the following, the notation **st** will be refer to one of these statistics on the sets $S_n(231)$ or $\mathcal{F}_n$. According to these definitions, it is straightforward to check that ϕ transports these statistics as related in Table 2.

$S_n(231)$	**des** = **ides**	**adj**	**lrM**	**rlm**	**inv**	**lmax**	**lsum**
$\mathcal{F}_n$	**ledg** = **redg**	**nod1**	**ordt**	**leav**	**vpat**	**dept**	**inpl**

Table 2: Correspondences of statistics by the bijection ϕ from $S_n(231)$ to $\mathcal{F}_n$.

Using the correspondence between these statistics and taking advantage of the recursive structure of a forest, we derive several bivariate generating functions for two sets of pattern-avoiding permutations with respect to the length and the statistics above.

THEOREM 3.1 *Let $F(z, y)$ be the bivariate generating function where the coefficient of $z^n y^k$ is the number of permutations $\pi \in S_n(231)$ with* **st**$(\pi) = k$. *Then, we have:*

- *if* **st** *is* **des, ides, or lmax**, *then* $F(z, y) = \frac{1-z+zy-\sqrt{z^2y^2-2\,z^2y+z^2-2\,zy-2\,z+1}}{2zy}$,
- *if* **st** *is* **adj**, *then* $F(z, y) = \frac{1-zy+z-\sqrt{z^2y^2+2\,z^2y-3\,z^2-2\,zy-2\,z+1}}{2z}$,
- *if* **st** *is* **lrM**, *then* $F(z, y) = \frac{2}{2-y+y\sqrt{1-4z}}$,

- *if* **st** *is* **rlm**, *then* $F(z,y) = \frac{1+z-zy-\sqrt{z^2y^2-2\,z^2y+z^2-2\,zy-2\,z+1}}{2z}$.

Whenever **st** *is* **inv** *or* **lsum**, *the generating function satisfies the functional equation*

$$F(z,y) = \frac{1}{1-z(F(zy,y)-1)-z}.$$

Proof. Since a forest $f \in \mathcal{F}_n$ is a collection of ordered trees, we have $F(z,y) = \frac{1}{1-T(z,y)}$ where $T(z,y)$ is the generating function for the number of ordered trees with respect to the length and the statistic **st**. Now, using the fact that a nonempty ordered tree is a node connected to the roots of the trees of a forest, we easily derive functional equations for each statistic **st**:
- for **ledg**, **redg** and **dept**, $T(z,y) = zy(F(z,y)-1)+z$;
- for **nod1**, $T(z,y) = zyT(z,y) + z(F(z,y)-T(z,y))$;
- for **ordt**, $T(z,y) = zyF(z,1)$;
- for **leav**, $T(z,y) = z(F(z,y)-1)+zy$;
- for **vpat** and **inpl**, $T(z,y) = z(F(zy,y)-1)+z$.

A simple calculation (using Maple for instance) completes the proof. □

THEOREM 3.2 *The sets* $S_n(231, \underline{51}\overline{4}23)$, $n \geq 1$, *are enumerated by the number of single-source directed animals on the square lattice (A005773, [19]). Let* $G(z,y)$ *be the bivariate generating function where the coefficient of* $z^n y^k$ *is the number of permutations* $\pi \in S_n(231, \underline{51}\overline{4}23)$ *with* $\mathbf{st}(\pi) = k$. *Then, we have:*

- *if* **st** *is* **des, ides, or lmax**, *then* $G(z,y) = \frac{2yz}{3\,yz-1+\sqrt{y^2z^2-4\,yz^2-2\,yz+1}}$,
- *if* **st** *is* **adj**, *then* $G(z,y) = \frac{2z}{2z-1+yz+\sqrt{y^2z^2-2\,yz-4\,z^2+1}}$,
- *if* **st** *is* **lrM**, *then* $G(z,y) = \frac{2z}{yz-y+2\,z+y\sqrt{-3\,z^2-2\,z+1}}$,
- *if* **st** *is* **rlm**, *then* $G(z,y) = \frac{2z}{3\,z-1+\sqrt{-4\,yz^2+z^2-2\,z+1}}$.

Whenever **st** *is* **inv** *or* **lsum**, *the generating function satisfies the functional equation*

$$\begin{cases} R(z,y) &= z + zR(yz,y) + zR(yz,y)^2 \\ G(z,y) &= \frac{1}{1-R(z,y)} \end{cases}.$$

Proof. Let $\mathcal{G}_n \subset \mathcal{F}_n$ be the set of forests of ordered binary trees, *i.e.*, ordered trees where each node has at most two children. Let us prove that we have $\phi^{-1}(\mathcal{G}_n) = S_n(231, \underline{51}\overline{4}23)$. Let π be a permutation in $S_n(231, \underline{51}\overline{4}23)$ and $f_\pi = \phi(\pi)$. Since π avoids $\underline{51}\overline{4}23$, any pattern $\underline{41}23$ can be expanded into a pattern $\underline{51}423$ which implies that the corresponding forest f_π does not contain a node with more than two children. Conversely, if the forest f_π belongs to $\mathcal{G}_n$, then the degree of any node of f is at most two. Let us suppose that π contains the pattern $\underline{41}23$ on $\pi_i\pi_{i+1}\pi_j\pi_k$, $i+1 < j < k$. If there does not

exist ℓ, $i+1<\ell<j$ such that $\pi_k<\pi_\ell<\pi_i$, by construction the forest f_π has the node π_i with at least three children π_{i+1}, π_j and π_k. So, any pattern $\underline{41}23$ in π can be expanded into a pattern $\underline{51}423$, which proves that π avoids $\underline{51}\overline{4}23$.

Let $G(z,y)$ be the generating function where the coefficient of $z^n y^k$ is the number of forests $f\in\mathcal{G}_n$ with $\mathbf{st}(f)=k$. Since a forest $f\in\mathcal{G}_n$ is a collection of the ordered binary trees, we have $G(z,y)=\frac{1}{1-R(z,y)}$ where $R(z,y)$ is the generating function for the number of ordered binary trees with respect to the length and the parameter **st**. Now, using the fact that a nonempty ordered binary tree is a node connected to the roots of at most two ordered binary trees, we can easily derive functional equations for each statistic **st**:

- for **ledg**, **redg** and **dept**, $R(z,y)=z+zyR(z,y)+zyR(z,y)^2$;
- for **nod1**, $R(z,y)=z+zyR(z,y)+zR(z,y)^2$;
- for **ordt**, $R(z,y)=zy+zyR(z,1)+zyR(z,1)^2$;
- for **leav**, $R(z,y)=zy+zR(z,y)+zR(z,y)^2$;
- for **vpat** and **inpl**, $R(z,y)=z+zR(z,y)+zR(z,y)^2$.

A simple calculation (using Maple for instance) completes the proof. □

Notice that for **inv** and **lsum**, functional equations provide generating functions as continued fractions instead of closed forms.

4 Enumeration of $S_n(123)^\sim$

In this section we prove that the set $S_n(123)^\sim$ is enumerated by the number of single-source directed animals on the square lattice (A005773, [19]). To achieve this, we construct a bijection between $S_n(123)^\sim$ and the set of forests of ordered binary trees, *i.e.*, trees where nodes have at most two ordered children (if a node has only one child then the corresponding link is called 0-*edge* the corresponding link, and if a node has two children then the two corresponding links are called 0-*edge* and 1-*edge*, which defines an order on siblings).

A *run* of pure descents (also called *run* for short) in $\pi=\pi_1\cdots\pi_n\in S_n$ is a maximal subsequence $\pi_i\pi_{i+1}\cdots\pi_j$, $1\le i\le j\le n$, of successive pure descents, *i.e.* (π_k,π_{k+1}) is a pure descent for $i\le k\le j-1$, and the two pairs $(\pi_{i-1}\pi_i)$, $(\pi_j\pi_{j+1})$ are not pure descents (a run contains at least one entry, that is π_i). To any run R of $\pi\in S_n$, we associate the interval $I(R)=[a,b]\subseteq[n]$ where a and b are the extremities of R, that is $a=\min R$ and $b=\max R$.

In a permutation $\pi\in S_n(123)$, there do not exist three runs R,S and T such that $I(S)\subset I(R)\subset I(T)$ (otherwise a pattern 123 would be created on the three entries $\min T,\min R$ and $\min S$). So, whenever there are two runs S, R such that $I(S)\subset I(R)$, we will say that S is a *secondary run*, and S appears necessarily in π at the right of R. A run R that is not secondary will be called *primary*. The family of intervals $I(R)$ associated to the primary runs of $\pi\in S_n(123)$ forms a partition of $[n]$. We denote by $p\ge 1$ the cardinality of this partition, and let I_i, $1\le i\le p$, be the ith interval (considered in decreasing order), and let P_i be its associated primary run ($I(P_i)=I_i$). For $1\le i\le p$, let L_i be the restriction of π to the interval I_i. It can be decomposed as $L_i=P_iS_i^1S_i^2\cdots S_i^{s_i}$ where P_i is the ith primary run of π, S_i^j is the jth secondary run of the interval I_i and s_i is the number of secondary runs in L_i.

We say there is a *break* between two consecutive secondary runs S_i^j and S_i^{j+1} in L_i if $\min S_i^j=1+\max S_i^{j+1}$. We refer to Figure 3 for an illustration of such a decomposition.

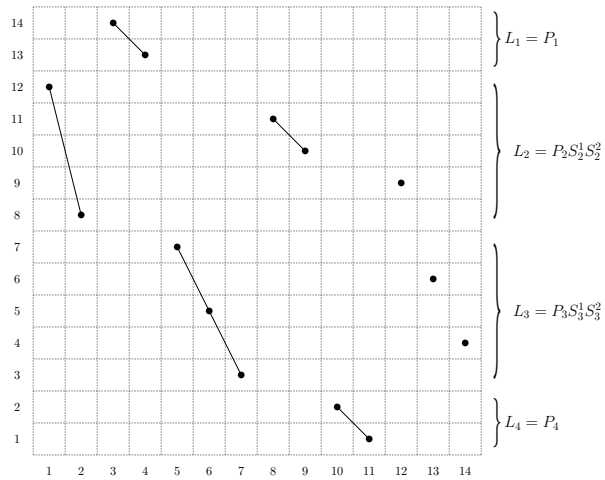


Figure 3: Illustration of the decomposition into runs of the permutation $\pi = 12\ 8\ 14\ 13\ 7\ 5\ 3\ 11\ 10\ 2\ 1\ 9\ 6\ 4$; $P_1 = 14\ 13$, $P_2 = 12\ 8$, $S_2^1 = 11\ 10$, $S_2^2 = 9$, $P_3 = 7\ 5\ 3$, $S_3^1 = 6$, $S_3^2 = 4$, $P_4 = 2\ 1$. A break occurs between the two consecutive secondary runs S_2^1 and S_2^2.

With the above definitions we have Lemma 4.1.

Lemma 4.1 *Let $\pi \in S_n(123)$ and $1 \leq i \leq p$. If there is a break between two consecutive secondary runs S_i^j and S_i^{j+1} in L_i, then there exists a unique primary run P_k between S_i^j and S_i^{j+1} in the one-line notation of π, and we necessarily have $k > i$.*

Proof. If there is a break between S_i^j and S_i^{j+1}, then we have $\min S_i^j = 1 + \max S_i^{j+1}$ and the pair $(\min S_i^j, \max S_i^{j+1})$ is not a pure descent. So, there exists an entry x of π between S_i^j and S_i^{j+1}, *i.e.* S_i^j and S_i^{j+1} are not contiguous. As π avoids 123, x is necessarily less than $\min P_i$ where P_i is the primary run of L_i, and it does not belong to a secondary run. For a contradiction, let us assume that there are two entries x and y, $x > y$, between S_i^j and S_i^{j+1}, that do not belong to the same primary run (we take y maximal such that $x > y$). Obviously, x is on the left of y, otherwise it would create a 123 pattern on the entries $yx \min S_i^{j+1}$. Let P_k (resp. P_ℓ, $\ell > k$) be the primary run that contains x (resp. y). Since $x > y$ and P_k is on the left of P_ℓ, the two primary runs are not contiguous, and there exists a value z between P_k and P_ℓ. The maximality of y implies that z is either below P_ℓ or above P_k, which creates a 123 pattern in both cases. Thus, we obtain the desired contradiction. □

Lemma 4.1 allows us to define an injective map α from the set $\mathcal{B}$ of breaks to the set $\mathcal{P}$ of primary runs, where the image of a break under α is the unique primary run defined in Lemma 1. Moreover, it is easy to check that the map α is increasing, *i.e.*, if $B_1, B_2, \ldots, B_r$ are the breaks of $\mathcal{B}$ ordered in decreasing order (from top to bottom in the graphical representation of π), then the two primary runs $P_k = \alpha(B_i)$ and $P_\ell = \alpha(B_j)$, $1 \leq i < j \leq r$, satisfy $k < \ell$ (*i.e.*, $P_k > P_\ell$, which means that P_k is above P_ℓ in the graphical representation of π). The existence of this increasing map α allows us to define another increasing map β (possibly equal to α) from $\mathcal{B}$ to $\mathcal{P}$:

- $\beta(B_1)$ is the highest primary run below B_1 (it always exists since $\alpha(B_1)$ is below B_1).
- Let us assume that β is defined on $U_i = \{B_1, \ldots, B_i\}$, $i \geq 1$, and β is increasing such that $\beta(B_j) \leq \alpha(B_j)$ for $1 \leq j \leq i$. Setting $V_i = \beta(U_i)$, we define $\beta(B_{i+1})$ by the highest primary run below B_{i+1} that does not lie in V_i (it always exists since $\alpha(B_{i+1})$ is below B_{i+1} and $\alpha(B_{i+1})$ cannot lie in V_i since $\alpha(B_{i+1})$ is below $\alpha(B_i)$ and thus, also below $\beta(B_i)$).

A crucial property of β is that it depends only on the set of primary and secondary runs, which means that two permutations in the same equivalence class provide the same map β.

Using the map β, we construct a forest $\chi(\pi)$ of ordered binary trees from the graphical representation of π by adding 0-edges and 1-edges between some entries of π using the following process.

(i) If (π_i, π_{i+1}) is a pure descent, then we add a 0-edge between π_i and π_{i+1}.

(ii) If there is break between S_i^j and S_i^{j+1}, then we add a 0-edge between $\min S_i^j$ and $\max S_i^{j+1}$.

(iii) If S_i^j is a secondary run and there is no break just before S_i^j, then we add a 1-edge between x and $\max S_i^{j+1}$ where x is the smallest entry greater than $\max S_i^{j+1}$ in the primary run P_i.

(iv) If there is a break B between S_i^j and S_i^{j+1}, then we add a 1-edge between $\max S_i^j$ and $\max \beta(B)$.

At the end of this process, we read the different connected components (rotated clockwise by $\frac{\pi}{4}$) from top to bottom, and we draw the corresponding trees so that any 0-edge points to the left child and 1-edge points to the right child. See Figure 4 for an illustration of this construction. Black line (resp. blue dash-dotted line, red dotted line, green dashed line) edges come from (i) (resp. (ii), (iii), (iv)). In what follows, an edge e in $\chi(\pi)$ will be denoted (a, b) where a and b are the extremities of e such that a is the parent of b.

Since any node in $\chi(\pi)$ has at most two children, $\chi(\pi)$ is a forest of ordered binary trees. Let T be a binary tree of $\chi(\pi)$. For any node $v \in T$ we denote by $r(v)$ (resp. $l(v)$) the number of 1-edges (resp. 0-edges) in the path connecting the root of T with v. We say that a node v is isolated when it has no siblings.

Remark 4.2 Let $e = (a, b)$ be a 0-edge in $\chi(\pi)$.

- (a, b) is a pure descent in a primary run of π if and only if $r(a)$ is even.
- (a, b) is a pure descent in a secondary run of π if and only if a is isolated and $r(a)$ is odd.
- There is a break between a and b if and only if a is not isolated and $r(a)$ is odd.

Let $\mathrm{rath}(v)$ be the binary word consisting of edge labels in the path from the root to v. Using a lexicographical order over such binary words (e.g. $101 > 011$, $1 > 01$), we define a total order on the set of nodes V in $\chi(\pi)$. For $a, b \in V$, we set

$$a < b \Longleftrightarrow \begin{cases} \text{either } a \text{ belongs to a tree before that of } b \text{ in the forest } \chi(\pi), \\ \text{or} \begin{cases} r(a) < r(b) \text{ or} \\ r(a) = r(b) \text{ and } \mathrm{rath}(a) > \mathrm{rath}(b) \end{cases} \end{cases} \qquad (\bigstar)$$

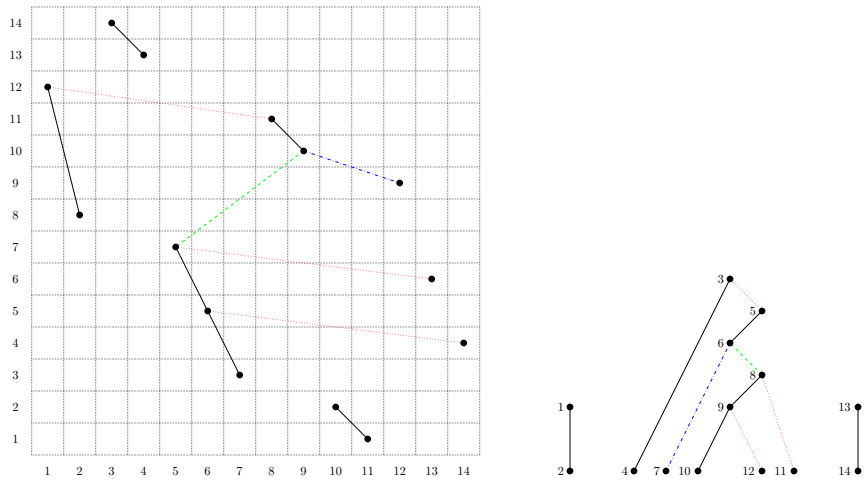


Figure 4: The permutation $\pi = 12\,8\,14\,13\,7\,5\,3\,11\,10\,2\,1\,9\,6\,4$ and the corresponding forest with nodes labeled using (★) order relation.

where $r(a)$ and is defined before Remark 4.2. We extend this order for paths $v_1v_2\cdots$ where v_i and v_{i+1} are nodes of $\chi(\pi)$ connected by a 0-edge: two disjoint paths $v_1v_2v_3\cdots$ and $u_1u_2u_3\cdots$ are compared by their heads, i.e.

$$(v_1v_2v_3\cdots) < (u_1u_2u_3\cdots) \iff v_1 < u_1.$$

Extracting from the forest $\chi(\pi)$ certain subsets of disjoint paths and taking into account the above order relation, we obtain the following.

Remark 4.3 The three statements hold:

- The ith primary run P_i in π (ordered from the top) corresponds to the ith maximal path $v_1v_2v_3\cdots$ of consecutive nodes joined by 0-edges in $\chi(\pi)$ where $r(v_1)$ is even.
- The ith secondary run in π (ordered from the top) corresponds to the ith maximal path $v_1v_2v_3\cdots$ of consecutive isolated nodes joined by 0-edges in $\chi(\pi)$ where $r(v_1)$ is odd.
- The ith break in π (ordered from the top) corresponds to the the ith 0-edge (a, b) in $\chi(\pi)$ such that $r(a)$ is odd and a is isolated.

Consequence Let (a, b) be a pure descent in a primary run of π, and $e = (a, b)$ its associated 0-edge in $\chi(\pi)$. Then, the number of entries of π in the interval (b, a), *i.e.* $a - b - 1$, is equal to the number of nodes in the maximal path of 0-edges starting on the right child of a.

Proposition 4.4 *Let π and π' be two permutations in $S_n(123)$.*
- If π and π' belong to the same equivalence class, then $\chi(\pi) = \chi(\sigma)$.
- If π and π' belong to different equivalence classes, then $\chi(\pi) \neq \chi(\sigma)$.

Proof. As we have seen above, the map β depends only on the set of primary and secondary runs. Thus, our construction applied on two permutations lying in the same class provides the same forest.

Moreover, if two permutations π and π' do not belong to the same class then their sets of primary and secondary runs necessarily differ. Due to the statements of Remark 2 and the above consequence, we deduce easily that $\chi(\pi)$ and $\chi(\pi')$ are different. □

THEOREM 4.5 *The sets $S_n(123)^{\sim}$, $n \geq 1$, are enumerated by the numbers of single-source directed animals on the square lattice (A005773, [19]).*

Proof. Proposition 1 proves that χ is injective. So, it suffices to show the surjectivity of χ, *i.e.*, any forest of ordered binary trees is the image by χ of a permutation avoiding 123.

First, we prove that any binary tree can be obtained from a permutation $\pi \in S_n(123)$ by the above construction. Let T be a binary tree with n nodes. By Remark 2, a maximal path P of nodes connected by 0-edges in T such that $r(P)$ is even corresponds to a primary run of π. Moreover, if a 0-edge e corresponds to a pure descent (a, b) in a primary run of π, $b(e) = a - b - 1$ is the number of nodes in the maximal path (possibly reduced to one node) of 0-edges starting on the right child of a.

Then the primary runs of π are entirely determined by the sequence $b_1 b_2 \cdots b_k$ with $b_1 = n$ and $b_i = b(e)$ where e is the ith 0-edge of T (using the (★) order relation) such that $r(e)$ is even. If p is the number of primary runs and P_i is the ith primary run of π then the sequence $P_1 P_2, \ldots P_p$ is decreasing.

Consequently, secondary runs of π take values from $[n] \setminus \cup_i^p P_i$, and the breaks correspond to the 0-edges $e = (a, b)$ where a is non-isolated and $r(a)$ is odd, which entirely determines secondary runs and breaks. If q is the number of secondary runs and S_i is the ith secondary run (using the (★) order relation) then the sequence $S_1 S_2 \cdots S_q$ is decreasing.

Now we construct a permutation π avoiding 123 by a shuffle of the two decreasing sequences $P_1 P_2 \cdots P_p$ and $S_1 S_2 \cdots S_q$. We read $P_1 S_1 S_2 \cdots S_q$ from left to right, and whenever we meet a break between S_i and S_{i+1} we insert between them the first primary run not yet inserted (this is exactly the correspondence given by the increasing map β defined above). Obviously, the sequence obtained at the end of the process is a permutation avoiding the pattern 123 since it is a shuffle of two decreasing sequences. Finally, the image of π by χ provides the tree T, which means that $\chi(S_n(123))$ contains the set of all ordered binary trees of size n.

So, let us assume that f is a forest of ordered binary trees $T_1, T_2, \ldots, T_k$. For $1 \leq i \leq k$, we construct the permutation π_i from the tree T_i by the previous process, *i.e.*, $\pi_i = \chi^{-1}(T_i)$. Let π be the permutation obtained by the skew sum $\pi_1 \ominus \pi_2 \ominus \ldots \ominus \pi_k$ where $\pi \ominus \pi'$ is the permutation σ such that

$$\sigma(i) = \begin{cases} \pi(i) + m' & \text{for } 1 \leq i \leq m, \\ \pi'(i - m) & \text{for } m + 1 \leq i \leq m + m' \end{cases}$$

where m (resp. m') is the length of π (resp. π').

Now we read $\pi_1 \ominus \pi_2 \ominus \ldots \ominus \pi_k$ from left to right. Whenever a pure descent is created between π_i and π_{i+1}, it is easy to see that π_i is necessarily a decreasing sequence. In this case we permute π_i and the first primary run of π_{i+1}. At the end of the process, the permutation π satisfies $\chi(\pi) = f$. □

For instance, the previous construction applied on the forest illustrated in Figure 4 provides the permutation $\pi = 12\,8\,14\,13\,11\,10\,7\,5\,3\,9\,6\,4\,2\,1$. Indeed, we have $\pi_1 = \chi^{-1}(T_1) = 2\,1$, $\pi_2 = \chi^{-1}(T_2) =$

$10\,6\,9\,8\,5\,3\,1\,7\,4\,2$ and $\pi_3 = \chi^{-1}(T_3) = 2\,1$. Since a pure descent $(13, 12)$ is created in the permutation $\pi_1 \ominus \pi_2 \ominus \pi_3 = \mathbf{14\,13}\,12\,8\,11\,10\,7\,5\,3\,9\,6\,4\,\mathbf{2\,1}$, we permute the two primary runs $12\,8$ and $14\,13$, which gives $\pi = 12\,8\,\mathbf{14\,13}\,11\,10\,7\,5\,3\,9\,6\,4\,2\,1$.

5 Going further

We conclude this paper by giving several open questions and possible research directions.

We experimentally obtained the numbers of classes in $S_n(132)^\sim$ and $S_n(213)^\sim$ for small values of n, $1 \leq n \leq 9$. For $S_n(132)^\sim$, we obtain the sequence $1, 2, 4, 10, 26, 66, 169, 437, 1130$ and for $S_n(213)^\sim$, we obtain the sequence $1, 2, 4, 9, 22, 56, 146, 388, 1048$. The first sequence does not appear in [19], while the second sequence seems to be $A152225$ which corresponds to the number of Dyck paths of semilength n with no peaks at height $0 \mod 3$ and no valleys at height $2 \mod 3$. Is it possible to obtain the generating functions for these sets and to construct a bijection with Dyck paths?

In [2], the authors give a one-to-one correspondence between the set $\mathcal{F}_n$ of forests of ordered trees and the set $S_n(321, 4\bar{1}523)$ that transports various parameters. However, they do not give an interpretation for the number of inversions, the degree of the root less one and the internal-path-length. In Section 2, we exhibit a bijection between $\mathcal{F}_n$ and $S_n(231, \underline{51}\bar{4}23)$, which gives a new set of pattern-avoiding permutations enumerated as the single-source directed animals on the square lattice. This bijection has the advantage that it transports many parameters (see Section 3), and in particular the three previous parameters. Is it possible to give an interpretation of these parameters in term of the single-source directed animals?

In Section 4, we prove that $S_n(123)^\sim$ is enumerated by the number of directed animals (or equivalently directed polyominoes). Is it possible to give an interpretation of the equivalence relation in term of polyominoes?

Acknowledgement. We would like to thank the anonymous referees for their very careful reading of this paper and their helpful comments and suggestions.

References

[1] E. Babson and E. Steingrímsson, *Generalized permutation patterns and a classification of Mahonian statistics*, Sém. Lothar. Combin., 44 (2000) Article B44b.

[2] E. Barcucci, A. Del Lungo, E. Pergola and R. Pinzani, *Directed animals, forests and permutations*, Discrete Math., 204 (1999) 41–71.

[3] J.-L. Baril, T. Mansour and A. Petrossian, *Equivalence classes of permutations modulo excedances*, J. Comb., 5 (2014) 453–469.

[4] J.-L. Baril and A. Petrossian, *Equivalence classes of permutations modulo descents and left-to-right maxima*, Pure Math. Appl. (PU.M.A.), 25 (2015) 19–29.

[5] J.-L. Baril and S. Kirgizov, *The pure descent statistics on permutations*, Discrete Math., 340 (2017) 2550–2558.

[6] F. Bergeron, N. Bergeron, R. B. Howlett and D. E. Taylor, *A decomposition of the descent algebra of a finite Coxeter group*, J. Algebraic Combin., 1 (1992) 23–44.

[7] M. Bóna, *Exact enumeration of* 1342*-avoiding permutations: a close link with labeled trees and planar maps*, J. Combin. Theory Ser. A, 80 (1997) 257–272.

[8] M. Bóna, *Combinatorics of permutations*, Chapman and Hall/CRC, Second Edition, 2012.

[9] M. Bousquet-Mélou, *New enumerative results on two-dimensional directed animals*, Proc. 7th FPSAC, Université de Marne-la-Vallée, Paris, 1995, 103–116.

[10] M. Bousquet-Mélou and S. Butler, *Forest-like permutations*, Ann. Comb., 11 (2007) 335–354.

[11] I. Gessel and G. Viennot, *Binomial determinants, paths, and hook length formulae*, Adv. Math., 58 (1985) 300–321.

[12] I. Gessel, *Symmetric functions and P-recursiveness*, J. Combin. Theory Ser. A, 53 (1990) 257–285.

[13] S. Kitaev, *Patterns in permutations and words*, Springer-Verlag, 2011.

[14] D. E. Knuth, *The art of computer programming - Vol. 1*, Addison-Wesley, Reading MA, 1973, Third edition, 1997.

[15] P. A. MacMahon, *Combinatory Analysis. Vols. 1 and 2*, Cambridge Univ. Press, Cambridge, UK, 1915 (reprinted by Chelsea, New-York, 1955).

[16] T. Mansour, *Combinatorics of Set Partitions*, Chapman & Hall/CRC, 2013.

[17] L. Pudwell, *Enumeration schemes for permutations avoiding barred patterns*, Electron. J. Combin., 17 (2010) #R29.

[18] R. Simion and F. W. Schmidt, *Restricted permutations*, European J. Combin., 6 (1985) 383–406.

[19] N. J. A. Sloane, *On-Line Encyclopedia of Integer Sequences*, Published electronically at http://oeis.org/.

[20] R. P. Stanley, *Enumerative Combinatorics - Vol. 1*, Cambridge University Press, 1997.

[21] J. West, *Permutations with forbidden subsequences and stack-sortable permutations*, Ph.D. Thesis, MIT, 1990.

[22] R. M. Wilson and J. H. Lint, *A course in combinatorics - Vol. I*, Cambridge University Press, 2002.

Discrete Mathematics 341 (2018) 1620–1628

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Dyck paths with a first return decomposition constrained by height

Jean-Luc Baril *, Sergey Kirgizov, Armen Petrossian

LE2I UMR-CNRS 6306, Université de Bourgogne, B.P. 47 870, 21078 DIJON-Cedex, France

ARTICLE INFO

Article history:
Received 6 June 2017
Received in revised form 16 October 2017
Accepted 2 March 2018
Available online 20 March 2018

Keywords:
Enumeration
Dyck and Motzkin paths
First return decomposition
Statistics
Height
Peak

ABSTRACT

We study the enumeration of Dyck paths having a first return decomposition with special properties based on a height constraint. We exhibit new restricted sets of Dyck paths counted by the Motzkin numbers, and we give a constructive bijection between these objects and Motzkin paths. As a byproduct, we provide a generating function for the number of Motzkin paths of height k with a flat (resp. with no flats) at the maximal height.
© 2018 Elsevier B.V. All rights reserved.

1. Introduction and notations

A *Dyck path* of semilength $n \geq 0$ is a lattice path starting at $(0, 0)$, ending at $(2n, 0)$, and never going below the x-axis, consisting of up steps $U = (1, 1)$ and down steps $D = (1, -1)$. Let $\mathcal{D}_n$, $n \geq 0$, be the set of all Dyck paths of semilength n, and let $\mathcal{D} = \cup_{n\geq 0}\mathcal{D}_n$. The cardinality of $\mathcal{D}_n$ is given by the nth Catalan number, which is the general term $\frac{1}{n+1}\binom{2n}{n}$ of the sequence A000108 in the On-line Encyclopedia of Integer Sequences of N.J.A. Sloane [17]. A large number of various classes of combinatorial objects are enumerated by the Catalan numbers such as planar trees, Young tableaux, stack sortable permutations, Dyck paths, and so on. A list of over 60 types of such combinatorial classes has been compiled by Stanley [18]. In combinatorics, many papers deal with Dyck paths. Most of them consist in enumerating Dyck paths according to several parameters, such as length, number of peaks or valleys, number of double rises, number of returns to the x-axis (see for instance [2,8–16,19]). Other studies investigate restricted classes of Dyck paths avoiding some patterns or having a specific structure. For instance, Barcucci et al. [1] consider non-decreasing Dyck paths which are those having a non-decreasing sequence of heights of valleys (see also [6,7]), and it is well known [5] that Dyck paths avoiding the triple rise UUU are enumerated by the Motzkin numbers (see A001006 in [17]).

Any non-empty Dyck path $P \in \mathcal{D}$ has a unique first return decomposition [8] of the form $P = U\alpha D\beta$ where α and β are two Dyck paths in $\mathcal{D}$. See Fig. 1 for an illustration of this decomposition.

Based on this decomposition, we construct a new collection of subsets of $\mathcal{D}$ as follows. Given a function $s : \mathcal{D} \to \mathbb{N}$, called *statistic*, and a comparison operator $\diamond$ on $\mathbb{N}$ (for instance $\geq$ or $>$), the set $\mathcal{D}^{s,\diamond}$ is the union of the empty Dyck path with all Dyck paths P having a first return decomposition $P = U\alpha D\beta$ satisfying the conditions:

$$\begin{cases} \alpha, \beta \in \mathcal{D}^{s,\diamond}, \\ s(U\alpha D) \diamond s(\beta). \end{cases} \tag{1}$$

* Corresponding author.
E-mail addresses: barjl@u-bourgogne.fr (J.-L. Baril), sergey.kirgizov@u-bourgogne.fr (S. Kirgizov), armen.petrossian@u-bourgogne.fr (A. Petrossian).

https://doi.org/10.1016/j.disc.2018.03.002
0012-365X/© 2018 Elsevier B.V. All rights reserved.

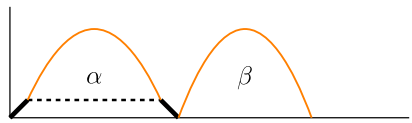


Fig. 1. First return decomposition $U\alpha D\beta$ of a Dyck path $P \in \mathcal{D}$.

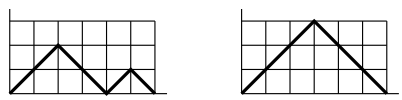


Fig. 2. The two Dyck paths in $\mathcal{D}_3^{h,>}$.

Table 1
Cardinalities of $\mathcal{D}_n^{h,\diamond}$ according to the $\diamond$-constraint.

$\diamond$-constraint	Sequence	OEIS	a_n, $1 \le n \le 9$
$h(U\alpha D) > h(\beta)$		A045761	1, 1, 2, 3, 6, 12, 24, 50, 107
$h(U\alpha D) \ge h(\beta)$	Motzkin	A001006	1, 2, 4, 9, 21, 51, 127, 323, 835

For $n \ge 0$, we denote by $\mathcal{D}_n^{s,\diamond}$ the set of Dyck paths of semilength n in $\mathcal{D}^{s,\diamond}$. Thus, we have $\mathcal{D}^{s,\diamond} = \bigcup_{n\ge 0}\mathcal{D}_n^{s,\diamond}$.

For instance, if the operator $\diamond$ is $=$ and s is a constant statistic (*i.e.*, $s(P) = 0$ for any $P \in \mathcal{D}$), then we obviously have $\mathcal{D}_n^{s,\diamond} = \mathcal{D}_n$ for $n \ge 0$.

If s is the number of returns (*i.e.*, $s(P)$ is the number of down steps D that return the path P to the x-axis) and $s(U\alpha D)\diamond s(\beta)$ is $s(U\alpha D) \ge s(\beta)$, then it is straightforward to see that $\mathcal{D}_n^{s,\ge}$ consists of Dyck paths built over the grammar $S \to \epsilon \mid USD \mid USDUSD$. So, the generating function $S(x)$ for the cardinalities of $\mathcal{D}_n^{s,\ge}$, $n \ge 0$, satisfies the functional equation $S(x) = 1 + xS(x) + x^2S(x)^2$. The solution of this equation is the well-known generating function for the Motzkin numbers (A001006 in [17]).

In this paper, we focus on the sets $\mathcal{D}^{h,\diamond}$ where the statistic h is the maximal height of a Dyck path, *i.e.*, $h(P)$ is the maximal ordinate reached by the path P.

In Section 2, we deal with the case in which operator $\diamond$ is a strict inequality $>$. We prove that the cardinalities of the sets $\mathcal{D}_n^{h,>}$, $n \ge 0$, are given by the sequence A045761 in [17]. This sequence corresponds to the coefficients of the series $\lim_{k\to\infty}P_k(x)$ where $P_k(x)$ is a polynomial recursively defined by $P_0(x) = x$, $P_1(x) = x^2$, $P_k(x) = P_{k-1}(x) + P_{k-2}(x)$ if k is even, and $P_k(x) = P_{k-1}(x) \cdot P_{k-2}(x)$ if k is odd.

In Section 3, we focus on the set $\mathcal{D}^{h,\ge}$ where $h(U\alpha D) \ge h(\beta)$ (the operator $\diamond$ is $\ge$). Using generating functions, we prove that the cardinalities of the $\mathcal{D}_n^{h,\ge}$, $n \ge 0$, are given by the Motzkin numbers (A001006 in [17]). Moreover, we give a constructive one-to-one correspondence ϕ between Dyck paths in $\mathcal{D}_n^{h,\ge}$ and Motzkin paths of length n. Also, we show how ϕ transforms peaks UD into peaks UD and flats F in Motzkin paths. Finally, we deduce generating function for the total number of peaks in $\mathcal{D}_n^{h,\ge}$.

Table 1 presents the two main enumerative results of $\mathcal{D}_n^{h,\diamond}$ obtained in Sections 2 and 3.

2. Enumeration of $\mathcal{D}_n^{h,>}$

In this section, we enumerate the set $\mathcal{D}_n^{h,>}$ of Dyck paths of semilength $n \ge 0$ with a first return decomposition satisfying $h(U\alpha D) > h(\beta)$ where h is the maximal height of a Dyck path. For instance, we have $\mathcal{D}_1^{h,>} = \{UD\}$, $\mathcal{D}_2^{h,>} = \{UUDD\}$, and $\mathcal{D}_3^{h,>} = \{UUDDUD, UUUDDD\}$ (see Fig. 2).

Let $A_k(x) = \sum_{n\ge 0}a_{n,k}x^n$ (resp. $B_k(x) = \sum_{n\ge 0}b_{n,k}x^n$) be the generating function where the coefficient $a_{n,k}$ (resp. $b_{n,k}$) is the number of Dyck paths in $\mathcal{D}_n^{h,>}$ with a maximal height equal to k (resp. of at most k). So, we have $B_k(x) = \sum_{i=0}^{k}A_i(x)$ and the generating function $B(x)$ for the set $\mathcal{D}^{h,>}$ is given by $B(x) = \lim_{k\to+\infty}B_k(x)$.

Any Dyck path of height k in $\mathcal{D}^{h,>}$ is either empty, or of the form $U\alpha D\beta$ where α is a Dyck path in $\mathcal{D}^{h,>}$ of height $k-1$ and $\beta \in \mathcal{D}^{h,>}$ such that $h(\beta) \le k-1$. So, we deduce easily the following functional equations:

$$\begin{cases} A_0(x) &= B_0(x) = 1, \\ A_k(x) &= xA_{k-1}(x) \cdot B_{k-1}(x) \text{ for } k \ge 1. \end{cases} \tag{2}$$

Theorem 1. *We have for $k \ge 0$,*

$$B_k(x) = \frac{P_{2k}(x)}{x}$$

Table 2
Number $a_{n,k}$ of Dyck paths of height k in $\mathcal{D}_n^{h,>}$, $1 \le n \le 10$ and $1 \le k \le 9$.

k	n									
	1	2	3	4	5	6	7	8	9	10
1	1									
2		1	1							
3			1	2	2	2	1			
4				1	3	5	8	11	13	15
5					1	4	9	18	33	56
6						1	5	14	33	71
7							1	6	20	54
8								1	7	27
9									1	...
$\sum$	1	1	2	3	6	12	24	50	107	...

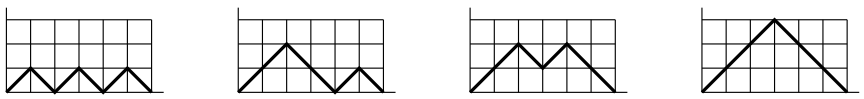


Fig. 3. The four Dyck paths in $\mathcal{D}_3^{h,\ge}$.

where P_k is the polynomial recursively defined by $P_0(x) = x$, $P_1(x) = x^2$, $P_{2k}(x) = P_{2k-1}(x) + P_{2k-2}(x)$ and $P_{2k+1}(x) = P_{2k}(x) \cdot P_{2k-1}(x)$. As consequence, we have for $k \ge 1$

$$A_k(x) = \frac{P_{2k}(x) - P_{2k-2}(x)}{x},$$

and $B(x)$ is generating function of the sequence A045761 *in* [17].

Proof. We proceed by induction on k. For $k = 0$, the property holds since $P_0(x) = x = xB_0(x)$. Assuming the property for $0 \le i \le k-1$, we prove it for k. Taking into account that $A_k(x) = B_k(x) - B_{k-1}(x)$ in Eq. (2), we obtain

$$xB_k(x) \quad = xB_{k-1}(x) + x^2 B_{k-1}(x)\big(B_{k-1}(x) - B_{k-2}(x)\big).$$

With the recurrence hypothesis, we have

$$\begin{aligned} xB_k(x) \quad &= P_{2k-2}(x) + P_{2k-2}(x)\big(P_{2k-2}(x) - P_{2k-4}(x)\big) \\ &= P_{2k-2}(x) + P_{2k-2}(x)P_{2k-3}(x) \\ &= P_{2k-2}(x) + P_{2k-1}(x) = P_{2k}(x). \end{aligned}$$

An induction on k completes the proof and the expression of $A_k(x)$ is deduced from $A_k(x) = B_k(x) - B_{k-1}(x)$. □

For instance, we have $B_2(x) = 1 + x + x^2 + x^3$, $B_3(x) = 1 + x + x^2 + 2x^3 + 2x^4 + 2x^5 + 2x^6 + x^7$, and the first ten terms of $B(x)$ are $1 + x + x^2 + 2x^3 + 3x^4 + 6x^5 + 12x^6 + 24x^7 + 50x^8 + 107x^9$. We refer to Table 2 for an overview of the coefficients $a_{n,k}$ for $1 \le n \le 10$ and $1 \le k \le 9$.

Notice that the family of sets $\mathcal{D}^{h,>}$, $n \ge 1$, seems to be the first example of combinatorial objects enumerated by the sequence A045761 in [17].

3. Enumeration of $\mathcal{D}_n^{h,\ge}$

3.1. Enumeration using generating functions

In this section, we enumerate the set $\mathcal{D}_n^{h,\ge}$ of Dyck paths of semilength $n \ge 0$ with a first return decomposition satisfying $h(U\alpha D) \ge h(\beta)$ where $\alpha, \beta \in \mathcal{D}^{h,\ge}$ and h is the maximal height of a Dyck path. For instance, we have $\mathcal{D}_1^{h,\ge} = \{UD\}$, $\mathcal{D}_2^{h,\ge} = \{UDUD, UUDD\}$, and $\mathcal{D}_3^{h,\ge}$ consists of the four Dyck paths $UDUDUD$, $UUDDUD$, $UUDUDD$ and $UUUDDD$ (see Fig. 3).

Let $C_k(x) = \sum_{n\ge 0} c_{n,k}x^n$ (resp. $D_k(x) = \sum_{n\ge 0} d_{n,k}x^n$) be the generating function where the coefficient $c_{n,k}$ (resp. $d_{n,k}$) is the number of Dyck paths in $\mathcal{D}_n^{h,\ge}$ with a maximal height equal to k (resp. of at most k). So, we have $D_k(x) = \sum_{i=0}^{k} C_i(x)$ and the generating function $D(x)$ for the set $\mathcal{D}_n^{h,\ge}$ is given by $D(x) = \lim_{k\to+\infty} D_k(x)$.

Any Dyck path of height k in $\mathcal{D}^{h,\ge}$ is either empty, or of the form $U\alpha D\beta$ where α (resp. β) is a Dyck path in $\mathcal{D}^{h,\ge}$ of height $k-1$ (resp. of height at most k). So, we deduce the following functional equations:

$$\begin{cases} C_0(x) & = D_0(x) = 1, \\ C_k(x) & = xC_{k-1}(x) \cdot D_k(x) \text{ for } k \ge 1. \end{cases} \tag{3}$$

Theorem 2 provides recursive expressions for the two generating functions $D_k(x)$ and $C_k(x)$. As a consequence, $D(x)$ can be expressed as an infinite product of terms $\frac{1}{1-xC_k(x)}$.

Theorem 2. *We have $D_0(x) = C_0(x) = 1$, $C_1(x) = \frac{x}{1-x}$ and*

$$D_k(x) = \prod_{i=0}^{k-1}\big(1 - xC_i(x)\big)^{-1} \text{ for } k \geq 1,$$

$$C_k(x) = \frac{C_1(x)^k}{\prod_{i=1}^{k-1}\big(1 - xC_i(x)\big)^{k-i}} \text{ for } k \geq 2,$$

and

$$D(x) = \prod_{i=0}^{\infty}\big(1 - xC_i(x)\big)^{-1}.$$

Proof. Since we have $C_k(x) = D_k(x) - D_{k-1}(x)$, Eq. (3) implies

$$D_k(x) = \frac{D_{k-1}(x)}{1 - xC_{k-1}(x)},$$

and starting from $D_0(x) = 1$, a straightforward induction on k provides $D_k(x) = \prod_{i=0}^{k-1}\big(1 - xC_i(x)\big)^{-1}$. Moreover, from Eq. (3) and $C_1(x) = \frac{x}{1-x}$, we deduce

$$\frac{C_k(x)}{C_{k-1}(x)} = xD_k(x) = C_1(x)\prod_{i=1}^{k-1}\big(1 - xC_i(x)\big)^{-1}.$$

An induction on k completes the proof. □

Lemma 1. *For $k \geq 1$, we have $D_{k-1}(x) = \frac{D_k(x)-1}{x(xD_k(x)+1)}$.*

Proof. We proceed by induction on k. Since $D_0(x) = 1$ and $D_1(x) = \frac{1}{1-x}$, it is easy to check that $D_0(x) = \frac{D_1(x)-1}{x(xD_1(x)+1)}$.

Assuming $D_{i-1}(x) = \frac{D_i(x)-1}{x(xD_i(x)+1)}$ for $1 \leq i \leq k-2$, we prove the result for $i = k-1$. From Eq. (3) and the recurrence hypothesis on $D_{k-2}(x)$ we obtain

$$D_k(x) = \frac{D_{k-1}(x)}{1 - x\left(D_{k-1}(x) - \frac{D_{k-1}(x)-1}{x\left(xD_{k-1}(x)+1\right)}\right)}.$$

Isolating $D_{k-1}(x)$, we obtain $D_{k-1}(x) = \frac{D_k(x)-1}{x(xD_k(x)+1)}$ which completes the induction. □

Theorem 3. *The sets $\mathcal{D}_n^{h,\geq}$, $n \geq 0$, are enumerated by the Motzkin numbers.*

Proof. Lemma 1 induces $x^2D_k(x)D_{k-1}(x) + xD_{k-1}(x) = D_k(x) - 1$ for $k \geq 1$. By taking the limits when k converges to infinity, one gets $x^2D(x)^2 + (x-1)D(x) + 1 = 0$, which is the functional equation of the generating function of Motzkin numbers. □

Now, we will show how $D_k(x)$, $k \geq 0$, can be expressed as a closed form. For this, let us define the function

$$f : u \mapsto \frac{1}{1 - \frac{x}{1-\frac{x}{1-x^2u}}}.$$

For $n \geq 1$, we denote by f^n the function recursively defined by $f^n(u) = f\big(f^{n-1}(u)\big)$ anchored with $f^0(u) = u$. A simple calculation (using Maple for instance) proves that the map f satisfies Remark 1.

Remark 1. If $X = \frac{Y-1}{x(xY+1)}$ then the map f satisfies

$$f\left(\frac{Y}{1 - x(Y - X)}\right) = \frac{f(Y)}{1 - x\big(f(Y) - f(X)\big)}.$$

Table 3
Number $c_{n,k}$ of Dyck paths of height k in $\mathcal{D}_n^{h,\geq}$, $1 \leq n \leq 10$ and $1 \leq k \leq 9$.

k	n									
	1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1	1
2		1	2	4	7	12	20	33	54	88
3			1	3	8	19	43	94	201	423
4				1	4	13	37	99	254	634
5					1	5	19	62	187	536
6						1	6	26	95	316
7							1	7	34	137
8								1	8	43
9									1	...
$\sum$	1	2	4	9	21	51	127	323	835	...

Theorem 4. *For $k \geq 0$, we have*

$$D_k(x) = f^{\lfloor \frac{k}{4} \rfloor}\big(D_{k \bmod 4}(x)\big)$$

with the initial cases $D_0(x) = 1$, $D_1(x) = \frac{1}{1-x}$, $D_2(x) = \frac{1}{1-x-x^2}$ and $D_3(x) = \frac{1}{1-\frac{x}{1-x}}$. □

Proof. We proceed by induction on k. Since we have $D_0(x) = f^0\big(D_0(x)\big)$, $D_1(x) = f^0\big(D_1(x)\big)$, $D_2(x) = f^0\big(D_2(x)\big)$ and $D_3(x) = f^0\big(D_3(x)\big)$, the basic case holds.

Assuming the result for $0 \leq i \leq k$, we prove it for $k+1$.

From Eq. (3) we have $D_{k+1}(x) = \frac{D_k(x)}{1-x\left(D_k(x)-D_{k-1}(x)\right)}$. Using the recurrence hypothesis for $D_k(x)$ and $D_{k-1}(x)$, we obtain for $k \geq 4$:

$$\begin{aligned} D_{k+1}(x) &= \frac{f^{\lfloor \frac{k}{4} \rfloor}\big(D_{k \bmod 4}(x)\big)}{1 - x\left(f^{\lfloor \frac{k}{4} \rfloor}\big(D_{k \bmod 4}(x)\big) - f^{\lfloor \frac{k-1}{4} \rfloor}\big(D_{k-1 \bmod 4}(x)\big)\right)} \\ &= \frac{f\left(f^{\lfloor \frac{k-4}{4} \rfloor}\big(D_{k-4 \bmod 4}(x)\big)\right)}{1 - x\left(f\left(f^{\lfloor \frac{k-4}{4} \rfloor}(D_{k-4 \bmod 4}(x))\right) - f\left(f^{\lfloor \frac{k-5}{4} \rfloor}(D_{k-5 \bmod 4}(x))\right)\right)}. \end{aligned}$$

The recurrence hypothesis implies

$$D_{k+1}(x) = \frac{f(D_{k-4}(x))}{1 - x\big(f(D_{k-4}(x)) - f(D_{k-5}(x))\big)},$$

and using Remark 1, we deduce

$$\begin{aligned} D_{k+1}(x) &= f\left(\frac{D_{k-4}(x)}{1 - x\big(D_{k-4}(x) - D_{k-5}(x)\big)}\right) = f(D_{k-3}(x)) \\ &= f^{1+\lfloor \frac{k-3}{4} \rfloor}(D_{k-3 \bmod 4}(x)) = f^{\lfloor \frac{k+1}{4} \rfloor}(D_{k+1 \bmod 4}(x)). \end{aligned}$$

The induction is completed. □

For instance, we have $D_{13}(x) = \frac{1-6x+10x^2-9x^4+2x^5+x^6}{1-7x+15x^2-5x^3-15x^4+9x^5+3x^6-x^7}$ and the first terms of its Taylor expansion are $1 + x + 2x^2 + 4x^3 + 9x^4 + 21x^5 + 51x^6 + 127x^7 + 323x^8 + 835x^9 + 2188x^{10} + 5798x^{11} + 15511x^{12} + 41835x^{13} + 113633x^{14} + 310557x^{15} + 853333x^{16}$. See Table 3.

Notice that, by taking limits in Theorem 4, we retrieve the continued fraction of the Motzkin numbers (P. Barry [3])

$$D(x) = \cfrac{1}{1 - \cfrac{x}{1 - \cfrac{x}{1 - \cfrac{x^2}{1 - \cfrac{x}{1 - \cfrac{x}{1 - \cfrac{x^2}{1 - \cfrac{x}{\cdots}}}}}}}}.$$

3.2. A constructive bijection

In this section, we exhibit a constructive bijection between $\mathcal{D}_n^{h,\geq}$ and the set $\mathcal{M}_n$ of Motzkin paths of length n, *i.e.*, lattice paths starting at $(0, 0)$, ending at $(n, 0)$, never going below the x-axis, consisting of up-steps $U = (1, 1)$, down steps $D = (1, -1)$ and flat steps $F = (1, 0)$. We set $\mathcal{M} = \cup_{n \geq 0} \mathcal{M}_n$.

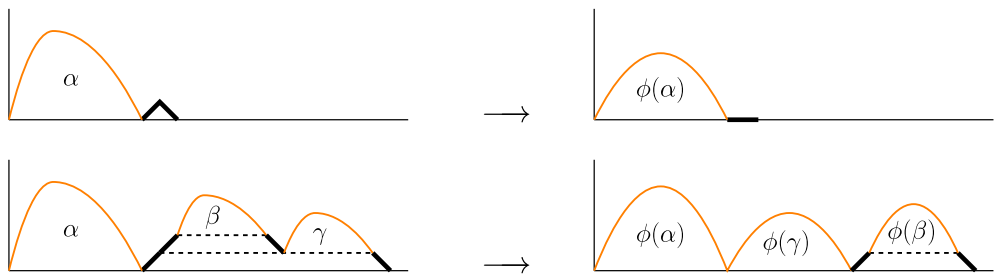


Fig. 4. Illustration of the bijection between $\mathcal{D}_n^{h,\geq}$ and $\mathcal{M}_n$.

Let us define recursively the map ϕ from $\mathcal{D}^{h,\geq}$ to $\mathcal{M}$ as follows. For $P \in \mathcal{D}^{h,\geq}$, we set

$$\phi(P) = \begin{cases} \epsilon & \text{if } P = \epsilon, \\ \phi(\alpha)F & \text{if } P = \alpha UD, \\ \phi(\alpha)\phi(\gamma)U\phi(\beta)D & \text{if } P = \alpha UU\beta D\gamma D. \end{cases}$$

Due to the recursive definition, the image by ϕ of a Dyck path of semilength n is a Motzkin path of length n. For instance, the images of *UDUDUD*, *UUDDUD*, *UUDUDD*, *UUUDDD*, *UUUUDDDDUUUDDUDD* are respectively *FFF*, *UDF*, *FUD*, *UFD* and *UUDDFUFD*. We refer to Fig. 4 for an illustration of this mapping.

Moreover, we easily deduce the two following facts.

Fact 1. *If $\alpha, \beta \in \mathcal{D}_n^{h,\geq}$ and $\alpha\beta \in \mathcal{D}_n^{h,\geq}$, then we have $\phi(\alpha\beta) = \phi(\alpha)\phi(\beta)$.*

Fact 2. *We say that a peak UD is odd whenever the maximal sub-path of the form U^kD that contains it has an odd number of U-steps, i.e., k is odd. Then, ϕ transforms peaks (resp. odd peaks) of paths of $\mathcal{D}^{h,\geq}$ into peaks and flats (resp. flats) of Motzkin paths.*

Theorem 5. *The map $\phi : \mathcal{D}_n^{h,\geq} \to \mathcal{M}_n$ defined above is a bijection satisfying for any $P \in \mathcal{D}_n^{h,\geq}$,*

$$h(\phi(P)) = \left\lfloor \frac{h(P)}{2} \right\rfloor.$$

Proof. We proceed by induction on n. Obviously, for $n = 1$, we have $\phi(UD) = F$ and $h(F) = 0 = \lfloor \frac{h(UD)}{2} \rfloor$. For $k \leq n$, we assume that ϕ is a bijection from $\mathcal{D}_k^{h,\geq}$ to $\mathcal{M}_k$ such that $h(\phi(P)) = \lfloor \frac{h(P)}{2} \rfloor$ for any $P \in \mathcal{D}_k^{h,\geq}$, and we prove the result for $n+1$.

Using the enumerative result of Theorem 3, it suffices to prove that ϕ is surjective from $\mathcal{D}_{n+1}^{h,\geq}$ to $\mathcal{M}_{n+1}$. So, let M be a Motzkin path in $\mathcal{M}_{n+1}$. We distinguish two cases: (i) $M = \sigma F$ with $\sigma \in \mathcal{M}_n$, and (ii) $M = \sigma U\pi D$ where σ and π are two Motzkin paths in $\mathcal{M}$.

(i) Using the recurrence hypothesis, there is $P \in \mathcal{D}_n^{h,\geq}$ such that $\sigma = \phi(P)$ and $h(\sigma) = \lfloor \frac{h(P)}{2} \rfloor$. So, the Dyck path *PUD* belongs to $\mathcal{D}_{n+1}^{h,\geq}$ and $\phi(PUD) = \sigma F$ which proves that M belongs to the image by ϕ of $\mathcal{D}_{n+1}^{h,\geq}$. Moreover we have $h(\phi(PUD)) = h(\sigma F) = h(\sigma) = \lfloor \frac{h(P)}{2} \rfloor = \lfloor \frac{h(PUD)}{2} \rfloor$.

(ii) We suppose $M = \sigma U\pi D$. Let us define the longest suffix σ_s of σ (possibly empty) such that $\sigma_s \in \mathcal{M}$ and $h(\phi^{-1}(\sigma_s)) \leq 1 + h(\phi^{-1}(\pi))$ (σ_s exists since the empty path ϵ satisfies the inequality, and the recurrence hypothesis ensures the existence and the uniqueness of $\phi^{-1}(\sigma_s)$ and $\phi^{-1}(\pi)$). Let σ_r be the Motzkin path obtained from σ by deleting the suffix σ_s, and let $S \in \mathcal{D}_n^{h,\geq}$ (resp. $R \in \mathcal{D}_n^{h,\geq}$) such that $\phi(S) = \sigma_s$ and $h(\sigma_s) = \lfloor \frac{h(S)}{2} \rfloor$ (resp. $\phi(R) = \sigma_r$ and $h(\sigma_r) = \lfloor \frac{h(R)}{2} \rfloor$). Also there is $T \in \mathcal{D}_n^{h,\geq}$ such that $\phi(T) = \pi$ with $h(\pi) = \lfloor \frac{h(T)}{2} \rfloor$.

Due to the maximality of σ_s, σ_r is either empty or $1 + h(T) < h(\phi^{-1}(\sigma_r\sigma_s))$. Using Fact 1 we obtain $h(\phi^{-1}(\sigma_r\sigma_s)) = h(RS) = h(R)$, and the last inequality can be written as $1 + h(T) < h(R)$.

- If $\sigma_r = \epsilon$ then the condition $h(S) \leq 1 + h(T)$ implies that *UUTDSD* belongs to $\mathcal{D}_n^{h,\geq}$ and we have $\phi(UUTDSD) = \phi(S)U\phi(T)D = \sigma_s U\pi D = \sigma U\pi D$. Moreover, we have $h(\phi(UUTDSD)) = h(\phi(S)U\phi(T)D) = \max\{\phi(S), 1 + \phi(T)\} = \max\{\lfloor \frac{h(S)}{2} \rfloor, 1 + \lfloor \frac{h(T)}{2} \rfloor\}$. With $h(S) \leq 1 + h(T)$, we deduce $h(\phi(UUTDSD)) = 1 + \lfloor \frac{h(T)}{2} \rfloor = \lfloor \frac{h(T)+2}{2} \rfloor = \lfloor \frac{h(UUTDSD)}{2} \rfloor$ as desired.
- If $\sigma_r \neq \epsilon$ then we have $h(S) \leq 1 + h(T) < h(R)$ which implies that the Dyck path *RUUTDSD* belongs to $\mathcal{D}_{n+1}^{h,\geq}$. Moreover, we have $h(\phi(RUUTDSD)) = h(\phi(R)\phi(S)U\phi(T)D) = \max\{\phi(R), \phi(S), 1 + \phi(T)\} = \max\{\lfloor \frac{h(R)}{2} \rfloor, \lfloor \frac{h(S)}{2} \rfloor, 1 + \lfloor \frac{h(T)}{2} \rfloor\}$. From $h(S) \leq 1 + h(T) < h(R)$, we obtain $\lfloor \frac{h(S)+1}{2} \rfloor \leq \lfloor \frac{h(T)+2}{2} \rfloor \leq \lfloor \frac{h(R)}{2} \rfloor$ which induces that $h(\phi(RUUTDSD)) = \lfloor \frac{h(R)}{2} \rfloor = \lfloor \frac{h(RUUTDSD)}{2} \rfloor$ as desired.

The induction is completed. □

Corollary 1. *Let P be a Dyck path in $\mathcal{D}_n^{h,\geq}$, $n \geq 1$. The Motzkin path $\phi(P)$ contains a flat step F at height $h(\phi(P))$ if and only if $h(P)$ is odd.*

Proof. We proceed by induction on n. For $n = 1$, we have $P = UD$ and $\phi(P) = F$ and the result holds since $h(P) = 1$ is odd. Assuming the result for $i \leq n$, we prove it for $n + 1$. We distinguish two cases: (i) $P = \alpha UD$ and (ii) $P = \alpha UU\beta D\gamma D$, where α, β and γ belong to $\mathcal{D}^{h,\geq}$.

(i) For $\alpha \neq \epsilon$, the recurrence hypothesis means that $\phi(\alpha)$ contains a flat step F at height $h(\phi(\alpha))$ if and only if $h(\alpha)$ is odd. Since we have $\phi(P) = \phi(\alpha)F$ and $h(P) = h(\alpha)$, $\phi(P)$ contains a flat step F a height $h(\phi(P)) = h(\phi(\alpha))$ if and only if $h(P)$ is odd.

(ii) If $P = \alpha UU\beta D\gamma D$ then we have $h(\alpha) \geq 2 + h(\beta) \geq 1 + h(\gamma)$ and $\phi(P) = \phi(\alpha)\phi(\gamma)U\phi(\beta)D$.

- As we have done in (i), a flat step F appears at height $h(\phi(P)) = h(\phi(\alpha))$ in $\phi(\alpha)$ if and only if $h(\alpha) = h(P)$ is odd.
- $\phi(P)$ contains a flat step at height $h(\phi(P))$ in $\phi(\beta)$ if and only if a flat step appears at height $h(\phi(\beta))$ in $\phi(\beta)$ and $h(\phi(P)) = h(\phi(\beta)) + 1$, *i.e.*, $\lfloor \frac{h(P)}{2} \rfloor = \lfloor \frac{h(\beta)}{2} \rfloor + 1$. Using the recurrence hypothesis, this is equivalent to $h(\beta)$ is odd, which means $2\lfloor \frac{h(P)}{2} \rfloor = h(\beta) + 1$. Since $h(P) \geq h(\beta) + 2$, this is equivalent to $h(P) = h(\beta) + 2$ and thus $h(P)$ is odd.
- Let us prove by contradiction that a flat step cannot occur at height $h(\phi(P))$ in $\phi(\gamma)$. Indeed, this should imply the following:

  $$h(\phi(P)) = h(\phi(\gamma)) \geq 1 + h(\phi(\beta)).$$

  With Theorem 5 we obtain

  $$\left\lfloor \frac{h(\gamma)}{2} \right\rfloor \geq 1 + \left\lfloor \frac{h(\beta)}{2} \right\rfloor.$$

  Due to the recurrence hypothesis, $h(\gamma)$ is odd, so

  $$\left\lfloor \frac{1 + h(\gamma)}{2} \right\rfloor - 1 \geq \left\lfloor \frac{2 + h(\beta)}{2} \right\rfloor, \text{ and } \left\lfloor \frac{h(U\gamma D)}{2} \right\rfloor > \left\lfloor \frac{h(UU\beta DD)}{2} \right\rfloor,$$

  which implies

  $$h(U\gamma D) > h(UU\beta DD).$$

  This last inequality contradicts $P \in \mathcal{D}^{h,\geq}$.

The induction is completed. □

As a byproduct, we derive in Corollary 2 three generating functions for the number of Motzkin paths of height k according to a constraint on the maximal height of a flat. The two first results do not seem to appear in the literature, while the third one is presented in [4]. See Tables 4 and 5 for numerical data.

Corollary 2. *The generating function $\overline{M}_k(x)$ where the coefficient of x^n is the number Motzkin paths in $\mathcal{M}_n$ of height exactly k and where there is a flat F at height k is*

$$\overline{M}_k(x) = D_{2k+1}(x) - D_{2k}(x).$$

The generating function $\widehat{M}_k(x)$ where the coefficient of x^n is the number Motzkin paths in $\mathcal{M}_n$ of height exactly k and where there is no flat F at height k is

$$\widehat{M}_k(x) = D_{2k}(x) - D_{2k-1}(x).$$

The generating function $M_k(x)$ where the coefficient of x^n is the number Motzkin paths in $\mathcal{M}_n$ of height exactly k is

$$M_k(x) = D_{2k+1}(x) - D_{2k-1}(x).$$

Proof. The proof is directly deduced from Theorem 5 and Corollary 1. □

For instance, we have $\overline{M}_1(x) = \frac{x^3}{(1-2x)(1-x-x^2)}$, and the coefficient of x^n is given by $2^n - F(n+2)$ where $F(n)$ is the nth Fibonacci number. The first terms of its Taylor expansion are $x^3 + 3x^4 + 8x^5 + 19x^6 + 43x^7 + 94x^8 + 201x^9 + 423x^{10} + 880x^{11}$ (see sequence A008466 in [17]). Also, $\widehat{M}_1(x) = \frac{x^2}{(1-x-x^2)(1-x)}$ and the coefficient of x^n is $F(n) - 1$. The first terms of its Taylor expansion are $x^2 + 2x^3 + 4x^4 + 7x^5 + 12x^6 + 20x^7 + 33x^8 + 54x^9 + 88x^{10} + 143x^{11}$ (see sequence A000071 in [17]). Notice that the two sequences defined by $\sum_{k\geq 0}\overline{M}_k(x)$ and $\sum_{k\geq 0}\widehat{M}_k(x)$ do not appear in [17]. We leave open the question of finding closed forms for their generating functions.

Corollary 3. *The bivariate generating function $Q(x, y)$ where the coefficient of $x^n y^k$ is the number of Dyck paths of $\mathcal{D}_n^{h,\geq}$ containing exactly k peaks UD is*

$$\frac{1 - x^2y + x^2 - xy - \sqrt{x^4y^2 - 2x^4y + 2x^3y^2 + x^4 - 2x^3y + x^2y^2 - 2x^2y - 2x^2 - 2xy + 1}}{2x^2}.$$

Table 4
Number of Motzkin paths of length n and height k with a flat at height k, $1 \le n \le 10$ and $0 \le k \le 4$.

k	n									
	1	2	3	4	5	6	7	8	9	10
0	1	1	1	1	1	1	1	1	1	1
1			1	3	8	19	43	94	201	423
2					1	5	19	62	187	536
3							1	7	34	137
4									1	9
$\sum$	1	1	2	4	10	25	64	164	424	1106

Table 5
Number of Motzkin paths of length n and height k with no flats at height k, $1 \le n \le 10$ and $1 \le k \le 5$.

k	n									
	1	2	3	4	5	6	7	8	9	10
1		1	2	4	7	12	20	33	54	88
2				1	4	13	37	99	254	634
3						1	6	26	95	316
4								1	8	43
5										1
$\sum$	0	1	2	5	11	26	63	159	411	1082

Proof. By Fact 2, the number of peaks UD in Dyck paths of $\mathcal{D}_n^{h,\ge}$ is equal to the number of peaks UD and flats F in Motzkin paths in $\mathcal{M}_n$. Since a non empty Motzkin path in $\mathcal{M}$ is either αF or $\alpha U \beta D$ where $\alpha, \beta \in \mathcal{M}$, the generating function $Q(x, y)$ satisfies the functional equation $Q(x, y) = 1 + xyQ(x, y) + x^2yQ(x, y) + x^2(Q(x, y) - 1)Q(x, y)$. A simple calculation (with Maple for instance) completes the proof. □

It is interesting to notice that $Q(x, y) - 1 = yQ'(x, y)$ where $Q'(x, y)$ is the bivariate generating function where the coefficient of $x^n y^k$ is the number of Motzkin n-paths with k weak valleys (see A110470 in [17]).

As a consequence, we deduce in Corollary 4 the popularity of peaks in $\mathcal{D}_n^{h,\ge}$.

Corollary 4. *The generating function where the coefficient of x^n is the total number of peaks in all Dyck paths of $\mathcal{D}_n^{h,\ge}$ is*

$$\frac{1 - x^2 - (1 + x)\sqrt{1 - 2x - 3x^2}}{2x\sqrt{1 - 2x - 3x^2}}.$$

Proof. Using Corollary 3, the result is given by $\left.\frac{\partial Q(x,y)}{\partial y}\right|_{y=1}$ □

The popularity of peaks in $\mathcal{D}_n^{h,\ge}$, $n \ge 1$, is given by the sequence A025566 in [17], which is the first difference of the sequence A005773 that counts the directed animals of size n. The first terms are $x + 3x^2 + 8x^3 + 22x^4 + 61x^5 + 171x^6 + 483x^7 + 1373x^8 + 3923x^9 + 11257x^{10}$.

Corollary 5. *The bivariate generating function $R(x, y)$ where the coefficient of $x^n y^k$ is the number of Dyck paths of $\mathcal{D}_n^{h,\ge}$ containing exactly k odd peaks UD is*

$$\frac{1 - xy - \sqrt{1 - 2xy - 4x^2 + x^2y^2}}{2x^2}.$$

Proof. By Fact 2, the number of odd peaks on $\mathcal{D}_n^{h,\ge}$ is also the number of flats on $\mathcal{M}_n$. Since a non empty Motzkin path in $\mathcal{M}$ is either αF or $\alpha U \beta D$ where $\alpha, \beta \in \mathcal{M}$, the generating function $R(x, y)$ satisfies the functional equation $R(x, y) = 1 + xyR(x, y) + x^2R(x, y)^2$. A simple calculation (with Maple for instance) completes the proof. □

Corollary 6. *The generating function where the coefficient of x^n is the total number of odd peaks in all Dyck paths of $\mathcal{D}_n^{h,\ge}$ is*

$$\frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x\sqrt{1 - 2x - 3x^2}}.$$

Proof. Using Corollary 5, the result is given by $\left.\frac{\partial R(x,y)}{\partial y}\right|_{y=1}$ □

The popularity of odd peaks is given by the sequence A005717 in [17], where the nth term is the nth Motzkin number times n. The first terms are $x + 2x^2 + 6x^3 + 16x^4 + 45x^5 + 126x^6 + 357x^7 + 1016x^8 + 2907x^9 + 8350x^{10}$.

4. Final remarks

In this paper, we study the enumeration of Dyck paths having a first return decomposition with special properties based on a height constraint. For future research, it would be interesting to investigate other statistics on Dyck paths such as number of peaks, valleys, zigzag or double rises, etc. More generally, can we extend this study for other combinatorial objects such as Motzkin and Ballot paths or permutations?

Acknowledgments

We would like to thank the anonymous referees for their very careful reading of this paper and their helpful comments and suggestions.

References

[1] E. Barcucci, A. Del Lungo, S. Fezzi, R. Pinzani, Nondecreasing Dyck paths and q-Fibonacci numbers, Discrete Math. 170 (1–3) (1997) 211–217.
[2] J.-L. Baril, A. Petrossian, Equivalence classes of Dyck paths modulo some statistics, Discrete Math. 338 (4) (2015) 655–660.
[3] P. Barry, On sequences with $\{-1, 0, 1\}$ Hankel transforms. preprint, 2012, available electronically at https://arxiv.org/abs/1205.2565.
[4] C. Brennan, A. Knopfmacher, The height of two types of generalised Motzkin paths, J. Statist. Plann. Inference 143 (2013) 2112–2120.
[5] D. Callan, Two bijections for Dyck path parameters. preprint, 2004, available electronically at http://www.arxiv.org/abs/math.CO/0406381.
[6] E. Czabarka, R. Florez, L. Junes, Some enumeration on non-decreasing Dyck paths, Electron. J. Combin. 22 (1) (2015).
[7] A. Denise, R. Simion, Two combinatorial statistics on Dyck paths, Discrete Math. 137 (1–3) (1995) 155–176.
[8] E. Deutsch, Dyck path enumeration, Discrete Math. 204 (1999) 167–202.
[9] T. Mansour, Counting peaks at height k in a Dyck path, J. Integer Seq. 5 (2002) 02.1.1.
[10] T. Mansour, Statistics on Dyck paths, J. Integer Seq. 9 (2006) 06.1.5.
[11] T. Mansour, M. Shattuck, Counting Dyck paths according to the maximum distance between peaks and valleys, J. Integer Seq. 15 (2012) 12.1.1.
[12] D. Merlini, R. Sprugnoli, M.C. Verri, Some statistics on Dyck paths, J. Statist. Plann. Inference 101 (2002) 211–227.
[13] A. Panayotopoulos, A. Sapounakis, On the prime decomposition of Dyck paths, J. Combin. Math. Combin. Comput. 40 (2002) 33–39.
[14] P. Peart, W.J. Woan, Dyck paths with no peaks at height k, J. Integer Seq. 4 (2001) 01.1.3.
[15] A. Sapounakis, I. Tasoulas, P. Tsikouras, Counting strings in Dyck paths, Discrete Math. 307 (23) (2007) 2909–2924.
[16] A. Sapounakis, I. Tasoulas, P. Tsikouras, Some strings in Dyck paths, Australas. J. Combin. 39 (2007) 49–72.
[17] N.J.A. Sloane, The On-line Encyclopedia of Integer Sequences, available electronically at http://oeis.org.
[18] R.P. Stanley, Enumerative Combinatorics, Vol. 2, Cambridge University Press, 1999.
[19] Y. Sun, The statistic "number of udu's" in Dyck paths, Discrete Math. 287 (2004) 177–186.

Discrete Mathematics 341 (2018) 2608–2615

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Descent distribution on Catalan words avoiding a pattern of length at most three

Jean-Luc Baril, Sergey Kirgizov, Vincent Vajnovszki

LE2I, Université de Bourgogne Franche-Comté, B.P. 47 870, 21078 DIJON-Cedex, France

ARTICLE INFO

Article history:
Received 18 March 2018
Received in revised form 24 May 2018
Accepted 1 June 2018
Available online 23 June 2018

Keywords:
Enumeration
Catalan word
Pattern avoidance
Descent
Popularity

ABSTRACT

Catalan words are particular growth-restricted words over the set of non-negative integers, and they represent still another combinatorial class counted by the Catalan numbers. We study the distribution of descents on the sets of Catalan words avoiding a pattern of length at most three: for each such a pattern p we provide a bivariate generating function where the coefficient of $x^n y^k$ in its series expansion is the number of length n p-avoiding Catalan words with k descents. As a byproduct, we enumerate the set of Catalan words avoiding p, and we provide the popularity of descents on this set.

© 2018 Elsevier B.V. All rights reserved.

1. Introduction and notation

A length n *Catalan word* is a word $w_1 w_2 \dots w_n$ over the set of non-negative integers with $w_1 = 0$, and

$$0 \le w_i \le w_{i-1} + 1,$$

for $i = 2, 3, \dots, n$. We denote by $\mathcal{C}_n$ the set of length n Catalan words, and $\mathcal{C} = \cup_{n\ge 0}\mathcal{C}_n$. For example, $\mathcal{C}_2 = \{00, 01\}$ and $\mathcal{C}_3 = \{000, 001, 010, 011, 012\}$. It is well known that the cardinality of $\mathcal{C}_n$ is given by the nth Catalan number $\frac{1}{n+1}\binom{2n}{n}$, see for instance [13, exercise 6.19.u, p. 222], which is the general term of the sequence A000108 in the On-line Encyclopedia of Integer Sequences (OEIS) [12]. See also [10] where Catalan words are considered in the context of the exhaustive generation of Gray codes for growth-restricted words.

A *pattern* p is a word satisfying the property that if x appears in p, then all integers in the interval $[0, x-1]$ also appear in p. We say that a word $w_1 w_2 \dots w_n$ contains the pattern $p = p_1 \dots p_k$ if there is a subsequence $w_{i_1} w_{i_2} \dots w_{i_k}$ of w, $i_1 < i_2 < \cdots < i_k$, which is order-isomorphic to p. For example, the Catalan word 01012312301 contains seven occurrences of the pattern 110 and four occurrences of the pattern 210. A word *avoids* the pattern p whenever it does not contain any occurrence of p. We denote by $\mathcal{C}_n(p)$ the set of length n Catalan words avoiding the pattern p, and $\mathcal{C}(p) = \cup_{n\ge 0}\mathcal{C}_n(p)$. For instance, $\mathcal{C}_4(012) = \{0000, 0001, 0010, 0011, 0100, 0101, 0110, 0111\}$, and $\mathcal{C}_4(101) = \{0000, 0001, 0010, 0011, 0012, 0100, 0110, 0111, 0112, 0120, 0121, 0122, 0123\}$.

A *descent* in a word $w = w_1 w_2 \dots w_n$ is an occurrence $w_i w_{i+1}$ such that $w_i > w_{i+1}$, and we denote by $d(w)$ the number of descents of w. The distribution of the number of descents has been widely studied on several classes of combinatorial objects such as permutations and words, since descents have some particular interpretations in fields as Coxeter groups or theory of lattice paths [4,6]. More specifically, there are natural one-to-one correspondences between descents in Catalan words and some patterns in other classical Catalan structures, and below we give two such examples.

E-mail addresses: barjl@u-bourgogne.fr (J. Baril), sergey.kirgizov@u-bourgogne.fr (S. Kirgizov), vvajnov@u-bourgogne.fr (V. Vajnovszki).

https://doi.org/10.1016/j.disc.2018.06.001
0012-365X/© 2018 Elsevier B.V. All rights reserved.

Let $\delta \mapsto w$ be the bijection which maps a semilength n Dyck word over $\{u, d\}$ into a length n Catalan word defined as: w is the sequence of the lowest ordinate of the up steps u in the Dyck word δ, in lattice path representation. Under this bijection, occurrences of consecutive patterns *ddu* in Dyck words correspond to descents in Catalan words. Similarly, Mäkinen's bijection [9] gives a one-to-one correspondence between descents in Catalan words (called *left-distance sequences* by the author of [9]) and particular nodes (left-child nodes having a right child) in binary trees.

A *statistic st* on a finite set S is an association of an integer to each element of S, and the *popularity* of *st* is $\sum_{x\in S} st(x)$, which is the cardinality of S times the expectation of *st*. The number of occurrences of a pattern or the number of descents are examples of statistics on words. See [5] where the notion of popularity was introduced in the context of pattern based statistics, and [1,7,11,2] for some related results.

The main goal of this paper is to study the descent distribution on Catalan words (see Table 1 for some numerical values). More specifically, for each pattern p of length at most three, we give the distribution of descents on the sets $\mathcal{C}_n(p)$ of length n Catalan words avoiding p. We denote by $C_p(x, y) = \sum_{n,k\geq 0} c_{n,k} x^n y^k$ the bivariate generating function for the cardinality of words in $\mathcal{C}_n(p)$ with k descents. Plugging $y = 1$

- into $C_p(x, y)$, we deduce the generating function $C_p(x)$ for the set $\mathcal{C}_n(p)$, and
- into $\frac{\partial C_p(x,y)}{\partial y}$, we deduce the generating function for the popularity of descents in $\mathcal{C}_n(p)$.

The proofs in this paper are based mainly on functional equations, and alternative bijective proofs would be of interest.

From the definition at the beginning of this section it follows that a Catalan word is either the empty word, or it can uniquely be written as $0(w' + 1)w''$, where w' and w'' are in turn Catalan words, and $w' + 1$ is obtained from w' by adding one to each of its entries. We call this recursive decomposition *first return decomposition* of a Catalan word, and it will be crucial in our further study. It follows that $C(x)$, the generating function for the cardinality of $\mathcal{C}_n$, satisfies:

$$C(x) = 1 + x \cdot C^2(x),$$

which corresponds precisely to the sequence of Catalan numbers.

The remainder of the paper is organized as follows. In Section 2, we study the distribution of descents on the set $\mathcal{C}$ of Catalan words. As a byproduct, we deduce the popularity of descents in $\mathcal{C}$. We consider also similar results for the obvious cases of Catalan words avoiding a pattern of length two. In Section 3, we study the distribution and the popularity of descents on Catalan words avoiding each pattern of length three.

2. The sets $\mathcal{C}$ and $\mathcal{C}(p)$ for $p \in \{00, 01, 10\}$

Here we consider both unrestricted Catalan words and those avoiding a length two pattern.

We denote by $C(x, y)$ the bivariate generating function where the coefficient of $x^n y^k$ of its series expansion is the number of length n Catalan words with k descents. When we restrict to Catalan words avoiding a pattern p, the corresponding generating function is denoted by $C_p(x, y)$.

Theorem 1. *We have*

$$C(x, y) = \frac{1 - 2x + 2xy - \sqrt{1 - 4x + 4x^2 - 4x^2 y}}{2xy}.$$

Proof. Let $w = 0(w' + 1)w''$ be the first return decomposition of a non-empty Catalan word w with $w', w'' \in \mathcal{C}$. If w' (resp. w'') is empty then the number $d(w)$ of descents in w is the same as that of w'' (resp. w'); otherwise, we have $d(w) = d(w') + d(w'') + 1$ since there is a descent between $w' + 1$ and w''. So, we obtain the functional equation $C(x, y) = 1 + xC(x, y) + x(C(x, y) - 1) + xy(C(x, y) - 1)^2$ which gives the desired result. □

By the considerations in the introductory section, it turns out that the coefficient of $x^n y^k$ in the series expansion of $C(x, y)$ is also the number of semilength n Dyck words with k occurrences of the consecutive pattern *ddu*, which is given by the statistic St000386 in [3], see also A091894 in [12].

As expected, $C(x) = C(x, 1) = \frac{1-\sqrt{1-4x}}{2x}$ is the generating function for the Catalan numbers, and $\frac{\partial C(x,y)}{\partial y}|_{y=1}$ is the generating function for the descent popularity on $\mathcal{C}$, and we have the next corollary.

Corollary 1. *The popularity of descents on the set $\mathcal{C}_n$ is $\binom{2n-2}{n-3}$, and its generating function is $\frac{1-4x+2x^2-(1-2x)\sqrt{1-4x}}{2x\sqrt{1-4x}}$ (sequence A002694 in [12]).*

Catalan words of odd lengths encompass a smaller size Catalan structure. This result is stated in the next theorem, see the bold entries in Table 1.

Theorem 2. *Catalan words of length $2n + 1$ with n descents are enumerated by the nth Catalan number $\frac{1}{n+1}\binom{2n}{n}$.*

Table 1
Number $c_{n,k}$ of length n Catalan words with k descents for $1 \le n \le 10$ and $0 \le k \le 4$.

k	n									
	1	2	3	4	5	6	7	8	9	10
0	**1**	2	4	8	16	32	64	128	256	512
1			**1**	6	24	80	240	672	1792	4608
2					**2**	20	120	560	2240	8064
3							**5**	70	560	3360
4									**14**	252
$\sum$	1	2	5	14	42	132	429	1430	4862	16796

Table 2
Catalan words avoiding a pattern of length three.

Pattern p	Sequence $\lvert C_n(p)\rvert$	Generating function	OEIS [12]
012, 001, 010	2^{n-1}	$\frac{1-x}{1-2x}$	A011782
021	$(n-1)\cdot 2^{n-2}+1$	$\frac{1-4x+5x^2-x^3}{(1-x)(1-2x)^2}$	A005183
102, 201	$\frac{3^{n-1}+1}{2}$	$\frac{1-3x+x^2}{(1-x)(1-3x)}$	A007051
120, 101	F_{2n-1}	$\frac{1-2x}{1-3x+x^2}$	A001519
011	$\frac{n(n-1)}{2}+1$	$\frac{1-2x+2x^2}{(1-x)^3}$	A000124
000		$\frac{1-2x^2}{1-x-3x^2+x^3}$	
100	$\lceil \frac{(1+\sqrt{3})^{n+1}}{12} \rceil$	$\frac{1-2x-x^2+x^3}{1-3x+2x^3}$	A057960
110	$\frac{1}{2}\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n+1}{2k+1} 2^k - \frac{n-1}{2}$	$\frac{1-3x+2x^2+x^3}{(1-x)^2(1-2x-x^2)}$	
210		$\frac{1-5x+7x^2-x^3-x^4}{(1-2x)(1-4x+3x^2+x^3)}$	

Proof. Clearly, the maximal number of descents in a length n Catalan word is $\lfloor \frac{n-1}{2} \rfloor$. Let w be a Catalan word of length $2n+1$ with n descents. We necessarily have $w = 0(w'+1)w''$ with $w', w'' \neq \epsilon$, $d(w') = \lfloor \frac{|w'|-1}{2} \rfloor$, $d(w'') = \lfloor \frac{|w''|-1}{2} \rfloor$ and $d(w) = d(w') + d(w'') + 1$. Since the length of w is odd, $|w'|$ and $|w''|$ have the same parity. If $|w'|$ and $|w''|$ are both even, then $d(w) = \frac{|w'|-2}{2} + \frac{|w''|-2}{2} + 1 = \frac{|w'|+|w''|-2}{2} < \lfloor \frac{(|w'|+|w''|+1)-1}{2} \rfloor = \lfloor \frac{n-1}{2} \rfloor$ which gives a contradiction. So, $|w'|$ and $|w''|$ are both odd, and we have $d(w) = \frac{|w'|-1}{2} + \frac{|w''|-1}{2} + 1 = \lfloor \frac{(|w'|+|w''|+1)-1}{2} \rfloor = \lfloor \frac{n-1}{2} \rfloor$. Thus the generating function $A(x)$ where the coefficient of x^n is the number of Catalan words of length $2n+1$ with n descents satisfies $A(x) = 1 + xA(x)^2$ which is the generating function for the Catalan numbers. □

There are three patterns of length two, namely 00, 01 and 10, and Catalan words avoiding such a pattern do not have descents, thus the corresponding bivariate generating functions collapse into one variable ones.

Theorem 3. *For $p \in \{00, 01\}$, we have $C_p(x, y) = \frac{1}{1-x}$.*

Proof. If $p = 00$ (resp. $p = 01$) then $012\ldots n-1$ (resp. $00\ldots 0$) is the unique non-empty Catalan word of length n avoiding p, and the statement follows. □

Theorem 4. *We have $C_{10}(x, y) = \frac{1-x}{1-2x}$, which is the generating function for the sequence 2^{n-1} (sequence A011782 in [12]).*

Proof. A non-empty Catalan word avoiding the pattern 10 is of the form $0^k(w'+1)$ for $k \ge 1$, and with $w' \in \mathcal{C}(10)$. So, we have the functional equation $C_{10}(x) = 1 + \frac{x}{1-x}C_{10}(x)$, which gives $C_{10}(x) = \frac{1-x}{1-2x}$. □

3. The sets $\mathcal{C}(p)$ for a length three pattern p

Here we turn our attention to patterns of length three. There are thirteen such patterns, and we give the distribution and the popularity of descents on Catalan words avoiding each of them. Some of the obtained results are summarized in Tables 2 and 3.

Theorem 5. *For $p \in \{012, 001\}$, we have*

$$C_p(x, y) = \frac{1 - x + x^2 - x^2y}{1 - 2x + x^2 - x^2y}.$$

Table 3
Popularity of descents on Catalan words avoiding a pattern of length three.

Pattern p	Popularity of descents on $\mathcal{C}_n(p)$	Generating function	OEIS [12]
012, 001	$(n-2)\cdot 2^{n-3}$	$\frac{x^3}{(1-2x)^2}$	A001787
010	0	0	
021	$(n+1)(n-2)\cdot 2^{n-5}$	$\frac{x^3(1-x)}{(1-2x)^3}$	A001793
102, 201	$(n-2)\cdot 3^{n-3}$	$\frac{x^3}{(1-3x)^2}$	A027471
120, 101	$\sum_{k=1}^{n-2} k\cdot\binom{n+k-2}{2k}$	$\frac{x^3(1-x)}{(1-3x+x^2)^2}$	A001870
011	$\frac{(n-1)(n-2)}{2}$	$\frac{x^3}{(1-x)^3}$	A000217
000		$\frac{x^3(1-x)(1+2x)(1+x)}{(1-x-3x^2+x^3)^2}$	
100		$\frac{x^3(1-x-x^2)}{(1-3x+2x^3)^2}$	
110		$\frac{x^3(1-x-x^2)^2}{(1-x)^3(1-2x-x^2)^2}$	
210		$\frac{x^3(1-2x)}{(1-4x+3x^2+x^3)^2}$	

Proof. A non-empty word $w \in \mathcal{C}(012)$ has its first return decomposition $w = 01^k w''$ where $k \geq 0$ and $w'' \in \mathcal{C}(012)$. If $k = 0$ or $w'' = \epsilon$, then the number of descents in w is the same as that of w''; otherwise, we have $d(w) = d(w'') + 1$ (there is a descent between 1^k and w''). So, we obtain the functional equation $C_{012}(x, y) = 1 + xC_{012}(x, y) + \frac{x^2}{1-x} + \frac{x^2}{1-x}y(C_{012}(x, y) - 1)$ which gives the desired result.

A non-empty word $w \in \mathcal{C}(001)$ has the form $w = 0(w' + 1)0^k$ where $w' \in \mathcal{C}(001)$ and $k \geq 0$. If $k = 0$ or $w' = \epsilon$, then the number of descents in w is the same as that of w'; otherwise, we have $d(w) = d(w') + 1$. So, we obtain the functional equation $C_{001}(x, y) = 1 + x(C_{001}(x, y) - 1) + \frac{x}{1-x} + \frac{x^2}{1-x}y(C_{001}(x, y) - 1)$ which gives the desired result. □

Considering the previous theorem and the coefficient of x^n in $C_p(x, 1) = \frac{1-x}{1-2x}$ and in $\frac{\partial C_p(x,y)}{\partial y}|_{y=1} = \frac{x^3}{(1-2x)^2}$, we obtain the next corollary.

Corollary 2. *For $p \in \{012, 001\}$, we have $|\mathcal{C}_n(p)| = 2^{n-1}$, and the popularity of descents on the set $\mathcal{C}_n(p)$ is $(n-2)\cdot 2^{n-3}$ (sequence A001787 in [12]).*

As in the case of length two patterns, a Catalan word avoiding 010 does not have descents, and we have the next theorem.

Theorem 6. *If $p = 010$, then $C_p(x, y) = \frac{1-x}{1-2x}$ which is the generating function for the sequence given by 2^{n-1} (sequence A011782 in [12]).*

Proof. A non-empty word $w \in \mathcal{C}(010)$ can be written either as $w = 0w'$ with $w' \in \mathcal{C}(10)$, or as $w = 0(w' + 1)$ with $w' \in \mathcal{C}(010) \setminus \{\epsilon\}$. So, we deduce $C_{010}(x) = 1 + xC_{10}(x) + x(C_{010}(x) - 1)$, and the statement holds. □

Theorem 7. *For $p = 021$, we have*

$$C_p(x, y) = \frac{1 - 4x + 6x^2 - x^2y - 4x^3 + 3x^3y + x^4 - x^4y}{(1-x)(1-2x)(1-2x+x^2-x^2y)}.$$

Proof. Let w be a non-empty word in $\mathcal{C}(021)$, and let $0(w' + 1)w''$ be its first return decomposition with $w', w'' \in \mathcal{C}(021)$. Note that w' belongs to $\mathcal{C}(10)$. We distinguish two cases: (1) w' does not contain 1, and (2) otherwise.

In the case (1), $w' \in \mathcal{C}(01)$ (*i.e.*, $w' = 0^k$ for some $k \geq 0$), and $w'' \in \mathcal{C}(021)$. If $w' = \epsilon$ (resp. $w'' = \epsilon$), then the number of descents in w is the same as that of w'' (resp. w'); otherwise, we have $d(w) = d(w') + d(w'') + 1$. So, this case contributes to $C_p(x, y)$ with $xC_{01}(x, y) + x(C_{021}(x, y) - 1) + xy(C_{01}(x, y) - 1)(C_{021}(x, y) - 1)$.

In the case (2), $w' \in \mathcal{C}(10) \setminus \mathcal{C}(01)$ and $w'' \in \mathcal{C}(01)$. If $w'' = \epsilon$ then w and w' have the same number of descents; otherwise, we have $d(w) = d(w') + d(w'') + 1$. So, this case contributes to $C_p(x, y)$ with $x(C_{10}(x, y) - C_{01}(x, y)) + xy(C_{10}(x, y) - C_{01}(x, y))(C_{01}(x, y) - 1)$.

Taking into account these two disjoint cases, and adding the empty word, we deduce the functional equation $C_{021}(x, y) = 1 + xC_{01}(x, y) + x(C_{021}(x, y) - 1) + xy(C_{01}(x, y) - 1)(C_{021}(x, y) - 1) + x(C_{10}(x, y) - C_{01}(x, y)) + xy(C_{10}(x, y) - C_{01}(x, y))(C_{01}(x, y) - 1)$, which after calculation gives the result. □

Corollary 3. *For $p = 021$, we have $C_p(x) = \frac{1-4x+5x^2-x^3}{(1-2x)^2(1-x)}$ which is the generating function for the sequence $(n-1)\cdot 2^{n-2} + 1$ (sequence A005183 in [12]). The popularity of descents on the set $\mathcal{C}_n(p)$ is $(n+1)(n-2)\cdot 2^{n-5}$ with the generating function $\frac{x^3(1-x)}{(1-2x)^3}$ (sequence A001793 in [12]).*

Theorem 8. *For $p \in \{102, 201\}$, we have*

$$C_p(x, y) = \frac{1 - 3x + 3x^2 - 2x^2y - x^3 + x^3y}{(1 - x)\left(1 - 3x + 2x^2 - 2x^2y\right)}.$$

Proof. Let w be a non-empty word in $\mathcal{C}(102)$, and let $0(w' + 1)w''$ be its first return decomposition with $w', w'' \in \mathcal{C}(102)$. If w' is empty, then $w = 0w''$ for some $w'' \in \mathcal{C}(102)$ and we have $d(w) = d(w'')$. If w'' is empty, then $w = 0(w' + 1)$ for some $w' \in \mathcal{C}(102)$ and we have $d(w) = d(w')$. If w' and w'' are both non-empty, then $w' \in \mathcal{C}(102) \setminus \{\epsilon\}$ and $w'' \in \mathcal{C}(012) \setminus \{\epsilon\}$. We deduce the functional equation $C_{102}(x, y) = 1 + xC_{102}(x, y) + x(C_{102}(x, y) - 1) + xy(C_{102}(x, y) - 1)(C_{012}(x, y) - 1)$. Finally, by Theorem 5 we obtain the desired result.

Let w be a non-empty word in $\mathcal{C}(201)$, and let $0(w' + 1)w''$ be its first return decomposition with $w', w'' \in \mathcal{C}(201)$. If w' is empty, then $w = 0w''$ for some $w'' \in \mathcal{C}(201)$ and we have $d(w) = d(w'')$. If w'' is empty, then $w = 0(w' + 1)$ for some $w' \in \mathcal{C}(201)$ and we have $d(w) = d(w')$. If w' and w'' are both non-empty, then $d(w) = d(w') + d(w'') + 1$ and we distinguish two cases: (1) w' does not contain 1, and (2) otherwise. In the case (1), we have $w' \in \mathcal{C}(01) \setminus \{\epsilon\}$ and $w'' \in \mathcal{C}(201) \setminus \{\epsilon\}$; in the case (2), w' contains 1 and $w' \in \mathcal{C}(201) \setminus \mathcal{C}(01)$ and $w'' \in \mathcal{C}(01) \setminus \{\epsilon\}$. Combining the previous cases, the functional equation becomes $C_{201}(x, y) = 1 + xC_{201}(x, y) + x(C_{201}(x, y) - 1) + xy(C_{01}(x, y) - 1)(C_{201}(x, y) - 1) + xy(C_{201}(x, y) - C_{01}(x, y))(C_{01}(x, y) - 1)$, which gives the desired result. □

Corollary 4. *For $p \in \{102, 201\}$, we have $C_p(x) = \frac{1-3x+x^2}{(1-x)(1-3x)}$ which is the generating function of the sequence $\frac{3^{n-1}+1}{2}$ (sequence A007051 in [12]). The popularity of descents on the set $\mathcal{C}_n(p)$ is $(n - 2) \cdot 3^{n-3}$ with the generating function $\frac{x^3}{(1-3x)^2}$ (sequence A027471 in [12]).*

Theorem 9. *For $p \in \{120, 101\}$, we have*

$$C_p(x, y) = \frac{1 - 2x + x^2 - x^2y}{1 - 3x + 2x^2 - x^2y}.$$

Proof. Let w be a non-empty word in $\mathcal{C}(120)$, and let $0(w' + 1)w''$ be its first return decomposition where $w', w'' \in \mathcal{C}(120)$. If w'' is empty, then $w = 0(w' + 1)$ for some $w' \in \mathcal{C}(120)$ and we have $d(w) = d(w')$; if w' is empty, then $w = 0w''$ for some $w'' \in \mathcal{C}(120)$ and we have $d(w) = d(w')$; if w' and w'' are not empty, then $w' \in \mathcal{C}(01) \setminus \{\epsilon\}$, $w'' \in \mathcal{C}(120) \setminus \{\epsilon\}$ and $d(w) = d(w') + d(w'') + 1$. We deduce the functional equation $C_{120}(x, y) = 1 + xC_{120}(x, y) + +x(C_{120}(x, y) - 1) + xy(C_{01}(x, y) - 1)(C_{120}(x, y) - 1)$ which gives the result.

Let w be a non-empty word in $\mathcal{C}(101)$, and let $0(w' + 1)w''$ be its first return decomposition where $w', w'' \in \mathcal{C}(101)$. If w' is empty, then $w = 0w''$ for some $w'' \in \mathcal{C}(101)$ and $d(w) = d(w'')$; if w'' is empty, then $w = 0(w' + 1)$ for some $w' \in \mathcal{C}(101)$ and $d(w) = d(w'')$; if w' and w'' are not empty, then $w' \in \mathcal{C}(101) \setminus \{\epsilon\}$ and $w'' \in \mathcal{C}(01) \setminus \{\epsilon\}$ and $d(w) = d(w') + d(w'') + 1$. We deduce the functional equation $C_{101}(x, y) = 1 + xC_{101}(x, y) + x(C_{101}(x, y) - 1) + xy(C_{101}(x, y) - 1)(C_{01}(x, y) - 1)$ which gives the result. □

Corollary 5. *For $p \in \{120, 101\}$, we have $C_p(x) = \frac{1-2x}{1-3x+x^2}$ and the coefficient of x^n in its series expansion is the $(2n - 1)$th term of the Fibonacci sequence (see A001519 in [12]). The popularity of descents on the set $\mathcal{C}_n(p)$ is given by $\sum_{k=1}^{n-2} k \cdot \binom{n+k-2}{2k}$ which is the coefficient of x^n in the series expansion of $\frac{x^3(1-x)}{(1-3x+x^2)^2}$ (sequence A001870 in [12]).*

Theorem 10. *For $p = 011$, we have*

$$C_p(x, y) = \frac{1 - 2x + 2x^2 - x^3 + x^3y}{(1 - x)^3}.$$

Proof. Let w be a non-empty word in $\mathcal{C}(011)$, and let $0(w' + 1)w''$ be its first return decomposition where $w', w'' \in \mathcal{C}(011)$. If w' (resp. w'') is empty, then we have $d(w) = d(w'')$ (resp. $d(w) = d(w')$); if w' and w'' are non-empty, then $w' \in \mathcal{C}(00) \setminus \{\epsilon\}$ and $w'' \in \mathcal{C}(01) \setminus \{\epsilon\}$. We deduce the functional equation $C_{011}(x, y) = 1 + xC_{011}(x, y) + x(C_{00}(x, y) - 1) + xy(C_{00}(x, y) - 1)(C_{01}(x, y) - 1)$ which gives the result. □

Corollary 6. *For $p = 011$, we have $C_p(x) = \frac{1-2x+2x^2}{(1-x)^3}$ and the coefficient of x^n in its series expansion is $1 + \binom{n}{2}$ (sequence A000124 in [12]). The popularity of descents on the set $\mathcal{C}_n(p)$ is given by $\frac{(n-1)(n-2)}{2}$ which is the coefficient of x^n in the series expansion of $\frac{x^3}{(1-x)^3}$ (sequence A000217 in [12]).*

Theorem 11. *For $p = 000$, we have*

$$C_p(x, y) = \frac{1 - x^2 - x^2y}{1 - x - 2x^2 - x^2y + x^3 + x^4 - x^4y}.$$

Proof. Let w be a non-empty word in $\mathcal{C}(000)$, and let $0(w'+1)w''$ be its first return decomposition where $w', w'' \in \mathcal{C}(000)$. We distinguish two cases: (1) w'' is empty, and (2) otherwise.

In the case (1), we have $w = 0(w'+1)$ for some $w' \in \mathcal{C}(000)$ and $d(w) = d(w')$. So, the generating function $A(x, y)$ for the Catalan words in this case is $A(x, y) = xC_{000}(x, y)$.

In the case (2), we set $w'' = 0(w'''+1)$ for some $w''' \in \mathcal{C}(000)$ and we have $w = 0(w'+1)0(w'''+1)$. We distinguish three sub-cases: (2.a) w' is empty, (2.b) w' is non-empty and w''' is empty, and (2.c) w' and w''' are both non-empty.

In the case (2.a), we have $w = 00(w'''+1)$ with $w''' \in \mathcal{C}(000)$. So, the generating function for the Catalan words belonging to this case is $B_a(x, y) = x^2C_{000}(x, y)$.

In the case (2.b), we have $w = 0(w'+1)0$ with $w' \in \mathcal{C}(000) \setminus \{\epsilon\}$. So, the generating function for the corresponding Catalan words is $B_b(x, y) = x^2y(C_{000}(x, y) - 1)$.

In the case (2.c), we have $w = 0(w'+1)0(w'''+1)$ where w' and w''' are non-empty Catalan words such that $w'w'''$ is a Catalan word satisfying the case (2). If $w' = 0$, then $d(w'w''') = d(w''') = d(w) - 1$; if $w' \neq 0$, then $d(w'w''') = d(w') + d(w''') + 1 = d(w)$. So, the generating function for the corresponding Catalan words is $B_c(x, y) = x^2yB_a(x, y) + x^2(B_b(x, y) + B_c(x, y))$.

Considering $C_{000}(x, y) = 1 + A(x, y) + B_a(x, y) + B_b(x, y) + B_c(x, y)$, the obtained functional equations give the result. □

Corollary 7. *For $p = 000$, we have $C_p(x) = \frac{1-2x^2}{1-x-3x^2+x^3}$ and the generating function for the popularity of descents in the sets $\mathcal{C}_n(p)$, $n \geq 0$, is*

$$\frac{x^3(1-x)(1+2x)(1+x)}{(1-x-3x^2+x^3)^2}.$$

Note that the sequences defined by the two generating functions in Corollary 7 do not appear in [12].

Theorem 12. *For $p = 100$, we have*

$$C_p(x, y) = \frac{1-2x-x^2y+x^3}{1-3x+x^2-x^2y+2x^3}.$$

Proof. For $k \geq 1$, we define $\mathcal{A}_k \subset \mathcal{C}(100)$ as the set of Catalan words avoiding 100 with exactly k zeros, and let $A_k(x, y)$ be the generating function for $\mathcal{A}_k$.

A Catalan word $w \in \mathcal{A}_1$ is of the form $w = 0(w'+1)$ with $w' \in \mathcal{C}(100)$. Since we have $d(w) = d(w')$, the generating function $A_1(x, y)$ for these words satisfies $A_1(x, y) = xC_{100}(x, y)$.

A Catalan word $w \in \mathcal{A}_k$, $k \geq 3$, is of the form $w = 0^{k-2}w'$ with $w' \in \mathcal{A}_2$. Since we have $d(w) = d(w')$, the generating function $A_k(x, y)$ for these words satisfies $A_k(x, y) = x^{k-2}A_2(x, y)$.

A Catalan word $w \in \mathcal{A}_2$ has one of the three following forms:

(1) $w = 00(w'+1)$ with $w' \in \mathcal{C}(100)$; we have $d(w) = d(w')$, and the generating function for these Catalan words is $x^2C_{100}(x, y)$.

(2) $w = 0(w'+1)0$ with $w' \in \mathcal{C}(100) \setminus \{\epsilon\}$; we have $d(w) = d(w') + 1$, and the generating function for these Catalan words is $x^2y(C_{100}(x, y) - 1)$.

(3) $w = 0(w'+1)0(w''+1)$ where w' and w'' are non-empty and $w'w'' \in \mathcal{A}_k$ for some $k \geq 2$ (*i.e.*, $w'w'' = 0^{k-2}0(u+1)0(v+1)$ with $0(u+1)0(v+1) \in \mathcal{A}_2$). So, there are $(k-1)$ possible choices for w', namely $0, 0^2, \ldots, 0^{k-2}$, and $0^{k-2}0(u+1)$. If $w' = 0, 0^2, \ldots, 0^{k-2}$, then $d(w) = d(0(u+1)0(v+1)) + 1$; if $w' = 0^{k-2}0(u+1)$ and $u \neq \epsilon$, then $d(w) = d(0(u+1)0(v+1))$; if $w' = 0^{k-2}0(u+1)$ and $u = \epsilon$, then $d(w) = d(0(u+1)0(v+1)) + 1$. So, the generating function for these words is $x^2yA_2(x, y)\sum_{k\geq 2}(k-2)x^{k-2} + x^2(A_2(x, y) - x^2C_{100}(x, y))\sum_{k\geq 2}x^{k-2} + x^4yC_{100}(x, y)\sum_{k\geq 2}x^{k-2}$, which is $\frac{x^3y}{(1-x)^2}A_2(x, y) + \frac{x^2}{1-x}A_2(x, y) + \frac{x^4y-x^2}{1-x}C_{100}(x, y)$.

Taking into account all previous cases, we obtain the following functional equations:

(i) $A_1(x, y) = xC_{100}(x, y)$,
(ii) $A_2(x, y) = x^2C_{100}(x, y) + x^2y(C_{100}(x, y) - 1) + \frac{x^3y}{(1-x)^2}A_2(x, y) + \frac{x^2}{1-x}A_2(x, y) + \frac{x^4y-x^2}{1-x}C_{100}(x, y)$,
(iii) $A_k(x, y) = x^{k-2}A_2(x, y)$ for $k \geq 3$,
(iv) $C_{100}(x, y) = 1 + \sum_{k\geq 1}A_k(x, y)$.

A simple calculation gives the desired result. □

Corollary 8. *For $p = 100$, we have $C_p(x) = \frac{1-2x-x^2+x^3}{1-3x+2x^3}$, which is the generating function for the sequence $\lceil \frac{(1+\sqrt{3})^{n+1}}{12} \rceil$ (see A057960 in [12]), and the generating function for the popularity of descents in the sets $\mathcal{C}_n(p)$, $n \geq 0$, is*

$$\frac{x^3(1-x-x^2)}{(1-3x+2x^3)^2}.$$

Theorem 13. *For $p = 110$, we have*

$$C_p(x) = \frac{1 - 3x + 2x^2 + x^3 - x^4 + x^4y}{(1-x)\left(1 - 3x + x^2 + 2x^3 - x^3y\right)}.$$

Proof. Let w be a non-empty word in $\mathcal{C}(110)$, and let $0(w'+1)w''$ be its first return decomposition where $w', w'' \in \mathcal{C}(110)$. Then, w has one of the following forms:

- $w = 0(w'+1)$ where $w' \in \mathcal{C}(110)$; the generating function for these words is $xC_{110}(x, y)$.
- $w = 0w'$ where $w' \in \mathcal{C}(110) \setminus \{\epsilon\}$; the generating function for these words is $x(C_{110}(x, y) - 1)$.
- $w = 0(w'+1)w''$ with $w' \in \mathcal{C}(00) \setminus \{\epsilon\}$ and $w'' \in \mathcal{C}(10) \setminus \{\epsilon\}$; the generating function for these words is $xy(C_{00}(x, y) - 1)(C_{10}(x, y) - 1)$.
- The last form is $w = 0(w'+1)w''$ where $w' \in \mathcal{C}(00) \setminus \{\epsilon\}$ and $w'' \notin \mathcal{C}(10)$. So, we have $w = 012\ldots k0^{a_0}1^{a_1}\ldots(k-1)^{a_k}(w''' + k - 1)$ where $k \geq 1$, $a_i \geq 1$ for $0 \leq i \leq k$, and $w''' \in \mathcal{C}(110) \setminus \mathcal{C}(10)$; the generating function for these words is $y\sum_{k\geq 1}\frac{x^{2k+1}}{(1-x)^k}(C_{110}(x, y) - C_{10}(x, y))$.

Combining these different cases, we deduce the functional equation:

$$\begin{aligned} C_{110}(x, y) = \quad & 1 + xC_{110}(x, y) + x(C_{110}(x, y) - 1) + xy(C_{00}(x, y) - 1)(C_{10}(x, y) - 1) + \\ & y\sum_{k\geq 1}\frac{x^{2k+1}}{(1-x)^k}(C_{110}(x, y) - C_{10}(x, y)). \end{aligned}$$

Considering Theorems 3 and 4, the result follows. □

Corollary 9. *For $p = 110$, we have $C_p(x) = \frac{1-3x+2x^2+x^3}{(1-x)^2(1-2x-x^2)}$ and the generating function for the popularity of descents in the sets $\mathcal{C}_n(p)$, $n \geq 0$, is*

$$\frac{x^3(1 - x - x^2)^2}{(1-x)^3(1 - 2x - x^2)^2}.$$

Theorem 14. *For $p = 210$, we have*

$$C_p(x) = \frac{1 - 5x + 8x^2 - x^2y - 4x^3 + 3x^3y - x^4y}{(1-2x)(1 - 4x + 4x^2 - x^2y + x^3y)}.$$

Proof. Let w be a non-empty word in $\mathcal{C}(210)$, and let $0(w'+1)w''$ be its first return decomposition where $w', w'' \in \mathcal{C}(210)$. Then, w has one of the following forms:

- $w = 0(w'+1)$ where $w' \in \mathcal{C}(210)$; the generating function for these words is $xC_{210}(x, y)$.
- $w = 0w''$ where $w'' \in \mathcal{C}(210) \setminus \{\epsilon\}$; the generating function for these words is $x(C_{210}(x, y) - 1)$.
- $w = 0(w'+1)w''$ where $w' \in \mathcal{C}(01) \setminus \{\epsilon\}$ and $w'' \in \mathcal{C}(210) \setminus \{\epsilon\}$; the generating function for these sets is $xy(C_{01}(x, y) - 1)(C_{210}(x, y) - 1)$.
- $w = 01^{a_1}2^{a_2}\ldots k^{a_k}w''$ where $k \geq 2$, $a_i \geq 1$ for $1 \leq i \leq k$, and $w'' \in \mathcal{C}(10) \setminus \{\epsilon\}$; the generating function for these words is $y(C_{10}(x, y) - 1)\sum_{k\geq 2}\frac{x^{k+1}}{(1-x)^k}$.
- $w = 01^{a_1}2^{a_2}\ldots k^{a_k}0^{b_0}1^{b_1}\ldots(k-2)^{b_{k-2}}(w'' + k - 2)$ where $k \geq 2$, $a_i \geq 1$ for $1 \leq i \leq k$, $b_i \geq 1$ for $1 \leq i \leq k-2$, and $w'' \in \mathcal{C}(210) \setminus \mathcal{C}(10)$; the generating function for these words is $y(C_{210}(x, y) - C_{10}(x, y))\sum_{k\geq 2}\frac{x^{k+1}}{(1-x)^k}\frac{x^{k-1}}{(1-x)^{k-1}}$.

Combining these different cases, we deduce the functional equation:

$$\begin{aligned} C_{210}(x, y) = \quad & 1 + xC_{210}(x, y) + x(C_{210}(x, y) - 1) + xy(C_{01}(x, y) - 1)(C_{210}(x, y) - 1) + \\ & y(C_{10}(x, y) - 1)\sum_{k\geq 2}\frac{x^{k+1}}{(1-x)^k} + y(C_{210}(x, y) - C_{10}(x, y))\sum_{k\geq 2}\frac{x^{k+1}}{(1-x)^k}\frac{x^{k-1}}{(1-x)^{k-1}}. \end{aligned}$$

Finally, considering Theorem 4 the desired result follows. □

Corollary 10. *For $p = 210$, we have $C_p(x) = \frac{1-5x+7x^2-x^3-x^4}{(1-2x)(1-4x+3x^2+x^3)}$ and the generating function for the popularity of descents in the set $\mathcal{C}_n(p)$, $n \geq 0$, is*

$$\frac{x^3(1 - 2x)}{(1 - 4x + 3x^2 + x^3)^2}.$$

4. Final remarks

At the time of writing this paper, the enumerating sequences $(|\mathcal{C}_n(p)|)_{n\geq 0}$, for $p \in \{000, 110, 210\}$, are not recorded in [12], and it will be interesting to find alternative interpretations of these sequences.

Our initiating study on pattern avoidance on Catalan words can naturally be extended to patterns of length more than three, vincular patterns (see Section 1.3 in [8] for their definition) and/or multiple pattern avoidance. For example, some of the patterns we considered here hide larger length patterns (for instance, an occurrence of 210 in a Catalan word is a part of an occurrence of 01210), and some of our results can be restated in this light.

Acknowledgments

We would like to thank the two anonymous referees for their careful reading of this paper and their helpful comments and suggestions.

References

[1] M. Albert, C. Homberger, J. Pantone, Equipopularity classes in the separable permutations, Electron. J. Combin. 22 (2) (2015) P2.2 (electronic).
[2] J.-L. Baril, S. Kirgizov, V. Vajnovszki, Patterns in treeshelves, Discrete Math. 340 (12) (2017) 2946–2954.
[3] C. Berg, C. Stump, V. Pons, T. Scrimshaw, The combinatorial statistic finder, www.findstat.org, 2013.
[4] F. Bergeron, N. Bergeron, R.B. Howlett, D.E. Taylor, A decomposition of the descent algebra of a finite Coxeter group, J. Algebraic Combin. 1 (1992) 23–44.
[5] M. Bóna, Surprising symmetries in objects counted by Catalan numbers, Electron. J. Combin. 19 (1) (2012) P62 (electronic).
[6] I. Gessel, G. Viennot, Binomial determinants, paths, and hook length formulae, Adv. Math. 58 (1985) 300–321.
[7] C. Homberger, Expected patterns in permutation classe, Electron. J. Combin. 19 (3) (2012) P43 (electronic).
[8] S. Kitaev, Patterns in Permutations and Words, Springer, 2011.
[9] E. Mäkinen, Left distance binary tree representations, BIT Numer. Math. 27 (2) (1987) 163–169.
[10] T. Mansour, V. Vajnovszki, Efficient generation of restricted growth words, Inform. Process. Lett. 113 (2013) 613–616.
[11] K. Rudolph, Pattern popularity in 132-avoiding permutations, Electron. J. Combin. 20 (1) (2013) P8 (electronic).
[12] N.J.A. Sloane, The on-line encyclopedia of integer sequences. Available electronically at http://oeis.org.
[13] R.P. Stanley, Enumerative Combinatorics, Vol. 2, Cambridge University Press, 1999.

Computers & Graphics: X 2 (2019) 100006

Contents lists available at ScienceDirect

Computers & Graphics: X

journal homepage: www.elsevier.com/locate/cagx

Special Section on SMI 2019

Representation of NURBS surfaces by Controlled Iterated Functions System automata

Lucas Morlet*, Christian Gentil*, Sandrine Lanquetin, Marc Neveu, Jean-Luc Baril

Laboratoire Informatique de Bourgogne (LIB), Univ. Bourgogne Franche - Comté, 8 avenue Alain Savary, 21000 Dijon, France

ARTICLE INFO

Article history:
Received 9 March 2019
Revised 24 May 2019
Accepted 24 May 2019
Available online 7 June 2019

Keywords:
Geometric modeling
Iterative modeling
Non-Uniform Rational B-Splines
Iterated Functions Systems
CIFS automata

ABSTRACT

Iterated Function Systems (IFS) are a standard tool to generate fractal shapes. In a more general way, they can represent most of standard surfaces like Bézier or B-Spline surfaces known as self-similar surfaces. Controlled Iterated Function Systems (CIFS) are an extension of IFS based on automata. CIFS are basically multi-states IFS, they can handle all IFS shapes but can also manage multi self-similar shapes. For example CIFS can describe subdivision surfaces around extraordinary vertices whereas IFS cannot. Having a common CIFS formalism facilitates the development of generic methods to manage interactions (junctions, differences...) between objects of different natures.

This work focuses on a CIFS approach of Non-Uniform Rational B-Splines (NURBS) which are the main used representation of surfaces in CAGD systems. By analyzing the recursive generating process of basis functions, we prove the stationarity of NURBS computation. This implies that NURBS can be represented as a finite automaton: a CIFS. Subdivision transformations implied in the generating process are directly deduced from blossoming formulation and are expressed as a function of the initial nodal vector. We provide a method to construct the CIFS automata for NURBS of any-degree. Then NURBS-surfaces automata are deduced using a "tensor-product" of NURBS automata. This new representation of NURBS allows us to build a bridge between them and other surfaces already represented in CIFS formalism: fractals and subdivision surfaces.

© 2019 The Authors. Published by Elsevier Ltd.
This is an open access article under the CC BY-NC-ND license.
(http://creativecommons.org/licenses/by-nc-nd/4.0/)

1. Introduction

Iterative processes based on fractal geometry is a common way to produce complex objects like terrains, clouds, trees, textures [1,2]. Generally, these processes used stochastic parameters to model natural objects with non-strict repetition of the same pattern. These algorithms are widely used in Computer Graphics but few in Computer Aided Geometric Design (CAGD), even if resulting shapes present interesting properties like, for example, rough surfaces to improve heat exchange, porous volumes needing less material or simply for their aestetic value. The purpose of our research is to develop a generic iterative geometric modeler for industrial applications associated to additive manufacturing. A first application, related to energy saving which is a crucial issue for industry, consists of designing lighter objects while maintaining high mechanical properties. To address this challenge we suggest to consider multiscale controlled lacunary structures which fill a given geometric hull, with imposed functional surface. A second application is tree-like structures support design for additive manufacturing or for reinforcing a given functional surfaces (see Fig. 1). In both cases, imposed geometric inputs, geometric hulls or functional surfaces, come from a CAGD software and are generally represented by Non-Uniform Rational B-Splines (NURBS). From the past decades, NURBS imposed themselves into CAGD. They are a powerful tool to construct smooth surfaces while respecting usual constraints of industrial conception. Our iterative modeler have to interact with CAGD software and needs to integrate NURBS representation.

From a theoretical point of view, it could be interesting to have a common representation for fractals, subdivision surfaces and NURBS to identify possible interactions. From a practical point of view, we could take advantage of each representation, without conversions and approximations.

Few generic representations were proposed for iterated processes. Most of the time, an iterative process is translated in a specific algorithm with a specific data structure and implemented in a specific software for a specific application. The two main generic representations are L-system [3] and Iterated Function

* Corresponding authors.
E-mail addresses: lucas.morlet@u-bourgogne.fr (L. Morlet), christian.gentil@u-bourgogne.fr (C. Gentil).

https://doi.org/10.1016/j.cagx.2019.100006
2590-1486/© 2019 The Authors. Published by Elsevier Ltd. This is an open access article under the CC BY-NC-ND license. (http://creativecommons.org/licenses/by-nc-nd/4.0/)

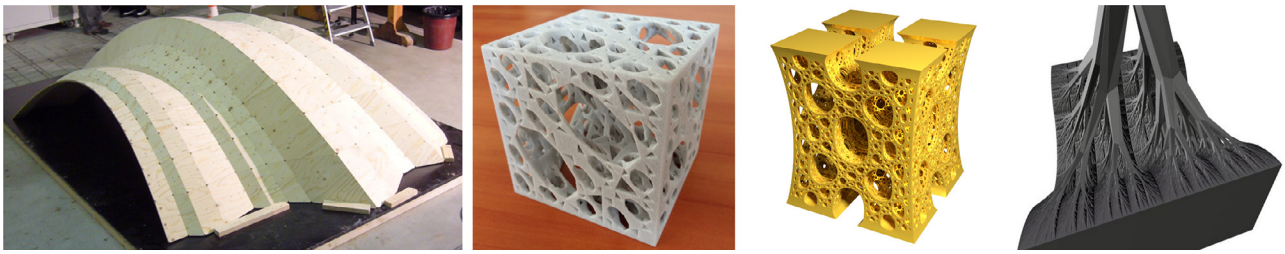


Fig. 1. Original creations of CIFS-based CAGD-modeling. From left to right: a wooden overlapping structure which is smooth in one direction and fractal in the second direction (©IBOIS, laboratory of EPLF), a lacunar structure build by 3D printing, a lacunar structure whose edges are B-Splines using topological optimization ($\approx$88% of material saved compared to a filled structure) and an arborescent fractal structure that support a bi-quadratic Bézier surface.

System (IFS) [4] which are quasi-equivalent [5]. L-system representation is based on rewriting rules while IFS representation is based on a self-similarity property. For our purpose we believe that IFS are more suitable according to their formulation which is easier for formal manipulation associated to CAGD. Iterated Function Systems (IFS) have been introduced by Hutchinson [4] and developed by Barnsley [6] to study self-similar shapes. Even if they are initially created for fractals, they can represent other types of objects: Goldman [7] and Schaeffer et al. [8] used this formalism to respectively generate Bèzier and B-Spline tensor-surfaces. In previous works, Zaïr and Tosan [9] introduce fractal modeling using free-form techniques which includes Bèzier and uniform B-spline curves and surfaces. Due to the presence of irregular vertices, subdivision surfaces are not self-similar and cannot be handled by IFS. To overpass this constraint, Morlet et al. [10] used Controlled Iterated Function Systems (CIFS) to generate low-degree uniform subdivision surfaces in an iterative way. CIFS are an extension of IFS based on automata whose states can represent different objects, self-similarities are no longer required. Every IFS can be described as a one-state CIFS-automaton.

In addition, to bring together usual CAGD representations in a common formalism, CIFS-representation provides specific tools. For instance, Podkorytov et al. [11] proved that interactions between two objects of different natures can be performed since they are both represented as CIFS-automata.

At this point, our CIFS-based CAGD-modeler does not handle NURBS, and this is the main purpose of this paper. We prove that NURBS tensor-surfaces of any degree can be represented as CIFS-automata and so can be integrated in our iterative modeler. Then we avail this integration for new applications.

2. Overview

First of all, we introduce in Section 3 all the notations and mathematical background that will be used in the paper. We explain two usually separate research fields: IFS and their extension CIFS in a first part and NURBS and their blossoming representation in a second one. Our work is placed at the cross of these two research fields.

In Section 4, we present how to create NURBS CIFS-automata. The automata for quadratic and cubic cases are fully described and a method to generate any-degree automaton is proposed.

Furthermore we explain how to generate "tensor-products" of complete automata in Section 5. Following this method we create bi-quadratic and bi-cubic NURBS tensor-surfaces CIFS-automata. Then we prove that any NURBS tensor-surfaces can be represented as a CIFS automaton.

In Section 6, we present several solutions that our formalism brings to CAGD-industry usual challenges. First, we use the formalism of Section 5 to generate blending tensor-surfaces (*i.e.* the two generative curves are of different natures). Then we show how to create a junction that respects constraints between a NURBS tensor-surface and a surface already defined as a CIFS-automaton. To finish we present the application of this formalism to the topological optimization field: we create a lacunar NURBS tensor-surface and an arborescent structure that support a NURBS tensor-surfaces.

To conclude, all contributions are summarized and research tracks for extension to any-degree non-tensor NURBS surface are proposed in Section 7.

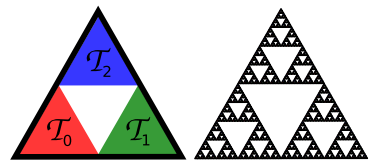


Fig. 2. IFS of Sierpinski triangle and its attractor. Transformations $\mathcal{T}_0, \mathcal{T}_1$, and $\mathcal{T}_2$ associate the black triangle to one of the three sub-triangles (respectively red, green, and blue). These transformations are homotheties of scale factor 0.5 where the center is one of the vertices of the initial triangle.

3. Background

3.1. Iterated Functions Systems (IFS)

According to Huchinson [4], an IFS is a set of n contracting transformations of a complete metric space $(\mathcal{X}, d)$: $\{\mathcal{T}_i\}_{i\in\Sigma}$ with $\Sigma = \{0, 1, \ldots, n-1\}$. Let $\mathcal{K}$ be a compact (*i.e.* a subset of an Euclidean space which is both closed and bounded), the Hutchinson operator $\mathcal{H}$ is defined by:

$$\mathcal{H}(\mathcal{K}) = \bigcup_{i=0}^{n-1} \mathcal{T}_i(\mathcal{K})$$

In a same way as a contractive transformation owns a unique fixed-point, there is a unique compact, $\mathcal{A}$, referred as the attractor of the IFS, which is defined as:

$$\mathcal{H}(\mathcal{A}) = \mathcal{A} \tag{1}$$

Due to the contractivity of the Huchinson operator, the attractor can be computed as the result of an infinite application of Hutchinson operator on any starting compact $\mathcal{K}$.

$$\mathcal{A} = \underbrace{\mathcal{H} \circ \ldots \circ \mathcal{H}}_{\infty}(\mathcal{K})$$

An usual example of IFS is the Sierpinski triangle presented in Fig. 2.

Every point belonging to the attractor can be indexed by at least one infinite word $\sigma = \sigma_0\sigma_1 \cdots \sigma_\ell \cdots$ of Σ ($\sigma \in \Sigma^\infty$) such that:

$$\lim_{\ell \longrightarrow \infty} \mathcal{T}_{\sigma_0} \circ \mathcal{T}_{\sigma_1} \circ \cdots \circ \mathcal{T}_{\sigma_\ell}(p) = p_\sigma, \ \forall p \in \mathcal{X}.$$

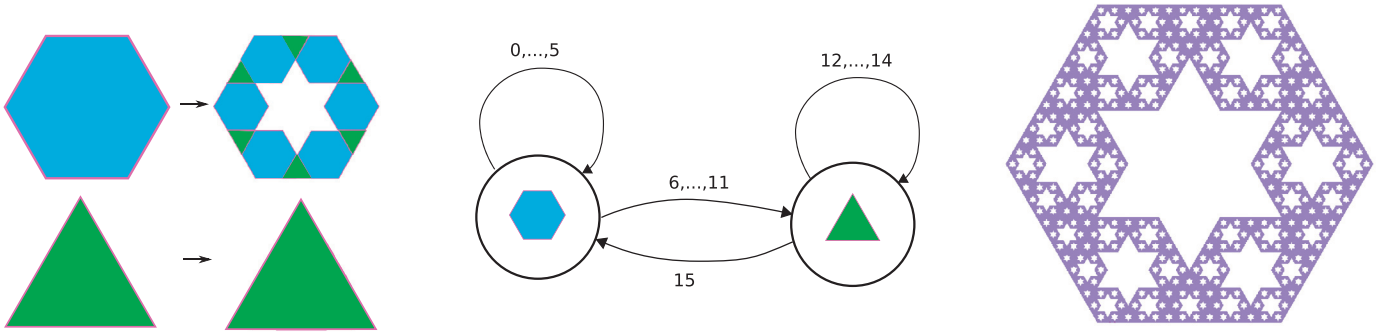


Fig. 3. Example of a CIFS automaton composed of two states: "blue hexagon" and "green triangle". On the left, the decomposition rules of each state. On the center, the CIFS automaton deduced from decomposition rules. On the right, the resulting attractor.

In other words, the infinite composition of the corresponding transformations leads to the point p_σ. σ is referred as an address of the point p_σ.

To any infinite word $\sigma \in \Sigma^\infty$ corresponds a point of $\mathcal{A}$. In particular, the address i^ω (composed of an infinite number of repetition of i), corresponding to the fixed point of $\mathcal{T}_i$ belongs to $\mathcal{A}$, $p_{i^\omega} \in \mathcal{A}$. This property is important, because if $\mathcal{T}_i$ are affine or linear transformations (which will be the case in the following) we can directly compute a point belonging to $\mathcal{A}$ in a finite time by calculating the fixed point of $\mathcal{T}_i$ by eigen-analysis. Furthermore, according to Eq. (1), $\mathcal{T}_j p_{i^\omega}$ belongs to $\mathcal{A}$ too. This means that we can compute a tesselation of $\mathcal{A}$ with all vertices belonging to $\mathcal{A}$ in finite time. The tesselation is calculated from a finite number of iterations of the Hutchinson operator. The starting compact $\mathcal{K}$ is set as a list of vertices of a 1D, 2D or 3D mesh, each vertex being a fixed points of a tranformation $\mathcal{T}_i, i \in \Sigma$. This approach is equivalent to Halstead et al. [12] and Stam [13] methods which directly compute a piece of subdivision surface from a given patch of control mesh.

3.2. Controlled Iterated Functions Systems (CIFS)

CIFS are an extension of IFS that provides more control. Instead of defining the attractor as all points P_σ, $\sigma \in \Sigma^\omega$, the attractor is restricted to points whose addresses are accepted by a given automaton.

A CIFS is defined by:

- an automaton $(\Sigma, Q, \delta, \natural)$, where Σ is an alphabet, Q a set of states, δ a transition function (δ: $Q \times \Sigma \to Q$), $\natural$ the initial state and all states of Q are final states;
- a set of spaces associated to the states : $(\mathcal{X}^q)_{q \in Q}$;
- a set of transformations associated to the transitions T_i^x : $E^{\delta(x,i)} \to E^x$.

To each state is associated an attractor $\mathcal{A}^q$ defined by:

$$\mathcal{A}^q = \bigcup_{i \in \Sigma^q} T_i^q(\mathcal{A}^{\delta(q,i)}) \tag{2}$$

where $\Sigma^q = \{i \in \Sigma, \delta(q,i) \in Q\}$.

The attractor of the CIFS is the attractor of the initial state, i.e. $\mathcal{A} = \mathcal{A}^\natural$.

From Eq. (2), we understand that a transformation associated to a transition copy the attractor of the destination state into the attractor of the starting state. Then interpretation and design of an automaton is intuitive. An example can be found in Fig. 3.

The last step to complete the needed material, is the introduction of free-form shapes modeling with CIFS. It is straightforward by building the attractors in barycentric space and then project it into the modeling space according to a set of control points. The final modeled shape is $F = \mathbf{P}\mathcal{A}$, where $\mathbf{P} = [p_0 p_1 \dots p_k]$ is a line vector composed of the list of control points. The projection in the modeling space is included in the automaton by adding an initial state and a transition with $\mathbf{P}$ as the associated transformation. For example a quadratic uniform spline is defined by the automaton of Fig. 4, where $\mathbf{P} = [p_0 p_1 p_2]$ (the three control points), T_0^q and T_1^q represent the two Chaikin matrices and K is composed of the fixed points of T_0^q and T_1^q:

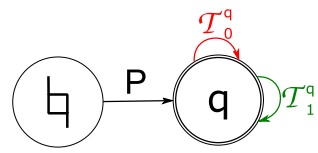


Fig. 4. CIFS automaton corresponding to Chainkin algorithm.

$$K = \begin{pmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix}.$$

The attractor associated to state q is the quadratic uniform spline basic functions and the Spline curve is $\mathcal{A}^\natural = \mathbf{P}\mathcal{A}^q$. A tesselation of the curve is obtained by computing, for a given level of iteration (approximation) ℓ: $\{\mathbf{P}\mathcal{T}_{\sigma_0} \circ \dots \circ \mathcal{T}_{\sigma_\ell} K, \sigma_i \in \Sigma\}$.

By using different states, CIFS provides more control than IFS. For example, B-Spline tensor surfaces can easily be represented by an IFS [8] whereas subdivision surfaces, around extraordinary vertices, cannot. Indeed, management of extraordinary patches is impossible with IFS formalism because an irregular patch is subdivided into several regular and irregular patches. At least two states are necessary: regular and irregular. More generally, transformations that change the nature of the object can only exist in CIFS formalism because different objects are simply represented by different states. Morlet et al. [10] prove that any low-degree uniform subdivision schemes can be represented by a CIFS automaton instead of a set of rules. They propose a generic efficient visualization of subdivision surfaces on GPU, based on tesselation of attractors.

3.3. Non-Uniform Rational B-Splines

B-Splines are a user-friendly tool to define smooth parametric curves by a list of points called control points that form the control polygon. A B-Spline curve $\mathcal{C}$ of degree d is defined from a control polygon composed of $d+1$ points $\mathbf{P} = [p_0 \dots p_d]$.

A B-Spline curve can also be defined by pieces: a curve composed of m pieces is defined by $\mathbf{P} = [p_0 \dots p_{d+m-1}]$ where every sub-polygon $[p_i \dots p_{i+d-1}]$ defines a B-Spline of degree d. Even if B-Splines are built to be defined by pieces, each piece is independently computed; so only one-piece curves are treated in this article.

B-Splines basis functions are computed given a knot vector. For a one-piece B-Spline curve of degree d, the knot vector is $\mathbf{T} = [t_0 = 0 \leq \cdots \leq t_{(2d-1)} = 1]$. Every point of the curve $\mathcal{C}(t)$ is computed by the well-known Cox–De-Boor formula [14].

What is important for curves computation is not the knot values but differences between two consecutive knots and the ratio between these differences. Thus, the nodal vector $\mathbf{T}$ is often replaced by the inter-nodal vector $\mathbf{U} = [u_0 \ldots u_{(2d-2)}]$ where $u_i = t_{i+1} - t_i$. In this article, the most convenient is used according to each situation.

Uniform B-Splines are often used (in this case it is convenient to set $\mathbf{T} = [t_0 = 0, t_1 = 1, < \cdots < t_{(2d-1)} = 2d-1]$ and then $u_0 = \cdots = u_{(2d-2)} = 1$).

Non-Uniform B-Splines (NUBS) offer more control over the curve. In this case u_i is computed according to user defined constraints (centripetal, chord length, ...)

The main problem NUBS have to face is their impossibility to handle conic shapes (*i.e.* circles, ellipses...). To overpass this issue a new concept is added: the rationality. NUBS then become Non-Uniform Rational B-Splines (NURBS) and a new coordinate is added to every control points. These coordinates increase or decrease points influence in a way similar to weight points. This rationality modifies the shape of the curves by adding an extra step in the computation which is independent from the non-uniformity. The rationality can be achieved simply by using control points with homogeneous coordinates. But it is not out purpose here, our work focuses only on the non-uniformity step and the term NURBS is used in a generic way.

3.4. Blossoming

NURBS are usually computed with the Cox-De-Boor formula [14] but Ramshaw [15] introduced an other way to understand NURBS called blossoming. A blossom of a curve of degree d is represented by a label composed of d arguments: $\{t_i \ldots t_{i+d-1}\}$. Blossoming is completely characterized by three properties: symmetry, diagonal, and multi-affinity. From the three previous properties, another one is deduced: the consecutivity.

Symmetry:

The order of arguments in a label have no consequence on blossoms. Thus, they can and will be always written in ascending order:

$$\{\ldots t_i \ldots t_j \ldots\} = \{\ldots t_j \ldots t_i \ldots\} \qquad (j < i)$$

Diagonal:

Every point of the limit curve is defined by a label composed of identical arguments. This repeated argument is also the curve parameter.

$$\mathcal{C}(t) = \{t \ldots t\}$$

Multi-affinity:

Every label can be defined as an affine combination of two others if these three labels only differ by one argument. Weights of the labels combination correspond to weights of the arguments affine combination:

$$\{\ldots t \ldots\} = \frac{b-t}{b-a}\{\ldots a \ldots\} + \frac{t-a}{b-a}\{\ldots b \ldots\}$$

Consecutivity:

A label whose arguments are consecutive nodes of the knot-vector corresponds to a control point; its index is the first index of arguments:

$$p_i : \{t_i, t_{i+1} \ldots t_{i+d-1}\}$$

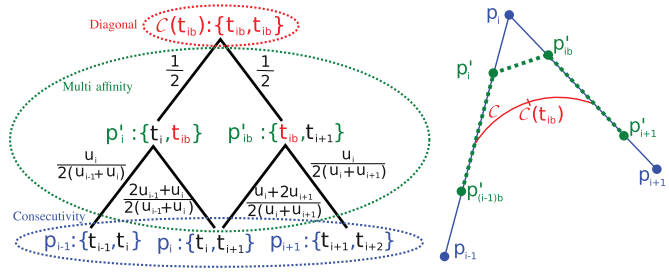


Fig. 5. On the left, the blossoming computation of the curve point $\mathcal{C}(t_{ib})$ where $t_{ib} = (t_i + t_{i+1})/2$. From this blossoming representation the influence of control points p_{i-1}, p_i, p_{i+1} are deduced: $\frac{1}{2} \cdot \frac{u_i}{2(u_{i-1}+u_i)}$, $\frac{1}{2} \cdot \frac{2u_{i-1}+u_i}{2(u_{i-1}+u_i)} + \frac{1}{2} \cdot \frac{u_i+2u_{i+1}}{2(u_i+u_{i+1})}$, and $\frac{1}{2} \cdot \frac{u_i}{2(u_i+u_{i+1})}$. On the right, the application of this result to compute this point of a quadratic NURBS (red) from a control polygon (blue). An intermediate polygon (green) that defines the same NURBS as the original one can also be computed according to the coefficients of the bottom branches of the blossoming. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Following these four properties, we can compute every point of the curve as a function of the control polygon. First, the diagonal property is used to create the label corresponding to the point of the curve and this label is placed as the root of the blossoming. Each label that does not respect the consecutive property have to be split into two new labels with respect to multi-affinity. The weights of affine combination are written on the branches that link the parent label to its two sons. A consecutive label is a leaf of the tree: it corresponds to a point of the control polygon. The weight of a path that links the root to a leaf is the product of branches weight it passes through. The influence of a control polygon point (leaf) on the curve point we compute (root) is the sum of paths weights joining the root to the leaf. An example of blossoming where a point of quadratic NURBS is computed is given in Fig. 5.

4. Representation of NURBS by CIFS-automata

In this section, we prove that NURBS with any-degree can be represented as a CIFS-automaton. The number of states is directly related to the degree of the NURBS. First the simplest cases (quadratic and cubic) are presented, then the generalization to any degree is explained and mathematically proven by generating words provided grammar rules.

4.1. NURBS and self-similarity

Let a NURBS be defined by a pair composed of a control polygon $\mathbf{P}$ and a knot-vector $\mathbf{T}$. If a new knot is inserted into $\mathbf{T}$, the new knot-vector $\mathbf{T}'$ can be associated to a new control polygon $\mathbf{P}'$ which defines the same NURBS as $(\mathbf{P}; \mathbf{T})$. An example can be found in Fig. 5 where the same curve is defined by the two pairs:

$$\begin{cases} [p_{i-1}, p_i, p_{i+1}] \\ [t_{i-1}, t_i, t_{i+1}, t_{i+2}] \end{cases} \text{and} \begin{cases} [p_{i-1}, p'_i, p'_{ib}, p_{i+1}] \\ [t_{i-1}, t_i, t_{ib}, t_{i+1}, t_{i+2}] \end{cases}$$

The usual way to exploit this property is the mid-knot insertion strategy [16,17]: a new knot is inserted in the middle of each consecutive knots of $\mathbf{T}$. Then all knots of $\mathbf{T}'$ are multiplied by two because there are not the knots but the ratios between them that influences the shape of the curve. To finish, extreme knots are dropped because they do not influence the curve anymore. The mid-knot insertion strategy is equivalent to inter-node doubling: each inter-node of $\mathbf{U}$ is written twice in a row. Due to modification on knot/inter-node vector, the control polygon $\mathbf{P}$ has to be modified to $\mathbf{P}'$. For instance:

$$\mathbf{T} = [a, b, c, d] \rightarrow \mathbf{T}' = \left[a, \frac{a+b}{2}, b, \frac{b+c}{2}, c, \frac{c+d}{2}, d\right]$$

$\rightarrow \mathbf{T}' = [2a, a+b, 2b, b+c, 2c, c+d, 2d]$

$\rightarrow \mathbf{T}' = [a+b, 2b, b+c, 2c, c+d]$

$\mathbf{U} = [b-a, c-b, d-c] \rightarrow \mathbf{U}' = [b-a, c-b, c-b, d-c]$

$\mathbf{P} = [p_0, p_1, p_2] \rightarrow \mathbf{P}' = [p'_0, p'_1, p'_2, p'_3]$

This new pair (**P'** ; **U'**) defines the same NURBS as (**P** ; **U**) but the curve is no longer a one-piece curve: it is now composed of two pieces. These two pieces are both one-piece curves with the same degree as the original one: there is a self-similarity. To avail this self-similarity, we define two transformations, $\mathcal{L}$ and $\mathcal{R}$, which associate the original NURBS to the left and the right part of the new NURBS. These transformations map a NURBS to another one. This implies they transform both the control polygon and the inter-nodal vector into new ones.

$\mathcal{L} : (\mathbf{P}\ ;\ \mathbf{U}) \mapsto (\mathbf{P}_{\mathcal{L}}\ ;\ \mathbf{U}_{\mathcal{L}})$

$\mathcal{R} : (\mathbf{P}\ ;\ \mathbf{U}) \mapsto (\mathbf{P}_{\mathcal{R}}\ ;\ \mathbf{U}_{\mathcal{R}})$

4.2. Quadratic NURBS CIFS-automaton

The length of the control polygon and the inter-nodal necessary to define a one-piece NURBS vector depends on the degree and so $\mathcal{L}$ and $\mathcal{R}$ have to be defined for every degree. For instance, in quadratic case, they are defined by this way:

$\mathcal{L} : (\mathbf{P}\ ;\ [a, b, c]) \mapsto (\mathbf{P}\mathcal{M}_{\mathcal{L}}(a, b, c)\ ;\ [a, b, b])$

$\mathcal{R} : (\mathbf{P}\ ;\ [a, b, c]) \mapsto (\mathbf{P}\mathcal{M}_{\mathcal{R}}(a, b, c)\ ;\ [b, b, c])$

where $M_{\mathcal{L}}$ and $M_{\mathcal{R}}$ are the matrices defined below. Starting from an inter-node vector **U** = [v, w, x], the two transformations $\mathcal{L}$ and $\mathcal{R}$ create two new inter-node vector configurations [v, w, w] and [w, w, x] which are both centered on the same inter-node value (the center value of the parent configuration: w). Each new configuration can be associated with a piece of the two-pieces NURBS, or for self-similarity reason two one-piece NURBS. Both new configurations produce two configurations: themselves (that already exist) and another new uniform configuration: [w, w, w]. At this point, no more configuration can be found: every built configuration is transposed as a state of the quadratic NURBS CIFS-automaton (see Fig. 6).

Transformations are always labeled in a same way but the coefficients of the associated matrices are not always the same: they are parameterized by the three inter-nodes of **U** relatively to the starting state of the transition.

These matrices are directly deduced from the weights of branches between the two bottom rows of Fig. 5:

$$\mathcal{M}_{\mathcal{L}}(u_1, u_2, u_3) = \begin{pmatrix} \frac{u_1+2u_2}{2(u_1+u_2)} & \frac{u_2}{2(u_1+u_2)} & 0 \\ \frac{u_1}{2(u_1+u_2)} & \frac{2u_1+u_2}{2(u_1+u_2)} & \frac{u_2+2u_3}{2(u_2+u_3)} \\ 0 & 0 & \frac{u_2}{2(u_2+u_3)} \end{pmatrix}$$

$$\mathcal{M}_{\mathcal{R}}(u_1, u_2, u_3) = \begin{pmatrix} \frac{u_2}{2(u_1+u_2)} & 0 & 0 \\ \frac{2u_1+u_2}{2(u_1+u_2)} & \frac{u_2+2u_3}{2(u_2+u_3)} & \frac{u_3}{2(u_2+u_3)} \\ 0 & \frac{u_2}{2(u_2+u_3)} & \frac{2u_2+u_3}{2(u_2+u_3)} \end{pmatrix}$$

Notice the two matrices have two rows in common. This ensures the continuity of the junction between two pieces of quadratic NURBS.

From both matrices, we can directly compute the position of the fixed-point and the tangent at this point using the following barycentric combinations $\mathcal{B}$ and τ:

$$\begin{cases} \mathcal{B}_{\mathcal{L}} = \left(\frac{u_2}{u_1+u_2}\ ;\ \frac{u_1}{u_1+u_2}\ ;\ 0\right)^{\top} \\ \tau_{\mathcal{L}} = (-1\ ;\ 1\ ;\ 0)^{\top} \end{cases}$$

$$\begin{cases} \mathcal{B}_{\mathcal{R}} = \left(0\ ;\ \frac{u_3}{u_2+u_3}\ ;\ \frac{u_2}{u_2+u_3}\right)^{\top} \\ \tau_{\mathcal{R}} = (0\ ;\ -1\ ;\ 1)^{\top} \end{cases}$$

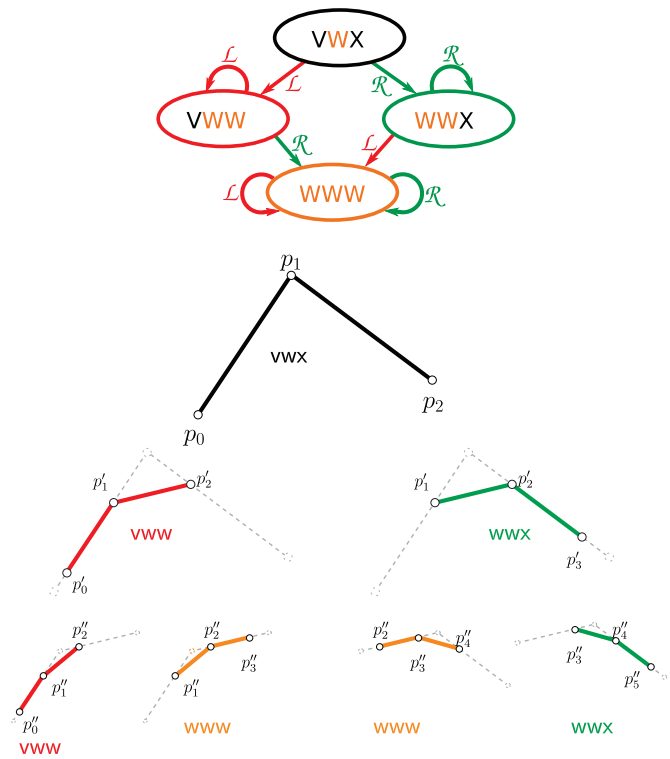


Fig. 6. The quadratic NURBS CIFS-automaton whose starting inter-node vector is [v,w,x]. The states are labeled as states of the corresponding configuration of inter-node vector. Below, the control polygon **P** is colored in black. After one iteration, the left and the right parts of the resulting polygons are respectively colored in red for the state vww and in green for the state wwx. For the second iteration, the two middle parts colored in brown define uniform B-Splines. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

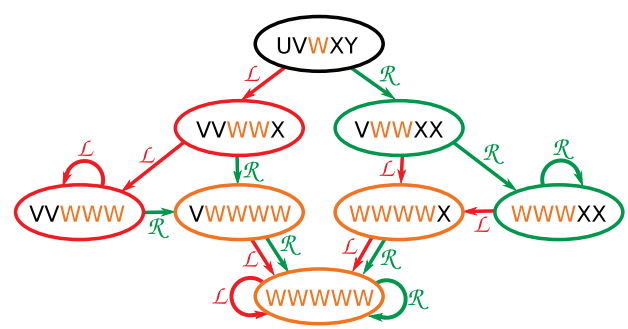


Fig. 7. The automaton used to generate cubic NURBS. Starting internode vector is [u,v,w,x,y] and states are labeled as states of the corresponding internode vector.

4.3. Cubic NURBS CIFS-automaton

Cubic NURBS are handled quite in the same way. They are at least defined by $\mathbf{P} = [p_0 \dots p_3]$ and $\mathbf{T} = [t_0 \dots t_5]$ or $\mathbf{U} = [u_0 \dots u_4]$. The automaton used to generate cubic NURBS is presented in Fig. 7. The transformation matrices, which are given in Appendix B are deduced from the blossoming form in Fig. 8.

4.4. High-degree NURBS CIFS-automaton

In CAGD-engineering, surfaces can be subject to constraints such as interpolating some points, continuity of junctions between surfaces *etc.* An usual way to solve these constraints for NURBS is degree elevation. This elevation involves a management of NURBS

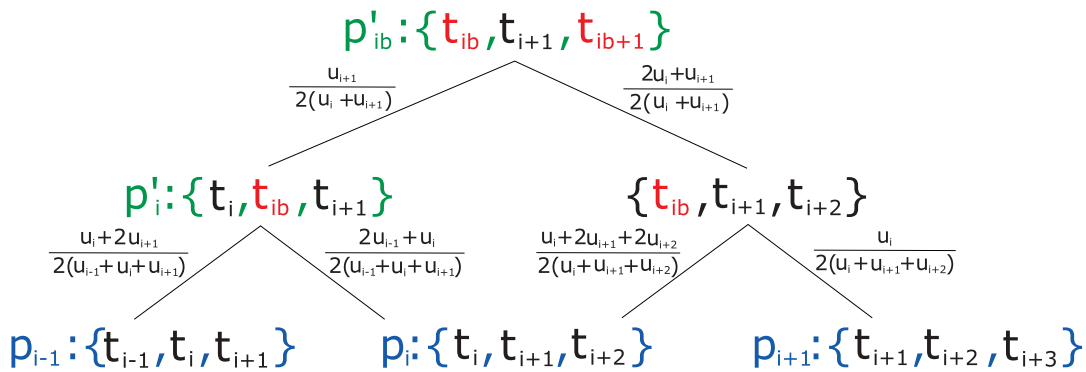


Fig. 8. The computation by blossoming of $\mathbf{P}'$ as a function of $\mathbf{P}$. The convention is $t_{jb} = (t_j + t_{j+1})/2$.

of any-degree, in particular of high-degree because a surface can be subject to many constraints.

A CIFS-automaton is available if and only if its number of states is finite. For NURBS CIFS-automata, each state corresponds to a possible configuration of inter-nodal vector. Before any iteration, this inter-nodal vector is composed of any values. Every iteration duplicates all inter-nodes and then deletes the extreme ones. Every sub-sequence of values (of a fixed length) is a possible state of the inter-nodal vectors. After some iterations, there only remains sub-sequences composed of at most two different values repeated several times. At this point, no more additional states can be found. High-degree automata are directly generated according to this algorithm. Mathematical proofs about the number of states based on a combinatorial approach of the generated words can be found in Appendix A.

5. Extension to NURBS tensor-surfaces

NURBS tensor-surfaces are created by a tensor-product of NURBS. They lie on a regular control grid and own two "orthogonal" inter-nodal vectors **U** and **V**. The automaton used to generate these surfaces can be viewed as a "tensor-product" of two curves-automata. In this section, the tensor-product of automata is explained and then used to generate NURBS tensor-surfaces CIFS automata.

5.1. Tensor-product of finite automata

Let **A** and **B** two finite automata whose states are respectively $\{\alpha_0 \dots \alpha_m\}$ and $\{\beta_0 \dots \beta_n\}$ and whose transitions are respectively $\{a_0 \dots a_p\}$ and $\{b_0 \dots b_q\}$. Let **C** be the automaton which is the "tensor-product" of **A** and **B**. The initial state of **C** is defined as the composition of the initial states of **A** (α_0) and **B** (β_0) and is labeled (α_0; β_0). For every pair of transitions a_i (that links α_0 to α_k) and b_j (β_0 to β_l), a new transition of **C**, $c_{(i,j)}$ is created between the initial state (α_0; β_0) to a potentially new state (α_k; β_l). If this state already exists, the transition is just added to the automaton; otherwise this new state is added to the automaton and its transitions are studied by the same process as the initial state.

From this generation, two properties are straightforward:

- **C** is finite and its numbers of states and transitions are respectively upper-bounded by $(m.n)$ and $(k.l)$.
- If **A** and **B** are complete (*i.e.* every transition is defined for every state), **C** is also complete.

5.2. NURBS-tensor surfaces CIFS automata

In the case where **A** and **B** are CIFS-automata that generate a curve and **C** an automaton of tensor-surface, the term "tensor-product" makes all its sense. Indeed, in this particular case, transformation matrices are tensor-products $\mathcal{M}_{c_{(i,j)}} = \mathcal{M}_{a_i} \otimes \mathcal{M}_{b_j}$. Furthermore, the left-eigenvector of matrices $\mathcal{M}_{c_{(i,j)}}$ are also tensor-product of left-eigenvectors of $\mathcal{M}_{a_i}$ and $\mathcal{M}_{b_j}$. In particular, the fixed point and its associated main-tangents and normal of a transformation γ defined as the tensor-product of two transformations α and β are:

$$\mathcal{B}_\gamma = \mathcal{B}_\alpha \otimes \mathcal{B}_\beta$$
$$\tau_\gamma = \mathcal{B}_\alpha \otimes \tau_\beta$$
$$\tau'_\gamma = \mathcal{B}_\beta \otimes \tau_\alpha$$

Let **A** be an "horizontal" NURBS CIFS-automaton whose transformations are $\mathcal{L}$ and $\mathcal{R}$ and **B** a "vertical" NURBS automaton whose transformations are respectively changed to $\mathcal{D}$ and $\mathcal{U}$. Four transformations are defined by the tensor-product of **A** and **B**:

$\otimes$	$\mathcal{L}$	$\mathcal{R}$
$\mathcal{U}$	$\mathcal{T}_3 = \mathcal{U} \otimes \mathcal{L}$	$\mathcal{T}_2 = \mathcal{U} \otimes \mathcal{R}$
$\mathcal{D}$	$\mathcal{T}_0 = \mathcal{D} \otimes \mathcal{L}$	$\mathcal{T}_1 = \mathcal{D} \otimes \mathcal{R}$

By using the tensor-product of automata presented in the previous sub-section, the bi-quadratic and bi-cubic automata are generated (see Fig. 9). Some examples of NURBS tensor-surfaces generated from this CIFS automata are given in Fig. 10.

Even if only low degree automata are presented, this method applies to any degree NURBS tensor-surfaces. The only pitfall to face is the explosion of the number of states that comes when the degree increases. This issue is inherent to NURBS: the recursive process of generation always implies difficulties related to the degree.

By generating several tensor automata from curves-automata of different degree, some properties appear. The first one is once a state is leaved, it is impossible to turn back to it. The second one is the number of stationary states (*i.e.* states that own at least one transition to themselves) is always 3 for curves-automata (labeled as left, right, and uniform); other states are transitory: they are active only once. The number of stationary states of a tensor-automaton is the product of the number of stationary states of the two original automata: it is 3 for curves, 9 for surfaces, 27 for volumes. Whatever the number of stationary states, there is always only one state, called the uniform state, which loops to itself for all transformations. This uniform state is the IFS that generates uniform B-Splines shapes (curve, surface, volume).

6. Applications

The representation of NURBS as CIFS-automata permits to push further some previous works. In this section, several CIFS-based

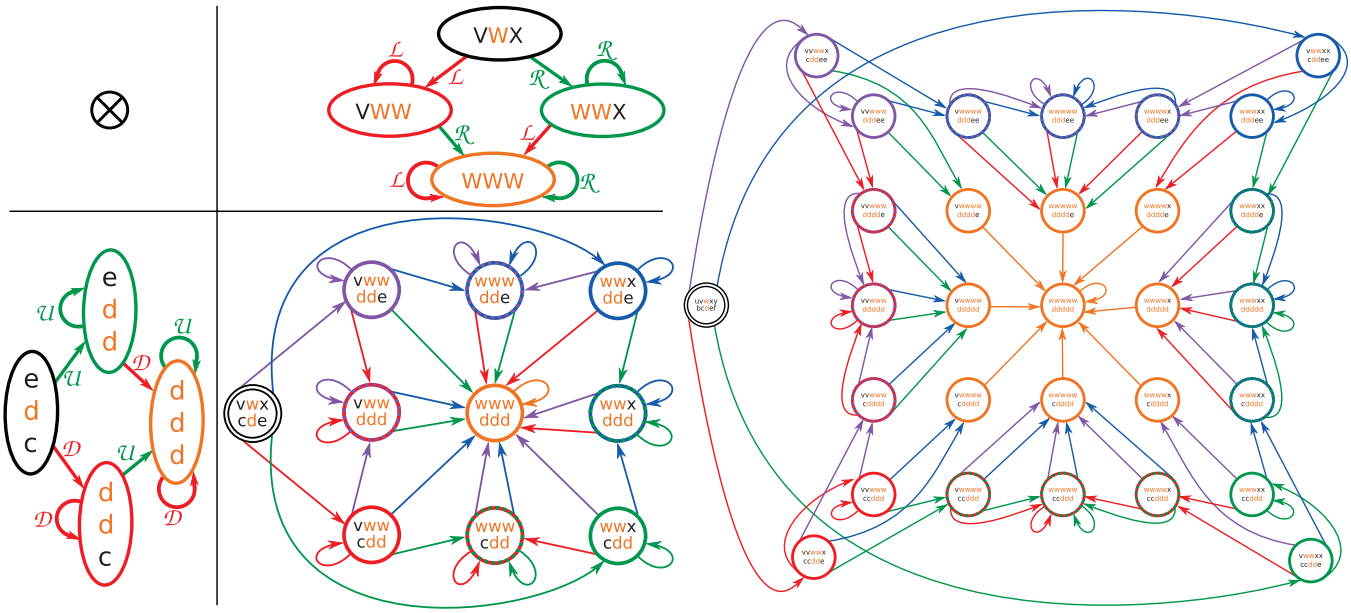


Fig. 9. On the left, the construction of bi-quadratic NURBS CIFS-automaton by tensor product of two quadratic NURBS CIFS-automata **A** and **B**. On the right the bi-cubic NURBS CIFS-automaton (without the input curve automata). Transformations $\mathcal{T}_0, \mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$ are respectively represented by red, green, blue, and purple arrows. Brown arrows represent any of these transformations. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

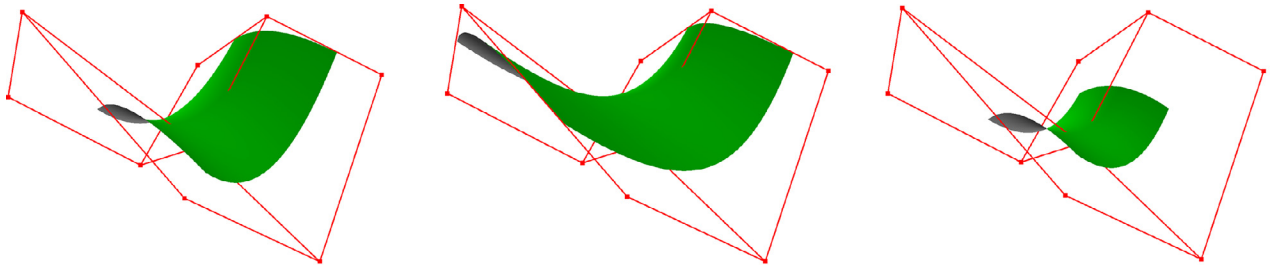


Fig. 10. Three examples of bi-quadratic NURBS tensor-surfaces with the same control grid but with different associated inter-nodal vectors. From left to right: $(1, 2, 3) \times (0, 3, 2)$, $(0, 1, 0) \times (1, 3, 8)$ and a uniform one $(1, 1, 1) \times (1, 1, 1)$.

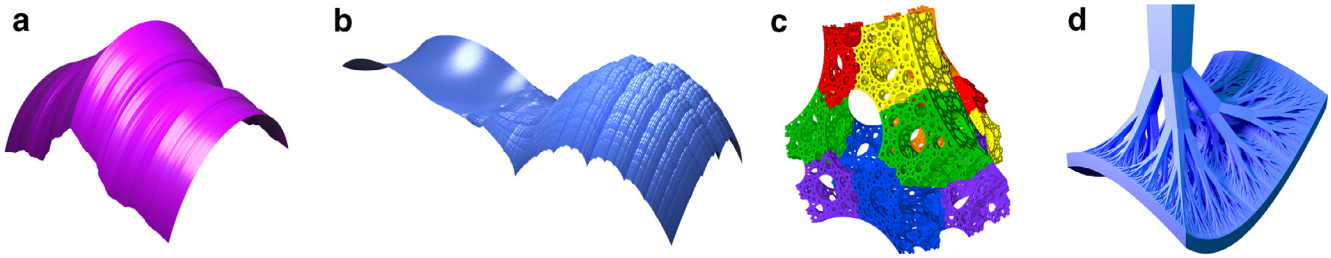


Fig. 11. From left to right: a tensor-surface generated from a NURBS and a Takagi curve, a junction between a NURBS bi-cubic tensor-surfaces and a Takagi tensor-surfaces, a lacunar volume with highlight subdivision process, and a tree iteratively defined such that its leaves correspond to the NURBS surface of Fig. 10 (left). All were designed using the iterative modeler MODITERE – LIB.

applications, currently limited to uniform B-Splines, are henceforth extended to non-uniform case.

6.1. Tensor surfaces with blending natures

Once CIFS-automaton tensor-product is defined, it can be used to generate tensor-surfaces from two generative curves which are not of the same nature, with the only condition that both are defined as CIFS-automata. For instance, in Fig. 11(a) is presented a tensor-surface generated from a cubic NURBS and a Takagi curve. Thanks to the overlaps of Takagi curve, the surface is mechanically more resistant than an usual NURBS tensor-surface while keeping its smoothness in one direction to evacuate fluids.

6.2. Junctions between surfaces of different natures

Podkorytov et al. [11] proves that a junction J between surfaces of different natures (A and B) is always possible since they are both represented as CIFS automata. These two surfaces are both bounded by a curve which is defined by a subset of the control mesh. For instance bi-quadratic NURBS tensor-surfaces, which are defined at least by a (3×3)-grid, are bounded by a quadratic NURBS which is defined by a (3×2)-grid. Both bounding-curves meshes are merged in a new CIFS of six transformations: two of them map the new mesh to a mesh of type A, two others map the mesh to a mesh of type B, and the two last ones map the mesh to a mesh of the new type J (see Fig. 12).

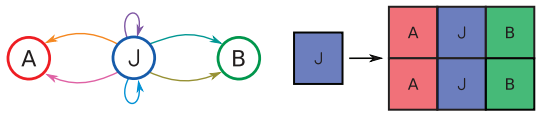


Fig. 12. Generic junction J between two surfaces of different types A and B. On left side, the CIFS automaton where A and B states are the usual CIFS automata of corresponding surfaces. On right side, the topological subdivision of J into six subsurfaces of the three types.

The transformations that create surface of type A or B are constrained to conserve the continuity at the junction with bounding surfaces. Central transformations are created depending on the four others in order to ensure a maximum continuity around the junction. If the central transformations are over-constrained, some phantom points can be added in the central mesh in order to release these constraints. An example can be found in Fig. 11(b) where a NURBS tensor-surface is joined to a Takagi tensor-surfaces.

6.3. Topological optimization

Some CAGD-pieces are subject to optimization while respecting constraints. For instance, they should be as light as possible and must efficiently evacuate heat without affecting their strength. A solution that directly proceeds on surfaces is the creation of lacunar surfaces (an example of lacunar NURBS-tensor surfaces is given in Fig. 11(c)). Another challenge that needs topological optimization is the creation of a support subject to the same constraints as described before. Arborescent structure that join a CIFS automaton have been proposed by Sokolov et al. [18] as a solution for uniform B-Splines tensor-surfaces. For instance, the tree in Fig. 11(d) is defined as a volumic subdivision bordered by a bi-quadratic NURBS tensor-surface. Constrains are defined to ensure that the subdivision of the volume is compatible with the subdivision of the surface.

7. Conclusion

In the present article, we show that NURBS curves and tensor-surfaces can be represented with a Controlled Iterated Function System (CIFS). As most of CAGD-systems are NURBS-based, adding these new surfaces to the CIFS-model is an important progress to a transition from usual CAGD-engineering to a CIFS-modeling.

We show the associated automaton is finite, and composed of limited number of stationary states. Furthermore, we provide two methods: the first one generates CIFS-automata that correspond to NURBS of any degree and the second one explains how to transpose tensor-product from curves to automata.

Once NURBS tensor-surfaces defined as CIFS automata, many interactions between them and object that are already defined in the formalism (fractals, subdivision surfaces...) become possible or easier.

The main issue NURBS have to face is the management of extraordinary vertices that are usually unavoidable in CAGD-engineering. Sederberg et al. [19] and Müller et al. [20] proposed another solution which is Non-Uniform Rational Subdivision Surfaces but have been restricted to low-degree (quadratic or cubic) surfaces. Thereafter, Cashman et al. [21] found a way to generalize [19] to high-degree surfaces but only for odd degree.

For this purpose, we need to manage non-tensor NURBS surfaces. Corresponding CIFS-automata should handle the evolution of both knot-vectors (which are no longer limited to two "orthogonal" ones) and the topology of the mesh. For quadratic and cubic cases, the shape (states and transitions) of automata seems to be at the cross of this paper for the non-uniformity and [10] for the management of extraordinary vertices. Coefficients of matrices associated with transitions should be deduced from [19].

Even if low-degree cases may be quite direct to integrate, a real issue will take place with higher degree. This comes from the fact that the neighborhood of vertices needed to compute a piece of surface is composed of a central vertex (for quadratic case) or a central face (for cubic case) and a unique ring of vertices that surrounds it. This ring is composed of vertices that share a face with the central vertex or a vertex of the central face. In these cases, the possibilities of irregularities are very limited and can be all listed. For higher degree, the patch owns several nested rings; thus the possibilities of irregularities are subject to combinatorial explosion. The solution proposed by Cashman et al. [21] for any odd-degree should be closely studied to find a solution to fill this gap.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Aknowledgments

We would like thank Gilles Gouaty for the implementation of automata, and Dominique Michelucci for his helpful comments and suggestions for scientific coherence and English correctness.

Appendix A. Mathematical proof

We fix $n = 2d - 1$ for $d \geq 1$. Let $\mathcal{M}_n$ be the set of words of length $n \geq 1$ over the alphabet $\{1, 2, \ldots, n\}$, *i.e.*, all words $w = w_1 w_2 \ldots w_n$ with $w_i \in \{1, 2, \ldots, n\}$ for $1 \leq i \leq n$. w_i corresponds to the ith letter of the word w and $\forall i, w_i \leq w_{i+1}$ The conjugate of a word $w = w_1 \ldots w_n$ is the word $\overline{w} = n + 1 - w_n, n + 1 - w_{n-1}, \ldots, n + 1 - w_1$.

From a word $w \in \mathcal{M}_n$, we construct the word $W \in \mathcal{M}_{2n}$ defined by $W = w_1 w_1 w_2 w_2 \ldots w_n w_n$. Note that $W_n = W_{n+1} = w_d$. Also, we define the maps $\mathcal{L}$ and $\mathcal{R}$ on $\mathcal{M}_n$ as follows:

$$\begin{cases} \mathcal{L}(w_1 \ldots w_n) = W_{n-d+1} \ldots W_{n-2} W_{n-1} W_n W_{n+1} \ldots W_{n+d-1}, \\ \mathcal{R}(w_1 \ldots w_n) = W_{n-d+2} \ldots W_{n-1} W_n W_{n+1} W_{n+2} \ldots W_{n+d}. \end{cases}$$

For instance, whenever $n = 7$ (or equivalently $d = 4$) and $w = 1234567$, we have $W = 11223344556677$, $\mathcal{L}(w) = 2334455$, $\mathcal{R}(w) = 3344556$, $\mathcal{L}(\mathcal{L}(w)) = 3334444$ and $\mathcal{R}(\mathcal{L}(w)) = 3344445$.

For $n \geq 1$, let us consider the set $\mathcal{S}_n$ of words obtained from $12 \ldots n$ after all possible combinations of $\mathcal{L}$ and $\mathcal{R}$. For instance, $\mathcal{S}_7 = \{1234567, 2334455, 3344556, 3334444, 3344445, 3444455, 4444555, 3444444, 3344444, 4444445, 4444455, 4444444\}$.

It is straightforward to check the following properties:

Fact 1. Any word w in $\mathcal{S}_n$ satisfies $w_d = d$.

Fact 2. For $1 \leq i \leq n - 1$, two consecutive letters w_i and w_{i+1} in a word $w \in \mathcal{S}_n$ satisfy $0 \leq w_{i+1} - w_i \leq 1$.

Fact 3. For any $w \in \mathcal{S}_n$, its conjugate $\overline{w}$ also lies in $\mathcal{S}_n$.

Fact 4. (i) d^n is the unique word w where $\mathcal{L}(w) = \mathcal{R}(w) = w$.

(ii) $(d-1)^{d-1} d^d$ is the unique word w where $\mathcal{L}(w) = w$ and $\mathcal{R}(w) \neq w$.

(iii) $d^d (d+1)^{d-1}$ is the unique word w where $\mathcal{R}(w) = w$ and $\mathcal{L}(w) \neq w$.

Fact 5. For any $w \in \mathcal{S}_n$, the number of letters d in $\mathcal{L}(w)$ (resp. $\mathcal{R}(w)$) is at least that of w.

Lemma 1. *Any word $w \in \mathcal{S}_n$ is a subword of $m_k = 1^{2^k} 2^{2^k} \ldots (n-1)^{2^k} n^{2^k}$ for some $k \geq 0$.*

Proof. Let w be a word obtained after applying $r \geq 0$ consecutive transformations $\mathcal{L}$ and/or $\mathcal{R}$, and let us prove by induction on r that w is a subword of $m_k = 1^{2^k} 2^{2^k} \dots (n-1)^{2^k} n^{2^k}$ for some k. Obviously, if $r = 1$ then w is one of the two words $\mathcal{L}(12\dots n)$ and $\mathcal{R}(12\dots n)$, which are subwords of $m_1 = 1122\dots nn$ (by definition of $\mathcal{L}$ and $\mathcal{R}$). Let us assume that w is subword of m_k with $w_d = d$, and let us prove that $\mathcal{L}(w)$ and $\mathcal{R}(w)$ are some subwords of m_{k+1}.

By definition, $w' = \mathcal{L}(w)$ (resp. $w' = \mathcal{R}(w)$) is obtained as a subword of $W = w_1 w_1 w_2 w_2 \dots w_n w_n$ after matching w'_d with $W_n = w_d$ (resp. $W_{n+1} = w_d$). Since w is a subword of m_k such that $w_d = d$, this implies that $W = w_1 w_1 \dots w_n w_n$ is a subword of m_{k+1}, and thus $w' = \mathcal{L}(w)$ (resp. $w' = \mathcal{R}(w)$) is a subword of m_{k+1}. □

Lemma 2. *Let w be a subword of length n of m_k, $k \geq 0$, such that $w_d = d$. Then, w belongs to $\mathcal{S}_n$.*

Proof. The proof is obtained by induction on k. Obviously, when $k = 0$ the unique length n subword of $12\dots(n-1)n$ is $12\dots n$, and it belongs to $\mathcal{S}_n$. For $k = 1$, w is a subword of $W = 1122\dots dd\dots nn$ with $w_d = W_n = W_{n+1} = d$. In the case where w_d matches W_n, $w = \mathcal{L}(12\dots n)$; otherwise w_d matches W_{n+1} and we have $w = \mathcal{R}(12\dots n)$. In the two cases, w lies into $\mathcal{S}_n$.

Now, let us assume the statement holds for $i \leq k$, and let us prove it holds for $k+1$. Let w be a length n subword of m_{k+1} such that $w_d = d$. Let $r \geq 1$ so that w_d matches the rth value of m_{k+1}. Since $w_d = d$ we necessarily have $r \geq (d-1)2^{k+1} \geq n$, and $r \leq n2^{k+1} - (d-1)2^{k+1} < n2^{k+1} - n$. These two last inequalities ensure that there is a length $2n$ subword W of m_{k+1} such that W_n or W_{n+1} matches w_d. In the case where w_d matches W_n, then we define the subword v of length n by deleting all values of odd ranks in W, and v satisfies $\mathcal{L}(v) = w$. In the case where w_d matches W_{n+1}, then we define the subword v of length n by deleting all values of even ranks in W, and v satisfies $\mathcal{R}(v) = w$. By construction, v is a subword of m_k with $v_d = d$ and after applying the recurrence hypothesis, v belongs to $\mathcal{S}_n$. As w is either $\mathcal{R}(v)$ or $\mathcal{L}(v)$, we deduce that w also is in $\mathcal{S}_n$. We conclude by induction. □

Theorem 1. *For $d \geq 1$, the cardinality of the set $\mathcal{S}_{2d-1}$ is $4(d-1)$.*

Proof. Let $r \geq 1$ be the integer such that $2^{r-1} < n = 2d - 1 < 2^r$, *i.e.* $r = \lceil \log_2(n) \rceil$. Using the previous two lemmas, it suffices to count the subwords w of length n in m_k, $k \geq 0$, satisfying $w_d = d$. From $k = 0$ to $r-1$, the number of such subwords in m_k is 2^k since such a word contains exactly 2^k letters d, and we have 2^k ways to fix $w_d = d$.

Now, let us count the subwords having exactly α letters d for $\alpha > 2^{r-1}$. It is straigthforward to see that such a word is a subword of m_r and for $\alpha < n$, there are two ways to choose such a word (either $w_d = d$ is the leftmost d in m_r or $w_d = d$ is the rightmost d in m_r). Then there are $2(n - 1 - 2^{r-1})$ possible words whenever $2^{r-1} < \alpha < n$. Finally, for $\alpha = n$, there is only one subword, that is d^n.

Combining all these subcases, the number of words in $\mathcal{S}_n$ is:

$$\sum_{k=0}^{r-1} 2^k + 2(n - 1 - 2^{r-1}) + 1 = 2^r - 1 + 2(n-1) - 2^r + 1 \tag{A.1}$$

$$= 2(2d-2) = 4(d-1) \tag{A.2}$$

□

Corollary 1. *From $12\dots n$, we can reach any word in $\mathcal{S}_n$ in at most $\lceil \log_2(n) \rceil$ transformations $\mathcal{L}$ and/or $\mathcal{R}$. We need $\lceil \log_2(n) \rceil$ transformations to reach the word d^n.*

Proof. Let $r \geq 0$ be the integer such that $2^{r-1} < n < 2^r$. We have $r = \lceil \log_2(n) \rceil$. Using the fact that d^n is obtained as a subword of 2^r, we need r transformations to reach it. From the proof of Theorem 1, all others words can be reached with at most r transformations, which gives the results. □

Corollary 2. *Let w be a word in $\mathcal{S}_n$ such that $w \notin \{d^n, (d-1)^{d-1}d^d, d^d(d+1)^{d-1}\}$. It is impossible to find a sequence ξ of $\mathcal{L}$ and $\mathcal{R}$ such as $w = \xi(w)$.*

Proof. Since w does not belong to $\{d^n, (d-1)^{d-1}d^d, d^d(d+1)^{d-1}\}$, Fact 4 proves that $\mathcal{R}(w)$ and $\mathcal{L}(w)$ differ from w. Let r be the number of letters d in w. Then, $\mathcal{L}(w)$ (resp. $\mathcal{R}(w)$) has at least one letter d more than w, and since $\mathcal{L}$ and $\mathcal{R}$ cannot decrease the number of d (see Fact 5), the proof is completed. □

Appendix B. Cubic matrices

$$M_{\mathcal{L}}(u_0 \ \dots \ u_4) = \begin{pmatrix} \frac{u_1+2u_2}{2(u_0+u_1+u_2)} & \frac{u_2}{2(u_1+u_2)}\frac{u_1+2u_2}{2(u_0+u_1+u_2)} & 0 & 0 \\ \frac{2u_0+u_1}{2(u_0+u_1+u_2)} & \frac{u_2}{2(u_1+u_2)}\frac{2u_0+u_1}{2(u_0+u_1+u_2)} + \frac{2u_1+u_2}{2(u_1+u_2)}\frac{u_1+2u_2+2u_3}{2(u_1+u_2+u_3)} & \frac{u_2+2u_3}{2(u_1+u_2+u_3)} & \frac{u_3}{2(u_2+u_3)}\frac{u_2+2u_3}{2(u_1+u_2+u_3)} \\ 0 & \frac{2u_1+u_2}{2(u_1+u_2)}\frac{u_1}{2(u_1+u_2+u_3)} & \frac{2u_1+u_2}{2(u_1+u_2+u_3)} & \frac{u_3}{2(u_2+u_3)}\frac{2u_1+u_2}{2(u_1+u_2+u_3)} + \frac{2u_2+u_3}{2(u_2+u_3)}\frac{u_2+2u_3+2u_4}{2(u_2+u_3+u_4)} \\ 0 & 0 & 0 & \frac{2u_2+u_3}{2(u_2+u_3)}\frac{u_2}{2(u_2+u_3+u_4)} \end{pmatrix}$$

$$M_{\mathcal{R}}(u_0 \ \dots \ u_4) = \begin{pmatrix} \frac{u_2}{2(u_1+u_2)}\frac{u_1+2u_2}{2(u_0+u_1+u_2)} & 0 & 0 & 0 \\ \frac{u_2}{2(u_1+u_2)}\frac{2u_0+u_1}{2(u_0+u_1+u_2)} + \frac{2u_1+u_2}{2(u_1+u_2)}\frac{u_1+2u_2+2u_3}{2(u_1+u_2+u_3)} & \frac{u_2+2u_3}{2(u_1+u_2+u_3)} & \frac{u_3}{2(u_2+u_3)}\frac{u_2+2u_3}{2(u_1+u_2+u_3)} & 0 \\ \frac{2u_1+u_2}{2(u_1+u_2)}\frac{u_1}{2(u_1+u_2+u_3)} & \frac{2u_1+u_2}{2(u_1+u_2+u_3)} & \frac{u_3}{2(u_2+u_3)}\frac{2u_1+u_2}{2(u_1+u_2+u_3)} + \frac{2u_2+u_3}{2(u_2+u_3)}\frac{u_2+2u_3+2u_4}{2(u_2+u_3+u_4)} & \frac{u_3+2u_4}{2(u_2+u_3+u_4)} \\ 0 & 0 & \frac{2u_2+u_3}{2(u_2+u_3)}\frac{u_2}{2(u_2+u_3+u_4)} & \frac{2u_2+u_3}{2(u_2+u_3+u_4)} \end{pmatrix}$$

References

[1] Peitgen H-O, Richter P. The beauty of fractals: Images of complex dynamical systems. Springer-Verlag, Heidelberg; 1986.

[2] Zhou H, Sun J, Turk G, Rehg JM. Terrain synthesis from digital elevation models. IEEE Trans Vis Comput Graph 2007;13(4):834–48.

[3] Prusinkiewicz P, Lindenmayer A. The algorithmic beauty of plants. New York, NY, USA: Springer-Verlag New York, Inc.; 1990. ISBN 0-387-97297-8.

[4] Hutchinson JE. Fractals and self similarity. Indiana Univ Math J 1981;30(5):713–47.

[5] Prusinkiewicz P, Hammel M. Language-restricted iterated function systems, koch constructions, and l-systems. In: Proceedings of the SIGGRAPH'94 course notes; 1994.

[6] Barnsley M. Fractals everywhere. Academic Press; 2014.

[7] Goldman R. The fractal nature of Bézier curves. In: Proceedings of the geometric modeling and processing, 2004. IEEE; 2004. p. 3–11.

[8] Schaefer S, Levin D, Goldman R. Subdivision schemes and attractors. In: Proceedings of the symposium on geometry processing. Citeseer; 2005. p. 171–80.

[9] Zair CE, Tosan E. Fractal modeling using free form techniques. Comput Graph Forum 1996;15:269–78.

[10] Morlet L, Neveu M, Lanquetin S, Gentil C. Barycentric combinations based subdivision shaders. Plzen, Czech Republic. J WSCG 2018;25.

[11] Podkorytov S, Gentil C, Sokolov D, Lanquetin S. Joining primal/dual subdivision surfaces. In: Proceedings of the eighth international conference mathematical methods for curves and surfaces, MMCS 2012, Oslo, Norway, June 28–July 3, 2012, Revised Selected Papers; 2012. p. 403–24. doi:10.1007/978-3-642-54382-1_23.

[12] Halstead M, Kass M, DeRose T. Efficient, fair interpolation using Catmull–Clark surfaces. In: Proceedings of the twentieth annual conference on computer graphics and interactive techniques. ACM; 1993. p. 35–44.

[13] Stam J. Exact evaluation of Catmull–Clark subdivision surfaces at arbitrary parameter values. In: Proceedings of the twenty-fifth annual conference on computer graphics and interactive techniques SIGGRAPH '98. New York, NY, USA: ACM; 1998. p. 395–404. ISBN 0-89791-999-8. doi:10.1145/280814.280945.

[14] Cox MG. The numerical evaluation of B-splines. IMA J Appl Math 1972;10(2):134–49.

[15] Ramshaw L. Blossoming: A connect-the-dots approach to splines. Digital Equipment Corporation Palo Alto; 1987.

[16] Boehm W. Inserting new knots into b-spline curves. Comput Aided Des 1980;12(4):199–201.

[17] Cohen E, Lyche T, Riesenfeld R. Discrete b-splines and subdivision techniques in computer-aided geometric design and computer graphics. Comput Graph Image Process 1980;14(2):87–111.

[18] Sokolov D, Gouaty G, Gentil C, Mishkinis A. Boundary controlled iterated function systems. In: Boissonnat J-D, Cohen A, Gibaru O, Gout C, Lyche T, Mazure M-L, et al., editors. Curves and surfaces. Cham: Springer International Publishing; 2015. p. 414–32.

[19] Sederberg TW, Zheng J, Sewell D, Sabin M. Non-uniform recursive subdivision surfaces. In: Proceedings of the twenty-fifth annual conference on Computer graphics and interactive techniques. ACM; 1998. p. 387–94.

[20] Müller K, Reusche L, Fellner D. Extended subdivision surfaces: building a bridge between NURBS and Catmull–Clark surfaces. ACM Trans Graph (TOG) 2006;25(2):268–92.

[21] Cashman TJ, Augsdörfer UH, Dodgson NA, Sabin MA. NURBS with extraordinary points: high-degree, non-uniform, rational subdivision schemes. ACM Trans Graph (TOG) 2009;28(3):46.

Discrete Mathematics 342 (2019) 997–1005

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Enumeration of Łukasiewicz paths modulo some patterns

Jean-Luc Baril *, Sergey Kirgizov, Armen Petrossian

LE2I, Université de Bourgogne, B.P. 47 870, 21078 DIJON-Cedex, France

ARTICLE INFO

Article history:
Received 18 April 2018
Received in revised form 5 December 2018
Accepted 6 December 2018
Available online 27 December 2018

Keywords:
Łukasiewicz path
Dyck path
Motzkin path
Equivalence relation
Patterns

ABSTRACT

For any pattern α of length at most two, we enumerate equivalence classes of Łukasiewicz paths of length $n \geq 0$ where two paths are equivalent whenever the occurrence positions of α are identical on these paths. As a byproduct, we give a constructive bijection between Motzkin paths and some equivalence classes of Łukasiewicz paths.

© 2018 Elsevier B.V. All rights reserved.

1. Introduction and notations

In the literature, lattice paths are widely studied. Their enumeration is a very active field in combinatorics, and they have many applications in other research domains as computer science, biology and physics [17,18]. Dyck and Motzkin paths are the most often considered. This is partly due to the fact that they are respectively counted by the famous Catalan and Motzkin numbers (see A000108 and A001006 in the on-line encyclopedia of integer sequences [27]). Almost always, these paths are enumerated according to several parameters and statistics (see for instance [5,13,14,16,19,20,23,24,29] for Dyck paths and [1,4,6,7,15,21,25] for Motzkin paths). Also, many one-to-one correspondences have been found between lattice paths and some combinatorial objects such as Young tableaux, pattern avoiding permutations, bargraphs, RNA shapes and so on [28]. Recently a new approach has been introduced for studying statistics on lattice paths. It consists in determining the cardinality of the quotient set generated by an equivalence relation based on the positions of a given pattern: *two paths belong to the same equivalence class whenever the positions of occurrences of a given pattern are identical on these paths.* Enumerating results are provided for the quotient sets of Dyck, Motzkin and Ballot paths for patterns of length at most three (see respectively [2,3] and [12]). The purpose of the present paper is to extend these studies for Łukasiewicz paths that naturally generalizes Dyck and Motzkin paths. As a byproduct, we show how Motzkin paths are in one-to-one correspondence with some equivalence classes of Łukasiewicz paths.

Throughout this paper, a *lattice path* is defined by a starting point $P_0 = (0, 0)$, an ending point $P_n = (n, 0)$, it consists of steps lying in $S = \{(1, i), i \in \mathbb{Z}\}$, and it never goes below the x-axis. The *length* of a path is the number of its steps. We denote by ϵ the empty path, *i.e.*, the path of length zero. Constraining the steps to lie into $\{(1, 1), (1, -1)\}$ (resp. $\{(1, 1), (1, 0), (1, -1)\}$), we retrieve the well known definition of Dyck paths (resp. Motzkin paths). *Łukasiewicz paths* are obtained when the steps belong to $\{(1, i) \in S, i \geq -1\}$. We refer to [9,22,28,30,31] for some combinatorial studies on Łukasiewicz paths. Let $\mathcal{L}_n$, $\mathcal{D}_n$, $\mathcal{M}_n$, $n \geq 0$, respectively, be the sets of Łukasiewicz, Dyck and Motzkin paths of length n, and $\mathcal{L} = \cup_{n\geq 0}\mathcal{L}_n$, $\mathcal{D} = \cup_{n\geq 0}\mathcal{D}_n$, $\mathcal{M} = \cup_{n\geq 0}\mathcal{M}_n$. For convenience, we set $D = (1, -1)$, $F = U_0 = (1, 0)$, $U = U_1 = (1, 1)$ and

* Corresponding author.
E-mail addresses: barjl@u-bourgogne.fr (J.-L. Baril), kerzolster@gmail.com (S. Kirgizov), armen.petrossian@u-bourgogne.fr (A. Petrossian).

https://doi.org/10.1016/j.disc.2018.12.005
0012-365X/© 2018 Elsevier B.V. All rights reserved.

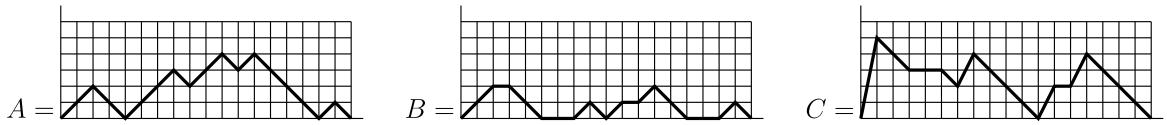


Fig. 1. From left to right, we show a Dyck path $A = UUDDUUUDUUDUDDDDUD$, a Motzkin path $B = UUFDDFFUDUFUDDFFUD$ and a Łukasiewicz path $C = U_5DDFFDU_2DDDDU_2FU_2DDDD$.

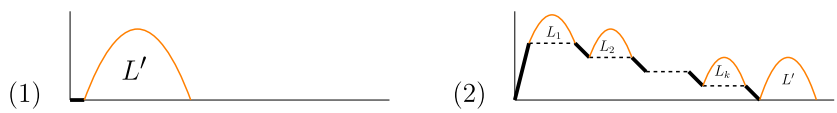


Fig. 2. The two forms of the decomposition of a non-empty Łukasiewicz path.

$U_i = (1, i)$ for $i \geq 2$. See Fig. 1 for an illustration of Dyck, Motzkin and Łukasiewicz paths of length 18. Note that Łukasiewicz paths can be interpreted as an algebraic language of words $w \in \{x_0, x_1, x_2, \ldots\}^\star$ such that $\delta(w) = -1$ and $\delta(w') \geq 0$ for any proper prefix w' of w where δ is the map from $\{x_0, x_1, x_2, \ldots\}^\star$ to $\mathbb{Z}$ defined by $\delta(w_1w_2\ldots w_n) = \sum_{i=1}^{n} \delta(w_i)$ with $\delta(x_i) = i-1$ (see [10,26]).

Any non-empty Łukasiewicz path $L \in \mathcal{L}$ can be decomposed (see [8]) into one of the two following forms: (1) $L = FL'$ with $L' \in \mathcal{L}$, or (2) $L = U_kL_1DL_2D\ldots L_kDL'$ with $k \geq 1$ and $L_1, L_2, \ldots, L_k, L' \in \mathcal{L}$ (see Fig. 2).

Due to this decomposition, the generating function $L(x)$ for the cardinalities of the sets $\mathcal{L}_n$, $n \geq 0$, satisfies the functional equation $L(x) = 1 + xL(x) + \sum_{k\geq 1} x^{k+1}L(x)^{k+1}$, or equivalently, $L(x) = \frac{1}{1-xL(x)}$. Then, $L(x) = \frac{1-\sqrt{1-4x}}{2x}$ and the coefficient of x^n in the series expansion of $L(x)$ is given by the nth Catalan number $\frac{1}{n+1}\binom{2n}{n}$ (see sequence A000108 in [27]).

A *pattern* of length one (resp. two) in a lattice path L consists of one step (resp. two consecutive steps). We will say that an *occurrence* of a pattern is at *position* $i \geq 1$ in L, whenever the first step of this occurrence appears at the ith step of the path. The *height* of an occurrence is the minimal ordinate reached by its points. For instance, the path $C = U_5DDF\mathbf{FD}U_2DDDDU_2FU_2DDDD$ (see Fig. 1) contains one occurrence of the pattern FD at position 5 and of height 2.

Following the recent studies [2,3,12], we define an equivalence relation on the set $\mathcal{L}$ for a given pattern α: *two Łukasiewicz paths of the same length are α-equivalent whenever the occurrences of the pattern α appear at the same positions in the two paths.* For instance, $UFFF\mathbf{FD}UDUDFFFFUDFF$ is FD-equivalent to the path C in Fig. 1 since the only one occurrence of the pattern FD (in boldface) appear at the same position in the two paths. Note that the height of the occurrences of α is not involved in this definition.

In this paper, for any pattern α of length at most two, we consider the above equivalence relation on the set $\mathcal{L}$, and for each of them we provide the cardinality of the quotient set with respect to the length. Three general methods are used:

- (M_0) we prove that any α-equivalence class contains at least one Motzkin path. Using $\mathcal{M}_n \subset \mathcal{L}_n$ for $n \geq 0$, we deduce that the number of α-equivalence classes in $\mathcal{M}_n$ is equal to that of $\mathcal{L}_n$. Since two of the authors have already determined this number for $\mathcal{M}_n$ (see [2]), we can conclude,
- (M_1) we define a set $\mathcal{A}_n$ of representatives of length $n \geq 0$ and we calculate $a_n = |\mathcal{A}_n|$,
- (M_2) we exhibit a one-to-one correspondence between a subset of Łukasiewicz paths (subset of representatives of the classes) and the set of equivalence classes by using combinatorial reasonings, and then, we evaluate algebraically the generating function for this subset.

The paper is organized as follows. In Section 2, we consider the case of patterns α studied using method (M_0), *i.e.*, $\alpha \in \{U, UU, UD, UF, DU, FU\}$. In Section 3, we focus on these ones that can be dealt using method (M_1), *i.e.*, $\alpha \in \{F, D, FD, DF, DD\}$. In Section 4, we complete our study by the remaining cases which are obtained using method (M_2). We refer to Table 1 for an exhaustive list of our enumerative results.

2. Modulo $\alpha \in \{U, UU, UD, UF, DU, FU\}$

In this section, we focus on the patterns that can be dealt using method (M_0).

Lemma 1. *For $n \geq 0$, let L be a Łukasiewicz path in $\mathcal{L}_n$ and $\alpha \in \{U, UU, UD, UF, DU, FU\}$. Then, there exists a Motzkin path $M \in \mathcal{M}_n$ such that M and L are α-equivalent.*

Proof. Let us assume that $\alpha \in \{U, UU, UD\}$. Any non-empty Łukasiewicz path can be decomposed into one of the two following forms: (i) $L = FL'$ with $L' \in \mathcal{L}$, and (ii) $L = U_kL_1DL_2D\ldots L_kDL'$ with $k \geq 1$ and $L_1, L_2, \ldots, L_k, L' \in \mathcal{L}$. For $n \geq 0$, we recursively define a map ϕ from $\mathcal{L}_n$ to $\mathcal{M}_n$ as follows:

$$\begin{cases} \phi(\epsilon) & = \epsilon, \\ \phi(FL') & = F\phi(L'), \\ \phi(UL_1DL') & = U\phi(L_1)D\phi(L'), \\ \phi(U_kL_1DL_2D\ldots L_kDL') & = F\phi(L_1)F\phi(L_2)F\ldots\phi(L_k)F\phi(L') \qquad \text{for } k \geq 2. \end{cases}$$

Table 1
Number of α-equivalence classes for Łukasiewicz paths. The last three sequences are recorded in OEIS [27] as generalized Catalan sequences.

Pattern α	Sequence	Sloane	a_n, $1 \le n \le 10$	Method
U	$\binom{n}{\lfloor \frac{n}{2} \rfloor}$	A001405	1, 2, 3, 6, 10, 20, 35, 70, 126, 252	
UU	$\frac{1-2x+x^2-\sqrt{(x^2+1)(1-3x^2)}}{2x(-1+2x-x^2+x^3)}$	A191385	1, 1, 1, 2, 3, 5, 7, 12, 18, 31	M_0
UD	Shift of Fibonacci	A000045	1, 2, 3, 5, 8, 13, 21, 34, 55, 89	
UF, FU	$\frac{2}{1-2x+\sqrt{1-4x^3}}$	A165407	1, 1, 2, 3, 4, 7, 11, 16, 27, 43	
DU	Shift of Fibonacci	A000045	1, 1, 1, 2, 3, 5, 8, 13, 21, 34	
F	$2^n - n$	A000325	1, 2, 5, 12, 27, 58, 121, 248, 503, 1014	
D	2^{n-1}	A011782	1, 1, 2, 4, 8, 16, 32, 64, 128, 256	M_1
FD, DF	Fibonacci	A000045	1, 1, 2, 3, 5, 8, 13, 21, 34, 55	
DD	$\frac{1-x}{1-2x+x^2-x^3}$	A005251	1, 1, 2, 4, 7, 12, 21, 37, 65, 114	
U_k	Motzkin	A001006	1, 2, 4, 9, 21, 51, 127, 323, 835, 2188	
FF	$\frac{1-3x+4x^2-5x^3+7x^4-7x^5+6x^6-3x^7+x^8}{(1-2x+x^2-x^3)(1-x)^2}$	New	1, 2, 2, 5, 9, 17, 32, 59, 107, 192	M_2
FU_k, U_kF	$\frac{1-x-\sqrt{1-2x+x^2-4x^3}}{2x^3}$	A023431	1, 1, 2, 4, 7, 13, 26, 52, 104, 212	
U_kD	$\frac{1-x-x^2-\sqrt{1-2x-x^2-2x^3+x^4}}{2x^3}$	A292460	1, 2, 4, 8, 17, 37, 82, 185, 423, 978	
DU_k	$\frac{1+x+x^2-\sqrt{1-2x-x^2-2x^3+x^4}}{2x}$	A004148	1, 1, 1, 2, 4, 8, 17, 37, 82, 185	

Clearly, $\phi(L)$ is a Motzkin path in $\mathcal{M}_n$, and whenever $\alpha \in \{U, UU, UD\}$ the occurrence positions of α in L and $\phi(L)$ are identical. Then, the equivalence class of L contains a Motzkin path $\phi(L)$.

Let us assume that $\alpha = DU$. Any non-empty Łukasiewicz path L can be written as follows:

$$L = K_0 \prod_{i=1}^{r} (DU)^{a_i} K_i$$

with $r \ge 0$, $a_i \ge 1$ for $1 \le i \le r$, and where K_i, $0 \le i \le r$, are some parts that do not contain any pattern DU. Note that K_0 and K_r necessarily contain at least one step. From $L \in \mathcal{L}_n$, we define the Motzkin path

$$M = UF^{b_0-1} \left(\prod_{i=1}^{r-1} (DU)^{a_i} F^{b_i} \right) (DU)^{a_r} DF^{b_r-1} \in \mathcal{M}_n$$

where $b_i = |K_i|$ for $0 \le i \le r$. Since the occurrence positions of DU in L and M are identical, M is a Motzkin path in the same class as L.

Let us assume that $\alpha \in \{FU, UF\}$. Any non-empty Łukasiewicz path L can be written as follows:

$$L = K_0 \prod_{i=1}^{r} \alpha^{a_i} K_i$$

with $r \ge 0$, $a_i \ge 1$ for $1 \le i \le r$, and where K_i, $0 \le i \le r$, are some parts that do not contain any pattern α. From $L \in \mathcal{L}_n$, we define the Motzkin path

$$M = F^{b_0} \prod_{i=1}^{r} \alpha^{a_i} D^{c_i} F^{b_i - c_i} \in \mathcal{M}_n$$

where $b_0 = |K_0|$, and for $1 \le i \le r$, $b_i = |K_i|$ and $c_i = \min\{b_i, a_i + \sum_{j=1}^{i-1}(a_j - c_j)\}$. Less formally, for i from 0 to r, K_i is replaced with $D^{c_i} F^{|K_i|-c_i}$ where the value c_i is the maximal number of down steps D that can be placed so that M remains a lattice path. This ensures that M has the same occurrence positions of α as L, which means that M is a Motzkin path in the same class as L. □

Using Lemma 1 and the fact that $\mathcal{M} \subset \mathcal{L}$, we directly deduce the following theorem.

Theorem 1. *For $\alpha \in \{U, UU, UD, UF, DU, FU\}$ and $n \ge 0$, the number of α-equivalence classes in $\mathcal{L}_n$ also is that of $\mathcal{M}_n$.*

Since two of the authors have already determined the number of α-equivalence classes in $\mathcal{M}_n$, we refer to their paper [2] for a detailed description of the different proofs, and we report the results in Table 1.

3. Modulo $\alpha \in \{F, D, FD, DF, DD\}$

In this section, we focus on the patterns that can be dealt with method (M_1) which consists in defining a set $\mathcal{A}_n$ of representatives and counting directly $a_n = |\mathcal{A}_n|$.

Theorem 2. *The number of F-equivalence classes in $\mathcal{L}_n$, $n \geq 0$, is given by $2^n - n$ (see sequence A000325 in [27]).*

Proof. We define the set $\mathcal{A}_n$ as the set of Łukasiewicz paths of length n where the first non-F step (if any) is of the form U_i, $i \geq 1$, and the rest non-F steps are D steps. Since $\mathcal{A}_n$ is in one-to-one correspondence with binary words of length n without a single one, we deduce that $a_n = 2^n - n$. □

Theorem 3. *The number of D-equivalence classes in $\mathcal{L}_n$, $n \geq 1$, is given by 2^{n-1} (see sequence A011782 in [27]).*

Proof. We define the set $\mathcal{A}_n$ of Łukasiewicz paths of length n where the first step is of the form U_i, $i \geq 0$, and the rest of non-D steps are F's. Since $\mathcal{A}_n$ is in one-to-one correspondence with binary words of length $n-1$, we deduce that $a_n = 2^{n-1}$. □

Theorem 4. *The number of FD-equivalence (resp. DF-equivalence) classes in $\mathcal{L}_n$, $n \geq 0$, is given by the Fibonacci number f_n defined by $f_0 = 1, f_1 = 1, f_2 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$ (see sequence A000045 in [27]).*

Proof. For the pattern FD (the pattern DF is similar), we define the set $\mathcal{A}_n$, $n \geq 3$, as the set of Łukasiewicz paths of length n where the first step is of the form U_i, $i \geq 0$, and the rest of the steps are either F or D steps provided that every D step is immediately preceded by an F step. Clearly, $\mathcal{A}_n$ is decomposed into two sets of paths starting with U_iFF or with U_iFD, which are counted by a_{n-1} and a_{n-2} respectively. □

Theorem 5. *The number of DD-equivalence classes in $\mathcal{L}_n$, $n \geq 0$, is given by the nth term g_n of the sequence defined by $g_0 = 1$, $g_1 = 1, g_2 = 1, g_3 = 2$ and $g_n = g_{n-1} + g_{n-2} + g_{n-4}$ for $n \geq 4$ (see sequence A005251 in [27]).*

Proof. We define the set $\mathcal{A}_n$, $n \geq 4$, as the set of Łukasiewicz paths of length n where the first step is of the form U_i, $i \geq 0$, and the rest of the steps are either F or D steps and there are no isolated D steps. Clearly, $\mathcal{A}_n$ is decomposed into sets of paths starting with U_iF, or with U_iDDDF (together with the path U_3DDD for $n = 4$), or with U_iD^j, $j \geq 2$, $j \neq 3$ which are counted by a_{n-1}, a_{n-4} and a_{n-2} respectively. □

4. Other patterns

In this section, we extend the definition of α-equivalence whenever α is a set S of patterns: two paths L and L' are S-equivalent if for any pattern $\alpha \in S$, the occurrence positions of α are the same in L and L'. We investigate the cases where S is $\{U_k, k \geq 1\}$, $\{DU_k, k \geq 1\}$, $\{U_kD, k \geq 1\}$, $\{FU_k, k \geq 1\}$, and $\{U_kF, k \geq 1\}$. For short, these S-equivalences will be written U_k-equivalence (resp. DU_k-, U_kD-, FU_k-, U_kF-equivalence). Also, we study the FF-equivalence relation in $\mathcal{L}$. For all these cases, we use the method (M_2) that consists in exhibiting subsets of representatives of equivalence classes, and determining algebraically their cardinalities.

4.1. Modulo $S = \{U_k, k \geq 1\}$

Let $\mathcal{B}$ be the set of Łukasiewicz paths without any flat steps at positive height. For instance, we have $U_3DDDFUD \in \mathcal{B}$ and $U_3FDDDUD \notin \mathcal{B}$. Let $\bar{\mathcal{B}} \subset \mathcal{B}$ be the set of Łukasiewicz paths without any flat steps.

Lemma 2. *There is a bijection between $\mathcal{B}$ and the set of U_k-equivalence classes of $\mathcal{L}$.*

Proof. Let L be a non-empty Łukasiewicz path in $\mathcal{L}$. Let us prove that there exists a Łukasiewicz path $L' \in \mathcal{B}$ (with the same length as L) such that L and L' are equivalent. We write

$$L = K_0 \prod_{i=1}^{r} \alpha_i K_i$$

with $r \geq 0$, where K_i is a part that does not contain any up steps for $0 \leq i \leq r$, and $\alpha_i \in \{U_k, k \geq 1\}$ for $1 \leq i \leq r$. From $L \in \mathcal{L}$, we define the Łukasiewicz path

$$L' = F^{b_0} \prod_{i=1}^{r} \alpha_i D^{c_i} F^{b_i - c_i}$$

with $b_0 = |K_0|$, and for $1 \leq i \leq r$, $b_i = |K_i|$, $c_i = \min\{b_i, a_i + \sum_{j=1}^{i-1}(a_j - c_j)\}$ where $\alpha_i = U_{a_i}$. Less formally, for i from 0 to r, K_i is replaced with $D^{c_i}F^{|K_i|-c_i}$ where c_i is the maximal number of down steps that can be placed so that L' remains a Łukasiewicz

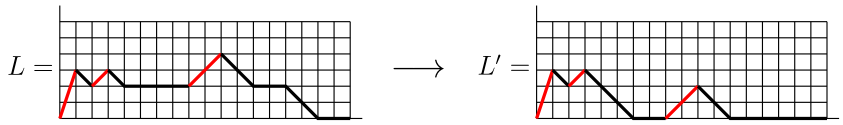


Fig. 3. Illustration of the example described in the proof of Lemma 2.

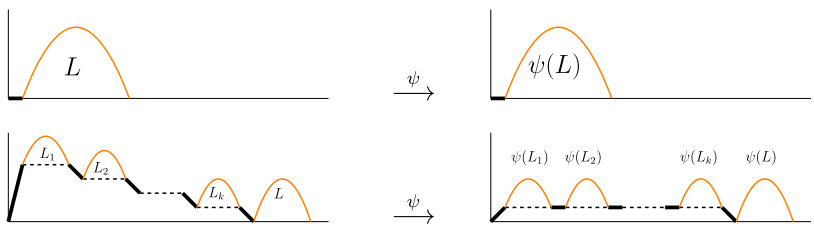


Fig. 4. Illustration of the map ψ from $\mathcal{L}$ to $\mathcal{M}$.

path. Clearly, L' belongs to $\mathcal{B}$ (it does not contain any flat at positive height), and for any $k \geq 1$ the occurrence positions of U_k are the same as for L, *i.e.*, $L' \in \mathcal{B}$ is in the same class as L. For instance, if $L = U_3DUDFFFFUUDDFFDDFF$, then we obtain $L' = U_3DUDDDFFUUDDFFFFFF$ (see Fig. 3 for an illustration of this example).

Since the positions of the up steps U_k, $k \geq 1$, remain fixed inside a class, and any flat of $L' \in \mathcal{B}$ lies necessarily on the x-axis, there are no other paths in $\mathcal{B}$ in the same class as L. □

Theorem 6. *The generating function for the set of U_k-equivalence classes of $\mathcal{L}$ with respect to the length is given by*

$$\frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x^2},$$

which generates the Motzkin numbers (A001006 in [27]).

Proof. Using Lemma 2, it suffices to obtain the generating function $B(x)$ for the set $\mathcal{B}$. A non-empty Łukasiewicz path $L \in \mathcal{B}$ can be written either $L = FL'$ where $L' \in \mathcal{B}$, or $L = U_kL_1DL_2D \dots L_kDL'$ for $k \geq 1$ where $L_1, L_2, \dots, L_k \in \bar{\mathcal{B}}$ are some Łukasiewicz paths without flats, and $L' \in \mathcal{B}$. So we obtain the functional equation $B(x) = 1 + xB(x) + xB(x)\sum_{k\geq 1} x^k\bar{B}(x)^k$ where $\bar{B}(x)$ is the generating function for the set $\bar{\mathcal{B}}$ of Łukasiewicz paths without flats. Using the classical decomposition of a Łukasiewicz path, $\bar{B}(x)$ satisfies $\bar{B}(x) = 1 + \sum_{k\geq 2} x^k\bar{B}(x)^k$, or equivalently $\bar{B}(x) = \frac{1}{(1+x)(1-x\bar{B}(x))}$. A simple calculation provides the result. □

Let us define recursively a map ψ from $\mathcal{L}$ to the set of Motzkin paths $\mathcal{M}$ as follows:

$$\begin{cases} \psi(\epsilon) & = \epsilon, \\ \psi(FL) & = F\psi(L), \\ \psi(U_kL_1DL_2D \dots L_kDL) & = U\psi(L_1)F\psi(L_2)F \dots \psi(L_k)D\psi(L), \end{cases}$$

where $L, L_1, L_2, \dots, L_k$ are some Łukasiewicz paths. See Fig. 4 for an illustration of the bijection ψ. For instance, the image by ψ of $U_4FU_2DFDDDU_2U_1DDDFDDFU_2FDU_2DDD$ is $UFUFFDFFUUDFDFFDFUFFUFDD$. Obviously, the map ψ preserves the length of the paths.

We easily deduce the two following facts.

Fact 1. *If $L, L' \in \mathcal{L}$, then $LL' \in \mathcal{L}$ and we have $\psi(LL') = \psi(L)\psi(L')$.*

Fact 2. *There is a one-to-one correspondence between:*

(a) steps $\{U_k, k \geq 1\}$ in L and steps U in $\psi(L)$;
(b) $\{U_kU_\ell, k, \ell \geq 1\}$ in L and UU in $\psi(L)$;
(c) steps F on the x-axis in L and steps F on the x-axis in $\psi(L)$;
(d) $\{U_kD, k \geq 2\} \cup \{U_kF, k \geq 1\}$ in L and UF in $\psi(L)$;
(e) peaks UD in L and peaks UD in $\psi(L)$.

Theorem 7. *For any $n \geq 0$, the map ψ induces a bijection from $\mathcal{B}_n$ to $\mathcal{M}_n$.*

Proof. We proceed by induction on n. Obviously, for $n = 0$ we have $\psi(\epsilon) = \epsilon$. We assume that ψ is a bijection from $\mathcal{B}_k$ to $\mathcal{M}_k$, $0 \leq k \leq n$, and we prove the result for $n + 1$. Using the enumerating result of Theorem 6, it suffices to prove that ψ is surjective. So, let M be a Motzkin path in $\mathcal{M}_{n+1}$. We distinguish two cases:

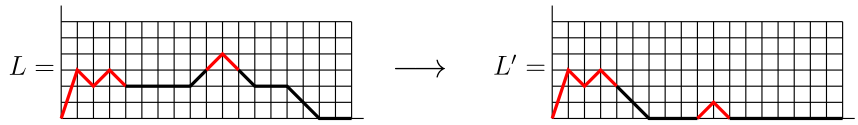


Fig. 5. Illustration of the example described in the proof of Lemma 3.

(i) $M = FM'$ with $M' \in \mathcal{M}_n$. Using the recurrence hypothesis, there is $L' \in \mathcal{B}_n$ such that $M' = \psi(L')$. So, the Łukasiewicz path $L = FL'$ lies into $\mathcal{B}_{n+1}$ and satisfies $\psi(L) = M$ which proves that M belongs to the image by ψ of $\mathcal{B}_{n+1}$.

(ii) $M = UM'DM''$ where M' and M'' are two Motzkin paths in $\mathcal{M}$. We can uniquely write $M' = M_0 \prod_{i=1}^{r} FM_i$ with $r \geq 0$ and where M_i is a (possibly empty) Motzkin path without flat F on the x-axis. Using the recurrence hypothesis, there are $B_0, B_1, \ldots, B_r \in \mathcal{B}$ such that $\psi(B_i) = M_i$, $0 \leq i \leq r$. Also let $B \in \mathcal{B}$ such that $\psi(B) = M''$. Since B_i (resp. B) belongs to $\mathcal{B}$, it does not contain any flat at positive height. Since $M_i = \psi(B_i)$, Fact 2(c) implies that B_i does not contain any flat on the x-axis. So, B_i does not contain any flat steps. So, let us define

$$L = U_{r+1}\left(\prod_{i=0}^{r} B_i D\right) B.$$

Clearly, L lies in $\mathcal{B}_{n+1}$ and satisfies $\psi(L) = M$; then, M belongs to the image by ψ of $\mathcal{B}_{n+1}$, and the map ψ from $\mathcal{B}_n$ to $\mathcal{M}_n$ is a bijection. □

4.2. Modulo $s = \{u_k d, k \geq 1\}$, and $S = \{DU_k, k \geq 1\}$

Let $\mathcal{C} \subset \mathcal{B}$ be the set of Łukasiewicz paths without any flat steps at positive height and such that any up step U_k, $k \geq 1$, is immediately followed by a down step D. For instance, we have $U_3DDDFUD \in \mathcal{C}$ and $U_3FDDDUD \notin \mathcal{C}$. We set $\bar{\mathcal{C}} = \bar{\mathcal{B}} \cap \mathcal{C}$.

Lemma 3. *There is a bijection between $\mathcal{C}$ and the set of U_kD-equivalence classes of $\mathcal{L}$.*

Proof. Let L be a non-empty Łukasiewicz path in $\mathcal{L}$. Let us prove that there exists a Łukasiewicz path $L' \in \mathcal{C}$ (with the same length as L) such that L and L' belong to the same class. We write

$$L = K_0 \prod_{i=1}^{r} \left(U_{k_i} D K_i\right),$$

where $k_i \geq 1$ for $1 \leq i \leq r$, and $K_0, K_1, K_2, \ldots, K_r$, $r \geq 0$, are some parts (possibly empty) without pattern U_kD for any $k \geq 1$.

We define the Łukasiewicz path

$$L' = F^{b_0} \prod_{i=1}^{r} \left(U_{k_i} D D^{a_i} F^{b_i - a_i}\right),$$

with $b_i = |K_i|$, $0 \leq i \leq r$, and for $1 \leq i \leq r$, $a_i = \min\{b_i, k_i - 1 + \sum_{j=1}^{i-1}(k_j - 1 - a_j)\}$. Less formally, for i from 0 to r, K_i is replaced with $D^{a_i}F^{b_i - a_i}$ where the value a_i is the maximal number of down steps that can be placed between the two occurrences $U_{k_i}D$ and $U_{k_{i+1}}D$ so that L' remains a lattice path. Clearly, $L' \in \mathcal{C}$ and L' belongs to the same class as L.

For instance, from $L = U_3DUDFFFFUUDDFFDDFF$, we obtain the path $L' = U_3DUDDDFFFUDFFFFFFF$ (see Fig. 5 for an illustration of this example).

Now we will prove that any U_kD-equivalence class contains at most one element in $\mathcal{C}$. For a contradiction, let L and L' be two different Łukasiewicz paths in $\mathcal{C}$ belonging to the same class. We write $L = QR$ and $L' = QS$ where R and S start with two different steps. Since L and L' lie in the same class, the two first steps of R and S cannot be U_kD for $k \geq 1$. Moreover, since L (resp. L') lies into $\mathcal{C}$, the two first steps of R (resp. S) cannot constitute a pattern U_kF for $k \geq 1$. Then, R and S cannot start with any up step U_k, $k \geq 1$.

Without loss of generality, let us assume that the first step of R is a down step D and then, the first step of S is a flat step F. This means that the last point of Q has its ordinate equal to zero (otherwise L' could not belong to $\mathcal{C}$). As the first step of R is D, the height of this step is -1 which gives a contradiction and completes the proof. □

Theorem 8. *The generating function for the set of U_kD-equivalence classes of $\mathcal{L}$ with respect to the length is given by*

$$\frac{1 - x - x^2 - \sqrt{1 - 2x - x^2 - 2x^3 + x^4}}{2x^3},$$

which generates the sequence A292460 in [27] that is a shift of the generalized Catalan sequence defined by $g_0 = 1$, and $g_{n+1} = g_n + \sum_{k=1}^{n-1} g_k g_{n-1-k}$ for $n \geq 0$ (see A004148 in [27]).

Proof. Using Lemma 3, it suffices to obtain the generating function $C(x)$ for the set $\mathcal{C}$. A non-empty Łukasiewicz path $L \in \mathcal{C}$ can be written either $L = FL'$ where $L' \in \mathcal{C}$, or $L = U_kDL_1DL_2D \dots L_{k-1}DL'$ for $k \geq 1$ where $L_1, L_2, \dots, L_{k-1}$ are some Łukasiewicz paths without flats in $\bar{\mathcal{C}}$, and $L' \in \mathcal{C}$. So we obtain the functional equation $C(x) = 1 + xC(x) + x^2C(x)\sum_{k\geq 0} x^k\bar{C}(x)^k$ where $\bar{C}(x)$ is the generating function for the set $\bar{\mathcal{C}}$ of Łukasiewicz paths without flats in $\mathcal{C}$. On the other hand, every $L \in \mathcal{C}$ is decomposed uniquely according to the number $k \geq 0$ of its flat steps as follows: $L = L_0 \prod_{i=1}^{k}(FL_i)$, where $L_i \in \bar{\mathcal{C}}$ for $i \geq 0$. Then by adding a peak $U_{k+1}D$ at the beginning of L and changing each F to D, we obtain a unique element of $\bar{\mathcal{C}}\backslash\{\epsilon\}$, viz. $U_{k+1}DL_0 \prod_{i=1}^{k}(DL_i)$. So, we have $\bar{C}(x) - 1 = x^2C(x)$ and a simple calculation provides the result. □

Theorem 9. *For any $n \geq 0$, the map ψ induces a bijection from $\mathcal{C}_n$ to the set of Motzkin paths in $\mathcal{M}_n$ that avoid the pattern UU.*

Proof. Theorem 7 ensures that ψ is a bijection from $\mathcal{B}_n$ to $\mathcal{M}_n$ for $n \geq 0$. We have $\mathcal{C} \subset \mathcal{B}$, and the paths in $\mathcal{C}$ are those in $\mathcal{B}$ that avoid the patterns U_kU_ℓ, $k, \ell \geq 1$ and U_kF, $k \geq 1$. Using Fact 2(b), the map ψ transforms occurrences of U_kU_ℓ, $k, \ell \geq 1$, into occurrences of UU. Since a path in $\mathcal{C}$ avoids any occurrence of U_kF for $k \geq 1$, the image by ψ of $\mathcal{C}_n$ is the subset of Motzkin paths in $\mathcal{M}_n$ that avoid any occurrence of the pattern UU. □

Let $\mathcal{L}'_n$, $n \geq 2$, be the set of Łukasiewicz paths of length n starting by U and ending by D. For $n \geq 0$, we define the bijection θ from $\mathcal{L}_n$ to $\mathcal{L}'_{n+2}$ as follows: $\theta(L)$ is obtained from L by replacing any occurrence U_kD by an occurrence DU_k for $k \geq 1$, and by adding a step U at the beginning and a step D at the ending. It is straightforward to verify that θ induces a bijection $\bar{\theta}$ between the set of U_kD-equivalence classes of $\mathcal{L}_n$ and the set of DU_k-equivalence classes of $\mathcal{L}'_{n+2}$. Then, Theorem 10 is deduced from Theorem 8.

Theorem 10. *The generating function for the set of DU_k-equivalence classes of $\mathcal{L}$ with respect to the length is given by*

$$\frac{1 + x + x^2 - \sqrt{1 - 2x - x^2 - 2x^3 + x^4}}{2x},$$

which generates the generalized Catalan sequence defined by $u_0 = u_1 = 1$, and for $n \geq 2$ $u_n = g_{n-2}$ where g_n is defined in Theorem 8. (see A004148 in [27]).

4.3. Modulo $S = \{U_kF, k \geq 1\}$, and $S = \{FU_k, k \geq 1\}$

Let $\mathcal{E} \subset \mathcal{L}$ be the set of Łukasiewicz paths such that any up step U_k, $k \geq 1$, is immediately followed by a flat step F, and any flat step F of positive height belongs to a pattern U_kF, $k \geq 1$. For instance, we have $U_3FDDFDUD \in \mathcal{E}$ and $U_3DDDFUFD \notin \mathcal{E}$. Let $\bar{\mathcal{E}} \subset \mathcal{E}$ be the set of Łukasiewicz paths in $\mathcal{E}$ without flat step on the x-axis.

Lemma 4. *There is a bijection between $\mathcal{E}$ and the set of U_kF-equivalence classes of $\mathcal{L}$.*

Proof. The proof is obtained *mutatis mutandis* as for Lemma 3 by replacing U_kD with U_kF. □

Theorem 11. *The generating function for the set of U_kF-equivalence classes of $\mathcal{L}$ with respect to the length is given by*

$$\frac{1 - x - \sqrt{1 - 2x + x^2 - 4x^3}}{2x^3},$$

which generates the generalized Catalan sequence defined by $h_0 = 1$, and for $n \geq 0$, $h_{n+1} = h_n + \sum_{k=0}^{n-2} h_kh_{n-2-k}$ (see A023431 in [27]).

Proof. Using Lemma 4, it suffices to obtain the generating function $E(x)$ for the set $\mathcal{E}$. A non-empty Łukasiewicz path $L \in \mathcal{E}$ can be written either $L = FL'$ where $L' \in \mathcal{E}$, or $L = U_kFL_1DL_2D \dots L_kDL'$ for $k \geq 1$ where $L_1, L_2, \dots, L_k$ are some Łukasiewicz paths in $\bar{\mathcal{E}}$, and $L' \in \mathcal{E}$. So we obtain the functional equation $E(x) = 1 + xE(x) + x^3E(x)\sum_{k\geq 0} x^k\bar{E}(x)^{k+1}$ where $\bar{E}(x)$ is the generating function for the set $\bar{\mathcal{E}}$. Using the classical decomposition of a Łukasiewicz path, we have $\bar{E}(x) = 1 + \sum_{k\geq 3} x^k\bar{E}(x)^{k-1}$. A simple calculation provides the result. □

Let ξ be the map from $\mathcal{L}$ to himself where $\xi(L)$ is obtained from L by replacing any occurrence U_kF by an occurrence FU_k for $k \geq 1$. It is straightforward to verify that ξ induces a bijection $\bar{\xi}$ between the set of U_kF-equivalence classes and the set of FU_k-equivalence classes. Then, we have Theorem 12.

Theorem 12. *The generating function for the set of FU_k-equivalence classes of $\mathcal{L}$ with respect to the length also is the generating function given in Theorem 11.*

Theorem 13. *For $n \geq 0$, the map ψ is a bijection from $\mathcal{E}_n$ to the subset $\mathcal{M}'_n$ of Motzkin paths in $\mathcal{M}_n$ that avoid UU and UD.*

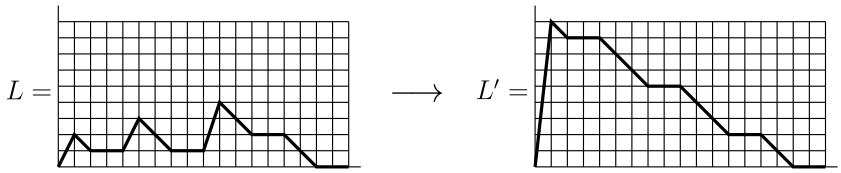


Fig. 6. Illustration of the example described in the proof of Lemma 5.

Proof. We proceed by induction on n. Obviously, for $n = 0$, we have $\psi(\epsilon) = \epsilon$. For $0 \le k \le n$, we assume that ψ is a bijection from $\mathcal{E}_k$ to the subset $\mathcal{M}'_k$ and we prove the result for $n + 1$. Since the set of length n Motzkin paths avoiding UU and UD is enumerated by the value h_n defined in Theorem 11 (see A023431 in [27]), it suffices to prove that ψ is surjective. So, let M be a Motzkin path in $\mathcal{M}'_{n+1}$. We distinguish two cases:

(i) $M = FM'$ with $M' \in \mathcal{M}'_n$. Using the recurrence hypothesis, there is $L' \in \mathcal{E}_n$ such that $M' = \psi(L')$. So, the Łukasiewicz path $L = FL' \in \mathcal{E}_{n+1}$ satisfies $\psi(L) = M$ which proves that M belongs to the image by ψ of $\mathcal{E}_{n+1}$.

(ii) $M = UM'DM''$ where M' and M'' are two Motzkin paths in $\mathcal{M}'$. Since $M \in \mathcal{M}'_{n+1}$, we have $M' \neq \epsilon$ and M' does not start with U, which implies that M' starts with F. Using the recurrence hypothesis, there are $L' \in \mathcal{E}$ and $L'' \in \mathcal{E}$ such that $\psi(L') = M'$ and $\psi(L'') = M''$. Since M' starts with a flat step, L' also starts with a flat step. So, $L = UL'DL''$ belongs to $\mathcal{E}_{n+1}$ and satisfies $\psi(L) = M$ which proves that ψ from $\mathcal{B}_n$ to $\mathcal{M}_n$ is bijective. □

4.4. Modulo FF

Let $\mathcal{F}$ be the set consisting of the union of $\{\epsilon, F\}$ with the set of Łukasiewicz paths containing at most one up step U_k, $k \ge 1$ and such that any flat step F is contained into a pattern FF. For instance, $FFU_3DFFDDFFDFF \in \mathcal{F}$ and $FFU_3FFDDFDFFF \notin \mathcal{F}$.

Lemma 5. *There is a bijection between $\mathcal{F}$ and the set of FF-equivalence classes of $\mathcal{L}$.*

Proof. Let L be a non-empty Łukasiewicz path in $\mathcal{L}$. Let us prove that there exists a Łukasiewicz path $L' \in \mathcal{F}$ (with the same length as L) such that L and L' belong to the same class. We write

$$L = K_1F^{a_1}K_2F^{a_2}K_3 \dots K_rF^{a_r}K_{r+1},$$

with $r \ge 0$ and $a_i \ge 2$ for $1 \le i \le r$, such that $K_1, K_2, \dots, K_r, K_{r+1}$ are some parts without pattern FF, $k \ge 1$, and $K_2, \dots, K_r$ are not empty and do not have any F in first and last position, and K_1 has no flat in last position, and K_{r+1} has no flat in first position.

If $L = F^n$ with $n \ge 0$, then its equivalence class is reduced to a singleton. Now let us assume that $L \neq F^n$. We distinguish two cases:

(1) K_1 is not empty. We define the Łukasiewicz path

$$L' = U_bD^{b_1-1}F^{a_1}D^{b_2}F^{a_2}D^{b_3} \dots D^{b_r}F^{a_r}D^{b_{r+1}}$$

with $b_i = |K_i|$, $1 \le i \le r + 1$, and $b = \sum_{i=1}^{r+1} b_i - 1$.

(2) K_1 is empty which means that $L = F^{a_1}K_2F^{a_2}K_3 \dots K_rF^{a_r}K_{r+1}$. We define the Łukasiewicz path

$$L' = F^{a_1}U_bD^{b_2-1}F^{a_2}D^{b_3} \dots D^{b_r}F^{a_r}D^{b_{r+1}}$$

with $b_i = |K_i|$, $2 \le i \le r + 1$, and $b = \sum_{i=2}^{r+1} b_i - 1$.

Less formally, we obtain L' from L by replacing any K_i (excepted the first) with a run of down steps $D^{|K_i|}$, and by replacing the first K_i (K_1 or K_2 according to the case (1) or (2)) with $U_bD^{|K_1|-1}$ (or $U_bD^{|K_2|-1}$) where the up step U_b balances all down steps in L', *i.e.*, b is the number of down steps in L'. Clearly, $L' \in \mathcal{F}$ and L' belongs to the same class as L. For instance, if $L = U_2DFFU_2DDFFU_3DDFFDDFF$, then $L' = U_9DFFDDDFFDDDFFDDFF$ (see Fig. 6).

The definition of $\mathcal{L}$ implies that there is only one path of $\mathcal{F}$ in the same class as L, which completes the proof. □

Theorem 14. *The generating function for the set of FF-equivalence classes of $\mathcal{L}$ with respect to the length is given by*

$$\frac{1 - 3x + 4x^2 - 5x^3 + 7x^4 - 7x^5 + 6x^6 - 3x^7 + x^8}{\left(1 - 2x + x^2 - x^3\right)(1 - x)^2}.$$

(Note that the associated sequence does not yet appear in [27]).

Proof. Using Lemma 5, it suffices to obtain the generating function $F(x)$ for the set $\mathcal{F}$. A non-empty Łukasiewicz path $L \in \mathcal{F}$ can be written either (i) $L = F^k$ for $k \ge 0$, or (ii) $L = F^{i_0}U_kF^{i_1}D^{j_1}F^{i_2}D^{j_2} \dots F^{i_\ell}D^{j_\ell}F^{i_{\ell+1}}$ with $\ell \ge 1$, $i_0 = 0$ or $i_0 \ge 2$, $i_1 = 0$ or $i_1 \ge 2$, $i_{\ell+1} = 0$ or $i_{\ell+1} \ge 2$, $i_m \ge 2$ for $2 \le m \le \ell$, and $j_m \ge 1$ for $1 \le m \le \ell$.

The generating function for the Łukasiewicz paths satisfying (i) is given by $\frac{1}{1-x}$.

For Łukasiewicz paths satisfying (ii), we give the generating function for each part of L, and we multiply them:

- For F^{i_0}, with $i_0 = 0$ or $i_0 \geq 2$, the generating function is $1 + \frac{x^2}{1-x}$;
- For $F^{i_{\ell+1}}$, with $i_{\ell+1} = 0$ or $i_{\ell+1} \geq 2$, the generating function is $1 + \frac{x^2}{1-x}$;
- For $U_k F^{i_1} D^{j_1}$, with $i_1 = 0$ or $i_1 \geq 2$ and $j_1 \geq 1$, the generating function is $x(1 + \frac{x^2}{1-x})\frac{x}{1-x}$;
- For $F^{i_2} D^{j_2} \ldots F^{i_\ell} D^{j_\ell}$, with $i_m \geq 2$, and $j_m \geq 1$, the generating function is $\frac{1}{1-\frac{x^3}{(1-x)^2}}$.

Considering all these cases, we deduce:

$$F(x) = \left(1 + \frac{x^2}{1-x}\right)^3 x^2 (1-x)^{-1} \left(1 - \frac{x^3}{(1-x)^2}\right)^{-1} + (1-x)^{-1}$$

which completes the proof. □

5. Concluding remarks

Extending recent works on Dyck and Motzkin paths [2,3], the goal of this paper is to calculate the number of Łukasiewicz paths modulo the positions of a given pattern, *i.e.* the number of possible sets $I = \{i_1, i_2, \ldots, i_k\}$ where $i_1, i_2, \ldots, i_k$ are the occurrence positions of the pattern in Łukasiewicz paths. Can one do the same study for other lattice paths such as meanders, bridges and excursions, or Schroeder and Riordan paths?

From our study, we can deduce a lower bound for the maximal cardinality of a class by calculating the average of cardinalities of the classes, *i.e.*, the total number of Łukasiewicz paths divided by the number of classes. Is it possible to calculate the exact value of the maximal cardinality for a class, and for which set I it is reached? Also, it would be interesting to study some properties of the number of Łukasiewicz paths (of a given length) having I as set of positions of the pattern. One can think this number is a polynomial with respect to the length n. If this is true, then we could give properties of these coefficients and roots, which would be a counterpart for lattice paths of the study of descent polynomial on the symmetric group S_n (see MacMahon [11]).

Acknowledgments

We would like to thank the anonymous referees for their very careful reading of this paper and their helpful comments and suggestions.

References

[1] E. Barcucci, R. Pinzani, R. Sprugnoli, The motzkin family, Pure Math. Appl. A 2 (3–4) (1992) 249–279.
[2] J.-L. Baril, A. Petrossian, Equivalence classes of Motzkin paths modulo a pattern of length at most two, J. Integer Seq. 18 (2015) 15.7.1.
[3] J.-L. Baril, A. Petrossian, Equivalence of dyck paths modulo some statistics, Discrete Math. 338 (2015) 655–660.
[4] C. Brennan, S. Mavhungu, Peaks and valleys in motzkin paths, Quaest. Math. 33 (2) (2010) 171–188.
[5] E. Deutsch, Dyck path enumeration, Discrete Math. 204 (1999) 167–202.
[6] R. Donaghey, L.W. Shapiro, Motzkin numbers, J. Combin. Theory Ser. A 23 (1977) 291–301.
[7] D. Drake, R. Gantner, Generating functions for plateaus in Motzkin paths, J. Chungcheong Math. Soc. 25 (2012) 475–489.
[8] P. Flajolet, R. Sedgewick, Analytic Combinatorics, Cambridge University Press, 2009.
[9] I.M. Gessel, S. Ree, Lattice paths and Faber polynomials, in: Advances in Combinatorial Methods and Applications to Probability and Statistics, Birkhauser Verlag, Boston, 1997.
[10] M. Lothaire, Combinatorics on words, in: Encyclopedia of Mathematics and Its Applications, vol. 17, Addison-Wesley, Reading, Massachusetts, 1983, Chapter 11, 219.
[11] P.A. MacMahon, Combinatory analysis, vol. I, II, Dover Phoenix Editions, Dover Publications Inc, Mineola, NY, 2004.
[12] K. Manes, A. Sapounakis, I. Tasoulas, P. Tsikouras, Equivalence classes of ballot paths modulo strings of length 2 and 3, Discrete Math. 339 (10) (2016) 2557–2572.
[13] T. Mansour, Counting peaks at height k in a dyck path, J. Integer Seq. 5 (2002) 02.1.1.
[14] T. Mansour, Statistics on Dyck paths, J. Integer Seq. 9 (2006) 06.1.5.
[15] T. Mansour, M. Schork, Y. Sun, Motzkin numbers of higher rank: generating function and explicit expression, J. Integer Seq. 10 (2007) 07.7.4.
[16] D. Merlini, R. Sprugnoli, M.C. Verri, Some statistics on dyck paths, J. Stat. Plann. Inference 101 (2002) 211–227.
[17] S.G. Mohanty, Lattice Path Counting and Applications, Academic Press, New York, 1979.
[18] T.V. Narayana, Lattice path combinatorics with statistical applications, in: Math. Expositions, vol. 23, Univ. of Toronto Press, Toronto, 1979.
[19] A. Panayotopoulos, A. Sapounakis, On the prime decomposition of Dyck paths, J. Combin. Math. Combin. Comput. 40 (2002) 33–39.
[20] P. Peart, W.J. Woan, Dyck paths with no peaks at height k, J. Integer Seq. 4 (2001) 01.1.3.
[21] H. Prodinger, S. Wagner, Minimal and maximal plateau lengths in Motzkin paths, in: DMTCS Proceedings, AH, 2007, pp. 353–362.
[22] G.N. Raney, Functional composition patterns and power series reversion, Trans. Amer. Math. Soc. 94 (1960) 411–451.
[23] A. Sapounakis, I. Tasoulas, P. Tsikouras, Counting strings in Dyck paths, Discrete Math. 307 (23) (2007) 2909–2924.
[24] A. Sapounakis, I. Tasoulas, P. Tsikouras, Some strings in Dyck paths, Australas. J. Combin. 39 (2007) 49–72.
[25] A. Sapounakis, P. Tsikouras, Counting peaks and valleys in k-colored Motzkin words, Electron. J. Combin. 12 (2005) #R16.
[26] M.P. Schützenberger, Le théorème de Lagrange selon G.N. Raney, in: Séminaire IRIA, Rocquencourt, 1971, pp. 199–205.
[27] N.J.A. Sloane, The On-line encyclopedia of integer sequences. Available electronically at http://oeis.org.
[28] R.P. Stanley, Enumerative Combinatorics, vol. 2, Cambridge University Press, 1999.
[29] Y. Sun, The statistic number of udu's in dyck paths, Discrete Math. 287 (2004) 177–186.
[30] A. Varvak, Lattice path encodings in a combinatorial proof of a differential identity, Discrete Math. 308 (2008) 5834–5840.
[31] G. Viennot, Une théorie combinatoire des polynômes orthogonaux généraux. Notes of lectures given at University of Quebec in Montreal, 1983.

Mathematics Magazine

ISSN: 0025-570X (Print) 1930-0980 (Online) Journal homepage: https://www.tandfonline.com/loi/umma20

The Pizza-Cutter's Problem and Hamiltonian Paths

Jean-Luc Baril & Céline Moreira Dos Santos

To cite this article: Jean-Luc Baril & Céline Moreira Dos Santos (2019) The Pizza-Cutter's Problem and Hamiltonian Paths, Mathematics Magazine, 92:5, 359-367, DOI: 10.1080/0025570X.2019.1664232

To link to this article: https://doi.org/10.1080/0025570X.2019.1664232

Published online: 20 Dec 2019.

Submit your article to this journal

Article views: 292

View related articles

View Crossmark data

Full Terms & Conditions of access and use can be found at
https://www.tandfonline.com/action/journalInformation?journalCode=umma20

The Pizza-Cutter's Problem and Hamiltonian Paths

JEAN-LUC BARIL
CÉLINE MOREIRA DOS SANTOS
Université de Bourgogne Franche-Comté
France
barjl@u-bourgogne.fr
celine.moreira@u-bourgogne.fr

The *pizza-cutter's problem* was introduced and solved by Steiner in 1826 (see [**14**]); it is considered as a doorstep to Euler's well-known formula $v + f - e = 2$ where v is the number of vertices, e the number of edges, and f the number of faces in a connected planar graph. The goal of the pizza-cutter's problem is to maximize the number of pieces that can be made with n straight cuts through a circular pizza, regardless of the size and shape of the pieces. Determining the maximum number of pieces of pizza is the same as determining the maximum number of regions formed by n lines in the plane, which appears in the literature as *Steiner's plane-cutting problem* [**1**, **2**, **16**, **17**]. If ℓ_n denotes this number then it satisfies the recurrence relation $\ell_n = \ell_{n-1} + n$ for $n \geq 1$ anchored by $\ell_0 = 1$, which induces the closed form $\ell_n = \frac{n(n+1)}{2} + 1$ (see A000124 in [**13**]). Indeed, from a solution to the problem with $n - 1$ lines that forms ℓ_{n-1} regions, we add an nth line that is not parallel to any of the others, and such that $n - 1$ new intersection points are created. Then, this line crosses n different regions, and each of them is divided into two regions which induces the above recursive formula.

Historically, the problem of line arrangements in the plane is studied by considering oriented matroids, more specifically known as non degenerate dissection types (see [**4**, **5**, **7**, **11**] for the literature and [**6**, **8**] for some databases). In this paper, we consider this problem from the point of view of graph theory. We refer to a solution of the Steiner's plane-cutting problem as an *S-solution*. For each S-solution, we consider the associated graph $G = (V, E)$ with vertex set V and the edge set E such that

- V is the set of regions; and
- $(p, q) \in E$ if and only if the two regions p and q are *adjacent*, i.e., if they share a common boundary that is not a corner, where corners are the points shared by three or more regions.

Of course there are many ways to cut the plane into a maximal number of regions with n lines, but G always has $|V| = \ell_n$ and $|E| = n^2$. In the case where two solutions produce two isomorphic graphs, we say that these solutions are *isomorphic*; otherwise they are called *non-isomorphic*. See Figure 1 for an illustration of two non-isomorphic S-solutions with their corresponding graphs. Finding the number of classes of non-isomorphic solutions for the plane-cutting problem still remains an open problem for $n \geq 10$. For $1 \leq n \leq 9$, it is known that these numbers are given by the sequence A090338 in [**13**]: 1, 1, 1, 1, 6, 43, 922, 38609, 3111341; see [**8**].

Math. Mag. **92** (2019) 359–367. doi:10.1080/0025570X.2019.1664232 © Mathematical Association of America
MSC: Primary 05C45, Secondary 05C10; 52C30
Color versions of one or more of the figures in the article can be found online at www.tandfonline.com/umma.

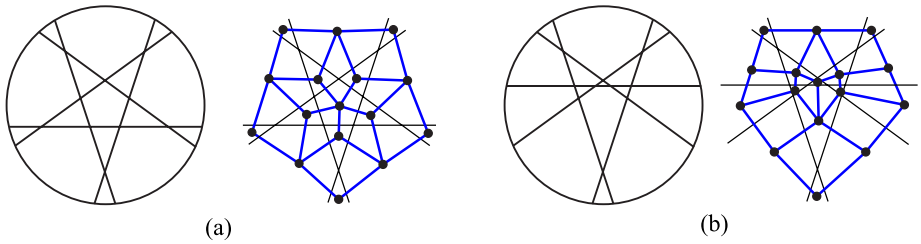


Figure 1 Two non-isomorphic S-solutions for $n = 5$ with their associated graphs drawn in blue.

A graph will be called *traceable* whenever it contains a *Hamiltonian path*, i.e., if there is a path that visits each vertex exactly once. This concept was introduced in 1856 in [15] to study whether a polyhedron contains a path that reaches each vertex once and only once. More generally, the problem of determining whether a graph is traceable is NP-complete and has many applications; see [9]. In particular, this problem appears in network theory where it is crucial to connect points so that the total length of connecting lines is a minimum. On the other hand, determining the traceability can often be a simple way to prove that two graphs are not isomorphic. Then it becomes natural to ask the following question. *For $n \geq 1$, does an S-solution exist such that its corresponding graph is traceable (respectively not traceable)?* A traceable solution to the pizza-cutter's problem means that we can eat up all pieces of the pizza such that any two pieces eaten consecutively are adjacent.

In the next section, we show how from an S-solution we can label each region with a binary string. This induces a graph where the vertex set is the set of labels, and two binary strings are adjacent if their Hamming distance is one. We prove that the traceability of the associated graph is equivalent to that of the graph on labels. Then, we construct an S-solution where the associated graph is not traceable for all $n \geq 5$. In the final section, we adapt this construction in order to obtain an S-solution for all n such that the graph is traceable. To our knowledge, no such precise constructions have previously been published. We conclude by formulating some open problems.

Binary string interpretation

A binary string s of length n is a word $s_1 s_2 \ldots s_n$ on the alphabet $\{0, 1\}$. The value s_i, $1 \leq i \leq n$, will be called the ith digit of s. A substring t of s is a word made up of consecutive digits of s. A run of 1's in s is a maximal substring of s of the form 1^k where $k \geq 1$, i.e., a run of 1's is a substring constituted of 1's that cannot be extended to a larger substring of 1's in s. For a binary string set B, we denote by B' (respectively B'') the subset of B of strings with an odd (respectively even) number of 1's.

The Hamming distance between two n-length binary strings s and t is the number of i, $1 \leq i \leq n$, such that s_i is different from t_i. A Gray code for a set of binary strings $B \subseteq \{0, 1\}^n$ is an ordered list $\mathcal{B}$ for B, such that the Hamming distance between any two consecutive strings in $\mathcal{B}$ is exactly one. A Gray code $\mathcal{B}$ for the set B may be viewed as a Hamiltonian path in the restriction of the hypercube Q_n to the set B. Note that no Gray code is possible for B whenever $\big||B'| - |B''|\big| > 1$.

Now, let us consider an S-solution with n lines numbered from 1 to n. We label each region with a binary string of length n where the ith digit is either 0 or 1 depending on whether the region is on one side or the other of the ith line. See Figure 2 for three illustrations of such a labeling. Of course, there are $n!$ possible ways to label n lines from 1 to n, and two half-planes are delimited by each line. Therefore, for an

S-solution there are at most $2^n \cdot n!$ possible sets of labels. In the following, such a set will be called *admissible* for a given S-solution.

Lemma 1. Let us consider an S-solution for $n \geq 1$, $G = (V, E)$ its associated graph and W an admissible set of binary strings for this solution. Let $H = (W, F)$ be the graph where the vertex set is W and two elements are adjacent in H if and only if their Hamming distance is one. Then G and H are isomorphic; and thus, G is traceable if and only if H is traceable.

Proof. It is straightforward to see that the two following assertions are equivalent: (1) two regions r and s are adjacent; and (2) the Hamming distance of the binary strings labeling r and s is one. ■

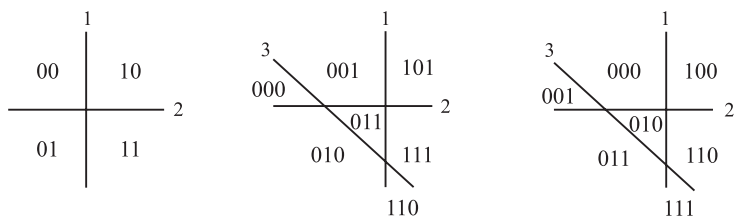


Figure 2 Regions labeled using admissible sets of binary strings. The leftmost and rightmost labeling provide the sets L_2 and L_3, while the central one does not generate L_3.

Remark 1. With the hypotheses of Lemma 1, a necessary condition for the traceability of the graph G is that the cardinalities of W' and W'' differ by at most one.

We end this section by introducing a set that will be crucial in what follows. Let L_n be the set of binary strings of length n containing at most one run of 1's. Any string $s_1 s_2 \ldots s_n \in L_n$, $n \geq 1$, can be written either $s = 0 s_2 \ldots s_n$ where $s_2 \ldots s_n \in L_{n-1}$, or $s = 1^k 0^{n-k}$ with $1 \leq k \leq n$. So, we have $|L_n| = |L_{n-1}| + n$ which induces $|L_n| = \ell_n$. Now, we denote by L'_n (respectively L''_n) the subset of L_n constituted of strings in L_n with an odd (respectively even) number of 1's. For instance, $L_3 = \{000, 001, 010, 100, 110, 011, 111\}$, $L'_3 = \{001, 010, 100, 111\}$ and $L''_3 = \{000, 110, 011\}$.

An S-solution where $G = (V, E)$ is not traceable

In this part, we construct an S-solution such that for each $n \geq 5$, its associated graph $G = (V, E)$ is not traceable. For this, we prove that the set L_n of n-length binary strings with at most one run of 1's is admissible for this solution and that the cardinality of their two subsets L'_n and L''_n differ by at least 2. Using Lemma 1 and Remark 1, we conclude that G is not traceable.

Lemma 2. For $n \geq 1$, there is an S-solution such that the set L_n of binary strings of length n with at most one run of 1's is admissible.

Proof. We proceed by induction on the number n of lines. The case $n = 1$ is trivial since we label the two half-planes by 0 and 1 and $L_1 = \{0, 1\}$.

Assume now that there is an S-solution of $n-1$ lines such that the regions can be labeled with the binary strings of the set L_{n-1}. Since there are exactly $n-1$ binary strings ending in a one in L_{n-1}, the $(n-1)$th line splits the plane into two half-planes such that one of them contains exactly the $n-1$ regions labeled $0^{n-2}1, 0^{n-3}1^2, \ldots, 01^{n-2}, 1^{n-1}$. Then, we necessarily have the leftmost configuration illustrated in Figure 3 where all previous binary strings appear on the same half-plane defined by the line $n-1$ (line 5 in Figure 3). Now, it suffices to place the nth line (line 6 in Figure 3) such that: it crosses the region 0^{n-1} and all other regions labeled 0^k1^{n-1-k} for $0 \le k \le n-2$ (the process is illustrated in Figure 3). Note that we can always add this line since it can be obtained from the $(n-1)$th line by a rotation centered on a point placed on the border between the regions 0^{n-1} and $0^{n-2}1$, and with an angle small enough to allow the nth line to intersect the first $(n-2)$ lines (as the $(n-1)$th line). Then, the labels of the newly created regions are obtained by adding 1 to the right of 0^{n-1} and 0^k1^{n-1-k} for $0 \le k \le n-2$, and adding 0 to the right of all other labels in L_{n-1}. Finally, the set of the obtained labels is exactly the set L_n of binary strings of length n with at most one run of 1's, and the proof is obtained by induction. ■

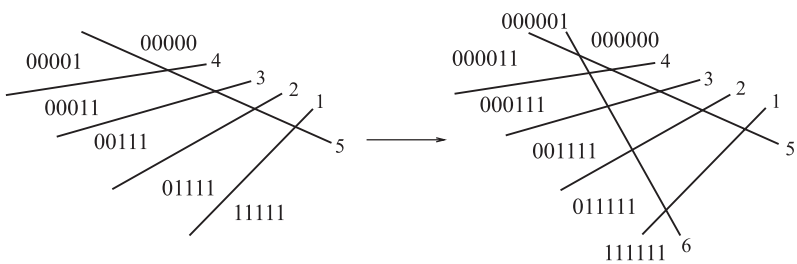


Figure 3 An illustration for the induction in the proof of Lemma 2.

Let $\{\phi_n\}_n \ge 0$ be the parity difference integer sequence corresponding to the binary strings with at most one run of 1's, i.e., $\phi_n = |L'_n| - |L''_n|$ for $n \ge 0$.

Lemma 3. For $n \ge 1$, we have $\phi_n = \lfloor \frac{n-1}{2} \rfloor$.

Proof. For $1 \le i \le n$, we denote by L_n^i the subsets of L_n made of strings with exactly i ones. Thus, it follows trivially that $|L_n^i| = n - i + 1$ for $1 \le i \le n$, and $|L_n^0| = 1$. Moreover, for i odd, $1 \le i \le n-1$, we have $|L_n^i| - |L_n^{i+1}| = 1$. Since $L'_n = \bigcup_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} L_n^{2i-1}$ and $L''_n = \bigcup_{i=0}^{\lfloor \frac{n}{2} \rfloor} L_n^{2i}$, we distinguish two cases. If n is odd, then $\phi_n = |L'_n| - |L''_n| = |L_n^n| - |L_n^0| + \sum_{i=1}^{\lfloor \frac{n-1}{2} \rfloor} (|L_n^{2i-1}| - |L_n^{2i}|) = \lfloor \frac{n-1}{2} \rfloor$. If n is even, then $\phi_n = \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} (|L_n^{2i-1}| - |L_n^{2i}|) - |L_n^0| = \lfloor \frac{n-1}{2} \rfloor$. ■

Theorem 1. For each $n \ge 5$, there exists an S-solution such that its associated graph is not traceable.

Proof. Figure 4 demonstrates a graph for $n = 5$ that is not traceable. For $n \ge 5$, Lemma 3 implies that $\phi_n = \lfloor \frac{n-1}{2} \rfloor \ge 2$. The combination of Remark 1 and Lemma 2 extends the result for $n > 5$. ■

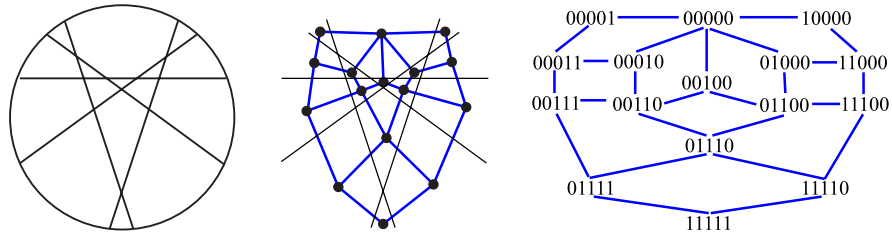


Figure 4 An S-solution where its associated graph is not traceable.

An S-solution where $G = (V, E)$ is traceable

In this part, for each $n \geq 1$, we construct an S-solution such that its associated graph is traceable.

From the set L_n defined previously (as the set of binary strings o flength n containing at most one run of 1's), we define the set K_n by replacing all strings $0^{4i}00100^{n-4(i+1)} \in L_n$ with $0^{4i}01010^{n-4(i+1)}$ for $0 \leq i \leq \lfloor \frac{n}{4} \rfloor - 1$. For instance, we obtain K_5 (respectively K_8) from L_5 (respectively L_8) by replacing 00100 (respectively 00100000 and 00000010) with 01010 (respectively 01010000 and 00000101).

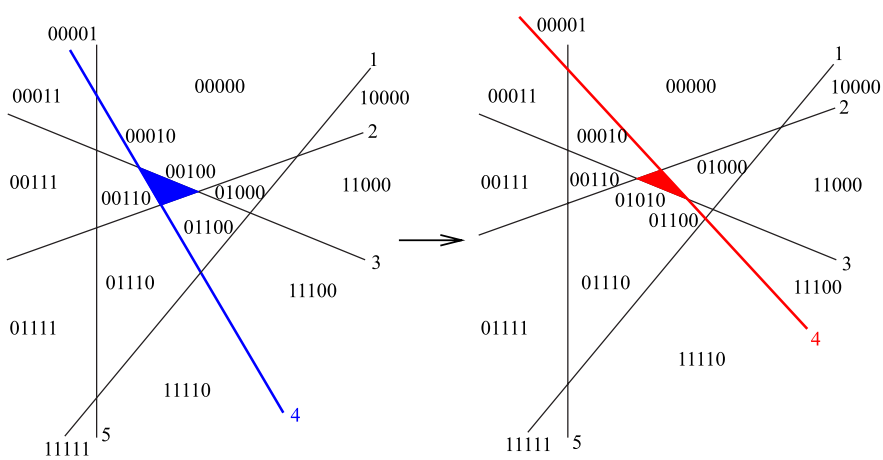


Figure 5 Construction in the proof of Lemma 4.

Lemma 4. For $n \geq 1$, there is an S-solution such that the set K_n is admissible.

Proof. Let us take the S-solution constructed in the proof of Lemma 2. Then we modify the position of each line labeled $4i$, $1 \leq i \leq \lfloor \frac{n}{4} \rfloor$ in the following way. For i from 1 to $\lfloor \frac{n}{4} \rfloor$, the line labeled $4i$ is moved so that in this new position, the half-plane delimited by this line and containing the point of intersection of the lines $4i - 1$ and $4i - 2$ does not contain any other points of intersection between lines from 1 to $4i - 1$. See Figure 5 for an illustration of the process. Then, a simple observation provides that the labels in L_n are preserved up to the labels $0^{4i}00100^{n-4(i+1)}$, $0 \leq i \leq \lfloor \frac{n}{4} \rfloor - 1$, that are replaced with $0^{4i}01010^{n-4(i+1)}$ which transforms the set L_n into the set K_n. Thus K_n is admissible. ■

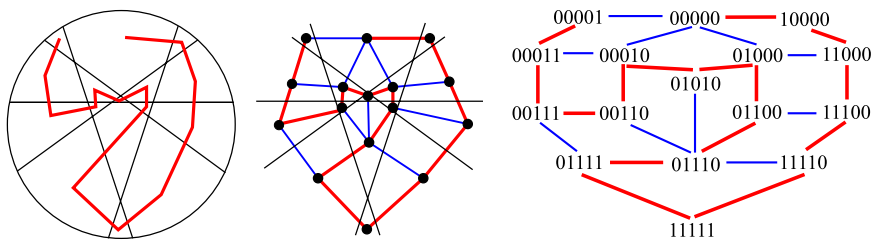


Figure 6 An S-solution where its associated graph is traceable. The red edges constitute a Hamiltonian path.

Theorem 2. For $n \geq 1$, there exists an S-solution such that its associated graph is traceable.

Proof. Due to Lemmas 1 and 4, it suffices to prove that the set K_n can be ordered in a list $\mathcal{K}_n$ such that two consecutive elements differ by one digit, i.e., $\mathcal{K}_n$ is in Gray code order. In order to facilitate the reading of the (somewhat theoretical) proof, we invite the reader to follow it by setting $n = 7$ or $n = 8$ before referring to Table 1. (We use color, different fonts, and boxed and underlined items to make proof easier to follow.)

Let $\mathcal{S}_n$, $n \geq 0$, be the list of the $n+1$ binary strings defined as follows: the ith binary element of the list is $1^{i-1}0^{n-i+1}$, $1 \leq i \leq n+1$. For instance, the list $\mathcal{S}_4$ is 0000, 1000, 1100, 1110, 1111. For $n = 0$, the list $\mathcal{S}_n$ is reduced to the empty string. Obviously, two consecutive elements of $\mathcal{S}_n$ differ by exactly one digit and the first and last elements of $\mathcal{S}_n$ are respectively 0^n and 1^n.

Using the lists $\mathcal{S}_n$, $n \geq 0$, we define an ordered list $\mathcal{L}_n$ of the set L_n by

$$\mathcal{L}_n = 0^n \odot \bigodot_{i=0}^{n-1} 0^i 1 \cdot \mathcal{S}^i_{n-i-1},$$

where $\odot$ is the concatenation operator of lists, and where $\mathcal{S}^i_n$ is the reverse list of $\mathcal{S}_n$ (i.e., the list $\mathcal{S}_n$ considered from the last to the first element) whenever i is odd, and the list $\mathcal{S}_n$ otherwise. See Table 1 for an illustration of the two lists $\mathcal{L}_7$ and $\mathcal{L}_8$.

In the list $\mathcal{L}_n$, it is straightforward to see that two consecutive elements differ by at most one digit except for the transitions between the sublists $0^i 1 \cdot \mathcal{S}^i_{n-i-1}$ and $0^{i+1}1 \cdot \mathcal{S}^{i+1}_{n-i-2}$ for i odd and $1 \leq i \leq n-2$. In these cases, the transitions move two digits since (when i is odd) the last element of $0^i 1 \cdot \mathcal{S}^i_{n-i-1}$ is $\mathbf{0^i 1 0^{n-i-1}}$ and the first element of $0^{i+1}1 \cdot \mathcal{S}^{i+1}_{n-i-2}$ is $\boxed{0^{i+1}10^{n-i-2}}$. Moreover, the first and last elements of the list $\mathcal{L}_n$ are respectively 0^n and $0^{n-1}1$. Now we modify the list $\mathcal{L}_n$ in order to construct a list $\mathcal{K}_n$ in Gray code order for the set K_n.

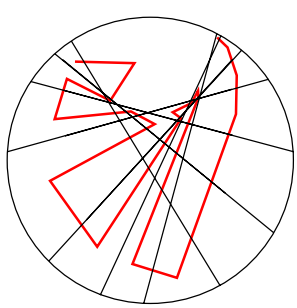
For all odd i such that $i = 1 \mod 4$, $1 \leq i \leq n-3$, we replace the string $\boxed{0^i 0100^{n-i-3}}$ with $\boxed{0^i 1010^{n-i-3}}$ and we change the place of $\underline{0^i 0010^{n-i-3}}$ by inserting it just after $\boxed{0^i 1010^{n-i-3}}$ and thus just before $0^i 0110^{n-i-3}$. See Table 1 for an illustration of this process for the lists $\mathcal{K}_7$ and $\mathcal{K}_8$.

By construction, the four binary strings $\mathbf{0^i 1000^{n-i-3}}$, $\boxed{0^i 1010^{n-i-3}}$, $\underline{0^i 0010^{n-i-3}}$ and $0^i 0110^{n-i-3}$ are consecutive in the list $\mathcal{K}_n$ and the three transitions differ by only one digit.

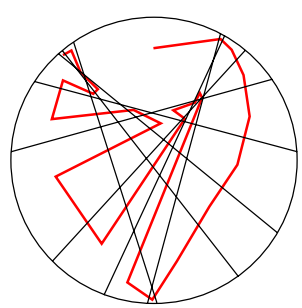
On the other hand, since we change the position of the binary strings of the form $\underline{0^i 0010^{n-i-3}}$, for $i = 1 \mod 4$, $1 \leq i \leq n-3$, we create a new transition between its

TABLE 1: The Lists $\mathcal{L}_n$ and $\mathcal{K}_n$ for $n = 7, 8$ and the Hamiltonian path on the S-solutions associated to the sets K_7 and K_8.

	$\mathcal{L}_7$	$\mathcal{K}_7$
	0000000	1000000
$i = 0$	1000000	1100000
	1100000	1110000
	1110000	1111000
	1111000	1111000
	1111000	1111000
	1111000	1111110
	1111110	1111111
	1111111	0111111
$i = 1$	0111111	0111110
	0111110	0111100
	0111100	0111000
	0111000	0110000
	0110000	**0100000**
	0100000	*0010000*
$i = 2$	*0010000*	0001000
	0011000	0011000
	0011100	0011100
	0011110	0011110
	0011111	0011111
$i = 3$	0001111	0001111
	0001110	0001110
	0001100	0001100
	0001000	0000100
$i = 4$	0000100	0000110
	0000110	0000111
	0000111	0000111
$i = 5$	0000011	0000010
	0000010	0000000
$i = 6$	0000001	0000001



	$\mathcal{L}_8$	$\mathcal{K}_8$
	00000000	00000000
$i = 0$	10000000	10000000
	11000000	11000000
	11100000	11100000
	11110000	11110000
	11111000	11111000
	11111100	11111100
	11111110	11111110
	11111111	11111111
$i = 1$	01111111	01111111
	01111110	01111110
	01111100	01111100
	01111000	01111000
	01110000	01110000
	01100000	01100000
	01000000	**01000000**
$i = 2$	*00100000*	*01010000*
	00110000	00010000
	00111000	00110000
	00111100	00111000
	00111110	00111100
$i = 3$	00011111	00111111
	00011110	00011111
	00011100	0011110
	00011000	00011100
	00010000	00011000
$i = 4$	00001000	00001000
	00001100	00001100
	00001110	00001110
	00001111	00001111
$i = 5$	00000111	00000111
	00000110	00000110
	00000100	**00000100**
$i = 6$	*00000010*	*00000101*
	00000011	00000001
$i = 7$	00000001	00000011



predecessor 0^i00110^{n-i-4} and its successor 0^i00010^{n-i-4} (if it exists) that moves only one digit. Notice that if $i = n-3$ then the string $\underline{0^i0010^{n-i-3}}$ has no successor in the list $\mathcal{L}_n$ and after moving its position, the last element of $\mathcal{K}_n$ becomes $0^{n-2}11$.

If n is even, then the last transition of two digits in $\mathcal{L}_n$ occurs between $0^{n-3}100$ and $0^{n-2}10$ which means that all transitions of two digits have been treated above, and the list $\mathcal{K}_n$ is in Gray code order. So, the first and last elements are respectively 0^n and $0^{n-2}11$ for $n = 0 \mod 4$, and 0^n and $0^{n-1}1$ for $n = 2 \mod 4$.

If n is odd, then the last transition of two digits in $\mathcal{L}_n$ occurs between $0^{n-2}10$ and $0^{n-1}1$. We distinguish two subcases. If $n \neq 3 \mod 4$, then the string $0^{n-2}10$ is moved by the above process and the obtained list is in Gray code order. So, the first and last elements are respectively 0^n and $0^{n-1}1$ (the Gray code is cyclic). However, if $n = 3 \mod 4$, then we insert the first element 0^n between $0^{n-2}10$ and $0^{n-1}1$ and we obtain a Gray code. Here, the first and last elements are respectively 10^{n-1} and $0^{n-1}1$.

Finally, for all $n \geq 1$ the constructed list $\mathcal{K}_n$ is in Gray code order. ■

Remark 2. For $n = 1, 2 \mod 4$, the Hamming distance between the first and last elements of the list $\mathcal{K}_n$ is one. Thus the associated graph becomes Hamiltonian (see Figure 6).

Going further

In this paper, we use a constructive method in order to prove that the pizza-cutter's problem admits an S-solution where its associated graph is traceable. Is it possible to provide a similar result using probabilistic method as studied in [**3**, **12**]? For a given n, can we find the number of isomorphism classes of S-solutions for $n \geq 10$? How many classes induce a traceable graph? For a given S-solution, can we characterize its corresponding admissible sets? More generally, can we make the same study for the space-cutting problem where the dimension of the space is greater than two?

Acknowledgment We would like to thank the anonymous referees for their very careful reading of this paper and their helpful comments and suggestions.

REFERENCES

[1] Alexanderson, G.L., Wetzel, J.E. (1978). Simple partitions of space. *Math. Mag.* 51(3): 220–225.
[2] Banks, R.B. (1999). *Slicing Pizzas, Racing Turtles, and Further Adventures in Applied Mathematics*. Princeton, NJ: Princeton University Press.
[3] Ben-Shimon, S., Krivelevich, M., Sudakov, B. (2011). On the resilience of Hamiltonicity and optimal packing of Hamilton cycles in random graphs. *SIAM J. Discrete Math.* 25(3): 1176–1193.
[4] Björner, A., Las Vergnas, M., Sturmfels, B., White, N., Ziegler, G. (1993). *Oriented Matroids*, 2nd ed. Cambridge: Cambridge University Press.
[5] Bokowski, J., Guedes de Oliveira, A. (2000). On the generation of oriented matroids. *Discret. Comput. Geom*. 24(2/3): 197–208.
[6] Christ, T. (2019). Database of combinatorially different simple line arrangements. https://geometry.inf.ethz.ch/christt/linearr/.
[7] Finschi, L. (2001). A graph theoretical approach for reconstruction and generation of oriented matroids. Thesis. https://www.math.ethz.ch/ifor/publications.html, Zurich.
[8] Finschi, L., Fukuda, K. (2019). Oriented matroids database. http://www.om.math.ethz.ch.
[9] Garey, M.R., Johnson, D.S. (1979). *Computers and Intractability*, Vol. 174. New York: Freeman.
[10] Graham, R.L., Knuth, D.E., Patashnik, O. (1990). *Concrete Mathematics*. New York: Addison Wesley.
[11] Ringel, G. (1956). Teilungen der ebene durch geraden oder topologische geraden. *Math. Z.* 64: 79–102.
[12] Shang, Y. (2015). On the Hamiltonicity of random bipartite graphs. *Indian J. Pure Appl. Math.* 46(2): 163–173.
[13] Sloane, N.J.A. (2019). The on-line encyclopedia of integer sequences. https://oeis.org.

[14] Steiner, J. (1826). Einige Gesetze über die Theilung der Ebene und des Raumes. *J. für die Reine Angewandte Mathematik*. 1: 349–364.

[15] Kirkman, T.P. (1856) On the enumeration of x-edra having triedral summits and an $(x-1)$-gonal base. *Philos. Trans. R. Soc. London*. 146: 399–411.

[16] JWetzel, J.E. (1978). On the division of the plane by lines. *Amer. Math. Monthly*. 85: 647–656.

[17] Zimmerman, S. (2001). Slicing space. *College Math. J.* 32: 126–128.

Summary. The pizza-cutter's problem is to determine the maximum number of pieces that can be made with n straight cuts through a circular pizza, regardless of the size and shape of the pieces. For a solution to this problem, we consider the graph $G = (V, E)$ where the vertex set V is the set of pieces and $(p, q) \in E$ if and only if the two pieces p and q are adjacent. We prove that there exists a solution where the graph G contains (respectively does not contain) a Hamiltonian path. Finally we present some open questions.

JEAN-LUC BARIL (MR Author ID: 709339) received his Ph.D. degree in pure mathematics from Bordeaux University, France, in 1996. He is currently full professor at the University of Burgundy, France, and member of LIB EA 7534. His main research involves combinatorial problems.

CÉLINE MOREIRA DOS SANTOS (MR Author ID: 695593) received her Ph.D. degree in mathematics from Caen University, France, in 2002. She is currently associate professor at the University of Burgundy, France, and member of LIB EA 7534. Her main research involves combinatorial problems.

#A46 INTEGERS 19 (2019)

MOTZKIN PATHS WITH A RESTRICTED FIRST RETURN DECOMPOSITION

Jean-Luc Baril[1]
LIB, Univ. Franche-Comté, France
barjl@u-bourgogne.fr

Sergey Kirgizov
LIB, Univ. Franche-Comté, France
kerzolster@gmail.com

Armen Petrossian
LIMOS, CNRS, Univ. Clermont Auvergne, France
armpetrossian@gmail.com

Received: 12/13/18, Accepted: 8/22/19, Published: 9/4/19

Abstract

Recently, the authors introduced new families of Dyck paths having a first decomposition constrained by the height or by the number of returns. In this work we extend the study to Motzkin paths and 2-colored Motzkin paths. For these new sets, we provide enumerative results by giving bivariate generating functions with respect to the length and another parameter, and we construct one-to-one correspondences with several restricted classes of ordered trees. We also deal with Schröder and Riordan paths. As a byproduct, we present a bijective proof of $M(x)^2 = \frac{1}{1-2x}M(\frac{x^2}{1-2x})$, where $M(x)$ is the generating function of Motzkin numbers.

1. Introduction and Notations

A *Motzkin path* of length $n \geq 0$ is a lattice path starting at point $(0,0)$, ending at $(n,0)$, and never going below the x-axis, consisting of up steps $U = (1,1)$, down steps $D = (1,-1)$ and flat steps $F = (1,0)$. Let $\mathcal{M}_n$ be the set of Motzkin paths of length n, and $\mathcal{M} = \bigcup_{n\geq 0} \mathcal{M}_n$. A *Dyck path* of semilength n is a Motzkin path of length $2n$ with no flat steps. Let $\mathcal{D}_n$ be the set of Dyck paths of semilength n, and $\mathcal{D} = \bigcup_{n\geq 0} \mathcal{D}_n$. Dyck (resp. Motzkin) paths of semilength (resp. length) n are enumerated by the n-th Catalan number $c_n = \frac{1}{n+1}\binom{2n}{n}$ (resp. by the n-th Motzkin number $\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \cdot c_k$) which is the general term of the sequence A000108 (resp. A001006) in the On-line Encyclopedia of Integer Sequences [20].

[1]Corresponding author

Many studies on Motzkin and Dyck paths appear in the literature. Generally, they consist in enumerating these paths according to several parameters, such as length, height, number of occurrences of a pattern, number of returns to the x-axis (see for instance [10, 16, 17, 19] for Dyck paths and [2, 5, 7, 14] for Motzkin paths). Also, many bijections have been found between these paths and various combinatorial objects such as Young tableaux, pattern avoiding permutations, RNA shapes and so on. See [21] for an overview.

Recently in [3], the authors introduced and enumerated the subset $\mathcal{D}^{s,\diamond}$ of Dyck paths having a restricted first return decomposition. More precisely, given a function $s : \mathcal{D} \to \mathbb{N}$, called *statistic*, and a comparison operator $\diamond$ on $\mathbb{N}$, the set $\mathcal{D}^{s,\diamond}$ is the union of the empty Dyck path with all Dyck paths P having a first return decomposition $P = U\alpha D\beta$ satisfying the conditions:

$$\begin{cases} \alpha, \beta \in \mathcal{D}^{s,\diamond}, \\ s(U\alpha D) \diamond s(\beta). \end{cases} \tag{1}$$

For $n \geq 0$, we denote by $\mathcal{D}_n^{s,\diamond}$ the set of Dyck paths of semilength n in $\mathcal{D}^{s,\diamond}$.

Whenever $\diamond$ equals $\geq$ and $s(P)$ is the maximal height $h(P)$ reached by P (resp. $s(P)$ is the number of returns in P), they prove algebraically and bijectively that $\mathcal{D}_n^{s,\diamond}$ is in one-to-one correspondence with the set $\mathcal{M}_n$ of Motzkin paths of length n. So, its generating function is

$$M(x) = \frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x^2}$$

and the first coefficients of x^n, $n \geq 0$, in the Taylor expansion are $1, 1, 2, 4, 9, 21, 51, 127$ (see Motzkin sequence `A001006` in [20]).

The purpose of the present paper is to extend the study to Motzkin paths that naturally generalize Dyck paths. Let $M \in \mathcal{M}$ be a non-empty Motzkin path, it can be uniquely written either $M = U\alpha D\beta$ or $M = F\alpha$ with $\alpha, \beta \in \mathcal{M}$. This decomposition will be called the *first return decomposition* of M (see Figure 1 for an illustration of this decomposition).

Figure 1: First return decompositions $U\alpha D\beta$ and $F\alpha$ of a Motzkin path.

Based on this decomposition and in the same way as for Dyck paths in [3], we construct a new collection of subsets of $\mathcal{M}$ as follows. Given a statistic function $s : \mathcal{M} \to \mathbb{N}$, and a comparison operator $\diamond$ on $\mathbb{N}$, the set $\mathcal{M}^{s,\diamond}$ is the union of the empty Motzkin path with all Motzkin paths M having a first return decomposition satisfying one of the two following conditions:

(C_1) $M = U\alpha D\beta$ with $\alpha, \beta \in \mathcal{M}^{s,\diamond}$ and $s(U\alpha D) \diamond s(\beta)$, or

(C_2) $M = F\alpha$ with $\alpha \in \mathcal{M}^{s,\diamond}$ and $s(F) \diamond s(\alpha)$.

For $n \geq 0$, we denote by $\mathcal{M}_n^{s,\diamond}$ the set of Motzkin paths of length n in $\mathcal{M}^{s,\diamond}$. For instance, if the operator $\diamond$ is $=$ and s is a constant statistic (*i.e.*, $s(M) = 0$ for any $M \in \mathcal{M}$), then we obviously have $\mathcal{M}_n^{s,\diamond} = \mathcal{M}_n$ for $n \geq 0$.

In this paper, we focus on the sets $\mathcal{M}^{s,\diamond}$, where the statistic $s(P)$ is either the number $r(P)$ of returns (a return is a down step D that touches the x-axis) or the maximal height $h(P)$ reached by the path.

Due to the above definition of $\mathcal{M}^{s,\diamond}$ whenever $s \in \{r, h\}$, a Motzkin path in $\mathcal{M}^{s,\geq}$ cannot contain any occurrence of FU because (C_2) implies that $0 = s(F) \geq s(\alpha)$, and thus $\alpha = F^k$ for some $k \geq 0$. Conversely, Motzkin paths in $\mathcal{M}^{s,\geq}$ can be constructed from Dyck paths in $\mathcal{D}^{s,\geq}$ by possibly adding flat steps before down steps or at the end. Since the two sets $\mathcal{D}_n^{r,\geq}$ and $\mathcal{D}_n^{h,\geq}$ are in bijection with the set $\mathcal{M}_n$ (see [3]), the generating function $M_s(x)$ of $\mathcal{M}^{s,\geq}$ for $s \in \{h, r\}$ satisfies

$$M_s(x) = \frac{1}{1-x} \cdot M\left(\frac{x^2}{1-x}\right).$$

The first coefficients of the Taylor expansion are $1, 1, 2, 3, 6, 11, 22, 43, 87, 176, 362, 748$, which correspond to the sequence `A026418` in [20].

In the same way, we define the set $\mathcal{R}^{s,\diamond}$ (resp. $\mathcal{S}^{s,\diamond}$) of Riordan paths (resp. of Schröder paths), *i.e.*, the set of Motzkin paths in $\mathcal{M}^{s,\diamond}$ with no flat steps on the x-axis (resp. where any maximal run of flats is of even length). As above, whenever $s \in \{h, r\}$, the generating functions $R_s(x)$ and $S_s(x)$ of $\mathcal{R}^{s,\diamond}$ and $\mathcal{S}^{s,\diamond}$ satisfy respectively

$$R_s(x) = M\left(\frac{x^2}{1-x}\right) \qquad \text{and} \qquad S_s(x) = \frac{1}{1-x^2} \cdot M\left(\frac{x^2}{1-x^2}\right).$$

The first coefficients of the Taylor expansion of $R_s(x)$ and $S_s(x)$ are respectively $1, 1, 1, 3, 5, 11, 21, 44, 89, 186$ and $1, 0, 2, 0, 5, 0, 14, 0, 42, 0, 132$, and the even ranks of this last sequence generate the Catalan numbers (`A097331` in [20]). Note that $S_s(x)$ is the binomial transform [18] of the Motzkin generating function, evaluated on the monomial x^2. So, we have $S_s(x) = C(x^2)$, where $C(x)$ is the generating function for the Catalan numbers [13].

The paper is organized as follows. In Section 2, we deal with the case where $s = r$ and $\diamond$ is $\geq$. We provide enumerating results for the cardinality of the set $\mathcal{M}^{r,\geq}$ according to the length and the number of returns. We give a constructive bijection ϕ between this set and the set of ordered trees with no branches of length one, and we show how ϕ transports several statistics. As a byproduct, we treat the case of Riordan and Schröder paths.

In Section 3, we focus on the set $\mathcal{M}^{h,\geq}$ (the operator $\diamond$ is $\geq$, and the statistic is the height). We provide a closed form of the generating function for the set of Motzkin

paths having a given height $k \geq 0$ in $\mathcal{M}^{h,\geq}$, and we deduce a continued fraction for the generating function of $\mathcal{M}^{h,\geq}$. We give a constructive one-to-one correspondence ψ between Motzkin paths in $\mathcal{M}^{h,\geq}$ and ordered trees with no branches of length one, and we show how ψ transports several parameters. So, we deduce a constuctive bijection $\psi^{-1} \circ \phi$ from $\mathcal{M}^{r,\geq}$ to $\mathcal{M}^{h,\geq}$.

In Section 4, we extend our study to the set $\bar{\mathcal{M}}$ of 2-Motzkin paths, *i.e.*, Motzkin paths where flat steps are of two kinds: straight and wavy. Using similar reasonings already done in Sections 2 and 3, we give enumerative results for the set $\bar{\mathcal{M}}_n^{s,\geq}$, $s \in \{r, h\}$, and we present a constructive bijection between $\bar{\mathcal{M}}^{s,\geq}$ and the set of ordered unary-binary trees where the root has exactly two children. As a byproduct, we obtain a bijective proof of the equality $M(x)^2 = \frac{1}{1-2x} M(\frac{x^2}{1-2x})$.

Finally, we conclude by presenting possible extensions of this work.

2. Motzkin Paths Constrained by the Number of Returns

In this section, the statistic s is the number of down steps D that touch the x-axis (called number of returns), and the comparison operator $\diamond$ is $\geq$.

Let $A(x, y) = \sum_{k,n \geq 0} a_{n,k} x^n y^k$ be the generating function where the coefficient $a_{n,k}$ of $x^n y^k$ is the number of Motzkin paths with k returns in $\mathcal{M}_n^{r,\geq}$.

Theorem 2.1. *The generating function $A(x, y)$ is given by*

$$A(x, y) = A_0(x) + A_1(x)y + A_2(x)y^2$$

where

$$A_0(x) = \frac{1}{1-x}, \qquad A_1(x) = \frac{1 - x - x^2 - \sqrt{1 - 2x - x^2 + 2x^3 - 3x^4}}{2x^2(1-x)}, \text{ and}$$

$$A_2(x) = \frac{1 - 2x - x^2 + 2x^3 - x^4 - (1 - x - x^2)\sqrt{1 - 2x - x^2 + 2x^3 - 3x^4}}{2x^4(1-x)}.$$

Proof. A Motzkin path M in $\mathcal{M}_n^{r,\geq}$ is either of the form $M = F^n$, or $M = U\alpha D\beta$, where $\alpha, \beta \in \mathcal{M}^{r,\geq}$ and $r(U\alpha D) = 1 \geq r(\beta)$. Using this last inequality, β is necessarily of one of two following forms: (i) $\beta = F^k$, or (ii) $\beta = U\gamma DF^k$ with $\gamma \in \mathcal{M}^{r,\geq}$ and $k \geq 0$. Then, we obtain the functional equation

$$A(x, y) = \frac{1}{1-x} + \frac{x^2 y}{1-x} A(x, 1) + \frac{x^4 y^2}{1-x} A(x, 1)^2.$$

A straightforward calculation provides the results. □

Notice that we retrieve the above generating function $M_r(x)$ for the sets $\mathcal{M}_n^{r,\geq}$, $n \geq 0$, by calculating $A(x,1)$:

$$M_r(x) = A(x,1) = \frac{1 - x - x^2 - \sqrt{1 - 2x - x^2 + 2x^3 - 3\,x^4}}{2x^4}.$$

The first terms of the Taylor expansion are $1 + x + 2x^2 + 3x^3 + 6x^4 + 11x^5 + 22x^6 + 43x^7 + 87x^8 + 176x^9 + 362x^{10}$ (see `A026418` in [20]). Table 1 presents the first values of coefficient $a_{n,k}$.

$k \backslash n$	1	2	3	4	5	6	7	8	9	10
0	1	1	1	1	1	1	1	1	1	1
1		1	2	4	7	13	24	46	89	176
2				1	3	8	18	40	86	185
$\sum$	1	2	3	6	11	22	43	87	176	362

Table 1: Number $a_{n,k}$ of Motzkin paths with k returns in $\mathcal{M}_n^{r,\geq}$, $1 \leq n \leq 10$ and $0 \leq k \leq 2$. For $k \geq 3$ and $n \geq 1$, we have $a_{n,k} = 0$.

The end of this section is dedicated to establishing a bijective link between $\mathcal{M}_n^{r,\geq}$ and a restricted set of ordered trees with n edges.

An *ordered tree* is a rooted tree where the order of subtrees matters. A *leaf* is a vertex with no child. According to the terminology used in [11], a vertex which is not a leaf or the root is called a *node*. A *branch node* is a node with at least two children, and a *branch* is a path where the extremities are either the root or a leaf or a branch node, and such that the other vertices are not branch nodes. The *length* of a branch is the number of its vertices minus one (or equivalently, the number of its edges). For $n \geq 0$, let $\mathcal{T}_n$ be the set of ordered trees with n edges, and let $\mathcal{T}_n^\star$ be the set of ordered trees with n edges and having no branches of length one. Any non-empty ordered tree $T \in \mathcal{T}_n$ can be decomposed in the form (L, R), where R consists of the root of T connected with its rightmost subtree, and L is the root of T connected with the remaining subtrees. In the following, L (resp. R) will be called the *left part* (resp. *right part*) of T. Note that the rightmost subtree of T is obtained from the right part R of T by deleting its root. Also, T is obtained from L and R by merging the roots of L and R (see Figure 2). According to this decomposition, we obtain directly the functional equation $T(x) = 1 + xT(x)^2$, where $T(x)$ is the generating function for the number of ordered trees with respect to the number of edges. As expected, the number of ordered trees with n edges is given by the n-th Catalan number.

By convenience, we adopt the following notation for an ordered tree T. We use the above decomposition $T = (L, R)$, and in the case where L is only the root, we

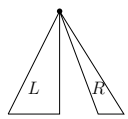


Figure 2: Decomposition (L, R) of T, where R is the root of T connected with its rightmost subtree, and L is the root of T connected with the remaining subtrees.

simplify this notation by writting $T = eR'$, where R' is the unique subtree of T (e refers to the edge between the root and the root of R'). For instance, e (resp. ee) is the root connected to a leaf (resp. the root connected to a node in turn connected to a leaf).

Now, we recursively define a map ϕ from $\mathcal{M}_n^{r,\geq}$ to the set $\mathcal{T}_{n+2}^{\star}$ for $n \geq 0$. Let M be a Motzkin path in $\mathcal{M}_n^{r,\geq}$, and $\alpha, \beta \in \mathcal{M}^{r,\geq}$:

(i) if $M = \epsilon$, then $\phi(M) = ee$,

(ii) if $M = \alpha F$ with $\phi(\alpha) = (L, R)$, then $\phi(M) = (L, eR)$,

(iii) if $M = U\alpha D$, then $\phi(M) = (\phi(\alpha), ee)$,

(iv) if $M = U\alpha DU\beta D$ with $\phi(\alpha) = (L, R)$, then $\phi(M) = (L, ee(R, \phi(\beta)))$.

An illustration of this map is given in Figure 3, and see Figure 5 for an example.

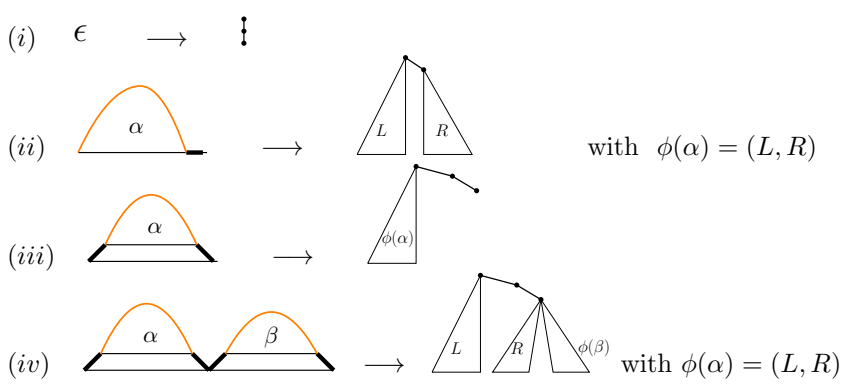


Figure 3: An illustration of the bijection ϕ.

Theorem 2.2. *For $n \geq 0$, the map ϕ is a bijection from $\mathcal{M}_n^{r,\geq}$ to $\mathcal{T}_{n+2}^{\star}$ satisfying $r(M) = \delta(\phi(M))$, where $\delta(T) = 0$ if T is a branch, and in other cases $\delta(T) = 1$ if the right part in the decomposition of T is a branch, and $\delta(T) = 2$ otherwise.*

Proof. We proceed by induction on n. Obviously, for $n = 1$, we have $\phi(F) = eee$ and $r(F) = 0 = \delta(eee)$. For $k \leq n$, we assume that ϕ is a bijection from $\mathcal{M}_k^{r,\geq}$ to $\mathcal{T}_{k+2}^{\star}$ such that $r(M) = \delta(\phi(M))$ for any $M \in \mathcal{M}_k^{r,\geq}$ and we prove the result for $n+1$. Since $\mathcal{M}_{n+1}^{r,\geq}$ and $\mathcal{T}_{n+3}^{\star}$ have the same cardinality (see [20]), it suffices to prove that if $M, M' \in \mathcal{M}_{n+1}^{r,\geq}$ with $\phi(M) = \phi(M')$ then $M = M'$. By a simple observation of Figure 3, the image by ϕ of paths satisfying (iii) are the trees in $\mathcal{T}_{n+2}^{\star}$ with a branch of length two as right part. Paths satisfying (ii) are sent on trees having the right part that starts with a branch of length at least three. Path satisfying (iv) are sent on trees having the right part that starts with a branch of length two and that contains a branch node. So, $\phi(M) = \phi(M')$ implies that M and M' belong to the same case (i), (ii), (iii), or (iv), and the recurrence hypothesis induces $M = M'$. Moreover, if $M = F^{n+1}$, then $\phi(M) = e^{n+3}$ and $r(M) = 0 = \delta(\phi(M))$; if $M = U\alpha D$, then $\phi(M) = (\phi(\alpha), ee)$ and $r(M) = 1 = \delta(\phi(M))$; if $M = U\alpha DU\beta D$ then $\phi(M) = (L, ee(R, \phi(\beta)))$ and $r(M) = 2 = \delta(\phi(M))$ which completes the proof. □

The bijection ϕ establishes correspondences between several statistics on Motzkin paths and ordered trees. For instance, the number of flats is translated into the difference between the number of edges and two times the number of branches. The number of up steps equals the number of branches minus one. The number of DU equals the number of branch nodes. Table 2 summarizes the main correspondences. We do not give formal proofs since they do not present any particular difficulties.

Corollary 2.3. *The restriction of ϕ to $\mathcal{D}_n^{r,\geq}$ establishes a one-to-one correspondence with ordered trees having $n+1$ branches in $\mathcal{T}_{2n+2}^{\star}$, which in turn are in bijection with ordered trees with $n+1$ edges.*

Corollary 2.4. *The restriction of ϕ to $\mathcal{S}_{2n}^{r,\geq}$ establishes a one-to-one correspondence with ordered trees in $\mathcal{T}_{2n+2}^{\star}$ having all its branches of even length.*

Corollary 2.5. *The restriction of ϕ to $\mathcal{R}_n^{r,\geq}$ establishes a one-to-one correspondence with ordered trees in $\mathcal{T}_{n+2}^{\star}$ where the rightmost branch starting from the root is of length 2.*

Since many bijections are already known [4, 9, 15, 22] between ordered trees and Dyck paths, it becomes easy to obtain constructive bijections between the two sets $\mathcal{D}_n$ and $\mathcal{S}_{2n}^{r,\geq}$ using Corollary 2.4.

3. Motzkin Paths Constrained by Height

In this section, we enumerate the set $\mathcal{M}_n^{h,\geq}$ of Motzkin paths of length $n \geq 0$ with a first return decomposition satisfying the two conditions (C_1) and (C_2) where $\diamond$

$M \in \mathcal{M}_n^{r,\geq}$	$\phi(M) \in \mathcal{T}_{n+2}^{\star}$
Number of returns	$\delta(\phi(M))$ (see Theorem 2.2)
Number of up steps	Number of branches minus one
Number of flats steps	$n+2-2\times$ Number of branches
Number of flats on the x-axis	Length minus 2 of the rightmost branch starting from root
Maximal length of a run of flats	Maximal length of a branch minus 2
Number of maximal runs of flats of even length	Number of branches of even length ≥ 4
Number of maximal runs of flats of odd length	Number of branches of odd length ≥ 3
Number of valleys DU	Number of branch nodes
Number of peaks UF^kD	Number of branch nodes plus one
Number of up steps minus number of valleys DU	Number of leaves minus one

Table 2: Statistic correspondences by the bijection ϕ

is the height statistic h. For $k \geq 0$, let $A_k(x) = \sum_{n\geq 0} a_{n,k}x^n$ (resp. $B_k(x) = \sum_{n\geq 0} b_{n,k}x^n$) be the generating function where the coefficient $a_{n,k}$ (resp. $b_{n,k}$) is the number of Motzkin paths in $\mathcal{M}_n^{h,\geq}$ having a maximal height equal to k (resp. of at most k). So, we have $B_k(x) = \sum\limits_{i=0}^{k} A_i(x)$ and the generating function for the set $\mathcal{M}^{h,\geq}$, namely $M_h(x)$, is given by $M_h(x) = \lim\limits_{k\to\infty} B_k(x)$.

Any non empty Motzkin path M of height $k \geq 0$ in $\mathcal{M}_n^{h,\geq}$, $n \geq 1$, is either F^n, or $U\alpha D\beta$, where α (resp. β) is a Motzkin path in $\mathcal{M}^{h,\geq}$ of height $k-1$ (resp. of height at most k). So we have

$$\begin{cases} A_0(x) & = B_0(x) = \frac{1}{1-x} \\ A_k(x) & = x^2 A_{k-1}(x) \cdot B_k(x) \end{cases}.$$

On the other hand, any Motzkin path of height k in $\mathcal{M}^{h,\geq}$ can be constructed from a Dyck path of height k in $\mathcal{D}^{h,\geq}$ by possibly adding flat steps before down steps, or at the end. Using the work in [3] on $\mathcal{D}^{h,\geq}$, we easily deduce $A_k(x) = \frac{1}{1-x}C_k(x^2/(1-x))$ and $B_k(x) = \frac{1}{1-x}D_k(x^2/(1-x))$, where $C_k(x)$ (resp. $D_k(x)$) is the generating function for Dyck paths of height k (resp. at most k) in $\mathcal{D}^{h,\geq}$. Therefore, we directly obtain Theorem 3.1 and Lemma 3.2 (the proofs are obtained *mutatis mutandis* as in [3]).

Theorem 3.1. *We have* $A_0(x) = B_0(x) = \frac{1}{1-x}$, $B_1(x) = \frac{1}{1-x-x^2}$, *and*

$$B_k(x) = \frac{1}{1-x} \cdot \prod_{i=0}^{k-1} (1 - x^2 A_i(x))^{-1} \text{ for } k \geq 1,$$

$$M_h(x) = \frac{1}{1-x} \cdot \prod_{i=0}^{\infty} (1 - x^2 A_i(x))^{-1},$$

$$A_k(x) = \frac{x^{2k}}{(1-x)^{k+1}} \cdot \prod_{i=0}^{k-1} (1 - x^2 A_i(x))^{i-k}.$$

$k \backslash n$	1	2	3	4	5	6	7	8	9	10
0	1	1	1	1	1	1	1	1	1	1
1		1	2	4	7	12	20	33	54	88
2				1	3	8	18	39	81	165
3						1	4	13	35	88
4								1	5	19
5										1
$\sum$	1	2	3	6	11	22	43	87	176	362

Table 3: Number $a_{n,k}$ of Motzkin paths of height k in $\mathcal{M}_n^{h,\geq}$, $1 \leq n \leq 10$ and $0 \leq k \leq 5$.

Lemma 3.2. *For* $k \geq 1$, *we have*

$$B_k(x) = \frac{1 + x^2 B_{k-1}(x)}{1 - x - x^4 B_{k-1}(x)}.$$

Note that by taking the limits when k converges to infinity, one gets $x^4 M_h(x)^2 + (x^2 + x - 1)M_h(x) + 1 = 0$, and we retreive that $M_h(x) = \frac{1}{1-x} \cdot M\left(\frac{x^2}{1-x}\right)$. Now we show how $B_k(x), k \geq 0$, can be expressed as a closed form. Let us define the function

$$f : u \mapsto \frac{1}{1 - x - x^2 - x^4 u}.$$

For $n \geq 1$, we denote by f^n the function recursively defined by $f^n(u) = f\big(f^{n-1}(u)\big)$ anchored with $f^0(u) = u$. Lemma 3.2 induces that for any $k \geq 2$, $B_k(x) = f(B_{k-2}(x))$, which induces Theorem 3.3.

Theorem 3.3. *For $k \geq 1$, we have*

$$B_k(x) = f^{\lfloor \frac{k}{2} \rfloor}\left(B_{k \bmod 2}(x)\right)$$

with the initial cases $B_0(x) = \frac{1}{1-x}$ and $B_1(x) = \frac{1}{1-x-x^2}$.

Table 3 presents the first values of $a_{n,k}$. Whenever $k = 1$, we have $B_1(x) = 1/(1-x-x^2)$ which is the generating function of the sequence of Fibonacci (see `A000045` in [20]). Note that, by taking limits in Theorem 3.3, we retrieve the continued fraction given by P. Barry for the sequence `A026418` in [20]:

$$M_h(x) = \cfrac{1}{1 - x - x^2 - \cfrac{x^4}{1 - x - x^2 - \cfrac{x^4}{1 - x - x^2 - \cfrac{x^4}{\cdots}}}}.$$

Now, we conclude this section by giving a constructive bijection between $\mathcal{M}_n^{h,\geq}$ and the set $\mathcal{T}_{n+2}^{\star}$, $n \geq 0$. Let M be a Motzkin path in $\mathcal{M}_n^{h,\geq}$ and $\alpha, \beta \in \mathcal{M}^{h,\geq}$, we recursively define the map ψ as follows:

(i) if $M = \epsilon$, then $\psi(M) = ee$,

(ii) if $M = \alpha F$ with $\psi(\alpha) = (L, R)$, then $\psi(M) = (L, eR)$,

(iii) if $M = \alpha U F^k D$ with $k \geq 0$ and $\psi(\alpha) = (L, R)$, then $\psi(M) = ((L, e^k R), ee)$,

(iv) if $M = \alpha U U \beta D \gamma D$ with $\psi(\alpha\gamma) = (L, R)$, then $\psi(M) = (L, ee(R, \psi(\beta)))$.

An illustration of this map is given in Figure 4. Notice that the two first cases (i) and (ii) are identical to those of ϕ. See Figure 5 for an example.

Theorem 3.4. *The map ψ is a bijection from $\mathcal{M}_n^{h,\geq}$ to $\mathcal{T}_{n+2}^{\star}$.*

Proof. We proceed by induction on n. Obviously, for $n = 1$, we have $\psi(F) = eee$. For $k \leq n$, we assume that ψ is a bijection from $\mathcal{M}_k^{h,\geq}$ to $\mathcal{T}_{k+2}^{\star}$ and we prove the result for $n+1$. Since $\mathcal{M}_{n+1}^{h,\geq}$ and $\mathcal{T}_{n+3}^{\star}$ have the same cardinality, it suffices to prove that if $M, M' \in \mathcal{M}_{n+1}^{h,\geq}$ with $\psi(M) = \psi(M')$ then $M = M'$. By a simple observation of Figure 4 and in a same way as for the proof of Theorem 2.2, the condition $\psi(M) = \psi(M')$ implies that M and M' belong to the same case (i), (ii), (iii), or (iv). For the first two cases, it is straightforward that the recurrence hypothesis induces $M = M'$. For the case (iii), we have $\psi(M) = \psi(M') = ((L, e^k R), ee)$. The first branch of R is necessarily of length two, otherwise this would mean that the first branch of R is of length at least three, and thus α would have a flat step at its end, which is impossible for $M \in \mathcal{M}^{h,\geq}$ satisfying (iii). Thus, k is entirely

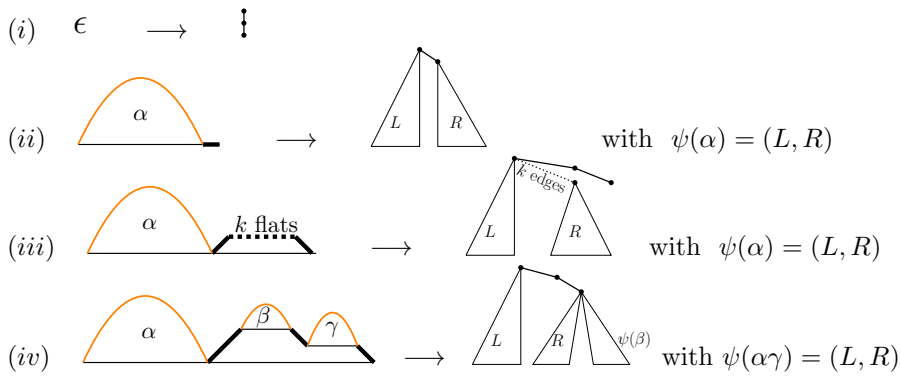


Figure 4: An illustration of the bijection ψ.

determined by $\psi(M) = \psi(M')$, and the recurrence hypothesis induces $M = M'$. For the case (iv), we have $M = \alpha UU\beta D\gamma D$, $M' = \alpha' UU\beta' D\gamma' D$ and $\psi(M) = \psi(M')$. This implies that $\psi(\beta) = \psi(\beta')$, $\psi(\alpha\gamma) = \psi(\alpha'\gamma')$, and the recurrence hypothesis gives $\beta = \beta'$ and $\alpha\gamma = \alpha'\gamma'$. Since $h(\alpha) \geq h(\beta) + 2 \geq h(\gamma) + 1$, $\alpha\gamma$ has a unique decomposition satisfying these inequalities. So we conclude $\alpha = \alpha'$ and $\gamma = \gamma'$. $\square$

The bijection ψ establishes correspondences between several statistics on Motzkin paths and ordered trees. Table 4 summarizes the main correspondences. We do not succeed to determine the statistic on $\mathcal{T}_n^\star$ which is the image by ψ of h. Note that using a straightforward proof by induction, the number of branch nodes of $\psi(M)$ equals the sum of $\lfloor \frac{k}{2} \rfloor$ on all maximal runs U^k in M. As a byproduct and using Corollary 3.5, Tables 2 and 4 induce several bijective correspondences between statistics on $\mathcal{M}_n^{r,\geq}$ and $\mathcal{M}_n^{h,\geq}$. See Figure 5 for an example of such a correspondence.

Corollary 3.5. *The bijection $\psi^{-1} \circ \phi$ induces one-to-one correspondences between $\mathcal{M}_n^{r,\geq}$ and $\mathcal{M}_n^{h,\geq}$, $\mathcal{D}_n^{r,\geq}$ and $\mathcal{D}_n^{h,\geq}$, $\mathcal{S}_n^{r,\geq}$ and $\mathcal{S}_n^{h,\geq}$, $\mathcal{R}_n^{r,\geq}$ and $\mathcal{R}_n^{h,\geq}$. These bijections preserve the number of up steps, the number of flats, the number of flats on the x-axis, the number of maximal runs of flats of even length, the maximal length of a run of flats, and it transports the number of valleys DU into the sum of $\lfloor \frac{k}{2} \rfloor$ on all maximal runs U^k.*

4. Going Further With 2-Motzkin Paths

In this part, we extend the previous study to 2-Motzkin paths. A 2-Motzkin path is a Motzkin path where flat steps can be of two kinds: F for straight and K for wavy. The number of 2-Motzkin paths with n steps is given by the $(n+1)$-th Catalan number $\frac{1}{n+2}\binom{2n+2}{n+1}$. We refer to [8, 12] for a constructive bijection between

$M \in \mathcal{M}_n^{h,\geq}$	$T = \psi(M) \in \mathcal{T}_{n+2}^{\star}$
Height	Open question
Number of up steps	Numbers of branches minus one
Number of flats	$n+2-2\times$ Number of branches
Number of flats on the x-axis	Length minus 2 of the rightmost branch starting from root
Number of maximal runs of flats of even length	Number of branches of even length ≥ 4
Number of maximal runs of flats of odd length	Number of branches of odd length ≥ 3
Maximal length of a run of flats	Maximal length of a branch minus 2
$\sum_{\text{Maximal } U^k} \lfloor \frac{k}{2} \rfloor$	Number of branch nodes
Number of up steps minus $\sum_{\text{Maximal } U^k} \lfloor \frac{k}{2} \rfloor$	Number of leaves minus one

Table 4: Statistic correspondences by the bijection ψ

2-Motzkin paths of length n and Dyck paths of semilength $n+1$. For a statistic $s \in \{r, h\}$, we define the set $\bar{\mathcal{M}}_n^{s,\geq}$ of 2-Motzkin paths of length n satisfying the two conditions (C_1) and (C_2) given in Introduction and we set $\bar{\mathcal{M}}^{s,\geq} = \bigcup_{n\geq 0} \bar{\mathcal{M}}_n^{s,\geq}$. As we did for Motzkin paths, a 2-Motzkin path in $\bar{\mathcal{M}}^{s,\geq}$ can be constructed from a Dyck path in $\mathcal{D}^{s,\geq}$ by possibly adding flat steps (F or K) before any down step, or at the end. So, the generating function $\bar{M}_s(x)$ for the set $\bar{\mathcal{M}}^{s,\geq}$ is:

$$\bar{M}_s(x) = \frac{1}{1-2x} \cdot M\left(\frac{x^2}{1-2x}\right).$$

The first coefficients of its Taylor expansion are $1, 2, 5, 12, 30, 76, 196, 512, 1353, 3610, 9713, 26324$, and these numbers correspond to the first differences of Motzkin numbers, which are also called generalized Ballot numbers (see `A002026` in [20]).

4.1. Enumerative Results

Below, we state enumerative results for $s \in \{r, h\}$. We do not give the proofs since it suffices to reread Sections 2 and 3 by replacing all fractions $\frac{1}{1-x}$ with $\frac{1}{1-2x}$ in all functional equations in order to taking into account the two kinds of flats. Theorem 4.1 and Table 5 deal with the statistic of the number of returns, while Theorem 4.2 and Table 6 treat the case of the height.

Let $A(x,y) = \sum_{k,n\geq 0} a_{n,k} x^n y^k$ be the generating function where the coefficient

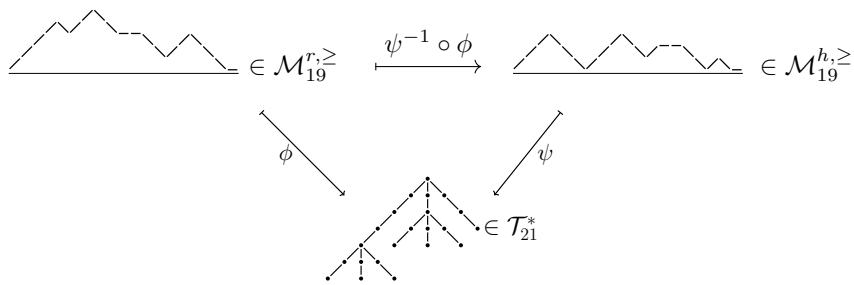


Figure 5: Example of one-to-one correspondence between two Motzkin paths in $\mathcal{M}_{19}^{r,\geq}$ and $\mathcal{M}_{19}^{h,\geq}$ passing by a tree in $\mathcal{T}_{21}^{\star}$.

$a_{n,k}$ of $x^n y^k$ is the number of 2-Motzkin paths with k returns in $\bar{\mathcal{M}}_n^{r,\geq}$.

Theorem 4.1. *The generating function $A(x,y)$ is given by*

$$A(x,y) = A_0(x) + A_1(x)y + A_2(x)y^2$$

with

$$A_0(x) = \frac{1}{1-2x}, \qquad A_1(x) = \frac{1-2x-x^2-\sqrt{1-4x+2x^2+4x^3-3x^4}}{2x^2\,(1-2\,x)}, \text{ and}$$

$$A_2(x) = \frac{\left(1-2x-x^2-\sqrt{1-4x+2x^2+4x^3-3x^4}\right)^2}{4x^4\,(1-2\,x)}.$$

$k\backslash n$	1	2	3	4	5	6	7	8	9	10
0	2	4	8	16	32	64	128	256	512	1024
1		1	4	13	38	106	288	772	2056	5465
2				1	6	26	96	325	1042	3224
$\sum$	2	5	12	30	76	196	512	1353	3610	9713

Table 5: Number $a_{n,k}$ of 2-Motzkin paths with k returns in $\bar{\mathcal{M}}_n^{r,\geq}$, $1 \leq n \leq 10$ and $0 \leq k \leq 2$. For $k \geq 3$ and $n \geq 1$, we have $a_{n,k} = 0$.

For $k \geq 0$, let $A_k(x) = \sum_{n\geq 0} a_{n,k}x^n$ (resp. $B_k(x) = \sum_{n\geq 0} b_{n,k}x^n$) be the generating function where the coefficient $a_{n,k}$ (resp. $b_{n,k}$) is the number of 2-Motzkin paths of height k (resp. at most k) in $\bar{\mathcal{M}}_n^{h,\geq}$.

Theorem 4.2. *We have* $A_0(x) = B_0(x) = \frac{1}{1-2x}$, $B_1(x) = \frac{1}{1-2x-x^2}$, *and*

$$B_k(x) = \frac{1}{1-2x} \cdot \prod_{i=0}^{k-1} (1 - x^2 A_i(x))^{-1} \text{ for } k \geq 1,$$

$$\bar{M}_h(x) = \frac{1}{1-2x} \cdot \prod_{i=0}^{\infty} (1 - x^2 A_i(x))^{-1},$$

$$A_k(x) = \frac{x^{2k}}{(1-2x)^{k+1}} \cdot \prod_{i=0}^{k-1} (1 - x^2 A_i(x))^{i-k}.$$

$k \backslash n$	1	2	3	4	5	6	7	8	9	10
0	2	4	8	16	32	64	128	256	512	1024
1		1	4	13	38	105	280	729	1866	4717
2				1	6	26	96	324	1032	3159
3						1	8	43	190	748
4								1	10	64
5										1
$\sum$	2	5	12	30	76	196	512	1353	3610	9713

Table 6: Number $a_{n,k}$ of Dyck paths of height k in $\bar{\mathcal{M}}_n^{h,\geq}$, $1 \leq n \leq 10$ and $0 \leq k \leq 5$.

Note that $B_1(x)$ is the generating function of the sequence of Pell numbers (`A000129` in [20]) which count the left factors of Grand Schröder paths.

Lemma 4.3. *For* $k \geq 1$, *we have*

$$B_k(x) = \frac{1 + x^2 B_{k-1}(x)}{1 - 2x - x^4 B_{k-1}(x)}.$$

Note that by taking the limits when k converges to infinity, one gets $x^4 \bar{M}_h(x)^2 + (x^2 + 2x - 1)\bar{M}_h(x) + 1 = 0$, and we retrieve that $\bar{M}_h(x) = \frac{1}{1-2x} \cdot M(\frac{x^2}{1-2x})$. Now, we show how $B_k(x)$ can be expressed as a closed form. Let us define the function

$$g : u \mapsto \frac{1}{1 - 2x - x^2 - x^4 u}.$$

Lemma 4.3 induces that for any $k \geq 2$, $B_k(x) = g(B_{k-2}(x))$, which induces Theorem 4.4.

Theorem 4.4. *For $k \geq 1$, we have*

$$B_k(x) = g^{\lfloor \frac{k}{2} \rfloor}\left(B_{k \bmod 2}(x)\right)$$

with the initial cases $B_0(x) = \frac{1}{1-x}$ and $B_1(x) = \frac{1}{1-2x-x^2}$.

By taking limits in Theorem 4.4, we obtain the continued fraction for the sequence A002026 in [20]:

$$\bar{M}_h(x) = \cfrac{1}{1 - 2x - x^2 - \cfrac{x^4}{1 - 2x - x^2 - \cfrac{x^4}{1 - 2x - x^2 - \cfrac{x^4}{\cdots}}}}.$$

4.2. Constructive Bijection

In this part, we construct a bijection between $\bar{\mathcal{M}}_n^{s,\geq}$, $s \in \{r, h\}$, and the subset $\mathcal{B}_{n+2}$ of $\mathcal{T}_{n+2}$ consisting of ordered trees with $n+2$ edges where the root has two children, and where every node has at most two children. In order to do this, we firstly extend the previous bijections ϕ and ψ respectively on the sets $\bar{\mathcal{M}}_n^{r,\geq}$ and $\bar{\mathcal{M}}_n^{h,\geq}$ by setting $\phi(\alpha K) = \psi(\alpha K) = (L, \bar{e}R)$ whenever $\phi(\alpha) = \psi(\alpha) = (L, R)$, and where e is a straight edge and $\bar{e}$ is a wavy edge. So, these two maps establish one-to-one correspondences between $\bar{\mathcal{M}}_n^{s,\geq}$, $s \in \{r, h\}$, and the set $\mathcal{T}_{n+2}^{\star\star}$ of ordered trees with $n+2$ edges having no branches of length one, and such that any branch is constituted of edges of two kinds (straight and wavy) except for its two last edges which must be straight. Secondly, we recursively define a map χ between $\mathcal{T}_{n+2}^{\star\star}$ and the subset $\mathcal{B}_{n+2}$.

For $n \geq 2$, let T be an ordered tree in $\mathcal{T}_n^{\star\star}$ and $\alpha, \beta \in \mathcal{T}^{\star\star} = \bigcup_{n\geq 2} \mathcal{T}_n^{\star\star}$ (which induces that α and β are non-empty). Let $\mathcal{T}^{\star\star(1)}$ be the set of ordered trees in $\mathcal{T}^{\star\star}$ where the root has only one child, and let $\mathcal{T}^{\star\star(2)} = \mathcal{T}^{\star\star} \backslash \mathcal{T}^{\star\star(1)}$.

We define χ as follows:
In the case where $T \in \mathcal{T}^{\star\star(1)}$:

(i) if $T = ee$, then $\chi(T) = (e, e)$,

(ii) if $T = ee\alpha$ with $\alpha \in \mathcal{T}^{\star\star(2)}$, then $\chi(T) = (e\chi(\alpha), e)$,

(iii) if $T = e\alpha$ with $\alpha \in \mathcal{T}^{\star\star(1)}$, then $\chi(T) = (L, e^{k+1})$ for $\chi(\alpha) = (L, e^k)$, $k \geq 1$,

(iv) if $T = \bar{e}\alpha$ with $\alpha \in \mathcal{T}^{\star\star(1)}$, then $\chi(T) = (e(L, e^{k-1}), e)$ for $\chi(\alpha) = (L, e^k)$, $k \geq 1$.

In the case where $T \in \mathcal{T}^{\star\star(2)}$:

(*v*) if $T = (\alpha, \beta)$ with $\alpha \in \mathcal{T}^{\star\star}$, $\beta \in \mathcal{T}^{\star\star(1)}$, then $\chi(T) = (L, e^k\chi(\alpha))$ for $\chi(\beta) = (L, e^k), k \geq 1$.

An illustration of this map is given in Figure 6.

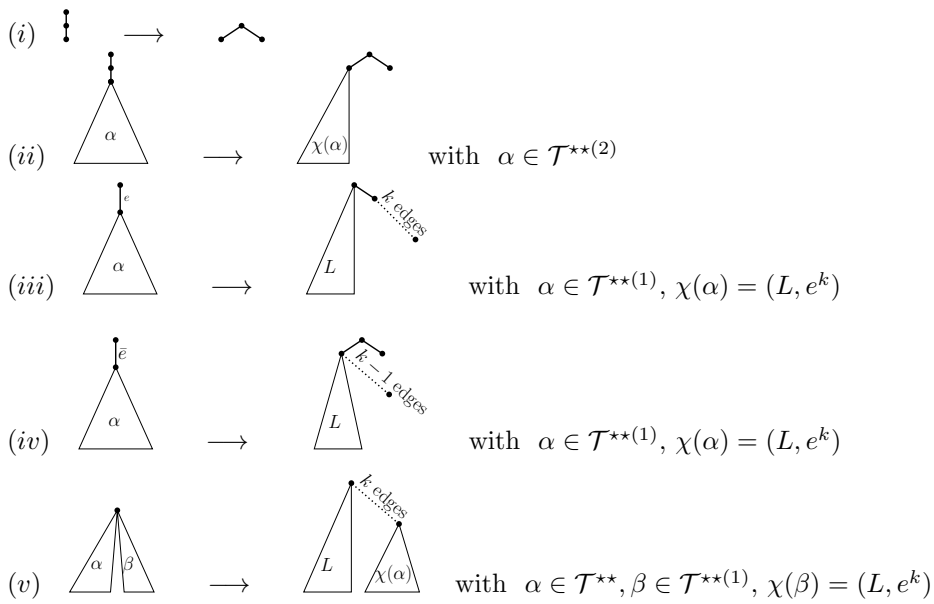


Figure 6: An illustration of the bijection χ.

It is easy to observe the following facts.

Fact 1. *The image by χ of $T \in \mathcal{T}^{\star\star(1)}$ does not have any branch node in its right part, that is, its right branch is of the form e^k for $k \geq 1$.*

Fact 2. *The image of $T \in \mathcal{T}^{\star\star(2)}$ has at least one branch node in its right part.*

Fact 3. *The image of a tree satisfying (iii) has a branch of length at least two as right part of the root, while the image of a tree satisfying (i), (ii) or (iv) has a branch of length one.*

Fact 4. *The image of a tree satisfying (ii) has a left subtree where the root has two subtrees such that the rightmost contains at least one branch node.*

Fact 5. *The image of a tree satisfying (iv) has a left subtree where either the root has two subtrees such that the rightmost is a branch (whenever $k \geq 2$), or the root has only one subtree (whenever $k = 1$).*

Theorem 4.5. *For $n \geq 2$, the map χ is a bijection from $\mathcal{T}_n^{\star\star}$ to $\mathcal{B}_n$.*

Proof. Considering Facts 1-5, the equality $\chi(T) = \chi(T')$ implies that T and T' belong to the same case (i), (ii), (iii), (iv) or (v). The injectivity is obtained by a simple induction. Now, it suffices to check that $\mathcal{B}_n$ and $\mathcal{T}_n^{\star\star}$ have the same cardinality. Let $E(x)$ be the generating function for $\mathcal{B}_n$, $n \geq 2$, and $E_1(x)$ the generating function for the set $\mathcal{B}^1 = \bigcup_{n\geq 2} \mathcal{B}_n^1$ of trees in $\mathcal{B}$ having a branch as right part. We set $\mathcal{B}^2 = \mathcal{B}\backslash\mathcal{B}^1$ and $E_2(x) = E(x) - E_1(x)$. A tree T in $\mathcal{B}^1$ consists of a branch of length at least one for its right part, and a unary-binary tree connected to the root for its left subtree. So, we have $E_1(x) = \frac{x^2}{1-x}M(x)$, where $M(x)$ is the Motzkin generating function for the unary-binary ordered trees. A tree T in $\mathcal{B}^2$ consists of a right part which is a branch of length at least one connected to a tree in $\mathcal{B}$, and a unary-binary tree connected to the root as left subtree. So, we deduce $E_2(x) = \frac{x^2}{1-x}E(x)M(x)$. A simple calculation of $E(x)$ proves that $\mathcal{B}_n$ and $\mathcal{T}_n^{\star\star}$ have the same cardinality for $n \geq 2$, which completes the proof. □

Remark 4.6. The set $\mathcal{B}$ is clearly in bijection with the square of the set of unary-binary trees (*i.e.*, ordered trees where any node, including the root, has at most two children). Then the generating function for $\mathcal{B}$ is given by $x^2M(x)^2$, where $M(x)$ is the generating function for the Motzkin numbers. So, the bijection $\psi \circ \chi$ establishes a bijective proof of the (apparently new) equality

$$M(x)^2 = \frac{1}{1-2x} \cdot M\left(\frac{x^2}{1-2x}\right).$$

Corollary 4.7. *The restriction of χ to $\mathcal{T}_n^\star$ establishes a one-to-one correspondence with ordered trees in $\mathcal{B}_n$ such that any left node has zero or two children, and any left leaf has a right sibling having at most one child.*

Proof. The restriction of χ to $\mathcal{T}_n^\star$ is defined using cases (i), (ii), (iii) and (v). Let $T \in \mathcal{T}_n^\star$ and $T' = \chi(T)$. By a simple induction, we observe that any left node of T' has zero or two children, and any left leaf has a right sibling having at most one child. Now, it suffices to prove that these trees are counted by the sequence `A026418` in [20] (as for $\mathcal{T}_n^\star$). Let $E(x)$ be the generating function for these trees. Since such a tree is of one of the forms (e, e), (eS, e), (eS, eS') and (L, eR) whenever $S = (L, R)$ and S' are also such trees, then $E(x)$ satisfies the functional equation: $E(x) = x^2 + x^2E(x) + xE(x) + x^2E(x)^2$. A simple calculation provides $E(x) = x^2M(x^2/(1-x))/(1-x)$ which achieves the proof. □

Corollary 4.8. *The restriction of χ to the subset of $\mathcal{T}_{2n}^\star$ consisting of ordered trees having n branches establishes a one-to-one correspondence with binary ordered trees in $\mathcal{B}_{2n}$ such that any node has zero or two children, and any left leaf has a right leaf as sibling.*

Proof. The proof is obtained as for the previous corollary without considering the case (iii). □

5. Conclusion and Future Works

In Corollary 3.5, we obtain a constructive bijection between $\mathcal{D}_n^{r,\geq}$ and $\mathcal{D}_n^{h,\geq}$. Unfortunately, this bijection does not transport simply the two statistics r and h. Is there another more natural bijection that behaves well with these statistics? On the other hand, we have seen that Schröder paths in $\mathcal{S}_{2n}^{r,\geq}$ and $\mathcal{S}_{2n}^{h,\geq}$ are enumerated by the n-th Catalan numbers. Is it possible to obtain direct bijections between these sets and the set of Dyck paths of semilength n, by not passing by a set of ordered trees? More generally, can we obtain bijective correspondences between these sets and other combinatorial classes such as permutations?

It would be interesting to study several parameters or statistics on the sets $\mathcal{M}_n^{s,\geq}$ and $\mathcal{D}_n^{s,\geq}$ for $s \in \{r, h\}$. For instance, the distribution of a given pattern is a subject widely studied these last years (in particular the avoidance of a pattern). Also, we could enumerate the restricted sets of paths having a non-decreasing height sequence (see [1, 6, 7]).

Can one develop algorithm for uniform random generation of an element of $\mathcal{M}_n^{s,\geq}$? Can one develop an efficient algorithm for the exhaustive generation of these objects?

References

[1] E. Barcucci, A. Del Lungo, A. Fezzi, and R. Pinzani, Nondecreasing Dyck paths and q-Fibonacci numbers, *Discrete Math.*, **170** (1997), 211–217.

[2] E. Barcucci, R. Pinzani, and R. Sprugnoli, The Motzkin family, *Pure Math. Appl. Ser. A*, **2(3-4)** (1991), 249–279.

[3] J.-L. Baril, S. Kirgizov, and A. Petrossian, Dyck paths with a first return decomposition constrained by height, *Discrete Math.*, **341** (2018), 1620–1628.

[4] S. Benchekroun and P. Moszkowski, A new bijection between ordered trees and legal bracketings, *Europ. J. Combinatorics*, **17** (1996), 605–611.

[5] C. Brennan and S. Mavhungu, Peaks and valleys in Motzkin paths, *Quaestiones Mathematicae*, **33(2)** (2010), 171–188.

[6] E. Czabarka, R. Florez, and L. Junes, Some enumerations on non-decreasing Dyck paths, *Electron. J. Combin.*, **22** (2015), 1–22.

[7] E. Czabarka, R. Florez, L. Junes, and J.L. Ramirez, Enumerations of peaks and valleys on non-decreasing Dyck paths, *Discrete Math.*, **341(10)** (2018), 2789–2807.

[8] M.P. Delest and G. Viennot, Algebraic languages and polyominoes enumeration, *Theoretical Computer Sciences*, **34** (1984), 169–206.

[9] N. Dershowitz and S. Zacks, Enumerations of ordered trees, *Discrete Math.*, **31** (1980), 9–28.

[10] E. Deutsch, Dyck path enumeration, *Discrete Math.*, **204** (1999), 167–202.

[11] E. Deutsch, Ordered trees with prescribed root degrees, node degrees, and branch length, *Discrete Math.*, **282(1-3)** (2004), 89–94.

[12] E. Deutsch and L.M. Shapiro, A bijection between ordered trees and 2-Motzkin paths and its many consequences, *Discrete Math.*, **256(3)** (2002), 655–670.

[13] R. Donaghey, Alternating permutations and binary increasing trees, *Journal of Combinatorial Theory, Series A*, **18(2)** (1975), 141–148.

[14] R. Donaghey and L.W. Shapiro, Motzkin numbers, *J. Combin. Theory Ser. A*, **23** (1977), 291–301.

[15] D.E. Knuth, *The art of computer programming, vol. 1: Fundamental Algorithms*, Addison-Wesley, 1968, third edition, 1997, 1968.

[16] T. Mansour, Statistics on Dyck paths, *J. Integer Sequences*, **9** (2006), 06.1.5.

[17] D. Merlini, R. Sprugnoli, and M.C. Verri, Some statistics on Dyck paths, *J. Statis. Plann. Inference*, **101** (2002), 211–227.

[18] H. Prodinger, Some information about the binomial transform, *The Fibonacci Quarterly*, **32** (1994), 412–415.

[19] I. Sapounakis, I. Tasoulas, and P. Tsikouras, Counting strings in Dyck paths, *Discrete Math.*, **307(23)** (2007), 2909–2924.

[20] N.J.A. Sloane, The On-line Encyclopedia of Integer Sequences, *Available electronically at* `http://oeis.org`.

[21] R.P. Stanley, *Enumerative combinatorics*, Cambridge University Press, 1999.

[22] D. Zeilberger, A bijection from ordered trees to binary trees that sends the pruning order to the Strahler number, *Discrete Math.*, **82** (1990), 89–92.

Theoretical Computer Science 814 (2020) 249–258

Contents lists available at ScienceDirect

Theoretical Computer Science

www.elsevier.com/locate/tcs

Popularity of patterns over d-equivalence classes of words and permutations

Jean-Luc Baril, Vincent Vajnovszki *

LIB, Université de Bourgogne Franche-Comté, B.P. 47 870, 21078 Dijon Cedex, France

ARTICLE INFO

Article history:
Received 13 November 2019
Accepted 6 February 2020
Available online 17 February 2020
Communicated by D. Perrin

Keywords:
Patterns in words and permutations
Distribution
Popularity
Descent set equivalence

ABSTRACT

Two same length words are d-equivalent if they have same descent set and same underlying alphabet. In particular, two same length permutations are d-equivalent if they have same descent set. The popularity of a pattern in a set of words is the overall number of copies of the pattern within the words of the set. We show the far-from-trivial fact that two patterns are d-equivalent if and only if they are equipopular over any d-equivalence class, and this equipopularity does not follow obviously from a trivial equidistribution.

© 2020 Elsevier B.V. All rights reserved.

1. Introduction and notation

We consider words over the set of positive integers and permutations are particular words. For $q \geq 1$, $[q]$ denotes the alphabet $\{1, 2, \ldots, q\}$ and the *underlying alphabet* of a word is the set of symbols occurring in the word. For instance, the 4-ary words 4313 and 4212 have different underlying alphabet, namely $\{1, 3, 4\}$ and $\{1, 2, 4\}$. A *descent* in a word $w_1 w_2 \ldots w_n$ is a position i with $w_i > w_{i+1}$ and the *descent set* of w, denoted $\mathrm{Des}(w)$, is the set of all such i; *ascent* and *ascent set*, $\mathrm{Asc}(w)$, are defined similarly. Two same length words are *d-equivalent* if they have the same descent set and the same underlying alphabet. For instance the words 31443 and 21332 have the same descent set but not the same underlying alphabet (thus they are not d-equivalent), whereas 31443 and 41131 are d-equivalent (the common underlying alphabet is $\{1, 3, 4\}$). If v and w are d-equivalent and v is a permutation, it follows that so is w. A *d-equivalence class* is a maximal set of d-equivalent words. For a word $w_1 w_2 \ldots w_n$ the *reverse* $\mathrm{r}(w)$ of w is the word $w = w_n w_{n-1} \ldots w_1$; and the *complement* $\mathrm{c}(w)$ of w is the word $(q - w_1 + 1)(q - w_2 + 1) \ldots (q - w_n + 1)$ with q the maximal entry of w.

A *pattern* is a word with the property that if i occurs in it, then so does j, for any j with $1 \leq j \leq i$, and the *reduction* of w, $red(w)$, is the unique pattern order isomorphic with w. The pattern $\pi = \pi_1 \pi_2 \ldots \pi_k$ occurs in the word $w = w_1 w_2 \ldots w_n$, $k \leq n$, if w has a (possibly scattered) subword $w_{i_1} w_{i_2} \ldots w_{i_k}$ order isomorphic with π, see for instance Kitaev's seminal book [6] on this topic. The number of occurrences of the pattern π in the word w is denoted by $(\pi)w$. For a set of words S, $(\pi)w$ becomes an integer valued statistic on S and the overall number of occurrences of π within the words of S is called the *popularity* of π in S; more formally, the popularity of π in S is $\sum_{w \in S} (\pi)w$.

The equidistribution of two patterns implies their equipopularity and recently a growing interest is shown in patterns that have same popularity but not same distribution on particular classes of words or permutations, see for instance [1,2,5,8].

* Corresponding author.
E-mail addresses: barjl@u-bourgogne.fr (J.-L. Baril), vvajnov@u-bourgogne.fr (V. Vajnovszki).

https://doi.org/10.1016/j.tcs.2020.02.005
0304-3975/© 2020 Elsevier B.V. All rights reserved.

In this paper we show that two patterns are d-equivalent if and only if they have the same popularity on any d-equivalence class. Specializing to permutations, we obtain that two same length permutations have same descent set if and only if they have the same popularity on any descent-set equivalence class of permutations.

2. Preliminary notions and results

2.1. d-Equivalence vs. f-equivalence

Here we show that any two d-equivalent patterns can be obtained from each other by a sequence of f-transformations, which are 'small changes' preserving the d-equivalence class.

2.1.1. Lexicographically smallest pattern in a d-equivalence class

For our purposes we need the lexicographically minimal pattern d-equivalent with a given pattern π, which in turn requires two other particular patterns α and ω that we define below.

The *descent word* of a length n word w is the binary word $b = b_1b_2\dots b_n$ where $b_i = 1$ if and only if i is a descent in w (and so, b_n is a redundant 0). The minimal arity of a pattern having descent word b is one more than the maximal number of consecutive 1s in b, and we denote by $\alpha(b)$ the lexicographically smallest pattern having minimal arity and descent word b. It is easy to see that the pattern $\alpha = \alpha(b)$ is defined as: for each i, $1 \le i \le n$,

$$\alpha_i = \min\{j : j \ge i,\ b_j = 0\} - i + 1.$$

Example 1. With $n = 9$ and $\pi = \mathbf{43}2\mathbf{4}112\mathbf{3}1$, we have $b = 110100010$ and $\alpha = \mathbf{32}1\mathbf{2}111\mathbf{2}1$ (the descents are in bold).

The maximal arity of a pattern having descent word $b = b_1b_2\dots b_n$ is n, and we denote by $\omega(b)$ the lexicographically smallest n-ary pattern having descent word b, which is necessarily a length n permutation. We divide the descent word b into *runs*: the length maximal factors of the form $11\cdots10$ with at least one occurrence of 1 are *descent runs* and, for convenience, we call the remaining length maximal 0s factors (if any) *ascent runs*. We define an order relation on $\{1, 2, \dots, n\}$: for two integers i and j, $1 \le i, j \le n$, we say that i *precedes* j, with respect to the binary word b, if

- b_i and b_j are in two distinct runs in b, and $i < j$, or
- b_i and b_j are in the same ascent run in b, and $i < j$, or
- b_i and b_j are in the same descent run in b, and $i > j$.

The desired permutation $\omega = \omega(b)$ is precisely that induced by this order relation:

$$\omega_i = \text{the rank of } i \text{ in } \{1, 2, \dots, n\},\ \text{in the precedence order,}$$

and ω is at the same time the lexicographically minimal word of maximal arity (that is n) having descent word b and, as we will see below, defines an order in which we cover the entries of a pattern with descent word b.

Example 2. If b is as in the previous example, then $b = 110.10.00.10$ (runs are separated by dots) and $\omega = \mathbf{32}1\mathbf{5}467\mathbf{9}8$ (the descents are in bold).

Now for an arbitrary arity $q \le n$ (not necessarily its minimal, or its maximal value n), we construct the lexicographically minimal pattern $\beta = \beta(q, b)$ where each symbol in $[q]$ occurs at least once and i is a descent in β if and only if $b_i = 1$, and in this construction the above defined patterns α and ω are involved. Moreover, if q reaches its minimal value, then $\beta = \alpha$ and if $q = n$, then $\beta = \omega$. The pattern β is obtained by covering its entries in $\beta_{\omega_1}, \beta_{\omega_2}, \dots, \beta_{\omega_n}$ order and the first entries (in this order) are taken from α and the last ones are increasing integers to guarantee that all symbols in $[q]$ occur in β. Formally:

$$\beta_{\omega_i} = \begin{cases} \alpha_{\omega_i} & \text{if} \max\{\alpha_{\omega_1}, \alpha_{\omega_2}, \dots, \alpha_{\omega_i}\} \ge q - (n - i) \\ q - (n - i) & \text{elsewhere.} \end{cases}$$

With these notations, it follows that if $\beta \ne \alpha$ and $k = \min\{i : \beta_{\omega_i} \ne \alpha_{\omega_i}\}$ then

$$\beta_{\omega_i} = \begin{cases} \alpha_{\omega_i} & \text{if } i < k \\ q - (n - i) & \text{elsewhere,} \end{cases}$$

and the entries $\beta_{\omega_k}, \beta_{\omega_{k+1}}, \dots, \beta_{\omega_n}$ are consecutive integers in increasing order.

Example 3. Continuing the previous example with $b = 110100010$, if $q = 7$, then the above construction gives $\beta = \mathbf{3212}\mathbf{1}4576$; and if $q = 8$, then it gives $\beta = \mathbf{321}4\mathbf{1}5687$ (the entries taken from α are in bold).

For a pattern π with descent word b, by a slight abuse of notation we denote by $\alpha(\pi)$ the pattern $\alpha(b)$, by $\omega(\pi)$ the pattern (permutation) $\omega(b)$; and in addition if π has arity q, then we denote by $\beta(\pi)$ the pattern $\beta(q, b)$. Note that

- the pattern $\alpha(\pi)$ is lexicographically minimal in its d-equivalence class and so are $\beta(\pi)$ and $\omega(\pi)$,
- the four patterns π, $\alpha(\pi)$, $\beta(\pi)$ and $\omega(\pi)$ have the same descent set,
- π and $\beta(\pi)$ are d-equivalent, but π, $\alpha(\pi)$ and $\omega(\pi)$ are not necessarily d-equivalent since they can have different underlying alphabet (or equivalently in this case, different arity).

2.1.2. f-Equivalent patterns

For later use we need the following rather technical notion: for two d-equivalent patterns π and σ we say that σ is an *f-transformation* of π if σ can be obtained from π by either

- increasing or decreasing by 1 an entry in π, or
- interchanging in π two entries with consecutive values.

Actually, the f-transformation is a symmetric binary relation on a set of d-equivalent patterns and two patterns are said *f-equivalent* if they belong to the same equivalence class with respect to the transitive closure of f-transformation. Below we prove that the notions of d-equivalence and f-equivalence coincide, which is stated in Corollary 1 of Theorem 1.

The order induced by $\omega(\pi)$ is related to the descent word of π, however we have the following.

Proposition 1. *Let π be a pattern. If $\omega = \omega(\pi)$ and i, k are such that $\pi_i > \pi_k$, then $\omega_i < \omega_k$ implies $i < k$.*

Proof. If $\pi_i > \pi_k$ and $\omega_i < \omega_k$, then ω_i and ω_k are not in the same descent run of π, so $i < k$. □

The next proposition says that, under certain conditions, decreasing an entry in the pattern π produces a d-equivalent pattern lexicographically smaller than π.

Proposition 2. *Let π be a pattern. If $\omega = \omega(\pi)$, $\beta = \beta(\pi)$ and*

- *there is an i such that $\pi_{\omega_j} = \beta_{\omega_j}$ for any j, $1 \le j < i$,*
- *$\pi_{\omega_i} > \beta_{\omega_i}$,*
- *the entry π_{ω_i} occurs at least twice in π,*

then the word σ with $\sigma_{\omega_j} = \pi_{\omega_j}$ for any j except $\sigma_{\omega_i} = \pi_{\omega_i} - 1$ is a pattern d-equivalent with π, lexicographically smaller than π.

Proposition 3. *Let π be a length n pattern. If $\omega = \omega(\pi)$, $\beta = \beta(\pi)$ and*

- *there is an i such that $\pi_{\omega_j} = \beta_{\omega_j}$ for any j, $1 \le j < i$,*
- *$\pi_{\omega_i} > \beta_{\omega_i}$,*
- *the entry π_{ω_i} occurs once in π,*
- *the entry $\pi_{\omega_i} - 1$ occurs at least once in the set $\{\pi_{\omega_{i+1}}, \pi_{\omega_{i+2}}, \dots, \pi_{\omega_n}\}$,*

then there is a k with $\omega_k \in \{\omega_{i+1}, \omega_{i+2}, \dots, \omega_n\}$ and $\pi_{\omega_k} = \pi_{\omega_i} - 1$, and the word σ with $\sigma_{\omega_j} = \pi_{\omega_j}$ for any j, except $\sigma_{\omega_i} = \pi_{\omega_i} - 1$ and $\sigma_{\omega_k} = \pi_{\omega_k} + 1$ $(= \pi_{\omega_i})$ is a pattern d-equivalent with π, lexicographically smaller than π.

Proof. Let ω_a be the largest element of the set $\{\omega_{i+1}, \omega_{i+2}, \dots, \omega_n\}$ with $\pi_{\omega_a} = \pi_{\omega_i} - 1$. It is enough to choose $k = a$. □

Note that, in the two propositions above σ is obtained by an f-transformation of π.

Proposition 4. *Let π be a q-ary length n pattern. If $\omega = \omega(\pi)$, $\beta = \beta(\pi)$ and i is such that each of $\beta_{\omega_i}, \beta_{\omega_{i+1}}, \dots, \beta_{\omega_n}$ occurs once in β, then*

1. *$\beta_{\omega_i}, \beta_{\omega_{i+1}}, \dots, \beta_{\omega_n}$ is a sequence of consecutive integers ending by q.*

In addition, if $\pi_{\omega_j} = \beta_{\omega_j}$ for any j, $1 \le j < i$, then

2. *each of $\pi_{\omega_i}, \pi_{\omega_{i+1}}, \dots, \pi_{\omega_n}$ occurs once in π.*

Proof. If condition 1. is violated, then β is not the lexicographically smallest pattern in its d-equivalence class, which is a contradiction. If condition 2. is violated, then π cannot be a q-ary pattern, again a contradiction. □

Proposition 5. *Let π be a length n pattern. If $\omega = \omega(\pi)$, $\beta = \beta(\pi)$ and*

- *there is an i such that $\pi_{\omega_j} = \beta_{\omega_j}$ for any j, $1 \le j < i$,*
- $\pi_{\omega_i} > \beta_{\omega_i}$,
- *the entry π_{ω_i} occurs once in π,*
- *the entry $\pi_{\omega_i} - 1$ does not occur in the set $\{\pi_{\omega_{i+1}}, \pi_{\omega_{i+2}}, \dots, \pi_{\omega_n}\}$,*

then there is a pattern σ which is f-equivalent with π and lexicographically smaller than π.

Proof. First we prove that at least one of the entries $\pi_{\omega_{i+1}}, \pi_{\omega_{i+2}}, \dots, \pi_{\omega_n}$ occurs at least twice in π. Indeed, if these entries occur once in π so are the entries in the set $P = \{\pi_{\omega_i}, \pi_{\omega_{i+1}}, \pi_{\omega_{i+2}}, \dots, \pi_{\omega_n}\}$. But since π and β have the same arity and $\pi_{\omega_j} = \beta_{\omega_j}$ for any $j < i$ it follows that the two sets P and $\{\beta_{\omega_i}, \beta_{\omega_{i+1}}, \dots, \beta_{\omega_n}\}$ are equal, and by the point 1. of Proposition 4, they are formed by consecutive integers. This is a contradiction since π_{ω_i} is not the minimal element of P (otherwise $\pi_{\omega_i} = \beta_{\omega_i}$) and $\pi_{\omega_i} - 1$ does not occur in P.

Now we prove the statement according to the following two (non exclusive) cases: (i) there is an integer larger than π_{ω_i} in the set $\{\pi_{\omega_{i+1}}, \pi_{\omega_{i+2}}, \dots, \pi_{\omega_n}\}$ that occurs at least twice in π, or (ii) there is at least one integer smaller than π_{ω_i} in the set $\{\pi_{\omega_{i+1}}, \pi_{\omega_{i+2}}, \dots, \pi_{\omega_n}\}$.

Case (i). If there is an integer larger than π_{ω_i} in the set $\{\pi_{\omega_{i+1}}, \pi_{\omega_{i+2}}, \dots, \pi_{\omega_n}\}$ that occurs twice in π, let $v \ne \pi_{\omega_i}$ be the smallest of them and let $\{\omega_{k_1}, \omega_{k_2}, \dots\} \subset \{\omega_1, \omega_2, \dots, \omega_n\}$ with $k_1 < k_2 \cdots$ be the set of occurrences of v in π. The entry $v - 1$ occurs once in π and let define a as: if $\pi_{\omega_{k_1}}$ is not in the same descent run as $v - 1$, then $a = \omega_{k_1}$, and $a = \omega_{k_2}$ otherwise. It follows that the pattern σ with $\sigma_{\omega_j} = \pi_{\omega_j}$ for all j, except $\sigma_{\omega_a} = \pi_{\omega_a} - 1$, is lexicographically smaller than π and is obtained from π by an f-transformation.

Case (ii). If there is an integer smaller than π_{ω_i} in the set $\{\pi_{\omega_{i+1}}, \pi_{\omega_{i+2}}, \dots, \pi_{\omega_n}\}$, then let $v \ne \pi_{\omega_i}$ be the largest of them and let ω_a be the largest element of $\{\omega_{i+1}, \omega_{i+2}, \dots, \omega_n\}$ with $\pi_{\omega_a} = v$. Necessarily v occurs at least twice in π; otherwise, since $v + 1$ does not occur in $\{\pi_{\omega_{i+1}}, \pi_{\omega_{i+2}}, \dots, \pi_{\omega_n}\}$, the pattern σ with $\sigma_{\omega_j} = \pi_{\omega_j}$ for any j, except $\sigma_{\omega_a} = v + 1$ and $\sigma_{\omega_k} = v$ for an appropriate $k < i$ with $\pi_{\omega_k} = v + 1$, is d-equivalent with π and lexicographically smaller than π, which is in contradiction with $\pi_{\omega_j} = \beta_{\omega_j}$ for any $j < i$.

Since v occurs at least twice in π it follows that the pattern τ with $\tau_{\omega_j} = \pi_{\omega_j}$ for any j, except $\tau_{\omega_a} = \pi_{\omega_a} + 1$, is d-equivalent with π (and lexicographically larger than π) and the entry τ_{ω_a} occurs at least twice in τ. Now two subcases can occur: $\tau_{\omega_a} = \pi_{\omega_i} - 1$ $(= \pi_{\omega_a} + 1)$ or $\tau_{\omega_a} < \pi_{\omega_i} - 1$.

When $\tau_{\omega_a} = \pi_{\omega_i} - 1$, since $\tau_{\omega_i} = \pi_{\omega_i}$ it follows that τ is a pattern satisfying Proposition 3, and the pattern σ with $\sigma_{\omega_j} = \tau_{\omega_j}$ for any j, except $\sigma_{\omega_i} = \tau_{\omega_i} - 1$ $(= \pi_{\omega_i} - 1)$ and $\sigma_{\omega_a} = \tau_{\omega_a} + 1$ $(= \pi_{\omega_i})$, is d-equivalent with τ (and thus with π) and is lexicographically smaller than π. The patterns π and σ are f-equivalent, and the statement holds.

When $\tau_{\omega_a} < \pi_{\omega_i} - 1$, since the entry τ_{ω_a} occurs twice in τ we can find as previously a pattern $\tau^{(2)}$ d-equivalent with τ (and so with π) where the entry $\tau_{\omega_a} + 1$ occurs twice in $\tau^{(2)}$. Iterating this procedure, we obtain a sequence of d-equivalent patterns $\tau = \tau^{(1)}, \tau^{(2)}, \dots, \tau^{(k)}$ with $\tau^{(k)}_{\omega_j} = \pi_{\omega_j}$ for all $j \le i$ and $\tau^{(k)}_{\omega_b} = \pi_{\omega_i} - 1$ for some $b > i$. As above, the pattern σ with $\sigma_{\omega_j} = \tau^{(k)}_{\omega_j}$ for any j, except $\sigma_{\omega_i} = \tau^{(k)}_{\omega_i} - 1$ $(= \pi_{\omega_i} - 1)$ and $\sigma_{\omega_b} = \tau^{(k)}_{\omega_b} + 1$ $(= \pi_{\omega_i})$ is d-equivalent with $\tau^{(k)}$ (and thus with π) and lexicographically smaller than π. Moreover, each $\tau^{(p)}$ is obtained from $\tau^{(p-1)}$ by an f-transformation and the statement holds. □

By Propositions 2, 3 and 5 we have the following theorem.

Theorem 1. *Any pattern π is f-equivalent with $\beta(\pi)$.*

Proof. Let π be a pattern with $\pi \ne \beta(\pi)$. Then π is in one of the cases stated in Propositions 2, 3 or 5, and according to these propositions there exists a pattern f-equivalent (and thus d-equivalent) with π and lexicographically smaller than π, and eventually π is f-equivalent with the lexicographically smallest pattern in its d-equivalence class, that is with $\beta(\pi)$. □

Corollary 1. *Two patterns are d-equivalent if and only if they are f-equivalent.*

Proof. By definition, f-equivalence implies d-equivalence. Conversely, if two patterns π and σ are d-equivalent, then $\beta(\pi) = \beta(\sigma)$, and by the previous theorem π is f-equivalent with $\beta(\pi)$ and σ is f-equivalent with $\beta(\sigma)$. Finally π and σ are f-equivalent. □

2.2. Bijection ψ

In the following we need a bijection on $[q]^n$ onto itself, that we denote by ψ, and satisfying:

(a) ψ preserves the underlying alphabet,

(b) the number of occurrences of the largest entry is the same in w and in $\psi(w)$, and the same holds for the smallest entry in w and in $\psi(w)$,
(c) ψ transforms descent set into ascent set, that is, for any word w $\operatorname{Des} w = \operatorname{Asc} \psi(w)$.

In particular when w is a permutation, the complement transformation c satisfies the three properties above, which in general is not longer true for arbitrary words, and we propose a bijection ψ which satisfies these properties for any words, not necessarily permutations. Its construction is based on the bijection ϕ on words defined in [4], which in turn is built on Foata and Schützenberger [3] bijection j on permutations. The bijection $\phi : [q]^n \to [q]^n$ in [4] satisfies for any word w:

(i) $\phi(w)$ is a rearrangement of the symbols of w,
(ii) $\operatorname{Des} w = \{n - i : i \in \operatorname{Des} \phi(w)\}$, and
(iii) $\operatorname{Ides} w = \operatorname{Ides} \phi(w)$.

See [7] for the definition of the set valued statistic Ides that we will not use here and for a weaker version of ϕ satisfying only (ii) and (iii) above. Note that from (i) it follows that ϕ preserves the underlying alphabet.

Based on the properties (i) and (ii) of ϕ it is easy to check that $\psi : [q]^n \to [q]^n$ defined as

$$\psi = \mathrm{r} \circ \phi \tag{1}$$

satisfies the above desiderata (a)–(c). Indeed, properties (a) and (b) follow from (i), and property (c) follows from (ii). Property (iii) is a deep and remarkable feature of ϕ (that we will not make use of it) and in some sense our bijection ψ is over endowed. For instance, $\phi(1321) = 3211$, $\phi(1232) = 3122$ and $\phi(3321) = 3213$ (see [4]), and thus $\psi(1321) = 1123$, $\psi(1232) = 2213$ and $\psi(3321) = 3123$.

2.3. Pattern trace and word substitution

For a word $w = w_1 w_2 \dots w_n$ and a set $S = \{i_1, i_2, \dots, i_p\} \subseteq \{1, 2, \dots, n\}$ we denote by w_S the subword $w_{i_1} w_{i_2} \dots w_{i_p}$ of w.

Let $t = t_1 t_2 \dots t_k$ be a length k word over $[q] \cup \{\square\}$, $q \geq 1$, and $I(t)$ be the set $\{\ell : 1 \leq \ell \leq k,\ t_\ell \neq \square\}$. We say that t is *a trace* of the pattern $\pi = \pi_1 \pi_2 \dots \pi_k$ if t_i and t_j have the same relative order ($<$, $=$, or $>$) as π_i and π_j have whenever $i, j \in I(t)$. Equivalently, t is a trace of π if the words $t_{I(t)}$ and $\pi_{I(t)}$ are order-isomorphic. In particular, when t does not contain $\square$, then $red(t) = \pi$; and t formed only by $\square$'s is a trace of any pattern. It can happen that t is a trace of several patterns. For instance, for two same length patterns π and σ, if the trace t of π is such that $\pi_\ell \neq \sigma_\ell$ implies $t_\ell = \square$, then t is a trace of σ as well.

With t a trace of a pattern π and $I(t)$ as above, for a word $w = w_1 w_2 \dots w_n$ and a set $A \subset \{1, 2, \dots, n\}$ of positions in w we say that t is a *trace of π in w at A* if $t_{I(t)} = w_A$ (and so, $|I(t)| = |A|$), and a trace t of π in w at A can be seen as a partial occurrence of the pattern π in w with $\square$ playing the role of 'wild' symbol. It can happen that several occurrences of π in a word w have trace t at A, and we denote by $(t, A, \pi)w$ the number of these occurrences, and thus (t, A, π) becomes an integer valued statistic on words.

Example 4. If $\pi = 1332$ and $\sigma = 2331$ are two patterns, then $t = \square 44 \square$ and $t' = \square 55 \square$ are traces of both π and σ. Furthermore, if $w = 154543$, then

- 1443 is an occurrence in w of π with trace $t = \square 44 \square$ at $A = \{3, 5\}$,
- 1554 and 1553 are occurrences in w of π with trace $t' = \square 55 \square$ at $A = \{2, 4\}$, and $(t', A, \pi)w = 2$.

See Table 1 in Appendix for other examples.

For a word $w = w_1 w_2 \dots w_n$ and two pairs of integer $a < b$ and $c < d$ we denote by $w \mid ([a, b], [c, d])$ the length-maximal subword $w_{i_1} w_{i_2} \dots w_{i_k}$ of w with $\{i_1, i_2, \dots, i_k\} \subseteq [a, b]$ and $\{w_{i_1}, w_{i_2}, \dots, w_{i_k}\} \subseteq [c, d]$. Alternatively, $w \mid ([a, b], [c, d])$ is the length-maximal subword of $w_{[a,b]}$ with entries in $[c, d]$.

If the word $w_{i_1} w_{i_2} \dots w_{i_k} = w \mid ([a, b], [c, d])$ has m different symbols and $u = u_1 u_2 \dots u_k$ is a word with the underlying alphabet $[m]$, then there is a unique word $v = v_1 v_2 \dots v_n$ with

- $v_\ell = w_\ell$ for any ℓ with $\ell \notin [a, b]$ or $w_\ell \notin [c, d]$, and
- $red(v \mid ([a, b], [c, d]) = u$, and $v \mid ([a, b], [c, d])$ and $w \mid ([a, b], [c, d])$ have the same underlying alphabet.

Indeed, v is obtained from w by replacing the subword $w_{i_1} w_{i_2} \dots w_{i_k}$ of w by an appropriate word order-isomorphic with u. With these notations we call v the *$([a, b], [c, d])$-substitution by u in w*. In particular, if $u = red(w \mid ([a, b], [c, d])$, then the $([a, b], [c, d])$-substitution by u in w is w itself, and we have the following easy to understand fact.

Fact 1. If $red(w \mid ([a, b], [c, d])$ and u are two d-equivalent words, then so are w and the $([a, b], [c, d])$-substitution by u in w.

See Example 5 where the $([3,7],[1,4])$-substitution by 3321 in $w = 21\mathit{1}43\mathit{6}15441$ is $v = 21\mathit{44}3\mathit{6}15441$ (the replaced elements are in italic and represented by $\times$ in the corresponding diagrams).

3. Proof of the main results

The main result of this article is Theorem 4. Prior to its proof, Lemmata 1 and 2 below establish some equidistribution results and the Corollary 2 of Theorems 2 and 3 says that if two patterns are an f-transformation of each other, then the patterns have the same popularity on any d-equivalence class.

Lemma 1. *Let $\pi = \pi_1\pi_2\dots\pi_k$ and $\sigma = \sigma_1\sigma_2\dots\sigma_k$ be two d-equivalent patterns with $\pi_\ell = \sigma_\ell$ for any ℓ, except $\sigma_i = \pi_i + 1$ for some i. Let also t be a trace of both π and σ with one $\square$ symbol and A be a $k-1$ element subset of $\{1,2,\dots,n\}$. Then on any d-equivalence class the statistics (t, A, π) and (t, A, σ) have the same distribution.*

Proof. For any d-equivalence class we give a bijection $w \mapsto v$ with $(t, A, \pi)w = (t, A, \sigma)v$.

Since π and σ differ in position i it follows that $t_i = \square$. In addition, since π and σ are patterns, the entry π_i occurs at least twice in π (otherwise $\sigma = \pi_1\dots(\pi_i+1)\dots\pi_k$ is not longer a pattern) and so does σ_i in σ. Let x be the symbol in t playing the role of π_i and y be that playing the role of σ_i, and we define the interval $[c,d] = [x,y]$ and the interval $[a,b]$ as follows. If $A = \{p_1, p_2, \dots, p_{k-1}\} \subset \{1,2,\dots,n\}$, then

- if $i = 1$, then $[a,b] = [1, p_1 - 1]$,
- if $i = k$, then $[a,b] = [p_{k-1}+1, n]$,
- elsewhere, $[a,b] = [p_{i-1}+1, p_i - 1]$.

Now let u be the word $red(w\,|\,([a,b],[c,d])$ and u' be the word $(\mathrm{c}\circ\psi)(u)$, with ψ defined in relation (1) and c the complement operation. The desired word v is the $([a,b],[c,d])$-substitution by u' in w. Indeed, u and u' have the same descent set and same underlying alphabet, and thus they are d-equivalent. By Fact 1 the transformation $w \mapsto v$ turns the word $w = w_1w_2\dots w_n$ into a d-equivalent word $v = v_1v_2\dots v_n$. In addition, by property (b) of ψ, it follows that the number of the largest entries in u is the same as that of the smallest entries in $u' = (\mathrm{c}\circ\psi)(u)$, and vice versa. Thus, for any j, $p_{i-1} < j < p_i$ (with the convention $p_0 = 0$ and $p_k = n+1$) $w \mapsto v$ transforms any occurrence

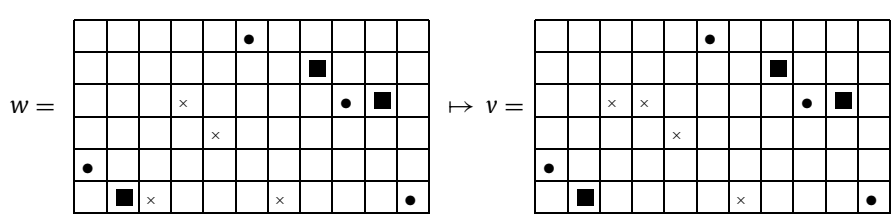
$$w_{p_1}\dots w_{p_{i-1}}w_jw_{p_i}\dots w_{p_{k-1}} = t_1\dots t_{i-1}xt_{i+1}\dots t_k$$

of π in w with trace t at A into an occurrence

$$v_{p_1}\dots v_{p_{i-1}}v_jv_{p_i}\dots v_{p_{k-1}} = t_1\dots t_{i-1}yt_{i+1}\dots t_k,$$

of σ in v with trace t at A. This transformation is reversible, indeed the $([a,b],[c,d])$-substitution by u in v gives the word w, and so it is a bijection. $\square$

Example 5. We represent words as diagrams identifying words $w = w_1w_2\dots w_n$ with the set of points $\{(i, w_i) : 1 \le i \le n\}$. Let w be the word 21143615441 (the left-hand side diagram in this representation), $\pi = 1132$ and $\sigma = 1232$ be two patterns as in Lemma 1, $t = 1\square 54$ be a common trace of π and σ, and A be the set $\{2,8,10\}$. In the diagram representation of w, the entries 1, 5 and 4 of t occurring in positions belonging to A are represented by ■ symbol. Following the notations in the proof of Lemma 1, the interval $[c,d]$ is $[1,4]$, $[a,b]$ is $[3,7]$, $w\,|\,([a,b],[c,d])$ is the subword 1431 (represented by $\times$ symbols in the left-hand side diagram), $u = red(w\,|\,([a,b],[c,d])$ is the word 1321, $u' = (\mathrm{c}\circ\psi)(u)$ is $\mathrm{c}(1123) = 3321$, see the examples at the end of the Section 2.2. Finally, $v = 21443615441$ in the right-hand side diagram is the image of w through the bijection in the proof of Lemma 1, and we have $(t, A, \pi)w = (t, A, \sigma)v$. Indeed, π occurs twice in w with trace t at A, namely in positions 2, 3, 8, 10 and in positions 2, 7, 8, 10; and so does σ in v with trace t at A, namely in positions 2, 3, 8, 10 and in positions 2, 4, 8, 10.



The next lemma is the counterpart of Lemma 1 where the patterns differ in two positions, with the additional requirement that the two different entries occur once in each pattern.

Lemma 2. *Let $\pi = \pi_1\pi_2\dots\pi_k$ and $\sigma = \sigma_1\sigma_2\dots\sigma_k$ be two d-equivalent patterns such that there are i and j, $i < j$, with*

- *$\pi_\ell = \sigma_\ell$ for any ℓ, except $\pi_i = \sigma_j$ and $\pi_j = \sigma_i$,*
- *$\pi_j = \pi_i + 1$,*
- *each of π_i and π_j occurs once in π (or, equivalently, σ_i and σ_j occur once in σ).*

Let also $t = t_1t_2\dots t_k$ be a trace of both π and σ with two $\square$ symbols and A be a subset of $\{1, 2, \dots, n\}$ of cardinality $k-2$. Then on any d-equivalence class the statistics (t, A, π) and (t, A, σ) have the same distribution.

Proof. To a certain extent the proof is similar to that of Lemma 1 by giving a bijection $w \mapsto v$ with $(t, A, \pi)w = (t, A, \sigma)v$ on any d-equivalence class.

Since π and σ differ in positions i and j and t contains two $\square$ symbols, it follows that $t_i = t_j = \square$, and since π and σ are d-equivalent i and j are not consecutive positions in t. We define three intervals $[a, b]$, $[a', b']$ and $[c, d]$. Let $A = \{p_1, p_2, \dots, p_{k-2}\}$ be the $k-2$ element subset.

- If π_i is the smallest entry in π then $c = 1$. Otherwise let π_u be the largest entry in π smaller than π_i, and x be the entry in t playing the role of π_u, and finally $c = x + 1$. Similarly, if π_j is the largest entry in π then $d = n$. Otherwise let π_u be the smaller entry in π larger than π_j, and x be the entry in t playing the role of π_u, and finally $d = x - 1$.
- If $i = 1$, then $a = 1$, otherwise $a = p_{i-1} + 1$; and $b = p_i - 1$.
- $a' = p_{j-2} + 1$; and if $j = k - 2$, then $b' = n$, otherwise $b' = p_{j-1} - 1$.

Now we define the announced bijection $w \mapsto v$, where v is obtained by constructing the words w', w'' and $w''' = v$ by applying the following steps.

1. Let u be the word $red(w \mid ([a, b], [c, d])$ and $u' = \psi(u)$, with ψ defined in relation (1), and w' be the $([a, b], [c, d])$-substitution by u' in w;
2. let u be the word $red(w' \mid ([a', b'], [c, d])$ and $u' = \psi(u)$, and w'' be the $([a', b'], [c, d])$-substitution by u' in w';
3. let u be the word $red(w'' \mid ([a, b'], [c, d])$ and $u' = \mathrm{c}(u)$, with c the complement operation, and w''' be the $([a, b'], [c, d])$-substitution by u' in w'';

and finally $v = w'''$. Note that the first two steps can be performed in arbitrary order since the substitution operations act on different entries of w (on the disjoint intervals $[a, b]$ and $[a', b']$). As in the proof of Lemma 1, taking in consideration the properties of ψ, $w \mapsto v$ transforms any occurrence of π in w with trace t at A into an occurrence of σ in v with trace t at A, and $w \mapsto v$ is reversible and so it is a bijection. $\square$

Note that in the previous proof, unlike in that of Lemma 2, the property (b) of the bijection ψ is not used.

See Table 1 in Appendix for an example of the equidistribution stated in Lemma 2.

Example 6. Let w be the word 217349648815371 (the left-hand side diagram below), $\pi = 125134$ and $\sigma = 135124$ be two patterns as in Lemma 2, $t = 1\square 81\square 7$ be a common trace of π and σ, and A be the set $\{2, 9, 11, 14\}$. In the diagram representation of w, the entries $1, 8, 1, 7$ of t occurring in positions belonging to A are represented by $\blacksquare$ symbols. Following the notations in the proof of Lemma 2, the interval $[a, b]$ is $[4, 8]$, $[a', b']$ is $[12, 13]$, $[c, d]$ is $[3, 6]$, $w \mid ([a, b], [c, d])$ is the subword 3464 (represented by $\times$ symbols in the left-hand side diagram), $w \mid ([a', b'], [c, d])$ is the subword 53 (represented by $+$ symbols), $red(w \mid ([a, b], [c, d]) = red(3464)$ is 1232 and $\psi(1232) = 2213$, $red(w \mid ([a', b'], [c, d]) = red(53)$ is 21 and $\psi(21) = 12$. Finally, $v = 217559638816471$ in the left-hand side diagram is the image of w through the bijection in the proof of Lemma 2, and we have $(t, A, \pi)w = (t, A, \sigma)v$. Indeed, π occurs three times in w with trace t at A, namely in positions $2, 4, 9, 11, 12, 14$, in positions $2, 5, 9, 11, 12, 14$, and in positions $2, 8, 9, 11, 12, 14$; and so does σ in v with trace t at A, namely in positions $2, 4, 9, 11, 13, 14$, in positions $2, 5, 9, 11, 13, 14$, and in positions $2, 7, 9, 11, 13, 14$.

$w =$

					•								
								■	•				
		•										■	
						×							
											+		
				×			×						
			×									+	
•													
	■									■			•

$\mapsto v =$

					•								
								■	•				
		•										■	
						×					+		
			×	×									
												+	
							×						
•													
	■									■			•

Theorem 2. *Let $\pi = \pi_1\pi_2\dots\pi_k$ and $\sigma = \sigma_1\sigma_2\dots\sigma_k$ be two d-equivalent patterns with $\pi_\ell = \sigma_\ell$ for any ℓ, except $\sigma_i = \pi_i + 1$ for some i. Then π and σ have the same popularity on any d-equivalence class.*

Proof. By Lemma 1, for any

- integer p,
- trace t with one $\square$ symbol in position i of both π and σ, and
- cardinality $k-1$ subset A of $\{1, 2, \ldots, n\}$,

on any d-equivalence class we have

$$|\{w : (t, A, \pi)w = p\}| = |\{w : (t, A, \sigma)w = p\}|.$$

For t and A fixed, summing over all w in a d-equivalence class we have

$$\sum_w (t, A, \pi)w = \sum_w (t, A, \sigma)w.$$

Further, for a fixed A, summing over all possible traces t at A of both π and σ we have

$$\sum_t \sum_w (t, A, \pi)w = \sum_t \sum_w (t, A, \sigma)w.$$

Note that in this equality there are no 'double counting' since different traces result in occurrences of π and σ with different values for the entries. Finally summing over all cardinality $k-1$ set A we have

$$\sum_A \sum_t \sum_w (t, A, \pi)w = \sum_A \sum_t \sum_w (t, A, \sigma)w.$$

Again, there are no 'double counting' since different sets A result in occurrences of π and σ in different positions. The two sides of the last equality give precisely the popularity of π and σ, respectively, on a d-equivalence class, and the statement follows. $\square$

Theorem 3. *Let $\pi = \pi_1\pi_2\ldots\pi_k$ and $\sigma = \sigma_1\sigma_2\ldots\sigma_k$ be two d-equivalent patterns with $\pi_\ell = \sigma_\ell$ for any ℓ, except $\pi_i = \sigma_j$ and $\pi_j = \sigma_i$ for some i and j, and $\pi_j = \pi_i + 1$. Then π and σ have the same popularity on any d-equivalence class.*

Proof. We distinguish two cases: at least one of the symbols π_i and π_j occurs twice in π, or each of these symbols occurs exactly once in π.

In the first case, suppose that π_i occurs twice in π and let $\tau = \tau_1\tau_2\ldots\tau_k$ be the pattern with $\tau_\ell = \tau_\ell$ for any ℓ, except $\tau_i = \pi_i + 1$ $(= \pi_j = \sigma_i)$. Since π and σ are d-equivalent so are π and τ (and thus τ and σ). By Theorem 2 it follows that τ has the same popularity as π. But $\tau_\ell = \sigma_\ell$ for any ℓ, except $\tau_j = \sigma_j + 1$ $(= \pi_j)$ and again by Theorem 2 it follows that τ has the same popularity as σ, and the statement follows.

In the second case (π_i and π_j occur once in π), applying Lemma 2 and reasoning as in the proof of Theorem 2, we have the desired equipopularity. $\square$

Recall that an f-transformation turns a pattern π into another d-equivalent one σ by making 'small changes' as in Theorems 2 and 3, and we have the next consequence of these theorems.

Corollary 2. *If the pattern σ is an f-transformation of the pattern π, then π and σ have the same popularity on any d-equivalence class.*

See Table 2 in Appendix for an example of the equipopularity stated in Corollary 2. Combining Corollaries 1 and 2 we obtain the next theorem.

Theorem 4. *Two patterns are d-equivalent if and only if they have the same popularity on any d-equivalence class.*

Proof. '$\Rightarrow$' If the patterns π and σ are d-equivalent, then they are f-equivalent, and thus there is a sequence of patterns $\pi = \tau^{(1)}, \tau^{(2)}, \ldots, \tau^{(k)} = \sigma$ such that $\tau^{(p+1)}$ is an f-transformation of $\tau^{(p)}$, $1 \le p \le k-1$. Thus $\tau^{(p+1)}$ and $\tau^{(p)}$ are equipopular on any d-equivalence class and so are π and σ.

'$\Leftarrow$' By contraposition: if the patterns π and σ are not d-equivalent, then within the words of the d-equivalence class containing (once) π the pattern σ does not occur, or vice versa. Indeed π and σ differ by their length and/or their arity, and/or their descent set; and so there is a d-equivalence class on which π and σ are not equipopular. $\square$

Two same length words are *descent-equivalent* if they have same descent set (and not necessarily same underlying alphabet), and so d-equivalence implies descent-equivalence. A q-ary descent-equivalence class is a maximal set of same length descent-equivalent q-ary words, for instance $\{121, 131, 132, 231, 232\}$ is a 3-ary descent-equivalence class. And we have the next easy to see corollary.

Corollary 3. *Two patterns are d-equivalent if and only if they have the same popularity on any q-ary descent-equivalence class.*

Finally, permutations are particular words (and particular patterns) for which the notions of d-equivalence and descent-equivalence coincide. Specializing the previous results to permutations we have the following straightforward result.

Corollary 4. *Two permutations are descent-equivalent if and only if they have the same popularity on any descent-equivalence class of permutations.*

See Table 2 in Appendix for an example of equipopularity of two descent-equivalent permutations.

Declaration of competing interest

There is no particular conflict of interest concerning the current paper.

Appendix A

Here we give two examples of equidistribution and equipopularity considered through this article.

Table 1
The equidistribution of the statistics (t, A, π) and (t, A, σ) over the set of length eight words with underlying alphabet $\{1, 2, \ldots, 5\}$ and descent set $\{2, 3, 5, 6, 7\}$, for: $\pi = 1332$, $\sigma = 2331$, $t = \square\mathbf{44}\square$ and $A = \{3, 6\}$. Only words w with $(t, A, \pi)w \neq 0$ or $(t, A, \sigma)w \neq 0$ are shown. The occurrences of the symbols **4** in positions belonging to A (and playing the role of 3 in the occurrences of π and of σ) are in bold. There are six words w for which $(t, A, \pi)w = 1$, as many as for which $(t, A, \sigma)w = 1$; and there are three words w for which $(t, A, \pi)w = 2$, as many as for which $(t, A, \sigma)w = 2$.

w	$(t, A, \pi)w$	$(t, A, \sigma)w$
15**4**1543**2**	2	0
15**4**2543**1**	1	0
15**4**2543**2**	2	0
15**4**3542**1**	1	0
15**4**3543**2**	2	0
25**4**1543**1**	1	1
25**4**1543**2**	1	0
25**4**2543**1**	1	1
25**4**3542**1**	0	1
25**4**3543**1**	1	1
35**4**1542**1**	0	2
35**4**1543**2**	0	1
35**4**2542**1**	0	2
35**4**2543**1**	0	1
35**4**3542**1**	0	2
…	0	0

Table 2
The equipopularity of the patterns 213 and 312 over the set of length five words with underlying alphabet $\{1, 2, \ldots, 5\}$ (that is, length five permutations) and descent set $\{1, 4\}$. Only words w with $(213)w \neq 0$ or $(312)w \neq 0$ are shown. The two patterns are not equidistributed over this set.

w	$(213)w$	$(312)w$
21354	3	0
21453	3	0
31254	4	1
31452	2	1
32451	2	0
41253	2	3
41352	2	2
42351	2	1
51243	0	5
51342	0	4
52341	0	3
…	0	0
popularity	20	20

References

[1] M. Albert, C. Homberger, J. Pantone, Equipopularity classes in the separable permutations, Electron. J. Comb. 22 (2) (2015) P2.2 (electronic).
[2] M. Bóna, Surprising symmetries in objects counted by Catalan numbers, Electron. J. Comb. 19 (1) (2012) P62 (electronic).
[3] D. Foata, M.P. Schützenberger, Major index and inversion number of permutations, Math. Nachr. 83 (1) (1978) 143–159.
[4] S. Fu, T. Hua, V. Vajnovszki, Mahonian STAT on rearrangement class of words, Discrete Appl. Math. 270 (2019) 134–141.
[5] C. Homberger, Expected patterns in permutation class, Electron. J. Comb. 19 (3) (2012) P43 (electronic).
[6] S. Kitaev, Patterns in Permutations and Words, Springer, 2011.
[7] S. Kitaev, V. Vajnovszki, Mahonian STAT on words, Inf. Process. Lett. 116 (2) (2016) 157–162.
[8] K. Rudolph, Pattern popularity in 132-avoiding permutations, Electron. J. Comb. 20 (1) (2013) P8 (electronic).

Bijections between directed animals, multisets and Grand-Dyck paths

Jean-Luc Baril*
LIB, Univ. Bourgogne Franche-Comté
France

David Bevan
University of Strathclyde
Glasgow, Scotland, United Kingdom

Sergey Kirgizov
LIB, Univ. Bourgogne Franche-Comté
France

Submitted: Jun 28, 2019; Accepted: Feb 29, 2020; Published: Apr 17, 2020
© The authors. Released under the CC BY-ND license (International 4.0).

Abstract

An n-multiset of $[k] = \{1, 2, \ldots, k\}$ consists of a set of n elements from $[k]$ where each element can be repeated. We present the bivariate generating function for n-multisets of $[k]$ with no consecutive elements. For $n = k$, these multisets have the same enumeration as directed animals in the square lattice. Then we give constructive bijections between directed animals, multisets with no consecutive elements and Grand-Dyck paths avoiding the pattern DUD, and we show how classical and novel statistics are transported by these bijections.

Mathematics Subject Classifications: 05A15,05A19

1 Introduction and motivation

For several decades, directed animals have been widely studied in the literature. They are special lattice point configurations, and their close links with certain problems of thermodynamics, in particular with the problem of directed percolation [9, 10], gives them an important place in the domains of theoretical physics and combinatorics. The problem of the enumeration of animals of a given area was stated for the first time by Harary in [16]. Then, Dhar, Phani and Barma [13] provided a first closed form for counting directed animals on the square and triangular lattices with respect to area. Later other proofs of this result were given using new combinatorial structures such as heaps of pieces [26] or gas models [1, 7, 12, 18]. In [2] Bacher provides a generating function for the total site

*Corresponding author: barjl@u-bourgogne.fr

perimeter on the square and triangular lattices, solving a conjecture proposed in 1996 by Conway [11]. But the problem of finding the generating function for the enumeration of directed animals according to the area and perimeter still remains open. According to [19], this function is not believed to be D-finite. Hoping to capture properties on the perimeter, other studies present one-to-one correspondences between directed animals and some restricted classes of already known combinatorial objects such as *guingois* trees [5], heaps of dimers [8], forests of 1-2 trees [4], permutations avoiding the patterns 321 and $4\bar{1}523$ [4], and lattice paths [6, 15].

The purpose of this paper is to present new combinatorial classes in bijection with directed animals on the square and triangular lattices. After preliminary discussions on multisets in Section 2, we exhibit in Section 3 a correspondence with Grand-Dyck paths avoiding the pattern DUD that is in turn in correspondence with multisets with no consecutive elements. Then, we show how these bijections transport classical and novel parameters on these classes, which opens a new way to explore statistics on directed animals in the well-known contexts of Dyck paths and multisets.

2 Preliminaries

An *n-multiset* on $[k] = \{1, 2, \ldots, k\}$ consists of a set of n elements from $[k]$ where we permit each element to be repeated [17, 23]. Throughout this paper, an n-multiset π will be represented by the unique sequence $\pi_1\pi_2 \ldots \pi_n$ of its elements ordered in non-decreasing order, *e.g.* the multiset $\{1, 1, 2, 2, 3, 3\}$ will be written 112233. For $n, k \geqslant 1$, let $\mathcal{M}_{n,k}$ be the set of n-multisets of $[k]$. We set $\mathcal{M}_n = \mathcal{M}_{n,n}$ and $\mathcal{M} = \bigcup_{n\geqslant 1} \mathcal{M}_n$. For instance, we have $\mathcal{M}_{3,2} = \{111, 112, 122, 222\}$ and $\mathcal{M}_3 = \mathcal{M}_{3,2} \cup \{113, 123, 133, 223, 233, 333\}$. The *graphical representation* of a multiset $\pi \in \mathcal{M}_{n,k}$ is the set of points in the plane at coordinates (i, π_i) for $i \in [n]$. Whenever none of the points (i, π_i) lie below the diagonal $y = x$, *i.e.*, $i \leqslant \pi_i$ for all $i \in [n]$, π will be called *superdiagonal*. Let $\mathcal{M}^s_{n,k}$ (resp. $\mathcal{M}^s_n$) be the set of superdiagonal n-multisets of $[k]$ (resp. of $[n]$) and $\mathcal{M}^s = \bigcup_{n\geqslant 1} \mathcal{M}^s_n$.

From a multiset $\pi \in \mathcal{M}_{n,k}$, we consider the path of length $n + k$ on its graphical representation with up and right moves along the edges of the squares that goes from the lower-left corner $(0, 0)$ to the upper-right corner (n, k) and leaving all the points (i, π_i), $i \in [n]$, to the right and remaining always as close to the line $x = n$ as possible (see the left part of Figure 1 for an example). Then, the number of n-multisets of $[k]$ is the number of possibilities to choose n right moves among $k + n - 1$ moves (the first is necessarily an up move), that is the binomial coefficient $\binom{k+n-1}{n}$ (see for instance [23]). Reading this path from left to right, we construct a lattice path of length $n + k$ from $(0, 0)$ to $(n + k, n - k)$ by replacing any up-move with up step $U = (1, 1)$ and any right-move with down step $D = (1, -1)$. Clearly, this path starts with U and consists of n up steps and k down steps. As a byproduct whenever $n = k$, this construction induces a bijection Φ between $\mathcal{M}_n$ and the set $\mathcal{GD}_n$ of Grand-Dyck paths of semilength n starting with an up-step, that is the set of paths from $(0, 0)$ to $(2n, 0)$ starting with U and consisting of U and D steps. Moreover, the image by Φ of $\mathcal{M}^s_n$ is the set $\mathcal{D}_n$ of Dyck paths of semilength n, *i.e.* the subset of paths in $\mathcal{GD}_n$ that do not cross the x-axis. See Figure 1 for two examples of this

construction.

Theorem 1. *The map Φ is a bijection from $\mathcal{M}_n$ to $\mathcal{GD}_n$, and the image of $\mathcal{M}_n^s$ is $\mathcal{D}_n$.*

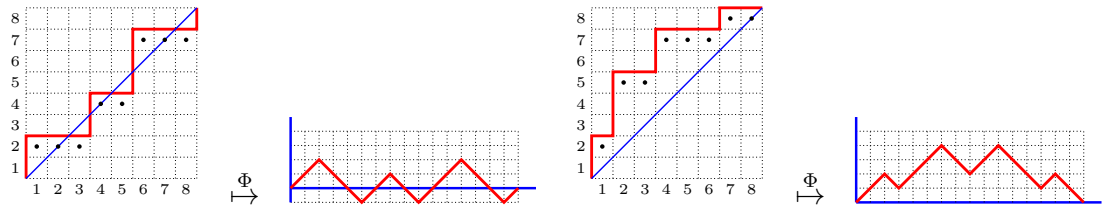


Figure 1: Illustration of the bijection Φ between multisets and lattice paths

Now, let us define the set $\mathcal{M}_{n,k}^\star$ of n-multisets of $[k]$ with no consecutive integers, *i.e.*, multisets π such that $\pi_{i+1} \neq \pi_i + 1$ for all $i \in [n-1]$. Then we set $\mathcal{M}_n^\star = \mathcal{M}_{n,n}^\star$, $\mathcal{M}_n^{s,\star} = \mathcal{M}_n^\star \cap \mathcal{M}_n^s$, $\mathcal{M}^\star = \bigcup_{n\geqslant 1} \mathcal{M}_n^\star$ and $\mathcal{M}^{s,\star} = \bigcup_{n\geqslant 1} \mathcal{M}_n^{s,\star}$. On the other hand, if $\mathcal{P}$ is a set of lattice paths consisting of U and D steps, then we denote by $\mathcal{P}^\star$ the subset of $\mathcal{P}$ consisting of paths that do not contain any occurrence of the pattern DUD.

Considering these notations, it is straightforward to obtain the following theorem.

Theorem 2. *The map Φ induces a bijection from $\mathcal{M}_n^\star$ to $\mathcal{GD}_n^\star$, and from $\mathcal{M}_n^{s,\star}$ to $\mathcal{D}_n^\star$.*

It is well known (see for instance [20, 21, 25]) that the cardinality of $\mathcal{D}_n^\star$ is given by the general term of Motzkin sequence A001006 in [22]. Then, using Theorem 2 the cardinality of the set $\mathcal{M}_n^{s,\star}$ is also counted by this sequence.

Now, using combinatorial arguments, we prove that the cardinality of $\mathcal{M}_n^\star$ is given by the general term of the sequence A005773 in [22] which also counts directed animals with a given area on the square lattice.

Theorem 3. *The ordinary generating function for n-multisets of $[n]$ with no consecutive integers is*

$$\frac{1 - 3\,z - \sqrt{1 - 2z - 3\,z^2}}{6\,z - 2}.$$

Proof. Let $f(z,u) = \sum_{n,k\geqslant 1} f_{n,k}\, z^n u^k$ be the bivariate generating function for the set $\mathcal{M}_{n,k}^\star$. That is, the coefficient $f_{n,k}$ is the number of n-multisets of $[k]$ with no consecutive integers. We build such a multiset by considering each integer from $\{1,\ldots,k\}$ in turn, and marking how many times it occurs in the multiset. Each of the k integers contributes a factor of u, and each occurrence in the multiset contributes a factor of z.

Using this approach, $f(z,u)$ can be written in the following form:

$$f(z,u) \;=\; \textsc{Seq}[u] \;\times\; u\;\textsc{Seq}^+[z] \;\times\; \textsc{Seq}\left[\,\textsc{Seq}^+[u]\,u\;\textsc{Seq}^+[z]\right] \;\times\; \textsc{Seq}[u], \qquad (1)$$

in which we make use of notation of Flajolet and Sedgewick [14]: $\textsc{Seq}[x] = 1/(1-x) = 1 + x + x^2 + \ldots$ represents the occurrence of zero or more items counted by x, and $\textsc{Seq}^+[x] = x/(1-x) = x + x^2 + \ldots$ represents the occurrence of one or more items.

$k\backslash n$	1	2	3	4	5	6	7	8	9
1	1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2	2
3	3	4	5	6	7	8	9	10	11
4	4	7	10	13	16	19	22	25	28
5	5	11	18	26	35	45	56	68	81
6	6	16	30	48	70	96	126	160	198

Table 1: The number of n-multisets of $[k]$ with no consecutive integers

The first term of (1), $\textsc{Seq}[u]$, represents the (possibly empty) initial sequence of integers not in the multiset. The second term, $u\,\textsc{Seq}^+[z]$, represents the first integer in the multiset, occurring one or more times.

Each subsequent integer, if any, that occurs one or more times in the multiset is preceded by at least one integer not in the multiset, since it does not contain consecutive integers. So each such additional integer in the multiset is represented by $\textsc{Seq}^+[u]u\,\textsc{Seq}^+[z]$. The third term of (1) thus represents all of the subsequent integers in the multiset. Finally, the fourth term, $\textsc{Seq}[u]$, represents the (possibly empty) final sequence of integers not in the multiset.

Expansion and simplification yields

$$f(z,u) \;=\; \frac{uz}{(1-u)\,(1-u-z+uz-u^2z)}.$$

Small values of $f_{n,k}$ are shown in Table 1.

As a consequence of the bijection Φ, the set of lattice paths of length $n+k$ starting at $(0,0)$, ending at $(n+k, n-k)$ consisting of n up steps and k down steps, starting with an up-step and avoiding the pattern DUD has a bivariate generating function given by $f(zu, z/u)$.

In order to obtain the generating function for the set $\mathcal{M}_n^\star = \mathcal{M}_{n,n}^\star$, we require the diagonal $\Delta(f)(z) = \sum_{n\geqslant 1} f_{n,n} z^n = [u^0]f(z/u,u)$, where $[u^0]g(u)$ is the constant coefficient of u in $g(u)$.

If $g(u) = g(u,z)$ is a formal Laurent series, then the constant term $[u^0]g(u)$ is given by the sum of the residues of $u^{-1}g(u)$ at those poles α of $g(u)$ for which $\lim_{z\to 0}\alpha(z) = 0$ (see [24, Section 6.3]).

In our case, $f(z/u,u)$ has a single pole $\alpha(z)$ for which $\alpha(0) = 0$, and the residue of $u^{-1}f(z/u,u)$ at $\alpha(z)$ is

$$\frac{1-3\,z-\sqrt{1-2z-3\,z^2}}{6\,z-2}$$

as required. □

As a consequence of the bijection Φ, we have the following.

Corollary 4. *The ordinary generating function for Grand-Dyck paths of semilength n starting with an up-step and avoiding the pattern DUD is counted by sequence A005773 in [22].*

Sequence A005773 in [22] counts diverse combinatorial objects, including various other types of lattice paths. For example, Banderier et al prove in [3] that Motzkin meanders (prefixes of Motzkin paths) are enumerated by this sequence. Sapanoukis et al. [21] state that Grand-Dyck paths of semilength n starting with an up-step and avoiding the pattern UDU are also counted by this sequence. We are unaware of a published proof of this, so present one very briefly here. We use the fact that the construction behind Φ induces a bijection between these Grand-Dyck paths avoiding UDU and n-multisets of $[n]$ in which no integer except n occurs exactly once.

Theorem 5. *The set of n-multisets of $[n]$ in which no integer except n occurs exactly once is counted by sequence A005773 in [22].*

Proof. Let $h(z,u)$ be the bivariate generating function for n-multisets of $[k]$ in which no integer except k occurs exactly once. We have

$$h(z,u) \;=\; \textsc{Seq}\left[u\left(\textsc{Seq}[z]-z\right)\right] \;\times\; u\;\textsc{Seq}[z] \;=\; \frac{u}{1-z-u\,(1-z+z^2)},$$

where the first term of the construction represents no occurrence or at least two occurrences of each integer from $\{1,\ldots,k-1\}$, and the second term represents zero or more occurrences of k. Extracting the diagonal then yields

$$[u^0]h(z/u,u) \;=\; \frac{1-3\,z-\sqrt{1-2z-3\,z^2}}{6\,z-2}$$

as required. □

Corollary 6. *The ordinary generating function for Grand-Dyck paths of semilength n starting with an up-step and avoiding the pattern UDU is counted by sequence A005773 in [22].*

3 From multisets to directed animals via Grand-Dyck paths

A *directed animal* A of area n (or equivalently with n nodes) in the triangular lattice is a subset of n points in the lattice containing $(0,0)$ and where any point in A can be reached from $(0,0)$ with up-moves $(0,1)$, right-moves $(1,0)$ and diagonal moves $(1,1)$ by staying always in A. Directed animals in the square lattice are those that do not use diagonal moves. See the left part of Figure 2 for an example of directed animal in the triangular lattice, and we refer to references in Introduction for several combinatorial studies on these objects. Let $\mathcal{Q}_n$ (resp. $\mathcal{T}_n$) be the set of directed animals with n nodes in the square (resp. triangular) lattice, then its cardinality is given by the nth term of the sequence A005773

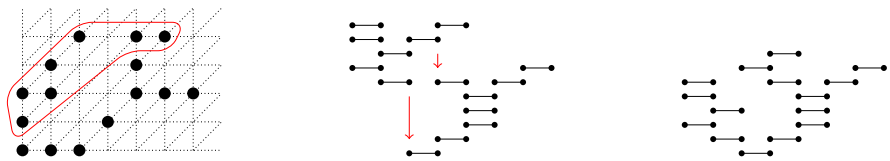
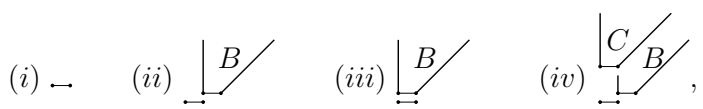


Figure 2: A directed animal and its associated heap.

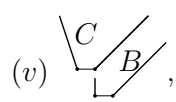
in [22] (resp. by the binomial coefficient $\binom{2n-1}{n}$). We set $\mathcal{Q} = \cup_{n\geqslant 1}\mathcal{Q}_n$, $\mathcal{T} = \cup_{n\geqslant 1}\mathcal{T}_n$ and obviously we have $\mathcal{Q} \subset \mathcal{T}$.

In the literature [8, 26, 27], directed animals are often viewed as heaps obtained by dropping vertically dimers such that each dimer (except the first) touches the one below by at least one of its extremities. Indeed, from $A \in \mathcal{T}$, we apply a counterclockwise rotation of 45 degree of its graphical representation and we replace each point of A with a dimer of width $\sqrt{2}/2$. See Figure 2 for an example of such a representation. Notice that directed animals in $\mathcal{Q}$ correspond to heaps of dimers where no dimer has another dimer directly above it (such a heap will be called *strict*). Let $\mathcal{T}^s$ (resp. $\mathcal{Q}^s$) be the set of all subdiagonal directed animals in $\mathcal{T}$ (resp. $\mathcal{Q}$), *i.e.*, directed animals where all its points (i, j) satisfy $j \leqslant i$.

Without losing accuracy, the sets $\mathcal{T}$ and $\mathcal{Q}$ will also be used to designate respectively the set of heaps of dimers and the set of strict heaps of dimers. Then, any heap $A \in \mathcal{T}^s$ has a unique factorization into one of the four following forms (see [8]):



where $B, C \in \mathcal{T}^s$. Moreover, any heap $A \in \mathcal{T}\backslash\mathcal{T}^s$ has a unique factorization:



where $B \in \mathcal{T}^s$ and $C \in \mathcal{T}$.

The factorization of $A \in \mathcal{Q}$ (resp. $A \in \mathcal{Q}^s$) is obtained after omitting the case (iii). Translating these factorizations using functional equations involving the generating functions $T(z)$ and $T^s(z)$ for $\mathcal{T}$ and $\mathcal{T}^s$ (resp. $Q(z)$ and $Q^s(z)$ for $\mathcal{Q}$ and $\mathcal{Q}^s$), we obtain

$$T^s(z) = \frac{1-2z-\sqrt{1-4z}}{2z}, \quad T(z) = \frac{1-4\,z-\sqrt{1-4z}}{8\,z-2},$$

$$Q^s(z) = \frac{1-z-\sqrt{1-2z-3\,z^2}}{2z}, \text{ and } Q(z) = \frac{1-3\,z-\sqrt{1-2z-3\,z^2}}{6\,z-2}.$$

The coefficients of z^n in the Taylor expansion of $Q^s(z)$ (resp. $Q(z)$, $T^s(z)$ and $T(z)$) generate a shift of the Motzkin sequence A001006 (resp. A005773, the Catalan sequence A000108 and A001700) in [22]).

Now, we construct a bijection from $\mathcal{M}_n$ to the set $\mathcal{T}_n$ of directed animals in the triangular lattice which transports $\mathcal{M}_n^\star$ into $\mathcal{Q}_n$. We proceed in two steps. Firstly, we define a bijection from $\mathcal{T}_n^s$ to $\mathcal{M}_n^s$ for $n \geqslant 1$, and secondly we extend it from $\mathcal{T}_n$ to $\mathcal{M}_n$. For the first step, and according to the above bijection Φ from $\mathcal{M}_n^s$ to $\mathcal{D}_n$, it suffices to define a one-to-one correspondence Ψ between $\mathcal{T}_n^s$ and $\mathcal{D}_n$. Letting A be a directed animal in $\mathcal{T}^s$, we define $\Psi(A)$ with respect to its four possible factorizations:

- if A satisfies (i) then $\Psi(A) = UD$,
- if A satisfies (ii) then $\Psi(A) = U\Psi(B)D$,
- if A satisfies (iii) then $\Psi(A) = \Psi(B)UD$,
- if A satisfies (iv) then $\Psi(A) = \Psi(C)U\Psi(B)D$.

Due to the recursive definition, the image by Ψ of a directed animal in $\mathcal{T}_n^s$ is a Dyck path of semilength n, and the image of an element of $\mathcal{Q}_n^s$ is a Dyck path with no pattern DUD, *i.e.* in $\mathcal{D}_n^\star$.

Theorem 7. *For $n \geqslant 1$, the map $\Phi^{-1} \cdot \Psi$ is a bijection from $\mathcal{T}_n^s$ to $\mathcal{M}_n^s$, and the image of $\mathcal{Q}_n^s$ is $\mathcal{M}_n^{s,\star}$.*

Proof. Since Φ^{-1} is a bijection from $\mathcal{D}_n$ to $\mathcal{M}_n^s$, it suffices to prove that Ψ is a bijection from $\mathcal{T}_n^s$ to $\mathcal{D}_n$. As these two last sets are both enumerated by the Catalan numbers, it suffices to prove the injectivity of Ψ. We proceed by induction on n. The case $n = 1$ holds trivially. We assume that Ψ is injective for $k \leqslant n$, and we prove the result for $n + 1$. By definition, the image by Ψ of animals satisfying (i) and (ii) are Dyck paths with only one return on the x-axis, *i.e.* with only one down step D that touches the x-axis. Animals satisfying (iii) are sent by Ψ to Dyck paths ending with DUD and with at least two return on the x-axis. Animals satisfying (iv) are sent to Dyck paths with at least two down steps at the end, and with at least two returns. Then, for $A, A' \in \mathcal{T}_{n+1}^s$, $\Psi(A) = \Psi(A')$ implies that A and A' belong to the same case (i), (ii), (iii) or (iv). The recurrence hypothesis induces $A = A'$ which completes the induction. Moreover, in the case where $A \in \mathcal{Q}_n^s$, it does not satisfy (iii) and this implies that $\Psi(A)$ is a Dyck path avoiding DUD. Finally, a cardinality argument proves that $\Psi(\mathcal{Q}_n^s) = \mathcal{M}_n^{s,\star}$. $\square$

Now we extend the map Ψ from $\mathcal{T}_n$ to $\mathcal{M}_n$ as follows. Let A be a directed animal in $\mathcal{T}_n \backslash \mathcal{T}_n^s$, then A can be factorized as (v) with $B \in \mathcal{T}^s$ and $C \in \mathcal{T}$. In the subcase where $C \in \mathcal{T} \backslash \mathcal{T}^s$, C satisfies the case (v), and let $D \in \mathcal{T}^s$, $E \in \mathcal{T}$ be the two parts of its factorization. According to these two cases, we set:

$$\Psi(A) = \begin{cases} \Psi(B)\Psi(C)^r & \text{if } C \in \mathcal{T}^s, \\ \Psi(B)\Psi(D)^r\Psi(E) & \text{otherwise,} \end{cases}$$

where P^r is obtained from P by reading the Dyck path P from right to left (for instance, if $P = UUDUUDDD$ then $P^r = DDDUUDUU$). Less formally, Ψ maps successive components from $\mathcal{T}^s$ to Dyck paths alternately above and below the x-axis. See Figure 3 for an illustration of the map Ψ.

Theorem 8. *For $n \geqslant 1$, the map $\Phi^{-1} \cdot \Psi$ is a bijection from $\mathcal{T}_n$ to $\mathcal{M}_n$, and the image of $\mathcal{Q}_n$ is $\mathcal{M}_n^\star$.*

Proof. Let us prove that Ψ is a bijection from $\mathcal{T}_n$ to $\mathcal{M}_n$. As these two sets have the same cardinality, it suffices to prove the injectivity of Ψ. Using Theorem 7, it remains to prove that directed animals $A \in \mathcal{T}_n \backslash \mathcal{T}_n^s$ are sent bijectively by Ψ to Grand-Dyck paths in $\mathcal{GD}_n \backslash \mathcal{D}_n$. Due to the definition of Ψ whenever $A \in \mathcal{T}_n \backslash \mathcal{T}_n^s$, we have either $\Psi(A) = \Psi(B)\Psi(C)^r$ or $\Psi(A) = \Psi(B)\Psi(D)^r\Psi(E)$ with $B, C, D \in \mathcal{T}^s$ and $E \in \mathcal{T}$. Then, the path $\Psi(A)$ starts with an up-step (the first step of the non-empty Dyck path $\Psi(B)$), and since the first step of $\Psi(C)^r$ (resp. $\Psi(D)^r$) is a down-step, $\Psi(A)$ crosses the x-axis which ensures that $\Psi(B)\Psi(C)^r$ (resp. $\Psi(B)\Psi(D)^r$) belongs to $\mathcal{GD}_n \backslash \mathcal{D}_n$. We complete the proof with a simple induction on n. Whenever $A \in \mathcal{Q}$, Theorem 7 ensures that $\Psi(B)$, $\Psi(C)$ and $\Psi(D)$ avoid the pattern DUD. By symmetry, the paths $\Psi(C)^r$ and $\Psi(D)$ avoid DUD which implies that the Grand-Dyck paths $\Psi(B)\Psi(C)^r$ and $\Psi(B)\Psi(D)^r$ do not contain DUD. By induction, $\Psi(A)$ belongs to $\mathcal{M}_n^\star$. $\square$

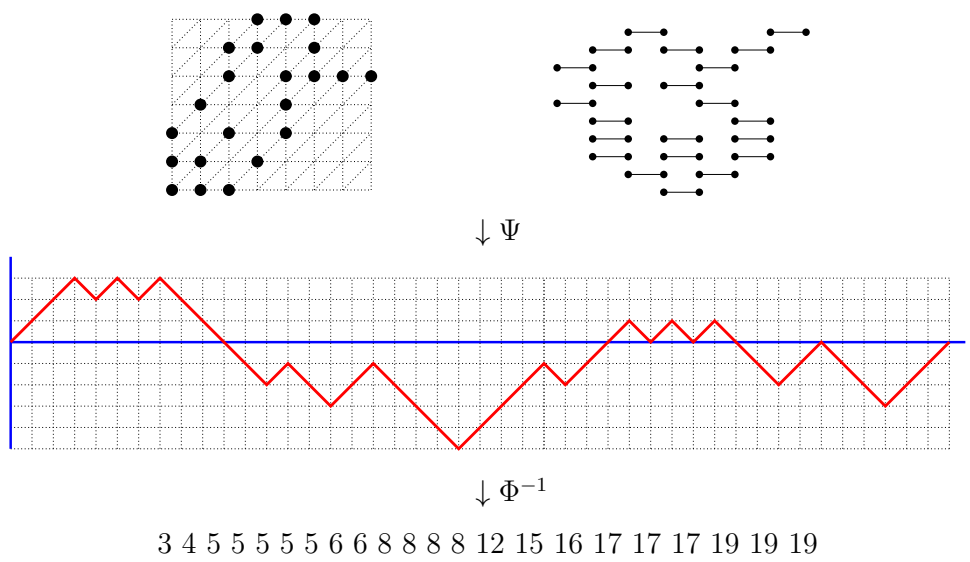


Figure 3: Bijection $\Phi^{-1}\Psi$ between directed animals and multisets via Grand-Dyck paths.

Now we define some statistics and parameters on $\mathcal{T}_n$, $\mathcal{M}_n$ and $\mathcal{GD}_n$, and we show how the bijections Φ, Ψ and $\Phi^{-1} \cdot \Psi$ establish correspondences between them. Table 2 summarizes these correspondences.

For a directed animal $A \in \mathcal{T}_n$, we set:

- **Area**(A) = number of points in A,
- **Lw**(A) = left width, *i.e.* $\max\{i \geqslant 0$ such that the line $y = x + i$ meets $A\}$,
- **Rw**(A) = right width, *i.e.* $\max\{i \geqslant 1$ such that the line $y = x - i + 1$ meets $A\}$,

- $\mathbf{Width}(A) = \mathbf{Lw}(A) + \mathbf{Rw}(A) =$ width,
- $\mathbf{Diag}(A) =$ number of in A, where $\times$ means a site without point in A,
- $\mathbf{Nbp}(A, i) =$ number of points of A on the line $y = x - i + 1$,

For a multiset $\pi \in \mathcal{M}_n$, we define $\delta(\pi_i) = 0$ if $\pi_i < i$ and 1 otherwise, and we set:

- $\mathbf{Length}(\pi) = n$,
- $\mathbf{Cross}(\pi) = \mathrm{card}\{i \in [n-1], \delta(\pi_i) \neq \delta(\pi_{i+1})\}$,
- $\mathbf{Adj}(\pi) =$ number of adjacencies, *i.e.*, $\mathrm{card}\{i \in [n-1], \text{such that } \pi_{i+1} = \pi_i + 1\}$,
- $\mathbf{Gap}(\pi, i) = |\pi_i - i| - c_i$ where $c_i = \mathrm{card}\{j \leqslant i-1, \delta(\pi_j) \neq \delta(\pi_{j+1})\}$,
- $\mathbf{Gap}(\pi) = \max_{i \in [n]} \mathbf{Gap}(\pi, i)$.

For a Grand-Dyck path $P \in \mathcal{GD}_n$, the height $\mathbf{h}(a, b)$ of a point $(a, b) \in P$ is the ordinate b, and $\mathbf{h}(P) = \max\{\mathbf{h}(a, b) : (a, b) \in P\}$. Here we consider a new height function defined by $\mathbf{Height}(a, b) = |b| - c_a$ where c_a is the number of the x-axis crossings before the line $x = a$, and we set:

- $\mathbf{Semilength}(P) =$ number of up-steps U,
- $\mathbf{Cross}(P) =$ number of crossings of the x-axis,
- $\mathbf{Height}(P) = \max_{(a,b) \in P} \mathbf{Height}(a, b)$,
- $\mathbf{Dud}(P) =$ number of pattern DUD,
- $\mathbf{Nbu}(P, i) =$ number of U having endpoint (a, b) satisfying $\mathbf{Height}(a, b) = i + 1$.

$A \in \mathcal{T}_n$	$P = \Psi(A) \in \mathcal{GD}_n$	$\pi = \Phi^{-1}(P) \in \mathcal{M}_n$
$\mathbf{Area}(A)$	$\mathbf{Semilength}(P)$	$\mathbf{Length}(\pi)$
$\mathbf{Lw}(A)$	$\mathbf{Cross}(P)$	$\mathbf{Cross}(\pi)$
$\mathbf{Rw}(A)$	$\mathbf{Height}(P)$	$\mathbf{Gap}(\pi)$
$\mathbf{Width}(A)$	$\mathbf{Cross}(P) + \mathbf{Height}(P)$	$\mathbf{Cross}(\pi) + \mathbf{Gap}(\pi)$
$\mathbf{Diag}(A)$	$\mathbf{Dud}(P)$	$\mathbf{Adj}(\pi)$
$\mathbf{Nbp}(A, i)$	$\mathbf{Nbu}(P, i)$	$\mathbf{Gap}(\pi, i)$

Table 2: Statistic correspondences by the bijections Ψ and Φ.

Theorem 9. *The bijections Φ and Ψ induce correspondences between statistics as summarized in Table 2.*

Proof. The statistic correspondences induced by Φ are easy to check. So, we only prove the correspondences generated by Ψ from directed animals to Grand-Dyck paths.

When $A \in \mathcal{T}^s$, we have $\mathbf{Lw}(A) = \mathbf{Cross}(\Psi(A)) = 0$. When $A \in \mathcal{T}\backslash\mathcal{T}^s$, A satisfies (v) with $B \in \mathcal{T}^s$ and $C \in \mathcal{T}$. Then $\mathbf{Lw}(A) = 1 + \mathbf{Lw}(C)$. We assume the recurrence hypothesis $\mathbf{Lw}(C) = \mathbf{Cross}(\Psi(C))$, which implies $\mathbf{Lw}(A) = 1 + \mathbf{Cross}(\Psi(C))$. Using the recursive definition of Ψ, we have $\mathbf{Cross}(\Psi(A)) = 1 + \mathbf{Cross}(\Psi(C))$ which gives by induction $\mathbf{Lw}(A) = \mathbf{Cross}(\Psi(A))$.

When $A \in \mathcal{T}^s$, it satisfies (i), (ii), (iii) or (iv), and the recursive definition of Ψ implies that $\mathbf{Rw}(A) = \mathbf{h}(\Psi(A)) = \mathbf{Height}(\Psi(A))$. Otherwise, if A is factorized as (v) with $B \in \mathcal{T}^s$ and $C \in \mathcal{T}^s$, then

$$\begin{aligned}\mathbf{Rw}(A) &= \max\{\mathbf{Rw}(B), \mathbf{Rw}(C) - 1\}\\ &= \max\{\mathbf{h}(\Psi(B)), \mathbf{h}(\Psi(C)) - 1\}\\ &= \max\{\mathbf{Height}(a,b), (a,b) \in \Psi(B)\Psi(C)^r\},\end{aligned}$$

which is equal to $\mathbf{Height}(\Psi(A))$. If A is factorized as (v) with $B \in \mathcal{T}^s$ and $C \in \mathcal{T}\backslash\mathcal{T}^s$, then C can be factorized as (v) with $D \in \mathcal{T}^s$ and $E \in \mathcal{T}$, and using an induction we have:

$$\begin{aligned}\mathbf{Rw}(A) &= \max\{\mathbf{h}(\Psi(B)), \mathbf{h}(\Psi(D)) - 1, \mathbf{Height}(\Psi(E)) - 2\}\\ &= \max\{\mathbf{Height}(a,b), (a,b) \in \Psi(B)\Psi(D)^r\Psi(E)\},\end{aligned}$$

which gives exactly $\mathbf{Height}(\Psi(A))$.

When $A \in \mathcal{T}^s$, it satisfies (i), (ii), (iii) or (iv), and the recursive definition of Ψ implies that $\mathbf{Nbp}(A, i) = \mathbf{Nbu}(\Psi(A), i)$. Otherwise, if A is factorized as (v) with $B \in \mathcal{T}^s$ and $C \in \mathcal{T}^s$, then $\mathbf{Nbp}(A, i) = \mathbf{Nbp}(B, i) + \mathbf{Nbp}(C, i+1)$, and using the recurrence hypothesis it is equal to

$$\mathbf{Nbu}(\Psi(B), i) + \mathbf{Nbu}(\Psi(C), i+1) = \mathbf{Nbu}(\Psi(B)\Psi(C)^r, i) = \mathbf{Nbu}(\Psi(A), i).$$

Whenever A is factorized as (v) with $D \in \mathcal{T}^s$ and $E \in \mathcal{T}$, a similar argument completes the proof. □

Due to the symmetry σ about the diagonal $y = x$, the two statistics $\mathbf{Lw}(\cdot) + 1$ and $\mathbf{Rw}(\cdot)$ have the same distribution on directed animals in $\mathcal{T}$ and $\mathcal{Q}$. Using Theorem 9 and Table 2, this induces that $\mathbf{Cross}(\cdot) + 1$ and $\mathbf{Height}(\cdot)$ also have the same distribution in $\mathcal{GD}$ and $\mathcal{GD}^\star$.

Acknowledgements

We would like to thank the anonymous referees for their helpful comments and suggestions.

References

[1] M. Albenque. A note on the enumeration of directed animals via gas considerations. *Ann. Appl. Probab.*, 19(5), 1860-1879, 2009.

[2] A. Bacher. Average site perimeter of directed animals on the two-dimensional lattices. *Discrete Math.*, 312(5), 1038–1058, 2012.

[3] C. Banderier, C. Krattenthaler, A. Krinik, D. Kruchinin, V. Kruchinin, D. Nguyen, and M. Wallner. Explicit Formulas for Enumeration of Lattice Paths: Basketball and the Kernel Method. In G.E. Andrews, C. Krattenthaler, and A. Krinik (eds), *Lattice Path Combinatorics and Applications*, 78–118, Springer, 2019.

[4] E. Barcucci, A. Del Lungo, E. Pergola, and R. Pinzani. Directed animals, forests and permutations. *Discrete Math.*, 204, 41–71, 1999.

[5] J. Betrema, and J.G. Penaud. Animaux et arbres guingois. *Theor. Comput. Sci.*, 117(1-2), 67–89, 1993.

[6] M. Bousquet-Mélou. Une bijection entre les polyominos convexes dirigés et les mots de Dyck bilatères. *RAIRO - Theoretical Informatics and Applications - Informatique Théorique et Applications*, 26(3), 205–219, 1992.

[7] M. Bousquet-Mélou. New enumerative results on two-dimensional directed animals. *Discrete Math.*, 180(1-3), 73–106, 1998.

[8] M. Bousquet-Mélou, and A. Rechnitzer. Lattice animals and heaps of dimers. *Discrete Math.*, 258, 235–274, 2002.

[9] S.R. Broadbent, and J.M. Hammersley. Percolation processes. I. Crystals and mazes. *Proc. Camb. Phil. Soc.*, 53, 629–641, 1957.

[10] A.R. Conway, and A.J. Guttmann. On two-dimensional percolation. *J. Phys. A: Math. Gen.*, 28, 891–904, 1995.

[11] A.R. Conway. Some exact results for moments of 2D-directed animals. *J. Phys. A: Math. Gen.*, 29(17), 5273–5283, 1996.

[12] D. Dhar. Equivalence of two-dimensional directed-site animal problem to Baxter's hard square lattice gas model. *Phys. Rev. Lett.*, 49, 959–962, 1982.

[13] D. Dhar, M.K. Phani, and M. Barma. Enumeration of directed site animals on two-dimensional lattices. *J. Phys. A: Math. Gen.*, 15, 279–284, 1982.

[14] P. Flajolet, and R. Sedgewick. Analytic combinatorics. Cambridge University Press, 2009.

[15] D. Gouyou-Beauchamps, and G. Viennot. Equivalence of two-dimensional directed animal problem to a one-dimensional path problem. *Adv. in Appl. Math.*, 9, 334–357, 1988.

[16] F. Harary. Graph Theory and Theoretical Physics. Ed. Harary, Academic Press, New York, 1967.

[17] D.E. Knuth. The art of computer programming. Vol. 4, Addison-Wesley, Reading MA, 1973, Third edition, 1997.

[18] Y. Le Borgne, and J.F. Marckert. Directed animals and gas models revisited. *Electron. J. Combin.*, 14, #R71, 2007.

[19] J.F. Marckert. Directed animals, quadratic systems and rewriting systems. *Electron. J. Combin.*, 19(3), #P45, 2012.

[20] D. Merlini, R. Sprugnoli, and M. Verri. Some statistics on Dyck paths. *J. Statist. Plann. and Infer.*, 101, 211–227, 2002.

[21] A. Sapanoukis, I. Tasoulas and P. Tsikouras. Counting strings in Dyck paths. *Discrete Math.*, 307(23), 2909–2924, 2007.

[22] N.J.A. Sloane. On-Line Encyclopedia of Integer Sequences. Published electronically at http://oeis.org/.

[23] R. Stanley. Enumerative Combinatorics, Vol. 1. Cambridge University Press, 1997.

[24] R. Stanley. Enumerative Combinatorics, Vol. 2. Cambridge University Press, 1999.

[25] Y. Sun. The statistic "number of udu's" in Dyck paths. *Discrete Math.*, 287, 177–186, 2004.

[26] G.X. Viennot. Heaps of pieces. *I. Basic definitions and combinatorial lemmas*, In Combinatoire énumérative (Montréal, Que., 1985), Vol. 1234 de Lecture notes in Math., 231–350, Springer, Berlin, 1986.

[27] G.X. Viennot. Multi-directed animals, connected heaps of dimers and Lorentzian triangulations. *J. Phys.: Conf. Series*, 42, 268–280, 2006.

Discrete Mathematics 343 (2020) 111995

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Pattern distributions in Dyck paths with a first return decomposition constrained by height

Jean-Luc Baril, Richard Genestier, Sergey Kirgizov *

LIB, Univ. Bourgogne Franche-Comté, France

ARTICLE INFO

Article history:
Received 8 November 2019
Received in revised form 14 May 2020
Accepted 15 May 2020
Available online 22 May 2020

Keywords:
Dyck and Motzkin paths
Pattern statistic
Distribution
Popularity

ABSTRACT

We provide generating functions for the popularity and the distribution of patterns of length at most three over the set of Dyck paths having a first return decomposition constrained by height.

© 2020 Elsevier B.V. All rights reserved.

1. Introduction and notation

Dyck paths with a constrained first return decomposition were introduced in [4] where the authors present both enumerative results using generating functions and a constructive bijection with the set of Motzkin paths. In [5], a similar study has been conducted for Motzkin, 2-colored Motzkin, Schröder and Riordan paths. In the literature, many papers deal with the enumeration of classical Dyck paths according to different parameters, e.g. length, number of peaks, valleys, double rises and other pattern occurrences [10,15,16,18–21,24]. Restricted classes of Dyck paths have also been considered, for instance Barcucci et al. [2] consider Dyck paths having a non-decreasing height sequence of valleys (see also [8,9]). Other papers deal with Motzkin paths using similar methods [3,6,11,12,17,22,25]. Motzkin and Catalan numbers appear alongside in many situations [11] and several one-to-one correspondences exist between restricted Dyck paths and Motzkin paths. For instance, Dyck paths avoiding a triple rise are enumerated by the Motzkin numbers [7].

In this paper, we focus on the distribution and the popularity of patterns of length at most three in constrained Dyck paths defined in [4]. Our method consists in showing how patterns are getting transferred from constrained Dyck paths to Motzkin paths, which settles us in a more suitable ground in order to provide generating functions for the distribution and the popularity.

A *Motzkin path* of length $n \geqslant 0$ is a lattice path consisting of flat steps $F = (1, 0)$, up steps $U = (1, 1)$ and down steps $D = (1, -1)$, starting at $(0, 0)$, ending at $(n, 0)$ and never going below the x-axis. For $n \geqslant 0$, we denote by $\mathcal{M}_n$ the set of all Motzkin paths of length n and we set $\mathcal{M} = \bigcup_{n\geqslant 0} \mathcal{M}_n$. A Motzkin path of length $2n$ with no flat steps is a *Dyck path* of semilength n. For $n \geqslant 0$, let $\mathcal{D}_n$ be the set of all Dyck paths of semilength n and $\mathcal{D} = \bigcup_{n\geqslant 0} \mathcal{D}_n$. The cardinality of $\mathcal{D}_n$ is given by the nth Catalan number $c_n = \frac{1}{n+1}\binom{2n}{n}$, which is the general term of the sequence A000108 in the On-line Encyclopedia of Integer Sequences of N.J.A. Sloane [23]. The cardinality of $\mathcal{M}_n$ is given by the nth Motzkin number $\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} c_k$ (see A000108 in [23]).

* Corresponding author.
E-mail addresses: barjl@u-bourgogne.fr (J.-L. Baril), Richard.Genestier@u-bourgogne.fr (R. Genestier), sergey.kirgizov@u-bourgogne.fr (S. Kirgizov).

https://doi.org/10.1016/j.disc.2020.111995
0012-365X/© 2020 Elsevier B.V. All rights reserved.

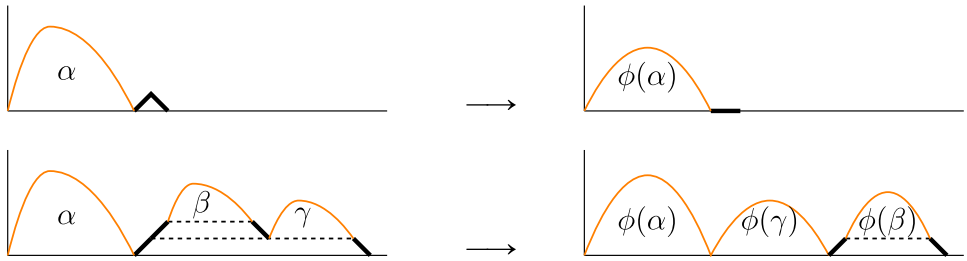


Fig. 1. Illustration of the bijection ϕ between $\mathcal{D}_n^{h,\geqslant}$ and $\mathcal{M}_n$.

Any non-empty Dyck path $P \in \mathcal{D}$ has a unique first return decomposition [10] of the form $P = U\alpha D\beta$ where α and β are two Dyck paths in $\mathcal{D}$. In [4], the authors introduced the set $\mathcal{D}^{h,\geqslant}$ constituted of the empty Dyck path and the Dyck paths in $\mathcal{D}$ having a first return decomposition satisfying

$$h(U\alpha D) \geqslant h(\beta)$$

where $\alpha, \beta \in \mathcal{D}^{h,\geqslant}$ and h returns the maximal height of a Dyck path. For $n \geqslant 0$, let $\mathcal{D}_n^{h,\geqslant}$ be the subset of Dyck paths of semilength n in $\mathcal{D}^{h,\geqslant}$. For instance, $\mathcal{D}_3^{h,\geqslant}$ consists of four Dyck paths *UDUDUD*, *UUDDUD*, *UUDUDD* and *UUUDDD*. In [4], the authors prove, using generating functions, that $\mathcal{D}_n^{h,\geqslant}$ and $\mathcal{M}_n$ have the same cardinality, and they present also the following bijection ϕ between these sets.

For $P \in \mathcal{D}^{h,\geqslant}$,

$$\phi(P) = \begin{cases} \epsilon & \text{if } P = \epsilon, \\ \phi(\alpha)F & \text{if } P = \alpha UD, \\ \phi(\alpha)\phi(\gamma)U\phi(\beta)D & \text{if } P = \alpha UU\beta D\gamma D. \end{cases}$$

For instance, the images by ϕ of *UDUDUD*, *UUDDUD*, *UUDUDD*, *UUUDDD*, *UUUUDDDDUUUDDUDD* are respectively *FFF*, *UDF*, *FUD*, *UFD* and *UUDDFUFD*. We refer to Fig. 1 for an illustration of this mapping.

A *statistic* on a set S of paths is an association of an integer in $\mathbb{Z}$ to each path in S. For instance the map that returns the number of steps is a statistic, and we denote by 1 (resp. 0) the constant statistic that sends any path to 1 (resp. 0). Let $\mathcal{S}$ be the set of all statistics on a set S. For $\mathrm{X}, \mathrm{Y} \in \mathcal{S}$, we define the statistic $\mathrm{X}+\mathrm{Y}$ so that $(\mathrm{X}+\mathrm{Y})(P) = \mathrm{X}(P) + \mathrm{Y}(P)$ for any $P \in S$, which endows $\mathcal{S}$ with a $\mathbb{Z}$-module structure. Let S and T be two sets of paths, and let $\mathcal{S}$ and $\mathcal{T}$ be the associated statistic sets. Two statistics $\mathrm{X} \in \mathcal{S}$ and $\mathrm{Y} \in \mathcal{T}$ have the *same distribution* if and only if there exists a bijection f from S to T such that for any $P \in S$ we have $\mathrm{X}(P) = \mathrm{Y}(f(P))$. In this case, we say that f transports the statistic X into Y, which can be shortly written with the statistic equation $f(\mathrm{X}) = \mathrm{Y}$ (or $\mathrm{X} = \mathrm{Y}$ whenever f is the identity).

A *pattern* X of length $k \geqslant 1$ occurs in a path P if and only if P contains X as a sequence of consecutive steps. Note that other variants of pattern definition exist in the literature (see for instance [1]). From a given pattern X and a set S of paths, we associate the *statistic* X from S to $\mathbb{N}$ such that $\mathrm{X}(P)$ is the number of occurrences of the pattern X in P. The *popularity* of a pattern X in S is the total number of occurrences of the pattern X in all paths $P \in S$, that is $\sum_{P\in S} \mathrm{X}(P)$.

For instance, if $P = UUUDDUDD$ then we have $\mathrm{UD}(P) = \mathrm{UU}(P) = 2$, $\mathrm{DDD}(P) = 0$. Moreover, if $S = \{UUDD, UDUD\}$ then the popularity of the pattern *UD* in S is 3. Also, for any subset S of $\mathcal{M}$, we have the statistic equation $\mathrm{U} = \mathrm{D}$ since for any path in S the number of up steps equals to the number of down steps. If we restrict S to $\mathcal{D}_n$ (resp. $\mathcal{M}_n$), then we have $\mathrm{U}+\mathrm{D} = 2\mathrm{n}$ (resp. $\mathrm{U}+\mathrm{F}+\mathrm{D} = \mathrm{n}$), where n is the constant statistic $P \mapsto n$.

Considering the above bijection ϕ from $\mathcal{D}_n^{h,\geqslant}$ to $\mathcal{M}_n$, the length of $\phi(P)$ is the semilength of P and we easily deduce the statistic equation:

$$\phi(\mathrm{U}) = \phi(\mathrm{D}) = \mathrm{U}+\mathrm{D}+\mathrm{F} = \mathrm{n}. \tag{1}$$

In this paper, we present statistic equations showing how $\phi\colon \mathcal{D}_n^{h,\geqslant} \to \mathcal{M}_n$ transports statistics associated to patterns of length at most three into linear combinations of other statistics. Then, this allows us to conduct our enumerative study in the more natural and simpler context of Motzkin paths, while a direct study on $\mathcal{D}_n^{h,\geqslant}$ is complicated due to the lack of an adequate recursive decomposition. So, we use statistic equations in order to derive bivariate generating functions for the distribution of patterns of length at most three, and we deduce the pattern popularity thereafter. Section 2 deals with patterns of length 2 while Section 3 deals with patterns of length 3. Many of the resulting sequences correspond to existing entries in the On-line Encyclopedia of Integer Sequences of N.J.A. Sloane [23] enumerating a quantity of already known combinatorial structures, and we also obtain new sequences not yet known in [23].

2. Patterns of length 2

In this part, we provide statistic equations showing how patterns of length two behave through $\phi : \mathcal{D}_n^{h,\geqslant} \to \mathcal{M}_n$, which allows us to deduce generating functions for the distribution and the popularity of such patterns. See Table 1 for an illustration of the distributions and Table 2 for the first terms of popularity sequences.

Table 1
Number of paths in $\mathcal{D}_n^{h,\geqslant}$ having k peaks UD, or equivalently $k-1$ valleys DU, or equivalently $n-k$ double rises UU.

k	n										
	1	2	3	4	5	6	7	8	9	10	11
1	1	1	1	1	1	1	1	1	1	1	1
2		1	2	4	6	9	12	16	20	25	30
3			1	3	9	19	38	66	110	170	255
4				1	4	16	44	111	240	485	900
5					1	5	25	85	260	676	1615
6						1	6	36	146	526	1602
7							1	7	49	231	959
8								1	8	64	344
9									1	...	...
Σ	1	2	4	9	21	51	127	323	835	2188	5798

Table 2
Popularity of length two patterns in $\mathcal{D}_n^{h,\geqslant}$ for $1 \leqslant n \leqslant 12$.

Pattern	Popularity sequence	OEIS
UD	1, 3, 8, 22, 61, 171, 483, 1373, 3923, 11257, 32418, 93644	A025566
DU	0, 1, 4, 13, 40, 120, 356, 1050, 3088, 9069, 26620, 78133	A025567
UU, DD	0, 1, 4, 14, 44, 135, 406, 1211, 3592, 10623, 31260, 92488	

Theorem 1. *For $n \geqslant 0$, the bijection ϕ from $\mathcal{D}_n^{h,\geqslant}$ to $\mathcal{M}_n$ transports statistics associated to patterns of length two as follows:*

$$\phi(\mathrm{UD}) = \mathrm{F} + \mathrm{UD}, \tag{2}$$

$$\phi(\mathrm{UU}) = \phi(\mathrm{DD}) = \mathrm{U} + \mathrm{UU} + \mathrm{UF}, \tag{3}$$

$$\phi(\mathrm{DU}) = \mathrm{FF} + \mathrm{FU} + \mathrm{DF} + \mathrm{DU}. \tag{4}$$

Proof. For Eq. (2), we refer to [4].

For Eq. (3), we observe that for any Dyck path of semilength n an up step U is always followed by U or D, which implies the equality $\mathrm{UU} + \mathrm{UD} = \mathrm{n}$ on $\mathcal{D}_n^{h,\geqslant}$. With a similar argument and $\mathrm{U} = \mathrm{D}$, we have $\mathrm{F} + \mathrm{U} + \mathrm{UD} + \mathrm{UU} + \mathrm{UF} = \mathrm{n}$ on $\mathcal{M}_n$. Using Eq. (2) we obtain $\phi(\mathrm{UU}) = \mathrm{n} - \phi(\mathrm{UD}) = \mathrm{n} - \big(\mathrm{F} + \mathrm{UD}\big) = \mathrm{U} + \mathrm{UU} + \mathrm{UF}$.

For Eq. (4), we observe that the statistic $\mathrm{UU} + \mathrm{UD} + \mathrm{DU} + \mathrm{DD}$ equals $2\mathrm{n} - 1$ on $\mathcal{D}_n^{h,\geqslant}$. Using the straightforward equality $\mathrm{UU} = \mathrm{DD}$ on $\mathcal{D}_n$, we obtain $\mathrm{DU} = 2\mathrm{n} - 1 - 2\mathrm{UU} - \mathrm{UD}$. Applying the bijection ϕ and using Eqs. (2) and (3) we obtain $\phi(\mathrm{DU}) = 2\mathrm{n} - 1 - 2(\mathrm{U} + \mathrm{UU} + \mathrm{UF}) - (\mathrm{F} + \mathrm{UD})$. On the other hand, on the set of Motzkin paths of length n we have the statistic equations $\mathrm{U} + \mathrm{F} + \mathrm{D} = \mathrm{n}$ and $\mathrm{FD} + \mathrm{FU} + \mathrm{FF} + \mathrm{DD} + \mathrm{DU} + \mathrm{DF} + \mathrm{UD} + \mathrm{UU} + \mathrm{UF} = \mathrm{n} - 1$, which induces $2\mathrm{n} - 1 = \mathrm{U} + \mathrm{F} + \mathrm{D} + \mathrm{FD} + \mathrm{FU} + \mathrm{FF} + \mathrm{DD} + \mathrm{DU} + \mathrm{DF} + \mathrm{UD} + \mathrm{UU} + \mathrm{UF}$. Also, for any Motzkin path we have $\mathrm{U} = \mathrm{D}$ which implies $\mathrm{UU} + \mathrm{UF} + \mathrm{UD} = \mathrm{UD} + \mathrm{FD} + \mathrm{DD}$, and thus $\mathrm{UU} + \mathrm{UF} = \mathrm{FD} + \mathrm{DD}$. So, combining all these equations, we obtain $\phi(\mathrm{DU}) = \mathrm{U}+\mathrm{F}+\mathrm{D}+\mathrm{FD}+\mathrm{FU}+\mathrm{FF}+\mathrm{DD}+\mathrm{DU}+\mathrm{DF}+\mathrm{UD}+\mathrm{UU}+\mathrm{UF}-\mathrm{U}-\mathrm{D}-\mathrm{UU}-\mathrm{UF}-\mathrm{FD}-\mathrm{DD}-\mathrm{F}-\mathrm{UD} = \mathrm{FF}+\mathrm{FU}+\mathrm{DF}+\mathrm{DU}$. □

Theorem 2. *The bivariate generating functions $F_p(x, y)$ where the coefficient of $x^n y^k$ is the number of Dyck paths in $\mathcal{D}_n^{h,\geqslant}$ containing exactly k occurrences of the pattern $p \in \{UD, UU, DD, DU\}$ are*

$$F_{UD}(x, y) = \frac{x^2 - x^2y - xy + 1 - \sqrt{-4x^2 + \left(x^2\,(y-1) + xy - 1\right)^2}}{2x^2},$$

$$F_{UU}(x, y) = F_{DD}(x, y) = \frac{x^2y^2 - x^2y - x + 1 - \sqrt{-4x^2y^2 + \left(x^2y\,(y-1) - x + 1\right)^2}}{2x^2y^2},$$

$$F_{DU}(x, y) = \frac{x^2y - x^2 - xy + 1 - \sqrt{-4x^2 + \left(x^2\,(y-1) + xy - 1\right)^2}}{2x^2y}.$$

Proof. For $p = UD$, Corollary 3 in [4] provides directly the bivariate generating function as a solution of a functional equation on Motzkin paths with respect to the number of occurrences of patterns F and UD (see Eq. (2)).

For $p = UU$, there are two ways to obtain the generating function. First, we know that $\mathrm{UU} + \mathrm{UD} = \mathrm{n}$ for Dyck paths of semilength n. So, we can obtain directly the generating function by calculating $F_{UD}(xy, \frac{1}{y})$. Moreover we know that $\phi(\mathrm{UU}) = \mathrm{U} + \mathrm{UU} + \mathrm{UF}$ (see Eq. (3)). So, we decompose the set $\mathcal{M}$ of all Motzkin paths in the following way:

$$\mathcal{M} = \epsilon + F\mathcal{M} + UD\mathcal{M} + U(\mathcal{M}\backslash\epsilon)D\mathcal{M}.$$

The generating function for $\epsilon + F\mathcal{M}$ is given by $1 + xM(x, y)$; the g.f. for $UD\mathcal{M}$ is $x^2yM(x, y)$ since UD contains one occurrence of U; the g.f. for $U(\mathcal{M}\backslash\epsilon)D\mathcal{M}$ is $x^2y^2(M(x, y) - 1)M(x, y)$ since a path in $U(\mathcal{M}\backslash\epsilon)D$ starts with U and its first two steps are either UU or UF. This induces the functional equation

$$M(x, y) = 1 + xM(x, y) + x^2yM(x, y) + x^2y^2(M(x, y) - 1)M(x, y),$$

which also gives the expected result.

For $p = DU$, the result is obtained by using the equality $\mathrm{DU} = \mathrm{UD} - 1$ and evaluating $F_{DU}(x, y) = 1 + \frac{F_{UD}(x,y)-F_{UD}(x,0)}{y}$. Note that we can obtain this result using Eq. (4). □

Table 1 gives the number of paths in $\mathcal{D}_n^{h,\geqslant}$ having k peaks UD for $1 \leqslant n \leqslant 11$ and $1 \leqslant k \leqslant 8$. Applying standard techniques from generating function theory, we verify that the second row corresponds to the sequence of quarter-squares numbers, $\lfloor(n^2/4)\rfloor$, which is the third row of Lozanić's triangle (see A034851 and [14]). The third row is a shift of the sequence A005994 corresponding to alkane numbers $l(7, n)$ from fifth row of Lozanić's triangle, which enumerates certain symmetries exhibited by chemical entities (alkane) consisting of hydrogen and carbon atoms arranged in a tree-like structure. Third diagonal corresponds to the squares, while fourth diagonal generates octahedral numbers, $n(2n^2 + 1)/3$ (see A005900).

Corollary 1. *For $n \geqslant 0$, the popularity of pattern $p \in \{UU, UD, DD, DU\}$ in $\mathcal{D}_n^{h,\geqslant}$ is given by the generating function $G_p(x)$:*

$$G_{UD}(x) = \frac{(x-1)\sqrt{-3x^2-2x+1} - 3x^2 - 2x + 1}{2x(3x-1)},$$

$$G_{UU}(x) = G_{DD}(x) = \frac{\sqrt{-3x^2-2x+1}\left(x^2+2x-2\right) + x^3 - 3x^2 - 4x + 2}{2x^2\sqrt{-3x^2-2x+1}},$$

$$G_{DU}(x) = \frac{\left(x^2-1\right)\sqrt{-3x^2-2x+1} - x^3 - 3x^2 - x + 1}{2x^2\sqrt{-3x^2-2x+1}}.$$

Proof. The generating function $G_p(x)$ of popularity is directly deduced from the bivariate generating function of pattern distribution

$$G_p(x) = \left.\frac{\partial F_p(x, y)}{\partial y}\right|_{y=1}. \quad \square$$

See Table 2 for the first terms of the popularity sequences. The popularity of the pattern UD generates a shift of the sequence A025566 in [23]. As suggested in [23], the same sequence enumerates the first differences of the directed animals sequence A005773, and also Motzkin paths of length $2n$ whose last weak valley occurs immediately after step n.

The popularity sequence for DU is the sequence A025567. As mentioned in [13] by Ferrari and Munarini, this sequence corresponds to the number of edges in Hasse diagram of Motzkin paths, where the partial order is defined by the coverings $FF \mapsto UD$, $FU \mapsto UF$, $DF \mapsto FD$, $DU \mapsto FF$. This is an immediate consequence of the fact that ϕ maps pattern DU from constrained Dyck paths to the patterns FF, FU, DF, DU in Motzkin paths.

The popularity sequence for UU (or DD) does not yet appear in [23].

3. Patterns of length 3

In this part we investigate how ϕ transports statistics associated to patterns of length three, and for each of them we provide generating functions for the distribution and popularity.

We need the following notations. For a step $X \in \{U, D, F\}$, an occurrence of the pattern X^+ inside a path P is an occurrence of X^k, where X^k consists of k consecutive repetitions of X, for some $k \geqslant 1$. The associated statistic of X^+ (denoted X^+) will be equal to $\sum_{k\geqslant 1} \mathrm{X}^k$ where X^k is the statistic giving the number of occurrences of X^k. More generally, for two possibly empty sequences of steps Y and Z we define the pattern YX^+Z as patterns of the form YX^kZ for $k \geqslant 1$, and its associated statistic as $\mathrm{YX^+Z} = \sum_{k\geqslant 1} \mathrm{YX^{k}Z}$. For instance, an occurrence of the pattern DU^+D can be an occurrence of DUD, $DUUD$, $DUUUD$, and so on... In the path $UUF\boldsymbol{DUUD}F\boldsymbol{DUDUUUD}DFFDD$ we count three occurrences of the pattern DU^+D. A path contains a dotted pattern $\bullet Y$ (resp. $Y\bullet$) if and only if it starts (resp. ends) with Y, and we use the notations $\bullet\mathrm{Y}$, $\mathrm{Y}\bullet$ for the associated statistics.

Any pattern UU in a Dyck path is immediately followed by an up step or a down step, implying the statistic equation $\mathrm{UU} = \mathrm{UUU} + \mathrm{UUD}$. Also, any pattern UU in a Dyck path P is either at the beginning of P or immediately preceded by an up or a down step, so $\mathrm{UU} = \bullet\mathrm{UU} + \mathrm{UUU} + \mathrm{DUU}$. Using similar arguments we obtain the two following systems of statistic equations.

$$(a)\begin{cases} \mathrm{UU} = \mathrm{UUU} + \mathrm{UUD} \\ \mathrm{UU} = \mathrm{UUU} + \mathrm{DUU} + \bullet\mathrm{UU} \\ \mathrm{UD} = \mathrm{UUD} + \mathrm{DUD} + \bullet\mathrm{UD} \\ \mathrm{DU} = \mathrm{DUU} + \mathrm{DUD} \end{cases}, \quad (b)\begin{cases} \mathrm{DD} = \mathrm{DDD} + \mathrm{UDD} \\ \mathrm{DD} = \mathrm{DDD} + \mathrm{DDU} + \mathrm{DD}\bullet \\ \mathrm{UD} = \mathrm{UDD} + \mathrm{UDU} + \mathrm{UD}\bullet \\ \mathrm{DU} = \mathrm{DDU} + \mathrm{UDU} \end{cases}.$$

Observe that for any $P \in \mathcal{D}_n^{h,\geqslant}$, $n \geqslant 1$, we have $\bullet\mathrm{UD}(P) = 1$ (resp. $\bullet\mathrm{UU}(P) = 0$) when $P = (UD)^n$, and $\bullet\mathrm{UD}(P) = 0$ (resp. $\bullet\mathrm{UU}(P) = 1$) otherwise. So, we have the statistic equations $\phi(\bullet\mathrm{UD}) = \delta_{F^n}$ and $\phi(\bullet\mathrm{UU}) = 1 - \phi(\bullet\mathrm{UD})$, where δ_{F^n} is the Dirac statistic defined by $\delta_{F^n}(P) = 1$ whenever $P = F^n$, and 0 otherwise.

Knowing the image through ϕ of only one statistic $\mathrm{X} \in \{\mathrm{UUU}, \mathrm{UUD}, \mathrm{DUU}, \mathrm{DUD}\}$ and using results from Section 2, we can obtain the expressions of the images of the three other statistics from the system (a). The same reasoning holds for the second system (b). So, we split Section 3 into two subsections, each dealing with one of the two systems.

3.1. The patterns UUU, UUD, DUU, DUD

Theorem 3. *For $n \geqslant 0$, the bijection ϕ from $\mathcal{D}_n^{h,\geqslant}$ to $\mathcal{M}_n$ transports the statistic* UUD *as follows:*

$$\phi(\mathrm{UUD}) = \mathrm{UF^+D} + \mathrm{UD}.$$

Proof. We proceed by induction on n. For $n = 1$, we have $\mathrm{UUD}(UD) = 0$. Since $\phi(UD) = F$, we obtain $\mathrm{UF^+D}(F)+\mathrm{UD}(F) = 0$, and the result holds. We assume the result for $k \leqslant n$, and we will prove it for $n + 1$ by distinguishing two main cases.

- Whenever $P = \alpha UD$, we have $\phi(P) = \phi(\alpha)F$, and $(\mathrm{UF^+D}+\mathrm{UD})(\phi(P)) = (\mathrm{UF^+D}+\mathrm{UD})(\phi(\alpha))$. Using induction hypothesis, $(\mathrm{UF^+D} + \mathrm{UD})(\phi(\alpha)) = \mathrm{UUD}(\alpha)$ which is also equal to $\mathrm{UUD}(\alpha UD)$, proving the first case.

- Whenever $P = \alpha UU\beta D\gamma D$, we have $\phi(P) = \phi(\alpha)\phi(\gamma)U\phi(\beta)D$, and $(\mathrm{UF^+D}+\mathrm{UD})(\phi(P)) = (\mathrm{UF^+D}+\mathrm{UD})(\phi(\alpha))+(\mathrm{UF^+D}+\mathrm{UD})(\phi(\gamma)U\phi(\beta)D)$. Using induction hypothesis, we obtain $(\mathrm{UF^+D} + \mathrm{UD})(\phi(P)) = \mathrm{UUD}(\alpha) + (\mathrm{UF^+D} + \mathrm{UD})(\phi(\gamma)U\phi(\beta)D)$. Now we distinguish three subcases in order to prove that $(\mathrm{UF^+D} + \mathrm{UD})(\phi(\gamma)U\phi(\beta)D) = \mathrm{UUD}(UU\beta D\gamma D)$.

If β is the empty path, then $(\mathrm{UF^+D} + \mathrm{UD})(\phi(\gamma)U\phi(\beta)D) = 1 + (\mathrm{UF^+D} + \mathrm{UD})(\phi(\gamma)) = 1 + \mathrm{UUD}(\gamma) = \mathrm{UUD}(UUD\gamma D)$.

If $\beta = (UD)^k$ for some $k \geqslant 1$, then $(\mathrm{UF^+D}+\mathrm{UD})(\phi(\gamma)U\phi((UD)^k)D) = (\mathrm{UF^+D}+\mathrm{UD})(\phi(\gamma)UF^kD) = (\mathrm{UF^+D}+\mathrm{UD})(\phi(\gamma))+1 = \mathrm{UUD}(\gamma) + 1 = \mathrm{UUD}(UU(UD)^kD\gamma D) = \mathrm{UUD}(UU\beta D\gamma D)$.

If β starts with a double rise UU, then $\phi(\beta)$ contains at least one up step ($\phi(\beta) \neq F^\ell$ for any $\ell \geqslant 0$). So, we have $(\mathrm{UF^+D} + \mathrm{UD})(\phi(\gamma)U\phi(\beta)D) = (\mathrm{UF^+D} + \mathrm{UD})(\phi(\gamma)) + (\mathrm{UF^+D} + \mathrm{UD})(\phi(\beta)) = \mathrm{UUD}(\gamma) + \mathrm{UUD}(\beta)$, which equals to $\mathrm{UUD}(UU\beta D\gamma D)$.

Considering these three cases, the second case is proved and the induction is completed. □

Theorem 4. *For $n \geqslant 0$, the bijection ϕ from $\mathcal{D}_n^{h,\geqslant}$ to $\mathcal{M}_n$ transports the statistic* UUU *as follows:*

$$\phi(\mathrm{UUU}) = \mathrm{UF^+D} + 2\big(\mathrm{UF^+U} + \mathrm{UU}\big).$$

Proof. Using Theorem 3 and the equation $\mathrm{UU} = \mathrm{UUD} + \mathrm{UUU}$ of system (a), we have $\phi(\mathrm{UUU}) = \phi(\mathrm{UU}) - \phi(\mathrm{UUD}) = \phi(\mathrm{UU}) - \mathrm{UF^+D} - \mathrm{UD}$, and using Theorem 1, we have $\phi(\mathrm{UUU}) = \mathrm{U}+\mathrm{UU}+\mathrm{UF}-\mathrm{UF^+D}-\mathrm{UD}$. Since we have $\mathrm{U} = \mathrm{UD}+\mathrm{UU}+\mathrm{UF}$ on $\mathcal{M}_n$, we obtain $\phi(\mathrm{UUU}) = 2(\mathrm{UU} + \mathrm{UF}) - \mathrm{UF^+D}$. Using the trivial equation $\mathrm{UF} = \mathrm{UF^+D} + \mathrm{UF^+U}$ for Motzkin paths, we complete the proof. □

Theorem 5. *For $n \geqslant 1$, the bijection ϕ from $\mathcal{D}_n^{h,\geqslant}$ to $\mathcal{M}_n$ transports the statistic* DUU *as follows:*

$$\phi(\mathrm{DUU}) = \mathrm{UF^+D} + \mathrm{UD} + \delta_{F^n} - 1,$$

where δ_{F^n} is the Dirac statistic defined by $\delta_{F^n}(P) = 1$ whenever $P = F^n$, and 0 otherwise.

Proof. Using the difference of the first two equations of (a), we obtain $\mathrm{DUU} = \mathrm{UUD} - \bullet\mathrm{UU}$, and finally, $\phi(\mathrm{DUU}) = \phi(\mathrm{UUD}) - \phi(\bullet\mathrm{UU}) = \mathrm{UF^+D} + \mathrm{UD} - \phi(\bullet\mathrm{UU})$. As discussed at the beginning of Section 3, $\phi(\bullet\mathrm{UU}) = 1 - \delta_{F^n}$, which completes the proof. □

Theorem 6. *For $n \geqslant 1$, the bijection ϕ from $\mathcal{D}_n^{h,\geqslant}$ to $\mathcal{M}_n$ transports the statistic* DUD *as follows:*

$$\phi(\mathrm{DUD}) = \mathrm{F} - \mathrm{UF^+D} - \delta_{F^n}$$

Proof. We have $\mathrm{DUD} = \mathrm{UD} - \mathrm{UUD} - \bullet\mathrm{UD}$, which implies that $\phi(\mathrm{DUD}) = \phi(\mathrm{UD}) - \phi(\mathrm{UUD}) - \phi(\bullet\mathrm{UD})$. From Theorem 1, Theorem 3 and the observations after the systems (a) and (b) we obtain $\phi(\mathrm{DUD}) = \mathrm{UD} + \mathrm{F} - \mathrm{UD} - \mathrm{UF^+D} - \delta_{F^n} = \mathrm{F} - \mathrm{UF^+D} - \delta_{F^n}$. □

Theorem 7. *The bivariate generating functions $F_p(x, y)$ where the coefficient of x^ny^k is the number of Dyck paths in $\mathcal{D}_n^{h,\geqslant}$ containing exactly k occurrences of the pattern $p \in \{UUU, UUD, DUU, DUD\}$ are given by the following expressions listed in the same order as written above.*

$$\frac{x^3y - x^3 - x^2y^2 + 2x - 1 + \sqrt{(x - xy - 1)(x^2 - xy + x - 1)(x^3 - x^3y + x^2y^2 + 2x^2y - 2xy - 2x + 1)}}{2x^2y^2(x - 1)},$$

$$\frac{x^2y - 2x^2 + 2x - 1 + \sqrt{\left(x^2y - 1\right)\left(x^2y - 4x^2 + 4x - 1\right)}}{2x^2\,(x - 1)},$$

Table 3
Popularity of $p \in \{UUU, UUD, DUU, DUD\}$ in $\mathcal{D}_n^{h,\geqslant}$ for $1 \leqslant n \leqslant 12$.

Pattern	Popularity sequence	OEIS
UUU	0, 0, 1, 5, 19, 65, 210, 658, 2023, 6147, 18534, 55594	
UUD	0, 1, 3, 9, 25, 70, 196, 553, 1569, 4476, 12826, 36894	A097861
DUU	0, 0, 0, 1, 5, 20, 70, 231, 735, 2289, 7029, 21384	A304011 ?
DUD	0, 1, 4, 12, 35, 100, 286, 819, 2353, 6780, 19591, 56749	

Table 4
Number of paths from $\mathcal{D}_n^{h,\geqslant}$ having k occurrences of the considered pattern.

k	n								
	1	2	3	4	5	6	7	8	9
0	1	2	3	5	8	13	21	34	55
1			1	3	8	18	38	76	147
2				1	4	14	40	104	250
3					1	5	21	71	215
4						1	6	30	119
5							1	7	40

(a) *UUU*

k	n								
	1	2	3	4	5	6	7	8	9
0	1	1	3	5	8	13	21	34	55
1		1	2	3	7	13	30	58	130
2			1	3	6	16	35	91	199
3				1	4	10	30	75	216
4					1	5	15	50	140
5						1	6	21	77

(b) *DUD*

k	n								
	1	2	3	4	5	6	7	8	9
0	1	1	1	1	1	1	1	1	1
1		1	3	7	15	31	63	127	255
2				1	5	18	56	160	432
3						1	7	34	138
4								1	9

(c) *UUD*

$k\backslash n$	1	2	3	4	5	6	7	8	9
0	1	2	4	8	16	32	64	128	256
1				1	5	18	56	160	432
2						1	7	34	138
3								1	9
4									

(d) *DUU*

$$\frac{2x - x^2y - 1 + \sqrt{(x^2y-1)\,(x^2y - 4x^2 + 4x - 1)}}{2x^2y\,(x-1)},$$

$$\frac{x^3y - x^3 - x^2y^2 + 2xy - 1 + \sqrt{(xy - x - 1)(x^2 + yx - x - 1)(x^3y - x^3 + x^2y^2 + 2x^2y - 2xy - 2x + 1)}}{2x^2(xy-1)}.$$

Proof. For $p = UUU$ and using Theorem 4, we have $\phi(\mathrm{UUU}) = \mathrm{UF^+D} + 2(\mathrm{UF^+U} + \mathrm{UU})$. So, we decompose Motzkin paths according to the patterns UF^+D, UF^+U, and UU in order to exhibit a functional equation having $F_p(x, y)$ as solution:

$$\mathcal{M} = \epsilon + F\mathcal{M} + UD\mathcal{M} + U\mathcal{M}_0D\mathcal{M} + U(\mathcal{M}\backslash(\mathcal{M}_0 \cup \epsilon))D\mathcal{M},$$

where $\mathcal{M}_0$ is the subset of $\mathcal{M}$ consisting of paths of the form F^k for $k \geqslant 1$. The generating function for $\epsilon + F\mathcal{M} + UD\mathcal{M}$ is $1 + xF_{UUU}(x, y) + x^2F_{UUU}(x, y)$; the g.f. for $U\mathcal{M}_0D\mathcal{M}$ is $x^2y\frac{x}{1-x}F_{UUU}(x, y)$ since a path in $U\mathcal{M}_0D$ is an occurrence of UF^+D; the g.f. for $U(\mathcal{M}\backslash(\mathcal{M}_0 \cup \epsilon))D\mathcal{M}$ is $x^2y^2\left(F_{UUU}(x, y) - \frac{1}{1-x}\right)F_{UUU}(x, y)$ since a path in $U(\mathcal{M}\backslash(\mathcal{M}_0 \cup \epsilon))D$ starts with either an occurrence of UU or an occurrence of UF^+U.

So the functional equation is:

$$F_{UUU}(x, y) = 1 + xF_{UUU}(x, y) + x^2F_{UUU}(x, y) + x^2y\frac{x}{1-x}F_{UUU}(x, y) + \\ + x^2y^2\left(F_{UUU}(x, y) - \frac{1}{1-x}\right)F_{UUU}(x, y).$$

A simple calculation (with Maple for instance) provides the result.

All other generating functions are obtained using a similar method, so we do not give the proofs here. □

The generating function $G_p(x)$ for the popularity of the pattern $p \in \{UUU, UUD, DUU, DUD\}$ is obtained directly by evaluating $\frac{\partial F_p(x,y)}{\partial y}\Big|_{y=1}$. Table 3 provides the first terms of the associated sequences. See also Table 4 for an illustration of the distribution of $p \in \{UUU, UUD, DUU, DUD\}$.

The popularity sequence for UUD in $\mathcal{D}_n^{h,\geqslant}$ is the sequence A097861, corresponding to the number of humps in all Motzkin paths of length n (a hump equals UF^+D or UD in our notation). The popularity for DUU seems to generate the sequence A304011, but we did not succeed in proving this fact, so we leave it as a conjecture.

Dyck paths from $\mathcal{D}_n^{h,\geqslant}$ avoiding UUU, DUU respectively generate Fibonacci numbers and integer squares. Those avoiding DUD seem to correspond to A007562 (number of planted trees where non-root, non-leaf nodes at even distance from root are of degree 2). At the present time, there is no closed form for the generating function of sequence A007562. Note that

any path avoiding *DUU* has at most one occurrence of *UUD*. Also, Dyck paths containing two occurrences of *UUD* in $\mathcal{D}_n^{h,\geqslant}$ generate a shift of sequence A001793, which corresponds to a subsequence in the triangle of coefficients of Chebyshev's polynomials which is sequence A053120.

3.2. The patterns *DDD*, *DDU*, *UDD*, *UDU*

Theorem 8. *For $n \geqslant 0$, the bijection ϕ from $\mathcal{D}_n^{h,\geqslant}$ to $\mathcal{M}_n$ transports the statistic* UDU *as follows:*

$$\phi(\mathrm{UDU}) = \mathrm{FF} + \mathrm{FUD}$$

Proof. We proceed by induction on n. For $n = 1$, we have $\mathrm{UDU}(UD) = 0$. With $\phi(UD) = F$, we obtain $\mathrm{FF}(F) + \mathrm{FUD}(F) = 0$, and the result holds. We assume the result for $k \leqslant n$, and we will prove it for $n + 1$.

- Whenever $P = \alpha UD$, we have $\phi(P) = \phi(\alpha)F$, and $(\mathrm{FF} + \mathrm{FUD})(\phi(P)) = \mathrm{FF}(\phi(\alpha)F) + \mathrm{FUD}(\phi(\alpha)F) = \mathrm{FF}(\phi(\alpha)F) + \mathrm{FUD}(\phi(\alpha))$. We distinguish two cases: (i) $\phi(\alpha)$ ends with F, and (ii) otherwise. In the case (i), α ends with *UD*, and thus $(\mathrm{FF}+\mathrm{FUD})(\phi(P)) = 1+\mathrm{FF}(\phi(\alpha))+\mathrm{FUD}(\phi(\alpha))$. Using the induction hypothesis we have $(\mathrm{FF}+\mathrm{FUD})(\phi(P)) = 1+\mathrm{UDU}(\alpha) = \mathrm{UDU}(\alpha UD) = \mathrm{UDU}(P)$. In the case (ii), α does not end with *UD*, and thus $\phi(\alpha)$ does not end with F. So, we have $(\mathrm{FF}+\mathrm{FUD})(\phi(P)) = \mathrm{FF}(\phi(\alpha))+\mathrm{FUD}(\phi(\alpha))$, and using the induction hypothesis $(\mathrm{FF}+\mathrm{FUD})(\phi(P)) = \mathrm{UDU}(\alpha) = \mathrm{UDU}(\alpha UD) = \mathrm{UDU}(P)$.

- Whenever P does not end with *UD*, we have $P = \alpha UU\beta D\gamma D$, and $(\mathrm{FF} + \mathrm{FUD})(\phi(P)) = (\mathrm{FF} + \mathrm{FUD})(\phi(\alpha)\phi(\gamma)U\phi(\beta)D)$. Note that α cannot end with *UD*, otherwise it would contradict $P \in \mathcal{D}_n^{h,\geqslant}$. This means that $\phi(\alpha)$ cannot end with F. So, all the possible occurrences of *FF* in $\phi(P)$ belong necessarily to $\phi(\alpha)$, $\phi(\beta)$ and $\phi(\gamma)$, which implies that $\mathrm{FF}(\phi(P)) = \mathrm{FF}(\phi(\alpha)) + \mathrm{FF}(\phi(\beta)) + \mathrm{FF}(\phi(\gamma))$. On the other hand, the possible occurrences of *FUD* in $\phi(P)$ belong necessarily to $\phi(\alpha)$, $\phi(\beta)$, $\phi(\gamma)$, and eventually at the junction of $\phi(\gamma)$ and $U\phi(\beta)D$ whenever $\phi(\gamma)$ ends with F and $\phi(\beta) = \epsilon$. So, we distinguish two cases: (a) $\phi(\gamma)$ ends with F and $\beta = \epsilon$, and (b) otherwise. In the case (a), we have $\mathrm{FUD}(\phi(P)) = 1 + \mathrm{FF}(\phi(\alpha)) + \mathrm{FF}(\phi(\beta)) + \mathrm{FF}(\phi(\gamma)) + \mathrm{FUD}(\phi(\alpha)) + \mathrm{FUD}(\phi(\beta)) + \mathrm{FUD}(\phi(\gamma))$, and using the induction hypothesis, we obtain $\mathrm{FUD}(\phi(P)) = 1 + \mathrm{UDU}(\alpha) + \mathrm{UDU}(\beta) + \mathrm{UDU}(\gamma) = \mathrm{UDU}(\alpha UU\beta D\gamma D) = \mathrm{UDU}(P)$. In the case (b), we have $\mathrm{FUD}(\phi(P)) = \mathrm{FF}(\phi(\alpha)) + \mathrm{FF}(\phi(\beta)) + \mathrm{FF}(\phi(\gamma)) + \mathrm{FUD}(\phi(\alpha)) + \mathrm{FUD}(\phi(\beta)) + \mathrm{FUD}(\phi(\gamma))$, and using the induction hypothesis, we obtain $\mathrm{FUD}(\phi(P)) = \mathrm{UDU}(\alpha) + \mathrm{UDU}(\beta) + \mathrm{UDU}(\gamma) = \mathrm{UDU}(\alpha UU\beta D\gamma D) = \mathrm{UDU}(P)$. So, the proof is complete. □

Theorem 9. *For $n \geqslant 0$, the bijection ϕ from $\mathcal{D}_n^{h,\geqslant}$ to $\mathcal{M}_n$ transports the statistic* UDD *as follows:*

$$\phi(\mathrm{UDD}) = \mathrm{FD} + \mathrm{UD} + \mathrm{FUU} + \mathrm{FUF}.$$

Proof. Considering the third equation of system (b), we have $\phi(\mathrm{UDD}) = \phi(\mathrm{UD}) - \phi(\mathrm{UDU}) - \phi(\mathrm{UD}\bullet)$. Using Theorems 1 and 8, we obtain $\phi(\mathrm{UDD}) = \mathrm{F} + \mathrm{UD} - \mathrm{FF} - \mathrm{FUD} - \phi(\mathrm{UD}\bullet)$. In any Motzkin path P, a flat step is either at the end of P, or followed by F, or D, or *UU*, or *UD*, or *UF*, that is $\mathrm{F} = \mathrm{F}\bullet + \mathrm{FF} + \mathrm{FD} + \mathrm{FUU} + \mathrm{FUD} + \mathrm{FUF}$. Then, we obtain $\phi(\mathrm{UDD}) = \mathrm{F}\bullet + \mathrm{UD} + \mathrm{FD} + \mathrm{FUU} + \mathrm{FUF} - \phi(\mathrm{UD}\bullet)$. Using the definition of ϕ in Introduction, it is clear that ϕ transports $\mathrm{UD}\bullet$ into $\mathrm{F}\bullet$, and the result holds. □

Theorem 10. *For $n \geqslant 0$, the bijection ϕ from $\mathcal{D}_n^{h,\geqslant}$ to $\mathcal{M}_n$ transports the statistic* DDU *as follows:*

$$\phi(\mathrm{DDU}) = \mathrm{DF} + \mathrm{DU} + \mathrm{FUU} + \mathrm{FUF}.$$

Proof. Combining the fourth equation $\mathrm{DU} = \mathrm{DDU} + \mathrm{UDU}$ of system (b) with Theorems 1 and 8, we obtain $\phi(\mathrm{DDU}) = \mathrm{FF} + \mathrm{FU} + \mathrm{DF} + \mathrm{DU} - \mathrm{FF} - \mathrm{FUD} = \mathrm{FU} + \mathrm{DF} + \mathrm{DU} - \mathrm{FUD}$. The proof is completed using the straightforward equation $\mathrm{FU} = \mathrm{FUU} + \mathrm{FUF} + \mathrm{FUD}$ on Motzkin paths. □

Theorem 11. *For $n \geqslant 0$, the bijection ϕ from $\mathcal{D}_n^{h,\geqslant}$ to $\mathcal{M}_n$ transports the statistic* DDD *as follows:*

$$\phi(\mathrm{DDD}) = 2(\mathrm{UU} + \mathrm{UF}) - \mathrm{FD} - \mathrm{FUU} - \mathrm{FUF}.$$

Proof. Combining the first equation $\mathrm{DD} = \mathrm{DDD} + \mathrm{UDD}$ of system (b) with Theorems 1 and 9, we obtain $\phi(\mathrm{DDD}) = \mathrm{U} + \mathrm{UU} + \mathrm{UF} - \mathrm{FD} - \mathrm{UD} - \mathrm{FUU} - \mathrm{FUF}$. Using $\mathrm{U} = \mathrm{UF} + \mathrm{UD} + \mathrm{UU}$ on Motzkin paths, the result holds. □

Theorem 12. *The bivariate generating functions $F_p(x, y)$ where the coefficient of $x^n y^k$ is the number of Dyck paths in $\mathcal{D}_n^{h,\geqslant}$ containing exactly k occurrences of the pattern $p \in \{UDU, UDD, DDU, DDD\}$ are given by the following expressions listed in the same order as written above.*

$$\frac{1 + x(x^2 - x^2y - y) - \sqrt{A}}{2x^2\,(x - xy + 1)},$$

$$\frac{1 + x(x^2y - x^2 - xy + x - 1) - \sqrt{B}}{2x^2\,(xy - x + 1)^2},$$

Table 5
Number of paths from $\mathcal{D}_n^{h,\geqslant}$ having k occurrences of the considered pattern.

k	n								
	1	2	3	4	5	6	7	8	9
0	1	2	3	6	11	22	43	87	176
1			1	2	7	16	43	102	251
2				1	2	10	25	80	208
3					1	2	13	34	130
4						1	2	17	46
5							1	2	21

(a) *DDD*

k	n								
	1	2	3	4	5	6	7	8	9
0	1	1	2	4	8	17	37	82	185
1		1	1	3	7	17	41	102	252
2			1	1	4	10	28	73	200
3				1	1	5	13	41	113
4					1	1	6	16	56
5						1	1	7	19

(b) *DUD*

k	n								
	1	2	3	4	5	6	7	8	9
0	1	2	3	4	5	6	7	8	9
1			1	5	14	31	59	102	164
2					2	14	57	174	444
3							4	39	209
4									9

(c) *DDU*

k	n								
	1	2	3	4	5	6	7	8	9
0	1	1	1	1	1	1	1	1	1
1		1	3	6	10	15	21	28	36
2				2	10	31	75	156	292
3						4	30	129	417
4								9	89

(d) *UDD*

$$\frac{1+x(2x^2y^2-3x^2y+x^2+xy-x-1)-\sqrt{C}}{2x^2y\,(xy-x+1)},$$
$$\frac{1-x\left(x^2y^2-x^2y-xy^2+x+1\right)-\sqrt{D}}{2x^2\,(xy-x-y)^2}$$

where

$$\begin{aligned}
A &= (x+1)\left(x^2y-x^2+xy-x-1\right)\left(x^3y-x^3-2x^2y+2x^2+xy+2x-1\right),\\
B &= (x+1)\left(x^2y-x^2+1\right)\left(x^3y-x^3-3x^2y+3x^2-3x+1\right),\\
C &= (x+1)\left(x^2y-x^2+1\right)\left(x^3y-x^3-3x^2y+3x^2-3x+1\right),\\
D &= (xy+1)\left(x^2y-x^2-xy+x-1\right)\left(x^3y^2-x^3y-x^2y^2-2x^2y+3x^2+2xy+x-1\right).
\end{aligned}$$

Proof. Let $A_p(x,y)$ (resp. $B_p(x,y)$) be the bivariate generating function where the coefficient of x^ny^k is the number of Motzkin paths of length n ending with a flat step (resp. down step) having exactly k occurrences of the patterns related to the statistic $\phi(\mathrm{p})$ obtained in the r.h.s. of equations of Theorems 8, 9, 10, 11. Using a refinement of the classical decompositions of non-empty Motzkin paths by taking into account the occurrences of the considered patterns, we deduce functional equations for $A_p(x,y)$ and $B_p(x,y)$. The method being classic, we do not give any more details. The solutions are obtained by a simple calculation.

For $p = UDU$:
$$\begin{cases} F_p(x,y) &= 1 + A_p(x,y) + B_p(x,y)\\ A_p(x,y) &= x + xyA_p(x,y) + xB_p(x,y)\\ B_p(x,y) &= x^2 + x^2yA_p(x,y) + x^2B_p(x,y) + x^2F_p(x,y)(F_p(x,y)-1); \end{cases}$$
For $p = UDD$:
$$\begin{cases} F_p(x,y) = 1 + A_p(x,y) + B_p(x,y)\\ A_p(x,y) = xF_p(x,y)\\ B_p(x,y) = x^2yF_p(x,y) + x^2yA_p(x,y) + x^2B_p(x,y) +\\ \qquad +x^2B_p(x,y)^2 + x^2yA_p(x,y)B_p(x,y) + x^2y^2A_p(x,y)^2 + x^2yA_p(x,y)B_p(x,y); \end{cases}$$
For $p = DDU$:
$$\begin{cases} F_p(x,y) = 1 + A_p(x,y) + B_p(x,y)\\ A_p(x,y) = x + xA_p(x,y) + xyB_p(x,y)\\ B_p(x,y) = x^2F_p(x,y) + x^2yA_p(x,y)(F_p(x,y)-1) + x^2A_p(x,y) +\\ \qquad +x^2yB_p(x,y)F_p(x,y); \end{cases}$$
For $p = DDD$:
$$\begin{cases} F_p(x,y) = 1 + A_p(x,y) + B_p(x,y)\\ A_p(x,y) = xF_p(x,y)\\ B_p(x,y) = x^2F_p(x,y) + x^2y^2B_p(x,y) + x^2y^2A_p(x,y)/y + x^2y^2A_p(x,y)^2/y^2 +\\ \qquad +x^2y^2A_p(x,y)B_p(x,y)/y + x^2y^2B_p(x,y)^2 + x^2y^2A_p(x,y)B_p(x,y)/y. \end{cases}$$

Table 6
Popularity of 3-length patterns in $\mathcal{D}_n^{h,\geqslant}$ for $1 \leqslant n \leqslant 12$.

Pattern	Popularity sequence	OEIS
DDD	0, 0, 1, 4, 14, 46, 145, 448, 1365, 4124, 12387, 37060	
DDU	0, 0, 1, 5, 18, 59, 185, 567, 1715, 5146, 15363, 45715	
UDD	0, 1, 3, 10, 30, 89, 261, 763, 2227, 6499, 18973, 55428	
UDU	0, 1, 3, 8, 22, 61, 171, 483, 1373, 3923, 11257, 32418	A025566

The above equations are intentionally left in non-simplified forms, in order to allow the reader to easily retrieve the refined decompositions from the classical decomposition of Motzkin paths $\mathcal{M} = \epsilon + \mathcal{M}F + \mathcal{M}U\mathcal{M}D$. $\square$

Generating function $G_p(x)$ for the popularity of a pattern p is obtained directly by evaluating $\frac{\partial F_p(x,y)}{\partial y}\Big|_{y=1}$. Table 6 provides the first terms of the generated sequences. See also Table 5 for an illustration of the distribution of $p \in \{UDU, UDD, DDU, DDD\}$.

Unlike what happens for classical Dyck paths, the popularity of UDU in $\mathcal{D}_{n+1}^{h,\geqslant}$ is equal to the popularity of UD in $\mathcal{D}_n^{h,\geqslant}$, while the corresponding distributions are different. Dyck paths from $\mathcal{D}_n^{h,\geqslant}$ avoiding a pattern UDU (resp. DDD) are counted by the Generalized Catalan numbers (resp. by the numbers of ordered trees with n edges and having no branches of length 1), which corresponds to the sequence A004148 (resp. A026418).

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments

We would like to thank the anonymous referees for their helpful comments and suggestions.

References

[1] A. Bacher, A. Bernini, L. Ferrari, B. Gundy, R. Pinzani, J. West, The Dyck pattern poset, Discrete Math. 321 (2014) 12–23.
[2] E. Barcucci, A. Del Lungo, S. Fezzi, R. Pinzani, Nondecreasing Dyck paths and q-Fibonacci numbers, Discrete Math. 170 (1–3) (1997) 211–217.
[3] E. Barcucci, R. Pinzani, R. Sprugnoli, et al., The Motzkin family, Pure Math. Appl. 2 (3–4) (1992) 249–279.
[4] J.-L. Baril, S. Kirgizov, A. Petrossian, Dyck paths with a first return decomposition constrained by height, Discrete Math. 341 (6) (2018) 1620–1628.
[5] J.-L. Baril, S. Kirgizov, A. Petrossian, Motzkin paths with a restricted first return decomposition, Integers 19 (2019) A46.
[6] C. Brennan, S. Mavhungu, Peaks and valleys in Motzkin paths, Quaest. Math. 33 (2) (2010) 171–188.
[7] D. Callan, Two bijections for Dyck path parameters, ArXiv preprint math/0406381, 2004.
[8] E. Czabarka, R. Flórez, L. Junes, Some enumerations on non-decreasing Dyck paths, Electron. J. Combin. 22 (1) (2015) 1–3.
[9] A. Denise, R. Simion, Two combinatorial statistics on Dyck paths, Discrete Math. 137 (1–3) (1995) 155–176.
[10] E. Deutsch, Dyck path enumeration, Discrete Math. 204 (1–3) (1999) 167–202.
[11] R. Donaghey, L.W. Shapiro, Motzkin numbers, J. Combin. Theory Ser. A 23 (3) (1977) 291–301.
[12] D. Drake, R. Gantner, Generating functions for plateaus in Motzkin paths, 2011, ArXiv preprint, arXiv:1109.3273.
[13] L. Ferrari, E. Munarini, Enumeration of edges in some lattices of paths, J. Integer Seq. 17 (1) (2014) 5.
[14] S.M. Losanitsch, Die isomerie-arten bei den homologen der paraffin-reihe, Berichte der deutschen chemischen Gesellschaft 30 (2) (1897) 1917–1926.
[15] T. Mansour, Counting peaks at height k in a Dyck path, J. Integer Seq. 5 (1) (2002) 1.
[16] T. Mansour, Statistics on Dyck paths, J. Integer Seq. 9 (1) (2006) 5.
[17] T. Mansour, M. Schork, Y. Sun, Motzkin numbers of higher rank: generating function and explicit expression, J. Integer Seq. 10 (7) (2007) 4.
[18] D. Merlini, R. Sprugnoli, M.C. Verri, Some statistics on Dyck paths, J. Stat. Plan. Inference 101 (1–2) (2002) 211–227.
[19] A. Panayotopoulos, A. Sapounakis, On the prime decomposition of Dyck words, J. Combin. Math. Combin. Comput. 40 (2002) 33–40.
[20] P. Peart, W.J. Woan, Dyck paths with no peaks at height k, J. Integer Seq. 4 (1) (2001) 3.
[21] A. Sapounakis, I. Tasoulas, P. Tsikouras, Counting strings in Dyck paths, Discrete Math. 307 (23) (2007) 2909–2924.
[22] A. Sapounakis, P. Tsikouras, Counting peaks and valleys in k-colored Motzkin paths, Electron. J. Combin. 12 (1) (2005) 16.
[23] N.J.A. Sloane, The on-line encyclopedia of integer sequences, 2019, https://oeis.org.
[24] Y. Sun, The statistic "number of udu's" in Dyck paths, Discrete Math. 287 (1–3) (2004) 177–186.
[25] S. Wagner, H. Prodinger, Minimal and maximal plateau lengths in motzkin paths, Discrete Math. Theoret. Comput. Sci. (2007).

ecajournal.haifa.ac.il

Enumerative Combinatorics and Applications *ECA* **1:3** (2021) Article #S2R19

Transformation à la Foata for Special Kinds of Descents and Excedances

Jean-Luc Baril and Sergey Kirgizov

LIB, Université Bourgogne Franche-Comté, B.P. 47 870, 21078 Dijon-Cedex, France
Email: barjl@u-bourgogne.fr, sergey.kirgizov@u-bourgogne.fr

Received: January 6, 2021, **Accepted**: March 17, 2021, **Published**: March 26, 2021
The authors: Released under the CC BY-ND license (International 4.0)

Abstract: A pure excedance in a permutation $\pi = \pi_1\pi_2\dots\pi_n$ is a position $i < \pi_i$ such that there is no $j < i$ with $i \leq \pi_j < \pi_i$. We present a one-to-one correspondence on the symmetric group that transports pure excedances to descents of a special kind. As a byproduct, we prove that the popularity of pure excedances equals those of pure descents on permutations, while their distributions are different.

Keywords: Cycle; Descent; Distribution; Excedance; Permutation; Popularity; Statistic
2020 Mathematics Subject Classification: 05A05; 05A15; 05A19

1. Introduction and notations

The distribution of the number of descents has been widely studied on several classes of combinatorial objects such as permutations [14], cycles [7, 8], and words [3, 10]. Many interpretations of this statistic appear in several fields as Coxeter groups [4, 11] or lattice path theory [12]. One of the most famous result involves the *Foata fundamental transformation* [9] to establish a one-to-one correspondence between descents and excedances on permutations. This bijection provides a more straightforward proof than those of MacMahon [14] for the equidistribution of these two Eulerian statistics.

In this paper, we present a bijection *à la Foata* on the symmetric group that exchanges pure excedances with special kind of descents defined as a mesh pattern p_2 [6] (see below for the definition of this pattern). Then, we deduce that the popularities (but not the distributions) of pure descents [2] and pure excedances are the same. This common popularity is given by the generalized Stirling number $n! \cdot (H_n - 1)$ (see Sequence A001705 in [15]) where $H_n = \sum_{k=1}^{n} \frac{1}{k}$ is the nth harmonic number. Finally, we conjecture the existence of a bijection on the symmetric group that exchanges pure excedances and p_2 while preserving the number of cycles.

Let S_n be the set of permutations of length n, *i.e.*, all bijections from $[n] = \{1, 2, \dots, n\}$ into itself. The one-line representation of a permutation $\pi \in S_n$ is $\pi = \pi_1\pi_2\dots\pi_n$ where $\pi_i = \pi(i)$, $1 \leq i \leq n$. For $\sigma \in S_n$, the *product* $\sigma \cdot \pi$ is the permutation $\sigma(\pi_1)\sigma(\pi_2)\dots\sigma(\pi_n)$. A *$\ell$-cycle* $\pi = \langle i_1, i_2, \dots, i_\ell\rangle$ in S_n is a n-length permutation satisfying $\pi(i_1) = i_2, \pi(i_2) = i_3, \dots, \pi(i_{\ell-1}) = i_\ell, \pi(i_\ell) = i_1$ and $\pi(j) = j$ for $j \in [n]\backslash\{i_1, i_2, \dots, i_\ell\}$. For $1 \leq k \leq n$, we denote by $C_{n,k}$ the set of all n-length permutations admitting a decomposition in a product of k disjoint cycles. The set $C_{n,k}$ is counted by the signless Stirling numbers of the first kind $c(n, k)$ defined by

$$c(n, k) = (n - 1)\, c(n - 1, k) + c(n - 1, k - 1)$$

where $c(n, k) = 0$ if $n = 0$ or $k = 0$, except $c(0, 0) = 1$ (see [16, 17] and Sequence A132393 in [15]). These numbers also enumerate n-length permutations π having k *left-to-right maxima*, *i.e.*, positions $i \in [n]$ such that $\pi_j < \pi_i$ for $j < i$ (see [16]), and permutations $\pi \in S_n$ with $k - 1$ *pure descents*, *i.e.*, descents $\pi_i > \pi_{i+1}$ where there is no $j < i$ such that $\pi_j \in [\pi_{i+1}, \pi_i]$ (see [2]). Note that a pure descent can be viewed as an occurrence of the mesh pattern $(21, L_1)$ where $L_1 = \{1\} \times [0, 2] \cup \{(0, 1)\}$. Indeed, for a k-length permutation σ and a subset $R \subseteq [0, k] \times [0, k]$, an occurrence of the mesh pattern (σ, R) in a permutation π is an occurrence of σ in π with the additional restriction that no element of π lies inside the shaded regions defined by R, where $(i, j) \in R$ means the square having bottom left corner (i, j) in the graphical representation $\{(i, \sigma_i), i \in [k]\}$ of σ. For instance, an occurrence of the mesh pattern p_1 in Figure 1 corresponds to an occurrence of a pure descent. See [6] for a more detailed definition of mesh patterns.

Regarding this interpretation of pure descents in terms of mesh patterns, we define other kinds of descents by the mesh patterns $p_i = (21, L_i)$, $p'_i = (21, R_i)$ with $L_i = \{1\} \times [0, 2] \cup \{(0, i)\}$ and $R_i = \{1\} \times [0, 2] \cup \{(2, i)\}$ for $0 \leq i \leq 2$. Modulo the trivial symmetries on permutations (reverse and complement), it is straightforward to see that p_0, p_1 and p_2 are respectively in the same distribution class as p'_2, p'_1 and p'_0. Then, we deal with only mesh

patterns p_i, $i \in [0,2]$. We refer to Figure 1 for a graphical illustration. On the other hand, we define a *pure excedance* as an occurrence of an excedance, *i.e.* $\pi_i > i$, with the additional restriction that there is no point (j, π_j) such that $1 \leq j \leq i-1$ with $i \leq \pi_j < \pi_i$. Although such a pattern (called *pex*) is not a mesh pattern, we can represent it graphically as shown in Figure 1.

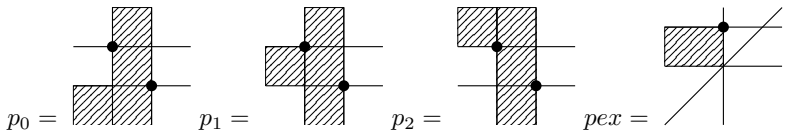


Figure 1: Illustration of the mesh patterns p_0, p_1, p_2 and *pex*; p_1 and *pex* correspond respectively to a pure descent and a pure excedance.

A *statistic* is an integer-valued function from a set $\mathcal{A}$ of n-length permutations (we use the boldface to denote statistics). For a pattern p, we define the pattern statistic $\mathbf{p} : \mathcal{A} \to \mathbb{N}$ where the image $\mathbf{p}\ \pi$ of $\pi \in \mathcal{A}$ by $\mathbf{p}$ is the number of occurrences of p in π. The *popularity* of p in $\mathcal{A}$ is the total number of occurrences of p over all objects of $\mathcal{A}$, that is $\sum_{a \in \mathcal{A}} \mathbf{p}\ a$ (see [5] for instance). Below, we present statistics that we use throughout the paper:

$$\begin{aligned}
&\mathbf{exc}\ \pi && = \text{number of excedances in } \pi,\\
&\mathbf{pex}\ \pi && = \text{number of pure excedances in } \pi,\\
&\mathbf{des}\ \pi && = \text{number of descents in } \pi,\\
&\mathbf{des}_i\ \pi && = \text{number of patterns } p_i \text{ in } \pi,\ 0 \leq i \leq 2,\\
&\mathbf{fix}\ \pi && = \text{number of fixed points in } \pi,\\
&\mathbf{cyc}\ \pi && = \text{number of cycles in the decomposition of } \pi,\\
&\mathbf{pcyc}\ \pi && = \text{number of pure cycles (i.e. cycles of length at least two) in } \pi,\\
& && = \mathbf{cyc}\ \pi - \mathbf{fix}\ \pi
\end{aligned}$$

We organize the paper as follows. In Section 2, we focus on patterns p_i, $0 \leq i \leq 2$. We prove that the statistics $\mathbf{des}_0$ and $\mathbf{des}_1$ are equidistributed by giving algebraic and bijective proofs. Next, we provide the bivariate exponential generating function for the distribution of p_2, and we deduce that p_2 has the same popularity as p_0 and p_1, without having the same distribution. In Section 3, we present a bijection on S_n that transports pure excedances into patterns p_2. Notice that the Foata's first transformation [9] is not a candidate for such a bijection. As a consequence, pure descents and pure excedances are equipopular on S_n, but they do not have the same distribution. Combining all these results, we deduce that patterns p_i, $0 \leq i \leq 2$, and *pex* are equipopular on the symmetric group S_n. Finally we present two conjectures about the equidistribution of $(\mathbf{cyc}, \mathbf{des}_2)$ and $(\mathbf{cyc}, \mathbf{pex})$, and that of $(\mathbf{des}, \mathbf{des}_2)$ and $(\mathbf{exc}, \mathbf{pex})$.

2. The statistics $\mathbf{des}_i$, $0 \leq i \leq 2$

For $0 \leq i \leq 2$, let $A^i_{n,k}$ be the set of n-length permutations having k occurrences of p_i, and denote by $a^i_{n,k}$ its cardinality. Let $A^i(x,y)$ be the bivariate exponential generating function $\sum_{n=0}^{\infty} \sum_{k=0}^{n-1} a^i_{n,k} \frac{x^n}{n!} y^k$. In [2,13], it is proved that $a^1_{n,k}$ equals the signless Stirling numbers of the first kind $c(n, k+1)$ (see Sequence A132393 in [15]). Indeed, a permutation $\sigma \in A^1_{n,k}$ can be uniquely obtained from an $(n-1)$-length permutation π by one of the two following constructions:

(i) if $\pi \in A^1_{n-1,k-1}$, then we increase by one all values of π greater than or equal to π_{n-1}, and we add π_{n-1} at the end;

(ii) if $\pi \in A^1_{n-1,k}$, then we increase by one all values of π greater than or equal to a given value $x \leq n$, $x \neq \pi_{n-1}$ and we add x at the end.

Then, we deduce the recurrence relation $a^1_{n,k} = a^1_{n-1,k-1} + (n-1)a^1_{n-1,k}$ with $a^1_{n,0} = (n-1)!$ for $n \geq 1$, $a^1_{0,0} = 1$ and the bivariate exponential generating function is

$$A^1(x,y) = \frac{1}{y(1-x)^y} - \frac{1}{y} + 1$$

which proves that $a^1_{n,k} = c(n, k+1)$.

Below, we prove that $a^1_{n,k}$ also counts n-length permutations having k occurrences of the pattern p_0.

Theorem 2.1. *The number $a^0_{n,k}$ of n-length permutations having k occurrences of pattern p_0 equals $a^1_{n,k} = c(n, k+1)$.*

Proof. An n-length permutation $\sigma \in A^0_{n,k}$ can be uniquely obtained from an $(n-1)$-length permutation π by one of the two following constructions:

(i) if $\pi \in A^0_{n-1,k-1}$, then we increase by one all values of π and we add 1 at the end;

(ii) if $\pi \in A^0_{n-1,k}$, then we increase by one all values of π greater than or equal to a given value x, $1 < x \leq n$, and we add x at the end.

We deduce the recurrence relation $a^0_{n,k} = a^0_{n-1,k-1} + (n-1)a^0_{n-1,k}$ with the initial condition $a^0_{n,0} = (n-1)!$, and then $a^0_{n,k} = a^1_{n,k} = c(n, k+1)$. □

Now, we focus on the distribution of the pattern p_2. Table 1 provides exact values for small sizes.

Theorem 2.2. *We have*

$$A^2(x, y) = \frac{e^{x(1-y)}}{(1-x)^y},$$

and the general term $a^2_{n,k}$ satisfies for $n \geq 2$ and $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$

$$a^2_{n,k} = na^2_{n-1,k} + (n-1)a^2_{n-2,k-1} - (n-1)a^2_{n-2,k}$$

with the initial conditions $a^2_{n,0} = 1$ and $a^2_{n,k} = 0$ for $n \geq 0$ and $k > \lfloor \frac{n}{2} \rfloor$ (see Table 1 and Sequence A136394 in [15]).

Proof. Let $\sigma = \sigma_1\sigma_2 \ldots \sigma_n$ denote a permutation of length n having k occurrences of pattern p_2. Let $u_{n,k}$ (resp. $v_{n,k}$) be the number of such permutations satisfying $\sigma_n = n$ (resp. $\sigma_n < n$). Obviously, we have

$$a^2_{n,k} = u_{n,k} + v_{n,k}.$$

A permutation σ with $\sigma_n = n$ can be uniquely constructed from an $(n-1)$-length permutation π as $\sigma = \pi_1\pi_2 \ldots \pi_{n-1}n$. No new occurrences of p_2 are created, and we obtain

$$u_{n,k} = a^2_{n-1,k}.$$

A permutation σ satisfying $\sigma_n < n$ can be uniquely obtained from an $(n-1)$-length permutation π by adding a value $x < n$ on the right side of its one-line notation, after increasing by one all the values greater than or equal to x. This construction creates a new pattern p_2 if and only if π ends with $n-1$. Thus, we deduce

$$v_{n,k} = (n-1)u_{n-1,k-1} + (n-1)v_{n-1,k}.$$

Combining the equations, we obtain for $n \geq 2$ and $k \geq 1$

$$a^2_{n,k} = na^2_{n-1,k} + (n-1)a^2_{n-2,k-1} - (n-1)a^2_{n-2,k},$$

which implies the following differential equation

$$\frac{\partial A^2(x, y)}{\partial x} = (y-1)xA^2(x, y) + \frac{\partial \left(xA^2(x, y)\right)}{\partial x}, \text{ where } A^2(x, 0) = 1.$$

A simple calculation provides the claimed closed form for the generating function $A^2(x, y)$. □

Corollary 2.1. *For $0 \leq i \leq 2$, the patterns p_i are equipopular on S_n. Their popularity is given by the generalized Stirling number $n! \cdot (H_n - 1)$ (see Sequence A001705 in [15]) where $H_n = \sum_{k=1}^{n} \frac{1}{k}$ is the nth harmonic number.*

Proof. The generating function of the popularity is directly deduced from the bivariate generating function of pattern distribution by calculating

$$\left.\frac{\partial A^1(x, y)}{\partial y}\right|_{y=1} = \left.\frac{\partial A^2(x, y)}{\partial y}\right|_{y=1}.$$

□

The statistic $\mathbf{des}_2$ has a different distribution from $\mathbf{des}_0$ and $\mathbf{des}_1$, but the three patterns p_0, p_1, p_2 have the same popularity. Below we present a bijection on S_n that transports the statistic $\mathbf{des}_2$ to the statistics $\mathbf{pcyc} = \mathbf{cyc} - \mathbf{fix}$.

$k\backslash n$	1	2	3	4	5	6	7	8
0	1	1	1	1	1	1	1	1
1		1	5	20	84	409	2365	16064
2				3	35	295	2359	19670
3						15	315	4480
4								105
...								...
$\sum$	1	2	6	24	120	720	5040	40320

Table 1: Number of n-length permutations having k occurrences of p_2 for $0 \leq k \leq 4$ and $1 \leq n \leq 8$.

Theorem 2.3. *There is a one-to-one correspondence ϕ on S_n such that for any $\pi \in S_n$, we have*

$$\mathbf{des}_2\ \pi = \mathbf{pcyc}\ \phi(\pi).$$

Proof. Let π be a permutation of length n having k occurrences of p_2. We decompose

$$\pi = B_0\pi_{i_1}A_1B_1\pi_{i_2}A_2B_2\pi_{i_3}\dots\pi_{i_k}A_kB_k,$$

where

- $\pi_{i_1} < \pi_{i_2} < \dots < \pi_{i_k}$ are the tops of the occurrences of p_2, *i.e.* values $\pi_{i_j} > \pi_{i_j+1}$ such that there does not exist $\ell < i_j$ such that $\pi_\ell > \pi_{i_j}$,
- A_j is a maximal sequence such that all its values are lower than π_{i_j},
- for $0 \leq j \leq k$, B_j is an increasing sequence such that $\pi_{i_j} < \min B_j$ and $\max B_j < \pi_{i_{j+1}}$.

Now we construct an n-length permutation $\phi(\pi)$ with k pure cycles as follows:

$$\phi(\pi) = \langle \pi_{i_1}A_1\rangle \cdot \langle \pi_{i_2}A_2\rangle \cdots \langle \pi_{i_k}A_k\rangle.$$

For instance, if $\pi = 125346879$ then $\phi(\pi) = \langle 5,3,4\rangle \cdot \langle 8,7\rangle$. The map ϕ is clearly a bijection on S_n such that $\mathbf{des}_2\ \pi$ equals the number of pure cycles in $\phi(\pi)$. □

Note that ϕ^{-1} is closely related to the Foata fundamental transformation [9].

3. The statistic pex of pure excedances

In order to prove the equidistribution of **pex** and $\mathbf{des}_2$, regarding Theorem 2.3, it suffices to construct a bijection on S_n that transports pure excedances to pure cycles. Here, we first exhibit a bijection on the set D_n of n-length derangements (permutations without fixed points), then we extend it to the set of all permutations S_n.

Any permutation $\pi \in S_n$ is uniquely decomposed as a product of transpositions of the following form:

$$\pi = \langle t_1, 1\rangle \cdot \langle t_2, 2\rangle \cdots \langle t_n, n\rangle$$

where t_i are integers such that $1 \leq t_i \leq i$. The transposition array of π is defined by $T(\pi) = t_1t_2\dots t_n$, which induces a bijection $\pi \longmapsto T(\pi)$ from S_n to the product set $T_n = [1] \times [2] \times \cdots \times [n]$. By Lemma 1 from [1], the number of cycles of a permutation π is given by the number of fixed points in $T(\pi)$. Moreover, it is straightforward to check the two following properties:

- if $t_i = i$, then $\pi_i = i$ if and only if there is no number $j > i$ such that $t_j = t_i = i$;
- if $t_i = i$ and $\pi_i \neq i$, then i is the minimal element of a cycle of length at least two in π.

So, we deduce the following lemma.

Lemma 3.1. *The transposition array $t_1t_2\dots t_n \in T_n$ corresponds to a derangement if and only if: $t_i = i \Rightarrow$ there is a $j > i$ such that $t_j = i$.*

Given a derangement $\pi = \pi_1\pi_2\dots\pi_n \in D_n$ and its graphical representation $\{(i, \pi_i), i \in [n]\}$. We say that the square $(i, j) \in [n] \times [n]$ is *free* if all following conditions hold:

(i) Neither π_i nor i is a position of a pure excedance;

(ii) (i, j) is not on the first diagonal, *i.e.* $j \neq i$;

(iii) there does not exist $k > i$ such that $\pi_k = j$;

(iv) j is not a pure excedance such that $j < i$ and $\pi^{-1}(j) < i$;

(v) there does not exist $k < i$, with $\pi_k = j > i$ such that all values of the interval $[i, j-1]$ appear on the right of π_i in π.

Whenever at least one of the statements above is not satisfied, we say that the square (i, j) is *unfree*. Notice that if i and π_i are not the positions of a pure excedance, then the square (i, π_i) is always free. So, for a column i of the graphical representation of π such that i and π_i are not the positions of a pure excedance, we label by j the jth free square from the bottom to the top. We refer to Figure 2 for an example of this labeling.

Now we define the map λ from D_n to the set $T_n^\bullet$ of transposition arrays of length n satisfying the property of Lemma 3.1.

For a permutation $\pi = \pi_1\pi_2\dots\pi_n \in D_n$, we label its graphical representation as defined above, and $\lambda(\pi) = \lambda_1\lambda_2\dots\lambda_n$ is obtained as follows:

- if i is a pure excedance in π, then we set $\lambda_i = i$ and $\lambda_{\pi^{-1}(i)} = i$;
- otherwise, λ_i is the sum of the label of the free square (i, π_i) with the number of pure excedances $k < i$ such that $\pi^{-1}(k) < i$.

For instance, if $\pi = 6\ 8\ 12\ 5\ 4\ 7\ 3\ 2\ 11\ 1\ 9\ 10$ then we obtain $\lambda(\pi) = 1\ 1\ 2\ 4\ 4\ 2\ 1\ 1\ 9\ 1\ 9\ 10$ (see Figure 2).

Let us consider i, $1 \le i \le n$. If i is a pure excedance of π, then we fix $\lambda_i = i$ and $\lambda_{\pi^{-1}(i)} = i < \pi^{-1}(i)$. Otherwise, the square (i, i) is unfree, and all squares (i, π_k), $i+1 \le k \le n$, are unfree, which implies that the number of free squares in the ith column is less than or equal to i. This means that $\lambda(\pi)$ lies in T_n. Note that, by construction, all labeled squares do not correspond to any pure excedance. Now let us prove that the square (i, π_i) cannot be labeled i. Indeed, if $\pi_i < i$ then the label of (i, π_i) is necessarily at most $\pi_i \le i-1$; otherwise, if $\pi_i > i$ then the fact that i is not a pure excedance implies that there is $\pi_j \in [i, \pi_i - 1]$ with $j < i$. Let us choose the lowest j with this property. Using (v), the square (i, j) is unfree, which implies that the label of (i, π_i) is less than or equal to n minus the minimal number of unfree squares (i, j) in column i, that is $n - (n - i + 1) = i - 1$. Moreover, the transposition array $\lambda(\pi)$ has exactly **pex** π fixed points, and for any fixed point i there necessarily exists $j = \pi^{-1}(i) > i$ such that $\lambda_j = \lambda_i = i$. This implies that $\lambda(\pi) \in T_n^\bullet$.

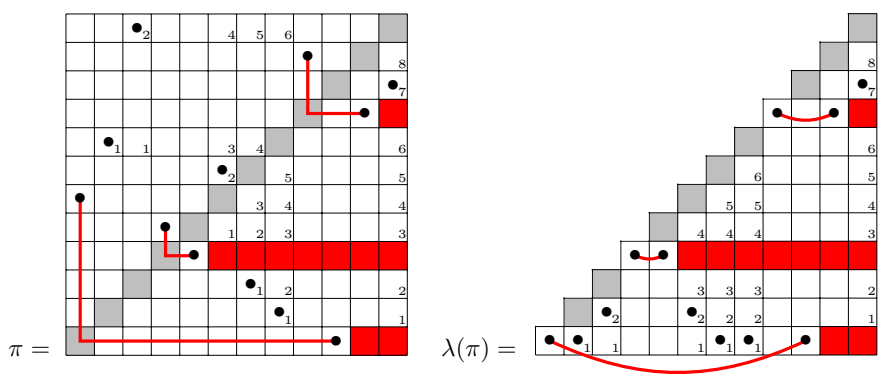


Figure 2: Illustration of the bijection λ for $\pi = 6\ 8\ 12\ 5\ 4\ 7\ 3\ 2\ 11\ 1\ 9\ 10$ and $\lambda(\pi) = 1\ 1\ 2\ 4\ 4\ 2\ 1\ 1\ 9\ 1\ 9\ 10$.

Theorem 3.1. *The map λ from D_n to $T_n^\bullet$ is a bijection such that*

$$\mathbf{pex}\ \pi = \mathbf{fix}\ \lambda(\pi).$$

Proof. Since the cardinality of $T_n^\bullet$ equals that of D_n, and the image of D_n by λ is contained in $T_n^\bullet$, it suffices to prove the injectivity.

Let π and σ, $\pi \neq \sigma$, be two derangements in D_n. If π and σ do not have the same pure excedances, then, by construction, $\lambda(\pi)$ and $\lambda(\sigma)$ do not have the same fixed points, and thus $\lambda(\pi) \neq \lambda(\sigma)$.

Now, let us assume that π and σ have the same pure excedances. If there is a pure excedance i such that $\pi^{-1}(i) \neq \sigma^{-1}(i)$ then the definition implies $\lambda(\pi) \neq \lambda(\sigma)$. Otherwise the two permutations have the same pure excedances i, and for each of them we have $\pi^{-1}(i) = \sigma^{-1}(i)$. Let j be the greatest integer such that $\pi_j \neq \sigma_j$ (without loss of generality, we assume $\pi_j < \sigma_j$). In this case, j is not a pure excedance for the two permutations. Thus, $\lambda(\pi)_j$ (resp. $\lambda(\sigma)_j$) is the sum of the label of (j, π_j) (resp. (j, σ_j)) with the number of pure excedances

$k < j$ such that $\pi^{-1}(k) < j$ (resp. $\sigma^{-1}(k) < j$). Since we have $\pi_j < \sigma_j$, the label of (j, π_j) is less than the label of (j, σ_j), and the number of pure excedances $k < j$ such that $\pi^{-1}(k) < j$ is less than or equal to the number of pure excedances $k < j$ such that $\sigma^{-1}(k) < j$. Then we have $\lambda(\pi)_j < \lambda(\sigma)_j$. Then λ is an injective map, and thus a bijection. □

Theorem 3.2. *There is a one-to-one correspondence ψ on the set D_n of n-length derangements such that for any $\pi \in D_n$,*

$$\mathbf{pex}\ \pi = \mathbf{cyc}\ \psi(\pi).$$

Proof. Considering Theorem 2.3 and Theorem 3.1, we define for any $\pi \in D_n$, $\psi(\pi) = \phi(\sigma)$ where σ is the permutation having $\lambda(\pi)$ as transposition array. □

Theorem 3.3. *The two bistatistics* $(\mathbf{pex}, \mathbf{fix})$ *and* $(\mathbf{pcyc}, \mathbf{fix})$ *are equidistribiuted on S_n.*

Proof. Considering Theorem 3.2, we define the map $\bar{\psi}$ on S_n. Let π' be the permutation obtained from π by deleting all fixed points and after rescaling as a permutation. Let $I = \{i_1, i_2, \ldots, i_k\}$ be the set of fixed points of π. Then, we set $\pi'' = \psi(\pi')$. So, $\sigma = \bar{\psi}(\pi)$ is obtained from π'' by inserting fixed points $i \in I$ after a shift of all other entries in order to produce a permutation in S_n. By construction, we have $\mathbf{pex}\ \pi = \mathbf{pcyc}\ \sigma$ and $\mathbf{fix}\ \pi = \mathbf{fix}\ \sigma$ which completes the proof. □

A byproduct of this theorem is

Corollary 3.1. *The statistics* $\mathbf{cyc}$ *and* $\mathbf{pex} + \mathbf{fix}$ *are equidistributed on S_n.*

Also, a direct consequence of Theorems 2.3 and 3.3 is

Theorem 3.4. *The two statistics* $\mathbf{pex}$ *and* $\mathbf{des}_2$ *are equidistributed on S_n.*

Notice that Foata's first transformation is not a candidate for proving the equidistribution of $\mathbf{pex}$ and $\mathbf{des}_2$, while it transports $\mathbf{exc}$ to $\mathbf{des}$. Combining Theorem 3.4 and Corollary 2.1 we have the following.

Corollary 3.2. *For $0 \leq i \leq 2$, the patterns p_i and pex are equipopular on S_n (see Sequence A001705 in [15]).*

Finally, we present two conjectures for future works.

Conjecture 3.1. *The two bistatistics* $(\mathbf{des}_2, \mathbf{cyc})$ *and* $(\mathbf{pex}, \mathbf{cyc})$ *are equidistributed on S_n.*

Conjecture 3.2. *The two bistatistics* $(\mathbf{des}_2, \mathbf{des})$ *and* $(\mathbf{pex}, \mathbf{exc})$ *are equidistributed on S_n.*

It is interesting to remark that $(\mathbf{des}, \mathbf{cyc})$ and $(\mathbf{exc}, \mathbf{cyc})$ are not equidistributed. Indeed, there are 3 permutations in S_3 having $\mathbf{exc} = 1$ and $\mathbf{cyc} = 2$, namely 132, 213, 321, but only 2 permutations with $\mathbf{des} = 1$ and $\mathbf{cyc} = 2$, videlicet 132 and 213. So, if the Conjectures 3.1 and 3.2 are true then their proofs are probably independent.

Acknowledgements

We would like to greatly thank Vincent Vajnovszki for having offered us Conjecture 3.2 and the anonymous referees for their helpful comments and suggestions.

References

[1] J.-L. Baril, *Statistics-preserving bijections between classical and cycle permutations*, Inform. Process. Lett. 113 (2013), 17–22.

[2] J.-L. Baril and S. Kirgizov, *The pure descent statistic on permutations*, Discrete Math. 340:10 (2017), 2250–2558.

[3] J.-L. Baril and V. Vajnovszki, *Popularity of patterns over d-equivalence classes of words and permutations*, Theoret. Comput. Sci. 814 (2020), 249–258.

[4] F. Bergeron, N. Bergeron, R. B. Howlett and D. E. Taylor, *A decomposition of the descent algebra of a finite Coxeter group*, J. Algebraic Combin. 1 (1992), 23–44.

[5] M. Bóna, *Surprising symmetries in objects counted by Catalan numbers*, Electron. J. Combin. 19:1 (2012), Article P62.

[6] P. Brändén and A. Claesson, *Mesh patterns and the expansion of permutation statistics as sums of permutation patterns*, Electron. J. Combin. 18:2 (2011), Article P5.

[7] S. Elizalde, *Descent sets of cyclic permutations*, Adv. in Appl. Math. 47.4 (2011), 688–709.

[8] S. Elizalde and J. M. Troyka, *The number of cycles with a given descent set*, Sém. Lothar. Combin. 80 (2018) Article #8.

[9] D. Foata and M. P. Schützenberger, *Théorie Géométrique des Polynômes Euleriens*, Lecture Notes in Math. 138, Springer-Verlag, Berlin, 1970.

[10] D. Foata and G.-N. Han, *Decreases and descents in words*, Sém. Lothar. Combin. 58 (2007), Article B58a.

[11] A. Garsia and C. Reutenauer, *A decomposition of Solomon's descent algebra*, Adv. Math. 77 (1989), 189–262.

[12] I. Gessel and G. Viennot, *Binomial determinants, paths, and hook length formulae*, Adv. Math. 58 (1985), 300–321.

[13] S. Kitaev and P.B. Zhang, *Distributions of mesh patterns of short lengths*, Adv. in Appl. Math. 110 (2019), 1–32.

[14] P.A. MacMahon, *Combinatory Analysis*, Volumes 1 and 2, Cambridge Univ. Press, Cambridge, UK, 1915 (reprinted by Chelsea, New York, 1955).

[15] N.J.A. Sloane, *The On-line Encyclopedia of Integer Sequences*, available electronically at `http://oeis.org`.

[16] R.P. Stanley, *Enumerating Combinatorics*, Volume 2, Cambridge University Press, 1999.

[17] R.M. Wilson and J.H. van Lint, *A course in combinatorics*, Volume I, Cambridge University Press, 2002.

Information Processing Letters 171 (2021) 106138

Contents lists available at ScienceDirect

Information Processing Letters

www.elsevier.com/locate/ipl

Catalan and Schröder permutations sortable by two restricted stacks

Jean-Luc Baril [a], Giulio Cerbai [b,1], Carine Khalil [a], Vincent Vajnovszki [a,*]

[a] LIB, Université de Bourgogne Franche-Comté, B.P. 47 870, 21078 Dijon Cedex, France
[b] Dipartimento di Matematica e Informatica "U. Dini", University of Firenze, Firenze, Italy

ARTICLE INFO

Article history:
Received 14 September 2020
Received in revised form 15 February 2021
Accepted 2 May 2021
Available online 19 May 2021
Communicated by Amit Chakrabarti

Keywords:
Stack sorting/two stacks in series
Pattern avoiding permutations/machines
Catalan and the Schröder numbers
Combinatorial problems

ABSTRACT

Pattern avoiding machines were introduced recently by Claesson, Cerbai and Ferrari as a particular case of the two-stacks in series sorting device. They consist of two restricted stacks in series, ruled by a right-greedy procedure and the stacks avoid some specified patterns. Some of the obtained results have been further generalized to Cayley permutations by Cerbai, specialized to particular patterns by Defant and Zheng, or considered in the context of functions over the symmetric group by Berlow. In this work we study pattern avoiding machines where the first stack avoids a pair of patterns of length 3 and investigate those pairs for which sortable permutations are counted by the (binomial transform of the) Catalan numbers and the Schröder numbers.

© 2021 Published by Elsevier B.V.

1. Introduction

Pattern avoiding machines were recently introduced in [7] in attempt to gain a better understanding of sortable permutations using stacks in series. They consist of two restricted stacks in series, equipped with a right-greedy procedure, where the first stack avoids a fixed pattern, reading the elements from top to bottom; and the second stack avoids the pattern 21 (which is a necessary condition for the machine to sort permutations). The authors of [7] provide a characterization of the avoided patterns for which sortable permutations do not form a class, and they show that those patterns are enumerated by the Catalan numbers. For specific patterns, such as 123 and the decreasing pattern of any length, a geometrical description of sortable permutations is also obtained. The pattern 132 has been solved later in [8]. Some of these results have been further generalized to Cayley permutations in [9]. More recently, Berlow [5] explores a single stack version of pattern avoiding machines, where the stack avoids a set of patterns and the sorting process is regarded as a function. Analogous machines, but based on the notion of consecutive patterns, have been introduced and discussed in [10].

In this work we study a variant of pattern-avoiding machines where the first stack avoids (σ, τ), a pair of patterns of length three. Following [7], we call it (σ, τ)-machine. More specifically, we restrict ourselves to those pairs of patterns for which sortable permutations are counted by either the Catalan numbers or two of their close relatives: the binomial transform of Catalan numbers and the Schröder numbers. For the pair $(132, 231)$ we show that sortable permutations are those avoiding 1324 and 2314, a set whose enumeration is given by the large Schröder numbers. Under certain conditions on the avoided patterns, the output of the first stack is bijectively related to its input (see [5,9]): it follows that for three pairs of patterns, namely $(123, 213)$, $(132, 312)$ and $(231, 321)$, sortable permutations are counted by the Cata-

* Corresponding author.
E-mail addresses: barjl@u-bourgogne.fr (J.-L. Baril), giulio.cerbai@unifi.it (G. Cerbai), carine.khalil@u-bourgogne.fr (C. Khalil), vvajnov@u-bourgogne.fr (V. Vajnovszki).

[1] G.C. is member of the INdAM Research group GNCS; he is partially supported by INdAM-GNCS 2020 project "Combinatoria delle permutazioni, delle parole e dei grafi: algoritmi e applicazioni".

https://doi.org/10.1016/j.ipl.2021.106138
0020-0190/© 2021 Published by Elsevier B.V.

lan numbers. This result was conjectured in [3] and settled independently in [4,5]. For the pair $(123, 132)$, we prove that sortable permutations are those avoiding the patterns 2314, 3214, 4213 and the generalized pattern $[24\bar{1}3$. We prove that sortable permutations are enumerated by the Catalan numbers by showing that the distribution of the first element is given by the well-known Catalan triangle. Finally, we show that for the pair $(123, 312)$ the corresponding counting sequence is the binomial transform of Catalan numbers.

This paper is the extended version of the conference presentation [4] and some of the presented results previously appear in [3].

2. Notations and some preliminary results

We start by recalling some classical definitions about pattern avoidance on permutations (see [12] for a more detailed introduction). Denote by $\mathfrak{S}_n$ the set of permutations of length n and let $\mathfrak{S} = \cup_{n\geq 0}\mathfrak{S}_n$. Given two permutations σ of length k and $\pi = \pi_1 \cdots \pi_n$, we say that π *contains* the pattern σ if π contains a subsequence $\pi_{i_1} \cdots \pi_{i_k}$, with $i_1 < i_2 < \cdots < i_k$, which is order isomorphic to σ. In this case, we say that $\pi_{i_1} \cdots \pi_{i_k}$ is an *occurrence* of the pattern σ in π. Otherwise, we say that π *avoids* σ.

We say that π contains an occurrence of the (generalized) pattern $[\sigma$ if π contains an occurrence of σ that involves the first element π_1 of π. For instance, an occurrence of $[12$ in π corresponds to a pair of elements $\pi_1\pi_i$, with $i > 1$ and $\pi_i > \pi_1$. A *barred pattern* $\tilde{\sigma}$ is a pattern where some entries are barred. Let σ be the classical pattern obtained by removing all the bars from $\tilde{\sigma}$. Let τ be the pattern which is order isomorphic to the non-barred entries of $\tilde{\sigma}$ (i.e. obtained from $\tilde{\sigma}$ by removing all the barred entries and suitably rescaling the remaining elements). A permutation π avoids $\tilde{\sigma}$ if each occurrence of τ in π can be extended to an occurrence of σ. For instance, a permutation π avoids the pattern $[24\bar{1}3$ if for any subsequence $\pi_1\pi_i\pi_j$, with $1 < i < j$ and $\pi_1 < \pi_j < \pi_i$, there is an index t, $i < t < j$, such that $\pi_1\pi_i\pi_t\pi_j$ is an occurrence of 2413.

Given a set of (generalized) patterns T, denote by $\mathtt{Av}_n(T)$ the set of permutations in $\mathfrak{S}_n$ avoiding each pattern in T. Similarly, let $\mathtt{Av}(T) = \cup_{n\geq 0}\mathtt{Av}_n(T)$. If $T = \{\sigma\}$ is a singleton, we write $\mathtt{Av}_n(\sigma)$ and $\mathtt{Av}(\sigma)$. In his celebrated book [13], Knuth gave the following characterization of stack sortable permutations, which is often considered the starting point of stack sorting and permutation patterns disciplines.

Proposition 1 *([13]). A permutation π is sortable using a classical stack (that is, a 21-avoiding stack) if and only if π avoids the pattern 231.*

Let T be a set of patterns. A T-stack is a stack that is not allowed to contain an occurrence of any pattern in T, reading its elements from top to bottom. Given a permutation π, denote by $\mathrm{out}^T(\pi)$ the permutation obtained after passing π through the T-avoiding stack by applying a greedy procedure, i.e. by always pushing the next element of the input, unless it creates an occurrence of a forbidden pattern inside the stack. Denote by $\mathrm{Sort}_n(T)$ the set of length n permutations that are sortable by the T-machine, that is, by passing π through the T-avoiding stack and then through the 21-avoiding stack. Permutations in $\mathrm{Sort}_n(T)$ are called *T-sortable*, and $\mathrm{Sort}(T)$ is the set of T-sortable permutations of any length. As a consequence of Proposition 1, $\mathrm{Sort}(T)$ consists precisely of those permutations π for which $\mathrm{out}^T(\pi)$ avoids 231. To ease notations, if T is either a singleton $T = \{\sigma\}$ or a pair of patterns $T = \{\sigma, \tau\}$, we will omit the curly brackets from the above notations. For instance, we will write $\mathrm{Sort}(\sigma, \tau)$ instead of $\mathrm{Sort}(\{\sigma, \tau\})$.

The authors of [7] showed that if π is a 12-sortable permutation of length n, then $\mathrm{out}^{12}(\pi) = n(n-1)\cdots 1$. Moreover, by Proposition 1 and applying the complement operation on the processed permutation, we have that $\mathrm{Sort}(12) = \mathtt{Av}(213)$. In order to refer to this result later, we state it below in a slightly more general form. A *partial permutation* of n is an injection $\pi : \{1, 2, \ldots, k\} \to \{1, 2, \ldots, n\}$, for some $0 \leq k \leq n$, and the integer k is said to be the length of π. We let a 12-stack act on a partial permutation π of n in the natural way by identifying π with the list of its images.

Proposition 2. *If π is a partial permutation of n which is 12-sortable, then $\mathrm{out}^{12}(\pi)$ is the decreasing rearrangement of the symbols of π. Moreover, π is 12-sortable if and only if it avoids 213.*

An entry π_i of a permutation π is a *left-to-right minimum* if $\pi_i < \pi_j$, for each $j < i$. The *left-to-right minima decomposition* (briefly *ltr-min decomposition*) of π is $\pi = m_1B_1m_2B_2\cdots m_tB_t$, where $m_1 > m_2 > \cdots > m_t$ are the ltr-minima of π and the *block* B_i contains the elements of π between m_i and m_{i+1}, for $i = 1, \ldots, t-1$. The last block B_t contains the elements that follow m_t in π. Note that $m_t = 1$. The notion of *left-to-right maximum* of a permutation π is defined similarly. The *ltr-max decomposition* of π is $\pi = M_1B_1M_2B_2\cdots M_tB_t$, where $M_1 < M_2 < \cdots < M_t$ are the ltr-maxima of π. In this case $M_t = n$, where n is the length of π.

Finally, the sequence $(c_n)_{n\geq 0}$, pervasive in this paper, is the sequence of Catalan numbers $c_n = \frac{1}{n+1}\binom{2n}{n}$ (A000108 in [15]).

3. Pair (132, 231)

This section is devoted to the analysis of the $(123, 231)$-machine.

Theorem 1. *Consider the $(132, \sigma)$-machine, where $\sigma = \sigma_1 \cdots \sigma_{k-1}\sigma_k \in \mathfrak{S}_k$, with $k \geq 3$ and $\sigma_{k-1} > \sigma_k$. Given a permutation π of length n, let $m_1B_1 \cdots m_tB_t = \pi$ be its ltr-min decomposition. Then:*

1. *Every time a ltr-minimum m_i is pushed into the $(132, \sigma)$-stack, the $(132, \sigma)$-stack contains the elements $m_{i-1}, \ldots, m_2, m_1$, reading from top to bottom. Moreover, we have*

$$\mathrm{out}^{132,\sigma}(\pi) = \tilde{B}_1 \cdots \tilde{B}_t m_t \cdots m_1,$$

where $\tilde{B}_i$ is a rearrangement of B_i.

2. *If π is $(132,\sigma)$-sortable, then $\tilde{B}_i$ is decreasing for each i. Moreover, for each $i \le t-1$, we have $B_i > B_{i+1}$ (i.e. $x > y$ for each $x \in B_i$, $y \in B_{i+1}$).*

Proof. 1. Let us consider the evolution of the $(132,\sigma)$-stack on input π. Note that, since $k \ge 3$, the element m_1 remains at the bottom of the $(132,\sigma)$-stack until the end of the process. Now, if B_1 is not empty then for each $x \in B_1$, the elements $m_2 x m_1$ form an occurrence of 132. Therefore the block B_1 is extracted before m_2 enters the $(132,\sigma)$-stack. After m_2 is pushed, the $(132,\sigma)$-stack contains $m_2 m_1$, reading from top to bottom. Since $m_2 < m_1$, but $\sigma_{k-1} > \sigma_k$ by hypothesis, m_2 cannot play the role of either σ_{k-1} in an occurrence of σ or of 3 in an occurrence of 132. Thus m_2 remains at the bottom of the $(132,\sigma)$-stack until the end of the sorting procedure. The thesis follows by iterating the same argument on each block B_i, for $i \ge 2$.

2. Suppose that π is $(132,\sigma)$-sortable. Assume, for a contradiction, that $\tilde{B}_i$ is not decreasing, for some i. Then there are two consecutive elements $x < y$ in $\tilde{B}_i$. Therefore, by what proved above, $\mathrm{out}^{132,\sigma}(\pi)$ contains an occurrence xym_t of 231, which is impossible due to Proposition 1. Finally, suppose that $x < y$, for $x \in B_i$ and $y \in B_{i+1}$. Then xym_t is an occurrence of 231 in $\mathrm{out}^{132,\sigma}(\pi)$, a contradiction. □

Theorem 1 and Proposition 2 guarantee that if $\pi = m_1 B_1 \cdots m_t B_t$ is the ltr-min decomposition of a $(132,231)$-sortable permutation π, then (with the notation above) $\tilde{B}_i = \mathrm{out}^{12}(B_i)$, for each i. However, this is true even when the sortability requirement is relaxed.

Lemma 1. *Let $\pi = m_1 B_1 \cdots m_t B_t$ be the ltr-min decomposition of a permutation π. Write $\mathrm{out}^{132,231}(\pi) = \tilde{B}_1 \cdots \tilde{B}_t m_t \cdots m_1$ as in Theorem 1. Then $\tilde{B}_i = \mathrm{out}^{12}(B_i)$, for each i.*

Proof. Consider the instant immediately after m_i is pushed into the $(132,231)$-stack and the non-empty block B_i has to be processed, for some i. By Theorem 1, at this point the $(132,231)$-stack contains $m_i, m_{i-1}, \ldots, m_1$, reading from top to bottom. We want to show that the behavior of the $(132,231)$-stack on B_i is equivalent to the behavior of an empty 12-stack on input B_i. We prove that the $(132,231)$-stack performs the pop operation of some $x \in B_i$ if and only if the 12-stack does the same. If either the next element of the input is m_{i+1} or x is the last element of π to be processed, then both the $(132,231)$-stack and the 12-stack perform a pop operation, as desired. Otherwise, suppose the next element of the input is y, for some y in the same block B_i, and the $(132,231)$-stack pops the element $x \in B_i$. This means that the $(132,231)$-stack contains two elements z, w, with z above w, such that yzw is an occurrence of either 132 or 231. Note that, since $z > w$, z is not a ltr-minimum. Therefore yz is an occurrence of 12 and the 12-stack performs a pop operation, as desired. Conversely, suppose that the 12-stack pops the element x, with $y \in B_i$ the next element of the input. This implies that the 12-stack contains an element z such that $z > y$. Therefore yzm_i is an occurrence of 231 and the $(132,231)$-stack performs a pop operation, as desired. □

Corollary 1. *Let $\pi = m_1 B_1 \cdots m_t B_t$ be the ltr-min decomposition of a permutation π. Then the following are equivalent.*

1. *B_i avoids 213 and $B_i > B_{i+1}$, for each i.*
2. *π is $(132,231)$-sortable.*
3. *$\pi \in \mathrm{Av}(1324, 2314)$.*

Proof. Combining the first point in Theorem 1 and Lemma 1 we have:

$$\mathrm{out}^{132,231}(\pi) = \mathrm{out}^{12}(B_1) \cdots \mathrm{out}^{12}(B_t) m_t \cdots m_1.$$

We will use this decomposition of $\mathrm{out}^{132,231}(\pi)$ throughout the rest of the proof.

[$1 \Rightarrow 2$] Suppose, for a contradiction, that $\mathrm{out}^{132,231}(\pi)$ contains an occurrence bca of 231. Note that, since $c > a$, while $m_t < \cdots < m_1$, c is not a ltr-minimum of π (and thus neither is b). Now, if b and c are in the same block B_j, then $\mathrm{out}^{12}(B_j)$ is not decreasing. Thus, by Proposition 2, B_j contains 213, which is a contradiction. Otherwise, if $b \in B_j$ and $c \in B_k$, with $j < k$, then we have a contradiction with the hypothesis $B_i > B_{i+1}$ for each i.
[$2 \Rightarrow 3$] Suppose, for a contradiction, that $\pi \notin \mathrm{Av}(1324, 2314)$. First, suppose that π contains an occurrence $acbd$ of 1324. Observe that b, c, d are not ltr-minima of π. Now, if b and d are in the same block B_j of π, for some j, then B_j contains an occurrence cbd of 213. Therefore $\mathrm{out}^{12}(B_j)$ contains an occurrence of 231 due to Proposition 2, which contradicts the hypothesis. Otherwise, if $b \in B_j$ and $d \in B_k$, for some $j < k$, then $\mathrm{out}^{132,231}(\pi)$ contains an occurrence bdm_k of 231, again a contradiction. The pattern 2314 can be addressed analogously, so we leave it to the reader.
[$3 \Rightarrow 1$] Let $\pi \in \mathrm{Av}(1324, 2314)$. If B_i contains an occurrence bac of 213, then π contains an occurrence $m_i bac$ of 1324, which is impossible. Otherwise, if π contains two elements $x \in B_j$, $y \in B_k$, with $x < y$ and $j < k$, then $m_j x m_k y$ is an occurrence of 2314, contradicting the hypothesis. □

The enumeration of $\mathrm{Av}(1324, 2314)$ (or a symmetry of these patterns) can be found for instance in [2,16]. Note that in [1], the authors provide a constructive bijection between these permutations and Schröder paths.

Corollary 2. *Permutations of length n in $Sort(132,231)$ are enumerated by the large Schröder numbers (sequence* A006318 *in [15]).*

4. The $(\sigma, \hat{\sigma})$-machine

For a permutation σ of length two or more, denote by $\hat{\sigma}$ the permutation obtained from σ by interchanging its first two entries. Let us regard a (σ,τ)-stack as an operator $\mathrm{out}^{\sigma,\tau} : \mathfrak{S} \to \mathfrak{S}$. By conveniently modifying the proof of Corollary 4.5 in [9] (stated in the context of Cayley permutations), we have that $\mathrm{out}^{\sigma,\tau}$ is a length preserving bijection on $\mathfrak{S}$ if and only if $\tau = \hat{\sigma}$. More generally, Berlow [5] showed that for a set T of patterns, out^T is a

length preserving bijection on $\mathfrak{S}$ if and only if T is closed under the $\hat{}$ operator. In order for the paper to be self-contained, we shall give the following result, which is easier to prove (although weaker): $\mathrm{out}^{\sigma,\hat{\sigma}}$ is a bijection for any pattern σ. An immediate consequence will be Theorem 2 below.

Let N^* be the set of finite length integer sequences. The action of the (σ,τ)-stack on input π can be naturally represented as a sequence of triples $(r;s;t) \in (N^*)^3$, where r is the current content of the output, s is the current content of the (σ,τ)-stack (read from top to bottom) and t is the current content of the input. The triple $(r;s;t)$ is said to be a *state of passing* of π through the (σ,τ)-stack. Clearly, r is a prefix of $\mathrm{out}^{\sigma,\tau}(\pi)$, t is a suffix of π, the initial state is $(\lambda;\lambda;\pi)$ and the final one is $(\mathrm{out}^{\sigma,\tau}(\pi);\lambda;\lambda)$, where λ is the empty sequence. Moreover a non-final state $(p_1p_2\cdots p_a; s_1s_2\cdots s_b; t_1t_2\cdots t_c)$ is followed by either the state

$$(p_1p_2\cdots p_as_1; s_2\cdots s_b; t_1t_2\cdots t_c),$$

if a pop operation is performed next, or

$$(p_1p_2\cdots p_a; t_1s_1s_2\cdots s_b; t_2\cdots t_c),$$

if a push operation is performed next.

For $p = p_1\cdots p_n \in \mathbb{N}^n$, we denote by p^{r} the *reverse* of p, that is $p^{\mathrm{r}} = p_n\cdots p_1$. We wish to show that the behavior of the $(\sigma,\hat{\sigma})$-stack on π is strictly related to its behavior on $\left(\mathrm{out}^{\sigma,\hat{\sigma}}(\pi)\right)^{\mathrm{r}}$. More precisely, if $o_1\cdots o_{2n}$ is the sequence of push/pop operations performed when π is passed through a $(\sigma,\hat{\sigma})$-stack, then $o'_{2n}\cdots o'_1$ is the sequence of push/pop operations performed when $\left(\mathrm{out}^{\sigma,\hat{\sigma}}(\pi)\right)^{\mathrm{r}}$ is passed through the $(\sigma,\hat{\sigma})$-stack, where o'_i is a push (resp. pop) operation if o_i is a pop (resp. push) operation. This can be equivalently expressed by saying that the state $(p;s;t)$ is followed by $(u;v;w)$ if and only if the state $(w^{\mathrm{r}};v;u^{\mathrm{r}})$ is followed by $(t^{\mathrm{r}};s;p^{\mathrm{r}})$.

Lemma 2. *Consider the action of the $(\sigma,\hat{\sigma})$-stack. Let $p,s,t \in \mathbb{N}^*$ and $x \in \mathbb{N}$.*

1. *If the state (p,xs,t) is followed by the state (px,s,t) (and thus a pop operation is performed) then the state $(t^{\mathrm{r}},s,xp^{\mathrm{r}})$ is followed by the state $(t^{\mathrm{r}},xs,p^{\mathrm{r}})$ (and thus a push operation is performed).*
2. *If the state (p,s,xt) is followed by the state (p,xs,t) (and thus a push operation is performed), then the state $(t^{\mathrm{r}},xs,p^{\mathrm{r}})$ is followed by the state $(t^{\mathrm{r}}x,s,p^{\mathrm{r}})$ (and thus a pop operation is performed).*

Proof. 1. Since xs is the content of the $(\sigma,\hat{\sigma})$-stack in the state (p,xs,t), we have that xs avoids σ and $\hat{\sigma}$. Thus a push operation is performed if s is the content of the $(\sigma,\hat{\sigma})$-stack and x is the next element of the input.

2. If p is empty, the statement holds. Otherwise, let $p = p_1\cdots p_a$ and $s = s_1\cdots s_b$. Observe that p_a is the last element that has been extracted from the $(\sigma,\hat{\sigma})$-stack before x enters. Therefore, when p_a is extracted, p_a plays the role of either σ_2 in an occurrence of σ or of $\hat{\sigma}_2$ in an occurrence of $\hat{\sigma}$. More precisely, one of the following four cases hold. We show the details for the first case only, the others being similar. Let z be the length of σ.
   - $s_\ell p_a s_{i_3}\cdots s_{i_z}$ is an occurrence of σ, for some $\ell \geq 1$ and $\ell < i_3 < \cdots < i_z$. Then $p_a s_\ell s_{i_3}\cdots s_{i_z}$ is an occurrence of $\hat{\sigma}$ and therefore a pop operation is performed when p_a is the next element of the input and xs is the content of the $(\sigma,\hat{\sigma})$-stack, as desired.
   - $s_\ell p_a s_{i_3}\cdots s_{i_z}$ is an occurrence of $\hat{\sigma}$, for some $\ell \geq 1$ and $\ell < i_3 < \cdots < i_z$.
   - $xp_a s_{i_3}\cdots s_{i_z}$ is an occurrence of σ, for some $i_3 < \cdots < i_z$.
   - $xp_a s_{i_3}\cdots s_{i_z}$ is an occurrence of $\hat{\sigma}$, for some $i_3 < \cdots < i_z$. $\square$

A straightforward consequence of the previous lemma is that the map $\left(\mathrm{out}^{\sigma,\hat{\sigma}}\right)^{\mathrm{r}} : \mathfrak{S} \to \mathfrak{S}$ is its own inverse, and thus a bijection. More specifically, for any permutation π, we have $\left(\mathrm{out}^{\sigma,\hat{\sigma}}\right)^{-1}(\pi) = \left(\mathrm{out}^{\sigma,\hat{\sigma}}(\pi^{\mathrm{r}})\right)^{\mathrm{r}}$. Since π is $(\sigma,\hat{\sigma})$-sortable if and only if $\mathrm{out}^{\sigma,\hat{\sigma}}(\pi)$ avoids 231 (and the reverse map is bijective), we have that $\mathrm{Sort}(\sigma,\hat{\sigma})$ is in bijection with $\mathtt{Av}(231)$. The next theorem follows.

Theorem 2. *For any pattern σ, $\mathrm{out}_n^{\sigma,\hat{\sigma}}$ is a bijection on $\mathfrak{S}_n$. Moreover, we have*

$$|\mathrm{Sort}_n(123,213)| = |\mathrm{Sort}_n(132,312)| = |\mathrm{Sort}_n(231,321)| = c_n,$$

the nth Catalan number.

5. Pair (123, 132)

We characterize $\mathrm{Sort}(123,132)$ in terms of pattern avoidance. Then we show that $(123,132)$-sortable permutations are enumerated by the Catalan numbers by exhibiting a link with the very well studied Catalan triangle.

Theorem 3. *A permutation π is $(123,132)$-sortable if and only if π avoids 2314, 3214, 4213 and $[24\bar{1}3$.*

Proof. Suppose that π is $(123,132)$-sortable. For a contradiction, suppose that π contains $\tau \in \{2314, 3214, 4213\}$. Pick an occurrence $\pi_i\pi_j\pi_k\pi_\ell$ of τ, with $i < j < k < \ell$, where ℓ is chosen minimal, and k, j, and i are chosen maximal, in this order.

If $\tau = 2314$, due to our choice of i,j,k,ℓ, we have $\pi_i < \pi_u < \pi_j$, for $k < u < \ell$. Now, when π_k is pushed in the $(123,132)$-stack, at least one of π_i and π_j has already been extracted: otherwise the $(123,132)$-stack would contain an occurrence of 132, which is forbidden. For each u, $k+1 \leq u \leq \ell$, we have $\pi_k < \pi_u$ and when π_u is pushed in the $(123,132)$-stack, π_k is still in the $(123,132)$-stack. Indeed, $\pi_u\pi_x\pi_y$ cannot be an occurrence of 123 nor of 132 with π_x above π_y, both in the tail of the $(123,132)$-stack

beginning by π_k. If π_i (resp. π_j) is extracted before π_k enters in the $(123, 132)$-stack, then $\pi_i\pi_{k+1}\pi_k$ (resp. $\pi_j\pi_\ell\pi_k$) creates a pattern 231 in $\mathrm{out}^{123,132}(\pi)$, a contradiction.

If $\tau = 3214$, due to our choice of i, j, k, ℓ, we have $\pi_i > \pi_u > \pi_j$, for $k < u < \ell$; and $\pi_u < \pi_{u+1}$, for $k \le u < \ell$. As above, when π_k is pushed in the $(123, 132)$-stack, at least one of π_i and π_j has already been extracted: otherwise the $(123, 132)$-stack would contain an occurrence of 123. Since $\pi_{k+1} > \pi_k$, the next step pushes π_{k+1} in the $(123, 132)$-stack. (i) Assume that π_i and π_j both had left the $(123, 132)$-stack. Then π_k is just below π_{k+1} in the $(123, 132)$-stack, and $\pi_k < \pi_j < \pi_{k+1}$. This implies that $\pi_j\pi_{k+1}\pi_k$ is an occurrence of 231 in $\mathrm{out}^{123,132}(\pi)$, a contradiction. (ii) Assume that π_i is still in the $(123, 132)$-stack and π_j had left this stack. Again, $\pi_j\pi_{k+1}\pi_k$ is an occurrence of 231 in $\mathrm{out}^{123,132}(\pi)$, a contradiction. (iii) Assume that π_j is still in the $(123, 132)$-stack and π_i had left this stack. Since $\pi_u < \pi_{u+1}$ for $k \le u \le \ell - 1$, the next steps of the process push successively all entries $\pi_{k+1}, \dots, \pi_\ell$ in the $(123, 132)$-stack. As above, $\pi_i\pi_\ell\pi_k$ is an occurrence of 231 in $\mathrm{out}^{123,132}(\pi)$, again a contradiction.

The case $\tau = 4213$ can be treated similarly.

Finally, suppose that π contains $[24\bar{1}3$. Equivalently, there are two indices $i < j$ such that $\pi_1\pi_i\pi_j$ is an occurrence of 132 and $\pi_k > \pi_1$ for each $i < k < j$. Observe that, by choosing j minimal and i maximal (in this order), we can assume $j = i + 1$. Now, if π_i is still in the $(123, 132)$-stack when π_{i+1} enters, then $\mathrm{out}^{123,132}(\pi)$ contains an occurrence $\pi_{i+1}\pi_i\pi_1$ of 231, which is impossible due to the sortability of π. Therefore π_i is extracted before π_{i+1} enters. This means that there are two elements π_u, π_v in the $(123, 132)$-stack, with $u < v$ (and thus π_v above π_u), such that $\pi_{i+1}\pi_v\pi_u$ is an occurrence of either 123 or 132. Choose u, v minimal amongst those indices,[2] so that π_u is still in the $(123, 132)$-stack when π_{i+1} enters. Notice that $\pi_{i+1} < \pi_u$ (and thus $u \ne 1$). Moreover, it must be $\pi_u < \pi_i$, otherwise $\pi_i\pi_u\pi_1$ would be an occurrence of 231 in the $(123, 132)$-stack, which is forbidden. But then $\pi_{i+1}\pi_u\pi_1$ is an occurrence of 231 in $\mathrm{out}^{123,132}(\pi)$, a contradiction.

Conversely, suppose that π is not $(123, 132)$-sortable. We shall prove that π contains at least one of the patterns 3214, 2314, 4213 or $[24\bar{1}3$. By hypothesis $\mathrm{out}^{123,132}(\pi)$ contains an occurrence bca of 231. Let $b = \pi_j$ and $c = \pi_k$, for some indices j, k. We distinguish two cases, according whether $j < k$ or $j > k$.

- Suppose that $j < k$ and thus π_j is extracted from the $(123, 132)$-stack before π_k enters. Then there are two elements π_u, π_v in the $(123, 132)$-stack, with $u < v$ (and thus π_v above π_u), such that $\pi_z\pi_v\pi_u$ is an occurrence of either 123 or 132, where π_z is the next element of the input. Notice that $\pi_j \ge \min\{\pi_u, \pi_v\}$, since otherwise $\pi_j\pi_v\pi_u$ would be an occurrence of either 123 or 132 in the $(123, 132)$-stack, which is impossible. Thus $\pi_k > \pi_j > \pi_z$. If $\pi_z\pi_v\pi_u$ is an occurrence of 123, then $\pi_u\pi_v\pi_z\pi_k$ is an occurrence of either 4213, if $\pi_u > \pi_k$, or 3214, if $\pi_u < \pi_k$. Finally, if $\pi_z\pi_v\pi_u$ is an occurrence of 132, then $\pi_u\pi_j\pi_z\pi_k$ is an occurrence of 2314.
- Suppose instead that $j > k$ and thus π_k is still in the $(123, 132)$-stack when π_j enters. Observe that $k \ne 1$, since π_1 is the last element of $\mathrm{out}^{123,132}(\pi)$. Therefore, when π_j enters the $(123, 132)$-stack, the $(123, 132)$-stack contains the elements $\pi_j\pi_k\pi_1$, reading from top to bottom. Notice that $\pi_j > \pi_1$, otherwise $\pi_j\pi_k\pi_1$ would realize an occurrence of the forbidden 132 inside the $(123, 132)$-stack. Moreover, for each entry π_t, with $k < t < j$, we have $\pi_t > \pi_1$. Otherwise $\pi_t\pi_k\pi_1$ would be an occurrence of 132 and π_k would be extracted before π_j, which is impossible due to our assumptions. Thus $\pi_1\pi_k\pi_j$ is an occurrence of $[24\bar{1}3$. This completes the proof. □

Proposition 3. *The distribution of the first element in* Sort(123, 132) *is given by the Catalan triangle (sequence* A009766 *in [15]).*

Proof. Let $A_n(k)$ be the set of $(123, 132)$-sortable permutations of length n and starting with k. Let $A_n^1(k)$ be the subset of $A_n(k)$ consisting of those permutations $\pi = \pi_1\pi_2\dots\pi_n$ where any occurrence $\pi_1\pi_i\pi_\ell$ of [231 with $\pi_\ell = \pi_1 - 1$ can be extended into an occurrence $\pi_1\pi_i\pi_j\pi_\ell$ of [3412. Set $A_n^2(k) = A_n(k)\backslash A_n^1(k)$ and let $k \ge 2$. We shall provide bijections $\alpha : A_n^1(k) \to A_n(k-1)$ and $\beta : A_n^2(k) \to A_{n-1}(k)$.

Define $\alpha : A_n^1(k) \to A_n(k-1)$ by $\alpha(\pi) = \pi'$, where π' is obtained from π by swapping the two entries π_1 and $\pi_1 - 1$ in π. Since $\pi \in A_n^1(k)$, it is easy to check that π' avoids $[24\bar{1}3$. In addition, swapping π_1 and $\pi_1 - 1$ does not affect the avoidance of the three patterns 3214, 2314, 4213, which implies (see Theorem 3) that $\alpha(\pi) \in A_n(k-1)$.

Next define $\beta : A_n^2(k) \to A_{n-1}(k)$ by $\beta(\pi) = \pi''$, where π'' is obtained from π by deleting the entry π_ℓ immediately before $k-1$ and by decreasing by one all entries of π greater than π_ℓ. Notice that $\beta(\pi) \in A_{n-1}(k)$. Let us now sketch the proof that β is bijective, leaving some technical details to the reader. We shall explicitly define the inverse map of β. Given $\pi \in A_{n-1}(k)$, choose an integer ℓ as follows:

- ℓ is the minimal entry $l = \pi_u > \pi_1$, with $1 < u < i$, such that there is an index v with $\pi_v < \pi_i$ and $u < v < i$, if such entry exists.
- Otherwise, set $\ell = n$.

The preimage π is obtained by inserting ℓ immediately before $\pi_i = k - 1$ and then increasing by one all the entries π_j of π with $\pi_j \ge \ell$.

Finally, setting $a_n^k = |A_n(k)|$, we have that $a_n^k = a_n^{k-1} + a_{n-1}^k$, for $2 \le k \le n$. Since $A_n(1) = \{123\cdots n\}$ and $A_n(n)$ is the set of length n permutations avoiding 213 and starting with n, the initial conditions are given by $a_n^1 = 1$ and $a_n^n = c_{n-1}$, where c_n is the nth Catalan number. Therefore, a_n^k generates the well-known Catalan triangle (see Table 1 and [6,11,14]). □

[2] In other words, pick the deepest such elements π_u, π_v in the $(123, 132)$-stack.

Table 1
The Catalan triangle $a_n^k = |A_n(k)|$, with $1 \le n \le 8$ and $1 \le k \le 6$.

$k \backslash n$	1	2	3	4	5	6	7	8
1	1	1	1	1	1	1	1	1
2		1	2	3	4	5	6	7
3			2	5	9	14	20	27
4				5	14	28	48	75
5					14	42	90	165
6						42	132	297
...							...	...
$\sum$	1	2	5	14	42	132	429	1430

Corollary 3. *Permutations of length n in* Sort$(123, 132)$ *are enumerated by the Catalan numbers.*

Proof. With the previous notations we have $|\mathrm{Sort}_n(123, 132)| = \sum_{k=1}^{n} a_n^k = c_n$, the nth Catalan number (see again [6,11,14]). □

6. Pair (123, 312)

We start by giving a ltr-max counterpart of Theorem 1.

Theorem 4. *Consider the $(312, \sigma)$-machine, where $\sigma = \sigma_1 \cdots \sigma_{k-1}\sigma_k \in \mathfrak{S}_k$, with $k \ge 3$ and $\sigma_{k-1} < \sigma_k$. Given a permutation π of length n, let $\pi = M_1 B_1 \cdots M_t B_t$ be its ltr-max decomposition. Then:*

1. *Everytime a ltr-maximum M_i is pushed into the $(312, \sigma)$-stack, the $(312, \sigma)$-stack contains the elements $M_i, M_{i-1}, \ldots, M_2, M_1$, reading from top to bottom. Moreover, we have*

   $$\mathrm{out}^{312,\sigma}(\pi) = \tilde{B}_1 \cdots \tilde{B}_t M_t \cdots M_1,$$

   where $\tilde{B}_i$ is a rearrangement of B_i.
2. *If π is $(312, \sigma)$-sortable, then $M_1, M_2, \ldots, M_t = n - t + 1, n - t + 2, \ldots, n$.*

Proof. 1. The proof is identical to that of the first part of Theorem 1.

2. If π is $(312, \sigma)$-sortable, then $\mathrm{out}^{312,\sigma}(\pi)$ avoids 231 by Proposition 1. Suppose, for a contradiction, that there is an element $j \in \{n - t + 1, \ldots, n\}$ which is not a ltr-maximum. Note that $j \ne \pi_1 = M_1$ and $j \ne n = M_t$. Then, $\mathrm{out}^{312,\sigma}(\pi)$ contains an occurrence jnM_1 of 231, which is a contradiction. □

Instantiating σ by 123 in the previous theorem we have the next result.

Theorem 5. *Let π be a $(123, 312)$-sortable permutation and let $\pi = M_1 B_1 \cdots M_t B_t$ be its ltr-max decomposition. Then:*

1. *B_i avoids 213 for each i.*
2. *$\tilde{B}_i = \mathrm{out}^{12}(B_i)$, for each i.*

Proof. Let $i \ge 2$. Notice that, as a consequence of Theorem 4, immediately after M_i has been pushed in the $(123, 312)$-stack, this stack contains the elements $M_i \cdots M_2 M_1$, reading from top to bottom. Moreover, these elements remain at the bottom of the $(123, 312)$-stack until the end of the sorting procedure, since they are the last elements of $\mathrm{out}^{123,312}(\pi)$. This fact will be used for the rest of the proof.

1. Suppose, for a contradiction, that B_i contains an occurrence of 213, for some i, and let bac be such an occurrence with a 'minimal', in the sense that there is no $a' < a$ where $ba'c$ is an occurrence of 213 in B_i. Therefore, since abM_i is an occurrence of 123, b is extracted from the $(123, 312)$-stack before a enters. In addition, when c enters into the $(123, 312)$-stack, a is still in this stack. Indeed, no entry in B_i between a and c together with a produces a forbidden pattern in the $(123, 312)$-stack. It follows that $\mathrm{out}^{123,312}(\pi)$ contains bca which is an occurrence of 231, yielding a contradiction with the sortability of π.

2. Let us consider the action of the $(123, 312)$-stack on the block B_i. We wish to show that the behavior of the $(123, 312)$-stack when processing B_i is equivalent to the behavior of an empty 12-stack on input B_i. In other words, we prove that the restriction of the $(123, 312)$-stack is triggered if and only if the next element of the input forms an occurrence of 12 together with some other element in the $(123, 312)$-stack. Immediately after M_i has been pushed (i.e. before the first element of B_i is processed), the $(123, 312)$-stack contains the elements $M_i \cdots M_2 M_1$, reading from top to bottom. Observe that B_i avoids 213 by what proved above, therefore the $(123, 312)$-stack cannot be triggered by an occurrence of 312 when processing B_i. Suppose that the next element of the input x forms an occurrence xy of 12 with some $y \in B_i$. Then xyM_i is an occurrence of 123 in the $(123, 312)$-stack, and so this stack behaves as a 12-stack. Conversely, suppose that the $(123, 312)$-stack is triggered by an occurrence of xyz of 123, where x is the next element of the input. Since $M_i > M_{i-1} > \cdots > M_1$, necessarily $y \in B_i$. Thus xy is an occurrence of 12 that triggers the 12-stack, as wanted. □

As a consequence of what proved so far in this section, for any $(123, 312)$-sortable permutation $\pi = M_1 B_1 \cdots M_t B_t$ of length n, we have $B_i \in \mathtt{Av}(213)$ and $M_1, \ldots, M_t = n - t + 1, \ldots, n$. Moreover, by Proposition 2, each $\tilde{B}_i$ in $\mathrm{out}^{123,312}(\pi) = \tilde{B}_1 \cdots \tilde{B}_t M_t \cdots M_1$ is decreasing. Therefore, for any three elements x, y, z, with $x \in B_i$, $y \in B_j$ and $z \in B_k$, with $i < j \le k$, xyz is not an occurrence of 231. Otherwise xyz would still be an occurrence of 231 in $\mathrm{out}^{123,312}(\pi)$, contradicting the fact that π is $(123, 312)$-sortable. From now on, we say that xyz is an occurrence of $2-3-1$ if $z < x < y$, with $x \in B_i$, $y \in B_j$ and $z \in B_k$, with $i < j < k$. Similarly, when $j = k$, we say that xyz is an occurrence of $2-31$.

Theorem 6. *Let $\pi = M_1 B_1 \cdots M_t B_t$ be the ltr-max decomposition of a permutation π of length n. Write $\mathrm{out}^{123,312}(\pi) = \tilde{B}_1 \cdots \tilde{B}_t M_t \cdots M_1$ as in Theorem 4. Then π is $(123, 312)$-sortable if and only if the following conditions are satisfied:*

1. *$M_j = n - t + j$, for each $j = 1, \ldots, t$.*
2. *B_i avoids 213 for each i (and thus $\tilde{B}_i$ is decreasing for each i).*

3. $\mathrm{out}^{123,312}(\pi)$ *avoids* $2-3-1$.
4. $\mathrm{out}^{123,312}(\pi)$ *avoids* $2-31$.

Proof. If π is $(123,312)$-sortable, then π satisfies all the above conditions as a consequence of what proved before in this section. Conversely, it is easy to check that if π satisfies the above conditions, then $\mathrm{out}^{123,312}(\pi)$ avoids 231. Thus π is $(123,312)$-sortable. □

Reformulating the third condition of Theorem 6 we obtain the following lemma, whose easy proof is omitted.

Lemma 3. *Let $\pi = M_1B_1\cdots M_tB_t$ be the ltr-max decomposition of the $(123,312)$-sortable permutation π. Write $\mathrm{out}^{123,312}(\pi) = \tilde{B}_1\cdots\tilde{B}_tM_t\cdots M_1$ as in Theorem 4. Then $\mathrm{out}^{123,312}(\pi)$ avoids $2-31$ if and only if for each $x\in B_i$, $y\in B_j$, with $i<j$, we have:*

- *if $y>x$, then $B_j>x$.*
- *If $y<x$, then $B_j<x$.*

In other words, Lemma 3 says that each block B_j of a $(123,312)$-sortable permutation π is bounded between two previous elements of π. The following result is obtained by restating this lemma and Theorem 6 in terms of pattern avoidance.

Theorem 7. *A permutation π is $(123,312)$-sortable if and only if π avoids the three generalized patterns* [132, [42531 *and* $[42\overline{1}\overline{5}3$.

Next we prove that $(123,312)$-sortable permutations are enumerated by the binomial transform of Catalan numbers. We shall exploit the above characterization in terms of patterns in order to provide a bijection with a certain set of partial permutations, whose enumeration is straightforward.

Recall from Section 2 that a partial permutation of n is an injection $\pi:\{1,2,\dots,k\}\to\{1,2,\dots,n\}$, for some $0\le k\le n$. The partial permutation of length zero will be denoted by λ. Denote by $\mathcal{A}_n$ the set of all partial permutations of n. For instance, we have $\mathcal{A}_3=\{\lambda,1,2,3,12,21,13,31,23,32,123,132,213,231,312,321\}$. There is a natural bijection between the set of permutations in $\mathfrak{S}_n$ avoiding the pattern [132 and $\mathcal{A}_{n-1}$. Indeed, from a length n permutation π avoiding [132, we associate the unique partial permutation $\alpha(\pi)\in\mathcal{A}_{n-1}$ defined as follows:

$$\alpha(\pi)_{\pi_i}=i-1,\ \text{for } \pi_i<\pi_1.$$

In other words, $\alpha(\pi)$ is obtained by recording the indices (minus one) of the elements $\pi_i<\pi_1$, from the smallest to the largest one. For instance, if $\pi=52461783$, then $\alpha(\pi)=4172$. Notice that $\alpha(\pi)=\lambda$ if and only if $\pi_1=1$. Let us now define two pattern containments on $\mathcal{A}_n$. Let $a=a_1a_2\cdots a_m$ be a partial permutation of n, with $m\le n$, and let $i<j<k$. Then $a_ia_ja_k$ is an occurrence of the pattern $31|2$ if it is an occurrence of 312 such that at least one value of the interval $[a_j,a_k]$ does not appear in a. Moreover, we say that $a_ia_ja_k$ is an occurrence of the pattern $\overline{2}1\overline{3}$ if it is an occurrence of 213 such that $a_i=a_k-1$.[3] By interpreting Theorem 7 in terms of partial permutations, we obtain easily:

Theorem 8. *A permutation π is $(123,312)$-sortable if and only if $\alpha(\pi)$ avoids $31|2$ and $\overline{2}1\overline{3}$.*

Let $\mathcal{A}_n(31|2,\overline{2}1\overline{3})$ be the set of partial permutations of n avoiding the two patterns $31|2$ and $\overline{2}1\overline{3}$, and $\mathcal{A}_n(213)$ be the set of partial permutations of n avoiding the classical pattern 213.

Theorem 9. *For any $n\ge 1$, there is a bijection ϕ between $\mathcal{A}_n(31|2,\overline{2}1\overline{3})$ and $\mathcal{A}_n(213)$.*

Proof. Let us define recursively the map ϕ from $\mathcal{A}_n(31|2,\overline{2}1\overline{3})$ to $\mathcal{A}_n(213)$. If $\pi=\lambda$, then we set $\phi(\pi)=\lambda$. Otherwise, π has a unique decomposition of the form $\pi=A\min(\pi)B$ where A and B are disjoint partial permutations of n. We distinguish three cases:

(i) If at least one of A or B is empty, then we set $\phi(\pi)=\phi(A)\min(\pi)\phi(B)$;
(ii) If both A and B are not empty and $\min(A)>\max(B)$, then we set $\phi(\pi)=\phi(A)\min(\pi)\phi(B)$. It is worth noting that the hypothesis that π avoids $31|2$ implies that any value $x\in[\min(\pi),\max(B)]$ occurs in B.
(iii) Suppose that both A and B are not empty and $\min(A)<\max(B)$. Since π avoids $\overline{2}1\overline{3}$, there exists $x\in[\min(A),\max(B)]$ such that x does not occur in π. We choose the smallest x with this property, so that any value of the interval $[\min(A),x]$ occurs in A. Moreover, since π avoids $31|2$, it must be $\max(A)=x-1$. An illustration of this case is depicted in Fig. 1. Let r be the maximum value of B that is lower than $\min(A)$ and consider the string B' obtained from B by decreasing by $x-r-1$ all entries greater than x. Similarly, let A' be obtained from A by increasing by $\max(B)-x+1$ all its entries. Obviously, A' and B' belong to $\mathcal{A}_n(31|2,\overline{2}1\overline{3})$, whereas $\pi'=A'\min(\pi)B'$ contains $31|2$. Then we set $\phi(\pi)=\phi(A')\min(\pi)\phi(B')$ (see again Fig. 1 for an illustration of this mapping). It is worth noting that the value $r+1$ does not occur in both A' and B', which implies that there exists $y\in[\min(\pi),r+1]$ such that y does not occur in $\phi(B')$.

Next we prove that ϕ is an injective map. We proceed by induction on the length of partial permutations. Let π be a partial permutation. Due to the remarks at the end of (ii) and (iii), the image of π under ϕ satisfying (ii) is a partial permutation π' such that any value $x\in[\min(\pi),\max(\phi(B))]$ occurs in $\phi(B)$, which is not true for a permutation π satisfying (iii). Then, for two non-empty partial permutations π and σ in $\mathcal{A}_n(31|2,\overline{2}1\overline{3})$,

[3] This is analogous to the notion of bivincular pattern on classical permutations.

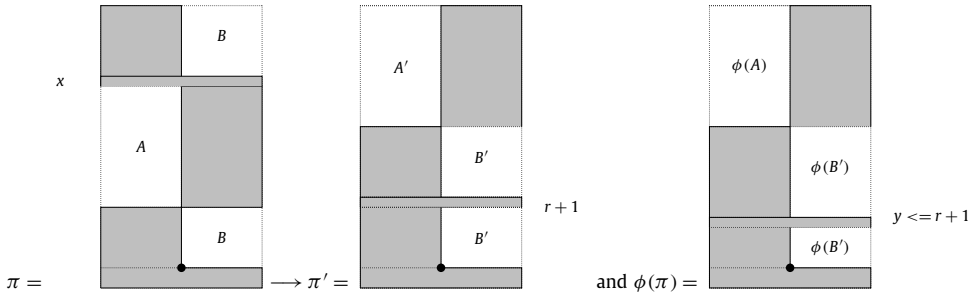


Fig. 1. Illustration of ϕ in the case (iii) of the proof of Theorem 8.

Table 2
Counting sequences not appearing in [15].

(σ,τ)	The sequence counting (σ,τ)-sortable permutations
(123,231)	1, 2, 6, 21, 79, 310, 1252, 5168, 21714, …
(132,213), (213,312)	1, 2, 5, 16, 261, 1206, 5882, 29861, 156366, …
(123,321)	1, 2, 4, 7, 14, 28, 56, 112, 224, 448, 896, 1792, …
(213,231)	1, 2, 6, 23, 101, 483, 2450, 12978, 71071, …
(213,321)	1, 2, 4, 12, 46, 200, 941, 4677, 24203, …
(231,312)	1, 2, 6, 23, 101, 484, 2471, 13254, 74026, …
(312,321)	1, 2, 4, 10, 28, 85, 274, 925, 3239, …

$\phi(\pi) = \phi(\sigma)$ implies that π and σ have the same length and they belong to the same case (i), (ii) or (iii). The recurrence hypothesis induces $\pi = \sigma$ which completes the induction.

Finally, observe that any partial permutation π avoiding 213 is of the form $A\min(\pi)B$ where $\min(A) > \max(B)$ and both A and B avoid 213. According to the geometrical shape of π (as in the proof of injectivity), π fits exactly in one of the cases (i), (ii) and (iii) in the definition of ϕ. Therefore the surjectivity of ϕ can be showed by using its recursive definition and induction on A and B. We leave the details to the reader. □

Now, it is easy to enumerate the set $\mathcal{A}_n(213)$. Indeed any partial permutation $\pi \in \mathcal{A}_n(213)$ can be obtained by choosing k integers from $\{1, 2, \ldots, n\}$ and then arranging them according to the partial order of a permutation in $\mathtt{Av}_k(213)$ (there are c_k such permutations). Therefore, we have:

$$|\mathcal{A}_n(213)| = \sum_{k=0}^{n} \binom{n}{k} c_k,$$

the binomial transform of Catalan numbers (sequence `A007317` in [15]). The enumeration of Sort(123, 312) follows immediately.

Corollary 4. *For each $n \geq 1$, we have:*

$$|\mathrm{Sort}_{n+1}(123, 312)| = \sum_{k=0}^{n} \binom{n}{k} c_k.$$

7. Final remarks

We have focused our study on pairs (σ, τ) of patterns of length three for which permutations sortable by (σ, τ)-machine are counted by Catalan numbers, by their binomial transform, or by Schröder numbers, thereby addressing six out of fifteen possible pairs of patterns of length three. Computer experiments have shown that for all nine remaining pairs of length three patterns, except for $(132, 321)$, the corresponding counting sequences are not recorded in [15] database (see Table 2), and we state the next conjecture.

Conjecture 1. *The sequence counting $(132, 321)$-sortable permutations is the sequence* `A102407` *in [15] counting particular Łukasiewicz paths and pattern avoiding Dyck paths.*
Moreover, the pairs $(132, 213)$ and $(213, 312)$ yield the same enumerating sequence.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

The authors are grateful to the anonymous referees for carefully reading the manuscript and providing many useful suggestions.

References

[1] J. Bandlow, E.S. Egge, K. Killpatrick, A weight-preserving bijection between Schröder paths and Schröder permutations, Ann. Comb. 6 (3–4) (2002) 235–248.

[2] E. Barcucci, A. Del Lungo, E. Pergola, R. Pinzani, Some combinatorial interpretations of q-analogs of Schröder numbers, Ann. Comb. 3 (1999) 171–190.

[3] J.-L. Baril, C. Khalil, V. Vajnovszki, Catalan and Schröder permutations sortable by two restricted stacks, https://arxiv.org/pdf/2004.01812, 2020.

[4] J.-L. Baril, G. Cerbai, C. Khalil, V. Vajnovszki, Permutations sortable by two restricted stacks: Catalan and Schröder cases, in: Permutation Patterns Virtual Workshop, June 30 – July 1, 2020.

[5] K. Berlow, Restricted stacks as functions, https://arxiv.org/pdf/2008.01164, 2020.

[6] Y. Cai, C. Yan, Counting with Borel's triangle, Discrete Math. 342 (2) (2019) 529–539.

[7] G. Cerbai, A. Claesson, L. Ferrari, Stack sorting with restricted stacks, J. Comb. Theory, Ser. A 173 (2020).

[8] G. Cerbai, A. Claesson, L. Ferrari, E. Steingrímsson, Sorting with pattern-avoiding stacks: the 132-machine, Electron. J. Comb. 27 (3) (2020).

[9] G. Cerbai, Sorting Cayley permutations with pattern-avoiding machines, https://arxiv.org/pdf/2003.02536, 2020.

[10] C. Defant, K. Zheng, Stack-sorting with consecutive-pattern-avoiding stacks, https://arxiv.org/pdf/2008.12297, 2020.

[11] D. Desantis, R. Field, W. Hough, B. Jones, R. Meissen, J. Ziefle, Permutation pattern avoidance and the Catalan triangle, Mo. J. Math. Sci. 25 (1) (2013) 50–60.

[12] S. Kitaev, Patterns in Permutations and Words, Springer, 2011.

[13] D.E. Knuth, The Art of Computer Programming, Vol. 1, Fundamental Algorithms, Addison-Wesley, 1968.

[14] L.W. Shapiro, A Catalan triangle, Discrete Math. 14 (1976) 83–90.

[15] N.J.A. Sloane, The on-line encyclopedia of integer sequences, http://oeis.org/.

[16] J. West, Generating trees and the Catalan and Schröder numbers, Discrete Math. 146 (1995) 247–262.

arXiv:1912.10447v3 [cs.DM] 14 Apr 2021

Catalan words avoiding pairs of length three patterns

Jean-Luc Baril Carine Khalil Vincent Vajnovszki

LIB, Université de Bourgogne Franche-Comté, Dijon, France

received 25th Dec. 2019, revised 31st Mar. 2021, accepted 1st Apr. 2021.

Catalan words are particular growth-restricted words counted by the eponymous integer sequence. In this article we consider Catalan words avoiding a pair of patterns of length 3, pursuing the recent initiating work of the first and last authors and of S. Kirgizov where (among other things) the enumeration of Catalan words avoiding a patterns of length 3 is completed. More precisely, we explore systematically the structural properties of the sets of words under consideration and give enumerating results by constructive bijections or bivariate generating functions with respect to the length and descent number. Some of the obtained enumerating sequences are known, and thus the corresponding results establish new combinatorial interpretations for them.

Keywords: Catalan/pattern-avoiding words, enumeration, constructive bijections, (bivariate) generating functions.

1 Introduction and notation

Catalan words are particular growth-restricted words and they represent still another combinatorial class counted by the Catalan numbers, see for instance [12, exercise 6.19.u, p. 222]. This paper contributes to a recent line of research on classical pattern avoidance on words subject to some growth restrictions (for instance, ascent sequences [2, 6], inversion sequences [5, 10, 14], restricted growth functions [4, 9]) by investigating connections between sequences on the On-line Encyclopedia of Integer Sequences [11] and Catalan words avoiding two patterns of length 3.

Through this paper we consider words over the set of non-negative integers and we denote such words by sequences (for instance $w_1w_2\dots w_n$) or by italicized boldface letter (for instance $\boldsymbol{w}$ and $\boldsymbol{u}$). The word $\boldsymbol{w} = w_1w_2\dots w_n$ is called a *Catalan word* if

$$w_1 = 0 \text{ and } 0 \le w_i \le w_{i-1} + 1 \text{ for } i = 2, 3, \dots, n.$$

Catalan words are in bijection with maybe the most celebrated combinatorial class having the same enumerating sequence: Dyck paths[(i)]. Indeed, in a length $2n$ Dyck path collecting for the up steps the ordinates of their starting points we obtain a length n Catalan word, and this construction is a bijection. See Figure 1 where this bijection is depicted for an example. We denote by $\mathcal{C}_n$ the set of length n Catalan words and $c_n = |\mathcal{C}_n|$ is the nth Catalan number $\frac{1}{n+1}\binom{2n}{n}$.

A *pattern* is a word with the property that if i occurs in it, then so does j, for any j with $0 \le j < i$. A pattern $\pi = \pi_1\pi_2\dots\pi_k$ is said to be contained in the word $\boldsymbol{w} = w_1w_2\dots w_n$, $k \le n$, if there is a sub-word of $\boldsymbol{w}$, $w_{i_1}w_{i_2}\dots w_{i_k}$, order-isomorphic with $\pi_1\pi_2\dots\pi_k$. If $\boldsymbol{w}$ does not contain π, we say that $\boldsymbol{w}$ *avoids* π, see for instance Kitaev's seminal book [7] on this topic.

For a pattern π, we denote by $\mathcal{C}_n(\pi)$ the set of length n Catalan words avoiding π, and $c_n(\pi) = |\mathcal{C}_n(\pi)|$ is the cardinality of $\mathcal{C}_n(\pi)$ and $\mathcal{C}(\pi) = \cup_{n\ge 0}\mathcal{C}_n(\pi)$. For example, $\mathcal{C}_n(101)$ is the set of length n Catalan words avoiding 101, that is, the set of words $\boldsymbol{w}$ in $\mathcal{C}_n$ such that there are no i, j and k, $1 \le i < j < k \le n$, with $w_i = w_k > w_j$. So, $\mathcal{C}_4(101) = \{0000, 0001, 0010, 0011, 0012, 0100, 0110, 0111, 0112, 0120, 0121, 0122, 0123\}$. Likewise, if π is the set of patterns $\{\alpha, \beta, \dots\}$, then $\mathcal{C}_n(\pi)$ and $\mathcal{C}_n(\alpha, \beta, \dots)$ denote both the set of length n Catalan words avoiding each pattern in π; and $c_n(\pi) = c_n(\alpha, \beta, \dots)$ and $\mathcal{C}(\pi) =$

(i) A Dyck path is a path in the first quadrant of the plane which begins at the origin, ends at $(2n, 0)$, and consists of up steps $(1, 1)$ and down steps $(1, -1)$.

 © 2021 by the author(s) Distributed under a Creative Commons Attribution 4.0 International License

$\mathcal{C}(\alpha, \beta, \dots)$ have similar meaning as above. A *descent* in a word $\boldsymbol{w} = w_1 w_2 \dots w_n$ is a position i, $1 \leq i \leq n-1$, with $w_i > w_{i+1}$. The (ordinary) generating function of a set of pattern avoiding Catalan words $\mathcal{C}(\pi)$ is the formal power series

$$C_\pi(x) = \sum_{n \geq 0} c_n(\pi) x^n = \sum_{\boldsymbol{w} \in \mathcal{C}(\pi)} x^{|\boldsymbol{w}|},$$

where $|\boldsymbol{w}|$ is the length of the word $\boldsymbol{w}$. In our case of generating function approach for counting classes of pattern avoiding Catalan words we consider the descent number as an additional statistic obtaining 'for free' the bivariate generating function

$$C_\pi(x, y) = \sum_{\boldsymbol{w} \in \mathcal{C}(\pi)} x^{|\boldsymbol{w}|} y^{\mathrm{des}(\boldsymbol{w})},$$

where $\mathrm{des}(\boldsymbol{w})$ is the number of descents of $\boldsymbol{w}$. With these notations, the coefficient of $x^n y^k$ in $C_\pi(x, y)$ is the number of Catalan words of length n avoiding π and having k descents, and for a set $\mathcal{S}$ of Catalan words $S(x)$ and $S(x, y)$ have similar meaning.

For a word $\boldsymbol{w} = w_1 w_2 \dots w_n$ and an integer a, we denote by $(\boldsymbol{w} + a)$ the word obtained from $\boldsymbol{w}$ by increasing by a each of its entries, that is, the word $(w_1 + a)(w_2 + a) \cdots (w_n + a)$. In our constructions we will often make use of two particular families of Catalan words: those avoiding 10 (i.e., with no descents) and we call these words *weakly increasing* (or *w.i* for short) *Catalan words*; and those avoiding 00 (and thus necessarily avoiding 10) and we call these words *strictly increasing* (or *s.i* for short) *Catalan words*. It is easy to see that for each length $n \geq 1$ there are 2^{n-1} w.i. Catalan words and one s.i. Catalan word.

The remaining of this paper is structured as follows. In the next section we characterize pattern avoiding ascent sequences which are Catalan words, establishing ties with some similar enumerative results for ascent sequences in [2]. In Section 3 we consider classes of Catalan words avoiding both a length two and a length three pattern. In the next sections we discuss Catalan words avoiding two patterns of length three, in increasing order of their complexity: obvious cases (Section 4), cases counted via recurrences (Section 5) and cases counted via generating functions (Section 6); these results are summarized in Table 2. We conclude with some remarks and further research directions.

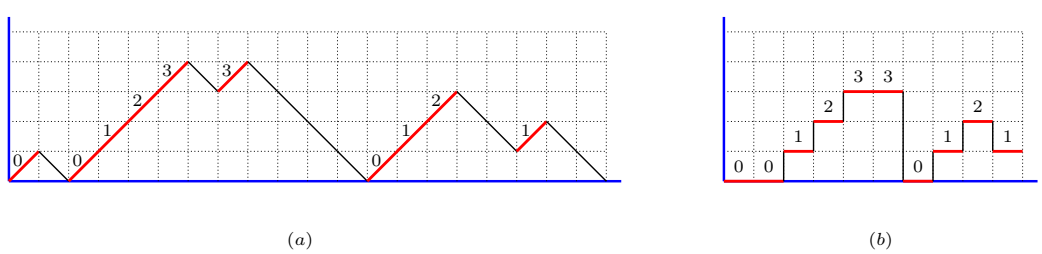


Figure 1: (a) A Dyck path where each up step is labeled by the ordinate of its starting point; and (b) its corresponding Catalan word 0012330121.

2 Catalan words vs. ascent sequences

An *ascent* in a word $\boldsymbol{w} = w_1 w_2 \dots w_n$ is a position i, $1 \leq i \leq n-1$, with $w_i < w_{i+1}$, and $\mathrm{asc}(\boldsymbol{w})$ denotes the number of ascents in $\boldsymbol{w}$. Closely related to Catalan words are ascent sequences introduced in [3] and defined as: the word $\boldsymbol{w} = w_1 w_2 \dots w_n$ is called an *ascent sequence* if

$$w_1 = 0 \text{ and } 0 \leq w_i \leq \mathrm{asc}(w_1 w_2 \dots w_{i-1}) + 1 \text{ for } i = 2, 3, \dots, n,$$

and $\mathcal{A}_n$ denotes the set of length n ascent sequences, and $\mathcal{A} = \cup_{n \geq 0} \mathcal{A}_n$. Similarly as for Catalan words, if π is a pattern, then $\mathcal{A}_n(\pi)$ is the set of length n ascent sequences avoiding π, and $\mathcal{A}(\pi) = \cup_{n \geq 0} \mathcal{A}_n(\pi)$. Clearly, $\mathcal{C}_n = \mathcal{A}_n$ for $n \leq 3$, and $\mathcal{C}_n \subset \mathcal{A}_n$ for $n \geq 4$, and this inclusion is strict, for instance $0102 \in \mathcal{A}_4 \setminus \mathcal{C}_4$. It turns out that, for particular patterns π, $\mathcal{A}_n(\pi)$ collapses to $\mathcal{C}_n(\pi)$ for any n, and this behaviour where the pattern 0102 plays a critical role is discussed below.

Proposition 1. *If $\boldsymbol{w} \in \mathcal{A} \setminus \mathcal{C}$, then $\boldsymbol{w}$ contains the pattern* 0102.

Proof: If $\boldsymbol{w} = w_1 w_2 \dots w_n$ is an ascent sequence which is not a Catalan word, then there is an i such that $w_i \geq w_{i-1} + 2$, and let k be the smallest such i. It follows that $w_i \leq w_{i-1} + 1$ for any i, $2 \leq i \leq k-1$, or equivalently $w_1 w_2 \dots w_{k-1}$ is a Catalan word. Thus, if $w_i > 0$, $2 \leq i \leq k-1$, then each symbol less than w_i occurs in the prefix $w_1 w_2 \dots w_{i-1}$. We distinguish two cases: (i) w_{k-1} is not the maximal symbol of the prefix $w_1 w_2 \dots w_{k-1}$, and (ii) otherwise.
(i) In this case there exist i and j, $1 \leq i < j < k-1$, such that $w_j = w_{k-1} + 1$ and $w_i = w_{k-1}$. It follows that $w_i w_j w_{k-1} w_k$ is an occurrence of 0102.
(ii) In this case the prefix $w_1 w_2 \dots w_{k-1}$ has a descent (otherwise, since $\boldsymbol{w}$ is an ascent sequence, the maximal possible value for w_k is $w_{k-1} + 1$), and let j be such a descent, that is $w_j > w_{j+1}$, $j+1 < k-1$. As noticed above, the symbol w_{j+1} already occurs in $w_1 w_2 \dots w_{j-1}$, say in position i. Thus, $w_i w_j w_{j+1} w_k$ is an occurrence of 0102. □

Proposition 2. *For $n \geq 4$ and a pattern π the followings are equivalent*

1. $\mathcal{A}_n(\pi) = \mathcal{C}_n(\pi)$,
2. 0102 *contains the pattern* π.

Proof: '2. $\Rightarrow$ 1.' We proceed by contraposition and considering $\mathcal{C}_n(\pi) \subseteq \mathcal{A}_n(\pi)$. If $\boldsymbol{w} \in \mathcal{A}_n(\pi) \setminus \mathcal{C}_n(\pi)$, then, by Proposition 1, $\boldsymbol{w}$ contains 0102, and so 0102 does not contain π.
'1. $\Rightarrow$ 2.' Again by contraposition: if 0102 does not contain π, then at least one of the words $01023 \cdots (n-2)$ or 01020^{n-4} belongs to $\mathcal{A}_n(\pi)$ and not to $\mathcal{C}_n(\pi)$, and so $\mathcal{A}_n(\pi) \neq \mathcal{C}_n(\pi)$. □

Since the only patterns of length three of 0102 are 001, 010, 012 and 102, we have the following consequence of Proposition 2.

Corollary 1. *For $n \geq 4$ and a pattern π of length three, $\mathcal{A}_n(\pi) = \mathcal{C}_n(\pi)$ if and only if $\pi \in \{001, 010, 012, 102\}$.*

Pattern avoidance in ascent sequences was initiated in [6], and in [2] ascent sequences avoiding a pair of patterns of length three are considered and exact enumeration for several such pairs are given. In light of Corollary 1 it can happen that if a pattern of the avoided pair is one of the four specified in this corollary, then the resulting ascent sequences are Catalan words as well. The pairs of avoided patterns for which ascent sequences and Catalan words coincide, and for which the enumeration has already been considered in [2] are highlighted in the summarizing Table 2. In order to keep the present article self-contained we fully consider these cases, our proofs being alternative to those in [2].

3 Avoiding a length two and a length three pattern

There are three patterns of length two, namely 00, 01 and 10, and we have:

Proposition 3. *The number of Catalan words avoiding a pattern of length two and a pattern π of length three is given by:*

$$c_n(00, \pi) = \begin{cases} 0 & \text{if } \pi = 012 \text{ and } n \geq 3, \\ 1 & \text{elsewhere;} \end{cases} \qquad c_n(01, \pi) = \begin{cases} 0 & \text{if } \pi = 000 \text{ and } n \geq 3, \\ 1 & \text{elsewhere;} \end{cases}$$

$$c_n(10, \pi) = \begin{cases} F_n & \text{if } \pi = 000, \\ n & \text{if } \pi \in \{001, 011, 012\}, \\ 2^{n-1} & \text{elsewhere,} \end{cases}$$

where F_n is the nth Fibonacci number defined by $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$, $n \geq 2$.

Proof: If $\boldsymbol{w} \in \mathcal{C}_n(00)$, then $\boldsymbol{w} = 012 \cdots (n-1)$. Thus, $\mathcal{C}_n(00, \pi) = \{012 \cdots (n-1)\}$ except when $\pi = 012$ and in this case $\mathcal{C}_n(00, \pi) = \varnothing$ for $n \geq 3$, and the counting relation for $\mathcal{C}_n(00, \pi)$ follows.
Similarly, if $\boldsymbol{w} \in \mathcal{C}_n(01)$, then $\boldsymbol{w} = 00 \cdots 0$. Thus, $\mathcal{C}_n(01, \pi) = \{000 \cdots 0\}$ except when $\pi = 000$ and in this case $\mathcal{C}_n(01, \pi) = \varnothing$ for $n \geq 3$.

Finally, a Catalan word avoids 10 if and only if it avoids 010. It follows that $c_n(10, \pi) = c_n(010, \pi)$, which falls in the case of avoidance of two length 3 patterns and the corresponding proofs are given in the next section, see also Table 2. □

4 Trivial cases

Superfluous patterns

If the pattern τ contains the pattern σ, then clearly $\mathcal{C}_n(\sigma,\tau)=\mathcal{C}_n(\sigma)$; but this phenomenon can occur even when σ and τ are not related by containment and in this case, following [2], we say that τ is a *superfluous pattern* for σ. For example, any word in $\mathcal{C}_n(012)$ is a binary word, and thus any pattern with at least three different symbols is a superfluous pattern for 012. In Table 1 are listed all pairs of superfluous patterns of length three. It is worth to mentioning that superfluousness is a transitive relation, for instance 201 is superfluous for 021 which in turn is superfluous for 011. So, a pattern can be superfluous for several other ones, for instance $\tau=201$ is superfluous for each of the patterns $001, 010, 011, 012, 021, 101, 120$. Also it is easy to see that if τ is superfluous for σ, then τ is larger lexicographically than σ.

σ	000	001	010	011	012	021	101	110	120
	100	101	021	021	021	201	102	210	201
		102	100	101	102	210	201		210
		201	101	102	120				
τ			102	110	201				
			110	201	210				
			120	210					
			201						
			210						

Table 1: In each column, each pattern τ is superfluous for the pattern σ.

Ultimately constant sequences

It can happen that the number of Catalan words avoiding a pair of length 3 patterns is constant for enough long words. The only two such cases are given below.

Proposition 4. $c_n(000,011)=\begin{cases}1 & \text{if } n=1,\\ 2 & \text{if } n=2,\\ 3 & \text{if } n\geq 3;\end{cases}$ *and* $c_n(000,012)=\begin{cases}1 & \text{if } n=1,\\ 2 & \text{if } n=2,\\ 3 & \text{if } n=3 \text{ or } n=4,\\ 0 & \text{if } n\geq 5.\end{cases}$

Proof: If $n\geq 3$, then $\mathcal{C}_n(000,011)=\{0\boldsymbol{x},\boldsymbol{x}0,\boldsymbol{x}(n-1)\}$ where $\boldsymbol{x}$ is the word $01\cdots(n-2)$, and the first point follows.
If a Catalan word avoids 012, then it is a binary word. In addition if its length is larger than 4 it necessarily contains three identical entries, and so $\mathcal{C}_n(000,012)=\varnothing$ for $n\geq 5$. Considering the initial values of $c_n(000,012)$ the second point follows. □

Counting sequence n

Proposition 5. *If π is one of the pairs of patterns $\{001,010\}$, $\{001,011\}$, $\{001,012\}$, $\{010,011\}$, $\{010,012\}$ or $\{011,012\}$, then $c_n(\pi)=n$.*

Proof: The proof is based on the easy to understand description given below for the corresponding sets of Catalan words:

π	$\mathcal{C}_n(\pi)$	π	$\mathcal{C}_n(\pi)$
$\{001,010\}$	$\{012\cdots jj\cdots j\ :\ 0\leq j\leq n-1\}$	$\{010,011\}$	$\{0^j12\cdots(n-j)\ :\ 1\leq j\leq n\}$
$\{001,011\}$	$\{012\cdots j0\cdots 0\ :\ 0\leq j\leq n-1\}$	$\{010,012\}$	$\{0^{j+1}1^{n-j-1}\ :\ 0\leq j\leq n-1\}$
$\{001,012\}$	$\{01^j0^{n-j-1}\ :\ 0\leq j\leq n-1\}$	$\{011,012\}$	$\{0^n\}\cup\{0^j10^{n-j-1}\ :\ 1\leq j\leq n-1\}$

□

5 Counting via recurrence

Counting sequence $2(n-1)$

Proposition 6. *If* $\pi = \{011, 100\}$ *or* $\pi = \{011, 120\}$*, then* $c_n(\pi) = 2(n-1)$ *for* $n \geq 2$ *(Sequence* A005843 *in [11]).*

Proof: If $\pi = \{011, 100\}$ and $\boldsymbol{w} \in \mathcal{C}_n(\pi)$, $n \geq 2$, then either

- $\boldsymbol{w} = 0^k\boldsymbol{u}$, $0 \leq k \leq n-1$, with $\boldsymbol{u}$ a s.i. Catalan word (of length at least one), or
- $\boldsymbol{w} = 0^k\boldsymbol{u}0$, $0 \leq k \leq n-3$, with $\boldsymbol{u}$ a s.i. Catalan word (of length at least two).

In the first case there are n possibilities for $\boldsymbol{w}$ and $n-2$ possibilities in the second case, and the result holds.
If $\pi = \{011, 120\}$ and $\boldsymbol{w} \in \mathcal{C}_n(\pi)$, $n \geq 2$, then either

- $\boldsymbol{w} = 0^k\boldsymbol{u}$, $0 \leq k \leq n-1$, with $\boldsymbol{u}$ a s.i. Catalan word, or
- $\boldsymbol{w} = 0^k10^{n-k-1}$, $1 \leq k \leq n-2$,

and as previously the result holds. □

Sequences involving 2^n

Proposition 7. *If* $\pi = \{000, 101\}$*, then* $c_n(\pi) = 2^{n-1}$ *for* $n \geq 1$ *(*A000079 *in [11]).*

Proof: If a Catalan word avoids 101, then it is unimodal (that is, it can be written not necessarily in a unique way as $\boldsymbol{uv}$ with $\boldsymbol{u}$ a weakly decreasing and $\boldsymbol{v}$ weakly decreasing word). In addition, if the word avoids 000, then its maximal value occurs at most twice, and when it occurs twice this happens in consecutive positions. We denote by $\mathcal{D}_n$ the subset of words in $\mathcal{C}_n(\pi)$ where the maximal entry occurs once and by $\mathcal{E}_n$ that where it occurs twice, $d_n = |\mathcal{D}_n|$, $e_n = |\mathcal{E}_n|$, and $c_n(\pi) = d_n + e_n$. Any word $\boldsymbol{w} = w_1 \cdots mm \cdots w_{n-1} \in \mathcal{E}_{n-1}$ with its maximal value m occurring twice can be extended into a word in $\mathcal{D}_n$ by one of the transformations:

$$\boldsymbol{w} \mapsto w_1 \cdots m(m+1)m \cdots w_{n-1}, \text{ and}$$

$$\boldsymbol{w} \mapsto w_1 \cdots mm(m+1) \cdots w_{n-1},$$

and any word $\boldsymbol{w} = w_1 \cdots m \cdots w_{n-1} \in \mathcal{D}_{n-1}$ with its maximal value m occurring once can be extended into a word in $\mathcal{D}_n$ by:

$$\boldsymbol{w} \mapsto w_1 \cdots m(m+1) \cdots w_{n-1}.$$

Conversely, any word in $\mathcal{D}_n$, $n \geq 2$, can uniquely be obtained from a word in $\mathcal{D}_{n-1}$ or in $\mathcal{E}_{n-1}$ by reversing one of the transformations above, so

$$d_n = 2 \cdot e_{n-1} + d_{n-1},$$

for $n \geq 2$.
Reasoning in a similar way we have

$$e_n = 2 \cdot e_{n-2} + d_{n-2},$$

Thus, for $n \geq 3$, $e_n = d_{n-1}$, and finally

$$c_n(\pi) = d_n + e_n = 2 \cdot (d_{n-1} + e_{n-1}) = 2 \cdot c_{n-1}(\pi),$$

and with the initial conditions $c_1(\pi) = 1$ and $c_2(\pi) = 2$, the result follows. □

Proposition 8. *If* $\pi = \{101, 210\}$ *or* $\pi = \{102, 120\}$*, then* $c_n(\pi) = (n-1) \cdot 2^{n-2} + 1$ *for* $n \geq 2$ *(*A005183 *in [11]).*

Proof: If $\pi = \{101, 210\}$ and $\boldsymbol{w} \in \mathcal{C}_n(\pi)$, $n \geq 3$, then either

- $\boldsymbol{w} = 0\boldsymbol{u}$ with $\boldsymbol{u} \in \mathcal{C}_{n-1}(\pi)$, or
- $\boldsymbol{w} = 0(\boldsymbol{u}+1)$ with $\boldsymbol{u} \in \mathcal{C}_{n-1}(\pi)$, or
- $\boldsymbol{w} = 0(\boldsymbol{u}+1)0^{n-k-1}$ with $\boldsymbol{u}$ a w.i. Catalan word of length k, $1 \leq k \leq n-2$.

The number of words in each of the first two cases is $c_{n-1}(\pi)$. The number of length k w.i. Catalan words is 2^{k-1}, so the number of words in the last case is

$$\sum_{k=1}^{n-2} 2^{k-1} = \sum_{k=0}^{n-3} 2^k = 2^{n-2} - 1.$$

Thus, $c_n(\pi) = 2c_{n-1}(\pi) + 2^{n-2} - 1$, and after calculation the result holds.

If $\pi = \{102, 120\}$ and $\boldsymbol{w} \in \mathcal{C}_n(\pi)$, $n \geq 3$, then either
- $\boldsymbol{w} = 0\boldsymbol{u}$ with $\boldsymbol{u} \in \mathcal{C}_{n-1}(\pi)$, or
- $\boldsymbol{w} = 0(\boldsymbol{u}+1)$ with $\boldsymbol{u} \in \mathcal{C}_{n-1}(\pi)$, or
- $\boldsymbol{w} = 01\boldsymbol{u}$ with $\boldsymbol{u}$ a length $(n-2)$ binary word other than $11\cdots1$.

The number of words in each of the first two cases is $c_{n-1}(\pi)$, and the number of words in the last case is $2^{n-2} - 1$. So $c_n(\pi) = 2c_{n-1}(\pi) + 2^{n-2} - 1$, and again the result holds. □

Proposition 9. *If π is one of the pairs of patterns $\{021, 100\}$, $\{021, 101\}$, $\{101, 110\}$ or $\{101, 120\}$, then $c_n(\pi) = 2^n - n$ for $n \geq 0$ (`A000325` in [11]).*

Proof:
If $\pi = \{021, 100\}$ and $\boldsymbol{w} \in \mathcal{C}_n(\pi)$, $n \geq 4$, then either
- $\boldsymbol{w}$ is a w.i. Catalan word, or
- $\boldsymbol{w} = \boldsymbol{u}0$ with $\boldsymbol{u}$ a w.i. Catalan word of length $(n-1)$ other than $00\cdots0$, or
- $\boldsymbol{w} = 0\boldsymbol{v}0(\boldsymbol{u}+1)$ where $\boldsymbol{v}$ is w.i. binary word ending by 1 and $\boldsymbol{u}$ is a w.i. Catalan word of length k, $1 \leq k \leq n-3$.

In the first case the number of words $\boldsymbol{w}$ is 2^{n-1} and in the second case the number of words $\boldsymbol{w}$ is $2^{n-2} - 1$. In last case the number of words $\boldsymbol{w}$ is

$$\sum_{k=1}^{n-3} (n-k-2) \cdot 2^{k-1} = 2^{n-2} - (n-1).$$

Combining these cases and considering the initial values of $c_n(\pi)$ the result holds.

If $\pi = \{021, 101\}$ and $\boldsymbol{w} \in \mathcal{C}_n(\pi)$, $n \geq 2$, then either
- $\boldsymbol{w} = \boldsymbol{u}0$ with $\boldsymbol{u} \in \mathcal{C}_{n-1}(\pi)$, or
- $\boldsymbol{w} = 0^{n-k}(\boldsymbol{u}+1)$ with $\boldsymbol{u}$ a w.i. Catalan word of length k, $1 \leq k \leq n-1$.

The number of words in the first case is $c_{n-1}(\pi)$ and the number of words in the second case is $\sum_{k=1}^{n-1} 2^{k-1} = 2^{n-1} - 1$. So $c_n(\pi) = c_{n-1}(\pi) + 2^{n-1} - 1$ and after calculation the result follows.

If $\pi = \{101, 110\}$ and $\boldsymbol{w} \in \mathcal{C}_n(\pi)$, $n \geq 3$, then either
- $\boldsymbol{w} = 0\boldsymbol{u}$ with $\boldsymbol{u} \in \mathcal{C}_{n-1}(\pi)$, or
- $\boldsymbol{w} = 0(\boldsymbol{u}+1)$ with $\boldsymbol{u} \in \mathcal{C}_{n-1}(\pi)$, or
- $\boldsymbol{w} = \boldsymbol{u}0^{n-k}$ with $\boldsymbol{u}$ a s.i. Catalan word of length k, $2 \leq k \leq n-1$.

The number of words $\boldsymbol{w}$ in each of the first two cases is $c_{n-1}(\pi)$. For the last case, for each k there is exactly one word $\boldsymbol{w}$, so their number is $(n-2)$ in this case. So $c_n(\pi) = 2c_{n-1}(\pi) + n - 2$ and after calculation the result holds.

If $\pi = \{101, 120\}$ and $\boldsymbol{w} \in \mathcal{C}_n(\pi)$, $n \geq 3$, then either
- $\boldsymbol{w} = 0\boldsymbol{u}$ with $\boldsymbol{u} \in \mathcal{C}_{n-1}(\pi)$, or
- $\boldsymbol{w} = 0(\boldsymbol{u}+1)$ with $\boldsymbol{u} \in \mathcal{C}_{n-1}(\pi)$, or
- $\boldsymbol{w} = 01^k0^{n-k-1}$, $1 \leq k \leq n-2$.

So, again $c_n(\pi) = 2c_{n-1}(\pi) + n - 2$. □

Proposition 10. *If $\pi = \{021, 120\}$, then $c_n(\pi) = (n+2) \cdot 2^{n-3}$ for $n \geq 3$ (`A045623` in [11]).*

Proof: If $\boldsymbol{w} \in \mathcal{C}_n(\pi)$, $n \geq 4$, then either
- $\boldsymbol{w} = 0\boldsymbol{u}0$ with $\boldsymbol{u}$ a length $(n-2)$ binary word, or
- $\boldsymbol{w} = 0(\boldsymbol{u}+1)$ with $\boldsymbol{u}$ a length $(n-1)$ w.i. Catalan word, or

– $\boldsymbol{w} = 0\boldsymbol{u}0(\boldsymbol{v}+1)$ with $\boldsymbol{u}$ a binary word and $\boldsymbol{v}$ w.i. Catalan word.

The number of words in each of the first two cases is 2^{n-2} and the number of words in the last case is

$$\sum_{k=1}^{n-2} 2^{n-k-2}2^{k-1} = (n-2)2^{n-3},$$

and so $c_n(\pi) = 2^{n-1} + (n-2)2^{n-3}$, which gives the desired result. □

If a Catalan word avoids both 102 and 110, then it has at most one descent. In the second part of the proof of the next proposition we need the following technical lemma which gives the number of Catalan words in $\mathcal{C}_n(102, 110)$ with one descent and avoiding the pattern 00 before the descent (note that in this case avoiding 00 is equivalent to avoiding equal consecutive entries). The set of these words is empty for $n \leq 2$, it is the single word set $\{010\}$ for $n = 3$, and $\{0100, 0101, 0120, 0121\}$ for $n = 4$.

Lemma 1. *Let $\mathcal{D}_n$ be the set of words in $\mathcal{C}_n(102, 110)$ having one descent and avoiding 00 before the descent. Then $|\mathcal{D}_n| = \frac{n}{6}(n-1)(n-2)$.*

Proof: A word belongs to $\mathcal{D}_n$, $n \geq 3$, if and only if it can be written as

$$012\ldots(k-1)a(\boldsymbol{u}+a),$$

with $2 \leq k \leq n-1$ (k is the position of the unique descent in the word), $a \in \{0, 1, \ldots k-2\}$, and $\boldsymbol{u}$ is a length $(n-k-1)$ w.i. Catalan word over $\{0, 1\}$. For each choice of k, there are $k-1$ choices for a, and for each choice for a there are $n-k$ choices for $\boldsymbol{u}$. It follows that

$$|\mathcal{D}_n| = \sum_{k=2}^{n-1}(k-1)\cdot(n-k),$$

and after calculation the statement holds. □

Proposition 11. *If $\pi = \{021, 102\}$ or $\pi = \{102, 110\}$, then $c_n(\pi) = 3 \cdot 2^{n-1} - \frac{1}{2}(n+1)(n+2) + n$ for $n \geq 1$ (`A116702` in [11]).*

Proof: If $\pi = \{021, 102\}$ and $\boldsymbol{w} \in \mathcal{C}_n(\pi)$, $n \geq 3$, then either

– $\boldsymbol{w} = 0\boldsymbol{u}$ with $\boldsymbol{u}$ a length $(n-1)$ binary word, or

– $\boldsymbol{w} = 0\cdots0(\boldsymbol{u}+1)0\cdots0$ with $\boldsymbol{u}$ a length k, $2 \leq k \leq n-1$, w.i. Catalan word other than $00\cdots0$ and $\boldsymbol{w}$ beginning by at least one 0.

The number of words in the first case is 2^{n-1} and the number of those in the second case is

$$\sum_{k=2}^{n-1}(2^{k-1}-1)\cdot(n-k) = 2\cdot 2^{n-1} - \frac{1}{2}(n+1)(n+2) + n,$$

and combining the two cases the result holds.

If $\pi = \{102, 110\}$ and $\boldsymbol{w} \in \mathcal{C}_n(\pi)$, then either

– $\boldsymbol{w}$ is a w.i. Catalan word, the number of such words is 2^{n-1}, or

– $\boldsymbol{w} \in \mathcal{D}_n$, with $\mathcal{D}_n$ defined in Lemma 1, or

– $\boldsymbol{w} = \boldsymbol{u}(\boldsymbol{v}+m)$, where $\boldsymbol{u}$ is a w.i. Catalan word of length k, $1 \leq k \leq n-3$, m is the maximal (rightmost) entry of $\boldsymbol{u}$, and $\boldsymbol{v}$ is a word belonging to $\mathcal{D}_{n-k}$.

Indeed, the first case corresponds to words with no descents, the second one to those with a descent and no occurrences of 00 before the descent, and the third one to those with both descent and occurrences of 00 before the descent (the rightmost such occurrence is in positions k and $k+1$). By Lemma 1, the number of words in the third case is

$$\sum_{k=1}^{n-3} 2^{k-1}\cdot(n-k)\cdot\left(\frac{1}{6}(n-k)^2 - \frac{1}{2}(n-k) + \frac{1}{3}\right) = 2^n - \frac{1}{6}(n+1)(n^2-n+6).$$

Finally, combining the three previous cases the desired result holds. □

Sequences involving binomial coefficients

In this part we use the notation

$$ab\cdots \underset{\substack{\uparrow\\ i}}{c}de\cdots f$$

to denote that the entry d is in position i in the word $ab\cdots cde\cdots f$.

Proposition 12. *If $\pi=\{001,210\}$, then $c_n(\pi)=\binom{n}{3}+n$ for $n\geq 3$ (*`A000125` *in [11]).*

Proof: If $\boldsymbol{w}\in\mathcal{C}_n(\pi)$, then it has at most one descent.
If $\boldsymbol{w}$ has no descents, then it has the form $\boldsymbol{w}=01\cdots(m-1)m\cdots m$ and there are n such words.
If $\boldsymbol{w}$ has one descent, then it has the form $\boldsymbol{w}=01\cdots(m-1)m\cdots mk\cdots k$ with $0\leq k<m<n-1$. It follows that there is a bijection between the family of 3-element subsets of $\{0,1,\ldots,n-1\}$ and the words in $\mathcal{C}_n(\pi)$ with one descent:

$$\{k,m,\ell\}\mapsto 0123\cdots(m-1)m\cdots \underset{\substack{\uparrow\\ \ell}}{m}k\cdots k.$$

Combining the two cases we have $c_n(\pi)=\binom{n}{3}+n$. □

Proposition 13. *If π is one of the pairs of patterns $\{001,021\}$, $\{001,110\}$, $\{001,120\}$, $\{012,100\}$, $\{012,101\}$ or $\{012,110\}$, then $c_n(\pi)=\binom{n}{2}+1$ for $n\geq 2$ (*`A000124` *in [11]).*

Proof: In any of the six cases for π, the set $\mathcal{C}_n(\pi)$ is in bijection with the family S of subsets of $\{2,\ldots,n\}$ with at most two elements. We give below explicit definitions for such bijections, where the empty set is mapped to $0\cdots 0\in\mathcal{C}_n(\pi)$ by each of them.

If $\pi=\{001,021\}$ and $\boldsymbol{w}\in\mathcal{C}_n(\pi)$, then either $\boldsymbol{w}=00\cdots 0$ or for some $m\geq 1$, $\boldsymbol{w}=0123\cdots(m-1)m^s$ with $s\geq 1$ or $\boldsymbol{w}=0123\cdots(m-1)m^s0^t$ with $s\geq 1$, $t\geq 1$, and the desired bijection $S\to\mathcal{C}_n(001,021)$ is

$$\{k\}\mapsto 0123\cdots(m-1)\underset{\substack{\uparrow\\ k=m+1}}{m}\cdots m;$$

$$\{k,j\}\mapsto 0123\cdots(m-1)\underset{\substack{\uparrow\\ k=m+1}}{m}\cdots m\underset{\substack{\uparrow\\ j}}{0}\cdots 0.$$

If $\pi=\{001,110\}$ and $\boldsymbol{w}\in\mathcal{C}_n(\pi)$, then either $\boldsymbol{w}=00\cdots 0$ or $\boldsymbol{w}=0123\cdots(m-1)m\cdots m$ with m the maximal entry of $\boldsymbol{w}$, or $\boldsymbol{w}=0123\cdots(m-1)m\ell\cdots\ell$ with $\ell<m$, and the desired bijection $S\to\mathcal{C}_n(001,110)$ is

$$\{k\}\mapsto 0123\cdots(m-1)\underset{\substack{\uparrow\\ k=m+1}}{m}\cdots m;$$

$$\{k,j\}\mapsto 0123\cdots(j-2)(k-2)\cdots(k-2).$$

If $\pi=\{001,120\}$ and $\boldsymbol{w}\in\mathcal{C}_n(\pi)$, then either $\boldsymbol{w}=00\cdots 0$, or for some $m\geq 1$, $\boldsymbol{w}=0123\cdots(m-1)m^s$ with $s\geq 1$ or $\boldsymbol{w}=0123\cdots(m-1)m^s(m-1)^t$ with $s,t\geq 1$, and the desired bijection $S\to\mathcal{C}_n(001,120)$ is

$$\{k\}\mapsto 0123\cdots(m-1)\underset{\substack{\uparrow\\ k=m+1}}{m}\cdots m;$$

$$\{k,j\}\mapsto 0123\cdots(m-1)\underset{\substack{\uparrow\\ k=m+1}}{m}\cdots m\underset{\substack{\uparrow\\ j}}{(m-1)}\cdots(m-1).$$

For the next three cases we need the following observation: a Catalan word avoids 012 if and only if it is a binary word (over $\{0,1\}$) beginning by a 0.

If $\pi=\{012,100\}$ and $\boldsymbol{w}\in\mathcal{C}_n(\pi)$, then either $\boldsymbol{w}=0^s1^t$ with $s\geq 1$, $t\geq 0$ or $\boldsymbol{w}=0^s1^t01^r$ with $s,t\geq 1$, $r\geq 0$, and the desired bijection $S\to\mathcal{C}_n(012,100)$ is

$$\{k\}\mapsto 0\cdots 0\underset{\substack{\uparrow\\ k}}{1}\cdots 1;$$

$$\{k,j\}\mapsto 0\cdots 0\underset{\substack{\uparrow\\ k}}{1}\cdots 1\underset{\substack{\uparrow\\ j}}{0}1\cdots 1.$$

If $\pi=\{012,101\}$ and $\boldsymbol{w}\in\mathcal{C}_n(\pi)$, then either $\boldsymbol{w}=0^s1^t$ with $s\geq 1$, $t\geq 0$ or $\boldsymbol{w}=0^s1^t0^r$ with $s,t\geq 1$, $r\geq 1$, and the desired bijection $S\to\mathcal{C}_n(012,101)$ is

$$\{k\}\mapsto 0\cdots 0\underset{\substack{\uparrow\\ k}}{1}\cdots 1;$$

$\{k,j\} \mapsto 0\cdots 0\underset{\underset{k}{\uparrow}}{1}\cdots 1\underset{\underset{j}{\uparrow}}{0}\cdots 0.$

If $\pi = \{012, 110\}$ and $\boldsymbol{w} \in \mathcal{C}_n(\pi)$, then either $\boldsymbol{w} = 0^s1^t$ with $s \geq 1$, $t \geq 0$ or $\boldsymbol{w} = 0^s10^t1^r$ with $s, t \geq 1$, $r \geq 0$, and the desired bijection $S \to \mathcal{C}_n(012, 110)$ is

$\{k\} \mapsto 0\cdots 0\underset{\underset{k}{\uparrow}}{1}\cdots 1;$

$\{k,j\} \mapsto 0\cdots 0\underset{\underset{k}{\uparrow}}{1}0\cdots 0\underset{\underset{j}{\uparrow}}{1}\cdots 1.$

□

Sequences involving Fibonacci(-like) numbers

As in Proposition 3, we consider the sequence of Fibonacci numbers $(F_n)_{n\geq 0}$ defined as $F_0 = 1$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.

Proposition 14. *If $\pi = \{000, 001\}$ or $\pi = \{000, 010\}$, then $c_n(\pi) = F_n$ for $n \geq 0$ (*`A00045` *in [11]).*

Proof: For $\pi = \{000, 001\}$ the proof is up to a certain point similar to that of Proposition 7. A word belonging to $\mathcal{C}_n(\pi)$ is unimodal and its maximal entry occurs once or twice in consecutive positions. Let $\mathcal{D}_n$ denote the subset of words in $\mathcal{C}_n(\pi)$ where the maximal entry occurs once and $\mathcal{E}_n$ denote that where it occurs twice. If $\boldsymbol{w} \in \mathcal{C}_n(\pi)$ has its maximal entry m, then the insertion of $(m+1)$ after the leftmost occurrence of m in $\boldsymbol{w}$ produces a word in $\mathcal{D}_{n+1}$, and the insertion of $(m+1)(m+1)$ produces a word in $\mathcal{E}_{n+2}$. It is easy to see that these transformations induce a bijection between $\mathcal{C}_n(\pi)$ and $\mathcal{D}_{n+1}$, and between $\mathcal{C}_n(\pi)$ and $\mathcal{E}_{n+2}$, and thus between $\mathcal{C}_{n-2}(\pi) \cup \mathcal{C}_{n-1}(\pi)$ and $\mathcal{C}_n(\pi)$. It follows that $c_n(\pi)$ satisfies a Fibonacci-like recurrence, and by considering the initial values for $c_n(\pi)$ the result holds.

For $\pi = \{000, 010\}$, a word $\boldsymbol{w} \in C_n(\pi)$ is characterized by: $\boldsymbol{w}$ is w.i. and $\boldsymbol{w}$ does not have three consecutive equal entries. So $\boldsymbol{w}$ can be represented by the binary word $b_1b_2\ldots b_{n-1}$ with no two consecutive 1s where $b_i = 1$ iff $w_i = w_{i-1}$. This representation is a bijection between $\mathcal{C}_n(\pi)$ and the set of binary words of length $(n-1)$ without two consecutive 1s, which cardinality is the Fibonacci number, see for instance [13]. □

Proposition 15. *If $\pi = \{001, 100\}$, then $c_n(\pi) = F_{n+1} - 1$ for $n \geq 1$.*

Proof: If $\boldsymbol{w} \in \mathcal{C}_n(\pi)$, $n \geq 3$, then either
- $\boldsymbol{w} = 0\cdots 0$, or
- $\boldsymbol{w} = 0(\boldsymbol{u}+1)$ with $\boldsymbol{u} \in \mathcal{C}_{n-1}(\pi)$, or
- $\boldsymbol{w} = 0(\boldsymbol{u}+1)0$ with $\boldsymbol{u} \in \mathcal{C}_{n-2}(\pi)$.

So, $c_n(\pi)$ satisfies the recurrence $c_n(\pi) = c_{n-1}(\pi) + c_{n-2}(\pi) + 1$ for $n \geq 3$, and solving it we have the desired result. □

In the next proposition we will make use of the following relation satisfied by the even index Fibonacci numbers: $F_{2n} = F_{2n-2} + \sum_{i=0}^{n-1} F_{2i}$ for $n \geq 1$.

Proposition 16. *If $\pi = \{100, 201\}$, then $c_n(\pi) = F_{2n-2}$ for $n \geq 1$ (*`A001519` *in [11]).*

Proof: If $\boldsymbol{w} \in \mathcal{C}_n(\pi)$, $n \geq 3$, then either
- $\boldsymbol{w} = 0\boldsymbol{u}$ with $\boldsymbol{u} \in \mathcal{C}_{n-1}(\pi)$, or
- $\boldsymbol{w} = 0(\boldsymbol{u}+1)$ with $\boldsymbol{u} \in \mathcal{C}_{n-1}(\pi)$, or
- $\boldsymbol{w} = 0(\boldsymbol{u}+1)0$ with $\boldsymbol{u} \in \mathcal{C}_{n-2}(\pi)$, or
- $\boldsymbol{w} = 01^{n-k-2}0(\boldsymbol{u}+1)$ with $\boldsymbol{u} \in \mathcal{C}_k(\pi)$ for some k, $1 \leq k \leq n-3$.

In both of the first two cases the numbers of words $\boldsymbol{w}$ is $c_{n-1}(\pi)$ and in the third case, this number is $c_{n-2}(\pi)$. In the last case, the number of words $\boldsymbol{w}$ is $\sum_{k=1}^{n-3} c_k(\pi)$. So, $c_n(\pi) = 2c_{n-1}(\pi) + c_{n-2}(\pi) + \sum_{k=1}^{n-3} c_k(\pi) = c_{n-1}(\pi) + \sum_{k=1}^{n-1} c_k(\pi)$, and with the initial conditions we have $c_n(\pi) = F_{2n-2}$. □

The sequence of Pell numbers $(p_n)_{n\geq 0}$ is defined as $p_0 = 0$, $p_1 = 1$ and $p_n = 2p_{n-1} + p_{n-2}$ for $n \geq 2$.

Proposition 17. *If $\pi = \{100, 101\}$, then $c_n(\pi)$ is the nth Pell number for $n \geq 1$ (*`A000129` *in [11]).*

Proof: If $\boldsymbol{w} \in \mathcal{C}_n(\pi)$, $n \geq 2$, then either

- $\boldsymbol{w} = 0\boldsymbol{u}$ with $\boldsymbol{u} \in \mathcal{C}_{n-1}(\pi)$, or
- $\boldsymbol{w} = 0(\boldsymbol{u}+1)$ with $\boldsymbol{u} \in \mathcal{C}_{n-1}(\pi)$, or
- $\boldsymbol{w} = 0(\boldsymbol{u}+1)0$ with $\boldsymbol{u} \in \mathcal{C}_{n-2}(\pi)$.

In both of the first two cases the numbers of words $\boldsymbol{w}$ is $c_{n-1}(\pi)$ and it is $c_{n-2}(\pi)$ in the last case. So, $c_n(\pi)$ satisfies Pell numbers recurrence and considering its initial values the statement holds. □

6 Counting via generating function

Here we give bivariate generating functions $C_\pi(x,y)$ where the coefficient of $x^n y^k$ is the number of Catalan words of length n having k descents and avoiding π, for each of the remaining pairs π of patterns of length 3. Plugging $y = 1$ in $C_\pi(x,y)$ we obtain $C_\pi(x) = C_\pi(x,1)$ where the coefficient of x^n is the number of Catalan words of length n avoiding π. All the obtained enumerating sequences are not yet recorded in [11], except that for: $\pi = \{100, 120\}$ and for $\pi = \{110, 120\}$ (see Corollary 3) and presumably for $\pi = \{100, 210\}$ (see Corollary 15). In almost all the proofs of the next propositions the desired generating function is the solution of a functional equation satisfied by it.

Proposition 18. *If* $\pi = \{000, 021\}$*, then*

$$C_\pi(x,y) = -\frac{x^4y + x^2y + 1}{x^2 + x - 1}.$$

Proof: Here we need the generating function for the Fibonacci numbers $C_{000,010}(x) = \frac{1}{1-x-x^2}$ for the set in Proposition 14. Note that words in $\mathcal{C}(000,010)$ have no descents.

Let $\boldsymbol{w}$ be a non-empty word in $\mathcal{C}(\pi)$. Then $\boldsymbol{w}$ has one of the following forms:

- $\boldsymbol{w} = 0(\boldsymbol{u}+1)$ where $\boldsymbol{u} \in \mathcal{C}(000,010)$; the generating function for these words is $x \cdot \frac{1}{1-x-x^2}$,
- $\boldsymbol{w} = 00(\boldsymbol{u}+1)$ where $\boldsymbol{u} \in C(000,010)$; the generating function for these words is $x^2 \cdot \frac{1}{1-x-x^2}$,
- $\boldsymbol{w} = 0(\boldsymbol{u}+1)0$ where $\boldsymbol{u} \in \mathcal{C}(000,010)$; the generating function for these words is $y \cdot x^2 \cdot \frac{1}{1-x-x^2}$,
- $\boldsymbol{w} = 0101(\boldsymbol{u}+2)$ where $\boldsymbol{u} \in \mathcal{C}(000,010)$; the generating function for these words is $y \cdot x^4 \cdot \frac{1}{1-x-x^2}$.

Combining these cases and adding 1 corresponding to the empty word we have

$$C_\pi(x,y) = 1 + x(1+x) \cdot \frac{1}{1-x-x^2} + yx^2(1+x^2)\left(\frac{1}{1-x-x^2}\right),$$

which after calculation gives the desired result. □

Corollary 2. *If* $\pi = \{000, 021\}$*, then*

$$C_\pi(x) = -\frac{x^4 + x^2 + 1}{x^2 + x - 1} = 1 + x + 3x^2 + 4x^3 + 8x^4 + 12x^5 + 20x^6 + 32x^7 + O(x^6).$$

Proposition 19. *If* $\pi = \{100, 120\}$*, then*

$$C_\pi(x,y) = -\frac{(x-1)^2}{x^3y - 2x^2 + 3x - 1}.$$

Proof: A word $\boldsymbol{w} \in \mathcal{C}(\pi)$ is in one of the following cases:

- $\boldsymbol{w}$ is a w.i. Catalan word,
- $\boldsymbol{w} = \boldsymbol{u}(m-1)(\boldsymbol{v}+m)$ where $\boldsymbol{u}$ is a w.i. Catalan word other than $00\cdots0$, m is the largest (last) entry of $\boldsymbol{u}$ and $\boldsymbol{v} \in C(\pi)$.

The generating function for the words of the first form is $\frac{1-x}{1-2x}$ and the generating function for the words of the second form is

$$\left(\frac{1-x}{1-2x} - \frac{1}{1-x}\right) \cdot x \cdot y \cdot C_\pi(x,y).$$

Combining these cases we deduce the functional equation below which solution gives the desired result:

$$C_\pi(x,y) = \left(\frac{1-x}{1-2x} - \frac{1}{1-x}\right) \cdot x \cdot y \cdot C_\pi(x,y) + \frac{1-x}{1-2x}.$$

□

Proposition 20. *If* $\pi = \{110, 120\}$*, then*

$$C_\pi(x,y) = -\frac{(x-1)^2}{x^3y - 2x^2 + 3x - 1}.$$

Proof: A word $\boldsymbol{w} \in \mathcal{C}(\pi)$ is in one of the following cases:

- $\boldsymbol{w}$ is a w.i. Catalan word,
- $\boldsymbol{w} = \boldsymbol{u}(m+1)\boldsymbol{v}$ where $\boldsymbol{u}$ is a non-empty w.i. Catalan word, m is the largest (last) entry of $\boldsymbol{u}$ and $\boldsymbol{v}$ is a word of the form $mm\cdots m(\boldsymbol{x}+m+1)$ with at least one m in its prefix and $\boldsymbol{x} \in \mathcal{C}(\pi)$.

The generating function for the words of the first form is $\frac{1-x}{1-2x}$.
For the second form, the generating function for the words $\boldsymbol{u}$ is $\frac{1-x}{1-2x} - 1 = \frac{x}{1-2x}$, and the generating function for the words $mm\cdots m(\boldsymbol{x}+m+1)$ is $\frac{x}{1-x}\cdot C_\pi(x,y)$. Thus, the generating function for the words of the second form is

$$\frac{x}{1-2x}\cdot x\cdot y\cdot \frac{x}{1-x}\cdot \mathcal{C}_\pi(x,y).$$

Combining these cases we deduce the functional equation

$$C_\pi(x,y) = \frac{x}{1-2x}\cdot x\cdot y\cdot \frac{x}{1-x}\cdot C_\pi(x,y) + \frac{1-x}{1-2x}.$$

□

The functional equations in the proofs of Propositions 19 and 20 are different but the resulting bivariate generating functions are the same. Instantiating y by 1 in $C_\pi(x,y)$ of these propositions we have the next corollary.

Corollary 3. *If* $\pi = \{100, 120\}$ *or* $\pi = \{110, 120\}$*, then*

$$C_\pi(x) = -\frac{(x-1)^2}{x^3 - 2x^2 + 3x - 1} = 1 + x + 2x^2 + 5x^3 + 12x^4 + 28x^5 + 65x^6 + 151x^7 + O(x^8),$$

and $c_n(\pi)$ *is the sequence* `A034943` *in [11].*

Proposition 21. *If* $\pi = \{021, 110\}$*, then*

$$C_\pi(x,y) = -\frac{x^5y + x^4y - x^4 - x^3y + 4x^3 - 6x^2 + 4x - 1}{(2x-1)(x-1)^3}.$$

Proof: A non-empty word $\boldsymbol{w} \in \mathcal{C}(\pi)$ is in one of the following cases:

- $\boldsymbol{w} = 0\boldsymbol{u}$ where $\boldsymbol{u} \in \mathcal{C}(\pi)$; the generating function for these words is $x\cdot C_\pi(x,y)$,
- $\boldsymbol{w} = 0(\boldsymbol{u}+1)$ where $\boldsymbol{u}$ is a non-empty w.i. Catalan word; the generating function for these words is $x\cdot\frac{x}{1-2x}$,
- $\boldsymbol{w} = 01\boldsymbol{u}$ where $\boldsymbol{u}$ is a non-empty w.i. Catalan word; the generating function for these words is $x^2\cdot y\cdot\frac{x}{1-2x}$,
- $\boldsymbol{w} = \boldsymbol{u}0\cdots 0$ where $\boldsymbol{u}$ is a s.i. Catalan word of length at least three and $\boldsymbol{w}$ ending by at least one 0; the generating function for these words is $y\cdot\frac{x^4}{(1-x)^2}$.

Combining these cases and considering the empty word which contributes with 1 to $C_\pi(x,y)$, we deduce the functional equation

$$C_\pi(x,y) = 1 + x\cdot C_\pi(x,y) + x\cdot\frac{x}{1-2x} + x^2\cdot y\cdot\frac{x}{1-2x} + y\cdot\frac{x^4}{(1-x)^2}.$$

□

Corollary 4. *If* $\pi = \{021, 110\}$*, then*

$$C_\pi(x) = -\frac{x^5 + 3x^3 - 6x^2 + 4x - 1}{(2x-1)(x-1)^3} = 1 + x + 2x^2 + 5x^3 + 12x^4 + 26x^5 + 53x^6 + 105x^7 + O(x^8).$$

Proposition 22. *If* $\pi = \{110, 201\}$*, then*

$$C_\pi(x,y) = \frac{x^4y - x^3 + 3x^2 - 3x + 1}{(x-1)(x^3y - 2x^2 + 3x - 1)}.$$

Proof: A non-empty word $\boldsymbol{w} \in \mathcal{C}(\pi)$ is in one of the following cases:

- $\boldsymbol{w} = 0\boldsymbol{u}$ where $\boldsymbol{u} \in \mathcal{C}(\pi)$; the generating function for these words is $x \cdot C_\pi(x,y)$,
- $\boldsymbol{w} = 0(\boldsymbol{u}+1)$ where $\boldsymbol{u}$ is a non-empty word in $\mathcal{C}(\pi)$; the generating function for these words is $x \cdot (C_\pi(x,y) - 1)$,
- $\boldsymbol{w} = \boldsymbol{u}0\cdots 0$ where $\boldsymbol{u}$ is a s.i. Catalan word of length at least 2 and $\boldsymbol{w}$ ending by at least one 0; the generating function for these words is $y \cdot \frac{x^3}{(1-x)^2}$,
- $\boldsymbol{w} = 010\cdots 0(\boldsymbol{u}+1)$ where $\boldsymbol{u}$ is a non-empty word in $\mathcal{C}(\pi)$ and $\boldsymbol{w}$ beginning by 010; the generating function for these words is $y \cdot x^3 \cdot \frac{1}{1-x} \cdot (C_\pi(x) - 1)$.

Combining these cases and adding 1 corresponding to the empty word we deduce the functional equation

$$C_\pi(x,y) = 1 + x \cdot C_\pi(x,y) + x \cdot (C_\pi(x,y) - 1) + y \cdot \frac{x^3}{(1-x)^2} + y \cdot x^3 \cdot \frac{1}{1-x} \cdot (C_\pi(x,y) - 1).$$

□

Corollary 5. *If* $\pi = \{110, 201\}$*, then*

$$C_\pi(x) = \frac{x^4 - x^3 + 3x^2 - 3x + 1}{(x-1)(x^3 - 2x^2 + 3x - 1)} = 1 + x + 2x^2 + 5x^3 + 13x^4 + 32x^5 + 76x^6 + 178x^7 + O(x^8).$$

Proposition 23. *If* $\pi = \{102, 201\}$*, then*

$$\mathcal{C}_\pi(x,y) = \frac{x^5y + x^4y^2 - x^5 - 5x^4y + 5x^4 + 6x^3y - 10x^3 - 2x^2y + 10x^2 - 5x + 1}{(-x+1)(x^2y - x^2 + 2x - 1)(x^2y - 2x^2 + 3x - 1)}.$$

Proof: A non-empty word $\boldsymbol{w} \in \mathcal{C}(\pi)$ is in one of the following cases:

- $\boldsymbol{w} = 0\boldsymbol{u}$ where $\boldsymbol{u} \in \mathcal{C}(\pi)$; the generating function for these words is $x \cdot C_\pi(x,y)$,
- $\boldsymbol{w} = 0(\boldsymbol{u}+1)$ where $\boldsymbol{u}$ is a non-empty word in $\mathcal{C}(\pi)$; the generating function for these words is $x \cdot (C_\pi(x,y) - 1)$,
- $\boldsymbol{w} = 0(\boldsymbol{v}+\mathbf{1})0\cdots 0$ where $\boldsymbol{u}$ is a non-empty word in $\mathcal{C}(\pi)$ and $\boldsymbol{w}$ ending by at least one 0; the generating function for these words is $y \cdot x^2 \cdot \frac{1}{1-x} \cdot (C_\pi(x,y) - 1)$,
- $\boldsymbol{w} = 01\cdots 1\boldsymbol{u}$ where $\boldsymbol{u}$ is a binary word beginning by a 0 and different from $0\cdots 0$, or equivalently, $\boldsymbol{u}$ a word in $\mathcal{C}(012)$ other than $0\cdots 0$; the generating function for these words is $\frac{x^2}{1-x} \cdot y \cdot \left(C_{012}(x,y) - \frac{1}{1-x}\right) = \frac{x^2 \cdot y}{1-x} \cdot \left(\frac{1-x+x^2-x^2y}{1-2x+x^2-x^2y} - \frac{1}{1-x}\right)$, see Theorem 4 in [1].

Combining these cases and adding 1 corresponding to the empty word we deduce the functional equation

$$\begin{aligned} C_\pi(x,y) = \quad & 1 + x \cdot C_\pi(x,y) + x \cdot (C_\pi(x,y) - 1) + \frac{x^2 \cdot y}{1-x} \cdot (C_\pi(x,y) - 1) \\ & + \frac{x^2 \cdot y}{1-x} \cdot \left(\frac{1 - x + x^2 - x^2y}{1 - 2x + x^2 - x^2y} - \frac{1}{1-x}\right). \end{aligned}$$

□

Corollary 6. *If* $\pi = \{102, 201\}$*, then*

$$C_\pi(x) = \frac{x^4 - 4x^3 + 8x^2 - 5x + 1}{(x-1)(2x-1)(x^2 - 3x + 1)} = 1 + x + 2x^2 + 5x^3 + 14x^4 + 40x^5 + 113x^6 + 314x^7 + O(x^8).$$

Proposition 24. *If* $\pi = \{100, 110\}$*, then*

$$C_\pi(x,y) = \frac{x^4y - x^4 + 2x^3 - 2x + 1}{(x-1)(x^3y - 2x^3 + x^2 + 2x - 1)}.$$

Proof: A non-empty word in $\mathcal{C}(\pi)$ has one of the following forms:

- $0\boldsymbol{u}$ where $\boldsymbol{u} \in \mathcal{C}(\pi)$; the generating function for these words is $x \cdot C_\pi(x,y)$,
- $0(\boldsymbol{u}+1)$ where $\boldsymbol{u}$ is a non-empty word in $\mathcal{C}(\pi)$; the generating function for these words is $x \cdot (C_\pi(x,y)-1)$,
- $\boldsymbol{u}(m+1)(m+2)\boldsymbol{v}$ where $\boldsymbol{u}$ and $\boldsymbol{v}$ are non-empty s.i. Catalan words, m is the largest entry of $\boldsymbol{u}$ and the length of $\boldsymbol{v}$ is less than or equal to that of $\boldsymbol{u}$; the generating function for these words is $y \cdot x^4 \cdot (x+1) \cdot \frac{1}{(1-x^2)^2}$,
- $\boldsymbol{u}(m+1)\boldsymbol{u}(\boldsymbol{v}+m+1)$ where $\boldsymbol{u}$ is a non-empty s.i. Catalan word, m is the largest entry of $\boldsymbol{u}$ and $\boldsymbol{v} \in \mathcal{C}(\pi)$; the generating function for these words is $y \cdot x^3 \cdot \frac{1}{1-x^2} \cdot C_\pi(x,y)$.

Combining these cases and adding 1 corresponding to the empty word we deduce the functional equation

$$C_\pi(x,y) = 1 + x \cdot C_\pi(x,y) + x \cdot (C_\pi(x,y)-1) + y \cdot x^4 \cdot \frac{(x+1)}{(1-x^2)^2} + y \cdot x^3 \cdot \frac{1}{1-x^2} \cdot C_\pi(x,y).$$

□

Corollary 7. *If* $\pi = \{100, 110\}$, *then*

$$C_\pi(x) = \frac{-2x^3+2x-1}{(x-1)(x^3-x^2-2x+1)} = 1 + x + 2x^2 + 5x^3 + 12x^4 + 28x^5 + 64x^6 + 145x^7 + O(x^8).$$

Proposition 25. *If* $\pi = \{000, 110\}$, *then*

$$C_\pi(x,y) = \frac{x^3y + x^3 - x^2 - x + 1}{(-x+1)(x^3 - x^2y - x^2 - x + 1)}.$$

Proof: A non-empty word in $\mathcal{C}(\pi)$ has one of the following forms:

- $0(\boldsymbol{u}+1)$ where $\boldsymbol{u} \in \mathcal{C}(\pi)$; the generating function for these words is $x \cdot C_\pi(x,y)$,
- $\boldsymbol{u}\boldsymbol{u}(\boldsymbol{v}+m+1)$ where $\boldsymbol{u}$ is a non-empty s.i. Catalan word, m is the largest (last) entry of $\boldsymbol{u}$ and $\boldsymbol{v} \in \mathcal{C}(\pi)$; the generating function for these words is $y \cdot \frac{x^2}{1-x^2} \cdot C_\pi(x,y)$,
- $\boldsymbol{u}(m+1)\boldsymbol{v}$ where $\boldsymbol{u}$ and m are as above, and $\boldsymbol{v}$ is a non-empty s.i. Catalan word of length less than that of $\boldsymbol{u}$; the generating function for these words is $y \cdot x^3 \cdot (1+x) \cdot \frac{1}{(1-x^2)^2}$.

Combining these cases and adding 1 corresponding to the empty word we deduce the functional equation

$$C_\pi(x,y) = 1 + x \cdot C_\pi(x,y) + y \cdot \frac{x^2}{1-x^2} \cdot C_\pi(x,y) + y \cdot x^3 \cdot (1+x) \cdot \frac{1}{(1-x^2)^2}.$$

□

Corollary 8. *If* $\pi = \{000, 110\}$, *then*

$$C_\pi(x) = \frac{2x^3 - x^2 - x + 1}{(-x+1)(x^3 - 2x^2 - x + 1)} = 1 + x + 2x^2 + 4x^3 + 8x^4 + 15x^5 + 28x^6 + 51x^7 + O(x^8).$$

Proposition 26. *If* $\pi = \{000, 102\}$ *or* $\pi = \{000, 201\}$, *then*

$$C_\pi(x,y) = \frac{yx^2 - 1}{yx^4 + yx^2 + x^2 + x - 1}.$$

Proof: If $\pi = \{000, 102\}$, then a non-empty word in $\mathcal{C}(\pi)$ has one of the following forms:

- $0(\boldsymbol{u}+1)$ where $\boldsymbol{u} \in \mathcal{C}(\pi)$; the generating function for these words is $x \cdot C_\pi(x,y)$,
- $00(\boldsymbol{u}+1)$ where $\boldsymbol{u} \in \mathcal{C}(\pi)$; the generating function for these words is $x^2 \cdot C_\pi(x,y)$,
- $0(\boldsymbol{u}+1)0$ where $\boldsymbol{u}$ is a non-empty word in $\mathcal{C}(\pi)$; the generating function for these words is $y \cdot x^2 \cdot (C_\pi(x,y)-1)$,
- $01(\boldsymbol{u}+2)01$ where $\boldsymbol{u} \in \mathcal{C}(\pi)$; the generating function for these words is $y \cdot x^4 \cdot C_\pi(x,y)$.

Similarly, if $\pi = \{000, 201\}$ and $\boldsymbol{w}$ is a non-empty word in $\mathcal{C}(\pi)$, then $\boldsymbol{w}$ has either one of the first three forms above, or

– $\boldsymbol{w} = 0101(\boldsymbol{u}+2)$ where $\boldsymbol{u} \in \mathcal{C}(\pi)$; the generating function for these words is $y \cdot x^4 \cdot C_\pi(x,y)$.

In both cases we obtain the functional equation

$$C_\pi(x,y) = 1 + x \cdot C_\pi(x,y) + x^2 \cdot C_\pi(x,y) + y \cdot x^2 \cdot (C_\pi(x,y) - 1) + y \cdot x^4 \cdot C_\pi(x,y).$$

□

Corollary 9. *If* $\pi = \{000, 102\}$ *or* $\pi = \{000, 201\}$*, then*

$$C_\pi(x) = \frac{x^2-1}{x^4+2x^2+x-1} = 1 + x + 2x^2 + 4x^3 + 9x^4 + 18x^5 + +38x^6 + 78x^7 + O(x^8).$$

Proposition 27. *If* $\pi = \{000, 120\}$*, then*

$$C_\pi(x,y) = -\frac{x^4y + x^3y + 1}{x^4y + x^2 + x - 1}.$$

Proof: A non-empty word in $\mathcal{C}(\pi)$ has one of the following forms:

– $0(\boldsymbol{u}+1)$ where $\boldsymbol{u} \in \mathcal{C}(\pi)$; the generating function for these words is $x \cdot C_\pi(x,y)$,

– $00(\boldsymbol{u}+1)$ where $\boldsymbol{u} \in \mathcal{C}(\pi)$; the generating function for these words is $x^2 \cdot C_\pi(x,y)$,

– $0101(\boldsymbol{u}+2)$ where $\boldsymbol{u} \in \mathcal{C}(\pi)$; the generating function for these words is $y \cdot x^4 \cdot C_\pi(x,y)$.

Apart from these general cases, there are two other fixed length ones:

– 010; the corresponding generating function is $y \cdot x^3$,

– 0110; the corresponding generating function is $y \cdot x^4$.

Combining these cases and adding 1 corresponding to the empty word we deduce the functional equation

$$C_\pi(x,y) = 1 + x \cdot C_\pi(x,y) + x^2 \cdot C_\pi(x,y) + y \cdot x^4 \cdot C_\pi(x,y) + y \cdot x^3 + y \cdot x^4.$$

□

Corollary 10. *If* $\pi = \{000, 120\}$*, then*

$$C_\pi(x) = -\frac{x^4+x^3+1}{x^4+x^2+x-1} = 1 + x + 2x^2 + 4x^3 + 8x^4 + 13x^5 + 23x^6 + 40x^7 + O(x^8).$$

Proposition 28. *If* $\pi = \{201, 210\}$*, then*

$$C_\pi(x,y) = \frac{x^4y - 2x^4 - 3x^3y + 7x^3 + x^2y - 9x^2 + 5x - 1}{(2x-1)(x-1)(x^2y - 2x^2 + 3x - 1)}.$$

Proof: A non-empty word $\boldsymbol{w}$ in $\mathcal{C}(\pi)$ has one of the following forms:

– $0\boldsymbol{u}$ where $\boldsymbol{u} \in \mathcal{C}(\pi)$; the generating function for these words is $x \cdot C_\pi(x,y)$,

– $0(\boldsymbol{u}+1)$ where $\boldsymbol{u}$ is a non-empty word in $\mathcal{C}(\pi)$; the generating function for these words is $x \cdot (C_\pi(x,y) - 1)$,

– $01\cdots1\boldsymbol{u}$ where $\boldsymbol{u}$ is a non-empty word in $\mathcal{C}(\pi)$ and 01 is a prefix of $\boldsymbol{w}$; the generating function for these words is $y \cdot \frac{x^2}{1-x} \cdot (C_\pi(x,y) - 1)$,

– $0(\boldsymbol{u}+1)0\ldots0$ where $\boldsymbol{u}$ is a w.i. Catalan word other than $0\cdots0$ and $\boldsymbol{w}$ ending by a 0; the generating function for these words is $y \cdot \frac{x^2}{1-x} \cdot (\frac{1-x}{1-2x} - \frac{1}{1-x})$.

Combining these cases and adding 1 corresponding to the empty word we deduce the functional equation

$$C_\pi(x,y) = 1 + x \cdot C_\pi(x,y) + x \cdot (C_\pi(x,y) - 1) + y \cdot \frac{x^2}{1-x} \cdot (C_\pi(x,y) - 1) + y \cdot \frac{x^2}{1-x} \cdot \left(\frac{1-x}{1-2x} - \frac{1}{1-x}\right).$$

□

Corollary 11. *If* $\pi = \{201, 210\}$, *then*

$$C_\pi(x) = \frac{x^4 - 4x^3 + 8x^2 - 5x + 1}{(x-1)(2x-1)(x^2-3x+1)} = 1 + x + 2x^2 + 5x^3 + 14x^4 + 40x^5 + 113x^6 + 314x^7 + O(x^8).$$

Proposition 29. *If* $\pi = \{102, 210\}$, *then*

$$C_\pi(x,y) =$$

$$\frac{2x^7y - 2x^7 + 13x^6 - 10x^6y - x^6y^2 - 36x^5 + 19x^5y + 55x^4 - 17x^4y - 50x^3 + 7x^3y + 27x^2 - yx^2 - 8x + 1}{(x-1)^3(2x-1)^2(x^2y - x^2 + 2x - 1)}.$$

Proof: A non-empty word $\boldsymbol{w}$ in $\mathcal{C}(\pi)$ has one of the following forms:

- $0\boldsymbol{u}$ where $\boldsymbol{u} \in \mathcal{C}(\pi)$; the generating function for these words is $x \cdot C_\pi(x,y)$,
- $0(\boldsymbol{u}+1)$ where $\boldsymbol{u}$ is a non-empty word in $\mathcal{C}(\pi)$; the generating function for these words is $x \cdot (C_\pi(x,y) - 1)$,
- $01\cdots1\boldsymbol{u}$ where $\boldsymbol{u}$ is a non-empty word in $\mathcal{C}(012)$, and $\boldsymbol{w}$ begins by 01; the generating function for these words is $y \cdot \frac{x^2}{1-x} \cdot (C_{012}(x,y) - 1) = y \cdot \frac{x^2}{1-x} \cdot (\frac{1-x+x^2-x^2y}{1-2x+x^2-x^2y} - 1)$ (see Theorem 4 in [1] for the generating function of $C_{012}(x,y)$),
- $0(\boldsymbol{u}+1)\boldsymbol{v}$ where $\boldsymbol{u}$ is a w.i. Catalan word of length at least 2 different from $0\cdots0$ and $\boldsymbol{v}$ is a non-empty word in $\mathcal{C}(010, 012)$ (see Proposition 5); the generating function for these words is $x \cdot (\frac{1-x}{1-2x} - \frac{1}{1-x}) \cdot y \cdot C_{010,012}(x,y) = y \cdot x \cdot \frac{x}{(1-x)^2} \cdot (\frac{1-x}{1-2x} - \frac{1}{1-x})$.

Combining these cases and adding 1 corresponding to the empty word we deduce the functional equation

$$\begin{aligned} C_\pi(x,y) = \quad & 1 + x \cdot C_\pi(x,y) + x \cdot (C_\pi(x,y) - 1) + y \cdot \frac{x^2}{1-x} \cdot \left(\frac{1-x+x^2-x^2y}{1-2x+x^2-x^2y} - 1 \right) + \\ & y \cdot \frac{x^2}{(1-x)^2} \cdot \left(\frac{1-x}{1-2x} - \frac{1}{1-x} \right). \end{aligned}$$

□

Corollary 12. *If* $\pi = \{102, 210\}$, *then*

$$\begin{aligned} C_\pi(x) &= \frac{2x^6 - 17x^5 + 38x^4 - 43x^3 + 26x^2 - 8x + 1}{(x-1)^3(2x-1)^3} \\ &= 1 + x + 2x^2 + 5x^3 + 14x^4 + 40x^5 + 111x^6 + 295x^7 + O(x^8). \end{aligned}$$

Proposition 30. *If* $\pi = \{100, 102\}$, *then*

$$C_\pi(x,y) = \frac{x^5y - x^4y - x^3 + 2x^3y + 3x^2 - x^2y - 3x + 1}{(x-1)(x^4y - x^3y - 2x^2 + x^2y + 3x - 1)}.$$

Proof: A non-empty word in $\mathcal{C}(\pi)$ has one of the following forms:

- $0\boldsymbol{u}$ where $\boldsymbol{u} \in \mathcal{C}(\pi)$; the generating function for these words is $x \cdot C_\pi(x,y)$,
- $0(\boldsymbol{u}+1)$ where $\boldsymbol{u}$ is a non-empty word in $\mathcal{C}(\pi)$; the generating function for these words is $x \cdot (C_\pi(x,y) - 1)$,
- $0(\boldsymbol{u}+1)0$ where $\boldsymbol{u}$ is as above; the generating function for these words is $y \cdot x^2 \cdot (C_\pi(x,y) - 1)$,
- $011\cdots1(\boldsymbol{u}+2)01$ where $\boldsymbol{u}$ is as above; the generating function for these words is $y \cdot \frac{x^4}{1-x}(C_\pi(x,y) - 1)$,
- $\boldsymbol{uv}$ where $\boldsymbol{u}$ and $\boldsymbol{v}$ are binary words of length at least 2 of the form $011\cdots1$; the generating function for these words is $y \cdot \frac{x^4}{(1-x)^2}$.

Combining these cases and adding 1 corresponding to the empty word we deduce the functional equation

$$C_\pi(x,y) = 1+x\cdot C_\pi(x,y)+x\cdot(C_\pi(x,y)-1)+y\cdot x^2\cdot(C_\pi(x,y)-1)+y\cdot\frac{x^4}{1-x}(C_\pi(x,y)-1)+y\cdot\frac{x^4}{(1-x)^2}.$$

□

Corollary 13. *If* $\pi = \{100, 102\}$*, then*

$$C_\pi(x) = \frac{x^5-x^4+x^3+2x^2-3x+1}{(x-1)(x^4-x^3-x^2+3x-1)} = 1+x+2x^2+5x^3+13x^4+34x^5+87x^6+220x^7+O(x^8).$$

In the proof of the next proposition we need the following lemma where the generating functions for some particular subsets of $\mathcal{C}(000, 210)$ are given.

Lemma 2. *The bivariate generating function corresponding to*

1. *the set* $\mathcal{A}$ *of words* $\boldsymbol{uu}$ *with* $\boldsymbol{u}$ *a non-empty s.i. Catalan word is* $A(x,y) = x^2 + yx^2 \cdot \frac{x^2}{1-x^2}$ *and* $A(x) = \frac{x^2}{1-x^2}$*;*
2. *the set* $\mathcal{B}$ *of words* $\boldsymbol{uv}$ *with* $\boldsymbol{u}$ *and* $\boldsymbol{v}$ *non-empty s.i. Catalan words and the length of* $\boldsymbol{v}$ *is less than or equal to that of* $\boldsymbol{u}$ *is* $B(x,y) = y \cdot \frac{x^2}{1-x^2} \cdot \frac{1}{1-x}$*;*
3. *the set* $\mathcal{D}$ *of words* $\boldsymbol{uv}$ *with* $\boldsymbol{u}$ *and* $\boldsymbol{v}$ *non-empty s.i. Catalan words and the length of* $\boldsymbol{v}$ *is less than that of* $\boldsymbol{u}$ *is* $D(x,y) = y \cdot \frac{x^2}{1-x^2} \cdot \frac{x}{1-x}$.

Proof: 1. For any even n there is exactly one word of this form, so the monovariate corresponding generating function is $\frac{x^2}{1-x^2}$; and only words of length larger than two have one descent.
2. The transformation $(\boldsymbol{uu}, \boldsymbol{x}) \mapsto \boldsymbol{uxu}$ where $\boldsymbol{uu} \in \mathcal{A}$ and $\boldsymbol{x}$ is a s.i. Catalan word defines a bijection between pairs of such words and $\mathcal{B}$, and thus $B(x,y) = y \cdot A(x) \cdot \frac{1}{1-x}$.
3. Similarly as point 2. □

Proposition 31. *If* $\pi = \{000, 210\}$*, then*

$$\mathcal{C}_\pi(x,y) = -\frac{(x+1)(x^3+x^3y-2x+1)}{(x^4-x^4y+x^3-2x^2-x+1)(x^2+x-1)}.$$

Proof: A non-empty word in $\mathcal{C}(000, 210)$ has one of the following forms:

- $\boldsymbol{u}$ with $\boldsymbol{u} \in \mathcal{D}$ and $\mathcal{D}$ as in Lemma 2; the generating function for these words is $D(x,y) = y \cdot \frac{x^2}{1-x^2} \cdot \frac{x}{1-x}$,
- $\boldsymbol{u}(m+1)(m+1)(\boldsymbol{x}+m+2)\boldsymbol{v}$ with $\boldsymbol{u}$ and $\boldsymbol{v}$ non-empty s.i. Catalan words and the length of $\boldsymbol{v}$ is less than or equal to that of $\boldsymbol{u}$, m is the largest entry of $\boldsymbol{u}$, and $\boldsymbol{x} \in \mathcal{C}(000, 010)$; the generating function for these words is $B(x,y) \cdot x^2 \cdot \frac{1}{1-x-x^2} = y \cdot \frac{x^2}{1-x^2} \cdot \frac{1}{1-x} \cdot x^2 \cdot \frac{1}{1-x-x^2}$ (see Lemma 2 and Proposition 14),
- $0(\boldsymbol{u}+1)$ where $\boldsymbol{u} \in \mathcal{C}(\pi)$; the generating function for these words is $x \cdot C_\pi(x,y)$,
- $\boldsymbol{uu}(\boldsymbol{v}+m+1)$ where $\boldsymbol{u}$ is a non-empty s.i. Catalan word, m the largest entry of $\boldsymbol{u}$ and $\boldsymbol{v} \in \mathcal{C}(\pi)$; the generating function for these words is $A(x,y) \cdot C_\pi(x,y)$.

Combining these cases and adding 1 corresponding to the empty word we deduce the functional equation

$$C_\pi(x,y) = 1+y\cdot\frac{x^2}{1-x^2}\cdot\frac{x}{1-x}+y\cdot\frac{x^4}{1-x^2}\cdot\frac{1}{1-x}\cdot\frac{1}{1-x-x^2}+x\cdot C(x,y)+\left(x^2+y\cdot\frac{x^4}{1-x^2}\right)\cdot C(x,y).$$

□

Corollary 14. *If* $\pi = \{000, 210\}$*, then*

$$C_\pi(x) = -\frac{(x+1)(2x^3-2x+1)}{(x^3-2x^2-x+1)(x^2+x-1)} = 1+x+2x^2+4x^3+9x^4+18x^5+37x^6+72x^7+O(x^8).$$

In the proof of the next proposition we need the following lemma where the generating functions for two subsets of $\mathcal{C}(100, 210)$ are given.

Lemma 3.

1. *The generating function corresponding to the set $\mathcal{E}$ of words $\boldsymbol{uv}$ with $\boldsymbol{u}$ a w.i. Catalan word, $\boldsymbol{v}$ a non-empty s.i. Catalan word and the largest entry of $\boldsymbol{v}$ is equal to that of $\boldsymbol{u}$ minus 1 is $\frac{x^3}{(x-1)(x^2+x-1)}$.*
2. *The generating function corresponding to the set $\mathcal{F}$ of words $\boldsymbol{uv}$ with $\boldsymbol{u}$ a w.i. Catalan word, $\boldsymbol{v}$ a non-empty s.i. Catalan word and the largest entry of $\boldsymbol{v}$ is less than that of $\boldsymbol{u}$ minus 1 is $\frac{x^3}{(x-1)(x^2+x-1)} \cdot \frac{x}{1-2x}$.*

Proof: 1. If $\mathcal{E}_n$ is the set of words of length n in $\mathcal{E}$, then $\mathcal{E}_n = \varnothing$ for $0 \leq i \leq 2$, $\mathcal{E}_3 = \{010\}$ and $\mathcal{E}_4 = \{0010, 0110\}$. With $\boldsymbol{u}$ and $\boldsymbol{v}$ as above, for any $n \geq 3$, the transformation

$\boldsymbol{uv} \mapsto \boldsymbol{u}a\boldsymbol{v}$, with a the maximal entry of $\boldsymbol{u}$, transforms a word in $\mathcal{E}_n$ into one in $\mathcal{E}_{n+1}$ where the maximal entry occurs at least twice,

$\boldsymbol{uv} \mapsto \boldsymbol{u}(a+1)\boldsymbol{v}(b+1)$, with a and b the maximal entries of $\boldsymbol{u}$ and of $\boldsymbol{v}$ respectively, transforms a word in $\mathcal{E}_n$ into one in $\mathcal{E}_{n+2}$ where the maximal entry occurs once.

Any word in $\mathcal{E}_n$, $n \geq 5$, except $0 \cdots 010$, can be obtained in a unique way from either a word in $\mathcal{E}_{n-1}$ or in $\mathcal{E}_{n-2}$ by one of these transformations. This yields the recurrence $|\mathcal{E}_n| = 1 + |\mathcal{E}_{n-1}| + |\mathcal{E}_{n-2}|$ for $n \geq 5$, and the desired generating function is precisely that of the sequence $(|\mathcal{E}_n|)_{n\geq 0}$.

2. Any pair of words $(\boldsymbol{w}, \boldsymbol{x})$ with $\boldsymbol{w} = \boldsymbol{uv} \in \mathcal{E}$ (with $\boldsymbol{u}$ and $\boldsymbol{v}$ as above) and $\boldsymbol{x}$ a non-empty w.i. Catalan word can be transformed into the word $\boldsymbol{uxv} \in \mathcal{F}$, and $(\boldsymbol{w}, \boldsymbol{x}) \mapsto \boldsymbol{uxv}$ is a bijection, so the generating function for $\mathcal{F}$ is that for $\mathcal{E}$ multiplied by $\frac{x}{1-2x}$.

□

Proposition 32. *If $\pi = \{100, 210\}$, then*

$$C_\pi(x, y) = \frac{1-x}{1-2x} - \frac{x^3 y}{(2x-1)(2x^3 - x^3 y + x^2 - 3x + 1)}.$$

Proof: First we consider only words in $\mathcal{C}(\pi)$ having at least one descent, and we denote by $G(x, y)$ the corresponding generating function, and clearly $C_\pi(x, y) = \frac{1-x}{1-2x} + G(x, y)$.

A word in $\mathcal{C}(\pi)$ with at least one descent has one of the following forms:

- $\boldsymbol{u}(\boldsymbol{\alpha} + s + 1)(\boldsymbol{v} + s + t + 1)$ where $\boldsymbol{u}$ and $\boldsymbol{v}$ are both w.i. Catalan words, $\boldsymbol{\alpha}$ belongs to the set $\mathcal{E}$ defined in Lemma 3, and s is the largest symbol of $\boldsymbol{u}$ (and for convenience -1 if $\boldsymbol{u}$ is empty) and t that of $\boldsymbol{\alpha}$; the generating function for these words is $y \cdot \frac{x^3}{(x-1)(x^2+x-1)} \cdot \left(\frac{1-x}{1-2x}\right)^2$,
- $\boldsymbol{u}(\boldsymbol{\alpha} + s + 1)$ where $\boldsymbol{u}$ and s are as above, and $\boldsymbol{\alpha}$ belongs to the set $\mathcal{F}$ defined in Lemma 3; the generating function for these words is $y \cdot \frac{1-x}{1-2x} \cdot \frac{x^3}{(x-1)(x^2+x-1)} \cdot \frac{x}{1-2x}$,
- $\boldsymbol{u}(\boldsymbol{\alpha} + s + 1)(\boldsymbol{v} + s + t + 1)$ where $\boldsymbol{u}$ and s are as above, $\boldsymbol{\alpha}$ belongs to $\mathcal{E}$, $\boldsymbol{v}$ is a word in $\mathcal{C}(\pi)$ with at least one descent, and t is the largest symbol of $\boldsymbol{\alpha}$; the generating function for these words is $y \cdot \frac{1-x}{1-2x} \cdot \frac{x^3}{(x-1)(x^2+x-1)} \cdot G(x, y)$.

It follows that $G(x, y)$ satisfies the functional equation

$$\begin{aligned} G(x,y) \quad = \quad & y \cdot \frac{x^3}{(x-1)(x^2+x-1)} \cdot \left(\frac{1-x}{1-2x}\right)^2 + y \cdot \frac{1-x}{1-2x} \cdot \frac{x^3}{(x-1)(x^2+x-1)} \cdot \frac{x}{1-2x} + \\ & y \cdot \frac{1-x}{1-2x} \cdot \frac{x^3}{(x-1)(x^2+x-1)} \cdot G(x,y). \end{aligned}$$

Finally, solving it and adding the generating function for the Catalan words with no descents (that is, w.i. Catalan words) the statement holds. □

Corollary 15. *If $\pi = \{100, 210\}$, then*

$$C_\pi(x) = \frac{x-1}{2x-1} - \frac{x^3}{(2x-1)(x-1)(x^2+2x-1)} = 1 + x + 2x^2 + 5x^3 + 13x^4 + 34x^5 + 88x^6 + 225x^7 + O(x^8).$$

Numerical evidences let us believe that $c_n(100, 210)$ is the sequence `A267905` in [11], however we failed to prove this formally.

$\sigma\backslash\tau$	000	001	010	011	012	021	100	101	102	110	120	201	210
000	-	P. 14	P. 14	$u.c.$	$u.c.$	*C. 2*	s	P. 7	*C. 9*	*C. 8*	*C. 10*	*C. 9*	*C. 14*
001	-	-	P. 5	P. 5	P. 5	P. 13	P. 15	s	s	P. 13	P. 13	s	P. 12
010	-	-	-	P. 5	P. 5	s	s	s	s	s	s	s	s
011	-	-	-	-	P. 5	s	P. 6	s	s	s	P. 6	s	s
012	-	-	-	-	-	s	P. 13	P. 13	s	P. 13	s	s	s
021	-	-	-	-	-	-	P. 9	P. 9	P. 11	*C. 4*	P. 10	s	s
100	-	-	-	-	-	-	-	P. 17	*C. 13*	*C. 7*	C. 3	P. 16	C. 15
101	-	-	-	-	-	-	-	-	s	P. 9	P. 9	s	P. 8
102	-	-	-	-	-	-	-	-	-	P. 11	P. 8	*C. 6*	*C. 12*
110	-	-	-	-	-	-	-	-	-	-	C. 3	*C. 5*	s
120	-	-	-	-	-	-	-	-	-	-	-	s	s
201	-	-	-	-	-	-	-	-	-	-	-	-	*C. 11*
210	-	-	-	-	-	-	-	-	-	-	-	-	-

Table 2: Pairs $\{\sigma, \tau\}$ where τ is superfluous for σ are marked by s and those yielding ultimately constant enumerating sequences (see Proposition 4) by $u.c.$. The references are to the propositions or the corollaries where the enumerating sequences or generating functions are given. Pairs referred by the same proposition or corollary form a Wilf-equivalence class and enumerating results that are not yet recorded in [11] are italicized. Highlighted pairs are already enumerated in [2] in the context of ascent sequences, see Section 2.

7 Final remarks

Catalan words are in bijection with Dyck paths (see Figure 1) and thus pattern avoiding Catalan words correspond to restricted Dyck paths. For instance, a Catalan word avoiding 012 corresponds to a Dyck path of height at most two. In this context, it can be of interest to investigate how our results on pattern avoiding Catalan words translate to corresponding restricted Dyck paths.

Even if in this article we restrict ourselves to the avoidance of two patterns of length 3, some classes considered here can be trivially extended to larger length patterns, for instance $\mathcal{C}(102, 201) = \mathcal{C}(01012, 01201)$. In this light, it can be of interest to explore Catalan words avoiding patterns of length 4 or more, triples of patterns or generalized patterns.

Acknowledgements

The authors are grateful to the anonymous referees for their careful reading of the paper and their many insightful comments and suggestions.

References

[1] J.-L. Baril, S. Kirgizov, and V. Vajnovszki, Descent distribution on Catalan words avoiding a pattern of length at most three, *Discrete Mathematics*, **341**(9), 2608-2615, 2018.

[2] A. Baxter and L. Pudwell, Ascent sequences avoiding pairs of patterns, *Electronic Journal of Combinatorics*, **22**(1), P1.58, electronic, 2015.

[3] M. Bousquet-Mélou, A. Claesson, M. Dukes, and S. Kitaev, (2 + 2)-free posets, ascent sequences and pattern avoiding permutations, *Journal of Combinatorial Theory, Series A*, **117** (7), 884–909, 2010.

[4] L.R. Campbell, S. Dahlberg, R. Dorward, J. Gerhard, T. Grubb, C. Purcell, and B.E. Sagan, Restricted growth function patterns and statistics, *Discrete Mathematics and Theoretical Computer Science*, **18**(2), electronic, 2016.

[5] S. Corteel, M.A. Martinez, C.D. Savage, and M. Weselcouch, Patterns in inversion sequences I, *Discrete Mathematics and Theoretical Computer Science*, **18**(2), electronic, 2016.

[6] P. Duncan and E. Steingrímsson, Pattern avoidance in ascent sequences, *The Electronic Journal of Combinatorics*, **18**(1), P226, electronic, 2011.

[7] S. Kitaev, *Patterns in Permutations and Words*, Springer, 2011.

[8] D.E. Knuth, *The Art of Computer Programming, volume 1, Fundamental Algorithms*. Addison-Wesley, 1968.

[9] Z. Lin and S. Fu, On 1212-avoiding restricted growth functions, *The Electronic Journal of Combinatorics*, **24**(1), P53, electronic, 2017.

[10] T. Mansour and M. Shattuck, Pattern avoidance in inversion sequences, *Pure Mathematics and Applications* **25**(2), 157-176, 2015.

[11] OEIS Foundation Inc. *On-Line Encyclopedia of Integer Sequences*, `http://oeis.org`, 2011.

[12] R.P. Stanley, *Enumerative Combinatorics*, volume 2. Cambridge University Press, 1999.

[13] V. Vajnovszki, A loopless generation of bitstrings without p consecutive ones, *Discrete Mathematics and Theoretical Computer Science*–Springer, 227-240, 2001.

[14] C. Yan and Z. Lin, Inversion sequences avoiding pairs of patterns, `https://arxiv.org/pdf/1912.03674.pdf`, electronic, 2019.

Discrete Mathematics Letters
www.dmlett.com

Discrete Math. Lett. **7** (2021) 5–10
DOI: 10.47443/dml.2021.0032

Research Article

Bijections from Dyck and Motzkin meanders with catastrophes to pattern avoiding Dyck paths

Jean-Luc Baril*, Sergey Kirgizov

Laboratoire d'Informatique de Bourgogne, Université de Bourgogne Franche-Comté, France

(Received: 29 March 2021. Received in revised form: 26 April 2021. Accepted: 27 April 2021. Published online: 28 April 2021.)

© 2021 the authors. This is an open access article under the CC BY (International 4.0) license (www.creativecommons.org/licenses/by/4.0/).

Abstract

In this note, we present constructive bijections from Dyck and Motzkin meanders with catastrophes to Dyck paths avoiding some patterns. As a byproduct, we deduce correspondences from Dyck and Motzkin excursions to restricted Dyck paths.

Keywords: bijection; Dyck and Motzkin paths; meander and excursion with catastrophes; pattern avoidance.

2020 Mathematics Subject Classification: 05A15, 05A19.

1. Introduction and notations

The domain of lattice paths provides a very fertile ground for the combinatorial community. They have many applications in computer science, queuing theory, biology and physics [17], and there are a multitude of one-to-one correspondences with various combinatorial objects such as directed animals, pattern avoiding permutations, bargraphs, RNA structures and so on [4, 8, 17]. A recurring problem in combinatorics is the enumeration of these paths with respect to their length and other statistics [2, 3, 6, 11–14, 16]. In the literature, Dyck and Motzkin paths are the most often considered, possibly because they are, respectively, counted by the famous Catalan and Motzkin numbers (see A108 and A1006 in the Sloane's On-line Encyclopedia of Integer Sequences [15]).

Throughout this note, a *lattice path* is defined by a starting point $(0,0)$, an ending point (n,k) with $n,k \geq 0$, it consists of steps lying in $S = \{(1,i) : i \in \mathbb{Z}, i \leq 1\}$, and it never goes below the x-axis. The length of a path is the number of its steps. We denote by ϵ the empty path, *i.e.*, the path of length zero. Constraining the steps to be in $\{(1,1),(1,-1)\}$ or $\{(1,1),(1,0),(1,-1)\}$, and fixing the end point on the x-axis, we retrieve the well-known definition of *Dyck and Motzkin paths* [17] respectively. Let $\mathcal{D}_n$ be the set of Dyck paths of semilength n, we define $\mathcal{D} = \cup_{n\geq 0}\mathcal{D}_n$. For short, we set $U = (1,1)$, $D = (1,-1)$, $F = (1,0)$ and $D_i = (1,-i)$ for $i \geq 2$.

Considering these notations, a *Motzkin meander with catastrophes* is a lattice path where possible steps are U, D, F and D_i for $i \geq 2$, such that all steps D_i end on the x-axis, and if we add the property that the path ends on the x-axis, we call it a *Motzkin excursion* with catastrophes (see [1]). *Dyck meanders* and *Dyck excursions* with catastrophes are those avoiding the step F. Let $\mathcal{M}_n$ (respectively $\mathcal{E}_n$) be the set of length n Dyck meanders (respectively excursions) with catastrophes, and we set $\mathcal{M} = \cup_{n\geq 0}\mathcal{M}_n$ (respectively $\mathcal{E} = \cup_{n\geq 0}\mathcal{E}_n$). The sets of Motzkin meanders and excursions with catastrophes are respectively denoted by adding prime superscripts, $\mathcal{M}'$ and $\mathcal{E}'$. As mentioned in Corollary 2.4 in [1], the cardinality of $\mathcal{M}_n$ is given by the sequence A274115 in [15], and the cardinality of $\mathcal{E}_n$ is given by the sequence A224747. For instance, we have $UUDFUUFD_3UDUUUDDUD_2UUFFUF \in \mathcal{M}'_{23}$ and $UUDUUDUD_3UDUUUDDUD_2 \in \mathcal{E}_{17}$, and we refer to Figure 1 for an illustration of these two paths. Since Motzkin meanders with catastrophes can be obtained from Dyck meanders with catastrophes by possibly adding flat steps F, the ordinary generating function (o.g.f.) for the cardinality of $\mathcal{M}'_n$ is given by

$$\frac{M(x/(1-x))}{1-x}$$

where

$$M(x) = \frac{2x}{2x + (x+1)\left(\sqrt{1-4x^2} - 1\right)}$$

is the o.g.f. for $\mathcal{M}_n$ (see [1]), which generates the $(n+1)$th term of A54391. Simarly, $\mathcal{E}'_n$ is counted by the nth term of A54391.

*Corresponding author (barjl@u-bourgogne.fr).

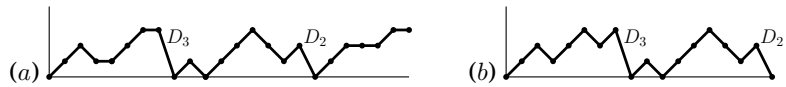


Figure 1: (a) A Motzkin meander with catastrophes in $\mathcal{M}'_{23}$, and (b) a Dyck excursion with catastrophes in $\mathcal{E}_{17}$.

Dyck meanders with catastrophes was first introduced by Krinik et al. in [9] in the context of queuing theory. They correspond to the evolution of the queue by allowing some resets modeled by a catastrophe step D_i for $i \geq 2$. Recently in [1], Banderier and Wallner provide many results about the enumeration and limit laws of these objects. Using algrebraic methods they prove that the set $\mathcal{M}_n$ of length n Dyck meanders with catastrophes has the same cardinality as the set of equivalence classes of semilength $n+1$ Dyck paths modulo the positions of the pattern DUU, which in turn (see [10]) is in one-to-one correspondence with the set $\mathcal{A}_n$ of semilength n Dyck paths avoiding occurrences at height $h > 0$ of the patterns UUU and DUD. They also provide a constructive bijection between $\mathcal{E}_n$ and the set of length n Motzkin paths having their flat steps F at height one.

The motivation of this work is to exhibit one-to-one correspondences between restricted Dyck paths (with no catastrophes) and the sets of paths with catastrophes $\mathcal{M}_n$, $\mathcal{E}_n$, $\mathcal{M}'_n$, and $\mathcal{E}'_n$. In Section 2 we present a constructive bijection between $\mathcal{M}_n$ and $\mathcal{A}_n$. Considering its restriction to excursions with catastrophes, we prove that $\mathcal{E}_n$ is in one-to-one correspondence with the set $\mathcal{A}'_n$ of Dyck paths in $\mathcal{A}_n$ where any occurrence UD on the x-axis appears before an occurrence of UUU (not necessarily contiguous to the occurrence UD). This bijection establishes a curious correspondence connecting Dyck meanders with catastrophes and equivalence classes modulo the positions of DUU in Dyck paths. In Section 3 we conduct the counterpart study for Motzkin meanders and excursions. More precisely, we exhibit a bijection between $\mathcal{M}'_n$ and the set $\mathcal{B}_{n+1}$ of semilength $n+1$ Dyck paths avoiding the pattern UUU at height $h \geq 2$, which also induces a bijection from the set $\mathcal{E}'_n$ of Motzkin excursions with catastrophes to the set $\mathcal{B}_n$. The following table gives an overview of all these correspondences.

Dyck meanders with cat.	$\mathcal{M}_n$	$\to$	$\mathcal{A}_n$	Dyck paths avoiding UUU and DUD at $h > 0$
Dyck excursions with cat.	$\mathcal{E}_n$	$\to$	$\mathcal{A}'_n$	$\mathcal{A}_n$ whose every UD on the x-axis appears before UUU
Dyck paths	$\mathcal{D}_n$	$\to$	$\mathcal{A}^\star_{2n}$	Dyck paths starting with UU and avoiding UUU and DUD
Motzkin meanders with cat.	$\mathcal{M}'_n$	$\to$	$\mathcal{B}_{n+1}$	Dyck paths avoiding UUU at $h \geq 2$
Motzkin excursions with cat.	$\mathcal{E}'_n$	$\to$	$\mathcal{B}_n$	
Motzkin paths	$\mathcal{M}otz_n$	$\to$	$\mathcal{B}'_{n+1}$	Dyck paths avoiding UUU at $h \geq 2$ and DU at $h = 1$.

2. Dyck meanders with catastrophes

In this section we exhibit a constructive bijection between the set $\mathcal{M}_n$ of length n Dyck meanders with catastrophes and the set $\mathcal{A}_n$ of semilength n Dyck paths having no occurrence of the consecutive three steps UUU and DUD at height $h > 0$ (or equivalently with a minimal ordinate $h > 0$). We set $\mathcal{A} = \cup_{n\geq 0}\mathcal{A}_n$. Let us define recursively the map ϕ from $\mathcal{M}$ to $\mathcal{D}$ as follows. For $P \in \mathcal{M}$, we set

$$\phi(P) = \begin{cases} \epsilon & \text{if } P = \epsilon, & (i) \\ UD\phi(\alpha) & \text{if } P = U\alpha, & (ii) \\ UUD\phi(\alpha)D\phi(\beta) & \text{if } P = U\alpha D\beta, & (iii) \\ U\phi(\alpha D)D\phi(\beta) & \text{if } P = U\alpha D_2\beta, & (iv) \\ UD\phi(\alpha D_{i-1})\phi(\beta) & \text{if } P = U\alpha D_i\beta \text{ and } i \geq 3, & (v) \end{cases}$$

where $\beta \in \mathcal{M}$, and α is either the empty path or a lattice path consisting of U- and D-steps such that α (respectively αD, αD_{k-1}) ends on the x-axis in the case (iii) (respectively (iv), (v)), and α does not necessarily end on the x-axis in the case (ii).

Due to the recursive definition, the image by ϕ of a length n Dyck meander with catastrophes is a Dyck path of semilength n. For instance, the images of U, UD, UUD_2, $UUDUUD_3$ are respectively

$$UD,\ UUDD,\ UUUDDD,\ UUDDUUDUUDDD.$$

We refer to Figure 2 for an illustration of this mapping.

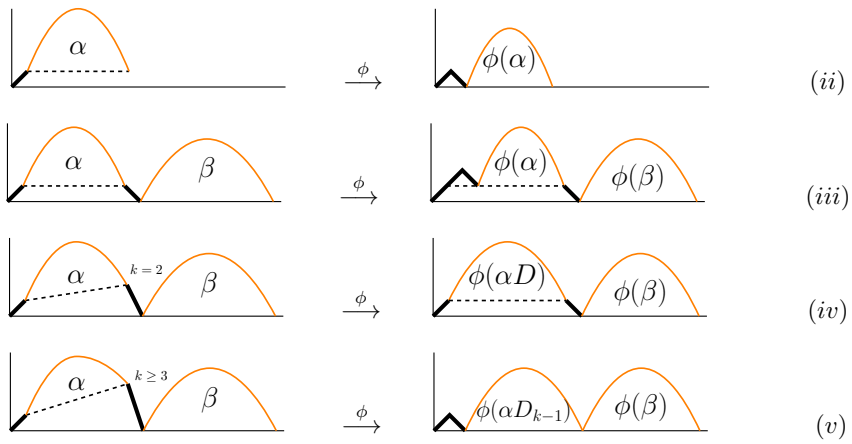


Figure 2: Illustration of the bijection ϕ between $\mathcal{M}_n$ and $\mathcal{A}_n$.

Lemma 2.1. *For any $n \geq 0$,*

- *if $P \in \mathcal{M}_n$ then we have $\phi(P) \in \mathcal{A}_n$,*
- *if $P \in \mathcal{D}_n$ then we have $\phi(P) \in \mathcal{A}^*_{2n}$, where $\mathcal{A}^*_0 = \{\epsilon\}$ and for $n \geq 1$, the set $\mathcal{A}^*_{2n}$ consists of semilength $2n$ Dyck paths avoiding the patterns UUU and DUD and starting with UUD.*

Proof. We proceed by induction on n. The case $n = 0$ is obvious. For $k \leq n$, we assume that for any $P \in \mathcal{D}_k$ we have $\phi(P) \in \mathcal{A}^*_{2k}$ and for any $P \in \mathcal{M}_k$ we have $\phi(P) \in \mathcal{A}_k$. Now, let us prove the result for $k = n + 1$.

Whenever $P \in \mathcal{D}_{n+1}$ we can write $P = U\alpha D\beta$ where $\alpha, \beta \in \mathcal{D}$. Thus, we have $\phi(P) = UUD\phi(\alpha)D\phi(\beta)$, and using the recurrence hypothesis on α and β, $\phi(P)$ is of semilength $2n + 2$, starts with UUD and avoids the pattern UUU. Moreover $\phi(\alpha)$ (respectively $\phi(\beta)$) is either empty or it starts with UUD, which implies that $\phi(P)$ avoids DUD, and thus $\phi(P) \in \mathcal{A}^*_{2n+2}$.

Now let us assume $P \in \mathcal{M}_{n+1}$.

- If $P = U\alpha$ with $\alpha \in \mathcal{M}_n$, then $\phi(P) = UD\phi(\alpha)$ and the recurrence hypothesis implies that $\phi(P)$ avoids UUU and DUD at height $h > 0$.
- If $P = U\alpha D\beta$ where $\alpha \in \mathcal{D}$ and $\beta \in \mathcal{M}$, then the first part of the proof implies that $\phi(\alpha)$ avoids UUU and DUD, and with the recurrence hypothesis on β, $\phi(P) = UUD\phi(\alpha)D\phi(\beta)$ belongs to $\mathcal{A}$.
- If $P = U\alpha D_2\beta$ and $\alpha D \in \mathcal{D}$, then using the first part of the proof $\phi(\alpha D)$ is not empty and avoids UUU and DUD. The recurrence hypothesis implies that $\phi(P) = U\phi(\alpha D)D\phi(\beta)$ belongs to $\mathcal{A}$.
- If $P = U\alpha D_i\beta$ where $i \geq 3$ and αD_{i-1} ends on the x-axis, then using a simple induction on $i \geq 2$, $\phi(\alpha D_{i-1})$ is not empty and avoids UUU and DUD. The recurrence hypothesis implies that $\phi(P) = U\phi(\alpha D_{i-1})D\phi(\beta) \in \mathcal{A}$.

The induction is completed. □

Theorem 2.1. *For $n \geq 0$, the map $\phi : \mathcal{M}_n \to \mathcal{A}_n$ is a bijection. Moreover, we have $\phi(\mathcal{D}_n) = \mathcal{A}^*_{2n}$.*

Proof. Due to the enumerative results in [1] (see Corollary 2.4) and the above lemma, it suffices to prove that ϕ is injective from $\mathcal{M}_n$ to $\mathcal{A}_n$. We proceed by induction on n. The case $n = 0$ is obvious. For $k \leq n$, we assume that ϕ is an injection from $\mathcal{M}_k$ to $\mathcal{A}_k$, and we prove the result for $k = n + 1$.

According to the definition of ϕ and Lemma 2.1, the image by ϕ of $P \in \mathcal{M}$ satisfying (ii) is a Dyck path starting by $(UD)^kR$ for some $k \geq 1$ where R is a Dyck path in $\mathcal{A}^*_{2i}$ for some $i \geq 0$, which means that R avoids UUU; a meander satisfying (iii) is sent by ϕ to a Dyck path in $\mathcal{A}^*_{2i}$ for some $i \geq 1$; a meander satisfying (iv) is sent to a Dyck path starting with $UUUD$; and a meander satisfying (v) is sent to a Dyck path starting with $(UD)^k$ for some $k \geq 1$ and such that it contains an occurrence UUU on the x-axis. Then, for $P, Q \in \mathcal{M}_{n+1}$, $\phi(P) = \phi(Q)$ implies that P and Q belong to the same case (i), (ii), (iii), (iv) or (v). So, the recurrence hypothesis induces $P = Q$ which completes the induction. Thus ϕ is injective. Since $\mathcal{M}_n$ and $\mathcal{A}_n$ have the same cardinality (see [1] and A274115 in [15]), ϕ is a bijection.

Considering the previous lemma, it suffices to check that $\mathcal{A}^*_{2n}$ is counted by the Catalan numbers in order to prove $\phi(\mathcal{D}_n) = \mathcal{A}^*_{2n}$. A Dyck path $P \in \mathcal{A}^*_{2n}$ is either empty or it consists of a sequence of $UUD\alpha D$ where α belongs to $\mathcal{A}^*_{2n-2}$.

Let $A^*(x)$ be the generating function for the cardinality of $\mathcal{A}^*_{2n}$ (with respect to the semilength). We obtain the following functional equations

$$A^*(x) = 1 + \frac{x^2 A^*(x)}{1 - x^2 A^*(x)}$$

which implies that $\mathcal{A}^*_{2n}$ is counted by the nth Catalan number. Therefore $\phi : \mathcal{D}_n \to \mathcal{A}^*_{2n}$ is a bijection. □

Remark 2.1. *In [10], it is proven that the set $\mathcal{A}_n$ is a representative set of the equivalence classes modulo the pattern DUU on Dyck paths, i.e. two Dyck paths P and Q are equivalent if and only if the positions of the occurrences DUU are the same in P and Q (see also [3]). So, the bijection ϕ establishes a direct correspondence between these classes and Dyck meanders with catastrophes.*

Let $\mathcal{A}'_n$ be the subset of $\mathcal{A}_n$ consisting of paths P such that any occurrence UD on the x-axis in P appears before an occurrence of UUU (not necessarily contiguous to the occurrence UD). The next theorem gives a bijection between $\mathcal{A}'_n$ and the set $\mathcal{E}_n$ of length n Dyck excursions with catastrophes.

Theorem 2.2. *For $n \geq 0$, we have $\phi(\mathcal{E}_n) = \mathcal{A}'_n$.*

Proof. Thanks to Theorem 2.1, it suffices to check that for any $P \in \mathcal{E}_n$, $\phi(P) \in \mathcal{A}'_n$, and $|\mathcal{A}'_n| = |\mathcal{E}_n|$. Any $P \in \mathcal{E}_n$ satisfies one of the cases (i), (iii), (iv) and (v) with $\beta \in \mathcal{E}$. We proceed by induction on the length in order to prove that $\phi(P) \in \mathcal{A}'_n$. The case (i) is obvious. Whenever P satisfies the cases (iii) or (iv), the only possibility for an occurrence of UD to appear at height zero in $\phi(P) = UUD\phi(\alpha)D\phi(\beta)$ (respectively $\phi(P) = U\phi(\alpha D)D\phi(\beta)$) is to be inside $\phi(\beta)$. Applying the recurrence hypothesis on β, $\phi(P) \in \mathcal{A}'_n$. For a path P satisfying the case (v), we have seen in the proof of Theorem 2.1 that $\phi(P)$ starts necessarily with $(UD)^k$ for $k \geq 1$ followed by UUU. Using the recurrence hypothesis for β, we obtain $\phi(P) \in \mathcal{A}'_n$. The induction is completed.

Now, let us prove that $|\mathcal{A}'_n| = |\mathcal{E}_n|$. Any path $P \in \mathcal{A}'_n$ satisfies one of the following two cases: (a) $P \in \mathcal{A}_n$ does not contain any occurrence UD on the x-axis, and (b) $P = QUDR$ where $Q \in \mathcal{A}$ and $R \in \mathcal{A}$ such that R contains at least one occurrence of UUU and avoids any occurrence UD on the x-axis. Let $\mathcal{K}$ (respectively $\overline{\mathcal{K}}$) be the set of Dyck paths in $\mathcal{A}$ satisfying (a) (respectively (b)), and let $K(x)$ and $\overline{K}(x)$ be the corresponding generating functions for their cardinalities with respect to the semilength. Obviously, the generating function $A'(x)$ for $\mathcal{A}' = \cup_{n\geq 0}\mathcal{A}_n$ satisfies

$$A'(x) = K(x) + \overline{K}(x).$$

A nonempty path $P \in \mathcal{K}$ can be decomposed $P = U\alpha D\beta$ where $\beta \in \mathcal{K}$ and α is a nonempty Dyck path avoiding UUU and DUD. Then either $\alpha \in \mathcal{A}^\star \backslash \{\epsilon\}$ or $\alpha = UD\alpha'$ with $\alpha' \in \mathcal{A}^\star$. Thus the generating function for $\mathcal{K}$ is given by

$$K(x) = 1 + x(A^\star(x) - 1 + xA^\star(x)) \cdot K(x).$$

Due to the form of a path $P \in \overline{\mathcal{K}}$, we deduce the functional equation

$$\overline{K}(x) = A(x)xR(x)$$

where $A(x)$ is the generating function for $\mathcal{A}$ and $R(x)$ is the generating function for the paths in $\mathcal{A}$ avoiding any occurrence UD on the x-axis and containing at least one occurrence of UUU. Due to Theorem 2.1, $A(x)$ is also the o.g.f. for Dyck meanders with catastrophes that is

$$A(x) = M(x) = \frac{2x}{2x + (x+1)\left(\sqrt{1-4x^2} - 1\right)}.$$

Then, we have $R(x) = K(x) - L(x)$ where $L(x)$ is the generating function for the set $\mathcal{L}$ of Dyck paths in $\mathcal{A}$ avoiding UD and UUU on the x-axis. Note that $\mathcal{L}$ is exactly the set $\mathcal{A}^\star$, then

$$R(x) = K(x) - A^\star(x).$$

Combining the previous equations, we obtain

$$A'(x) = \frac{2 - 3\,x - 2\,x^2 + x\sqrt{1 - 4\,x^2}}{2 - 2x - 4x^2 - 2x^3}$$

which is exactly the generating function of $\mathcal{E}_n$ found by [1]. □

3. Motzkin meanders with catastrophes

In this section we exhibit a constructive bijection between the set $\mathcal{M}'_n$ of length n Motzkin meanders with catastrophes and the set $\mathcal{B}_{n+1}$ of semilength $n+1$ Dyck paths avoiding the patterns UUU at height $h \geq 2$. Before defining this bijection we recall that there exists a one-to-one correspondence χ between length n Motzkin paths and semilength n Dyck paths avoiding UUU. From a Dyck path avoiding UUU, we replace each UUD with U, and we replace each remaining UD with F. For instance, the image by χ of $UFUDD$ is $UUDUDUUDDD$ (see [5, 7]).

Now, let us use χ in order to define recursively the map ψ from $\mathcal{M}'$ to $\mathcal{D}$ as follows. For $P \in \mathcal{M}'$, we set

$$\psi(P) = \begin{cases} UD & \text{if } P = \epsilon, \\ UD\psi(\alpha) & \text{if } P = F\alpha, \\ UU\chi(\alpha_1)DU\chi(\alpha_2)D\dots U\chi(\alpha_k)DD & \text{if } P = U\alpha_1U\alpha_2\dots U\alpha_k, \\ UU\chi(\alpha_1)DU\chi(\alpha_2)D\dots U\chi(\alpha_k)DD\psi(\beta) & \text{if } P = U\alpha_1U\alpha_2\dots U\alpha_kD_k\beta, \end{cases}$$

where $k \geq 1$, $\alpha, \beta \in \mathcal{M}'$, and $\alpha_1, \alpha_2, \dots, \alpha_k$ are some possibly empty Motzkin paths (considering that D_1 is defined to be D).

Clearly, the image by ψ of a length n Motzkin meander with catastrophes is a Dyck path of semilength $n+1$. For instance, the images by ψ of ϵ, F, UD, UUD_2, $UUDUUD_3$ are respectively

$$UD,\ UDUD,\ UUDDUD,\ UUDUDDUD,\ UUUUDDUDUDDDUD.$$

We refer to Figure 3 for an illustration of this mapping.

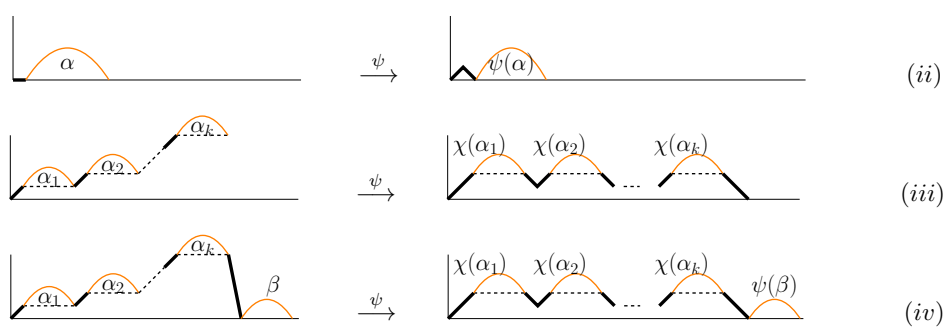


Figure 3: Illustration of the bijection ψ between $\mathcal{M}'_n$ and $\mathcal{B}_{n+1}$.

A simple observation provides the following results.

Theorem 3.1. *For $n \geq 0$, the map ψ, defined above, induces a bijection from $\mathcal{M}'_n$ to $\mathcal{B}_{n+1}$. Moreover, the image of the set of length n Motzkin paths is the set of semilength $n+1$ Dyck paths avoiding UUU at height $h \geq 2$ and the pattern DU at height one.*

Corollary 3.1. *For $n \geq 0$, $\psi(\mathcal{E}'_n)$ is the set of Dyck paths in $\mathcal{B}_{n+1}$ ending with UD, which implies that ψ induces a one-to-one correspondence from paths $P \in \mathcal{E}'_n$ to $\mathcal{B}_n$ after deleting the last two steps UD from $\psi(P)$.*

Acknowledgment

We would like to thank Cyril Banderier for suggesting us to explore constructive bijections between meanders with catastrophes and Dyck paths avoiding some patterns.

References

[1] C. Banderier, M. Wallner, Lattice paths with catastrophes, *Discrete Math. Theor. Comput. Sci.* **19** (2017) Art# 23.
[2] E. Barcucci, A. Del Lungo, S. Fezzi, R. Pinzani, Nondecreasing Dyck paths and q-Fibonacci numbers, *Discrete Math.* **170** (1997) 211–217.
[3] J.-L. Baril, A. Petrossian, Equivalence classes of Dyck paths modulo some statistics, *Discrete Math.* **338** (2015) 655–660.
[4] J.-L. Baril, D. Bevan, S. Kirgizov, Bijections between directed animals, multisets and Grand-Dyck paths, *Electron. J. Combin.* **27** (2020) Art# P2.10.
[5] D. Callan, Two bijections for Dyck path parameters, *arXiv*:math/0406381v2 [math.CO], (2004).
[6] E. Deutsch, Dyck path enumeration, *Discrete Math.* **204** (1999) 167–202.
[7] S. Elizalde, T. Mansour, Restricted Motzkin permutations, Motzkin paths, continued fractions, and Chebyshev polynomials, *Discrete Math.* **305** (2005) 170–189.
[8] D. E. Knuth, *The Art of Computer Programming*, Volume 1. Fundamental Algorithms, Addison-Wesley, Reading, 1973.
[9] A. Krinik, G. Rubino, D. Marcus, R. J. Swift, H. Kasfy, H. Lam, Dual processes to solve single server systems, *J. Statist. Plann. Inference* **135** (2005) 121–147.

[10] K. Manes, A. Sapounakis, I. Tasoulas, P. Tsikouras, Equivalence classes of ballot paths modulo strings of length 2 and 3, *Discrete Math.* **339** (2016) 2557–2572.
[11] T. Mansour, Statistics on Dyck paths, *J. Integer Seq.* **9** (2006) Art# 06.1.5.
[12] D. Merlini, R. Sprugnoli, M. C. Verri, Some statistics on Dyck paths, *J. Statist. Plann. Inference* **101** (2002) 211–227.
[13] A. Panayotopoulos, A. Sapounakis, On the prime decomposition of Dyck paths, *J. Combin. Math. Combin. Comput.* **40** (2002) 33–39.
[14] A. Sapounakis, I. Tasoulas, P. Tsikouras, Counting strings in Dyck paths, *Discrete Math.* **307** (2007) 2909–2924.
[15] N. J. A. Sloane, *The On-line Encyclopedia of Integer Sequences*, http://oeis.org
[16] Y. Sun, The statistic "number of udu's" in Dyck paths, *Discrete Math.* **287** (2004) 177–186.
[17] R. P. Stanley, *Enumerative Combinatorics*, Volume 2, Cambridge University Press, Cambridge, 1999.

Discrete Mathematics 344 (2021) 112464

Contents lists available at ScienceDirect

Discrete Mathematics

www.elsevier.com/locate/disc

Pattern statistics in faro words and permutations

Jean-Luc Baril [a,*], Alexander Burstein [b], Sergey Kirgizov [a]

[a] LIB, Université de Bourgogne Franche-Comté, B.P. 47 870, 21078 Dijon Cedex, France
[b] Howard University, Department of Mathematics, Washington, DC 20059, USA

ARTICLE INFO

Article history:
Received 13 October 2020
Received in revised form 22 April 2021
Accepted 3 May 2021
Available online xxxx

Keywords:
Word
Permutation
Faro shuffle
Pattern statistics
Popularity
Equidistribution

ABSTRACT

We study the distribution and the popularity of some patterns in *k-ary faro words*, i.e. words over the alphabet $\{1, 2, \ldots, k\}$ obtained by interlacing the letters of two nondecreasing words of lengths differing by at most one. We present a bijection between these words and dispersed Dyck paths (i.e. Motzkin paths with all level steps on the x-axis) with a given number of peaks. We show how the bijection maps statistics of consecutive patterns of faro words into linear combinations of other pattern statistics on paths. Then, we deduce enumerative results by providing multivariate generating functions for the distribution and the popularity of patterns of length at most three. Finally, we consider some interesting subclasses of faro words that are permutations, involutions, derangements, or subexcedent words.

© 2021 Elsevier B.V. All rights reserved.

1. Introduction and notations

The faro shuffle is a well-known technique to shuffle a deck of cards. The deck is split in two at the middle, and the cards from the two halves are combined back by taking alternatively the bottoms of stacks. Certain mathematical questions about the faro shuffle are considered for example in the works of Morris [23], Diaconis, Graham and Kantor [15]. Inspired by these studies and a solid body of modern combinatorial literature (see for instance Lothaire [21], Stanley [27], Bóna [13] and Kitaev [19] books) that explores enumerative and bijective aspects of patterns in various discrete structures, the present paper considers an unexpectedly overlooked combinatorial objects, which we call *faro words*. They are special kind of word shuffles, which are important in several algorithmic and combinatorial settings (see for example Barnes work [7] and references therein). In this paper, we present enumerative results and show how faro words and patterns therein are related to other structures such as Dyck paths, Motzkin paths and Dumont permutations.

1.1. Faro words and permutations

We deal with k-ary words $u_1u_2\ldots u_n$ over the integer alphabet $[1, k] = \{1, 2, \ldots, k\}$ endowed with the usual total order. A k-ary word is called *nondecreasing* if $u_i \leqslant u_{i+1}$ for all $i \in [1, n-1]$.

Definition 1.1. For two k-ary words u and v such that $0 \leqslant |u| - |v| \leqslant 1$, the *faro shuffle* of u and v is the k-ary word of length $|u| + |v|$ obtained by interlacing the letters of u and v as follows: $u_1v_1u_2v_2u_3v_3\ldots$ A *k-ary faro word* is a faro shuffle of two nondecreasing k-ary words.

* Corresponding author.
E-mail addresses: barjl@u-bourgogne.fr (J.-L. Baril), aburstein@howard.edu (A. Burstein), sergey.kirgizov@u-bourgogne.fr (S. Kirgizov).

https://doi.org/10.1016/j.disc.2021.112464
0012-365X/© 2021 Elsevier B.V. All rights reserved.

Let $\mathcal{S}_{n,k}$ be the set of k-ary faro words of length n. Its cardinality equals the product of two binomial coefficients $\binom{\lfloor n/2\rfloor+k-1}{k-1}\binom{\lceil n/2\rceil+k-1}{k-1}$, each of them being, respectively, the number of m-multisets of $[1,k]$ for $m=\lfloor\frac{n}{2}\rfloor$ and $m=\lceil\frac{n}{2}\rceil$. For example, we have $\mathcal{S}_{4,2}=\{1111, 1112, 1121, 1122, 1212, 1222, 2121, 2122, 2222\}$ and $|\mathcal{S}_{4,2}|=9$.

Definition 1.2. A *faro permutation* of length n is an n-ary faro word of length n that contains every letter in $[1,n]$ exactly once.

Let $\mathcal{P}_n$ be the set of length n faro permutations. For instance, we have $\mathcal{P}_3=\{123, 132, 213\}$. Since a faro permutation is entirely determined by the choice of its values on the odd indices, the cardinality of $\mathcal{P}_n$ is $\binom{n}{\lfloor n/2\rfloor}$.

A k-ary word $w=w_1w_2\dots w_n$ avoids a classical pattern (resp. consecutive pattern) $p=p_1\text{-}p_2\text{-}\cdots\text{-}p_k$ (resp. $p=p_1p_2\dots p_k$) if there does not exist a strictly increasing sequence of indices $i_1i_2\dots i_k$ (resp. with $i_{j+1}=i_j+1$ for $1\leqslant j\leqslant k-1$) such that $w_{i_1}w_{i_2}\dots w_{i_k}$ is order-isomorphic to p (see [19] for instance). Obviously, any faro word avoids the classical pattern 3-2-1. Let $Av_n(\sigma)$ denote the set of permutations avoiding a classical pattern σ, then we have $\mathcal{P}_n\subseteq Av_n(3\text{-}2\text{-}1)$ for $n\geqslant 0$, and $\mathcal{P}_n\neq Av_n(3\text{-}2\text{-}1)$ for $n\geqslant 3$ since $(n-1)n12\dots(n-2)\in Av_n(3\text{-}2\text{-}1)$ is not a faro word. Note that a faro permutation can contain all classical patterns of length 3 except 3-2-1 (e.g., 31425).

Remark 1.3. A k-ary word $w=w_1w_2\dots w_n$ is a faro word if and only if $w_i\leqslant w_{i+2}$ for any $i\in[1,n-2]$, which means that faro permutations are precisely those avoiding the three consecutive patterns 231, 321 and 312.

1.2. Dyck and dispersed Dyck paths

In order to study the distribution of patterns in faro words, we will exhibit one-to-one correspondences between these objects and some specific lattice paths in the first quadrant of the plane. Hence, we provide basic necessary definitions on lattice paths.

Definition 1.4. *Dispersed Dyck paths* (see [17]) are lattice paths starting at $(0,0)$, ending at $(n,0)$, consisting of level steps $F=(1,0)$, up step $U=(1,1)$ and down steps $D=(1,-1)$, and never going below the x-axis and where all level steps are on the x-axis.

Let $\mathcal{B}_n$ be the set of dispersed Dyck paths of length n (or, equivalently, consisting of n steps) and set $\mathcal{B}=\cup_{n\geqslant 0}\mathcal{B}_n$, where the empty path is denoted by ϵ. A *Dyck path* of semilength $n\geqslant 0$ is a dispersed Dyck path of length $2n$ with no level steps. Let $\mathcal{D}_n$ be the set of Dyck paths of semilength n and let $\mathcal{D}=\bigcup_{n\geqslant 0}\mathcal{D}_n$. Dispersed Dyck paths of length n are in straightforward bijection with prefixes of Dyck paths of length n, also known as ballot paths [8,28]. Indeed, we can obtain a ballot path from a dispersed Dyck path by replacing all level steps with up steps. Dyck and dispersed Dyck paths are counted by the Catalan and ballot numbers, respectively (see A000108 and A001405 in the Online Encyclopedia of Integer Sequences of N.J.A. Sloane [26], where the general terms are $c_n=\frac{1}{n+1}\binom{2n}{n}$ and $b_n=\binom{n}{\lfloor n/2\rfloor}$, respectively).

A path P avoids a pattern X if and only if P does not contain X as a sequence of consecutive steps (see for instance [14,22]). Note that other pattern definitions exist in the literature where steps are not necessarily consecutive [3]. We also need some notations similar to Kleene star and plus symbols of formal language theory. For a nonempty pattern X, an occurrence of the pattern X^+ in a path P is a maximal sequence of consecutive repetitions of X, i.e. a maximal subword of the form X^k for $k\geqslant 1$. The pattern X^* will be either an empty pattern or a pattern X^+. More generally, for two possibly empty patterns Y and Z such that Y does not end with X and Z does not start with X, the pattern YX^+Z (resp. YX^*Z) corresponds to an occurrence obtained by concatenation of Y, X^+ and Z (resp. Y, X^* and Z). For instance, the path $FUDUDFFUDF$ contains two occurrences of the pattern $F(UD)^+F$ and three occurrences of $F(UD)^*F$.

1.3. Statistics on words and lattice paths

Definition 1.5. A *statistic* $\mathbf{s}$ is an integer-valued function from a set $\mathcal{A}$ of words or paths.

To a given pattern p, we associate the pattern statistic $\mathbf{p}:\mathcal{A}\to\mathbb{N}$ such that $\mathbf{p}(a)$ is the number of occurrences of the pattern p in the object $a\in\mathcal{A}$ (we use the boldface to denote statistics). For example, the statistic giving the number of occurrences of the consecutive pattern 123 (resp. $UDUD$) in a word (resp. a lattice path) is denoted by **123** (resp. **UDUD**). We denote by $\hat{\mathbf{1}}$ (resp. $\hat{\mathbf{2}}, \hat{\mathbf{n}}$) the constant statistic returning the value 1 (resp. 2, n).

Definition 1.6. The *popularity* of a pattern p in $\mathcal{A}$ is the total number of occurrences of p over all objects of $\mathcal{A}$, that is $\mathbf{p}(\mathcal{A})=\sum_{a\in\mathcal{A}}\mathbf{p}(a)$ (see [5,10,18,19]).

For instance, for a dispersed Dyck path $P=FFUDFUUDUUUDDDD$ we have $\mathbf{FF}(P)=1$, $\mathbf{DDD}(P)=2$, $\mathbf{UD}(P)=3$, $\mathbf{UUUU}(P)=0$ and $\hat{\mathbf{1}}(P)=1$. Moreover, if $\mathcal{A}=\{UUDD, UDUD\}$ then the popularity of the pattern UD in $\mathcal{A}$ is $\mathbf{UD}(\mathcal{A})=3$.

Let $\mathbf{T}_{\mathcal{A}}$ be the set of all statistics defined on a set $\mathcal{A}$. For any pair of statistics $\mathbf{s}, \mathbf{t} \in \mathbf{T}_{\mathcal{A}}$, we define the statistic $\mathbf{s}+\mathbf{t}$ by $(\mathbf{s}+\mathbf{t})(a) = \mathbf{s}(a) + \mathbf{t}(a)$ for any $a \in \mathcal{A}$, which endows $\mathbf{T}_{\mathcal{A}}$ with a $\mathbb{Z}$-module structure. Let $\mathcal{B}$ be a set of combinatorial objects, and let $\mathbf{T}_{\mathcal{B}}$ be the corresponding set of statistics. We say that two statistics $\mathbf{s} \in \mathbf{T}_{\mathcal{A}}$ and $\mathbf{t} \in \mathbf{T}_{\mathcal{B}}$ have the *same distribution*, or are *equidistributed*, if there exists a bijection $f : \mathcal{A} \to \mathcal{B}$ such that $\mathbf{s}(a) = \mathbf{t}(f(a))$ for any $a \in \mathcal{A}$. In this case, with a slight abuse of the notation already used in [4], we write shortly $f(\mathbf{s}) = \mathbf{t}$ or $\mathbf{s} = \mathbf{t}$ whenever f is the identity. As a byproduct, for any constant statistic $\hat{\mathbf{n}}$, we have $f(\hat{\mathbf{n}}) = \hat{\mathbf{n}}$.

1.4. Outline of the paper

The paper is organized as follows. In Section 2, we present a constructive bijection f between the set $\mathcal{S}_{n,k}$ of k-ary faro words of length n and the set of dispersed Dyck paths of length $n+2k-2$ with $k-1$ peaks. We show where pattern statistics are transported by f, which provides a more suitable ground for studying the distribution of consecutive patterns. Thus, we derive enumerating results on the distribution and popularity of patterns in $\mathcal{S}_{n,k}$ by giving multivariate generating functions where the coefficient of $x^n y^k z^t$ is the number of k-ary faro words of length n having exactly t occurrences of a given pattern. In Section 3, we present a similar study for faro permutations. More precisely, we provide a bijection g between $\mathcal{P}_n$ and the set of dispersed Dyck paths of length n and show how g acts on pattern statistics of length at most three. Consequently, we deduce enumerative results for the distribution and the popularity of these patterns in $\mathcal{P}_n$. We also present a bijection between $\mathcal{P}_n$ and involutions avoiding the classical pattern 3-2-1. Finally, in Section 4, we prove that the set of subexcedent words in $\mathcal{S}_{n,n}$ is related to ternary trees and Dumont permutations of the second kind [12] avoiding the classical pattern 2-1-4-3, and we show why faro involutions and faro derangements are respectively enumerated by the Fibonacci and Catalan numbers.

2. Patterns in faro words

In this section we construct a bijection f between the set $\mathcal{S}_{n,k}$ of k-ary faro words of length n and a subset of dispersed Dyck paths, and show how f transports pattern statistics. Then, we deduce generating functions for the distribution and popularity of some patterns.

A *pair* in a faro word w is an occurrence $w_i w_{i+1}$ with $w_i > w_{i+1}$. Remark 1.3 implies that a letter cannot be part of two pairs since a faro word avoids the consecutive pattern 321. A *singleton* in w is a letter w_i not in any pair of w. Any faro word can be uniquely decomposed as a sequence of pairs and singletons, which are called *blocks* of faro words. For instance, the block decomposition of 111212131333 is $1^3(21)^2(31)3^3$.

Let $\mathcal{L}_k$ be the set of all possible blocks of a decomposition of a k-ary faro word, that is

$$\mathcal{L}_k = \{1, 2, \dots, k\} \cup \{ji : 1 \leqslant i < j \leqslant k\}.$$

Definition 2.1. We define an order relation $\preceq$ on $\mathcal{L}_k$ as follows: for $g, h, i, j \in \{1, 2, \dots, k\}$,

$$\begin{cases} i \preceq j, & \text{if } i \leqslant j, \\ i \preceq jh, & \text{if } i \leqslant h < j, \\ ig \preceq j, & \text{if } g < i \leqslant j, \\ ig \preceq jh, & \text{if } g < i \leqslant j \text{ and } g \leqslant h < j. \end{cases}$$

Remark 2.2. The order relation $\preceq$ can be defined less technically as follows: for $p, q \in \mathcal{L}_k$,

$$p \preceq q \iff pq \text{ is a faro word different from a pair.}$$

This order relation endows the set $\mathcal{L}_k$ with a poset structure, which we call *faro poset*. See Fig. 2.1 for an illustration of the Hasse diagram of $(\mathcal{L}_k, \preceq)$.

A *multichain* in a poset is a chain, i.e. *a totally ordered subset*, with repetitions allowed. Due to the simple structure of the faro poset, we easily deduce the following remarks.

Remark 2.3. There is a one-to-one correspondence between k-ary faro words and the multichains of $\mathcal{L}_k$. Indeed, Remark 2.2 implies that the block decomposition of a k-ary faro word w into pairs and singletons $w = b_1 b_2 \dots b_\ell$ unambiguously corresponds to the multichain $b_1 \preceq b_2 \preceq \cdots \preceq b_\ell$ in $\mathcal{L}_k$, and vice versa. For instance, the faro word $11313232343 = 11(31)(32)(32)3(43)$ corresponds to the multichain $1 \preceq 1 \preceq 31 \preceq 32 \preceq 32 \preceq 3 \preceq 43$ (see Fig. 2.1).

Remark 2.4. If a k-ary faro word w contains a singleton x in its decomposition into blocks, then it satisfies the following property: the set of pairs of the form ab, $b < a \leqslant x$, equals the set of pairs of the form cd, $d \leqslant x-1$.

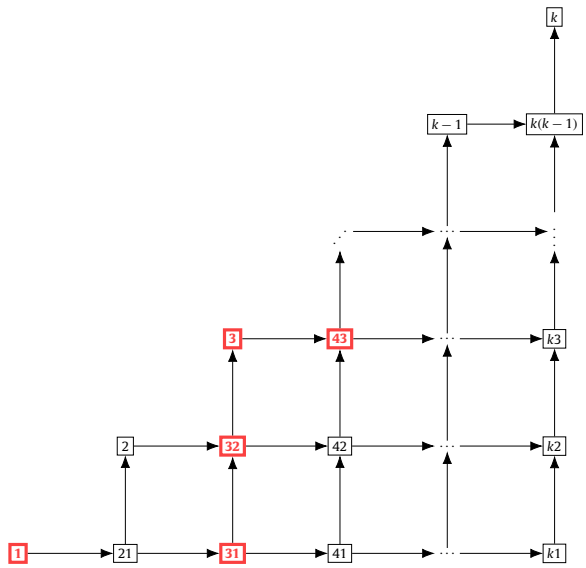


Fig. 2.1. The faro poset $(\mathcal{L}_k, \preceq)$. Red blocks represent the multichain associated to the k-ary faro word $11313232343 = 1^2(31)(32)^2 3(43)$. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

2.1. A bijection to the set of dispersed Dyck paths

As mentioned by E. Deutsch in [26] (see sequence A124428), the number of dispersed paths of length n with k peaks (a peak is an occurrence of the pattern UD) is given by

$$|\mathcal{B}_{n,k}| = \binom{\lfloor \frac{n}{2} \rfloor}{k}\binom{\lceil \frac{n}{2} \rceil}{k}.$$

Thus, we present a bijection f from the set $\mathcal{S}_{n,k}$ of k-ary faro words of length n to the set $B_{n+2(k-1),k-1}$ of dispersed Dyck paths of length $n+2(k-1)$ with exactly $k-1$ peaks. For a given $w \in \mathcal{S}_{n,k}$, we set

$$f(w) = F^{T_0}U^{T_1}D^{T_2}F^{T_3}\dots F^{T_{3(k-2)}}U^{T_{3(k-2)+1}}D^{T_{3(k-2)+2}}F^{T_{3(k-1)}},$$

where T_i is defined for $0 \leqslant i \leqslant 3(k-1)$ as follows:

- if $i = 3(x-1)$ then T_i is the number of occurrences of the singleton x in w;
- if $i = 3(x-1)-1$ then T_i is one plus the number of pairs xy, $y < x$, in w;
- if $i = 3(x-1)+1$ then T_i is one plus the number of pairs yx, $y > x$, in w.

It is worth noting that the image of a faro word $w \in \mathcal{S}_{n,k}$ depends on the arity k that we consider. Indeed, the image of the empty word ϵ is UD when $k = 2$, while $f(\epsilon) = UDUD$ for $k = 3$. We refer to Fig. 2.2 for one detailed example of this bijection, while Fig. 2.3 provides more additional examples. For instance, the images by f of the 5-ary words ϵ, 12345, 3141, 111111212222 are, respectively, $UDUDUDUD$, $FUDFUDFUDFUDF$, $UUUDUDDUDDUD$ and $FFFFFFUUDDFFFFUDUDUD$.

Remark 2.5. Clearly, the values T_i, $0 \leqslant i \leqslant 3(k-1)$, can be obtained from w by reading it from left to right and by determining if the current entry x belongs to either a pair xy or yx, or a singleton x. Moreover, values of T at indices $i = 0 \mod 3$ correspond to the lengths of maximal runs of consecutive level steps, and values at indices $i = 1 \mod 3$ (resp. $i = 2 \mod 3$) correspond to the lengths of maximal runs of consecutive up (resp. down) steps, which means that the sequence $T = T_0T_1\dots T_{3(k-1)}$ is a run-length-like encoding of the path $f(w)$. Thus, $f(w)$ can be constructed from w using a linear time algorithm.

Lemma 2.6. *The path $f(w)$ is necessarily a dispersed Dyck path of length $n+2(k-1)$ with exactly $k-1$ peaks.*

Proof. Since for any $i \neq 0 \mod 3$, $1 \leqslant i \leqslant 3(k-1)-1$ we have $T_i \geqslant 1$, the path w contains exactly $k-1$ peaks UD. Interpreting Remark 2.4 on the path $f(w)$, the number of up steps before a given level step equals the number of down steps before the same level step, which implies that any level step belongs to the x-axis. Let $d_x = \sum_{i=2}^{x+2} T_{3(i-1)-1}$ (resp. $u_x = \sum_{i=1}^{x+1} T_{3(x-1)+1}$) be the total number of down steps (resp. up steps) in the first $x+1$ maximal runs of down steps (resp. up steps). Due to the definition of f, d_x equals the number of pairs ij, $1 \leqslant j < i \leqslant x+2$, in w, and u_x equals the

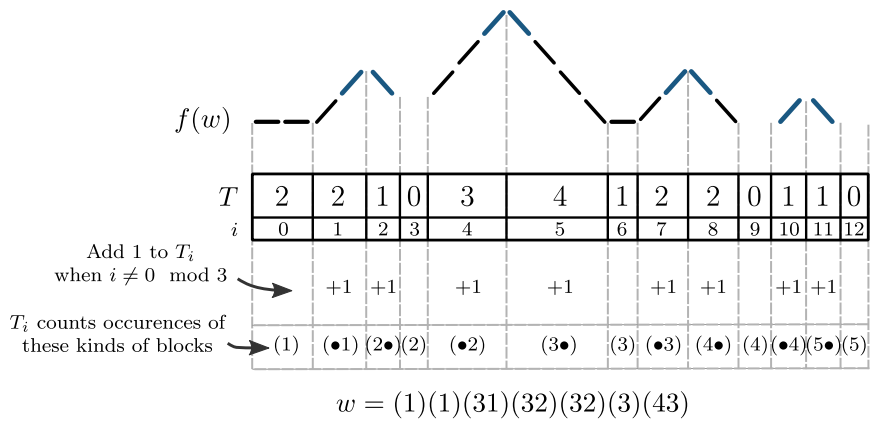


Fig. 2.2. The image by f of the 5-ary faro word $w = 11313232343$ is $f(w) = FFUUDUUUDDDDFUUDDUD$.

number of pairs ij, $1 \leqslant j \leqslant x+1$, $i \geqslant j+1$, which implies that $d_x \leqslant u_x$. Also by definition, the total number of up steps (resp. down steps) in $f(w)$ equals the total number of pairs in w, which completes the proof. □

Theorem 2.7. *The map f is a bijection from $\mathcal{S}_{n,k}$ to the set $\mathcal{B}_{n+2(k-1),k-1}$ of dispersed Dyck paths of length $n+2(k-1)$ with exactly $k-1$ peaks.*

Proof. Let us prove that if w and w' are two distinct k-ary faro words then we have $f(w) \neq f(w')$. Let $i \geqslant 1$ be the smallest positive integer such that $w_i \neq w'_i$. Without loss of generality, we assume $w_i < w'_i$. Let us consider the positions of w_i and w'_i in the block decomposition of w.

If w_i and w'_i are both in the pairs $w_i w_{i+1}$ and $w'_i w'_{i+1}$, then Remark 2.3 implies that a pair $w_i x$, $w_i > x$, cannot appear to the right of w'_i in w', which implies that $T_{3(w_i-1)-1} \neq T'_{3(w_i-1)-1}$, and thus $f(w) \neq f(w')$.

There remain the following cases:

(i) w_i or w'_i is a singleton in w,
(ii) w_i and w'_i are both in the pairs $w_{i-1}w_i$ and $w'_{i-1}w'_i = w_{i-1}w'_i$,
(iii) w_i belongs to the pair $w_{i-1}w_i$ and w'_i belongs to the pair $w'_i w'_{i+1}$,
(iv) w_i and w'_i are both in the pairs $w_i w_{i+1}$ and $w'_{i-1}w'_i$.

The fact that a faro word avoids 231 in case (i) and Remark 2.3 for cases (ii), (iii) (iv), imply that w_i cannot appear to the right of w'_i in w'. Then the number of w_i in w, i.e. $T_{3(w_i-1)} + T_{3(w_i-1)+1} + T_{3(w_i-1)-1}$, is different from the number of w_i in w', which is $T'_{3(w_i-1)} + T'_{3(w_i-1)+1} + T'_{3(w_i-1)-1}$. Therefore, there is $\delta \in \{-1, 0, 1\}$ such that $T_{3(w_i-1)+\delta} \neq T'_{3(w_i-1)+\delta}$, which implies that $f(w) \neq f(w')$.

Thus, f is an injective map, and using a cardinality argument (see A124428 in [26]), we conclude that f is a bijection from $\mathcal{S}_{n,k}$ to $\mathcal{B}_{n+2(k-1),k-1}$. □

Although it is not used in the paper, we could prove that from a given dispersed Dyck path $P \in \mathcal{B}_{n+2(k-1),k-1}$, $f^{-1}(P)$ can be obtained after applying the following procedure. We refer to Fig. 2.3 for several examples.

We set $s = 1$ as the initial value. We mark all D-steps preceded by an U-step and all the other D-steps are left unmarked. Reading the steps of P from left to right:

- If a D-step is encountered, then skip it.
- If an F-step is encountered, then write the singleton s. If the next step is not an F-step, then update $s = s+1$.
- If an U-step is encountered in the ith run of U-steps, then we distinguish two cases:

  (i) the next step is D; then we skip this UD-pattern by continuing from the step after D, if it exists.
  (ii) the next step is U; then we write the pair ji, where j is the least integer such that the $(j-1)$-th run of D-steps has at least one unmarked D-step. Mark the first unmarked D-step from the $(j-1)$-th run of D-steps.

2.2. Distribution and popularity of patterns

In this part, we first show how the bijection f transports pattern statistics on $\mathcal{S}_{n,k}$ into the context of dispersed Dyck paths. After, we deduce multivariate generating functions for the distribution and the popularity of patterns of length two by exploiting the classic recursive decomposition of dispersed Dyck paths.

```
                                  /\
      111111212222 ↦ ______/  \____/\/\/\

                                          /\
    144 ↦  _/\/\/\__/\    254 ↦ /\_/\/\/  \

                                       /\   /\
  12345 ↦ _/\_/\_/\_/\_    32343 ↦ /\/  \_/  \/\

            /\/\   /\                /\/\  /\/\
 31343 ↦  /    \_/  \/\     3153 ↦  /    \/    \

             /\/\
            /    \/\                 /\/\    /\
 3141 ↦   /        \/\      3154 ↦  /    \/\/  \

                  /\                      /\/\
               /\/  \/\/\             /\/\/    \
 1113152 ↦ ___/          \  5153 ↦  /          \

                           /\/\    /\
   1113133434444555 ↦ ___/    \__/  \____/\___
```

Fig. 2.3. Images of several 5-ary words under bijection f.

Theorem 2.8. *For $n \geqslant 0$, the bijection f from $\mathcal{S}_{n,k}$ to $\mathcal{B}_{n+2(k-1),k-1}$ maps statistics associated to patterns of length 2 as follows:*

$$f(\mathbf{11}) = \mathbf{FF},$$
$$f(\mathbf{21}) = \mathbf{UU} = \mathbf{DD},$$
$$f(\mathbf{12}) = \mathbf{DD(UD)^*UU} + \mathbf{DD(UD)^*D} + \mathbf{DD(UD)^*F} + \mathbf{F(UD)^+F} + \mathbf{F(UD)^*UU}$$
$$= \hat{\mathbf{n}} - \hat{\mathbf{1}} - \mathbf{UU} - \mathbf{FF}.$$

Proof. By Remark 2.3, any occurrence of the pattern 11 in a faro word w is formed by two consecutive singletons xx. From the definition of the bijection f, it follows that the number of occurrences of 11 in w equals the number of occurrences of FF in $f(w)$, that is $f(\mathbf{11}) = \mathbf{FF}$.

An occurrence of the pattern 21 in w is necessarily a pair in the decomposition of w. Since the length of a maximal run of consecutive up steps is equal to one plus the number of pairs yx in w for a given $x \in [1, n]$, the number of occurrences of 21 in w equals the number of occurrences of UU in $f(w)$. On the other hand, any nonempty dispersed Dyck path P is of the form either $P = FR$ or $P = UQDR$ where Q is a Dyck path and R a dispersed Dyck path. Reasoning by induction, we obtain that the number of occurrences of DD equals those of UU in any dispersed Dyck path, which implies $f(\mathbf{21}) = \mathbf{UU} = \mathbf{DD}$.

Now, let us prove the equation $f(\mathbf{12}) = \mathbf{DD(UD)^*UU} + \mathbf{DD(UD)^*D} + \mathbf{DD(UD)^*F} + \mathbf{F(UD)^+F} + \mathbf{F(UD)^*UU}$. An occurrence xy of the pattern 12 occurs in w as a subblock of one of the following:

(i) two distinct consecutive pairs $(ax)(yb)$,
(ii) two equal consecutive pairs $(yx)(yx)$,
(iii) a pair followed by a singleton $(ax)(y)$,
(iv) a singleton followed by a pair $(x)(ya)$,
(v) two distinct singletons $(x)(y)$.

For the case (i), we distinguish three subcases.

Subcase 1. The occurrence xy appears in a factor of the form $(ax)(yb)$ with $b \geqslant a$. This implies that neither a singleton $s \in [a, b]$ nor a pair pq with $p \in (a, b]$ or $q \in [a, b)$ can appear in w. Therefore, $T_{3(s-1)} = 0$ for $s \in [a, b]$, $T_{3(p-1)-1} = 1$ for $p \in (a, b]$ and $T_{3(q-1)+1} = 1$ for any $q \in [a, b)$. Thus, between the run of D-steps associated to $T_{3(a-1)-1} \geqslant 2$ and the run of U-steps associated to $T_{3(b-1)+1} \geqslant 2$, there are no level steps, and the runs of D-steps and U-steps are of length one, which creates $m = b - a \geqslant 0$ peaks UD. Hence, the occurrence xy is associated to an occurrence of the pattern $DD(UD)^*UU$.

Subcase 2. The occurrence xy appears in a factor of the form $(ax)(yb)$ with $b < a$ and $a < y$. This implies that neither a singleton $x \in [a, y)$ nor a pair pq with $p \in (a, y)$ or $q \in [a, y)$ can appear in the word w. Therefore, $T_{3(x-1)} = 0$ for

$x \in [a, y[$, $T_{3(p-1)-1} = 1$ for $p \in (a, y)$ and $T_{3(q-1)+1} = 1$ for any $q \in [a, y)$. Thus, between the run of D-steps associated to $T_{3(a-1)-1} \geqslant 2$ and the run of D-steps associated to $T_{3(y-1)+1} \geqslant 2$, there are no level steps, and the runs of D-steps and U-steps are of length one, which creates $m = y - a > 0$ peaks UD. Hence, the occurrence xy is associated to an occurrence of the pattern $DD(UD)^+D$.

Subcase 3. The occurrence xy appears in a factor of the form $(ax)(yb)$ with $b < a$ and $a \geqslant y$. By definition of a faro word, we necessarily have $a \leqslant y$. Thus, we deduce $a = y$. So, we have $T_{3(a-1)-1} \geqslant 3$, which counts all consecutive pairs $az, a > z$ in w. Due to Remark 2.3, all these pairs appear consecutively in w. Thus, the number of occurrences of the form $(ax)(ab)$, for x, b such that $x \leqslant b < a$ is equal to the number of $DDD = DD(UD)^0D$ patterns in the $(a-1)$-th run of D-steps in the corresponding dispersed Dyck path. Combining to the subcase 2, the occurrence xy is associated to an occurrence of the pattern $DD(UD)^*D$.

In the case (ii), we have a factor of the form $(ax)(yb)$ with $a = y$ and $x = b$ and the argument from Subcase 3 of case (i) applies. For the remaining cases, (iii) through (v), the occurrence xy of the pattern 12 is either created by a pair followed by a singleton $(ax)(y)$, or by a singleton followed by a pair $(x)(ya)$, or by two different singletons $(x)(y)$. Arguments similar to the ones given above, allow us to prove that an occurrence xy in w corresponds to an occurrence of:

- $DD(UD)^{y-a}F$ for the case $(ax)(y)$,
- $F(UD)^{a-x}UU$ for the case $(x)(ya)$, and
- $F(UD)^{y-x}F$ for the case $(x)(y)$.

Finally, in any n-length word we have $n-1$ occurrences of 2-length patterns, thus $\hat{\mathbf{n}} - \hat{\mathbf{1}} = \mathbf{11} + \mathbf{21} + \mathbf{12}$. Applying the bijection f to both parts of the equation, we obtain $f(\mathbf{12}) = \hat{\mathbf{n}} - \hat{\mathbf{1}} - f(\mathbf{11}) - f(\mathbf{21}) = \hat{\mathbf{n}} - \hat{\mathbf{1}} - \mathbf{UU} - \mathbf{FF}$. □

Theorem 2.9. *For $p \in \{11, 12, 21\}$, the trivariate generating functions $F_p(x, y, z)$ where the coefficient at $x^n y^k z^t$ is the number of k-ary faro words of length n containing exactly t occurrences of the pattern p are:*

$$F_{11}(x, y, z) = \frac{2y\,(xz - x - 1)}{-xyz + xy + x^3z - x^3 + y - x^2 + xz + x - 1 + (xz - x - 1)A_1},$$

$$F_{21}(x, y, z) = \frac{2y}{-y + x^2z - 2x + 1 + A_2},$$

$$F_{12}(x, y, z) = \frac{y\left(x^3z^2 - x^3z + x^2z + xyz - xy - 3xz + x + y - 1 + (xz - x + 1)A_2\right)}{\left(x^3z^2 - x^3z + x^2z - xyz + xy - xz - x - y + 1 + (xz - x + 1)A_2\right)(-1 + y)\,z} + \frac{y}{1 - y},$$

where $A_1 = \sqrt{x^4 - 2x^2y - 2x^2 + y^2 - 2y + 1}$ and $A_2 = \sqrt{x^4z^2 - 2x^2yz - 2x^2z + y^2 - 2y + 1}$.

Proof. We have $f(\mathcal{S}_{n,k}) = \mathcal{B}_{n+2(k-1),k-1}$. Thus, for any pattern p, the trivariate generating function $F_p(x, y, z)$ is given by $y \cdot B_p(x, \frac{y}{x^2}, z)$ where $B_p(x, y, z)$ is the trivariate generating function whose coefficient at $x^n y^k z^t$ is equal to the number of dispersed Dyck paths $P \in \mathcal{B}_{n,k}$ such that $\mathbf{q}(P) = t$, where $\mathbf{q} = f(\mathbf{p})$.

For $p = 21$, Theorem 2.8 has $f(\mathbf{21}) = \mathbf{UU}$. Therefore, we decompose the set $\mathcal{D}$ of Dyck paths as follows:

$$\mathcal{D} = \epsilon \uplus UD\mathcal{D} \uplus U(\mathcal{D} \setminus \epsilon)D\mathcal{D}.$$

We also decompose the set $\mathcal{B}$ of dispersed paths as follows:

$$\mathcal{B} = \epsilon \uplus F\mathcal{B} \uplus UD\mathcal{B} \uplus U(\mathcal{D} \setminus \epsilon)D\mathcal{B}.$$

If $D(x, y, z)$ is the generating function where $x^n y^k z^t$ is the number of Dyck paths of length n with k peaks and t occurrences of UU, then the above algebraic equation yields $D(x, y, z) = 1 + x^2yD(x, y, z) + x^2z(D(x, y, z) - 1)D(x, y, z)$. If $B_{21}(x, y, z)$ is the generating function whose coefficient at $x^n y^k z^t$ is the number of dispersed Dyck paths of length n with k peaks and t occurrences of UU, then the above decomposition of $\mathcal{B}$ yields the functional equation

$$B_{21}(x, y, z) = 1 + xB_{21}(x, y, z) + x^2yB_{21}(x, y, z) + x^2z(D(x, y, z) - 1)B_{21}(x, y, z),$$

which, in turn, yields the desired result.

For $p = 11$, Theorem 2.8 has $f(\mathbf{11}) = \mathbf{FF}$. Therefore, we decompose the set $\mathcal{D}$ of Dyck paths as follows:

$$\mathcal{D} = \epsilon \uplus UD\mathcal{D} \uplus U(\mathcal{D} \setminus \epsilon)D\mathcal{D}.$$

We also decompose the set $\mathcal{B}$ of dispersed Dyck paths as follows:

$$\mathcal{B} = \epsilon \uplus \mathcal{F} \uplus \mathcal{F}UD\mathcal{B} \uplus \mathcal{F}U(\mathcal{D} \setminus \epsilon)D\mathcal{B} \uplus UD\mathcal{B} \uplus U(\mathcal{D} \setminus \epsilon)D\mathcal{B},$$

where $\mathcal{F}$ is the infinite set of paths F^k for $k \geqslant 1$. Denote by $F(x, y, z)$ the generating function for $\mathcal{F}$, where its coefficient at $x^n y^k z^t$ is the number of n-length paths from $\mathcal{F}$ having k peaks and t occurrences of a pattern FF. Notice that $F(x, y, z) = \frac{x}{1-xz}$. If $D(x, y)$ is the generating function where the coefficient at $x^n y^k$ is the number of Dyck paths of length n with k peaks, then the above set decomposition yields $D(x, y) = 1 + x^2 y D(x, y) + x^2(D(x, y) - 1)D(x, y)$. Using the second set decomposition of $\mathcal{B}$, we obtain a functional equation

$$\begin{aligned} B_{11}(x, y, z) = 1 &+ x(1 + z(F(x, y, z) - 1)) + x^3(1 + z(F(x, y, z) - 1))B_{11}(x, y, z) \\ &+ x^3(1 + z(F(x, y, z) - 1))D(x, y)B_{11}(x, y, z) + x^2 y B_{11}(x, y, z) \\ &+ x^2(D(x, y) - 1)B_{11}(x, y, z), \end{aligned}$$

which provides the result.

For $p = 12$, we have, for any $P \in \mathcal{B}_{n,k}$, that $\mathbf{12}(P) = n - 1 - \mathbf{11}(P) - \mathbf{21}(P)$ (that is $\mathbf{12} = \hat{\mathbf{n}} - \hat{\mathbf{1}} - \mathbf{11} - \mathbf{21}$), and thus

$$F_{12}(x, y, z) = \frac{1}{z}\left(F_{11+21}\left(xz, y, \frac{1}{z}\right) - \frac{y}{1-y}\right) + \frac{y}{1-y}.$$

According to Theorem 2.8, we have $f(\mathbf{11} + \mathbf{21}) = \mathbf{FF} + \mathbf{UU}$. Therefore, we decompose the set $\mathcal{B}$ as before for the case of pattern 11, and construct a functional equation by taking into account the different occurrences of FF and UU, which yields the claimed result. □

Corollary 2.10. *For $n \geqslant 0$, the popularity of pattern $p \in \{11, 12, 21\}$ in $\mathcal{S}_{n,k}$ is given by the bivariate generating function $G_p(x, y)$:*

$$\begin{aligned} G_{11}(x, y) &= \frac{4x^2 y}{\left(1 - y - 2x + x^2 + A_1\right)^2}, \\ G_{21}(x, y) &= \frac{2x^2 y\left(1 + y - x^2 - A_1\right)}{\left(1 - y - 2x + x^2 + A_1\right)^2 A_1}, \\ G_{12}(x, y) &= \frac{2xy\left(A_3 + (x^3 - 2x^2 + 2xy - 2x - 2y + 2)A_1\right)}{\left(1 - y - 2x + x^2 + A_1\right)^2 (1 - y) A_1}, \end{aligned}$$

where $A_1 = \sqrt{x^4 - 2x^2 y - 2x^2 + y^2 - 2y + 1}$ and $A_3 = x^5 - 2x^4 - x^3 y - 3x^3 + 4x^2 y + 4x^2 - 2xy - 2y^2 + 2x + 4y - 2$.

Proof. Using Theorem 2.9, we obtain the result by calculating $\left(\frac{\partial}{\partial z} F_p(x, y, z)\right)\big|_{z=1}$ for $p \in \{11, 21, 12\}$. □

Corollary 2.11. *For $p \in \{11, 12, 21\}$, the bivariate generating functions $H_p(x, y)$ whose coefficient at $x^n y^k$ is the number of k-ary faro words of length n avoiding the pattern p are:*

$$\begin{aligned} H_{11}(x, y) &= \frac{2y(x+1)}{1 - x - y - xy + x^2 + x^3 + (1+x)\sqrt{(x^2 - 2x - y + 1)(x^2 + 2x - y + 1)}}, \\ H_{21}(x, y) &= \frac{y}{1 - x - y}, \\ H_{12}(x, y) &= \frac{y\left(-x^3 y + x^2 y - xy^2 + xy + y^2 - 2y + 1\right)}{xy^3 - 3xy^2 - y^3 + 3xy + 3y^2 - x - 3y + 1}. \end{aligned}$$

Proof. Note that $H_p(x, y) = F_p(x, y, 0)$, where $F_p(x, y, z)$ is as in Theorem 2.9. □

Now we discuss the two special cases of $k = 2$ and $k = n$, which correspond respectively to binary words and n-ary words of length n (see Table 1 for numerical values).

Case $k = 2$: using Corollary 2.10, we can easily prove that the popularity of the pattern 11 in $\mathcal{S}_{n,2}$ generates a shift of the sequence A212964 in [26], which also counts the number of 3-element subsets A of $\{1, \ldots, n+1\}$ such that all the sums $a_1 + a_2$ with $a_1 \leqslant a_2$ and $a_1, a_2 \in A$ are distinct. The popularity of 21 generates a shift of the sequence A006918 where the general term is given by $\binom{n+3}{3}/4$ if n is odd, and $n(n+2)(n+4)/24$ if n is even. The other patterns do not provide known sequences in [26].

Case $k = n$: the sequences of popularity of $p \in \{11, 21, 12\}$ are not listed in [26], and we have not succeeded in finding a closed form for the diagonal of $G_p(x, y)$. However, using the Maple package *gfun* [24], we conjecture that the popularity sequence for 11 satisfies a recurrence equation $Q_1(n)u_n + Q_2(n)u_{n+1} + Q_3(n)u_{n+2} + Q_4(n)u_{n+3} = 0$, where Q_1, Q_2, Q_3, Q_4

Table 1
Popularity of patterns p of length two in $\mathcal{S}_{n,2}$ and $\mathcal{S}_{n,n}$.

k	Pattern p	Popularity of p in $\mathcal{S}_{n,k}$ for $1 \leqslant n \leqslant 9$
2	11	0, 2, 6, 14, 26, 44, 68, 100, 140, …
	21	0, 1, 2, 5, 8, 14, 20, 30, 40, …
	12	0,1,4,8,14,22,32,45,60, …
n	11	0,2,12,80,490, 3192, 20076,13094,83655, …
	21	0,1,8,85, 574,4788,31800, 24489,162305, …
	12	0,1,16,135,1036, 7700,53964,38646,2636920, …

are some polynomial functions of degree at most 10, which suggests that the generating function of the diagonal is D-finite when $p = 11$. However, we have not succeeded in obtaining a closed form of the diagonal of $H_{11}(x, y)$. In contrast, a simple study of the residues (see [27] Section 6.3) of $H_{21}(x/y, y)$ at the pole $y_0 = (1 - \sqrt{1-4x})/2$ yields the generating function $(1 - \sqrt{1-4x})/(2\sqrt{1-4x})$ of the diagonal of $H_{21}(x, y)$, and its general term is, therefore, $\binom{2n-1}{n}$ (see sequence A001700). A similar study for the pattern 12 yields the diagonal $x(x^3 - 2x^2 + x + 1)/(1-x)^2$ (here, the pole is $y_0 = x$), which generates the sequence $u_1 = 1$, $u_2 = 3$, $u_n = n$ for $n \geqslant 3$.

Statistic correspondences for other patterns can be obtained using a method similar to that of Theorem 2.8. Therefore, we list directly (without proof) in Theorem 2.12 the f-images of all statistics associated to a pattern of length three. It is worth noting that the reverse-complement χ is a bijection on $\mathcal{S}_{n,k}$, which proves that the statistics **112** and **122** (resp. **121** and **212**, resp. **132** and **213**) have the same distribution on $\mathcal{S}_{n,k}$.

Theorem 2.12. *For $n \geqslant 0$, the bijection f from $\mathcal{S}_{n,k}$ to $\mathcal{B}_{n+2(k-1),k-1}$ translates statistics associated to patterns of length three as follows:*

$$f(\mathbf{111}) = \mathbf{FFF},$$
$$f(\mathbf{112}) = \mathbf{FF(UD)^+F} + \mathbf{FF(UD)^*UU},$$
$$f(\mathbf{122}) = \mathbf{F(UD)^+FF} + \mathbf{DD(UD)^*FF},$$
$$f(\mathbf{121}) = \mathbf{FUU} + \mathbf{UUU},$$
$$f(\mathbf{212}) = \mathbf{DDF} + \mathbf{DDD},$$
$$f(\mathbf{132}) = \mathbf{F(UD)^+UU} + \mathbf{U(UD)^+UU} + \mathbf{DD(UD)^*UU},$$
$$f(\mathbf{213}) = \mathbf{DD(UD)^+F} + \mathbf{DD(UD)^+D} + \mathbf{DD(UD)^*UU},$$
$$f(\mathbf{123}) = \mathbf{DD(UD)^*F(UD)^*UU} + \mathbf{DD(UD)^*F(UD)^+F} + \mathbf{F(UD)^+F(UD)^*UU} + \mathbf{F(UD)^+F(UD)^+F},$$
$$f(\mathbf{211}) = f(\mathbf{221}) = f(\mathbf{231}) = f(\mathbf{312}) = f(\mathbf{321}) = \hat{\mathbf{0}}.$$

It would be interesting to see how the method developed in [1,2] could be applied to obtain more pattern distributions in dispersed Dyck paths, but this is beyond the scope of the present paper. The multivariate generating functions for patterns of length three are quite technical, not particularly interesting and laborious to obtain. So, we decide to leave them as an exercise for the reader.

3. Patterns in faro permutations

We say that a k-ary faro word w of length n is *injective* (resp. *surjective*) if and only if any value in w appears only once in w (resp. any value $x \in [1, k]$ appears in w). A *faro permutation* of length n is an n-ary faro word that is both injective and surjective. Let $\mathcal{P}_n$ be the set of length n faro permutations. For instance, we have $\mathcal{P}_3 = \{123, 132, 213\}$. Since faro permutations are entirely determined by the choice of their values on the odd indices, the cardinality of $\mathcal{P}_n$ is $\binom{n}{\lfloor n/2 \rfloor}$. Note that faro permutations are permutations avoiding the three consecutive patterns 231, 321 and 312 (see Remark 1.3).

Theorem 3.1. *The bijection f maps surjective k-ary faro words of length n onto dispersed Dyck paths in $\mathcal{B}_{n+2(k-1),k-1}$ avoiding $UDUD$ that neither start nor end with UD.*

Proof. Using the definition of the bijection f and in particular the definition of the sequence T, surjective faro words are those that have a sequence T satisfying (i) $T_0 + T_1 > 1$, (ii) $T_{3(x-1)-1} + T_{3(x-1)} + T_{3(x-1)+1} > 2$ for $x \in [2, k-1]$, and (iii) $T_{3(k-1)-1} + T_{3(k-1)} > 1$. Since $T_1 \geqslant 1$, the condition (i) is equivalent to $T_0 \neq 0$, or $T_0 = 0$ and $T_1 > 1$, which means that $f(w)$ does not start with UD. Similarly, the condition (iii) is equivalent to the fact that $f(w)$ does not end with UD. Since

$T_{3(x-1)-1} \geqslant 1$ and $T_{3(x-1)+1} \geqslant 1$, the condition (ii) is equivalent to $T_{3(x-1)} > 0$, or $T_{3(x-1)} = 0$ and $T_{3(x-1)-1} + T_{3(x-1)+1} > 2$, which means that $f(w)$ does not contain any occurrence of $UDUD$. □

Theorem 3.2. *The bijection f maps injective k-ary faro words of length n into dispersed Dyck paths in $\mathcal{B}_{n+2(k-1),k-1}$ avoiding the patterns FF, DDD, UUU, DDF, FUU, and $DDUU$.*

Proof. Using the definition of f, injective faro words are those that have a sequence T satisfying (i) $T_{3(x-1)} < 2$, $x \in [1, k]$; (ii) $T_{3(x-1)-1} < 3$, $x \in [2, k]$; (iii) $T_{3(x-1)+1} < 3$, $x \in [1, k-1]$; (iv) $T_{3(x-1)-1} + T_{3(x-1)} < 3$, $x \in [2, k]$; (v) $T_{3(x-1)} + T_{3(x-1)+1} < 3$, for $x \in [1, k-1]$; and (vi) $T_{3(x-1)-1} + T_{3(x-1)+1} < 3$, for $x \in [2, k-1]$. It means that $f(w)$ avoids, respectively, the patterns FF, DDD, UUU, DDF, FUU and $DDUU$. □

Theorem 3.3. *The image by f of $\mathcal{P}_n$ is the subset $B'_{3n-2,n-1}$ of dispersed Dyck paths in $B_{3n-2,n-1}$ that neither start nor end with UD and where any two consecutive occurrences of UD are separated by exactly one step.*

Proof. The two previous theorems imply that $f(\mathcal{P}_n)$ is the set of dispersed Dyck paths in $B_{3n-2,n-1}$ that neither start nor end with UD and that avoid the patterns FF, DDD, UUU, $DDUU$, $UDUD$, DDF and FUU, which is exactly the dispersed Dyck paths that neither start nor end with UD and where any two consecutive occurrences of UD are separated by exactly one step. Indeed, in any dispersed Dyck path, the subpath between two consecutive occurrences of UD is necessarily of the form $D^iF^jU^k$ for $i, j, k \geqslant 0$. Then, the avoidance of FF, DDD and UUU implies that $i, j, k \leqslant 1$, and the avoidance of the other patterns implies that $i + j + k = 1$ as claimed. □

Thus, we deduce a one-to-one correspondence g between length n faro permutations and dispersed Dyck paths of length n, where $g(p)$ is obtained from $p \in \mathcal{P}_n$ by removing all occurrences of UD in $f(p)$. For instance, if $p = 1243576$ then $f(p) = FUDFUDUUDDUDFUDUUDD$ and $g(p) = FFUDFUD$.

Theorem 3.4. *For $n \geqslant 0$, the bijection g from $\mathcal{P}_n$ to $\mathcal{B}_n$ transports the pattern statistics as follows:*

$$\begin{aligned}
g(\mathbf{21}) &= \mathbf{U},\\
g(\mathbf{12}) &= \mathbf{DU} + \mathbf{DD} + \mathbf{DF} + \mathbf{FF} + \mathbf{FU},\\
&= \hat{\mathbf{n}} - \hat{\mathbf{1}} - \mathbf{U},\\
g(\mathbf{132}) &= \mathbf{FU} + \mathbf{UU} + \mathbf{DU},\\
g(\mathbf{213}) &= \mathbf{DF} + \mathbf{DD} + \mathbf{DU},\\
g(\mathbf{123}) &= \mathbf{DFU} + \mathbf{DFF} + \mathbf{FFU} + \mathbf{FFF},\\
&= \hat{\mathbf{n}} - \hat{\mathbf{2}} - \mathbf{FU} - \mathbf{UU} - 2\,\mathbf{DU} - \mathbf{DF} - \mathbf{DD},\\
g(\mathbf{231}) &= g(\mathbf{312}) = g(\mathbf{321}) = \hat{\mathbf{0}}.
\end{aligned}$$

Proof. For a faro permutation w, $g(w)$ is obtained from $f(w)$ by removing all peaks UD. In $f(w)$ consecutive occurrences of UD are separated by one letter exactly, and no U^k, D^k or F^k exists for $k \geqslant 3$. The last U in U^2 and the first D in D^2 must be a part of an occurrence of UD. The claimed statistic equations are obtained from Theorem 2.8 and Theorem 2.12 by deleting peaks UD and replacing all remaining U^2 (resp. D^2) with U (resp. D) in all considered pattern statistics. Only two (resp. three) patterns of length 2 (resp. 3) are possible in a faro permutation, namely 12 and 21 (resp. 123, 132 and 213). In any n-length word there is $n-1$ (resp. $n-2$) occurrences of patterns of length 2 (resp. 3). It follows that $g(\mathbf{12}) = \hat{\mathbf{n}} - \hat{\mathbf{1}} - \mathbf{U}$ and $g(\mathbf{123}) = \hat{\mathbf{n}} - \hat{\mathbf{2}} - \mathbf{FU} - \mathbf{UU} - 2\,\mathbf{DU} - \mathbf{DF} - \mathbf{DD}$. □

Theorem 3.5. *For $p \in \{21, 12, 132, 213, 123\}$, the bivariate generating functions $K_p(x, y, z)$, where the coefficient at $x^n y^k$ is the number of faro permutations of length n containing exactly k occurrences of the pattern p, are:*

$$\begin{aligned}
K_{21}(x, y) &= \frac{2}{1 - 2x + \sqrt{1 - 4x^2 y}},\\
K_{12}(x, y) &= \frac{1 + y + 2xy - 2xy^2 + (y-1)\sqrt{1 - 4x^2 y}}{y\left(1 - 2xy + \sqrt{1 - 4x^2 y}\right)},\\
K_{132}(x, y) &= \frac{1 + y + (y-1)\sqrt{1 - 4x^2 y}}{y\left(1 - 2x + \sqrt{1 - 4x^2 y}\right)},
\end{aligned}$$

$$K_{213}(x,y) = K_{132}(x,y),$$
$$K_{123}(x,y) = \frac{2+3x-3xy+2x^2-2x^2y-x(1-y)\sqrt{1-4x^2}}{1-2xy+\sqrt{1-4x^2}}.$$

Proof. For $p = 21$, Theorem 3.4 has $g(\mathbf{21}) = \mathbf{U}$. So, we decompose the set of Dyck paths as $\mathcal{D} = \epsilon \uplus U\mathcal{D}D\mathcal{D}$, the set of dispersed Dyck paths as $\mathcal{B} = \epsilon \uplus F\mathcal{B} \uplus U\mathcal{D}D\mathcal{B}$, and obtain the following system:

$$\begin{cases} D(x,y) = 1 + x^2yD^2(x,y), \\ B = 1 + xB(x,y) + x^2yD(x,y)B(x,y), \end{cases}$$

where $D(x,y)$ (resp. $B(x,y)$) is the generating function for the set of Dyck paths (resp. dispersed Dyck paths) with respect to the number of occurrences of U. Solving it, we obtain $K_{21}(x,y) = B(x,y)$.

Since only the two length 2 patterns (12 and 21) are possible in a faro permutation, we have $\mathbf{12} = \hat{\mathbf{n}} - \hat{\mathbf{1}} - \mathbf{21}$. Hence, $K_{12}(x,y) = (K_{21}(xy, \frac{1}{y}) - 1)/y + 1$.

Only tree patterns of length 3 are possible in a faro permutation, 123, 132 and 213, so we have $\mathbf{123} = \hat{\mathbf{n}} - \hat{\mathbf{2}} - \mathbf{132} - \mathbf{213}$. By Theorem 3.4, $g(\mathbf{132} + \mathbf{213}) = \mathbf{FU} + \mathbf{UU} + 2\ \mathbf{DU} + \mathbf{DF} + \mathbf{DD}$. We decompose the sets of Dyck and dispersed Dyck paths as follows:

$$\begin{cases} \mathcal{D} &= \epsilon \uplus UD \uplus U(\mathcal{D}\setminus\epsilon)D \uplus UD(\mathcal{D}\setminus\epsilon) \uplus U(\mathcal{D}\setminus\epsilon)D(\mathcal{D}\setminus\epsilon), \\ \mathcal{B} &= \epsilon \uplus \overline{\mathcal{B}} \uplus U(\mathcal{D}\setminus\epsilon)D \uplus U(\mathcal{D}\setminus\epsilon)D\overline{\mathcal{B}} \uplus U(\mathcal{D}\setminus\epsilon)D(\mathcal{B}\setminus(\epsilon\uplus\overline{\mathcal{B}})) \\ &\quad \uplus UD \uplus UD\overline{\mathcal{B}} \uplus UD(\mathcal{B}\setminus(\epsilon\uplus\bar{\mathcal{B}})), \\ \overline{\mathcal{B}} &= \mathcal{F} \uplus \mathcal{F}(\mathcal{B}\setminus(\epsilon\uplus\overline{\mathcal{B}})), \end{cases}$$

where $\mathcal{F}$ is the set of paths F^k, $k \geqslant 1$ and $\overline{\mathcal{B}}$ is the set of dispersed Dyck paths starting with a level step. From this decomposition we obtain the following system of functional equations:

$$\begin{cases} D(x,y) &= 1 + x^2 + 2x^2y^2\left(D(x,y)-1\right) + x^2y^4\left(D(x,y)-1\right)^2, \\ B(x,y) &= 1 + \overline{B}(x,y) + x^2y^2\left(D(x,y)-1\right) + x^2y^3\left(D(x,y)-1\right)\overline{B}(x,y) + \\ &\quad + x^2y^4\left(D(x,y)-1\right)\left(B(x,y)-\overline{B}(x,y)-1\right) + x^2 + x^2y\overline{B}(x,y) \\ &\quad + x^2y^2\left(B(x,y)-\overline{B}(x,y)-1\right), \\ \overline{B}(x,y) &= \frac{x}{1-x} + \frac{xy}{1-x}\left(B(x,y)-\overline{B}(x,y)-1\right), \end{cases}$$

where $D(x,y)$ (resp. $B(x,y)$, resp. $\overline{B}(x,y)$) is the generating function for the set of Dyck paths (resp. dispersed Dyck paths, resp. dispersed Dyck paths starting with F) with respect to the statistics $\mathbf{FU}+\mathbf{UU}+2\,\mathbf{DU}+\mathbf{DF}+\mathbf{DD}$. After solving this system, we obtain the result by evaluating $K_{123}(x,y) = 1 + x + (B(xy, \frac{1}{y}) - 1 - xy)/y^2$. Note that it is possible to look directly at $g(\mathbf{123}) = \mathbf{DFU} + \mathbf{DFF} + \mathbf{FFU} + \mathbf{FFF}$ rather than at $g(\mathbf{132} + \mathbf{213})$ as we did, but the decomposition will be more complicated.

Note that $K_{132}(x,y) = K_{213}(x,y)$, by taking the reverse-complement of faro permutations. We remark that $\mathbf{DF}+\mathbf{DD}+\mathbf{DU}$ corresponds to the number of occurrences of D except the last symbol if it is a D. So, if $\mathcal{D}$ is the set of Dyck paths and $\mathcal{B}$ the set of dispersed paths, then the classical decompositions $\mathcal{D} = \epsilon \uplus \mathcal{D}U\mathcal{D}D$, $\mathcal{B} = \epsilon \uplus \mathcal{B}F \uplus \mathcal{B}U\mathcal{D}D$ provide the following system of functional equations:

$$\begin{cases} D(x,y) &= 1 + x^2yD(x,y)^2, \\ B(x,y) &= 1 + xB(x,y) + x^2yB(x,y)D(x,y), \\ \overline{B}(x,y) &= 1 + xB(x,y) + x^2B(x,y)D(x,y), \end{cases}$$

where $D(x,y)$ and $B(x,y)$ (resp. $\overline{B}(x,y)$) are the generating functions for the sets of Dyck paths and dispersed Dyck paths with respect to the number of D (resp. number of D except the last symbol if it is a D). Solving the system, we obtain $K_{132}(x,y) = K_{213}(x,y) = \overline{B}(x,y)$. □

Corollary 3.6. *For $n \geqslant 0$, the popularity of pattern $p \in \{21, 12, 132, 213, 123\}$ in $\mathcal{P}_n$ is given by the generating function $L_p(x)$:*

$$L_{21}(x) = \frac{1-\sqrt{1-4x^2}}{2(1-2x)\sqrt{1-4x^2}},$$
$$L_{12}(x) = \frac{2x\left(-1+4x^2+x+\sqrt{1-4x^2}\right)}{(1-2x)(1+\sqrt{1-4x^2})\sqrt{1-4x^2}},$$

Table 2
Popularity of patterns p of length at most three in faro permutations.

Pattern p	Popularity of p in $\mathcal{P}_n$ for $1 \leqslant n \leqslant 11$	OEIS
21	0, 1, 2, 7, 14, 38, 76, 187, 374, 874, 1748, ...	A107373
12	0, 1, 4, 11, 26, 62, 134, 303, 634, 1394, 2872, ...	A340567
132, 213	0, 0, 1, 4, 10, 28, 61, 152, 318, 748, 1538, ...	A340568
123	0, 0, 1, 4, 10, 24, 53, 116, 246, 520, 1082, ...	A340569
231,312,321	0, 0, 0, 0, ...	

$$L_{132}(x) = \frac{x\left(-1+4x^2+2x+(1-2x)\sqrt{1-4x^2}\right)}{(1-2x)(1+\sqrt{1-4x^2})\sqrt{1-4x^2}},$$
$$L_{213}(x) = L_{132}(x),$$
$$L_{123}(x) = \frac{x(1+2x)(1-\sqrt{1-4x^2})}{(1-2x)(1+\sqrt{1-4x^2})}.$$

Proof. We evaluate $\frac{\partial K_p(x,y)}{\partial y}\big|_{y=1}$. □

Corollary 3.7. *For $n \geqslant 2$, the popularity of pattern $p \in \{21, 12, 132, 213, 123\}$ in $\mathcal{P}_n$ is given by $\mathbf{p}(\mathcal{P}_n)$:*

$$\mathbf{21}(\mathcal{P}_n) = \frac{n+1}{2}\binom{n}{\lfloor \frac{n}{2} \rfloor} - 2^{n-1} \sim \sqrt{\frac{n}{2\pi}} \cdot 2^n,$$
$$\mathbf{12}(\mathcal{P}_n) = (n-1)\binom{n}{\lfloor \frac{n}{2} \rfloor} - \mathbf{21}(\mathcal{P}_n) \sim \sqrt{\frac{n}{2\pi}} \cdot 2^n,$$
$$\mathbf{123}(\mathcal{P}_n) = 2^n - 2\binom{n-1}{\lfloor \frac{n-1}{2} \rfloor} - \binom{n}{\lfloor \frac{n}{2} \rfloor} \sim 2^n,$$
$$\mathbf{132}(\mathcal{P}_n) = \mathbf{213}(\mathcal{P}_n) = \frac{1}{2}\left((n-2)\binom{n}{\lfloor \frac{n}{2} \rfloor} - \mathbf{123}(\mathcal{P}_n)\right) \sim \sqrt{\frac{n}{2\pi}} \cdot 2^n.$$

Proof. Recall that $|\mathcal{P}_n| = \binom{n}{\lfloor n/2 \rfloor}$. Then we have $\mathbf{21}(\mathcal{P}_n) + \mathbf{12}(\mathcal{P}_n) = (n-1) \cdot \binom{n}{\lfloor n/2 \rfloor}$ and $\mathbf{132}(\mathcal{P}_n) + \mathbf{213}(\mathcal{P}_n) + \mathbf{123}(\mathcal{P}_n) = (n-2) \cdot \binom{n}{\lfloor n/2 \rfloor}$, and considering $\mathbf{132}(\mathcal{P}_n) = \mathbf{213}(\mathcal{P}_n)$, it suffices to prove the result for $\mathbf{21}(\mathcal{P}_n)$ and $\mathbf{123}(\mathcal{P}_n)$. Due to Corollary 3.6, we have

$$L_{21}(x) = \frac{W(x)}{2x} - \frac{1}{2(1-2x)}$$

where $W(x) = \frac{x}{(1-2x)\sqrt{1-4x^2}}$ is the generating function for the sequence A100071 in [26] which has the general term $\frac{n}{2} \cdot \binom{n-1}{\lfloor (n-1)/2 \rfloor}$. This induces directly $\mathbf{21}(\mathcal{P}_n) = \frac{n+1}{2} \cdot \binom{n}{\lfloor n/2 \rfloor} - 2^{n-1}$.

Similarly, if we expand the numerator of $L_{123}(x)$ given in Corollary 3.6, then we obtain four generating functions having the general terms respectively equal to $2^{n-1} - \binom{n-1}{\lfloor (n-1)/2 \rfloor}$, $-\binom{n-1}{\lfloor (n-1)/2 \rfloor}$, $2^{n-1} - \frac{1}{2} \cdot \binom{n}{\lfloor n/2 \rfloor}$ and $-\frac{1}{2} \cdot \binom{n}{\lfloor n/2 \rfloor}$, which implies the claimed result. Finally, asymptotics are easily obtained using $\binom{n}{\lfloor n/2 \rfloor} \sim \sqrt{\frac{2}{\pi n}} \cdot 2^n$. □

Using formulae from Corollary 3.7 the following remark can be easily verified.

Remark 3.8. The expected number of the occurrences of the pattern 21 (respectively 12, 132 and 213) in a randomly selected faro permutation of length n is asymptotically equivalent to $n/2$ when $n \to \infty$. In contrast, the expectation of 123 is asymptotically equivalent to $\sqrt{\pi n/2}$ and thus the probability that a random faro permutation contains an occurrence of 123 at a random position approaches 0 as n grows.

Table 2 provides the first values of the popularity of each pattern of length at most three in faro permutations.

4. Some particular subsets of $\mathcal{P}_n$ and $\mathcal{S}_{n,k}$

In this part, we study particular subsets of faro permutations and faro words which are in one-to-one correspondence with other sets of well-known combinatorial objects. Let us recall the definition of a *standard cycle notation* (s.c.n.) and Foata's first fundamental transformation ϕ (see [21]).

In the standard cycle notation (s.c.n.) of a permutation w each cycle starts with its largest element, and cycles are ordered from left to right in increasing order of their largest elements.

Foata's first fundamental transformation (see [21]) acts on a permutation w as follows. Write a permutation w in s.c.n., and then cyclically rearrange every cycle so that it ends with its largest element. Then, reverse each cycle and delete all parentheses. For instance, if $w = 7321564$, then the s.c.n. for w is $(32)(5)(6)(741)$, after rearrangement we have $(23)(5)(6)(417)$, and thus $\phi(w) = 3256714$. If w is an involution, it contains only cycles of length one or two, rearrangement and reversion are not needed, we directly obtain the image after dropping parentheses in s.c.n.

Theorem 4.1. *Foata's first fundamental transformation bijectively maps the set $\mathcal{I}_n$(3-2-1) of involutions of length n avoiding the classical pattern* 3-2-1 *onto the set $\mathcal{P}_n$ of faro permutations.*

Proof. Let us prove that the standard cycle notation of $w \in \mathcal{I}_n$(3-2-1) cannot contain any of the following consecutive cycles: $(x)(yz)$ with $z < x$, $(xy)(z)$ with $z < x$, $(x)(y)(z)$ with $z < x$, or $(xy)(zt)$ with $t < y$ or $z < x$. Assume that $w \in \mathcal{I}_n$(3-2-1) and assume towards contradiction that the standard cycle notation (s.c.n.) of w contains $(x)(yz)$ with $z < x$. Then we have $x < y$ and thus $z < x < y$, which means that the subsequence yxz (occurring at indices z, x, y) is an occurrence of 3-2-1 in w, a contradiction. Due to the definition of the s.c.n. of w, the case $(x)(y)(z)$ with $z < x$, the case $(xy)(z)$ with $z < x$ and the case $(xy)(zt)$ with $z < x$ do not occur since the cycles are arranged in increasing order of their first elements. If the s.c.n. of w contains $(xy)(zt)$ with $t < y$, then we have $t < y < x < z$, which implies that w contains an occurrence zxy (at indices t, y, x) of 3-2-1, a contradiction. Thus, $\phi(\mathcal{I}_n(\text{3-2-1})) \subset \mathcal{P}_n$. Since ϕ is injective, and $\mathcal{I}_n$(3-2-1) is also enumerated by ballot numbers b_n (see for instance [6,25]), we have $\phi(\mathcal{I}_n(\text{3-2-1})) = \mathcal{P}_n$. □

Remark 4.2. It is known that Foata's first transformation ϕ maps the statistic of the number of excedances (values w_i such that $w_i > i$) to the statistic **21** (number of descents $w_i > w_{i+1}$). Therefore, the generating functions $K_{21}(x, y)$ and $L_{21}(x)$ in Corollary 3.6 also give the distribution and the popularity of excedances in $\mathcal{I}_n$(3-2-1).

Remark 4.3. We could easily check that $g(\phi(w)) = \Phi(w)$ for $w \in \mathcal{I}_n$(3-2-1), where Φ is a bijection in [6] between involutions and labeled Motzkin paths, which also is a restriction of Biane's bijection [9], which in turn is closely related to Françon-Viennot bijection [16].

The next theorem deals with alternating faro permutations, i.e. permutations w satisfying $w_1 > w_2 < w_3 > \cdots$. Let $\mathcal{A}_n$ be the set of alternating faro permutations of length n.

Theorem 4.4. *There is a bijection between $\mathcal{A}_{2n}$ and the set of Dyck paths of length $2n$.*

Proof. Let w be a faro permutation of length $2n$. Then, w is alternating if and only if w does not contain any singleton in its block decomposition. Due to the definition of f, this means that $f(w)$ does not contain any F-steps and thus, $g(w)$ is a Dyck path of length $2n$, and vice versa. □

Theorem 4.5. *The set $\mathcal{A}_{2n}$ is exactly the set of length $2n$ faro derangements, i.e. faro permutations with no fixed point $w_i = i$ for $i \in [1, 2n]$.*

Proof. Let $w = w_1 w_2 \dots w_{2n-1} w_{2n}$ be a faro permutation of length $2n$, that is $w_i < w_{i+2}$ for $1 \leqslant i \leqslant 2n-2$. Then, w is alternating if and only if w does not contain any singleton in its block decomposition, or equivalently, w satisfies $w_i > w_{i+1}$ if i is odd, and $w_i < w_{i+1}$ otherwise. This is equivalent to w_i is greater than $w_1, w_2, \dots, w_{i-1}$ and w_{i+1} if i is odd, and w_i is smaller than w_{i-1}, $w_{i+1}, w_{i+2}, \dots, w_{2n}$ if i is even, which means that $w_i > i$ if i is odd and $w_i < i$ otherwise. Thus, we have $w_i \neq i$, and w is a derangement. This last implication also is an equivalence because it cannot occur $w_i < i$ with i odd, or $w_i > i$ with i even in a faro derangement w. □

Theorem 4.6. *Let $\mathcal{B}_n$ (resp. $\mathcal{B}'_n$) be the set of length n faro permutations avoiding the classical pattern* 2-3-1 *(resp. the pattern* 3-1-2*), then*

- *The cardinality of $\mathcal{B}_n$ is given by the Fibonacci sequence f_n defined by $f_n = f_{n-1} + f_{n-2}$ with $f_1 = 1$, $f_2 = 2$.*
- *We have $\mathcal{B}_n = \mathcal{B}'_n$.*
- *$\mathcal{B}_n$ is exactly the set of length n faro involutions.*

Proof. A faro permutation w avoiding the pattern 2-3-1 is of the form $1w'$ or $21w'$, where w' also is a faro permutation avoiding 2-3-1. Indeed, if a faro permutation w starts with $x > 2$, then w starts with $x1y$ for some $y > x$. Then the value 2 is to the right of $x1y$, which creates an occurrence $xy2$ of 2-3-1, a contradiction. Therefore, the cardinality f_n of length n faro permutations satisfies $f_n = f_{n-1} + f_{n-2}$ with $f_1 = 1$, $f_2 = 2$. Using the same argument, faro permutations avoiding 2-3-1 are also faro permutations avoiding 3-1-2.

For the third statement, due to the decomposition of $w \in \mathcal{B}_n$ (either $w = 1w'$ or $w = 21w'$ with $w' \in \mathcal{B}$), we conclude by induction that w is necessarily a faro involution. Conversely, a faro involution w avoids the classical pattern 321, which implies that $w = 1w'$ or $w = 21w'$ where w' is also a faro involution (if the first entry w_1 of w satisfies $w_1 \geqslant 3$, then we have $w_{w_1} = 1$; since w avoids 321, we necessarily have $w_i > w_1 \geqslant 3$ for $1 \leqslant i \leqslant w_1 - 1$, and thus $w_{w_1-2} > w_{w_1} = 1$, which is not possible in a faro permutation). A simple induction implies that $w \in \mathcal{B}_n$, which completes the proof. □

In the following, we consider (for convenience) faro words on the n-ary alphabet $[0, n-1]$, and we focus on the set of *subexcedent* faro words of length n, i.e. faro words $w_1 w_2 \dots w_n$ satisfying $w_i \leqslant i - 1$ for $1 \leqslant i \leqslant n$. We make a shift $[1, n] \to [0, n-1]$ on the alphabet in order to apply directly the results presented in [12].

Theorem 4.7. *There is a bijection between subexcedent faro words of length n and 2-1-4-3-avoiding Dumont permutations of the second kind of length $2n$.*

We will briefly recall the result given in [12] that enumerated 2-1-4-3-avoiding Dumont permutations of the second kind of length $2n$. Dumont permutations of the second kind of length $2n$ are permutations π that satisfy the following conditions for $i \in [n]$:

$$\pi(2i-1) \geqslant 2i-1, \qquad \pi(2i) \leqslant 2i-1.$$

In other words, the values in the odd positions are weak excedances, whereas the values in the even positions are deficiencies. In addition, if π avoids the pattern 2-1-4-3 (i.e. does not contain a subsequence $\pi(i_1)\pi(i_2)\pi(i_3)\pi(i_4)$ of length 4 such that $i_1 < i_2 < i_3 < i_4$ and $\pi(i_2) < \pi(i_1) < \pi(i_4) < \pi(i_3)$), then the values in the even positions of π are exactly $\{1, 2, \dots, n\}$, and the values in the odd positions of π are exactly $\{n+1, n+2, \dots, 2n\}$. Moreover, the subsequence of values of π in the even positions avoids the pattern 2-1-3 while the subsequence of values of π in the odd positions avoids the pattern 1-3-2. This allows [12] to construct a bijection as in Krattenthaler [20] from the even-position subsequence of π to north-east integer lattice paths from $(0, 0)$ to $(n, \lfloor n/2 \rfloor)$ staying on or below the line $y = x/2$, and from the odd-position subsequence of π to the same paths but ending at $(n+1, \lfloor (n+1)/2 \rfloor)$. Let $\{a_n\}_{n \geqslant 0}$ be the sequence A047749 [26], so that

$$a_{2n} = \frac{1}{2n+1}\binom{3n}{n}, \qquad a_{2n+1} = \frac{1}{n+1}\binom{3n+1}{n},$$

then the number of 2-1-4-3-avoiding Dumont permutations of the second kind of length $2n$ is $a_n a_{n+1}$. Thus, to prove Theorem 4.7, we only need to construct a bijection from subexcedent faro words of length n to ordered pairs of north-east lattice paths on or below the line $y = x/2$ from $(0, 0)$ to $(n, \lfloor n/2 \rfloor)$ and $(n+1, \lfloor (n+1)/2 \rfloor)$, respectively.

Proof of Theorem 4.7. Let π be a subexcedent faro word of length n. As in [12], let π_o and π_e be the odd-position and even-position subsequences of π. Then π_o and π_e are nondecreasing subsequences such that

$$\begin{aligned} \pi_o(i) &= \pi(2i-1) \in [0, 2i-2], \quad i \leqslant \left\lfloor \frac{n+1}{2} \right\rfloor, \\ \pi_e(i) &= \pi(2i) \in [0, 2i-1], \quad i \leqslant \left\lfloor \frac{n}{2} \right\rfloor. \end{aligned} \tag{4.1}$$

Conversely, any word π whose odd-position and even-position subsequences π_o and π_e satisfy the above properties is a subexcedent faro word of length n. Given sequences π_o and π_e as in (4.1), associate to them a pair of north-east lattice paths as follows. If π_o or π_e has a letter a_i in position i, map such an entry to the point $(i-1, a_i)$ in the integer lattice. Let $k = \lfloor \frac{n+1}{2} \rfloor$ for π_o and $k = \lfloor \frac{n}{2} \rfloor$ for π_e, and let $a_{k+1} = 2k$ for π_o and $a_{k+1} = 2k+1$ for π_e.

Now consider a north-east lattice path (as in Fig. 4.1 from $(0, 0)$ to (k, a_{k+1}) through vertices $(0, a_1), (1, a_2), \dots, (k-1, a_k)$ in that order so that each vertex is joined to the next one by a (possibly empty) sequence of east steps followed by a (possibly empty) sequence of north steps. In other words, consider the path

$$N^{a_1}, E, N^{a_2-a_1}, E, N^{a_3-a_2}, E, \dots, E, N^{a_{k+1}-a_k} \tag{4.2}$$

from $(0, 0)$ to (k, a_{k+1}), where $E = (1, 0)$ is the unit east step and $N = (0, 1)$ is the unit north step. Then this path lies on or below the line $y = 2x$ for π_o and on or below the line $y = 2x+1$ for π_e, and each such path corresponds to a unique π_o or a unique π_e.

Moreover, notice that if n is even, then

$$\begin{aligned} \left(\left\lfloor \frac{n+1}{2} \right\rfloor, 2\left\lfloor \frac{n+1}{2} \right\rfloor\right) &= \left(\left\lfloor \frac{n}{2} \right\rfloor, n\right) \\ \left(\left\lfloor \frac{n}{2} \right\rfloor, 2\left\lfloor \frac{n}{2} \right\rfloor + 1\right) &= \left(\left\lfloor \frac{n+1}{2} \right\rfloor, n+1\right), \end{aligned}$$

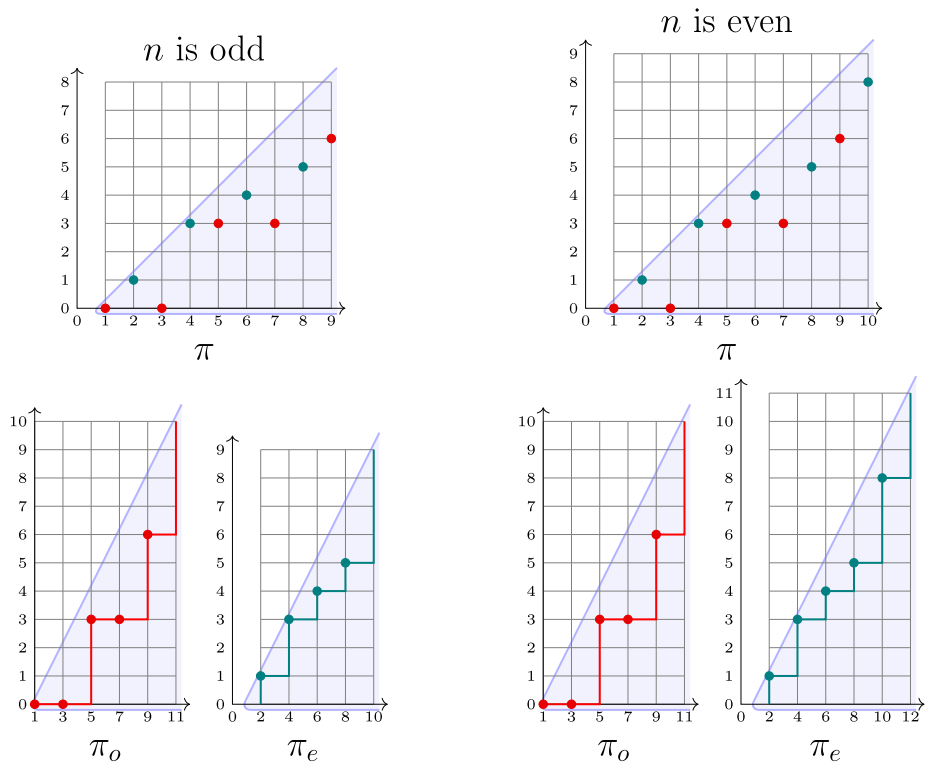


Fig. 4.1. An example of the bijection between a subexcedent faro word and pairs of ternary paths as in the proof of Theorem 4.7.

and if n is odd, then

$$\left(\left\lfloor \frac{n+1}{2} \right\rfloor, 2\left\lfloor \frac{n+1}{2} \right\rfloor\right) = \left(\left\lfloor \frac{n+1}{2} \right\rfloor, n+1\right)$$
$$\left(\left\lfloor \frac{n}{2} \right\rfloor, 2\left\lfloor \frac{n}{2} \right\rfloor + 1\right) = \left(\left\lfloor \frac{n}{2} \right\rfloor, n\right).$$

It is easy to see now that the pair of paths thus obtained for π_o and π_e are in bijection with the pair of paths in the proof of the [12, Theorem 3.5] (see also [12, Figure 6]), which yields a bijection between the subexcedent faro words of size n and 2-1-4-3-avoiding Dumont permutations of the second kind of size $2n$. □

The enumeration of subexcedent faro words may be refined by considering some natural statistics on such words. Together with the bijection of Theorem 4.7 to pairs of ternary paths (or 2-Dyck paths), a recent result [11] lets us find several equidistributed statistics on the odd-position and even-position subsequences of subexcedent faro words.

Recall that a *ternary* (or 2-*Dyck*) path is a sequence of unit steps $u = (1, 1)$ and $d = (1, -2)$ starting at $(0, 0)$ and staying in the first quadrant. A *peak* of a 2-Dyck path is an ud-block in that path, as well as the vertex between the two steps. Likewise, a *double descent* of a 2-Dyck path is a dd-block in that path, as well as the vertex between the two steps. Define the following statistics on 2-Dyck paths:

- $\mathbf{pk}_0$, the number of peaks at even height,
- $\mathbf{pk}_1$, the number of peaks at odd height,
- $\mathbf{dd}$, the number of double descents.

Then the following results hold.

Theorem 4.8 *([11]).*

- *On 2-Dyck paths ending at height 0, the tristatistic* $(\mathbf{pk}_0 - \mathbf{1}, \mathbf{pk}_1, \mathbf{dd})$ *is jointly equidistributed with any of its permutations.*
- *On 2-Dyck paths ending at height 1, the bistatistics* $(\mathbf{pk}_0, \mathbf{pk}_1)$ *and* $(\mathbf{pk}_1, \mathbf{pk}_0)$ *are jointly equidistributed.*

For a subexcedent faro word π of length $2n$, define the following statistics on its odd-position and even-position subsequences π_o and π_e:

- **eOdis**(π), the number of distinct *positive* even letters in π_o (we exclude 0 since π_o and π always start with 0);
- **oOdis**(π), the number of distinct odd letters in π_o;
- **aOrpt**$(\pi) = \{i \in [n-1] \mid \pi(2i-1) = \pi(2i+1)\}$, the number of letter repetitions in π_o (the "a" in **aOrpt** stands for "any parity");
- **eEdis**(π), the number of distinct even letters in π_e;
- **oEdis**(π), the number of distinct odd letters in π_e.

Then we have the following result.

Theorem 4.9. *On subexcedant faro words of length n,*

- *the tristatistic* (**eOdis**, **oOdis**, **aOrpt**) *is jointly equidistributed with any of its permutations.*
- *the bistatistics* (**eEdis**, **oEdis**) *and* (**oEdis**, **eEdis**) *are jointly equidistributed.*

Proof. For each of π_o and π_e, define k and $a_1, a_2, \ldots, a_k, a_k+1$ as in the proof of Theorem 4.7, and let

$$P = N^{a_1}, E, N^{a_2-a_1}, E, N^{a_3-a_2}, E, \ldots, E, N^{a_{k+1}-a_k}$$

be the corresponding north-east path as in (4.2) (when needed, we will distinguish the paths obtained from π_o and π_e as P_o and P_e, respectively). Map P to a lattice path obtained by reversing P and mapping unit steps $N \mapsto u = (1,1)$ and $E \mapsto d = (1,-2)$. In other words, consider the map

$$\phi : P \mapsto \phi(P) = u^{a_{k+1}-a_k}, d, u^{a_k-a_{k-1}}, d, \ldots, d, u^{a_2-a_1}, d, u^{a_1},$$

where $\phi(P)$ starts at $(0,0)$. Recall that P starts at $(0,0)$, stays in the first quadrant on or below $y = 2x$ for π_o and $y = 2x+1$ for π_e, and ends on $y = 2x$ for π_o and $y = 2x+1$ for π_e. Therefore, it is easy to see that $\phi(P)$ stays in the first quadrant and ends at height 0 for π_o and at height 1 for π_e. Moreover, each distinct letter of π_o or π_e (except for 0 in π_o) corresponds to a block EN in the corresponding path P, which in turn corresponds to a block ud of $\phi(P)$, i.e. to a peak of $\phi(P)$.

Furthermore, a repetition of a letter in positions i and $i+1$ of π_o means that $a_{i+1} = a_i$, and thus the i-th and $(i+1)$-st steps E in P are adjacent, which in turn corresponds to a block dd in $\phi(P)$. Therefore, **aOrpt**$(\pi) =$ **dd**(P_o).

Let ℓ be one of distinct letters of in π_o or π_e (for π_o, also assume $\ell > 0$). Suppose its rightmost occurrence is in position j. Then there are $k+1-\ell$ east steps and $a_{k+1} - a_\ell$ north steps in path P to the right of that point, so the height of the corresponding peak in $\phi(P)$ is

$$a_{k+1} - a_\ell - 2(k+1-\ell) \equiv a_{k+1} - a_\ell \pmod 2 \equiv a_\ell \pmod 2 + a_{k+1} \pmod 2.$$

It follows that, on π_o $(\mathbf{eOdis}, \mathbf{oOdis})(\pi) = (\mathbf{pk}_0 - 1, \mathbf{pk}_1)(\phi(P_o))$ if a_{k+1} is even, and $(\mathbf{eOdis}, \mathbf{oOdis})(\pi) = (\mathbf{pk}_1, \mathbf{pk}_0 - 1)(\phi(P_o))$ if a_{k+1} is odd. Likewise, $(\mathbf{eEdis}, \mathbf{oEdis})(\pi) = (\mathbf{pk}_0, \mathbf{pk}_1)(\phi(P_e))$ if a_{k+1} is even, and $(\mathbf{eEdis}, \mathbf{oEdis})(\pi) = (\mathbf{pk}_1, \mathbf{pk}_0)(\phi(P_e))$ if a_{k+1} is odd. However, the two statistics on the right-hand side of the equations are jointly equidistributed in each case by Theorem 4.8, and thus the parity of a_{k+1} is immaterial in each case. □

From Corollary 1.12 and Equation (2.7) of [11], we can also determine the joint distribution of all the statistics we defined on subexcedent faro words. For this result, we let $n_o = \lfloor \frac{n+1}{2} \rfloor$ and $n_e = \lfloor \frac{n}{2} \rfloor$ (so $n_o + n_e = n$). We also let **aErpt**(π) be the number of letter repetitions in π_e, i.e. **aErpt**$(\pi) = \{i \in [n-1] \mid \pi(2i) = \pi(2i+2)\}$.

Corollary 4.10. *The number of subexcedent faro words π of length n such that*

$$(\mathbf{eOdis}, \mathbf{oOdis}, \mathbf{aOrpt}, \mathbf{eEdis}, \mathbf{oEdis}, \mathbf{aErpt})(\pi) = (r_1, r_2, r_2, r_4, r_5, r_6)$$

is

$$\frac{1}{n_o}\binom{n_o}{r_1}\binom{n_o}{r_2}\binom{n_o}{r_3}\frac{r_4+r_5}{n_e(n_e+1)}\binom{n_e+1}{r_4}\binom{n_e+1}{r_5}\binom{n_e}{r_6}.$$

Note also that $r_1 + r_2 + r_3 = n_o - 1$ and $r_4 + r_5 + r_6 = n_e$.

Declaration of competing interest

No conflict of interest.

Acknowledgements

We would like to greatly thank anonymous referees for their helpful comments and suggestions.

References

[1] A. Asinowski, A. Bacher, C. Banderier, B. Gittenberger, Analytic combinatorics of lattice paths with forbidden patterns, the vectorial kernel method, and generating functions for pushdown automata, Algorithmica 82 (3) (2020) 386–428.
[2] A. Asinowski, C. Banderier, V. Roitner, Generating functions for lattice paths with several forbidden patterns, in: Séminaire Lotharingien de Combinatoire — Proceedings of the 32nd International Conference on "Formal Power Series and Algebraic Combinatorics", 84B, 2020.
[3] A. Bacher, A. Bernini, L. Ferrari, B. Gunby, R. Pinzani, J. West, The Dyck pattern poset, Discrete Math. 321 (2014) 12–23.
[4] J.-L. Baril, R. Genestier, S. Kirgizov, Pattern distributions in Dyck paths with a first return decomposition constrained by height, Discrete Math. 342 (9) (2020) 111995.
[5] J.-L. Baril, V. Vajnovszki, Popularity of patterns over d-equivalence classes of words and permutations, Theor. Comput. Sci. 814 (April 2020) 249–258.
[6] M. Barnabei, F. Bonetti, M. Silimbani, Restricted involutions and Motzkin paths, Adv. Appl. Math. 47 (1) (2011) 102–115.
[7] C.S. Barnes, Enumeration of the distinct shuffles of permutations, in: DMTCS Proceedings, vol. AK, 21st International Conference on Formal Power Series and Algebraic Combinatorics, FPSAC 2009, 2009, pp. 155–166.
[8] J. Bertrand, Solution d'un problème, C. R. Acad. Sci. 105 (1887) 369.
[9] P. Biane, Permutations suivant le type d'excédance et le nombre d'inversions et interprétation combinatoire d'une fraction continue de Heine, Eur. J. Comb. 14 (4) (July 1993) 277–284.
[10] M. Bóna, Surprising symmetries in objects counted by Catalan numbers, Electron. J. Comb. 19 (1) (March 2012).
[11] A. Burstein, Distribution of peak heights modulo k and double descents on k-Dyck paths, submitted for publication, https://arxiv.org/pdf/2009.00760.pdf, 2020.
[12] A. Burstein, S. Elizalde, T. Mansour, Restricted Dumont permutations, Dyck paths, and noncrossing partitions, Discrete Math. 306 (22) (2006) 2851–2869.
[13] M. Bóna, A Walk through Combinatorics: An Introduction to Enumeration and Graph Theory, 2nd ed edition, World Scientific Pub, Hackensack, NJ, 2006.
[14] E. Deutsch, Dyck path enumeration, Discrete Math. 204 (1–3) (1999) 167–202.
[15] P. Diaconis, R.L. Graham, W.M. Kantor, The mathematics of perfect shuffles, Adv. Appl. Math. 4 (2) (June 1983) 175–196.
[16] J. Françon, G. Viennot, Permutations selon leurs pics, creux, doubles montées et double descentes, nombres d'Euler et nombres de Genocchi, Discrete Math. 28 (1) (1979) 21–35.
[17] B. Hackl, C. Heuberger, H. Prodinger, Counting Ascents in Generalized Dyck Paths, in: 29th International Conference on Probabilistic, Combinatorial and Asymptotic Methods for the Analysis of Algorithms, vol. 110, AofA 2018, Dagstuhl, Germany, 2018, pp. 26:1–26:15.
[18] C. Homberger, Expected patterns in permutation classes, Electron. J. Comb. 19 (3) (2012), #P43.
[19] S. Kitaev, Patterns in Permutations and Words, Springer Science & Business Media, 2011.
[20] C. Krattenthaler, Permutations with restricted patterns and Dyck paths, Adv. Appl. Math. 27 (2001) 510–530.
[21] M. Lothaire, Combinatorics on Words, Encyclopedia of Mathematics and Its Applications., vol. 17, Addison-Wesley, 1983.
[22] T. Mansour, Statistics on Dyck paths, J. Integer Seq. 9 (1) (2006) 06.
[23] S.B. Morris, The basic mathematics of the faro shuffle, Delta Pi Epsil. J. 6 (2) (1975) 85–92.
[24] B. Salvy, P. Zimmermann. Gfun, A Maple package for the manipulation of generating and holonomic functions in one variable, ACM Trans. Math. Softw. 20 (2) (June 1994) 163–177.
[25] R. Simion, F. Schmidt, Restricted permutations, Eur. J. Comb. 6 (1985) 383–406.
[26] N.J.A. Sloane, et al., The On-line Encyclopedia of Integer Sequences, 2020, https://oeis.org.
[27] R. Stanley, Enumerative Combinatorics, vol. 1, Cambridge University Press, 1997.
[28] W.A. Whitworth, Arrangements of m things of one sort and n things of another sort, under certain conditions of priority, Mess. Math 8 (1878) 105–114.

Journal of Integer Sequences, Vol. 24 (2021),
Article 21.8.2

Counting Prefixes of Skew Dyck Paths

Jean-Luc Baril
LIB, Université de Bourgogne
B.P. 47 870
21078 Dijon Cedex
France
barjl@u-bourgogne.fr

José L. Ramírez and Lina M. Simbaqueba
Departamento de Matemáticas
Universidad Nacional de Colombia
Bogotá
Colombia
jlramirezr@unal.edu.co
lmsimbaquebam@unal.edu.co

Abstract

We present enumerative results on prefixes of skew Dyck paths by giving recursive relations, Riordan arrays, and generating functions, as well as closed formulas to count the total number of these paths with respect to the length, the height of its endpoint and the number of left steps.

1 Introduction

A *skew Dyck path* is a lattice path in the first quadrant of the xy-plane that starts at the origin, ends on the x-axis, and made of up-steps $U = (1, 1)$, down-steps $D = (1, -1)$, and left steps $L = (-1, -1)$ so that up and left steps do not overlap. We let $|P|$ denote the *length* of the path P, i.e., the number of its steps, which is an even non-negative integer. Let ϵ denote

the skew Dyck path of length zero. For example, Figure 2 shows the skew Dyck paths of length 6, or equivalently of semilength 3. The concept of skew Dyck path was introduced by Deutsch, Munarini, and Rinaldi [1]. Some additional results can be found in [2, 8], where the authors present enumerative results according to different parameters and some bijections with other combinatorial objects, as hex trees, tree-like polyhexes, and 3-Motzkin paths.

We let $\mathbb{S}$ denote the set of all skew Dyck paths, and let $\mathbb{S}_n$ denote the set of all skew Dyck paths of semilength n. Let s_n be the cardinality of $\mathbb{S}_n$. Any nonempty skew Dyck path P can be uniquely decomposed either $P = UT_1DT_2$ or $P = UT_3L$, where $T_1, T_2, T_3 \in \mathbb{S}$ and $T_3 \neq \epsilon$. Figure 1 shows a graphical representation of this decomposition.

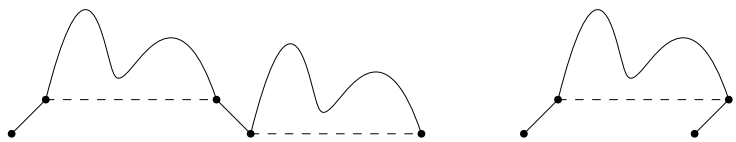


Figure 1: Decomposition of a nonempty skew Dyck path.

This decomposition induces the functional equation $S(x) = 1 + xS(x)^2 + x(S(x) - 1)$ for the generating function $S(x) := \sum_{P\in\mathbb{S}} x^{|P|/2} = \sum_{n\geq 0} s_n x^n$, and thus we have

$$S(x) = \frac{1 - x - \sqrt{1 - 6x + 5x^2}}{2x}. \tag{1}$$

The sequence s_n appears in the OEIS as A002212 [15], and its first few values are

$$1, \quad 1, \quad 3, \quad 10, \quad 36, \quad 137, \quad 543, \quad 2219, \quad 9285, \quad 39587.$$

From the above functional equation, we obtain $S(x) - 1 = x(S(x)^2 + S(x) - 1)$, and applying directly the Lagrange inversion Theorem, (see [5, 10] for instance), we deduce the explicit formulas:

$$s_n = \sum_{k=1}^{n} \binom{n-1}{k-1} c_k,$$

$$s_n = \sum_{k=1}^{n} \binom{n-1}{k-1} (-1)^{k-1} 5^{n-k} c_k,$$

$$s_n = \frac{1}{n} \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{n}{k} \binom{n-k}{k+1} 3^{n-2k-1},$$

where $c_n = \frac{1}{n+1}\binom{2n}{n}$ is the n-th Catalan number, which is the general term of the sequence A000108 in [15].

A prefix of a skew Dyck path is called *skew meander*. In a way, skew meanders appear as the counterpart of ballot paths (prefixes of Dyck paths [6]) for skew Dyck paths. The *height*

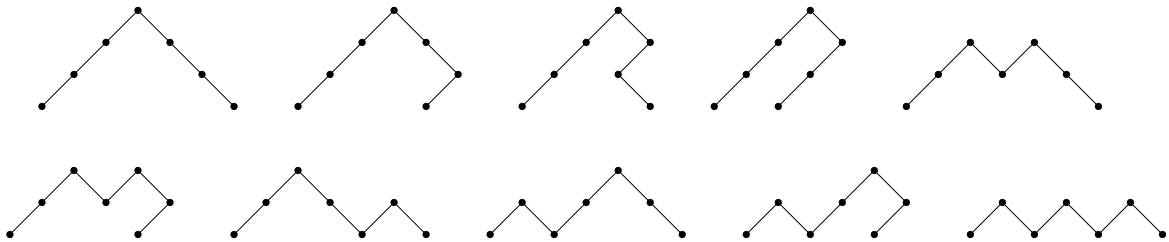


Figure 2: Skew Dyck paths of semilength 3.

$\alpha(P)$ of a skew meander P is the y-coordinate of the endpoint of the last step of P. For example, Figure 3 shows a skew meander of length 27 and height 3. Let $\mathbb{M}$ be the set of all skew meanders, and let $s(n,k)$ denote the number of skew meanders of length n and height k.

In this paper, we present enumerative results on skew meanders by giving recursive relations, Riordan arrays, generating functions, as well as closed formulas to count the total number of skew meanders of a given length with respect to the height and the number of left steps.

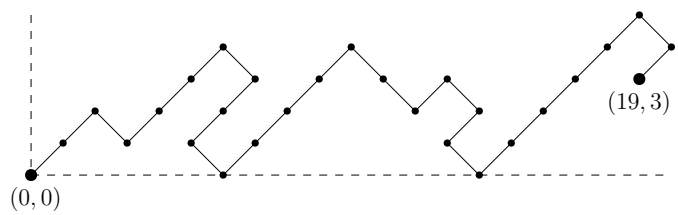


Figure 3: Skew meander of length 27 and height 3.

2 Skew meanders via Riordan arrays

The goal of this section is to study the distribution of the number of skew meanders of a fixed height. Let $\mathcal{S} = [s(n,k)]_{n,k\geq 0}$ denote the matrix where $s(n,k)$ is the number of skew meanders of length n and height k. The first few rows of the matrix $\mathcal{S}$ are

$$\mathcal{S} = \begin{pmatrix} 1 & & & & & & & & & \\ 0 & 1 & & & & & & & & \\ 1 & 0 & 1 & & & & & & & \\ 0 & 2 & 0 & 1 & & & & & & \\ 3 & 0 & 3 & 0 & 1 & & & & & \\ 0 & \mathbf{6} & 0 & 4 & 0 & 1 & & & & \\ 10 & 0 & 10 & 0 & 5 & 0 & 1 & & & \\ 0 & 21 & 0 & 15 & 0 & 6 & 0 & 1 & & \\ 36 & 0 & 37 & 0 & 21 & 0 & 7 & 0 & 1 & \\ 0 & 79 & 0 & 59 & 0 & 28 & 0 & 8 & 0 & 1 \\ & \vdots & & & \vdots & & & & \vdots & \end{pmatrix}.$$

The sequence $(s(n,1))_{n\geq 1}$ corresponds to the sequence A033321 that counts skew Dyck paths of semilength n with a down-step D. Figure 4 shows the six skew meanders of length 5 and height 1 corresponding to the bold red entry in the above matrix.

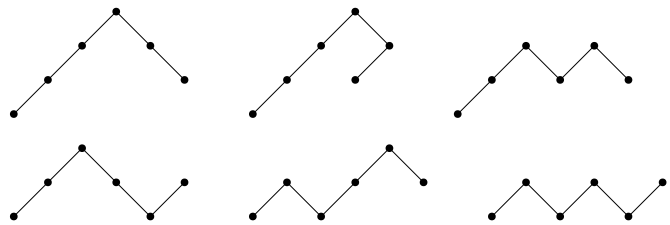


Figure 4: The six skew meanders of length 5 and height 1.

We will prove that the matrix $\mathcal{S}$ is a Riordan array. So, let us give some background on Riordan arrays [14]. We will say that an infinite column vector $(a_0, a_1, \dots)^T$ has generating function $f(x)$ if $f(x) = \sum_{n\geq 0} a_n x^n$, and we index rows and columns starting at 0. A *Riordan array* is an infinite lower triangular matrix whose k-th column has generating function $g(x)f(x)^k$ for all $k \geq 0$, for some formal power series $g(x)$ and $f(x)$, with $g(0) \neq 0$, $f(0) = 0$, and $f'(0) \neq 0$. Such a Riordan array is denoted by $(g(x), f(x))$. If we multiply this matrix by a column vector $(c_0, c_1, \dots)^T$ having generating function $h(x)$, then the resulting column vector has generating function $g(x)h(f(x))$. This property is known as the fundamental theorem of Riordan arrays, or as the summation property.

The product of two Riordan arrays $(g(x), f(x))$ and $(h(x), l(x))$ is defined by

$$(g(x), f(x)) * (h(x), l(x)) = (g(x)h(f(x)), l(f(x)))\,. \tag{2}$$

Under the operation "$*$", the set of all Riordan arrays is a group [14]. The identity element is $I = (1, x)$, and the inverse of $(g(x), f(x))$ is

$$(g(x), f(x))^{-1} = \left(1/\left(g \circ \overline{f}\right)(x), \overline{f}(x)\right), \tag{3}$$

where $\overline{f}(x)$ denotes the compositional inverse of $f(x)$.

Recall that for a skew meander P, $\alpha(P)$ is the height of P, $|P|$ is the length of P. For a non-negative integer k, we introduce the generating function

$$H_k(x) := \sum_{P\in\mathbb{M},\, \alpha(P)=k} x^{|P|} = \sum_{n\geq 0} s(n,k)x^n.$$

It is clear that $H_0(x) = S(x^2)$.

Theorem 1. *The matrix $\mathcal{S} = [s(n,k)]_{n,k\geq 0}$ is a Riordan array given by*

$$\mathcal{S} = \left(S(x^2), \frac{x}{1 - x^2 S(x^2)}\right), \tag{4}$$

where $S(x) = (1 - x - \sqrt{1 - 6x + 5x^2})/2x$.

Proof. Obviously, skew meanders of height zero are skew Dyck paths, and a skew meander P of height $k \geq 1$ can be decomposed either $P = UP_1$ or $UQDP_2$, where P_1 (resp. P_2) is a skew meander of height $k-1$ (resp. k), and Q is a skew Dyck path in $\mathbb{S}$. Figure 5 illustrates the two cases. Using the symbolic method (see [4]), we obtain the functional equation

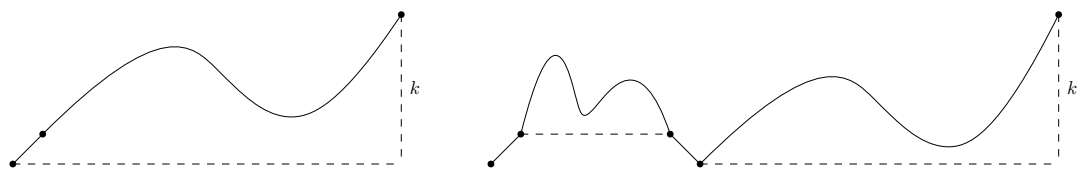


Figure 5: Decomposition of a skew meander of height at least one .

$$H_k(x) = xH_{k-1}(x) + x^2 S(x^2) H_k(x), \quad k \geq 1,$$

anchored with $H_0(x) = S(x^2)$. Then $H_k(x) = xH_{k-1}(x)/(1 - x^2 S(x^2))$, and by iterating this expression we deduce for $k \geq 0$,

$$H_k(x) = \left(\frac{x}{1 - x^2 S(x^2)} \right)^k S(x^2).$$

From the definition of Riordan array we obtain the desired result. □

Corollary 2. *The generating function for the total number of skew meanders of length n is given by*

$$\sum_{n \geq 0} \sum_{k=0}^{n} s(n,k) x^n = \frac{-2 + x + 4x^2 + x^3 + (2+x)\sqrt{1 - 6x^2 + 5x^4}}{2x(1 - 2x - x^2)}.$$

Proof. Multiplying the right-hand side of the equality (4) by the vector $(1, 1, 1, \dots)^T$, which has generating function $1/(1-x)$, and using the summation property, the resulting vector has generating function

$$\left(S(x^2), \frac{x}{1 - x^2 S(x^2)} \right) \frac{1}{1-x} = \frac{S(x^2)}{1 - \frac{x}{1 - x^2 S(x^2)}} = \frac{S(x^2)}{1 - x - x^2 S(x^2)}.$$

After simplification we obtain the desired result. □

The total number of skew meanders of length n, for $0 \leq n \leq 10$ are

$$1, \quad 1, \quad 2, \quad 3, \quad 7, \quad 11, \quad 26, \quad 43, \quad 102, \quad 175, \quad 416.$$

In Theorem 3, we will give a combinatorial formula for $s(n,k)$ involving binomial coefficients and the entries of the Catalan triangle introduced by Shapiro [13] in the context

of lattice paths on $\mathbb{Z}\times\mathbb{Z}$. It is defined by the matrix $[B(n,k)]_{n,k\geq 1}=[\frac{k}{n}\binom{2n}{n-k}]_{n,k\geq 1}$. For convenience, we extend this triangle by adding a row and column of indice zero by setting $B(0,0)=1$, $B(0,k)=B(k,0)=0$ if $k>0$. So, the first few rows are

$$\mathcal{B}:=[B(n,k)]_{n,k\geq 0}=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 4 & 1 & 0 & 0 & 0 & 0 \\ 0 & 14 & 14 & 6 & 1 & 0 & 0 & 0 \\ 0 & 42 & 48 & 27 & 8 & 1 & 0 & 0 \\ 0 & 132 & 165 & 110 & 44 & 10 & 1 & 0 \\ 0 & 429 & 572 & 429 & 208 & 65 & 12 & 1 \\ \vdots & & & \vdots & & & \vdots & \end{pmatrix},$$

which correspond to array A039598 in [15]. From the generating function of the columns of $\mathcal{B}$, see [16], we prove that the matrix $[B(n,k)]_{n,k\geq 0}$ is the Riordan array $\mathcal{B}=(1,xC(x)^2)$, where $C(x)=\frac{1-\sqrt{1-4x}}{2x}$ is the generating function of the Catalan numbers $c_n=\frac{1}{n+1}\binom{2n}{n}$. It is well known that $C(x)$ satisfies the functional equation $C(x)=xC(x)^2+1$. So, if we set $B(x):=C(x)-1$, then $B(x)=x(B(x)+1)^2=x\Phi(B(x))$, where $\Phi(z)=(z+1)^2$. Moreover,

$$[x^n]B(x)^k=[x^n](C(x)-1)^k=[x^n](xC(x)^2)^k=B(n,k).$$

From the Lagrange inversion theorem (see for instance [10]), we have

$$B(n,k)=[x^n](B(x))^k=\frac{k}{n}[z^{n-k}](z+1)^{2n}=\frac{k}{n}\binom{2n}{n-k},$$

which proves that $\mathcal{B}$ is the Riordan array $(1,xC(x)^2)$.

Theorem 3. *For $n,k\geq 0$, if $n+k\equiv 0 \pmod 2$, then*

$$s(n,k)=\sum_{j=0}^{\frac{n-k}{2}}\sum_{i=0}^{j+1}\sum_{\ell=0}^{\frac{n-k}{2}-j}\binom{j+k-1}{j}\binom{j+1}{i}\binom{\frac{n-k}{2}-j-1}{\ell-1}B(\ell,i),$$

and $s(n,k)=0$ otherwise.

Proof. Let $U(x):=S(x)-1$, where $S(x)$ is the generating function defined in (1). Note that $U(x)=x(U(x)^2+3U(x)+1)=x\Phi(U(x))$, where $\Phi(z)=z^2+3z+1$. From the Lagrange inversion theorem, we deduce

$$\begin{aligned} u_n^{(k)}:&=[x^n]U(x)^k=\frac{k}{n}[z^{n-k}]\Phi(z)^n=\frac{k}{n}[z^{n-k}](z^2+3z+1)^n \\ &=\frac{k}{n}[z^{n-k}]((z+1)^2+z)^n=\frac{k}{n}[z^{n-k}]\sum_{i=0}^{n}\binom{n}{i}(z+1)^{2i}z^{n-i} \end{aligned}$$

$$= \sum_{i=0}^{n} \binom{n}{i} \frac{k}{n} [z^{i-k}](z+1)^{2i} = \sum_{i=1}^{n} \binom{n-1}{i-1} \frac{k}{i} [z^{i-k}](z+1)^{2i}$$
$$= \sum_{i=0}^{n} \binom{n-1}{i-1} B(i,k).$$

If we set $g_n^{(k)} := [x^n]S(x^2)^k = (U(x^2)+1)^k$, then it is cleat that

$$g_n^{(k)} = \begin{cases} \sum_{i=0}^{k} \binom{k}{i} u_{n/2}^{(i)}, & \text{if } n \equiv 0 \pmod 2; \\ 0, & \text{otherwise.} \end{cases}$$

Now, by definition of a Riordan array, we have

$$\begin{aligned} s(n,k) &= [x^n]S(x^2)\frac{x^k}{(1-x^2S(x^2))^k} \\ &= [x^{n-k}]\sum_{j\geq 0} \binom{j+k-1}{j} S(x^2)^{j+1}x^{2j} \\ &= \sum_{j\geq 0} \binom{j+k-1}{j} [x^{n-k-2j}]S(x^2)^{j+1} \\ &= \sum_{j=0}^{\lfloor \frac{n-k}{2} \rfloor} \binom{j+k-1}{j} g_{n-k-2j}^{(j+1)}, \end{aligned}$$

which provides the desired result. □

By considering different expansions of the expression $(z^2+3z+1)^n$, and using similar proofs as for Theorem 3, we obtain the following two corollaries.

Corollary 4. *For $n,k \geq 0$, if $n+k \equiv 0 \pmod 2$, then*

$$s(n,k) = \sum_{j=0}^{\frac{n-k}{2}} \sum_{i=0}^{\frac{n-k}{2}-j} \binom{j+k-1}{j} \binom{\frac{n-k}{2}-j-1}{i-1} \binom{j+2i+1}{i} \frac{j+1}{j+2i+1},$$

and $s(n,k) = 0$ otherwise.

Corollary 5. *For $n,k \geq 0$, if $n+k \equiv 0 \pmod 2$, then*

$$s(n,k) = \sum_{j=0}^{\frac{n-k}{2}} \sum_{i=0}^{j+1} \binom{j+k-1}{j} \binom{j+1}{i} t_{\frac{n-k}{2}-j}^{(i)},$$

where $t_0^{(0)} = 1, t_n^{(k)} = 0$ if $n = 0, k \neq 0$, and $t_n^{(k)} = \sum_{i=0}^{\lfloor \frac{n-k}{2} \rfloor} \frac{k}{n}\binom{n}{i}\binom{n-i}{k+i}3^{n-k-2i}$ otherwise. Moreover, $s(n,k) = 0$ if $n+k \equiv 1 \pmod 2$.

3 A recurrence relation for $s(n,k)$

In this part, we will provide a recurrence relation for $s(n,k)$. For this, we consider the matrix $\mathcal{S}_2 := [s(2n-k,k)]_{n,k\geq 0} = [\widetilde{s}(n,k)]_{n,k\geq 0}$, which is a compression of $\mathcal{S}$ obtained by deleting some zeros. The first few rows of the matrix $\mathcal{S}_2$ are

$$\mathcal{S}_2 = [\widetilde{s}(n,k)]_{n,k\geq 0} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 10 & 6 & 3 & 1 & 0 & 0 & 0 & 0 \\ 36 & 21 & 10 & 4 & 1 & 0 & 0 & 0 \\ 137 & 79 & 37 & 15 & 5 & 1 & 0 & 0 \\ 543 & 311 & 145 & 59 & 21 & 6 & 1 & 0 \\ 2219 & 1265 & 589 & 241 & 88 & 28 & 7 & 1 \\ & \vdots & & & \vdots & & & \vdots \end{pmatrix}.$$

From the summation property for Riordan arrays (see Theorem 1), we obtain directly the following corollary.

Corollary 6. *The matrix $\mathcal{S}_2$ is a Riordan array given by $\mathcal{S}_2 = \left(S(x), \frac{x}{1-xS(x)}\right)$, and the generating function $S_2(x)$ for the row sums of the matrix $\mathcal{S}_2$ is given by*

$$\begin{aligned} S_2(x) := \sum_{n\geq 0} s_2(n)x^n = \sum_{n\geq 0}\sum_{k=0}^{n} \widetilde{s}(n,k)x^n &= \frac{1-2x-\sqrt{1-6x+5x^2}}{x} \\ &= 1+2x+6x^2+20x^3+72x^4+274x^5+1086x^6+4438x^7+18570x^8+\cdots \end{aligned}$$

The generating function $S_2(x)$ satisfies $S_2(x) = 2S(x)-1$, and $s_2(n)$ corresponds to the general term of the sequence A122737 which has a combinatorial interpretation using special polyominoes (bi-wall directed polygons).

A plane figure is a *polyomino* if it is a union of finitely many cells and the interior is connected. A polyomino P is a *polygon* if P has a connected border, and this border is a closed self-avoiding walk. We say that P is *directed* if each of its cells can be reached from its bottom left-hand corner by a path which is contained in the polyomino and uses only north and east steps. Let c be the bottom left-hand corner of P. A *clockwise tour* of P is a closed walk which, starting at c, goes along the boundary of P (in the clockwise direction), and ends when the corner c is reached again. A *bi-wall directed polygon* is a directed polygon such that its clockwise tour satisfies that once the first left step has been made, up steps occur never more [3]. For example, Figure 6 shows a bi-wall directed polygon of perimeter 66. The sequence $s_2(n)$ counts the number of bi-wall directed polygons with perimeter $2(n+2)$. We wonder how the bijective proof of this result will appear.

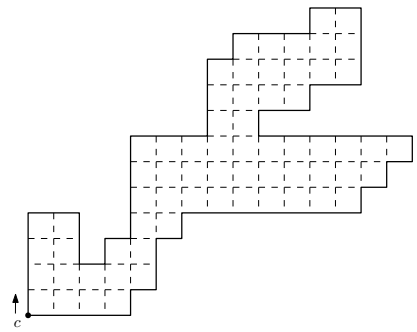


Figure 6: Bi-wall directed polyomino of perimeter 66.

Notice that, due to the relation $S_2(x) = 2S(x) - 1$, bi-wall directed polygons of perimeter $2(n+2)$ are in one-to-one correspondence with special directed polygons of perimeter $2(2n+1)$ having a clockwise tour that can be decomposed either $NPS\overline{P}$ or $PE\overline{P}W$ where N (resp. S, E, W) are the north- (resp. south-, east-, west-) steps, and P is obtained from a skew Dyck path Q of semilength n by replacing U with N, D with E, and L with W, and $\overline{P}$ is obtained from Q by reading Q from right to left and by replacing U with W, D with S, and L with E. For instance, from the skew Dyck path $P = UDUUDL$ we generate two directed polygons $NNENNESSESWWSW$ and $NENNESEESWWSWW$ illustrated in Figure 7. Finding a constructive bijection between this class of polyominoes and bi-wall directed polygons remains an interesting open question.

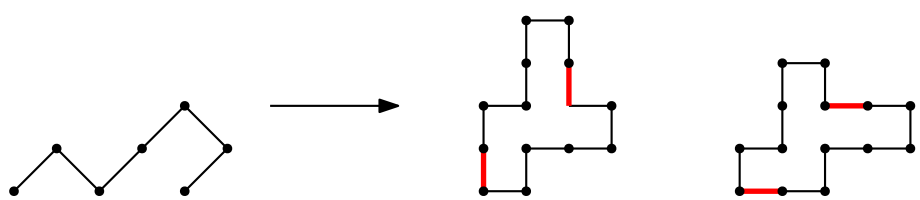


Figure 7: Construction of two directed polygons of perimeter 14 from a skew Dyck path of length 6.

Rogers [12] observed that every element $d_{n+1,k+1}$ of a Riordan array (not belonging to 0 row or 0 column) can be expressed as a fixed linear combination of the elements in the preceding row. The *A-sequence* is defined to be the sequence coefficients of this linear combination. Similarly, Merlini et al. [9] introduced the *Z-sequence*, which characterizes the elements in column 0, except for the element $d_{0,0}$. Therefore, the A-sequence, the Z-sequence and the element $d_{0,0}$ completely characterize a Riordan array. We summarize this characterization in the following theorem.

Theorem 7 ([9, 7])**.** *An infinite lower triangular array $\mathcal{D} = [d_{n,k}]_{n,k\geq 0} = (g(x), f(x))$ is a Riordan array if and only if $d_{0,0} \neq 0$ and there exist two sequences $(a_0, a_1, a_2, \dots)$, with*

$a_0 \neq 0$, and $(z_0, z_1, z_2, \dots)$ (called the A-sequence and the Z-sequence, respectively), such that

$$\begin{aligned} d_{n+1,k+1} &= a_0 d_{n,k} + a_1 d_{n,k+1} + a_2 d_{n,k+2} + \cdots && \text{for } n, k \geq 0, \\ d_{n+1,0} &= z_0 d_{n,0} + z_1 d_{n,1} + z_2 d_{n,2} + \cdots && \text{for } n \geq 0, \end{aligned}$$

or equivalently

$$g(x) = \frac{d_{0,0}}{1 - xZ(f(x))} \quad \text{and} \quad f(x) = x(A(f(x))),$$

where $A(x)$ and $Z(x)$ are the generating functions of the A-sequence and Z-sequence, respectively.

Below we describe the A-sequence and Z-sequence for the Riordan array $\mathcal{S}_2$.

Theorem 8. *For $n, k \geq 0$, we have*

$$\widetilde{s}(n+1, k+1) = \widetilde{s}(n, k) + \widetilde{s}(n, k+1) + \sum_{i \geq 2} 2^{i-2}\, \widetilde{s}(n, k+i),$$

Moreover, for $n \geq 0$

$$\widetilde{s}(n+1, 0) = \widetilde{s}(n, 0) + \sum_{i \geq 1} (2^{i-1} + 1) \widetilde{s}(n, i),$$

with the initial value $\widetilde{s}(0, 0) = 1$.

Proof. From Theorem 7 and Corollary 6, the generating function of the A-sequence is

$$A(x) = \frac{x}{\overline{f}(x)} = \frac{x}{\frac{x(1-2x)}{1-x-x^2}} = \frac{1 - x - x^2}{1 - 2x} = 1 + x + x^2 + 2x^3 + 4x^4 + 8x^5 + \cdots.$$

The generating function of the Z-sequence of $\mathcal{S}_2$ is

$$Z(x) = \frac{1}{\overline{f}(x)} \left(1 - \frac{d_{0,0}}{g(\overline{f}(x))}\right) = \frac{1 - x - x^2}{(1-x)(1-2x)} = 1 + 2x + 3x^2 + 5x^3 + 9x^4 + 17x^5 + 33x^6 + \cdots.$$

Combining the above relations, we obtain the following corollaries. □

Corollary 9. *For any integers $n, k \geq 0$, we have the following relations*

$$s(n, k) = s(n-1, k-1) + s(n-2, k) + \sum_{i=2}^{\lfloor \frac{n-1}{2} \rfloor} 2^{i-2}\, s(n-i-1, k+i-1),$$

$$s(n, 0) = s(n-2, 0) + \sum_{i=1}^{\lfloor \frac{n-1}{2} \rfloor} (2^{i-1} + 1) s(n-i-2, i).$$

with $s(0, 0) = 1 = s(1, 1)$, $s(1, 0) = 0$, and $s(n, k) = 0$ for $n < k$.

Corollary 10. *For any integers $n, k \geq 2$, we have the following relation*

$$s(n+1, k-1) - 2s(n, k) = s(n, k-2) - s(n-1, k-1) - s(n-2, k).$$

4 The number of left steps

In this section, we provide a close form for the number $a_h(n,m)$ of skew meanders of length n, height h, and with exactly m left steps $L = (-1,-1)$. Let $\ell(P)$ denote the number of left steps in a path P. Given a non-negative integer h, we let $\mathbb{M}_h$ denote the set of skew meanders of height h. We are interested in the bivariate generating function

$$L_h(x,y) := \sum_{P \in \mathbb{M}_h} x^{|P|} y^{\ell(P)} = \sum_{n,m \geq 0} a_h(n,m) x^n y^m.$$

Theorem 11. *The generating function $L_h(x,y)$ is given by the equations*

$$L_0(x,y) = \frac{1 - x^2 y - \sqrt{1 - 4x^2 - 2x^2 y + 4x^4 y + x^4 y}}{2x},$$

and for $h \geq 1$, we have

$$L_h(x,y) = \frac{x^h L_0(x,y)}{(1 - x^2 L_0(x,y))^h}.$$

Proof. From the decomposition given in Figure 1, we obtain the functional equation

$$L_0(x,y) = 1 + x^2 L_0(x,y)^2 + x^2 y (L_0(x,y) - 1).$$

Solving for $L_0(x,y)$, we have the first equation. If $h \geq 1$, then from the decomposition given in Figure 5 we have

$$L_h(x,y) = x L_{h-1}(x,y) + x^2 L_0(x,y) L_h(x,y).$$

Therefore

$$L_h(x,y) = \frac{x L_{h-1}(x,y)}{1 - x^2 L_0(x,y)} = \cdots = \frac{x^h L_0(x,y)}{(1 - x^2 L_0(x,y))^h}.$$

□

Let $a^{(j)}(n,m)$ be the coefficient of $x^n y^m$ in the generating function $L_0(x,y)^j$.

Lemma 12. *For $n, j \geq 1$, $n \equiv 0 \pmod 2$, and $0 \leq m \leq n/2 - 1$ we have*

$$a^{(j)}(n,m) = \sum_{i=0}^{j} \frac{i}{n/2 - m} \binom{j}{i} \binom{n/2 - 1}{m} \binom{2(n/2 - m)}{n/2 - m - i},$$

with the initial values $a^{(j)}(0,0) = 1$. Moreover, for $n \equiv 1 \pmod 2$, $a^{(j)}(n,m) = 0$.

Proof. If $f(x,y) = L_0(x^{1/2}, y) - 1$, then from Theorem 11 we have

$$f(x,y) = x(f(x,y)^2 + (2+y) f(x,y) + 1) = x\Phi(f(x,y)),$$

where $\Phi(z) = z^2 + (2+y)z + 1$. From the Lagrange inversion theorem, we deduce

$$\begin{aligned} f^{(k)}(n,m) &:= [x^n y^m] f(x,y)^k = \frac{k}{n}[z^{n-k}y^m]\Phi(z)^n \\ &= \frac{k}{n}[z^{n-k}y^m](z^2+(2+y)z+1) = \binom{n-1}{m} B(n-m,k). \end{aligned}$$

Since $B(n,k) = \frac{k}{n}\binom{2n}{n-k}$, we obtain

$$f^{(k)}(n,m) = \frac{k}{n-m}\binom{n-1}{m}\binom{2(n-m)}{n-m-k}.$$

If n is even then

$$\begin{aligned} a^{(j)}(n,m) &= [x^n y^m](f(x^2,y)+1)^j = [x^n y^m]\sum_{i=0}^{j}\binom{j}{i} f(x^2,y)^i \\ &= \sum_{i=0}^{j}\frac{i}{n/2-m}\binom{j}{i}\binom{n/2-1}{m}\binom{2(n/2-m)}{n/2-m-i}. \end{aligned}$$

Moreover, if n is odd this sequence is equal to zero. □

The summand on the right side of $a^{(j)}(2n,m)$ is denoted by $F(j,i)$, that is

$$F(j,i) := \frac{i}{n-m}\binom{j}{i}\binom{n-1}{m}\binom{2(n-m)}{n-m-i}.$$

By Zeilberger's creative telescoping method (cf. [11]) $F(j,i)$ satisfies the relation

$$(1+j)(j-2m+2n)F(j,i) - j(1+j-m+n)F(j+1,i) = G(j,i+1) - G(j,i), \tag{5}$$

with the certificate

$$R(j,i) = \frac{(-1+i)(1+j)(i-m+n)}{1-i+j}.$$

That is, $R(j,i) = F(j,i)/G(j,i)$ is a rational function in both variables. Summing both sides of (5) with respect to i and after simplification we obtain the following result.

Proposition 13. *For* $0 \le m \le n-1$

$$a^{(j)}(2n,m) = \frac{j}{2(n-m+j)}\binom{n-1}{m}\binom{2(n-m)}{n-m}\frac{(2m-2n-j+1)_j}{(m-n-j+1)_j},$$

where $(x)_n := x(x+1)\cdots(x+n-1)$ *and* $(x)_0 = 1$.

Theorem 14. *For $n, m, h \geq 0$ we have*

$$a_h(n,m) = \sum_{j=0}^{\frac{n-h}{2}} \binom{j+h-1}{j} a^{(j+1)}(n-h-2j, m).$$

Proof. For $h \geq 1$ we have from Theorem 11 that

$$\begin{aligned} a_h(n,m) &= [x^n y^m] \frac{x^h L_0(x,y)}{(1-x^2 L_0(x,y))^h} \\ &= [x^{n-h-2j} y^m] \sum_{j \geq 0} \binom{j+h-1}{j} L_0(x,y)^{j+1} \\ &= \sum_{j \geq 0} \binom{j+h-1}{j} a^{(j+1)}(n-h-2j, m). \end{aligned}$$

□

Figure 8 shows the skew meanders corresponding to $a_3(11,2) = 12$.

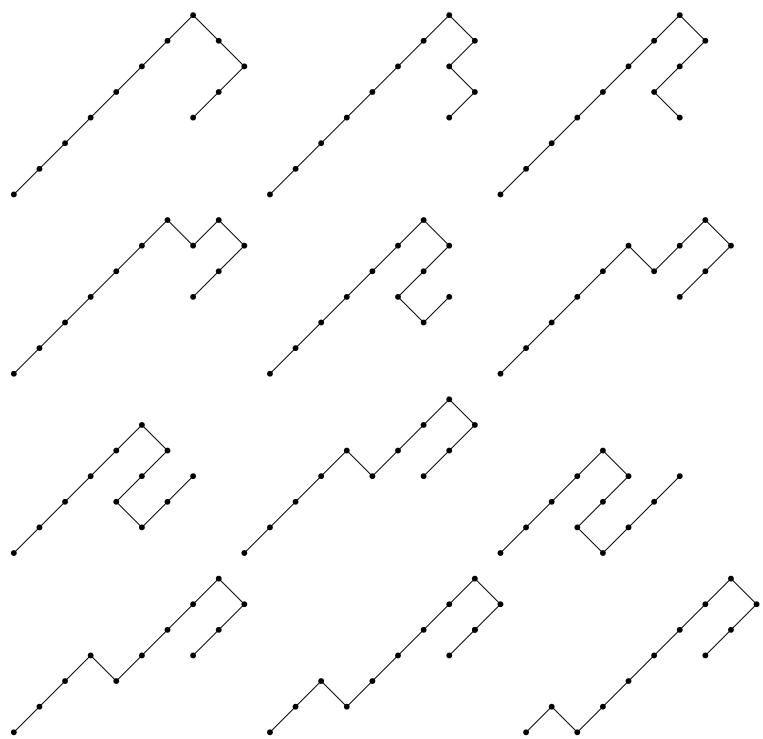


Figure 8: The skew meanders of length 11, height 3, with exactly 2 left steps.

References

[1] E. Deutsch, E. Munarini, and S. Rinaldi, Skew Dyck paths, *J. Statist. Plann. Inference* **140** (2010), 2191–2203.

[2] E. Deutsch, E. Munarini, and S. Rinaldi, Skew Dyck paths, area, and superdiagonal bargraphs, *J. Statist. Plann. Inference* **140** (2010), 1550–1526.

[3] S. Feretic, Generating functions for bi-wall directed polygons, in S. Rinaldi and S. G. Mohanty, eds., *Seventh Int. Conf. on Lattice Path Combinatorics and Applications*, 2010, pp. 147–151.

[4] P. Flajolet and R. Sedgewick, *Analytic Combinatorics*, Cambridge University Press, 2009.

[5] I. M. Gessel, Lagrange inversion, *J. Combin. Theory Ser. A* **144** (2016), 212–224.

[6] P. Kirschenhofer and H. Prodinger, Return statistics of simple random walks, *J. Statist. Plann. Inference* **54** (1996), 67–74.

[7] T.-X. He and R. Sprugnoli, Sequence characterization of Riordan arrays, *Discrete Math.* **309** (2009), 3962–3974.

[8] Q. L. Lu, Skew Motzkin paths, *Acta Math. Sin. (Engl. Ser.)* **33** (2017), 657–667.

[9] D. Merlini, D. G. Rogers, R. Sprugnoli, and M. C. Verri, On some alternative characterizations of Riordan arrays, *Canadian J. Math.* **49** (1997), 301–320.

[10] D. Merlini, R. Sprugnoli, and M. C. Verri, Lagrange inversion: when and how, *Acta Appl. Math*, **94** (2006), 233–249.

[11] M. Petkovšek, H. Wilf, and D. Zeilberger, *A=B*, A. K. Peters, Ltd. 1996.

[12] D. G. Rogers, Pascal triangles, Catalan numbers and renewal arrays, *Discrete Math.* **22** (1978), 301–310.

[13] L. Shapiro, A Catalan triangle, *Discrete Math.* **14** (1976), 83–90.

[14] L. W. Shapiro, S. Getu, W. Woan, and L. Woodson, The Riordan group, *Discrete Appl. Math.* **34** (1991) 229–239.

[15] N. J. A. Sloane et al., The On-Line Encyclopedia of Integer Sequences, 2021. Available at https://oeis.org/.

[16] W.-J. Woan, L. Shapiro, and D. G. Rogers, The Catalan numbers, the Lebesgue integral, and 4^{n-2}, *Amer. Math. Monthly* **104** (1997), 926–931.

2010 *Mathematics Subject Classification*: Primary 05A15; Secondary 05A19.
Keywords: skew Dyck path, generating function, Lagrange inversion, Riordan array.

(Concerned with sequences A000108, A002212, A033321, A039598, and A122737.)

Received May 18 2021; revised version received July 28 2021. Published in *Journal of Integer Sequences*, August 4 2021.

Return to Journal of Integer Sequences home page.

Discrete Mathematics Letters
www.dmlett.com

Discrete Math. Lett. **8** (2022) 60–65
DOI: 10.47443/dml.2021.0108

Research Article

Lattice paths with a first return decomposition constrained by the maximal height of a pattern

Jean-Luc Baril*, Sergey Kirgizov

Laboratoire d'Informatique de Bourgogne, Université de Bourgogne Franche-Comté, France

(Received: 5 October 2021. Received in revised form: 25 October 2021. Accepted: 26 October 2021. Published online: 29 October 2021.)

© 2021 the authors. This is an open access article under the CC BY (International 4.0) license (www.creativecommons.org/licenses/by/4.0/).

Abstract

We consider the system of equations $A_k(x) = p(x)A_{k-1}(x)(q(x) + \sum_{i=0}^{k} A_i(x))$ for $k \geqslant r+1$ where $A_i(x)$, $0 \leqslant i \leqslant r$, are some given functions and show how to obtain a close form for $A(x) = \sum_{k\geqslant 0} A_k(x)$. We apply this general result to the enumeration of certain subsets of Dyck, Motzkin, skew Dyck, and skew Motzkin paths, defined recursively according to the first return decomposition with a monotonically non-increasing condition relative to the maximal ordinate reached by an occurrence of a given pattern π.

Keywords: enumeration; Dyck and Motzkin paths; first return decomposition; statistics; height; pattern.

2020 Mathematics Subject Classification: 05A15, 05A19, 68R05.

1. Introduction and notations

Let $\mathcal{A}$ be a combinatorial class, that is a collection of similar objects (lattices, trees, permutations, words) endowed with a size function ℓ whose values are non-negative integers, so that the number $|\ell^{-1}(n)|$ of objects of a given size n is finite. With $\mathcal{A}$ we associate the generating function

$$A(x) = \sum_{a\in\mathcal{A}} x^{\ell(a)} = \sum_{n\geqslant 0} |\ell^{-1}(n)|x^n,$$

which appears to be the main source of interest for researchers working in enumerative combinatorics.

Given a function $s : \mathcal{A} \to \mathbb{N}$, called *statistic*, and an integer $k \geqslant 0$, we define the combinatorial class $\mathcal{A}_k$ consisting of objects $a \in \mathcal{A}$ such that $s(a) = k$, and associate the following generating function with this class:

$$A_k(x) = \sum_{a\in\mathcal{A}_k} x^{\ell(a)}, \text{ which implies } A(x) = \sum_{k\geqslant 0} A_k(x).$$

In three recent articles [1,2,6], the authors enumerate particular subclasses of lattice paths (Dyck, Motzkin, 2-Motzkin and Schröder paths) defined recursively according to the first return decomposition with a condition on the path height. More precisely, in the first paper, they focus on the Dyck path set $\mathcal{D}^{h,\geqslant}$ consisting of the union of the empty path with all Dyck paths P having a first return decomposition $P = U\alpha D\beta$ satisfying the conditions:

$$\begin{cases} \alpha, \beta \in \mathcal{D}^{h,\geqslant}, \\ h(U\alpha D) \geqslant h(\beta), \end{cases} \tag{1}$$

where h is the *height statistic*, i.e. the maximal ordinate reached by a path. In these studies, the authors need to consider a system of equations involving the generating functions $D_k(x)$, $k \geqslant 0$, for the subset of paths $P \in \mathcal{D}^{h,\geqslant}$ satisfying $h(P) = k$:

$$D_k(x) = x \cdot D_{k-1}(x) \cdot \sum_{i=0}^{k} D_i(x) \text{ for } k \geqslant 1 \tag{2}$$

anchored with initial condition $D_0(x) = 1$.

They deduce algebraically a close form of the generating function $D(x) = \sum_{k\geqslant 0} D_k(x)$ for $\mathcal{D}^{h,\geqslant}$ by solving System (2), and prove that $\mathcal{D}^{h,\geqslant}$ is enumerated by the Motzkin sequence, that is A001006 in the On-Line Encyclopedia of Integer Sequences (OEIS) [13]. The second paper is a similar study in the context of Motzkin and Schröder paths.

*Corresponding author (barjl@u-bourgogne.fr).

Inspired by these works, the motivation of the paper is to extend such studies by investigating the solutions of a more general system defined as follows. For an integer $r \geqslant 0$ and two functions $p(x)$ and $q(x)$, we consider the system of equations:

$$A_k(x) = p(x)A_{k-1}(x)\left(q(x) + \sum_{i=0}^{k} A_i(x)\right), \text{ for } k \geqslant r+1, \tag{3}$$

anchored with $A_i(x) = u_i(x)$ for $0 \leqslant i \leqslant r$ where $u_i(x)$, $0 \leqslant i \leqslant r$, are some given functions. For the sake of brevity, we set $u = u_r(x)$, $v = \sum_{i=0}^{r} u_i(x) - u$, $p = p(x)$ and $q = q(x)$.

Before considering how Equation (3) is related to particular sets of lattice paths defined in the same way as the set $\mathcal{D}^{h,\geqslant}$, let us recall some definitions and notations. A *skew Motzkin path* (see [7]) of length $n \geqslant 0$ is a lattice path in the quarter plane starting at point $(0,0)$, ending on the x-axis (but never going below), consisting of n steps of the four following types: up steps $U = (1,1)$, down steps $D = (1,-1)$, flat steps $F = (1,0)$ and left steps $L = (-1,-1)$, such that left and up steps never overlap. Let $\mathcal{SM}$ be the set of all skew Motzkin paths of any length. A *Motzkin path* is a skew Motzkin path with no left steps. Let $\mathcal{M}$ be the set of all Motzkin paths. A *Dyck path* is a Motzkin path with no flat steps. Let $\mathcal{D}$ be the set of all Dyck paths. A *skew Dyck path* is a skew Motzkin path with no flat steps. Let $\mathcal{SD}$ be the set of all skew Dyck paths. Obviously, we have $\mathcal{D} \subset \mathcal{SD} \subset \mathcal{SM}$ and $\mathcal{D} \subset \mathcal{M} \subset \mathcal{SM}$.

A *pattern* of length $n \geqslant 1$ in a lattice path P consists of n consecutive steps. The *height* of an *occurrence* of a pattern in a given path is the maximal ordinate reached by its points. For a pattern π, we define the statistic h_π on lattice paths so that $h_\pi(P)$ is the maximal height reached by the occurrences of π in P. The *amplitude* r_π of a pattern π is the height of the path π considered as a path touching the x-axis in the quarter plane. For instance, the path $P = UUDUUDUDDD$ contains the pattern UUD and $h_{UUD}(P) = 3$, while the amplitude of UUD is $r_{UUD} = 2$.

The paper is organized as follows. In Section 2, we show how we can obtain a close form for $A(x)$ defined by the system (3). In Section 3, we provide several concrete applications in the context of lattices paths (Dyck, Motzkin, skew Dyck, and skew Motzkin paths) for the statistic h_π giving the maximal ordinate reached by the occurrence of π in a path. Finally in Section 4, we provide some possible future research directions.

2. General result

For $k \geqslant 0$, we set $B_k(x) = \sum_{i=0}^{k} A_i(x)$, and thus we have $A(x) = \lim_{k\to\infty} B_k(x)$. The next theorem provides a close form of $A(x)$ whenever Equation (3) admits a solution, that is when $1 - A_k(x)p(x) \neq 0$ for all $k \geqslant r$.

Theorem 2.1. *If Equation (3) has a solution then for $k \geqslant r+1$ we have*

$$B_k(x) = \frac{a + bB_{k-1}(x)}{c + dB_{k-1}(x)}, \tag{4}$$

where

$$\begin{cases} a = p^2qv\,(q+u+v) - pqu - u - v \\ b = -p\,(pq+1)\,(q+v+u) \\ c = -p^2v\,(q+u+v) - qp - pv - 1 \\ d = p^2\,(q+v+u)\,. \end{cases}$$

Moreover $A(x) = \lim\limits_{k\to\infty} B_k(x) = \sum_{k=0}^{\infty} A_i(x)$ is a root of the polynomial

$$d \cdot A(x)^2 + (c-b)\cdot A(x) - a. \tag{5}$$

Proof. Using $B_r(x) = u + v$, we solve the following linear system for a, b, c, d

$$\begin{cases} B_{r+1}(x)(c + dB_r(x)) = a + bB_r(x) \\ B_{r+2}(x)(c + dB_{r+1}(x)) = a + bB_{r+1}(x) \\ B_{r+3}(x)(c + dB_{r+2}(x)) = a + bB_{r+2}(x), \end{cases}$$

and obtain (modulo a multiplicative factor) the claimed values. However, it remains to prove that

$$B_k(x) = \frac{a + bB_{k-1}(x)}{c + dB_{k-1}(x)}$$

for any $k \geqslant r+1$. We proceed by induction on k. Obviously, for $k = r+1$ it is true. So, we assume that the formula is true for $k \leqslant n$ and we prove that this is also true for $k = n+1$. More precisely, we will prove that $R = (c + dB_n(x))B_{n+1}(x) - (a + bB_n(x))$ equals zero.

Considering Equation (3), we have $B_{n+1}(x) = \frac{pqA_n(x)+B_n(x)}{1-pA_n(x)}$. The recurrence hypothesis provides $B_n(x) = \frac{a+bB_{n-1}(x)}{c+dB_{n-1}(x)}$, and we have $A_n(x) = B_n(x) - B_{n-1}(x)$. Replacing respectively $B_{n+1}(x)$, $B_n(x)$ and $A_n(x)$ in R, and multiplying it by $1 - p \cdot A_n(x) \neq 0$, we obtain

$$\left(qp\left(\frac{a+bS}{c+dS} - S\right) + \frac{a+bS}{c+dS}\right)\left(c + \frac{d\,(a+bS)}{c+dS}\right) - \left(a + \frac{b\,(a+bS)}{c+dS}\right)\left(1 - p\left(\frac{a+bS}{c+dS} - S\right)\right),$$

where $S = B_{n-1}(x)$. A simplification of this expression (whatever the value of S) implies that $R = 0$, which completes the induction. Finally, taking the limit when $k \to \infty$ of both sides of $B_k(x) = \frac{a+bB_{k-1}(x)}{c+dB_{k-1}(x)}$, and using the fact that $A(x) = \lim_{k\to\infty} B_k(x)$, one gets

$$d \cdot A(x)^2 + (c-b) \cdot A(x) - a = 0.$$

Notice that an alternative proof can be obtained by using Chebyshev polynomials of the second kind (see Proposition D.5 p. 508 in [10]). □

3. Some applications

In this section, we apply Theorem 2.1 to study special kinds of classes of lattices paths: Dyck, Motzkin, skew Dyck, and skew Motzkin paths. We extend the definition (1) to these paths using the maximal height reached by the occurrences of π, the statistic h_π instead of statistic h. Notice that the statistic h equals the statistic h_U on all considered paths, which allows us to focus only on h_π. We will deal with patterns π of length at most three.

Given a pattern π, we define the set $\mathcal{A}^{h_\pi,\geqslant}$ as the union of the empty path with paths P satisfying a recurrence condition (R), which depends on the first return decomposition of a path. The four cases are described in the following table with $\alpha, \beta, \gamma \in \mathcal{A}^{h_\pi,\geqslant}$.

Paths	First return decomposition with recurrence condition (R)
Dyck	$P = U\alpha D\beta$ with $h_\pi(U\alpha D) \geqslant h_\pi(\beta)$
Motzkin	$\begin{cases} P = U\alpha D\beta \\ P = F\gamma \end{cases}$ with $h_\pi(U\alpha D) \geqslant h_\pi(\beta)$ and $h_\pi(F) \geqslant h_\pi(\gamma)$
Skew Dyck	$\begin{cases} P = U\alpha D\beta \\ P = U\alpha L \end{cases}$ with $\begin{cases} h_\pi(U\alpha D) \geqslant h_\pi(\beta) \\ \text{and } \alpha \neq \epsilon \text{ for the last case} \end{cases}$
Skew Motzkin	$\begin{cases} P = U\alpha D\beta \\ P = F\gamma \\ P = U\alpha L \\ P = U\alpha LF\gamma \end{cases}$ with $\begin{cases} h_\pi(U\alpha D) \geqslant h_\pi(\beta) \text{ and } h_\pi(F) \geqslant h_\pi(\gamma) \\ \text{and } \alpha \neq \epsilon \text{ for the last two cases} \end{cases}$

We keep the notations used in the previous sections. The generating functions $A_k(x)$, $k \geqslant 0$, correspond to the sets $\mathcal{A}_k$ of Dyck paths $P \in \mathcal{A}^{h_\pi,\geqslant}$ so that $h_\pi(P) = k$. We set $A(x) = \sum_{k\geqslant 0} A_k(x)$ and $B_k(x) = \sum_{i=0}^{k} A_i(x)$.

In any case, the set $\mathcal{A}_0$ of paths $P \in \mathcal{A}^{h_\pi,\geqslant}$ such that $h_\pi(P) = 0$ is the set of paths having no occurrence of π at height at least one. In the case where the amplitude of π is at least one, $\mathcal{A}_0$ consists of the paths avoiding π. So, we can easily obtain $A_0(x)$ using the first return decomposition of such paths. For Dyck and Motzkin paths, these generating functions (for short g.f.) are already known (see [3,4,8,9,11,12]). For skew Dyck and Motzkin paths, we can obtain them by similar methods, which are classic and present no difficulty. So, we do not give here all the details to get $A_0(x)$.

Note that the maximal height of an occurrence of pattern π in P is necessarily greater or equal than its amplitude r_π. This means that the generating functions for the sets $\mathcal{A}_k$, $1 \leqslant k < r_\pi$ satisfy $A_k(x) = 0$. Finally, if $r_\pi \geqslant 1$ (which is equivalent to $\pi \neq F^k$ for all $k \geqslant 1$) then $\mathcal{A}_{r_\pi}$, is the set of paths $P \in \mathcal{A}^{h_\pi,\geqslant}$ having at least one occurrence of π and such that $h_\pi(P) = r_\pi$, i.e. all occurrences of π touch the x-axis in P. Similar methods as for $A_0(x)$ can be used in order to obtain $A_{r_\pi}(x)$. Theorems 3.1 and 3.2 focus on two examples of these methods in the context of Dyck paths.

3.1. Dyck and Motzkin paths

First we consider Dyck paths. For a given pattern π, Condition (R) implies that for $k \geqslant r_\pi + 1$ we have

$$A_k(x) = x \cdot A_{k-1}(x) \cdot \sum_{i=0}^{k} A_i(x)$$

which is a particular case of Equation (3) for $r = r_\pi$, $p(x) = x$, $q(x) = 0$, $u = A_{r_\pi}(x)$, and $v = \sum_{i=0}^{r} A_i(x) - u = A_0(x)$. From Theorem 2.1, we can easily deduce the following.

Corollary 3.1.

$$A(x) = \frac{x^2uv + x^2v^2 - ux + 1 - \sqrt{\Delta}}{2x^2\,(v+u)} \tag{6}$$

with $\Delta = u^2v^2x^4 + 2\,uv^3x^4 + v^4x^4 - 2\,u^2vx^3 - 2\,uv^2x^3 - 3\,u^2x^2 - 6\,x^2uv - 2\,x^2v^2 - 2\,ux + 1$.

In order to convince the reader that all generating functions $A(x)$ can be obtained easily by calculating algebraically u and v, we give here the methods for the two patterns UUD and DUU.

Theorem 3.1. *If* $\pi = UUD$ *then we have* $r_\pi = 2$, $u = A_2(x) = \frac{x^2}{(x-1)(x^2+x-1)}$, $A_1(x) = 0$, *and* $v = A_0(x) = \frac{1}{1-x}$.

Proof. As already mentioned above, the set $\mathcal{A}_0$ is the set of Dyck paths avoiding the pattern π. So, a nonempty path $P \in \mathcal{A}_0$ is of the form $P = UD\alpha$ where $\alpha \in \mathcal{A}_0$, which implies that $A_0(x) = \frac{1}{1-x}$. On the other hand, the set $\mathcal{A}_1$ is the set of Dyck paths $P \in \mathcal{A}^{h_\pi,\geqslant}$ such that $h_{UUD}(P) = 1$, which cannot be possible because the amplitude $r_\pi = 2$; so, $A_1(x) = 0$. Finally, the set $\mathcal{A}_2$ is the set of Dyck paths $P \in \mathcal{A}^{h_\pi,\geqslant}$ such that $h_{UUD}(P) = 2$, which means that P can be written $U\alpha D\beta$ with $\alpha \in \mathcal{A}_0\backslash\epsilon$ and $\beta \in \mathcal{A}_2 \cup \mathcal{A}_1 \cup \mathcal{A}_0$. So, we have the functional equation $A_2(x) = x(A_0(x) - 1)(A_2(x) + A_1(x) + A_0(x))$, which gives the expected result. □

Theorem 3.2. *If* $\pi = DUU$ *then we have* $r_\pi = 2$, $u = A_2(x) = \frac{x^3(1-x)}{(2\,x-1)^2}$, $A_1(x) = 0$, *and* $v = A_0(x) = \frac{x-1}{2x-1}$.

Proof. The set $\mathcal{A}_0$ is the set of Dyck paths avoiding the pattern DUU. So, a nonempty path $P \in \mathcal{A}_0$ is of the form $P = U\alpha D\beta$ where $\alpha \in \mathcal{A}_0$ and $\beta = (UD)^k$ for some $k \geqslant 0$, which implies that $A_0(x) = 1 + \frac{xA_0(x)}{1-x}$, and thus $A_0(x) = \frac{x-1}{2x-1}$. On the other hand, the set $\mathcal{A}_1$ is the set of Dyck paths $P \in \mathcal{A}^{h_\pi,\geqslant}$ such that $h_{DUU}(P) = 1$, which cannot be possible because the amplitude $r_\pi = 2$; so, $A_1(x) = 0$. Finally, the set $\mathcal{A}_2$ is the set of Dyck paths $P \in \mathcal{A}^{h_\pi,\geqslant}$ such that $h_{DUU}(P) = 2$, which means that P can be written $U\alpha DU\beta D$ with $\alpha \in \mathcal{A}_0$ and $\beta \in \mathcal{A}_0\backslash\{\epsilon\}$.

So, we have the functional equation $A_2(x) = x^2A_0(x)(A_0(x) - 1)$, which gives the expected result. □

So, we present in the following table the first values of the sequences corresponding to the cardinality of $\mathcal{A}^{h_\pi,\geqslant}$ for patterns π of length at most three.

Statistic h_π	$a_n,\ 1 \leqslant n \leqslant 9$	OEIS
$h, U, D, UD, UU,$ DD, UDD, UUD	1, 2, 4, 9, 21, 51, 127, 323	A001006
DU	1, 2, 4, 8, 17, 39, 94, 233, 588	
UUU, DDD	1, 2, 5, 13, 35, 97, 274, 786, 2282	
UDU, DUD	1, 2, 4, 9, 22, 56, 146, 389, 1053	
DUU, DDU	1, 2, 5, 13, 34, 89, 234, 621, 1669	

Now, we examine the set of Motzkin paths. For a given pattern π, Condition (R) implies that for $k \geqslant r_\pi + 1$ we have

$$A_k(x) = x^2 \cdot A_{k-1}(x) \cdot \sum_{i=0}^{k} A_i(x)$$

which is a particular case of Equation (3) for $r = r_\pi$, $p(x) = x^2$, $q(x) = 0$, $u = A_{r_\pi}(x)$, and $v = A_0(x)$. Then, the generating function $A(x)$ is the generating function found above in the context of Dyck paths (see (6)) applied on the variable x^2 (x is not replaced with x^2 in u and v). The following table gives the first values of generating functions $A(x)$ obtained for all patterns of length at most two.

Statistic h_π	$a_n,\ 1 \leqslant n \leqslant 9$	OEIS
h, U, D	1, 2, 3, 6, 11, 22, 43, 87, 176	A026418
F	1, 2, 4, 8, 17, 36, 78, 170, 374	
UU, DD	1, 2, 4, 9, 20, 46, 107, 253, 604	
UD	1, 2, 3, 7, 13, 29, 61, 138, 308	
DU	1, 2, 4, 9, 20, 46, 107, 252, 599	
UF, FD	1, 2, 4, 8, 17, 37, 82, 185, 422	
DF, FU	1, 2, 4, 8, 17, 36, 79, 175, 395	
FF	1, 2, 4, 9, 20, 47, 111, 268, 653	

Looking at the two tables above, one observes that some patterns have same sequences, for example DUU and DDU, UF and FD. We actually can generalize this as follows. Let $\pi = \pi_1\pi_2\dots\pi_k$ be a pattern of length k and $C(\pi)$ be the *reversed complement* of π, that is $C(\pi) = \overline{\pi}_k\overline{\pi}_{k-1}\cdots\overline{\pi}_1$ where

$$\overline{x} = \begin{cases} F \text{ if } x = F, \\ D \text{ if } x = U, \\ U \text{ if } x = D. \end{cases}$$

For instance we have $C(UFDD) = UUFD$.

Theorem 3.3. *If ${}_\pi A(x)$ is the generating function for the set $\mathcal{A}^{h_\pi,\geqslant}$, then we have*

$${}_\pi A(x) = {}_{C(\pi)}A(x).$$

Proof. For short we set $\sigma = C(\pi)$. Since the reversed complement operation preserves the amplitude, we have $r_\pi = r_\sigma$. Now, we will exhibit a bijection ϕ from $\mathcal{A}^{h_\pi,\geqslant}$ to $\mathcal{A}^{h_\sigma,\geqslant}$. Suppose $\pi \neq F^k, k \geqslant 1$, if a path $P \in \mathcal{A}^{h_\pi,\geqslant}$ does not contain π, we set $\phi(P) = C(P)$. It avoids σ and belongs to $\mathcal{A}^{h_\sigma,\geqslant}$. So, we have ${}_\pi A_0(x) = {}_\sigma A_0(x)$. If $\pi = F^k$ for some $k \geqslant 1$, we have $r_\pi = 0$ and we also set $\phi(P) = C(P)$. We easily deduce again that ${}_\pi A_0(x) = {}_\sigma A_0(x)$.

Now, let us consider the case where $r_\pi \geqslant 1$ and prove that g.f. ${}_\pi A_{r_\pi}$ for pattern π is actually equals ${}_\sigma A_{r_\sigma}$ for pattern σ. To do this we consider a path $P \in \mathcal{A}^{h_\pi,\geqslant}$ having at least one occurrence of π and such that $h_\pi(P) = r_\pi$. We can write P either as (i) $P = F\alpha$ or (ii) $P = U\alpha D\beta$ where $\alpha, \beta \in \mathcal{A}^{h_\pi,\geqslant}$.

Case (i). We assume $P = F\alpha$ and $h_\pi(F) \geqslant h_\pi(\alpha)$. Since the pattern π is not F^k (because $r_\pi \geqslant 1$), we necessarily have $h_\pi(F) = 0 = h_\pi(\alpha)$ which means that α avoids π. Therefore, the pattern π occurs once in P and it straddles F and α. We deduce that $C(P)$ belongs to $\mathcal{A}^{h_\sigma,\geqslant}$ and it has at least one occurrence of σ such that $h_\sigma(C(P)) = r_\sigma$. So in this case, we define the function ϕ with $\phi(P) = C(P)$.

Case (ii). We assume $P = U\alpha D\beta$ and $h_\pi(U\alpha D) \geqslant h_\pi(\beta)$. If $h_\pi(U\alpha D) = 0$ then we have $h_\pi(\beta) = 0$ and using the same argument as previous, $C(P)$ belongs to $\mathcal{A}^{h_\sigma,\geqslant}$ and it has exactly one occurrence of σ straddling $U\alpha D$ and β, and such that $h_\sigma(C(P)) = r_\sigma$. In this subcase, we set $\phi(P) = C(P)$. If $h_\pi(U\alpha D) = r_\pi$ then the occurrence of π in $U\alpha D$ is at the beginning or at the end (or touches both extremities) of $U\alpha D$ since it must touch the x-axis, but the occurrence does not straddle $U\alpha D$ and β. This means that α avoids π and β either avoids π (i.e. $h_\pi(\beta) = 0$, as $\pi \neq F^k$) or contains π with $h_\pi(\beta) = r_\pi$. In this subcase, we recursively define ϕ as $\phi(P) = UC(\alpha)D\phi(\beta)$. It belongs to $\mathcal{A}^{h_\sigma,\geqslant}$ and has at least one occurrence of σ such that $h_\sigma(\phi(P)) = r_\sigma$. So, ϕ is a one-to-one correspondence between paths in $\mathcal{A}^{h_\pi,\geqslant}$ having at least one occurrence of π such that $h_\pi(\phi(P)) = r_\pi$ and paths $\mathcal{A}^{h_\sigma,\geqslant}$ having at least one occurrence of σ such that $h_\sigma(\phi(P)) = r_\sigma$. So, we have ${}_\pi A_{r_\pi}(x) = {}_\sigma A_{r_\sigma}(x)$.

Since ${}_\pi A_k(x) = {}_\sigma A_k(x) = 0$ for $0 < k < r_\pi$, finally we have ${}_\pi A_k(x) = {}_\sigma A_k(x)$ for $0 \leqslant k \leqslant r_\pi$, and Equation (3) implies ${}_\pi A_k(x) = {}_\sigma A_k(x)$ for $k \geqslant r_\pi + 1$. □

3.2. Skew Dyck and Motzkin paths

Let us discuss the skew Dyck paths first. For a given pattern π, Condition (R) implies that for $k \geqslant r_\pi + 1$ we have

$$A_k(x) = x \cdot A_{k-1}(x) \cdot \left(\sum_{i=0}^{k} A_i(x) + 1\right)$$

which is a particular case of Equation (3) for $r = r_\pi$, $p(x) = x$, $q(x) = 1$, $u = A_{r_\pi}(x)$, and $v = A_0(x)$. Then, we deduce the following.

Corollary 3.2.

$$A(x) = \frac{uvx^2 + v^2x^2 - x^2u - xu - x^2 + 1 - \sqrt{\Delta}}{2x^2(1 + v + u)} \tag{7}$$

with $\Delta = u^2v^2x^4 + 2\,uv^3x^4 + v^4x^4 + 2\,u^2vx^4 + 6\,uv^2x^4 + 4\,v^3x^4 - 2\,u^2vx^3 + u^2x^4 - 2\,uv^2x^3 + 6\,uvx^4 + 6\,v^2x^4 - 2\,x^3u^2 - 4\,uvx^3 + 2\,ux^4 + 4\,vx^4 - 3\,u^2x^2 - 6\,uvx^2 - 2\,ux^3 - 2\,v^2x^2 + x^4 - 6\,x^2u - 4\,vx^2 - 2\,xu - 2\,x^2 + 1$.

The following table gives the first values of generating functions $A(x)$ obtained for all patterns of length at most two. Notice that none sequences appear in [13].

Statistic h_π	$a_n,\ 1 \leqslant n \leqslant 9$	OEIS
h, U, D, UU , UD	1, 3, 8, 23, 68, 211, 668, 2169, 7145	
L, DL	1, 3, 9, 28, 91, 307, 1062, 3748, 13429	
DD	1, 3, 9, 29, 96, 327, 1136, 4014, 14365	
DU	1, 3, 9, 27, 82, 255, 813, 2655, 8847	
LD	1, 3, 10, 35, 126, 463, 1728, 6529, 24916	
LL	1, 3, 10, 35, 128, 485, 1890, 7531, 30545	

In the context of skew Motzin paths we have

$$A_k(x) = x^2 \cdot A_{k-1}(x) \cdot \left(\sum_{i=0}^{k} A_i(x) + x \cdot A_0(x) + 1 \right)$$

which is a particular case of Equation (3) for $r = r_\pi$, $p(x) = x^2$, $q(x) = x \cdot A_0(x) + 1$, $u = A_{r_\pi}(x)$, which means that $v = 0$ when $r_\pi = 0$ and $v = A_0(x)$ otherwise. Then, the generating function $A(x)$ is the generating function for skew Dyck paths (see Equation (7)) applied on the variable x^2 (x is not replaced with x^2 in u and v), and we obtain the following results for patterns of length at most one. The sequences do not appear in [13].

Statistic h_π	$a_n,\ 1 \leqslant n \leqslant 9$	OEIS
h, U	1, 2, 4, 9, 20, 45, 101, 229, 524, 1211, 2820	
D	1, 2, 4, 10, 23, 55, 131, 318, 774, 1899, 4678	
F	1, 2, 5, 11, 27, 64, 157, 383, 946, 2347, 5854	
L	1, 2, 5, 12, 30, 76, 196, 513, 1359, 3639, 9831	

4. Conclusion and research directions

Except Dyck and Motzkin cases, where we obtain two sequences appearing in [13] for the pattern U, all the others have never been studied in the literature. So, this work provides a new catalog of sequences which is a possibly fertile ground in number theory, and more precisely in the analyse of their modular congruences (see [5] for instance). Another research direction could be the investigation of the distribution of a given pattern in the set $\mathcal{A}^{h_\pi,\geqslant}$. For instance, if $\pi = UU$ in the context of Dyck paths, then the paths in $\mathcal{A}^{h_\pi,\geqslant}$ avoiding the pattern UDU are enumerated by the generalized Catalan numbers (see A004148 in [13]), which also count peak-less Motzkin paths. It would be interesting to find a one-to-one correspondence between these objects.

In this paper, we focus on the generating $A(x)$ of $\mathcal{A}^{h_\pi,\geqslant}$, but we do not deal with the generating functions $A_k(x)$ for paths $P \in \mathcal{A}^{h_\pi,\geqslant}$ such that $h_\pi(P) = k$ since they can be easily obtained. Observing these g.f. for small k, we have identified some of them that are already known in OEIS [13]. For instance, the number of n-length Dyck paths $P \in \mathcal{A}^{h_{UU},\geqslant}$ such that $h_{UU}(P) = 2$ is the classical number of Fibonacci minus one (see A000071 in [13]). Such a result suggests us to look for new bijections with known classes counted by this sequence.

Finally, it would be interesting to extend this work to other classes of objects counted by generating functions satisfying System (3).

Acknowledgement

This work was partly supported by the project ANER ARTICO financed by Bourgogne-Franche-Comté region.

References

[1] J.-L. Baril, S. Kirgizov, A. Petrossian, Dyck paths with a first return decomposition constrained by height, *Discrete Math.* **341** (2018) 1620–1628.
[2] J.-L. Baril, S. Kirgizov, A. Petrossian, Motzkin paths with a restricted first return decomposition, *Integers* **19** (2019) #A46.
[3] C. Brennan, S. Mavhungu, Peaks and valleys in Motzkin paths, *Quaest. Math.* **33** (2010) 171–188.
[4] E. Deutsch, Dyck path enumeration, *Discrete Math.* **204** (1999) 167–202.
[5] E. Deutsch, B. Sagan, Congruences for Catalan and Motzkin numbers and related sequences, *J. Number Theory* **117** (2006) 191–215.
[6] R. Flórez, L. Junes, J. Ramírez, Enumerating several aspects of non-decreasing Dyck paths, *Discrete Math.* **342** (2019) 3079–3097.
[7] Q. L. Lu, Skew Motzkin paths, *Acta Math. Sinica* **33** (2017) 657–667.
[8] T. Mansour, Counting peaks at height k in a Dyck path, *J. Integer Seq.* **5** (2002) #02.1.1.
[9] T. Mansour, Statistics on Dyck paths, *J. Integer Seq.* **9** (2006) #06.1.5.
[10] T. Mansour, *Combinatorics of Set Partitions*, CRC Press, Boca Raton, 2012.
[11] T. Mansour, E. Deng, R. Du, Dyck paths and restricted permutations, *Discrete Appl. Math.* **154** (2006) 1593–1605.
[12] D. Merlini, R. Sprugnoli, M. Verri, Some statistics on Dyck paths, *J. Stat. Plan. Inference* **101** (2002) 211–227.
[13] N. Sloane, *The On-Line Encyclopedia of Integer Sequences*, Available electronically at https://oeis.org.

EQUIVALENCE CLASSES OF SKEW DYCK PATHS MODULO SOME PATTERNS

Jean-Luc Baril[1]
LIB, Univ. Franche-Comt, France
barjl@u-bourgogne.fr

José Luis Ramírez
Departamento de Matemticas, Universidad Nacional de Colombia, Bogot, Colombia
jlramirezr@unal.edu.co

Lina Maria Simbaqueba
Departamento de Matemticas, Universidad Nacional de Colombia, Bogot, Colombia
lmsimbaquebam@unal.edu.co

Received: 8/23/21, Accepted: 2/1/22, Published: 2/28/22

Abstract

For any pattern $p \in \{U, L, UU, UD, DU, DD\}$, we enumerate the equivalence classes of skew Dyck paths, where two skew Dyck paths of the same semilength are p-equivalent whenever the positions of the occurrences of the pattern p are the same. In this paper we use generating functions, bijective arguments, and recurrence relations to obtain the main results.

1. Introduction and Notation

A *skew Dyck path* is a lattice path in the first quadrant of the xy-plane that starts at the origin, ends on the x-axis, and is made of up-steps $U = (1, 1)$, down-steps $D = (1, -1)$, and left steps $L = (-1, -1)$ so that up and left steps do not overlap. Whenever we do not permit the step L, we retrieve the well known definition of Dyck paths (see [5]). We let $\mathcal{SD}$ denote the set of all skew Dyck paths, $\mathcal{D}$ the set of Dyck paths, and $|P|$ the *length* of the path P, i.e., the number of its steps, which is an even non-negative integer. Let λ be the skew Dyck path of length zero. For example, Figure 1 shows all skew Dyck paths of length 6, or equivalently of semilength 3.

[1]Corresponding author

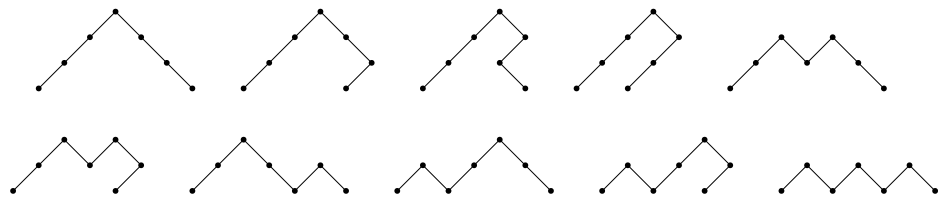


Figure 1: Skew Dyck paths of semilength 3.

The concept of skew Dyck path was introduced by Deutsch, Munarini, and Rinaldi [6]. Some additional studies can be found in [4, 7, 11], where the authors present enumerative results according to different parameters and some bijections with other combinatorial objects, as hex trees, tree-like polyhexes, and 3-Motzkin paths.

In the following, a *pattern* consists of consecutive steps in a path. We will say that a pattern is at *position* $i \geq 1$ in a path whenever the first step of the pattern appears at the i-th step of the path, the second step at the $(i+1)$-th step, and so on. The *height* of an occurrence of a pattern is the minimal ordinate reached by its points. For instance, the skew Dyck path $P = UDUUDL$ contains two occurrences of the pattern UD at positions 1 and 4, and the heights of these occurrences are respectively 0 and 1.

Recently in [1, 2, 3, 12], the authors investigate equivalence relations on the sets of Dyck paths, Motzkin paths, Łukasiewicz paths, and Ballot paths where two paths of the same length are equivalent whenever they coincide on all occurrences of a given pattern. In this paper, we extend these studies for skew Dyck paths by considering the analogous equivalence relation on $\mathcal{SD}$:

Two skew Dyck paths of the same semilength are p-equivalent whenever they have the same positions of the occurrences of the pattern p.

For instance, the skew Dyck path $UDUUDL$ is L-equivalent with $UUDUDL$ since the occurrences of L appear at the same positions in the two paths.

For some patterns p of length one or two, we provide generating functions for the number of p-equivalence classes in $\mathcal{SD}$ with respect to the semilength. The general method used consists in providing bijections between equivalence classes and some subsets of skew Dyck paths, and then, evaluating algebraically the generating functions for these subsets. We handle the cases $p \in \{U, L, UU, UD, DU, DD\}$, and we leave the other cases as open problems. As a byproduct, we characterize skew Dyck paths entirely fixed by the positions of its left steps L, and we count them using generating functions and recurrence relations. We also provide and conjecture asymptotic approximations for the number of p-equivalence classes of skew Dyck paths of semilength n.

2. Equivalence Classes Modulo Patterns of Length One

In this part, we focus on the patterns p of length one, that is $p \in \{U, D, L\}$. Table 1 presents the first few values for the number of p-equivalence classes. We do not succeed to solve the case D which is left as an open question (for this case, values in Table 1 are experimentally obtained).

Pattern	Sequence	OEIS([13])	a_n, $1 \leq n \leq 9$
U	Catalan	A000108	1, 2, 5, 14, 42, 132, 429, 1430, 4862
L		New	1, 2, 4, 9, 21, 50, 123, 308, 781
D	Open problem	New	1, 3, 10, 35, 129, 488, 1881, 7341, 28876

Table 1: Number of p-equivalence classes for skew Dyck paths.

2.1. The Pattern U

The number of U-equivalence classes is given by the Catalan numbers $c_n = \frac{1}{n+1}\binom{2n}{n}$ since we can establish a bijection between Dyck paths and the set of U-equivalence classes of skew Dyck paths. Indeed, each equivalence class of skew Dyck paths of length $2n$ can be represented by a word of length $2n$ using the symbols `1` and `0`. The symbol `1` represents a diagonal up step U, and the symbol `0` represents an absence of the step U, or a down step D in a Dyck path. For example, Figure 2 shows the case for the paths of semilength 2.

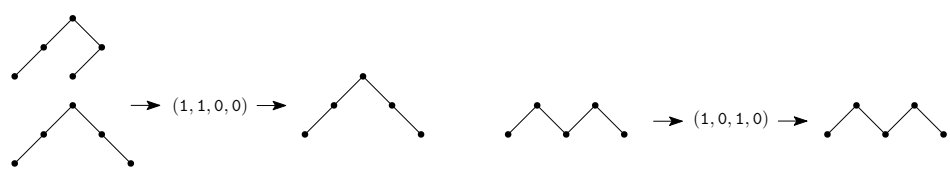


Figure 2: Bijection between Dyck paths and U-equivalence classes of $\mathcal{SD}$.

2.2. The Pattern L

In order to study the equivalence classes modulo L, we define the subclass $\mathcal{L}$ of skew Dyck paths avoiding the patterns UDD and DDD, and where all occurrences of UDU and DDU are at height 0.

Theorem 1. *There is a bijection between $\mathcal{L}$ and the set of L-equivalence classes of $\mathcal{SD}$.*

Proof. First, we will prove that for every $P \in \mathcal{SD}$ there exists $P' \in \mathcal{L}$ such that P and P' belong to the same equivalence class modulo left steps. Let us consider the

sequence of skew Dyck paths $P = P_0, P_1, \ldots, P_k = P'$ with $k \geq 1$, where for any i, $0 \leq i \leq k-1$, P_{i+1} is obtained from P_i by performing the first possible transformation among the four described below, until the path belongs to $\mathcal{L}$.

(1) Remove occurrences UDU at height greater than 0.

If P_i contains such a pattern, $P_i = \alpha UDU\beta$, then we define $P_{i+1} = \alpha DUU\beta$.

Notice that if P_{i+1} avoids UDU at height $h > 0$ and contains an occurrence of DD at height $k > 0$, then before it, there is necessarily an occurrence UUU at height $k-1$.

(2) Remove occurrences DDU at height greater than 0.

If P_i contains such a pattern, then $P_i = \alpha DDU\beta$ where α contains the pattern UUU. Considering the rightmost pattern UUU in α, we have $P_i = \alpha_1 UUU\alpha_2 DDU\beta$ and we define $P_{i+1} = \alpha_1 UDU\alpha_2 DUU\beta$.

(3) Remove occurrences UDD.

If P_i contains such a pattern, $P_i = \alpha UDD\beta$, then we define $P_{i+1} = \alpha DUD\beta$.

(4) Remove occurrences DDD.

If P_i contains such a pattern, then $P_i = \alpha DDD\beta$ where α has the pattern UUU. Considering the rightmost UUU in α, we have $P_i = \alpha_1 UUU\alpha_2 DDD\beta$ and we define $P_{i+1} = \alpha_1 UDU\alpha_2 DUD\beta$.

Since the process do not modify the positions of the left steps L, the paths P and P' belong to the same equivalence class. An example of this process is shown in Figure 3.

Now, let us prove that if P and P' with the same length ℓ both belong to $\mathcal{L}$ and are in the same equivalence class modulo L, then $P = P'$. Any $P \in \mathcal{L}$ can be decomposed

$$P = \left(\prod_{i=0}^{n-1} \alpha_i L^{k_i}\right) \alpha_n,$$

where L does not belong in α_i, $0 \leq i \leq n$, and $k_i \geq 1$ for $0 \leq i \leq n-1$.

First, if P does not contain L, then P is a Dyck path, and as $P \in \mathcal{L}$, $P = (UD)^{\ell/2}$. With a similar argument for P', we obtain directly $P = P'$.

The second case is when P and P' have at least one occurrence of L. Since P belongs to $\mathcal{L}$, we can determine the form of α_i.

- Case $i = 0$. We have $\alpha_0 = (UD)^{s_1} U^{s_2} D$ with $s_1 \geq 0$ and $s_2 \geq 2$.
- Case $1 \leq i < n$. The endpoint of $\alpha_{i-1} L^{k_{i-1}}$ must be at the height $h \geq 1$.
  - If $h = 1$, $\alpha_i = D(UD)^{t_1} U^{t_2} D$ with $t_1 \geq 0$ and $t_2 \geq 2$.

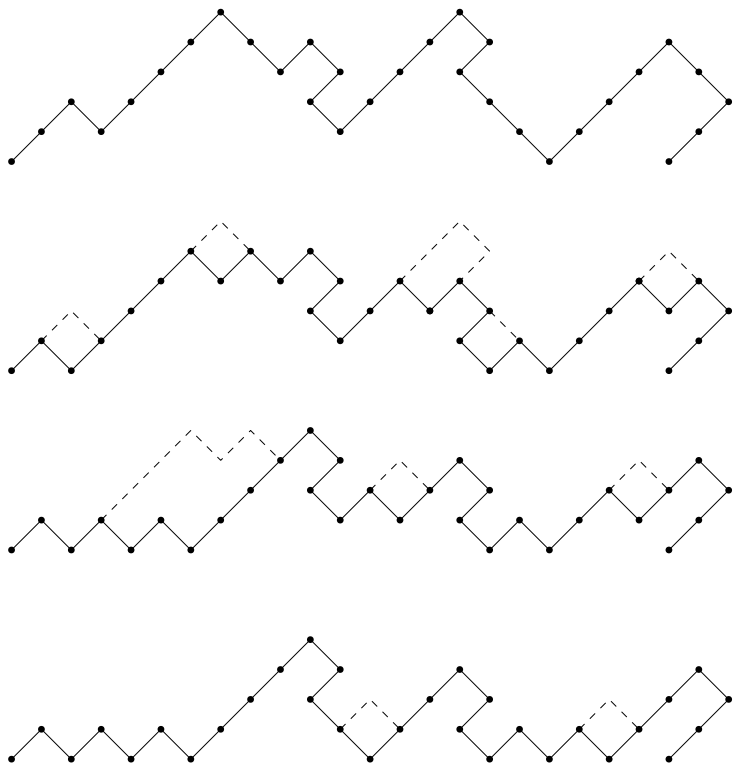


Figure 3: Construction of a path in $\mathcal{L}$ from a skew Dyck path.

- If $h = 2$, α_i is either D, $D^2(UD)^{t_3}U^{t_4}D$, or $DU^{t_5}D$ with $t_3 \geq 0$, $t_4 \geq 2$, $t_5 \geq 1$.
- If $h \geq 3$, α_i is either D, D^2, or DU^tD with $t \geq 1$.

- Case $i = n$. The endpoint of $\alpha_{n-1}L^{k_{n-1}}$ must be at height $h = 0, 1, 2$.
  - If $h = 0$, $\alpha_n = \lambda$.
  - If $h = 1$, $\alpha_n = D(UD)^{r_1}$ with $r_1 \geq 0$.
  - If $h = 2$, $\alpha_n = D^2(UD)^{r_2}$ with $r_2 \geq 0$.

Now, let us suppose that $P \neq P'$. Since P and P' are in the same class, we have $P' = \left(\prod_{i=0}^{n-1} \alpha'_i L^{k_i}\right) \alpha'_n$ with $|\alpha_j| = |\alpha'_j|$, and there exists j such that $\alpha_j \neq \alpha'_j$. We take the greatest j satisfying this condition.

Let us assume $j = n$. If $r = |\alpha_n| = |\alpha'_n|$ is even, then we have either $\alpha_n = \alpha'_n = \lambda$ or $\alpha_n = \alpha'_n = D^2(UD)^{\frac{r-2}{2}}$; if $r = |\alpha_n| = |\alpha'_n|$ is odd, then we have $\alpha_n = \alpha'_n = D(UD)^{\frac{r-1}{2}}$ which gives a contradiction with $P \neq P'$.

Let us assume $1 \le j < n$. In this case, the endpoints of α_j and α'_j are at the same height h. Since $P \neq P'$ and using the form of α_j defined above, we necessarily have $|\alpha_j| \ge 3$. So, let us suppose that α_j and α'_j are of the form DU^tD or $D(UD)^{t_1}U^{t_2}D$.

Let analyze the following two cases:

- $\alpha_j = DU^{t_1}D$ and $\alpha'_j = DU^{t_2}D$. Since $|\alpha_j| = |\alpha'_j|$, we have $\alpha_j = \alpha'_j$ which is a contradiction.
- $\alpha_j = DU^tD$ and $\alpha'_j = D(UD)^{t_1}U^{t_2}D$ with $t_1, t_2 \ge 1$. Due to the fact that they have the same length, we conclude that $t = 2t_1 + t_2$, and due to the fact that they belong to $\mathcal{L}$, we conclude that $t_2 = h + 1$, as the occurrence UDU must appear at height 0. As a result, $t > h + 1$ and consequently, the path α_j is not well defined because it crosses the x-axis. So, this case throws a contradiction.
- $\alpha_j = D(UD)^{t_1}U^{t_2}D$ and $\alpha'_j = D(UD)^{s_1}U^{s_2}D$ with $t_1, s_1 \ge 1$ $t_2, s_2 \ge 2$. As they have the same length, $2t_1 + t_2 = 2s_1 + s_2$. If $\alpha_j \neq \alpha'_j$ then without loss of generality we can suppose $t_1 < s_1$ and conclude that $t_2 \ge 2 + s_2$. This establishes a contradiction because α_j would have a pattern UDU at height greater than 2.

A similar reasoning allows us to conclude that $\alpha_0 = \alpha'_0$ and therefore, $P = P'$. □

Theorem 2. *The generating function of equivalence classes modulo L is given by $L(x)$, where $L(x)$ is a root of*

$$(4x-1)L^4(x) - 3L^3(x) - (7x-10)L^2(x) + (5x-8)L(x) - x + 2 = 0.$$

The series expansion of $L(x)$ is

$$1 + x + 2x^2 + 4x^3 + 9x^4 + 21x^5 + 50x^6 + 123x^7 + 308x^8 + 781x^9 + 2008x^{10} + O(x^{11}).$$

Proof. Let us define the following subsets of $\mathcal{SD}$:

- $\mathcal{A}$ is the subset of paths in $\mathcal{SD}$ avoiding UDU, DDU, UDD, and DDD, and not ending with an occurrence of UD or DD;
- $\mathcal{B}$ is the subset of paths in $\mathcal{SD}$ avoiding UDU, DDU, UDD, and DDD;
- $\mathcal{C}$ is the subset of paths in $\mathcal{SD}$ avoiding UDU, DDU, UDD, and DDD, and not ending with an occurrence of D.

In order to find the generating function of $\mathcal{L}$ we consider the first return decomposition of a path $P \in \mathcal{L}$: either P is empty, or $P = U\alpha D\beta$, or $P = U\gamma L$, with $\alpha \in \mathcal{A}$, $\beta \in \mathcal{L}$ and $\gamma \in \mathcal{B}\backslash\{\lambda\}$. Consequently, if $L(x), A(x), B(x)$, and $C(x)$ are respectively the generating functions for the sets $\mathcal{L}, \mathcal{A}, \mathcal{B}$, and $\mathcal{C}$, we obtain the functional equation (cf. [8])

$$L(x) = 1 + xA(x)L(x) + x(B(x) - 1).$$

A nonempty path $P \in \mathcal{A}$ is either $P = U\alpha D\beta$ or $P = U\gamma L$ with $\alpha \in \mathcal{C}\backslash\{\lambda\}$, $\beta \in \mathcal{A}$ and $\gamma \in \mathcal{B}\backslash\{\lambda\}$. Therefore, we have

$$A(x) = 1 + x(C(x) - 1)A(x) + x(B(x) - 1).$$

A nonempty path $P \in \mathcal{B}$ is either $P = UD$, or $P = U\alpha D\beta$ or $P = U\alpha' D$, or $P = U\gamma L$, with $\alpha \in \mathcal{C}\backslash\{\lambda\}$, β, $\gamma \in \mathcal{B}\backslash\{\lambda\}$, and $\alpha' \in \mathcal{A}\backslash\{\lambda\}$. Therefore we have

$$B(x) = 1 + x + x(C(x) - 1)(B(x) - 1) + x(A(x) - 1) + x(B(x) - 1).$$

A nonempty path $P \in \mathcal{C}$ is either $P = U\alpha D\beta$, or $P = U\gamma L$ with $\alpha, \beta \in \mathcal{C}\backslash\{\lambda\}$ and $\gamma \in \mathcal{B}\backslash\{\lambda\}$. Therefore we have

$$C(x) = 1 + x(C(x) - 1)^2 + x(B(x) - 1).$$

Using Gröbner basis on the polynomial equations for $L(x), A(x), B(x)$, and $C(x)$ we obtain the desired result. □

Remark 1. Since the generating function of equivalence classes modulo L satisfies an algebraic equation of order four, the counting sequence $a_n := [x^n]L[x]$ satisfies a recurrence relation with polynomial coefficients. This can be automatically solved with Kauers's algorithm [9]. In particular we obtain that a_n satisfies the recurrence relation

$$p_0(n)a_n + p_1(n)a_{n+1} + p_2(n)a_{n+2} + p_3(n)a_{n+3} + p_4(n)a_{n+4} + p_5(n)a_{n+5} + p_6(n)a_{n+6} = 0,$$

for $n \geq 6$, where $p_i(n)$ $(i = 0, 1, \ldots, 6)$ are polynomials in n. From the package `Asymptotics` for Mathematica, see [10], we conjecture that

$$a_n \sim c \cdot \frac{\left(\frac{3}{2} + \sqrt{2}\right)^n}{n^{3/2}},$$

where $c \approx 2.111031048$.

2.3. Equivalence Classes of Size One

In the previous section, we proved that equivalence classes of size one are in one-to-one correspondences with the set $\mathcal{B}$, which means that every skew Dyck path in $\mathcal{B}$ is entirely fixed by the position of its L steps. Consequently, the number of skew Dyck paths in $\mathcal{B}$ with exactly k occurrences of L is finite. In this section, we study the number of skew Dyck paths in $\mathcal{B}$ having exactly n left steps L. We take two points of view: first, we provide an expression of the generating function $B(y) = \sum_{n\geq 0} b_n y^n$, where b_n is the number of skew Dyck paths in $\mathcal{B}$ with n left steps L, and next we provide a recursive formula for b_n.

First, it is worth noticing that as a skew path in $\mathcal{B}$ avoids UDU, DDU, UDD, and DDD, the last left step of a path must be at the last four positions. Consequently, as shown in Figure 4, there are four paths with one left step.

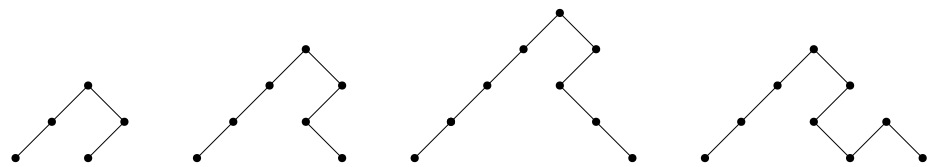


Figure 4: Skew Dyck paths in $\mathcal{B}$ with one occurrence of L.

2.3.1. Using Generating Functions

Theorem 3. *The generating function $B(y)$ for the number of skew Dyck paths in $\mathcal{B}$ with respect to the number of left steps L is a root of*

$$y^2B^4(y) + (y^3 + y^2)B^3(y) + (2y^2 + y)B^2(y) + (3y - 1)B(y) + 1 = 0.$$

The series expansion of $B(y)$ is

$$1+4y+24y^2+181y^3+1549y^4+14312y^5+139142y^6+1402646y^7+14527909y^8+O(x^9).$$

Proof. As mentioned above, a nonempty skew Dyck path in $\mathcal{B}\backslash\{UD\}$ ends with either L, LD, LDD, or $LDUD$. Let $B_1(y)$ (resp. $B_2(y)$, $B_3(y)$, $B_4(y)$) be the generating function for the number of skew Dyck paths in $\mathcal{B}$ ending with L (resp. LD, LDD, $LDUD$).

A skew Dyck path P ending with L can be written $P = \alpha U\beta L$ where $\alpha \in \mathcal{B}$ is either empty or ends with LD and $\beta \in \mathcal{B}$. So, we have the functional equation

$$B_1(y) = (1 + B_2(y))y(1 + B_1(y) + B_2(y) + B_3(y) + B_4(y)).$$

A skew Dyck path P ending with LD can be written $P = \alpha U\beta D$ where $\alpha \in \mathcal{B}$ is either empty or ends with LD and $\beta \in \mathcal{B}$ and ends with L. So, we have

$$B_2(y) = (1 + B_2(y))B_1(y).$$

A skew Dyck path P ending with LDD can be written $P = \alpha U\beta D$ where $\alpha \in \mathcal{B}$ is either empty or ends with LD and $\beta \in \mathcal{B}$ and ends with LD. So, we have

$$B_3(y) = (1 + B_2(y))B_2(y).$$

A skew Dyck path P ending with $LDUD$ can be written $P = \alpha UD$ where $\alpha \in \mathcal{B}$ ends with LD. So, we have

$$B_4(y) = B_2(y).$$

Using Gröbner basis on the polynomial equations for each generating function we obtain the desired result. □

Remark 2. Let b_n be the n-th coefficient of $B[y]$, that is, $b_n := [y^n]B[y]$. From a similar approach as in the Remark 1, we conjecture that

$$b_n \sim c \cdot \frac{\alpha^n}{n^{3/2}},$$

where $c \approx 0.6011640677$ and $\alpha = 0.650096$.

2.3.2. Using Recurrence Relation

Theorem 4. *Let $a_i(n)$ denote a family of sequences where $a_0(n) = n$, and for all $i \geq 1$*

$$a_i(n) = a_{i-1}(n) + a_{i-1}(n+1) + \sum_{k=0}^{n-i-3} a_{i-1}(n+2-k).$$

The number of skew Dyck paths in $\mathcal{B}$ with exactly $k \geq 2$ left steps, is given by $a_{k-1}(k+3)$.

Proof. Let us denote by $\mathcal{B}_l$ the set of all prefixes ending with L of skew Dyck paths in $\mathcal{B}$. Such a path will be called a *meander*.

Let us prove by induction on i that $a_i(n)$ is the number of meanders in $\mathcal{B}_l$ with $i+2$ occurrences of L and ending with an occurrence of L at height $n-i-5$.

First, let us assume that $i = 0$. A meander in $\mathcal{B}_l$ ending at height $n-5$, with two occurrences of L, ends with either LL, LDL, $LDDL$, or LDU^kDL, $1 \leq k \leq n-3$. Therefore, there are $n-3+3 = a_0(n)$ paths. Figure 5 shows the case when $n = 5$.

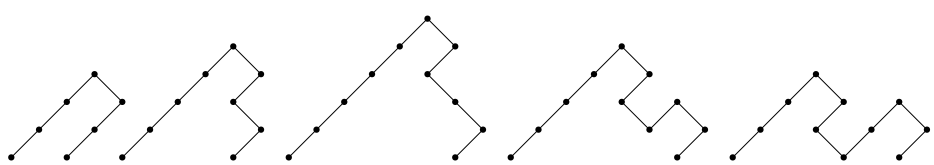


Figure 5: Skew Dyck paths in $\mathcal{B}$ with 2 occurrences of L and $s_1 = 1$

Now, assume that $a_j(n)$ satisfies the statement for $j \leq i$. Let us count the number $a_{i+1}(n)$ of meanders in $\mathcal{B}_l$ ending at height $n-i-6$, with $i+3$ occurrences of L. Taking into account all possible ways of a meander in $\mathcal{B}_l$ ends:

- there are $a_i(n)$ such paths ending with LL,
- there are $a_i(n+1)$ such paths ending with LDL,
- there are $a_i(n+2)$ such paths ending with $LDDL$,
- there are $a_i(n+2-k)$ paths ending with LDU^kDL, $1 \leq k \leq n-i-4$.

Consequently, we have:

$$a_{i+1}(n) = a_i(n) + a_i(n+1) + a_i(n+2) + \sum_{k=1}^{n-i-4} a_i(n+2-k)$$

which completes the induction.

Finally, as skew Dyck paths in $\mathcal{B}\backslash\{UD\}$ end with L, LD, LDD, or $LDUD$, this implies that the number of skew Dyck paths in $\mathcal{B}$ with exactly $k \geq 2$ left steps is given by

$$a_{k-1}(k+3) = a_{k-2}(k+3) + a_{k-2}(k+4) + a_{k-2}(k+5) + a_{k-2}(k+4). \quad \square$$

3. Equivalence Classes Modulo Patterns of Length Two

In this section, we focus on equivalence classes modulo patterns of length two. We start by giving a general result that allows us to solve the cases of patterns that do not contain occurrences of L and DD. Indeed, for these patterns the number of p-equivalence classes on the set $\mathcal{SD}$ of skew Dyck paths also is the same on the set $\mathcal{D}$ of Dyck paths, which is already given in [1]. We also deal with the pattern DD and leave as an open question the cases of patterns in $\{DL, LD, LL\}$. We refer to Table 2 for an overview of these numbers for small values of the length (the last three cases are obtained experimentally).

Pattern	Sequence	OEIS	a_n, $1 \leq n \leq 9$
UU	$\frac{1-x+\sqrt{1-2x-3x^2}}{1-3x+x^2+x^3+(1-x^2)\sqrt{1-2x-3x^2}}$	A244886	1, 2, 4, 9, 22, 56, 147, 393, 1065
UD	$\frac{(1-x)(1-5x+7x^2-x^3)}{(1-2x)^2(1-3x+x^2)}$	A244885	1, 2, 5, 14, 41, 121, 354, 1021, 2901
DU	$\frac{1-2x}{1-3x+x^2}$	A001519	1, 2, 5, 13, 34, 89, 233, 610, 1597
DD	$\frac{2(1+x)}{x+x^2+(2+x)\sqrt{1-2x-3x^2}}$	New	1, 2, 5, 12, 31, 81, 216, 583, 1590
DL	Open question	New	1, 2, 3, 6, 12, 23, 49, 102, 212
LD	Open question	New	1, 1, 2, 4, 7, 15, 31, 62, 136
LL	Open question	New	1, 1, 2, 4, 8, 15, 30, 63, 134

Table 2: Number of p-equivalence classes for skew Dyck paths.

Proposition 1. *If p is a pattern of length at least 2 avoiding L and DD, then the number of p-equivalence classes is the same in $\mathcal{SD}$ and $\mathcal{D}$.*

Proof. Let p be a pattern of length at least 2 avoiding L and DD, and let us consider $P \in \mathcal{SD}$. We will show that there exists a path $P' \in \mathcal{D}$ such that P and P' belong to the same equivalence class.

We can decompose $P = \alpha_0 \prod_{i=1}^{n} p^{d_i} \alpha_i$ or $P = \alpha_0$, where α_i is a sub-skew Dyck path avoiding p and $d_i \geq 1$. We can define $P' = \alpha'_0 \prod_{i=1}^{n} p^{d_i} \alpha'_i$ or $P' = \alpha'_0$, depending on the decomposition of P, where each α'_i is obtained from α_i by replacing all steps L with D. As p and α'_i avoid L, $P' \in \mathcal{D}$. Moreover, in a skew Dyck path, L cannot be contiguous with a step U, which implies that the operation of changing a step L by D does not create a pattern p whenever p avoids DD. Consequently, P and P' belong to the same p-equivalence class. □

3.1. Pattern DD

Let $\mathcal{E}$ denote the set of skew Dyck paths where all occurrences of UDU are at height 0 or 1, all occurrences of UD^kL, with $k \geq 1$, are at height 0, and the patterns $UUDU$ and $UUDL$ do not appear. For instance, Figure 6 shows two skew Dyck paths that do not belong to $\mathcal{E}$, whereas Figure 7 shows a skew Dyck path that belongs to $\mathcal{E}$.

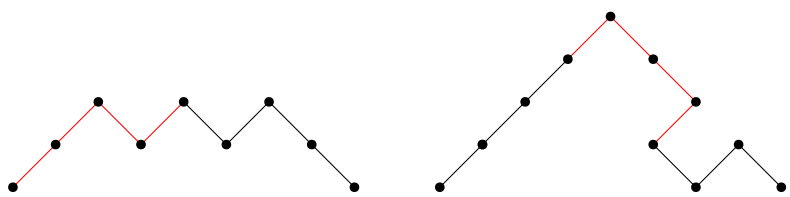


Figure 6: Skew Dyck paths that do not belong to $\mathcal{E}$.

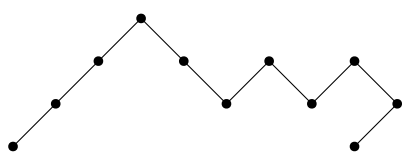


Figure 7: Skew Dyck path that belongs to $\mathcal{E}$.

Lemma 1. *For every skew Dyck path P, there exists a skew Dyck path $P' \in \mathcal{E}$ in the same equivalence class modulo DD.*

Proof. Let P be a skew Dyck path such that $P \notin \mathcal{E}$. Consider the sequence of skew Dyck paths $P = P_0, P_1, P_2, \ldots, P_{k-1}, P_k = P'$, $k \geq 1$, defined as follows:

For any i, $1 \leq i \leq k$, the skew Dyck path P_{i+1} is obtained from P_i by performing the first possible transformation among the four described below, until the path belongs to $\mathcal{E}$:

(1) Remove occurrences of $UUDU$.

If P_i has such a pattern, then $P_i = \alpha U^k DU\beta$ with $k \geq 2$ and α does not end with U. So, we define $P_{i+1} = \alpha UDU^k\beta$. Figure 8 shows a step of this process. Repeat this operation until the path does not contain $UUDU$.

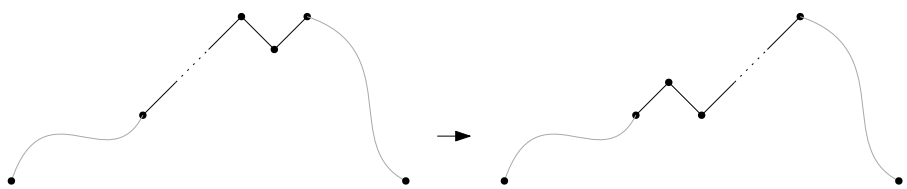


Figure 8: Removing $UUDU$.

(2) Remove occurrences $UUDL$ at height 0.

If P_i has such a pattern, $P_i = \alpha UUDL$, where α is a skew Dyck path, then we define $P_{i+1} = \alpha UDUD$. Figure 9 shows this process.

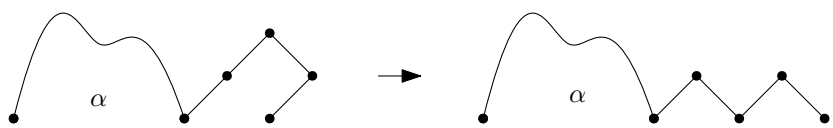


Figure 9: Removing $UUDL$ at height 0.

(3) Remove maximal occurrences of UD^kL^m, for $k, m \geq 1$, except when the occurrence is at height 0 with $m = 1$.

Since m is maximal, the chain L^m is followed by an occurrence of D^s for $s \geq 0$. We write $P_i = (\prod_{i=1}^{d-1} U^{s_i}\alpha_i)U^{s_d}D^kL^m\beta$ where α_i is nonempty and avoids U, and β is either empty or starts with D^s with $s \geq 1$. We consider three cases depending if $m \geq 3$, $m = 2$ and $m = 1$.

- Case $m \geq 3$. The maximum ordinate reached by the occurrence UD^kL^m is $p \geq m + k \geq 4$. Since the path reaches the ordinate $p \geq 4$, and since P_i avoids $UUDU$, there is at least one s_i such that $s_i \geq 3$; we choose the rightmost i.

  The path P_{i+1} is obtained by replacing U^{s_i} with $UDUU^{s_i-3}$, and by replacing UD^kL^m with UD^kUDL^{m-2}. This operation deletes the occurrence UD^kL^m and creates another occurrence UDL with maximum ordinate $p - 3$. See Figure 10.

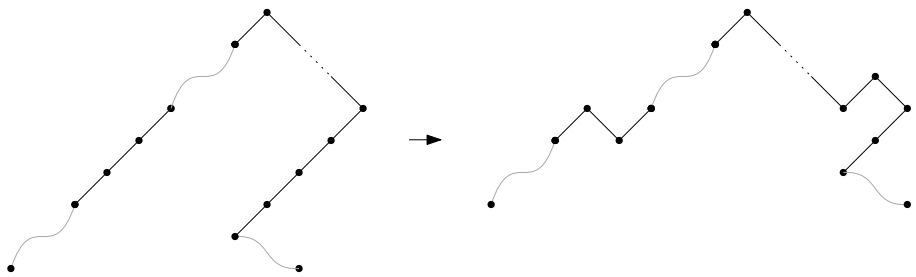


Figure 10: Removing UD^kL^m if $m \geq 3$.

- Case $m = 1$. The maximal ordinate reached by the occurrence $UD^kL^m = UD^kL$ is at least three, and (as above) there is some s_i such that $s_i \geq 3$; we choose the rightmost i.

  The path P_{i+1} is obtained by replacing U^{s_i} with $UDUU^{s_i-3}$, and by replacing $UD^kL^m = UD^kL$ with UD^kU. This operation deletes the occurrence UD^kL^m. See Figure 11.

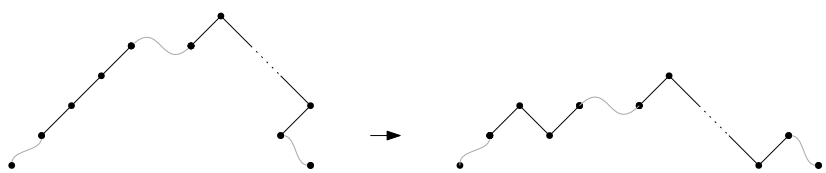


Figure 11: Removing UD^kL^m if $m = 1$.

- Case $m = 2$. The maximal ordinate p reached by the occurrence $UD^kL^m = UD^kL^2$ is at least three, and (as above) there is some s_i such that $s_i \geq 3$; we choose the rightmost i. Moreover, if $p \geq 5$ then either there is $s_i \geq 5$ or there are two indices i_0, i_1, such that $3 \leq i_0 \leq 4$ and $3 \leq i_1 \leq 4$; we choose the rightmost indices with these properties.

  If $s = 0$ then UD^kLL is at the end of P_i, and P_{i+1} is obtained by replacing UD^kLL with UD^kUD, and by replacing U^{s_i} with $UDUU^{s_i-3}$.

  Now, let us consider $s \geq 1$.

  - If UD^kLL is followed by D and P_i ends after D, then P_{i+1} is obtained by replacing UD^kLLD with UD^kUDL and by replacing the rightmost U^{s_i} for $s_i \geq 3$ with $UDUU^{s_i-3}$.
  - If UD^kLL is followed by DUU then we replace UD^kLLDU with UD^kUDUD and we replace the rightmost U^{s_i} for $s_i \geq 3$ with $UDUU^{s_i-3}$.

- If UD^kLL is followed by DUD then we replace UD^kLLDUD with UD^kUUDLD and we replace the rightmost U^{s_i} for $s_i \geq 3$ with $UDUU^{s_i-3}$.
- If UD^kLL is followed by DL then we replace UD^kLLDL with UD^kUUDL and either we replace the rightmost U^{s_i} for $s_i \geq 5$ with $UDUDUU^{s_i-5}$, or we replace $U^{s_{i_0}}$ with $UDUU^{s_{i_0}-3}$ and $U^{s_{i_1}}$ with $UDUU^{s_{i_1}-3}$ where i_0 and i_1 are defined above.
- If UD^kLL is followed by DD then we replace UD^kLLDD with UD^kUUDD and either we replace the rightmost U^{s_i} for $s_i \geq 5$ with $UDUDUU^{s_i-5}$, or we replace $U^{s_{i_0}}$ with $UDUU^{s_{i_0}-3}$ and $U^{s_{i_1}}$ with $UDUU^{s_{i_1}-3}$ where i_0 and i_1 are defined above.

All previous transformations either delete an occurrence UD^kL at height at least one, or decreases by at least one the maximal ordinate of one occurrence UD^kL, or decrease by one m in a subcase of $m = 2$. See Figure 12.

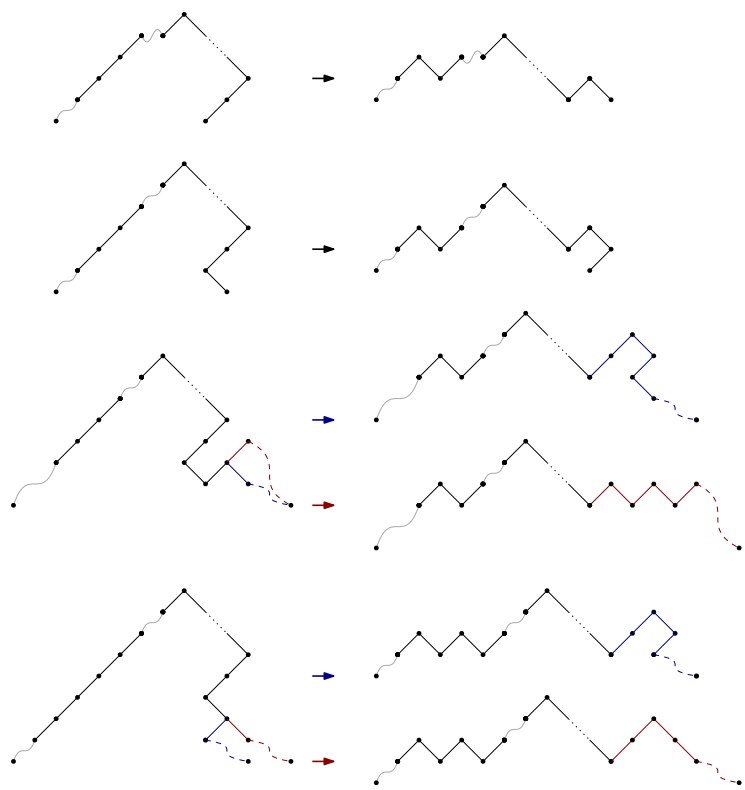


Figure 12: Removing UD^kL^m if $m = 2$.

(4) Remove UDU at height greater than 1. As the maximal ordinate p reached by the occurrence UDU satisfies $p \geq 3$, and as P_i avoids $UUDU$, there exists an occurrence U^3 at height $p-3$ at the left (we take the rightmost possible). The path P_{i+1} is obtained by exchanging the two occurrences U^3 and UDU, which decreases by at least one the height of the occurrence UDU.

After applying the previous process, we obtain a path $P' \in \mathcal{E}$. Since all transformations do not change the positions of occurrences DD, P and P' belong to the same equivalence class. An example of this process is shown in Figure 13.

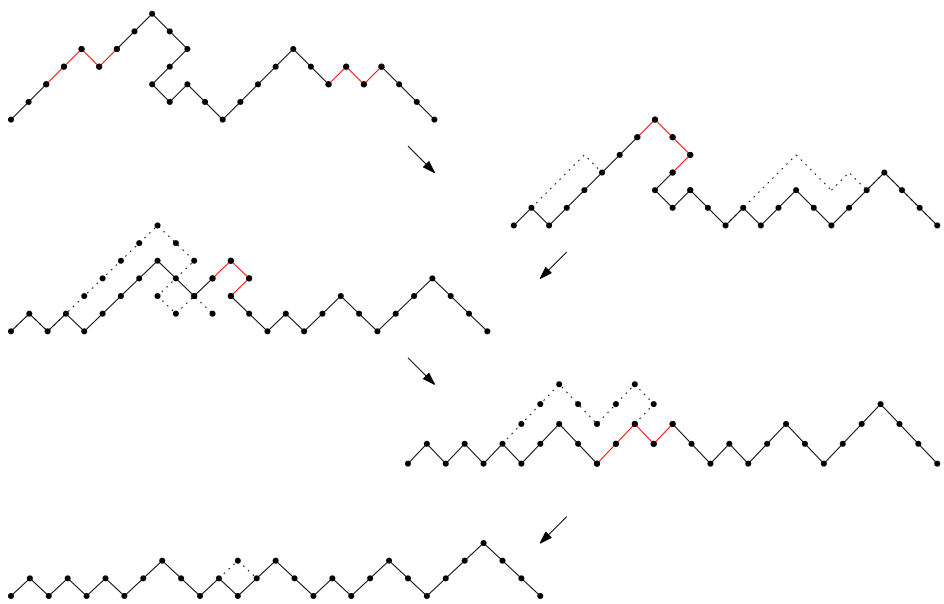


Figure 13: An example of the process described in the proof of Lemma 3.2

□

Theorem 5. *There is a bijection between $\mathcal{E}$ and the set of DD-equivalence classes of $\mathcal{SD}$.*

Proof. Considering Lemma 3.2, it suffices to prove that if P and P' have the same length in $\mathcal{E}$ and lie in the same equivalence class, then $P = P'$. We decompose $P = \alpha_0 \prod_{i=1}^{n} D^{k_{i-1}} \alpha_i$ (resp. $P' = \alpha'_0 \prod_{i=1}^{n} D^{k_{i-1}} \alpha'_i$) where α_i (resp. α'_i) do not contain the pattern DD and $k_i \geq 2$ (resp. $k'_i \geq 2$) are taken to be maximal.

First, if P and P' do not have the pattern DD, then $P = \alpha_0$ and $P' = \alpha'_0$. Moreover $P = \alpha_0$ avoids $UUDU$ and $UUDL$, which implies that $\alpha_0 = (UD)^m$ with

$m = |P|/2$. By a similar argument, we obtain $P = P'$.

Secondly, let us assume that P and P' have at least one occurrence of DD. Since P and P' belong to the same equivalence class, we have $|\alpha_i| = |\alpha'_i|$ for $0 \le i \le n$. Notice that we necessarily have $n \ge 1$. Now, we determine the form of α_i and α'_i.

- α_i and α'_i for $0 \le i \le n-1$.

  Since α_0 (resp. α'_0) avoids DD, $UUDU$, and $UUDL$, we can write $\alpha_0 = (UD)^{s_1}U^{s_2}$ with $s_2 \ge 2$ and $s_1 \ge 0$ (resp. $\alpha'_0 = (UD)^{s'_1}U^{s'_2}$ with $s'_2 \ge 2$ and $s'_1 \ge 0$). If $n > 1$, α_1 cannot starts with L (otherwise P contains UD^kL), thus it starts with U, and with the same argument as above it has the same form as α_0, i.e. $\alpha_1 = (UD)^{t_1}U^{t_2}$ with $t_2 \ge 2$ and $t_1 \ge 0$. Repeating this argument, α_i and α'_i are all of the same form for $1 \le i \le n-1$.

- α_n and α'_n.

  We have three cases depending on the final ordinate of

$$Q = \alpha_0 \left(\prod_{i=1}^{n-1} D^{k_{i-1}} \alpha_i \right) D^{k_{n-1}}.$$

  Case 1. The path Q ends at height 0. Since P avoids $UUDU$ and $UUDL$, α_n does not contain an occurrence UU. Therefore α_n is either the empty path λ or of the form $(UD)^s$.

  Case 2. The path Q ends at height 1. The only one possibility is $\alpha_n = (UD)^sL$.

  Case 3. The path Q ends at height greater than 1. This case is not possible because α_n does not start with D, avoids DD and avoids UD^kL at height greater than 0.

Now let us prove that that $\alpha_i = \alpha'_i$ for every i. With the reasoning done above, $\alpha_n \in \{\lambda, (UD)^sL, (UD)^s\}$ and $\alpha'_n \in \{\lambda, (UD)^tL, (UD)^t\}$. Since $|\alpha_n| = |\alpha'_n|$, we have $\alpha_n = \alpha'_n$.

For a contradiction we suppose that there exists $i < n$, such that $\alpha_i \ne \alpha'_i$ (we take the greatest j satisfying this condition). With the reasoning above, we have $\alpha_i = (UD)^{s_1}U^{s_2}$ and $\alpha'_i = (UD)^{s'_1}U^{s'_2}$ with $2s_1 + s_2 = 2s'_1 + s'_2$ since $|\alpha_i| = |\alpha'_i|$. Without loss of generality we can assume $s_1 < s'_1$ because α_i and α'_i are different. This implies that $s_2 \ge 2 + s'_2$. Since α_i and α'_i end at the same height in P and P', this means that P' contains an occurrence UDU at height at least 2 which is a contradiction.

In summary, $\alpha_i = \alpha'_i$ for every i and consequently, $P = P'$. □

Before proving Theorem 1, we need the preliminary results shown in Lemmas 2 and 3. Let $\mathcal{F}$ be the set of all Dyck paths where all occurrences of UDU are at height 0 and not starting with UDU; let $\mathcal{G}$ be the set of Dyck paths that do not

contain UDU and, let $\mathcal{H}$ be the set of Dyck paths where all occurrences of UDU are at height 0. It is well known (see [14]) that the generating function of $\mathcal{G}$, $G(x)$, is given by the expression $G(x) = 1+xM(x)$, where $M(x)$ is the generating function for the number of Motzkin paths, i.e., $G(x) = (1 + x - \sqrt{1 - 2x - 3x^2})/2x$.

Lemma 2. *The generating function of the set $\mathcal{H}$ is given by the expression*

$$H(x) = \frac{1}{1 - xG(x)}.$$

Proof. A Dyck path in $\mathcal{H}$ is either empty or of the form $U\alpha D\beta$, with $\alpha \in \mathcal{G}$ and $\beta \in \mathcal{H}$. So, $H(x)$ satisfies the relation $H(x) = xG(x)H(x) + 1$ that induces the required result. □

Lemma 3. *The generating function of the set $\mathcal{F}$ is given by*

$$F(x) = \frac{x^2 + x - 2 - x\sqrt{-3x^2 - 2x + 1}}{x - 1 - \sqrt{-3x^2 - 2x + 1}}.$$

Proof. A Dyck path in $\mathcal{F}$ is either empty, or UD, or $U\alpha D\beta$ where $\alpha \in \mathcal{G}\backslash\{\lambda\}$ and $\beta \in \mathcal{H}$. We conclude that $F(x)$ satisfies the relation $F(x) = 1 + x + x(G(x) - 1)H(x)$ which gives the required result. □

Theorem 6. *The generating function of the set $\mathcal{E}$ is given by*

$$E(x) = \frac{2(1 + x)}{x + x^2 + (2 + x)\sqrt{1 - 2x - 3x^2}}.$$

The series expansion of $E(x)$ is

$$1 + x + 2x^2 + 5x^3 + 12x^4 + 31x^5 + 81x^6 + 216x^7 + 583x^8 + 1590x^9 + O\left(x^{10}\right).$$

Proof. A skew Dyck path in $\mathcal{E}$ is empty, is $U\alpha D\beta$, or is $U\gamma L$, where $\alpha \in \mathcal{F}$, $\beta \in \mathcal{E}$ and $\gamma \in \mathcal{F}\backslash\{\lambda, UD\}$. So, $E(x)$ satisfies the relation $E(x) = 1 + xF(x)E(x) + x(F(x) - 1 - x)$, which is equivalent to

$$E(x) = \frac{1 + x(F(x) - 1 - x)}{1 - xF(x)}.$$

□

Let e_n be the number of DD-equivalence classes for skew Dyck paths. That is, $e_n = [x^n]E(x)$ for all $n \geq 0$. In Theorem 7 we give an asymptotic approximation for the sequence e_n. To accomplish this goal we use the singularity analysis method for finding an asymptotic expression of the coefficients of a generating function (see for example [8] for the details).

Theorem 7. *The sequence e_n has the asymptotic approximation*

$$e_n \sim 21\sqrt{\frac{3}{4\pi n^3}} \cdot 3^n.$$

Proof. The dominant singularity of the generating function $E(x)$ is $1/3$, that is, the smallest positive root of $1 - 2x - 3x^2$. Around the point $1/3$ the expansion of $E(x)$ is given by

$$E(x) = 6 - 21\sqrt{3(1-3x)} + O(1-3x).$$

The singularity analysis allows the transfer of the above equality to the asymptotic approximation of the coefficients. □

References

[1] J.-L. Baril and A. Petrossian, Equivalence of Dyck paths modulo some statistics, *Discrete Math.* **338** (2015), 655–660.

[2] J.-L. Baril and A. Petrossian, Equivalence classes of Motzkin paths modulo a pattern of length at most two, *J. Integer Seq.* **18** (2015), 15.7.1.

[3] J.-L. Baril, S. Kirgizov, and A. Petrossian, Enumeration of Łukasiewicz paths modulo some patterns, *Discrete Math.* **342**(4) (2019), 997–1005.

[4] J.-L. Baril, J.L. Ramírez, L.M. Simbaqueba, Counting prefixes of skew Dyck paths, *J. Integer Seq.* **24** (2021), 21.8.2.

[5] E. Deutsch, Dyck path enumeration, *Discrete Math.* **204**(1999), 167–202.

[6] E. Deutsch, E. Munarini, and S. Rinaldi, Skew Dyck paths, *J. Statist. Plann. Inference* **140** (2010), 2191–2203.

[7] E. Deutsch, E. Munarini, and S. Rinaldi, Skew Dyck paths, area, and superdiagonal bargraphs, *J. Statist. Plann. Inference* **140** (2010), 1550–1526.

[8] P. Flajolet and R. Sedgewick, *Analytic Combinatorics*, Cambridge University Press, 2009.

[9] M. Kauers, Guessing handbook. Technical Report 09-07, RISC-Linz, 2009.

[10] M. Kauers, A Mathematica package for computing asymptotic expansions of solutions of P-finite recurrence equations. Technical Report RISC-Linz, 2011.

[11] Q. L. Lu, Skew Motzkin paths, *Acta Math. Sin. (Engl. Ser.)* **33** (2017), 657–667.

[12] K. Manes, A. Sapounakis, I. Tasoulas, and P. Tsikouras, Equivalence classes of ballot paths modulo strings of length 2 and 3, *Discrete Math.* **339**(10) (2016), 2557–2572.

[13] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences. Available at https://oeis.org/.

[14] Y. Sun, The statistic “number of udu’s" in Dyck paths, *Discrete Math.* **287** (2004), 177–186.

Theoretical Computer Science 927 (2022) 120–132

Contents lists available at ScienceDirect

Theoretical Computer Science

www.elsevier.com/locate/tcs

Gray codes for Fibonacci q-decreasing words

Jean-Luc Baril, Sergey Kirgizov, Vincent Vajnovszki *

LIB, Université de Bourgogne Franche-Comté, B.P. 47 870, 21078 Dijon Cedex, France

ARTICLE INFO

Article history:
Received 20 October 2020
Received in revised form 1 April 2022
Accepted 1 June 2022
Available online 6 June 2022
Communicated by M. Sciortino

Keywords:
(Generalized) Fibonacci words
Fibonacci cubes/graphs
Popularity/frequency
Exhaustive generation
Gray codes

ABSTRACT

A length n binary word is q-decreasing, $q \geqslant 1$, if every of its maximal factors of the form 0^a1^b satisfies $a = 0$ or $q \cdot a > b$. In particular, in 1-decreasing words every run of 0s is immediately followed by a strictly shorter run of 1s. We show constructively that these words are in bijection with binary words having no occurrences of 1^{q+1}, and thus they are enumerated by the $(q+1)$-generalized Fibonacci numbers. We give some enumerative results and reveal similarities between q-decreasing words and binary words having no occurrences of 1^{q+1} in terms of the frequency of 1-bits. In the second part of our paper, we provide an efficient exhaustive generating algorithm for q-decreasing words in lexicographic order, for any $q \geqslant 1$, show the existence of 3-Gray codes and explain how a generating algorithm for these Gray codes can be obtained. Moreover, we give the construction of a more restrictive 1-Gray code for 1-decreasing words, which in particular settles a conjecture stated recently in the context of interconnection networks by Eğecioğlu and Iršič.

© 2022 Elsevier B.V. All rights reserved.

1. Introduction and preliminaries

The Fibonacci sequence origins have been traced back to the works of ancient Indian mathematician Ācārya Piṅgala dealing with rhythmic structure patterns in Sanskrit poetry [25], [16, p. 50]. Over time, the study of words and patterns became more abstract and systematic (see for instance Lothaire's books [17–19] and [7]). A considerable amount of questions concerning efficient enumeration and generation of words respecting certain properties (including pattern avoidance) were mathematically formulated and answered only relatively recently, the works closely related to the present study include [3,5,9,10,12,27–29].

In this paper we introduce q-decreasing words, a novel class of run-restricted binary words enumerated by the $(q+1)$-generalized Fibonacci numbers, $q \geqslant 1$. For $q = 1$ the subclass of such words that start with 0 was recently considered in the context of induced subgraphs of hypercubes [9,10]. The q-decreasing words could be interesting objects to study in other domains, for example in stringology, the study of string algorithms [8]. In Section 2 we present a bijection between this novel class of words and Fibonacci words, i.e., binary words avoiding consecutive 1s. Section 3 is devoted to the presentation of several generating functions and enumeration results. Finally, in Section 4, we show the existence of a 3-Gray code for any $q \geqslant 1$, give an efficient exhaustive generating algorithms and a much more involved construction for a 1-Gray code in the special case $q = 1$. In particular, the latter Gray code gives a Hamiltonian path in Fibonacci-run graphs whose existence is conjectured in [9].

* Corresponding author.
E-mail addresses: barjl@u-bourgogne.fr (J.-L. Baril), sergey.kirgizov@u-bourgogne.fr (S. Kirgizov), vvajnov@u-bourgogne.fr (V. Vajnovszki).

https://doi.org/10.1016/j.tcs.2022.06.003
0304-3975/© 2022 Elsevier B.V. All rights reserved.

The following set of notations is adopted. Let $\mathcal{B}$ denote the set of all finite-length binary words, i.e., strings over the alphabet $\{0, 1\}$, and $\mathcal{B}_n$, $n \geqslant 0$, be the set of all binary words of length n. For a given binary word w we use the notation w_i to mean the letter at position i.

A nonempty sequence of adjacent letters inside a word is called *factor*. A factor v repeated k times is denoted by v^k, for instance $(00)^2 1^2 = 000011$. For a given length n, the notation v^* is used to repeat factor v as many times as possible, until the length n is reached, possibly trimming extra symbols at the end; and the length n will be understood from the context. For example, if a word v of length $n = 7$ is equal to $(001)^*$ it means $v = 0010010$.

The set of all n-length binary words containing no occurrences of factor v is denoted by $\mathcal{B}_n(v)$. Let $\mathcal{B}(v) = \bigcup_{n=0}^{\infty} \mathcal{B}_n(v)$. The concatenation of two words w and v is denoted by $w \cdot v$ or simply by wv. If v is a binary word and $\mathcal{W}$ is a set of binary words, let $\mathcal{W} \cdot v = \{w \cdot v \mid w \in \mathcal{W}\}$, and $v \cdot \mathcal{W}$ is defined similarly. Whenever $\mathcal{A}$ and $\mathcal{C}$ are two subsets of $\mathcal{B}$, we define $\mathcal{A} \cdot \mathcal{C} = \{a \cdot c \mid a \in \mathcal{A}, c \in \mathcal{C}\}$.

Following [20] the nth k-generalized Fibonacci number is defined as

$$f_{n,k} = \begin{cases} 0 & \text{if } 0 \leqslant n \leqslant k-2, \\ 1 & \text{if } n = k-1, \\ \sum_{i=1}^{k} f_{n-i,k} & \text{otherwise.} \end{cases} \tag{1}$$

As noted in [28], the generating function $g_k(x) = \sum_{n=0}^{\infty} f_{n,k} x^n$ for k-generalized Fibonacci numbers is

$$g_k(x) = \frac{x^{k-1}}{1 - x - x^2 - \cdots - x^k} = \frac{x^{k-1} - x^k}{1 - 2x + x^{k+1}}. \tag{2}$$

Related constructions also appear in [11, p. 42] and in [1, p. 309].

Classical fact. *The number of words in $\mathcal{B}_n(1^k)$ equals $f_{n+k,k}$ for $k \geqslant 2$, moreover*

$$\mathcal{B}_n(1^k) = \begin{cases} \mathcal{B}_n & \text{if } n < k, \\ \bigcup_{i=0}^{k-1} 1^i 0 \cdot \mathcal{B}_{n-i-1}(1^k) & \text{otherwise.} \end{cases} \tag{3}$$

The classical fact comes, for instance, from [15, p. 286]. The binary words avoiding two consecutive 1s are counted by Fibonacci numbers, words without factor 111 are counted by Tribonacci numbers, etc. We call such words *(generalized) Fibonacci words*. The On-line Encyclopedia of Integer Sequences founded by N.J.A. Sloane [26] contains several corresponding sequences (see for example A000045 and A000073, after taking a binary complement). Gray codes for Fibonacci words are discussed in [28], and more generally, Gray codes for words avoiding a given factor in [3,6,24,27].

Lemma 1. *For $q \geqslant 1$, the bivariate generating function $B_q(x, y) = \sum_{n,k \geqslant 0} b_{n,k} x^n y^k$ where the coefficient $b_{n,k}$ of $x^n y^k$ is the number of Fibonacci words of length n having k 1s in $\mathcal{B}_n(1^{q+1})$ is*

$$B_q(x, y) = \frac{y\left(1 - (xy)^{q+1}\right)}{y - xy^2 - xy + (xy)^{q+2}}.$$

Proof. Alternatively to (3), the set $\mathcal{B}(1^{q+1})$ of (any length) binary words avoiding 1^{q+1} can be defined recursively as

$$\mathcal{B}(1^{q+1}) = \mathbb{1}_q \cup \bigcup_{i=0}^{q} 1^i 0 \cdot \mathcal{B}(1^{q+1})$$

where $\mathbb{1}_q = \bigcup_{i=0}^{q} \{1^i\}$. It follows that the bivariate generating function $B_q(x, y)$ satisfies the functional equation

$$B_q(x, y) = \sum_{i=0}^{q} x^i y^i + B_q(x, y) \sum_{i=0}^{q} x^{i+1} y^i,$$

and we have $B_q(x, y) = \frac{y\left(1-(xy)^{q+1}\right)}{y-xy^2-xy+(xy)^{q+2}}$. $\square$

It is not surprising that $B_q(x, 1) = \frac{g_{q+1}(x) - x^q}{x^{q+1}}$, see the (2). Indeed, both sides of this equality are generating functions for the sequence $(f_{n+q+1,q+1})_{n \geqslant 0}$ counting words in $\mathcal{B}(1^{q+1})$, which is the $(q+1)$th left shift of the sequence $(f_{n,q+1})_{n \geqslant 0}$, see (1) and the classical fact following it.

The *Hamming distance* between two binary words of the same length equals the number of positions at which they differ. A k-Gray code for a set $\mathcal{A} \subset \mathcal{B}_n$ is an ordered list $\mathbf{A}$ for $\mathcal{A}$, such that the Hamming distance between any two consecutive words in $\mathbf{A}$ is at most k, and we say that words in $\mathbf{A}$ are listed in *Gray code order*. Frank Gray's patent [12] discusses an early

Table 1
(a) The images of words in $\mathcal{B}_4(111)$ under the bijection ϕ. Words in $\mathcal{B}_4(111)$ are listed in a BRGC-like order, called local reflected order in [28], which yields a 1-Gray code order. (b) The set $\mathcal{W}_6^1$ in BRGC order together with the Hamming distance between consecutive words.

$u \in \mathcal{B}_4(111)$	$\phi(u) \in \mathcal{W}_4^2$
1100	0011
1101	1111
1001	1001
1000	0001
1010	0101
1011	1101
0011	1100
0010	0100
0000	0000
0001	1000
0101	1010
0100	0010
0110	1110

(a)

Words in $\mathcal{W}_6^1$ in BRGC order					
1	000000		*12*	111100	3
2	000001	1	*13*	111111	2
3	000011	1	*14*	111110	1
4	000010	1	*15*	111001	3
5	000110	1	*16*	111000	1
6	000100	1	*17*	100100	3
7	001001	3	*18*	100010	2
8	001000	1	*19*	100011	1
9	110000	3	*20*	100001	1
10	110001	1	*21*	100000	1
11	110010	2			

(b)

example and application of such a code for the set of n-length binary words. The concatenation of two ordered lists $\mathbf{L}_1$ and $\mathbf{L}_2$ is denoted by $\mathbf{L}_1 \circ \mathbf{L}_2$, and $\overleftarrow{\mathbf{L}}$ designates the reverse of the list $\mathbf{L}$. If $\mathbf{L}$ is a list of words, then $\langle \mathbf{L} \rangle^i = \mathbf{L}$ whenever i is even, and $\langle \mathbf{L} \rangle^i = \overleftarrow{\mathbf{L}}$ otherwise. First and last elements of $\mathbf{L}$ are denoted respectively by first($\mathbf{L}$) and last($\mathbf{L}$). Also, we denote by $\widehat{\mathbf{L}}$ the list obtained from $\mathbf{L}$ by deleting its first element.

For example, a list containing elements 00, 01 and 11 will be noted as $\mathbf{L} = 00 \circ 01 \circ 11$, and $\widehat{\mathbf{L}} = 00 \circ 01$.

Definition 1. A binary word is called *q-decreasing*, for $q \geqslant 1$, if any of its length maximal factors of the form $0^a 1^b$, $a > 0$, satisfies $q \cdot a > b$.

The set of q-decreasing words of length n is denoted by $\mathcal{W}_n^q$. For example we have $\mathcal{W}_4^1 = \{0000, 0001, 0010, 1000, 1001, 1100, 1110, 1111\}$. See also Table 1 for the sets $\mathcal{W}_4^2$ and $\mathcal{W}_6^1$. Let $\mathcal{W}^q = \bigcup_{n=0}^{\infty} \mathcal{W}_n^q$.

2. Bijection with classical Fibonacci words

In this section we prove that q-decreasing words, $q \geqslant 1$, are enumerated by $(q+1)$-generalized Fibonacci numbers defined in (1). We start with a definition and several propositions.

Definition 2. For any $q \geqslant 1$, we define the map ψ^q from $\mathcal{B}_n$ to $\mathcal{B}_{n+q+1}$ as

$$\psi^q(w) = \begin{cases} v001^{k+q} & \text{if } w = v01^k,\ v \in \mathcal{B}, k \geqslant 0, \\ 1^{n+q+1} & \text{otherwise.} \end{cases}$$

Less formally, ψ^q inserts a factor 01^q immediately after the last occurrence of 0, and it adds the suffix 1^{q+1} to the word containing no 0. For example $\psi^1(0) = 0\mathbf{01}$, $\psi^1(00011) = 000\mathbf{01}11$, $\psi^2(0011101) = 001110\mathbf{011}1$ and $\psi^5(1) = 1\mathbf{111111}$. The value of q will be clear from the context, so by slight abuse of notation ψ^q will be denoted ψ throughout the paper.

Proposition 1. *For $n \geqslant 0$, $q \geqslant 1$, ψ is an injective map from $\mathcal{B}_n$ to $\mathcal{B}_{n+q+1}$.*

Proof. For two n-length words $w \neq w'$ we show that $\psi(w) \neq \psi(w')$. It is clear that if one of the given words contains no 0 the injectivity holds. Otherwise we have two cases. If $w = v01^k$ and $w' = v'01^k$ then we have necessarily $v \neq v'$ and $v001^{k+q} \neq v'001^{k+q}$, so the images are different. If $w = v01^k$ and $w' = v'01^\ell$ with $k \neq \ell$, then $v001^{k+q} \neq v'001^{\ell+q}$ and again $\psi(w) \neq \psi(w')$. $\square$

In the following, we will use the restriction of ψ to the set of q-decreasing words, namely $\psi : \mathcal{W}_n^q \to \mathcal{W}_{n+q+1}^q$. It is possible due to Proposition 2 below.

Proposition 2. *For $n \geqslant 0$, $q \geqslant 1$, $\psi(\mathcal{W}_n^q)$ consists of all q-decreasing words of length $n+q+1$ ending with at least q ones.*

Proof. If $w = 1^n$, then $\psi(w) = 1^{n+q+1}$. Otherwise, we write $w = v0^a1^b$ where $a > b/q \geqslant 0$ and the word v is either empty or ends with 1. So $\psi(v0^a1^b) = v0^{a+1}1^{q+b}$. As we have $1+a > 1+b/q = (q+b)/q$, $\psi(w)$ is a q-decreasing word ending with at least q 1s. Similarly, any $(n+q+1)$-length q-decreasing word ending with at least q 1s can be obtained from a (unique) word in $\mathcal{W}_n^q$ by ψ. □

Now, we present a one-to-one correspondence between Fibonacci and q-decreasing words. Recall that, for $q \geqslant 1$, the set $\mathcal{B}_n(1^{q+1})$ of $(q+1)$-generalized Fibonacci words is the set of binary words of length n with no 1^{q+1} factors, see (3) for the recursive definition of these words according to their length.

Definition 3. We define the map $\phi : \mathcal{B}_n(1^{q+1}) \to \mathcal{B}_n$ as

$$\phi(w) = \begin{cases} 1^k & \text{if } w = 1^k \text{ and } k \in [0, q], \\ \psi\big(\phi(v)\big) & \text{if } w = 1^q0v, \\ \phi(v)01^k & \text{if } w = 1^k0v \text{ and } k \in [0, q-1]. \end{cases} \tag{4}$$

See Table 1(a) for the images of the words in $\mathcal{B}_4(111)$ through ϕ.

Theorem 1. *For $n \geqslant 0$, $q \geqslant 1$, ϕ maps bijectively $\mathcal{B}_n(1^{q+1})$ into $\mathcal{W}_n^q$.*

Proof. We proceed by induction on n. The classical decomposition in (3) gives rise to the three cases specified in (4), and we have:

(i) If $n \leqslant q$, then the word 1^n is sent by ϕ to 1^n;
(ii) Words of the form 1^q0v, where $v \in \mathcal{B}_{n-q-1}(1^{q+1})$, are sent to words ending by at least q 1s (see Proposition 2);
(iii) Words of the form 1^k0v, where $k \in [0, q-1]$ and $v \in \mathcal{B}_{n-k-1}(1^{q+1})$, are sent to words ending by at most $q-1$ 1s.

Using the bijectivity of ψ (see Propositions 1 and 2) and the induction hypothesis, it is routine to check that $\phi(w) \in \mathcal{W}_n^q$ for any $w \in \mathcal{B}_n(1^{q+1})$, and $\phi(w) \neq \phi(w')$ for any two different words $w, w' \in \mathcal{B}_n(1^{q+1})$.

Similarly, by induction on n, any word in $\mathcal{W}_n^q$ can be obtained by ϕ from a word in $\mathcal{B}_n(1^{q+1})$, and the statement holds. □

It follows that $(q+1)$-order Fibonacci words and q-decreasing words are equinumerous.

3. Some enumeration results

Here we provide a bivariate generating function $W_q(x, y) = \sum_{n,k\geqslant 0} w_{n,k}x^n y^k$, where $w_{n,k}$ is the number of n-length q-decreasing words having k 1s. This bivariate generating function is of a particular interest since it will help us (see Corollary 1) to prove that q-decreasing words satisfy a necessary condition for the existence of 1-Gray code, called *parity condition*. More precisely, if a set $\mathcal{A}$ of binary words admits a 1-Gray code, and $\mathcal{A}^+$ (resp. $\mathcal{A}^-$) denotes the subset of $\mathcal{A}$ having even (resp. odd) number of 1s, then the *parity difference* $|\mathcal{A}^+| - |\mathcal{A}^-|$ must be equal to either 0, 1, or -1. Indeed, the graph where the vertex set is $\mathcal{A}$ and edges connect vertices with Hamming distance one is bipartite with partite sets $\mathcal{A}^+$ and $\mathcal{A}^-$. A Hamiltonian path in this graph (or equivalently a 1-Gray code for $\mathcal{A}$) cannot exist unless it satisfies the parity condition. This parity condition is used for instance in [27] to investigate the possibility of 1-Gray code for a set of words avoiding a given factor.

In order to derive the expression of $W_q(x, y)$, we use the following decomposition of the set $\mathcal{W}^q$:

$$\mathcal{W}^q = \mathbb{1} \cup \mathcal{W}^q \cdot \mathcal{S}^q,$$

where $\mathbb{1} = \cup_{n=0}^{\infty}\{1^n\}$ and $\mathcal{S}^q$ corresponds to the set of all factors of the form 0^a1^b respecting q-decreasing property (i.e., $a > b/q \geqslant 0$) such that none of them is a concatenation of other factors respecting q-decreasing property. More precisely, a is the smallest integer strictly greater than b/q, i.e., $a = \lfloor b/q \rfloor + 1$. A factor from $\mathcal{S}^q$ will be called *q-prime factor*, and thus $\mathcal{S}^q$ is the set of such factors. For instance: $\mathcal{S}^1 = \{0, 001, 00011, 0000111, \ldots\}$, $\mathcal{S}^2 = \{0, 01, 0011, 00111, 0001111, 00011111, \ldots\}$.

Lemma 2. *For $q \geqslant 1$, the bivariate generating function $S_q(x, y) = \sum_{n,k\geqslant 0} s_{n,k}x^n y^k$ where the coefficient $s_{n,k}$ is the number of q-prime factors of length n having exactly k 1s is:*

$$S_q(x, y) = \frac{x\left(1 - (xy)^q\right)}{(xy-1)(x^{q+1}y^q - 1)}.$$

Proof. Any q-prime factor is of the form 0^a1^b with $a = \lfloor b/q \rfloor + 1$. So, if $b = kq + r$ with $k \geqslant 0$ and $r \in [0, q-1]$, then $a + b = k(q+1) + r + 1$ and we can write:

$$S_q(x, y) = \sum_{k=0}^{\infty} \sum_{r=0}^{q-1} x^{k(q+1)+r+1} y^{kq+r}.$$

A simple calculation results in the claimed formula. □

Theorem 2. *For $q \geqslant 1$, the bivariate generating function $W_q(x, y) = \sum_{n,k \geqslant 0} w_{n,k} x^n y^k$ where the coefficient $w_{n,k}$ is the number of n-length q-decreasing words containing exactly k 1s is given by:*

$$W_q(x, y) = \frac{1 - x^{q+1} y^q}{1 - xy - x + x^{q+2} y^{q+1}}.$$

Proof. Due to the decomposition $\mathcal{W}^q = \mathbb{1} \cup \mathcal{W}^q \cdot \mathcal{S}^q$, we have $W_q(x, y) = \frac{1}{1-xy} \cdot \frac{1}{1-S_q(x,y)}$, and the result holds after applying Lemma 2. □

Corollary 1. *For any $n \geqslant 0, q \geqslant 1$, the set $\mathcal{W}_n^q$ satisfies the parity condition.*

Proof. The generating function $D_q(x) = \sum_{n \geqslant 0} d_n x^n$ where the coefficient d_n is the parity difference corresponding to the set $\mathcal{W}_n^q$ is obtained by making the substitution $y = -1$ in $W_q(x, y)$:

$$D_q(x) = \frac{(-1)^q x^{q+1} - 1}{(-1)^q x^{q+2} - 1}.$$

When q is even, $D_q(x) = \frac{x^{q+1}-1}{x^{q+2}-1} = \sum_{n=0}^{\infty} \left(x^{n(q+2)} - x^{n(q+2)+q+1}\right)$, otherwise $D_q(x) = \frac{x^{q+1}+1}{x^{q+2}+1} = \sum_{n=0}^{\infty} (-1)^n \left(x^{n(q+2)} + x^{n(q+2)+q+1}\right)$. All involved coefficients are from $\{-1, 0, 1\}$, and the parity condition holds. □

The following two corollaries are obtained by respectively calculating the expressions: $W_q(x, 1)$, $\frac{\partial W_q(x,y)}{\partial y}\big|_{y=1}$ and $\frac{\partial W_q(xy,1/y)}{\partial y}\big|_{y=1}$.

Corollary 2. *For $q \geqslant 1$, the generating function $F_q(x) = \sum_{n \geqslant 0} f_n x^n$ where the coefficient f_n is the number of n-length q-decreasing words is given by:*

$$F_q(x) = \frac{1 - x^{q+1}}{1 - 2x + x^{q+2}}.$$

Note that, as predicted by Theorem 1, $F_q(x) = B_q(x, 1)$, see Lemma 1 and the remark following it.

The *frequency* of a symbol in a set of words is the overall number of the occurrences of the symbol in the words of the set.

Corollary 3. *For $q \geqslant 1$, the generating function $P_{q,1}(x) = \sum_{n \geqslant 0} p_n x^n$ where the coefficient p_n is the frequency of 1s in all n-length q-decreasing words is*

$$P_{q,1}(x) = \frac{x\left(1 - qx^q + qx^{q+1} - 2x^{q+1} + x^{2q+2}\right)}{\left(1 - 2x + x^{q+2}\right)^2}.$$

Similarly, the generating function for the frequency of 0s in all n-length q-decreasing words is

$$P_{q,0}(x) = \frac{x\left(1 - x^q\right)}{\left(1 - 2x + x^{q+2}\right)^2}.$$

The frequency of 1s in $\mathcal{B}_n(11)$ is equal to the number of edges in the Fibonacci cube [13] of order n, see [14] and comments to the sequence A001629 in [26]. The generating function $P_{1,0}(x)$ allows us to show that the frequency of 0s in $\mathcal{W}_n^1$ is a shift of the sequence A006478 enumerating the number of edges in the *Fibonacci hypercube* [21], i.e., in a polytope determined by the convex hull of the Fibonacci cube.

Despite the q-decreasing words and Fibonacci words have quite different definitions, they are equinumerous and share some common features. We end this section by showing that the relative frequency of 1s (defined formally below) in both sets have the same limit when n tends to infinity.

If ω_n (resp. β_n) is the ratio between the frequency of 1s and that of 0s in the words in $\mathcal{W}_n^1$ (resp. in $\mathcal{B}_n(11)$), then $\lim_{n\to\infty}\omega_n = \lim_{n\to\infty}\beta_n$. Indeed, extracting the coefficients of x^n in both $P_{1,1}$ and of $P_{1,0}$, their ratio tends to $2-\varphi \approx 0.3819660113$ when n tends to infinity, where φ is the golden ratio; and this is also the limit of β_n, which is obtained by investigating the ratio of the coefficients of x^n in $\frac{\partial B_q(x,y)}{\partial y}\big|_{y=1}$ and in $\frac{\partial B_q(xy,1/y)}{\partial y}\big|_{y=1}$, where $B_q(x,y)$ is from Lemma 1.

The *relative frequency of* 1*s* in a set of binary words is the ratio between the (cumulative) frequency of 1s and the overall number of bits in the words of the set. Alternatively, it is the expected value when a bit is randomly chosen in the words of the set. With the notations above, we have that the relative frequency of 1s in $\mathcal{W}_n^1$ is $\frac{1}{1+\frac{1}{\omega_n}}$ and in $\mathcal{B}_n(11)$ is $\frac{1}{1+\frac{1}{\beta_n}}$, and we have the next result.

Corollary 4. *The relative frequency of* 1*s in* $\mathcal{W}_n^1$ *and in* $\mathcal{B}_n(11)$ *both tend to* $\frac{2-\varphi}{3-\varphi}$ *when n tends to infinity, where* φ *is* $\frac{1+\sqrt{5}}{2}$.

More generally, for any $q \geqslant 1$, the overall number of bits in both sets $\mathcal{W}_n^q$ and $\mathcal{B}_n(1^{q+1})$ is $n \cdot f_{n+q+1,q+1}$, and due to the second rule in relation (4) defining the bijection $\phi : \mathcal{B}(1^{q+1}) \to \mathcal{W}^q$ we have that in $\mathcal{W}_n^q$ there are more 1s than in $\mathcal{B}_n(1^{q+1})$. However, the next corollary shows that the difference between the relative frequency of 1s in $\mathcal{W}_n^q$ and that in $\mathcal{B}_n(1^{q+1})$ tends to zero when n tends to infinity.

Corollary 5. *For any* $q \geqslant 1$, *if* $u_{n,q}$ *(resp.* $v_{n,q}$*) is the frequency of* 1*s in* $\mathcal{W}_n^q$ *(resp. in* $\mathcal{B}_n(1^{q+1})$*), then we have*

$$\lim_{n\to\infty} \frac{u_{n,q} - v_{n,q}}{n \cdot f_{n+q+1,q+1}} = 0.$$

Proof. Since, for any $q \geqslant 1$, $u_{n,q} - v_{n,q} \geqslant 0$, it suffices to prove that we have $u_{n,q} - v_{n,q} \leqslant f_{n+q+1,q+1}$. Using Corollary 3 and Lemma 1, the generating function $H(x)$ where the coefficient of x^n is $f_{n+q+1,q+1} + v_{n,q+1} - u_{n,q}$ is

$$\begin{aligned} H(x) &= B_q(x,1) + \frac{\partial B_q(x,y)}{\partial y}\bigg|_{y=1} - P_{q,1}(x) \\ &= \frac{1-2x^{q+1}}{1-2x+x^{q+2}}, \end{aligned}$$

which satisfies the functional equation $H(x) = 1 - 2x^{q+1} + 2xH(x) - x^{q+2}H(x)$. By a simple observation, $H(x)$ is also the generating function with respect to the length of binary words different from 0^{q+1} and 1^{q+1} and that do not start with 0^{q+2}. Then we have $u_{n,q} - v_{n,q} \leqslant f_{n+q+1,q+1}$. Dividing by $nf_{n+q+1,q+1}$, and taking the limit when n tends to infinity, we obtain the expected result.

Corollary 6. *For* $q \geqslant 1$ *the relative frequency of* 1*s in* $\mathcal{W}_n^q$ *and in* $\mathcal{B}_n(1^{q+1})$ *have a common non-zero limit when n tends to infinity.*

Proof. Corollary 4 says that, for $q = 1$, the relative frequency of 1s in $\mathcal{W}_n^q$ and in $\mathcal{B}_n(1^{q+1})$ have a common limit when n tends to infinity. For $q \geqslant 2$, Corollary 5 does not ensure that each of the relative frequency of 1s in $\mathcal{W}_n^q$ (that is $\frac{u_{n,q}}{n\cdot f_{n+q+1,q+1}}$) and in $\mathcal{B}_n(1^{q+1})$ (that is $\frac{v_{n,q}}{n\cdot f_{n+q+1,q+1}}$) has a limit when n tends to infinity. However, in [2], using asymptotic analysis it is shown that $\frac{v_{n,q}}{n\cdot f_{n+q+1,q+1}}$ converges to a non-zero value when n tends to infinity, and the limit can be approximated by numerical methods. From Corollary 5 it follows that so does $\frac{u_{n,q}}{n\cdot f_{n+q+1,q+1}}$, and the two limits are equal. □

4. Exhaustive generation and Gray codes for *q*-decreasing words

Here we show that q-decreasing words can be efficiently generated in lexicographic order and explain how the obtained generating algorithm can be turned into a 3-Gray code generating algorithm. Then, we give a more intricate construction of a 1-Gray code for the particular case $q = 1$. As a byproduct, this construction gives a positive answer for the existence a Hamiltonian path in Fibonacci-run graph conjectured in [9].

4.1. 3-*Gray codes and exhaustive generation*

Algorithm in Fig. 1 generates prefixes of q-decreasing words in the lexicographical order, and eventually all n-length q-decreasing words. The size n, parameter q and the array w of length $n+1$ are global variables and the main call is LexFib(1, n). For convenience, $w[0]$ is initialized by 1 and the parameter *delta* is the number of consecutive 1s that can be added to the current generated prefix without violating the q-decreasingness.

It can be seen that this recursive algorithm satisfies Baronaigien and Ruskey's *constant amortized time* (*CAT*) *principle* in [4] stated below. As noticed in [4], by considering the underlying computation tree it follows that any algorithm satisfying

```
procedure LexFib(pos, delta: integer)
   if (pos = n + 1) print w;
   else     w[pos] := 0;
            if (w[pos − 1] = 1) d := q − 1; else d:=delta + q; endif
            LexFib(pos + 1, d);
            if (delta > 0)
                w[pos] := 1; LexFib(pos + 1, delta − 1);
            endif
   endif
end procedure
```

Fig. 1. Lexicographic generation algorithm for q-decreasing words.

the CAT principle is efficient, and thus LexFib is an efficient generating algorithm. We refer the reader to [23, Section 1.7] for more about CAT generating algorithms.

CAT principle [4]: A recursive generation algorithm with the following properties runs in constant amortized time. (i) Every call results in the output of at least one object; (ii) Excluding the computation done by recursive calls the amount of computation of any call is proportional to the degree of the call (that is, the number of subsequent recursive calls produced by the call); (iii) The number of calls of degree one is $O(N)$, where N is the number of generated objects.

Generally, the bijection ϕ in (4) does not preserve *Graycodeness*, i.e., a Gray code for $(q+1)$-Fibonacci words is not necessarily mapped by ϕ to a Gray code for q-decreasing words. For instance, when $q = 1$ and $n = 2k+1$ the Gray code for Fibonacci words in [28] always contains two consecutive words $u = (10)^k1$ and $v = (10)^k0$, but their images $\phi(u) = 1^{2k+1}$ and $\phi(v) = 0^{k+1}1^k$ have arbitrarily large Hamming distance for enough large n. A similar phenomenon happens when $n = 2k$ and $q = 1$ with $u = (10)^{k-2}10$ and $v = (10)^{k-2}00$: $\phi(u) = 1^{2k}$ and $\phi(v) = 0^{k+1}1^{k-1}$. See also Table 1(a) for the image through ϕ of the 1-Gray code in [28] for $\mathcal{B}_4(111)$.

Below we show that BRGC order (that is, the order induced by Binary Reflected Gray Code in [12]) yields a 3-Gray code on $\mathcal{W}_n^q$. Much more interestingly, thanks to Corollary 1, the necessary condition for the existence of a 1-Gray code is satisfied, and we will provide such a Gray code for $\mathcal{W}_n^1$ in the following part.

In [29] the author introduces the notion of *absorbent set*, which (up to complement) is defined as: a binary word set $\mathcal{X} \subset \{0,1\}^n$ is called absorbent if for any $u \in \mathcal{X}$ and any k, $1 \leqslant k < n$, $u_1u_2\dots u_k0^{n-k}$ is also a word in $\mathcal{X}$. Corollary 1 from the same paper proves that when an absorbent set $\mathcal{X}$ is listed in BRGC order (that is, restricting BRGC to $\mathcal{X}$) the resulting listing yields a 3-Gray code. Clearly, $\mathcal{W}_n^q$ is an absorbent set and we have the following consequence.

Corollary 7. *For $n \geqslant 0$, $q \geqslant 1$, the restriction of BRGC order yields a 3-Gray code for $\mathcal{W}_n^q$.*

See for an example Table 1(b).

Applying reversing sublists technique [22] by adding a new parameter to LexFib which keeps track of the parity of the number of 1s in the current generated prefix, procedure LexFib can be turned into one generating the same class of words in BRGC order instead of the lexicographic one, and so according to Corollary 7 producing a 3-Gray code for $\mathcal{W}_n^q$. Obviously the obtained Gray code generating algorithm inherits CAT property.

At this point it is worth mentioning an alternative powerful framework to design Gray codes as sublists of BRGC. In [24] it is given the following definition (up to mirroring each word): a set of same length binary words is a *flip-swap language* (with respect to 1) if it is closed under the operations (i) flip the rightmost 1, and (ii) swap the rightmost 1 with the bit to its left. Theorem 2 in [24] states that when a flip-swap language is listed in BRGC order the resulting listing yields a 2-Gray code, and in the same paper there is given a variety of combinatorial classes that are flip-swap languages together with efficient generating algorithms for most of them. It is easy to see that absorbent sets coincide with languages closed under flip operation, and thus it is not surprising that, when listed in BRGC order, flip-swap languages yield more restrictive (that is, 2- instead of 3-) Gray codes than absorbent sets do. In particular, for $n \geqslant 0$ and $q \geqslant 1$, $\mathcal{W}_n^q$ is an absorbent set (or equivalently, a flip-closed language) but not a flip-swap language.

4.2. 1-Gray code for $\mathcal{W}_n^1$

In this section, we construct a 1-Gray code for the set $\mathcal{W}_n^q$, $n \geqslant 0$, when $q = 1$, which in particular gives a positive answer to a conjecture in [9]. For this purpose, we decompose $\mathcal{W}_n^1$ $n \geqslant 1$, as

$$\mathcal{W}_n^1 = \mathcal{Z}_n \cup 1 \cdot \mathcal{W}_{n-1}^1$$

where $\mathcal{W}_0^1$ consists of the empty word ϵ, and $\mathcal{Z}_n$ is the subset of words starting with 0 in $\mathcal{W}_n^1$. In turn, we decompose $\mathcal{Z}_n$ as

$$\mathcal{Z}_n = \{0^n\} \cup \bigcup_{r=3}^{n} \mathcal{D}_n^r$$

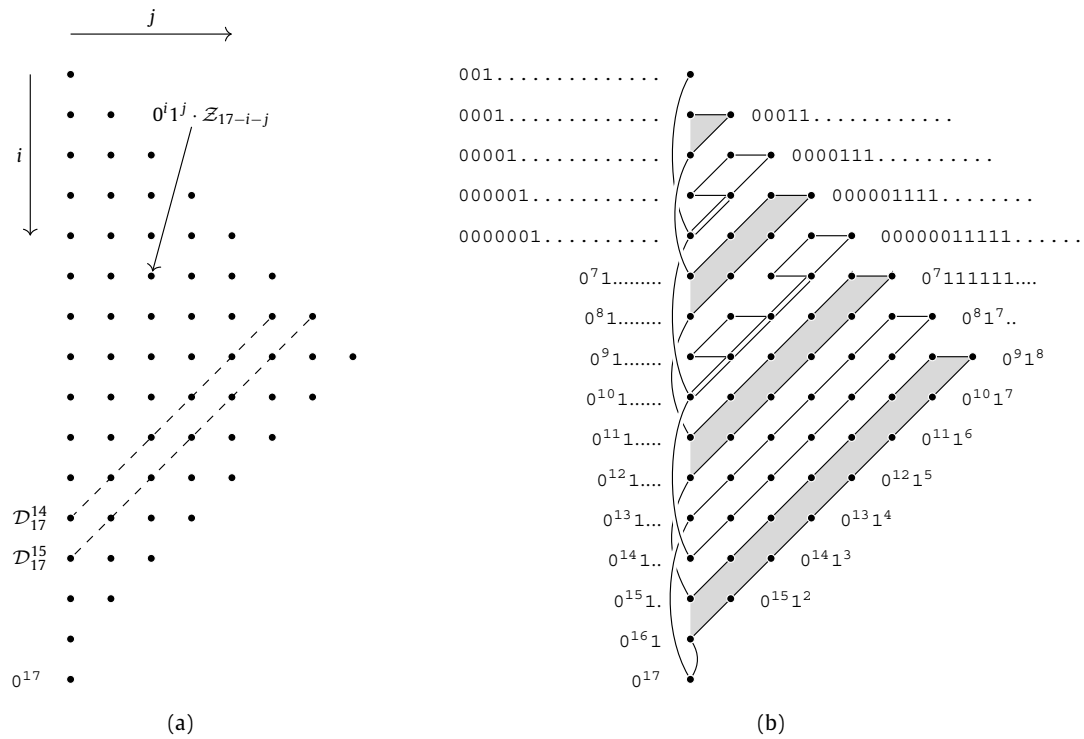


Fig. 2. (a) A decomposition of $\mathcal{Z}_{17}$ as a union of subsets $0^i1^j \cdot \mathcal{Z}_{17-i-j}$ (or equivalently a union of diagonals $\mathcal{D}_{17}^r$) and $\{0^{17}\}$. (b) An illustration of the 1-Gray code $\mathbf{Z}_{17}$. The pairs of consecutive diagonals dealt with Lemma 3 are shown in gray-filled area; the other pairs are dealt with Lemma 4. A point labeled 0^91..... (that is 0^91 followed by seven dots) corresponds to the set of words in $0^91 \cdot \mathcal{Z}_7$.

where $\mathcal{D}_n^r = \bigcup_{j=1}^{\lfloor \frac{r-1}{2} \rfloor} 0^{r-j}1^j \cdot \mathcal{Z}_{n-r}$. We refer to Fig. 2(a) for a graphical illustration of the decomposition of $\mathcal{Z}_n$ for $n = 17$ where the point at coordinates (i, j) corresponds to the set $0^i1^j \cdot \mathcal{Z}_{n-i-j}$, $1 \leqslant j < i \leqslant n-1$, $3 \leqslant i+j \leqslant n$, except the lowest point which corresponds to $\{0^n\}$. The sets $\mathcal{D}_n^r$, $3 \leqslant r \leqslant n$, correspond to the southwest-northeast diagonals of the graphic.

Let $\mathbf{Z}_0 = \epsilon$ be the list containing only the empty word ϵ, $\mathbf{Z}_1 = 0, \mathbf{Z}_2 = 00, \mathbf{Z}_3 = 000 \circ 001$. If $\mathbf{L}$ is a list and w is a word, then $w \cdot \mathbf{L}$ denotes the list where w is concatenated to every word from $\mathbf{L}$. According to the above definitions, it is straightforward to check the following lemma.

Lemma 3. *For any $1 \leqslant k < n$, we suppose that $\mathbf{Z}_k$ is a 1-Gray code for $\mathcal{Z}_k$ with $\mathrm{first}(\mathbf{Z}_k) = 0(001)^\star$ and $\mathrm{last}(\mathbf{Z}_k) = (001)^\star$. Given i and j such that $1 \leqslant j < i \leqslant n$, then*

- (i) *for $3 \leqslant i+j < n$, the list $\mathbf{L} = 0^i1^j \cdot \mathbf{Z}_{n-i-j}$ is a 1-Gray code for $0^i1^j \cdot \mathcal{Z}_{n-i-j}$ with $\mathrm{first}(\mathbf{L}) = 0^i1^j0(001)^\star$ and $\mathrm{last}(\mathbf{L}) = 0^i1^j(001)^\star$;*
  when $i+j = n$ the list $\mathbf{L}$ contains only the word 0^i1^j;
- (ii) *for $3 \leqslant i+j < n-1$, the list $\mathbf{L} = 0^i1^{j+1} \cdot \mathbf{Z}_{n-i-j-1} \circ 0^i1^j \cdot \mathbf{Z}_{n-i-j}$ is a 1-Gray code for $0^i1^{j+1} \cdot \mathcal{Z}_{n-i-j-1} \cup 0^i1^j \cdot \mathcal{Z}_{n-i-j}$ with $\mathrm{first}(\mathbf{L}) = 0^i1^{j+1}0(001)^\star$ and $\mathrm{last}(\mathbf{L}) = 0^i1^j(001)^\star$;*
  when $i+j+1 = n$, the list $\mathbf{L} = 0^i1^{j+1} \circ 0^i1^j0$;
- (iii) *for $3 \leqslant i+j < n$, the list $\mathbf{L} = 0^i1^j \cdot \mathbf{Z}_{n-i-j} \circ \overleftarrow{0^{i-1}1^{j+1} \cdot \mathbf{Z}_{n-i-j}}$ is a 1-Gray code for $0^i1^j \cdot \mathcal{Z}_{n-i-j} \cup 0^{i-1}1^{j+1} \cdot \mathcal{Z}_{n-i-j}$ with $\mathrm{first}(\mathbf{L}) = 0^i1^j0(001)^\star$ and $\mathrm{last}(\mathbf{L}) = 0^{i-1}1^{j+1}0(001)^\star$;*
  when $i+j = n$, the list $\mathbf{L} = 0^i1^j \circ \overleftarrow{0^{i-1}1^{j+1}}$;
- (iv) *for $3 \leqslant i+j \leqslant n$, the list $\mathbf{L} = \overleftarrow{0^i1^j \cdot \mathbf{Z}_{n-i-j}} \circ 0^{i-1}1^{j+1} \cdot \mathbf{Z}_{n-i-j}$ is a 1-Gray code for $0^i1^j \cdot \mathcal{Z}_{n-i-j} \cup 0^{i-1}1^{j+1} \cdot \mathcal{Z}_{n-i-j}$ with $\mathrm{first}(\mathbf{L}) = 0^i1^j(001)^\star$ and $\mathrm{last}(\mathbf{L}) = 0^{i-1}1^{j+1}(001)^\star$.*

Lemma 4. *Let us consider $r = 1 \mod 4$, $3 \leqslant r \leqslant n$. For any $1 \leqslant k < n$, we suppose that $\mathbf{Z}_k$ is a 1-Gray code for $\mathcal{Z}_k$ with $\mathrm{first}(\mathbf{Z}_k) = 0(001)^\star$ and $\mathrm{last}(\mathbf{Z}_k) = (001)^\star$.*

- (i) *If $r \neq n-1$, then there is a 1-Gray code Δ_n^r for $\mathcal{D}_n^r \cup \mathcal{D}_n^{r-1}$ such that $\mathrm{first}(\Delta_n^r) = 0^{r-2}1(001)^\star$ and $\mathrm{last}(\Delta_n^r) = 0^{r-1}1(001)^\star$.*
- (ii) *If $r = n-1$, then there is a 1-Gray code Δ_n^{n-1} for $\mathcal{D}_n^n \cup \mathcal{D}_n^{n-1} \cup \mathcal{D}_n^{n-2}$ such that $\mathrm{first}(\Delta_n^{n-1}) = 0^{n-3}100$ and $\mathrm{last}(\Delta_n^{n-1}) = 0^{n-1}1$.*

Proof. For the first assertion (i), it suffices to consider the list

$$\Delta_n^r = \bigcirc_{j=1}^{\frac{r-3}{2}} 0^{r-1-j}1^j \cdot \langle \mathbf{Z}_{n-r+1}\rangle^j \circ \overleftarrow{\bigcirc_{j=1}^{\frac{r-1}{2}} 0^{r-j}1^j \cdot \langle \mathbf{Z}_{n-r}\rangle^j}.$$

After considering assertions of Lemma 3, it remains to examine the transition between $w_0 = \mathrm{last}(0^{r-1-j_0}1^{j_0} \cdot \langle \mathbf{Z}_{n-r+1}\rangle^{j_0})$ for $j_0 = \frac{r-3}{2}$ and $w_1 = \mathrm{first}(\overleftarrow{0^{r-j_1}1^{j_1} \cdot \langle \mathbf{Z}_{n-r}\rangle^{j_1}})$ for $j_1 = \frac{r-1}{2}$. Since $r = 1 \mod 4$, we have necessarily j_0 is odd and $j_0 + 1 = j_1$ which implies that $w_0 = \mathrm{last}(0^{r-1-j_0}1^{j_0} \cdot \overleftarrow{\mathbf{Z}_{n-r+1}}) = 0^{r-1-j_0}1^{j_0}0(001)^\star$ and $w_1 = \mathrm{first}(\overleftarrow{0^{r-j_1}1^{j_1} \cdot \mathbf{Z}_{n-r}}) = 0^{r-j_1}1^{j_1}(001)^\star$, and they differ by exactly one bit.

For the second assertion (ii), we consider the list

$$\Delta_n^{n-1} = \bigcirc_{j=1}^{\frac{n-4}{2}} 0^{n-2-j}1^j 00 \circ 0^{\frac{n}{2}}1^{\frac{n-2}{2}}0 \circ \overleftarrow{\bigcirc_{j=1}^{\frac{n-4}{2}} \left\langle 0^{n-j-1}1^{j+1} \circ 0^{n-1-j}1^j 0\right\rangle^{j-1}} \circ 0^{n-1}1.$$

A simple study of each kind of transitions allows us to see that Δ_n^{n-1} is a 1-Gray code for $\mathcal{D}_n^n \cup \mathcal{D}_n^{n-1} \cup \mathcal{D}_n^{n-2}$ satisfying $\mathrm{first}(\Delta_n^{n-1}) = 0^{n-3}100$ and $\mathrm{last}(\Delta_n^{n-1}) = 0^{n-1}1$. An illustration of this Gray code for $n = 10$ (and thus $r = 9$) can be found in the last sketch of Fig. 5. □

In the following, we write $w\mathbf{L}$ instead of $w \cdot \mathbf{L}$ to be more concise.

Lemma 5. *Let us consider $r = 3 \mod 4$, $3 < n$ and $3 \leqslant r \leqslant n$. For any $1 \leqslant k < n$, we suppose that $\mathbf{Z}_k$ is a 1-Gray code for $\mathcal{Z}_k$ with $\mathrm{first}(\mathbf{Z}_k) = 0(001)^\star$ and $\mathrm{last}(\mathbf{Z}_k) = (001)^\star$.*

- (i) *If $r = 3$, then there is a 1-Gray code Δ_n^3 for $\mathcal{D}_n^3$ such that $\mathrm{first}(\Delta_n^3) = 0010(001)^\star$ and $\mathrm{last}(\Delta_n^3) = (001)^\star$.*
- (ii) *If $r = n-2$, then there is a 1-Gray code Δ_n^{n-2} for $\mathcal{D}_n^{n-2} \cup \mathcal{D}_n^{n-3}$ such that $\mathrm{first}(\Delta_n^{n-2}) = 0^{n-4}1000$ and $\mathrm{last}(\Delta_n^{n-2}) = 0^{n-3}100$.*
- (iii) *If $r = n-1$, then there is a 1-Gray code Δ_n^{n-1} for $\mathcal{D}_n^n \cup \mathcal{D}_n^{n-1} \cup \mathcal{D}_n^{n-2} \setminus \{0^{n-1}1\}$ such that $\mathrm{first}(\Delta_n^{n-1}) = 0^{n-3}100$ and $\mathrm{last}(\Delta_n^{n-1}) = 0^{n-2}10$.*
- (iv) *If $r = n$, then there is a 1-Gray code Δ_n^n for $\mathcal{D}_n^n \cup \mathcal{D}_n^{n-1}$ such that $\mathrm{first}(\Delta_n^n) = 0^{n-2}10$ and $\mathrm{last}(\Delta_n^n) = 0^{n-1}1$.*
- (v) *If $r \notin \{3, n-2, n-1, n\}$, then there is a 1-Gray code Δ_n^r for $\mathcal{D}_n^r \cup \mathcal{D}_n^{r-1}$ such that $\mathrm{first}(\Delta_n^r) = 0^{r-1}10(001)^\star$ and $\mathrm{last}(\Delta_n^r) = 0^{r-1}1(001)^\star$.*

Proof. For the case (i), we set: $\Delta_3 = 0^2 1\mathbf{Z}_{n-3}$.

For the case (ii), we set: $\Delta_n^{n-2} = \bigcirc_{j=1}^{\frac{n-5}{2}} 0^{n-3-j}1^j \langle \mathbf{Z}_3\rangle^{j-1} \circ \overleftarrow{\bigcirc_{j=1}^{\frac{n-3}{2}} 0^{n-2-j}1^j \mathbf{Z}_2}$. Since we have $\mathbf{Z}_2 = 00$ and $\mathbf{Z}_3 = 000 \circ 001$, it is straightforward to see that Δ_n^{n-2} is a 1-Gray code with $\mathrm{first}(\Delta_n^{n-2}) = 0^{n-4}1000$ and $\mathrm{last}(\Delta_n^{n-2}) = 0^{n-3}100$.

For the case (iii), we set:

$$\Delta_n^{n-1} = \bigcirc_{j=1}^{\frac{n-4}{2}} 0^{n-2-j}1^j \mathbf{Z}_2 \circ 0^{\frac{n}{2}}1^{\frac{n-2}{2}}\mathbf{Z}_1 \circ \overleftarrow{\bigcirc_{j=1}^{\frac{n-4}{2}} \left\langle 0^{n-1-j}1^{j+1} \circ 0^{n-1-j}1^j \mathbf{Z}_1\right\rangle^j}.$$

Knowing that $\mathbf{Z}_2 = 00$ and $\mathbf{Z}_1 = 0$, we can easily check that any pair of consecutive words differ by exactly one bit, which proves that Δ_n^{n-1} is a 1-Gray code.

For the case (iv), we set: $\Delta_n^n = \bigcirc_{j=1}^{\frac{n-3}{2}} 0^{n-1-j}1^j 0 \circ \overleftarrow{\bigcirc_{j=1}^{\frac{n-1}{2}} 0^{n-j}1^j}$. As previously the result can be obtained easily.

The case (v) is more challenging to handle. The set $\mathcal{D}_n^r \cup \mathcal{D}_n^{r-1}$ consists of the union of the following subsets: $K_1 = 0^{r-2}1\mathcal{Z}_{n-r+1}$, $K_2 = 0^{r-3}11\mathcal{Z}_{n-r+1}$, $\ldots$, $K_a = 0^{r-a-1}1^a\mathcal{Z}_{n-r+1}$ and $L_1 = 0^{r-1}1\mathcal{Z}_{n-r}$, $L_2 = 0^{r-2}11\mathcal{Z}_{n-r}$, $\ldots$, $L_{a+1} = 0^{r-a-1}1^{a+1}\mathcal{Z}_{n-r}$ with $a = \lfloor \frac{r-2}{2} \rfloor = \frac{r-3}{2}$. Let us denote by $\mathbf{K}_1, \mathbf{K}_2, \ldots, \mathbf{K}_a$ and $\mathbf{L}_1, \mathbf{L}_2, \ldots, \mathbf{L}_{a+1}$ the associated Gray codes obtained by replacing $\mathcal{Z}_k$ with the Gray code $\mathbf{Z}_k$.

Remark that for $1 \leqslant i \leqslant a-1$ (resp. $1 \leqslant i \leqslant a$) and for a given j, the jth word of $\mathbf{K}_i$ (resp. $\mathbf{L}_i$) and the jth word of $\mathbf{K}_{i+1}$ (resp. $\mathbf{L}_{i+1}$) differ by exactly one bit. Recall that $\widehat{\mathbf{A}}$ is obtained from a list $\mathbf{A}$ by deleting its first element. The words $\mathrm{first}(\mathbf{K}_i)$ and $\mathrm{last}(\mathbf{L}_{i+1})$ differ by one bit. We have $\mathrm{first}(\widehat{\mathbf{L}_{a+1}} \circ \mathbf{K}_a) = \mathrm{first}(\widehat{\mathbf{L}_{a+1}})$ and it differs obviously by one bit from $\mathrm{first}(\mathbf{L}_{a+1})$. We also have $r = 3 \mod 4$, so $\frac{r-3}{2} = a$ is even. Taking into account all these remarks, the list Δ_n^r defined below is a Gray code:

$$\Delta_n^r = \bigcirc_{i=1}^{a+1} \mathrm{first}(\mathbf{L}_i) \circ \bigcirc_{i=a}^{1} \left\langle \widehat{\mathbf{L}_{i+1}} \circ \mathbf{K}_i\right\rangle^i \circ \widehat{\mathbf{L}_1}$$

We refer to Fig. 3 for a graphical representation of this Gray code. □

Theorem 3. *For any $n \geqslant 0$, there exists a 1-Gray code $\mathbf{Z}_n$ for $\mathcal{Z}_n$ such that $\mathrm{first}(\mathbf{Z}_n) = 0(001)^\star$ and $\mathrm{last}(\mathbf{Z}_n) = (001)^\star$.*

Using initial conditions $\mathbf{Z}_0 = \epsilon, \mathbf{Z}_1 = 0, \mathbf{Z}_2 = 00, \mathbf{Z}_3 = 000 \circ 001$, where ϵ is the empty word, and the recursive constructions for lists Δ_n^r (Lemmas 3 and 4) we define the 1-Gray code $\mathbf{Z}_n$ as follows:

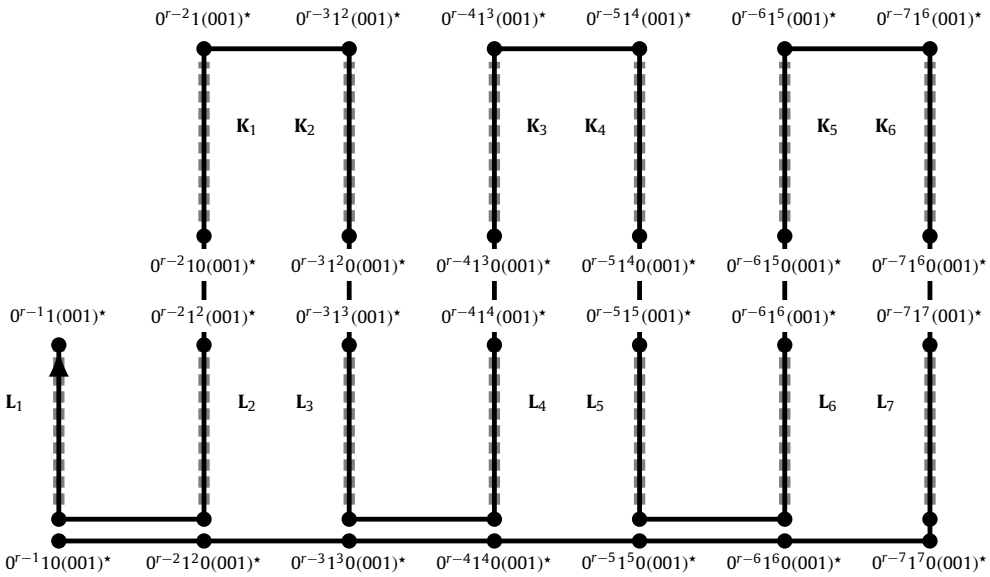


Fig. 3. An illustration of the Gray code Δ_n^r for the case (v) in the proof of Lemma 4 (we consider $a = 6$, $r = 15$). Vertical sequences of squares are Gray codes $\mathbf{K}_i$, $1 \leqslant i \leqslant a$, and $\mathbf{L}_i$, $1 \leqslant i \leqslant a+1$, so that the first and the last elements are respectively the bottom and top squares of the segments. The walk illustrates the Gray code Δ_n^r that starts with first($\mathbf{L}_1$) and ends with last($\mathbf{L}_1$).

$$\mathbf{Z}_n = \begin{cases} \Delta_n^5 \circ \Delta_n^9 \circ \cdots \circ \Delta_n^n \circ 0^n \circ \Delta_n^{n-2} \circ \cdots \circ \Delta_n^7 \circ \Delta_n^3 & \text{if } n = 1 \mod 4, \\ \Delta_n^5 \circ \Delta_n^9 \circ \cdots \circ \Delta_n^{n-1} \circ 0^n \circ \Delta_n^{n-3} \circ \cdots \circ \Delta_n^7 \circ \Delta_n^3 & \text{if } n = 2 \mod 4, \\ \Delta_n^5 \circ \Delta_n^9 \circ \cdots \circ \Delta_n^{n-2} \circ 0^n \circ \Delta_n^n \circ \cdots \circ \Delta_n^7 \circ \Delta_n^3 & \text{if } n = 3 \mod 4. \\ \Delta_n^5 \circ \Delta_n^9 \circ \cdots \circ \Delta_n^{n-3} \circ 0^{n-1}1 \circ 0^n \circ \Delta_n^{n-1} \circ \cdots \circ \Delta_n^7 \circ \Delta_n^3 & \text{if } n = 0 \mod 4. \end{cases}$$

Due to Lemmas 4 and 5, last(Δ_n^{4i+1}) differs by one bit from first(Δ_n^{4i+5}) and first(Δ_n^{4i+3}) differs by one bit from last(Δ_n^{4i+7}). Using 0^n (and $0^{n-1}1$ when $n = 0 \mod 4$) we connected all Δ_n^r and ensure that $\mathbf{Z}_n$ is a 1-Gray code. See Table 2 for a more detailed view on the structure of $\mathbf{Z}_n$. □

We refer to Fig. 5 for a graphical representation of $\mathbf{Z}_n$ for $4 \leqslant n \leqslant 10$, see also Fig. 2(b) for $n = 17$. An immediate consequence of Theorem 3 is the following (see Table 3 for the list $\mathbf{W}_6^1$).

Theorem 4. *For any $n \geqslant 1$, $\mathbf{W}_n^1 = 1 \cdot \mathbf{W}_{n-1}^1 \circ \mathbf{Z}_n$ is a 1-Gray code for $\mathcal{W}_n^1$ such that* first($\mathbf{W}_n^1$) $= 1^n$, last($\mathbf{W}_n^1$) $= (001)^\star$, *and where $\mathbf{W}_0^1$ is a list containing only the empty word.*

However, the efficient generation of this Gray code remains an open problem.

Eğecioğlu and Iršič introduce in [9] the *"run-constrained binary strings"*. These are binary words, in which every run of 1s is immediately followed by a strictly longer run of 0s. Using these strings of length $n+2$ as vertices, and connecting two vertices if they differ at only one position, the authors of [9] form the *Fibonacci-run graph* $\mathcal{R}_n$ as the induced subgraph of the hypercube. (As every non-empty run-constrained string must end with 00, authors of [9] actually drop the last 2 zeros, but we do not.) Fig. 4 gives small examples.

It turns out that the run-constrained binary strings are precisely the reverse of 1-decreasing words beginning with 0. In this light, the Gray code $\mathbf{Z}_n$ in Theorem 3 gives a Hamiltonian path in the Fibonacci-run graph. The next corollary settles a conjecture in [9].

Corollary 8. *For any $n \geqslant 1$, the Fibonacci-run graph $\mathcal{R}_n$ has a Hamiltonian path.*

Lemma 9.1 from [9] says that if $n \neq 1 \mod 3$, then $\mathcal{R}_n$ does not contain a Hamiltonian cycle. Our method gives a Hamiltonian path, which is not a cycle. The question of whether there is a Hamiltonian cycle for the case $n = 1 \mod 3$ remains open.

Finally, the validity of the parity condition stated in Corollary 1 and experimental investigations for small values, $0 \leqslant n \leqslant 5$ and $2 \leqslant q \leqslant 5$, suggest the following extension of Theorem 4.

Conjecture 1. *For any $n \geqslant 0$ and $q \geqslant 1$, there is a 1-Gray code for $\mathcal{W}_n^q$.*

Table 2
The structure of $\mathbf{Z}_n$.

		$n=1 \mod 4$	$n=2 \mod 4$	$n=3 \mod 4$	$n=0 \mod 4$
Lemma 4	$\Delta_n^5=$	$0001(001)^\star$ … $00001(001)^\star$	$0001(001)^\star$ … $00001(001)^\star$	$0001(001)^\star$ … $00001(001)^\star$	$0001(001)^\star$ … $00001(001)^\star$
	$\Delta_n^9=$	$00000\,001(001)^\star$ … $000000001(001)^\star$	$00000\,001(001)^\star$ … $000000001(001)^\star$	$00000\,001(001)^\star$ … $000000001(001)^\star$	$00000\,001(001)^\star$ … $000000001(001)^\star$
	…	…	…	…	…
	$\Delta_n^{n-4}=$	$0^{n-6}1(001)^\star$ … $0^{n-5}1(001)^\star$	$0^{n-7}1(001)^\star$ … $0^{n-6}1(001)^\star$	$0^{n-8}1(001)^\star$ … $0^{n-7}1(001)^\star$	$0^{n-9}1(001)^\star$ … $0^{n-8}1(001)^\star$
	$\Delta_n^{n}=$	$0^{n-2}10$ … $0^{n-1}1$	$0^{n-3}100$ … $0^{n-1}1$	$0^{n-4}1(001)^\star$ … $0^{n-3}1(001)^\star$	$0^{n-5}1(001)^\star$ … $0^{n-4}1(001)^\star$
		0^n	0^n	0^n	$0^{n-1}1$ 0^n
Lemma 5	$\Delta_n^{n-2}=$	$0^{n-4}1000$ … $0^{n-3}100$	$0^{n-4}10(001)^\star$ … $0^{n-4}1(001)^\star$	$0^{n-2}10$ … $0^{n-1}1$	$0^{n-3}100$ … $0^{n-2}10$
	$\Delta_n^{n-6}=$	$0^{n-7}10(001)^\star$ … $0^{n-7}1(001)^\star$	$0^{n-8}10(001)^\star$ … $0^{n-8}1(001)^\star$	$0^{n-5}10(001)^\star$ … $0^{n-5}1(001)^\star$	$0^{n-6}10(001)^\star$ … $0^{n-6}1(001)^\star$
	…	…	…	…	…
	$\Delta_n^{7}=$	$00000010(001)^\star$ … $0000\,001(001)^\star$	$00000010(001)^\star$ … $0000\,001(001)^\star$	$00000010(001)^\star$ … $0000\,001(001)^\star$	$00000010(001)^\star$ … $0000\,001(001)^\star$
	$\Delta_n^{3}=$	$0010(001)^\star$ … $(001)^\star$	$0010(001)^\star$ … $(001)^\star$	$0010(001)^\star$ … $(001)^\star$	$0010(001)^\star$ … $(001)^\star$

Table 3
The Gray code $\mathbf{W}_6^1$ for the set $\mathcal{W}_6^1$. The Hamming distance between two consecutive words is one.

1	111111	*8*	110010	*15*	000110
2	111110	*9*	100010	*16*	000010
3	111100	*10*	100011	*17*	000011
4	111000	*11*	100001	*18*	000001
5	111001	*12*	100000	*19*	000000
6	110001	*13*	100100	*20*	001000
7	110000	*14*	000100	*21*	001001

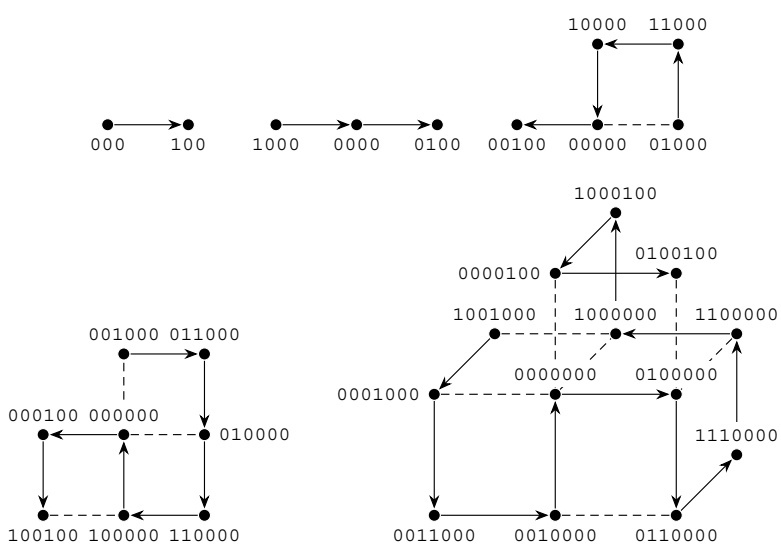


Fig. 4. Fibonacci-run graphs for small values of n. Vertices correspond to the reverse of words from $\mathcal{W}_n^1$ beginning with 0. The Hamiltonian path is provided by Corollary 8. If we read words from right to left, the path starts at $0(001)^\star$ and ends at $(001)^\star$.

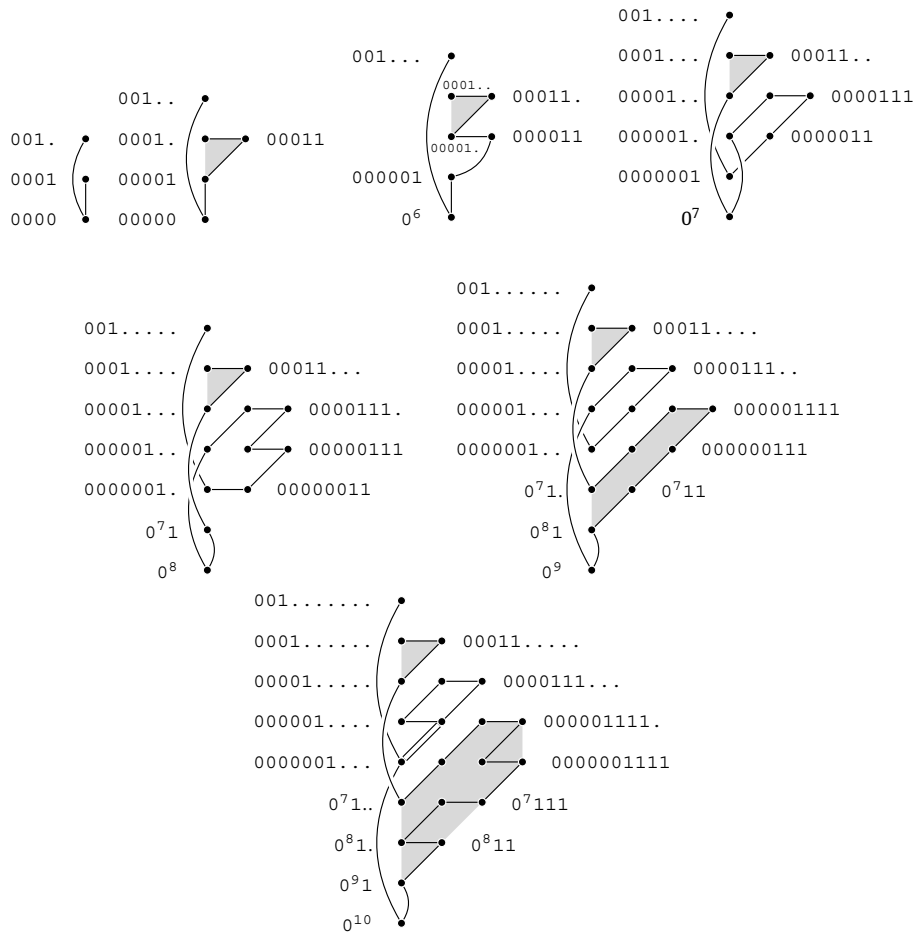


Fig. 5. An illustration of the recursive definition for the Gray codes $\mathbf{Z}_n$, $4 \leqslant n \leqslant 10$. A point labeled `001.....` (that is `0001` followed by seven dots) corresponds to the set of words in $001 \cdot \mathcal{Z}_7$.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgement

The authors are grateful to the anonymous referees for helping to rectify an error in the construction of the 1-Gray code in the last part of the paper and for providing numerous comments helping to improve the presentation of the paper.

References

[1] J. Arndt, Matters computational - ideas, algorithms, source code [the fxtbook], http://www.jjj.de/fxt/#fxtbook, 2010.
[2] J.-L. Baril, S. Kirgizov, V. Vajnovszki, Asymptotic bit frequency in Fibonacci words, arXiv:2106.13550 [.org/pdf], 2019.
[3] J.-L. Baril, V. Vajnovszki, Minimal change list for Lucas strings and some graph theoretic consequences, Theor. Comput. Sci. 346 (2–3) (2005) 189–199.
[4] D. Roelants van Baronaigien, F. Ruskey, Generating subsets with a given sum, J. Comb. Math. Comb. Comput. 14 (1993) 87–96.
[5] A. Bernini, Restricted binary strings and generalized Fibonacci numbers, in: International Workshop on Cellular Automata and Discrete Complex Systems, Springer, 2017, pp. 32–43.
[6] A. Bernini, S. Bilotta, R. Pinzani, A. Sabri, V. Vajnovszki, Reflected Gray codes for q-ary words avoiding a given factor, Acta Inform. 52 (7) (2015) 573–592.
[7] J. Berstel, D. Perrin, The origins of combinatorics on words, Eur. J. Comb. 28 (3) (2007) 996–1022.
[8] M. Crochemore, W. Rytter, Jewels of stringology, in: Text Algorithms, World Scientific Publishing Co., Inc., River Edge, NJ, 2003.
[9] Ö. Eğecioğlu, V. Iršič, Fibonacci-run graphs I: basic properties, Discrete Appl. Math. 295 (2021) 70–84.
[10] Ö. Eğecioğlu, V. Iršič, Fibonacci-run graphs II: degree sequences, Discrete Appl. Math. 300 (2021) 56–71.
[11] P. Flajolet, R. Sedgewick, Analytic Combinatorics, Cambridge University Press, 2009.
[12] F. Gray, Pulse code communication (Mar 17 1953), US Patent 2,632,058.
[13] W.-J. Hsu, Fibonacci cubes–a new interconnection topology, IEEE Trans. Parallel Distrib. Syst. 4 (1) (1993) 3–12.
[14] S. Klavžar, On median nature and enumerative properties of Fibonacci-like cubes, Discrete Math. 299 (1) (2005) 145–153.
[15] D.E. Knuth, The Art of Computer Programming, Volume 3: Sorting and Searching, 2nd ed., Addison Wesley Longman Publishing Co., Inc., 1998.

[16] D.E. Knuth, The Art of Computer Programming, Volume 4, Fascicle 4, Generating All Trees–History of Combinatorial Generation, Addison-Wesley Professional, 2006.
[17] M. Lothaire, Combinatorics on Words, vol. 17, Cambridge University Press, 1997.
[18] M. Lothaire, Algebraic Combinatorics on Words, vol. 90, Cambridge University Press, 2002.
[19] M. Lothaire, Applied Combinatorics on Words, vol. 105, Cambridge University Press, 2005.
[20] E.P. Miles Jr., Generalized Fibonacci numbers and associated matrices, Am. Math. Mon. 67 (8) (1960) 745–752.
[21] F.J. Rispoli, S. Cosares, The Fibonacci hypercube, Australas. J. Comb. 40 (2008) 187–196 [electronic only].
[22] F. Ruskey, Simple combinatorial Gray codes constructed by reversing sublists, in: 4th ISAAC, International Symposium on Algorithms and Computation, in: Lecture Notes in Computer Science, vol. 762, 1993, pp. 201–208.
[23] F. Ruskey, Combinatorial Generation, Electronic available book, 2003.
[24] J. Sawada, A. Williams, D. Wong, Inside the binary reflected Gray code: flip-swap languages in 2-Gray code order, in: T. Lecroq, S. Puzynina (Eds.), Combinatorics on Words 2021, in: Lecture Notes in Computer Science, vol. 12847, 2021.
[25] P. Singh, The so-called Fibonacci numbers in ancient and medieval India, Hist. Math. 12 (3) (1985) 229–244.
[26] N.J.A. Sloane, The on-line encyclopedia of integer sequences, available electronically at http://oeis.org.
[27] M.B. Squire, Gray codes for A-free strings, Electron. J. Comb. (1996) #R17.
[28] V. Vajnovszki, A loopless generation of bitstrings without p consecutive ones, in: Combinatorics, Computability and Logic, Springer, 2001, pp. 227–240.
[29] V. Vajnovszki, Gray code order for Lyndon words, Discret. Math. Theor. Comput. Sci. 9 (2) (2007) 145–151.

AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **84(3)** (2022), Pages 398–418

Dyck paths with catastrophes modulo the positions of a given pattern

Jean-Luc Baril Sergey Kirgizov

LIB, Université de Bourgogne Franche-Comt
B.P. 47 870, 21078 Dijon Cedex
France
barjl@u-bourgogne.fr sergey.kirgizov@u-bourgogne.fr

Armen Petrossian

ESEO Paris
B.P. 47 870, 21078 Dijon Cedex
France
armen.petrossian@eseo.fr

Abstract

For any pattern p of length at most 2, we provide generating functions and asymptotic approximations for the number of p-equivalence classes of Dyck paths with catastrophes, where two paths of the same length are p-equivalent whenever the positions of the occurrences of the pattern p are the same.

1 Introduction and notation

A *Dyck path with catastrophes* is a lattice path in the first quadrant of the xy-plane that starts at the origin, ends on the x-axis, and is made of up-steps $U = (1, 1)$, down-steps $D = (1, -1)$, and catastrophe steps $C_k = (1, -k)$, $k \geqslant 2$, so that catastrophe steps always end on the x-axis. Depending on the context, we can use the symbol C to design a catastrophe step, and by convenience we use $C_1 = D$. We let $\mathcal{E}$ denote the set of all Dyck paths with catastrophes, and $\mathcal{D}$ be the set of *Dyck paths*, i.e. the paths in $\mathcal{E}$ that do not contain any catastrophe steps C_k, $k \geqslant 2$. The *length* $|P|$ of a path P is the number of its steps. The empty path is denoted by ϵ. See Figure 1 for an example of a Dyck path with catastrophes of length 14. A *pattern* consists of consecutive steps in a path. We will say that *an occurrence of a pattern* (or for short *a pattern*) is at *position* $i \geqslant 1$ in a path whenever the first step of the pattern appears at the i-th step of the path, the second step at the $(i + 1)$-st step, and so on. The

ISSN: 2202-3518 ©The author(s). Released under the CC BY-ND 4.0 International License

height of an occurrence of a pattern is the minimal ordinate reached by its points. For instance, the Dyck path with catastrophes $P = UUC_2UUUDUDDUC_2UD$ contains three occurrences of the pattern UU at positions 1, 4 and 5, and the heights of these occurrences are respectively 0, 0 and 1.

Figure 1: Dyck path with catastrophes $UUC_2UUUDUDDUC_2UD$.

The concept of a Dyck path with catastrophes was first introduced by Krinik et al. in [9] in the context of queuing theory. These paths correspond to the evolution of a queue by allowing some resets modeled by a catastrophe step C_k, $k \geqslant 2$. Then, Banderier and Wallner [1] provided enumerative results and limit laws of these objects. They showed how any non empty path $P \in \mathcal{E}$ can be decomposed either as $P = U\alpha D\beta$, or $P = U\alpha_1 U\alpha_2 \dots U\alpha_k C_k \beta$ for some $k \geqslant 2$, where $\alpha, \alpha_1, \alpha_2, \dots, \alpha_k$ are Dyck paths in $\mathcal{D}$ and $\beta \in \mathcal{E}$. They deduced a functional equation for the generating function $E(x) = \sum_{n \geqslant 0} e_n x^n$ where e_n is the number of paths of length n in $\mathcal{E}$, with the solution

$$E(x) = \frac{2x - 1 + \sqrt{1 - 4\,x^2}}{x - 1 + (1 + x)\sqrt{1 - 4\,x^2}}.$$

The sequence $(e_n)_{n \geqslant 0}$ corresponds to A224747 in the On-line Encyclopedia of Integer Sequences (OEIS) [12], and the first values for $n \geqslant 0$ are $1, 0, 1, 3, 5, 12, 23, 52, 105, 232, 480$. More recently, Baril and Kirgizov [4] exhibited a one-to-one correspondence between Dyck paths with catastrophes of length n and Dyck paths of length $2n$ avoiding UUU and DUD at height at least one, and where every occurrence of UD on the x-axis appears before (but not necessarily contiguous with) an occurrence of UUU.

On the other hand, in [2–6, 10] the authors investigated equivalence relations on the sets of Dyck paths, Motzkin paths, skew Dyck paths, Łukasiewicz paths, and Ballot paths where two paths of the same length are equivalent whenever they coincide on all occurrences of a given pattern. The main goal of this study consists in extending these studies for Dyck paths with catastrophes by considering the analogous equivalence relation on $\mathcal{E}$.

Definition 1.1 Two Dyck paths of the same length and with catastrophes are p-equivalent whenever they have the same positions of the occurrences of the pattern p.

For instance, the path $UUDUUC_3$ is U-equivalent to UUC_2UUC_2 since the occurrences of U appear at the same positions in the two paths.

Then, one may naturally split the set $\mathcal{E}$ into *p-equivalence classes*, which are constructed so that paths P and Q in $\mathcal{E}$ belong to the same p-equivalence class if, and only if, they are p-equivalent.

In this paper, we provide ordinary generating functions (o.g.f. for short) for the number of p-equivalence classes in $\mathcal{E}$ with respect to the length whenever p is a pattern of length at most two. Our method consists in providing one-to-one correspondences between equivalence classes and certain subsets of $\mathcal{E}$ (called subsets of *representative elements*) and enumerating them using algebraic techniques. Remark that only one pattern of size 2, namely DD, gives a non-rational generating function. For this pattern, the construction of a set of representative elements and its enumeration are quite intricate and handled in the last subsection of the paper. We refer to Table 1 for an overview of our results.

Pattern	OEIS [12]	Sequences for a_n, $2 \leqslant n \leqslant 10$, and formulæ	Theorem	O.g.f.
U	Shift of A037952	1, 1, 3, 4, 10, 15, 35, 56, 126, ... $\binom{n-1}{\lfloor \frac{n-2}{2} \rfloor}$	Thm 2.2	Algebraic
D	New	1, 1, 3, 5, 11, 21, 42, 84, 162, ... $\begin{cases} \binom{n}{\frac{n-3}{2}} & \text{if } n \text{ is odd,} \\ \binom{n}{\frac{n-4}{2}} + \binom{n}{\frac{n}{2}} / \left(\frac{n}{2}+1\right) & \text{otherwise} \end{cases}$	Thm 2.4	
C	A212804	1, 1, 2, 3, 5, 8, 13, 21, 34, ... $\begin{cases} c_n = c_{n-1} + c_{n-2}, \\ c_0 = 1, c_1 = 0 \end{cases}$	Thm 2.6	Rational
UU	A347493	1, 1, 3, 4, 8, 13, 24, 41, 75, ... $\begin{cases} f_n = f_{n-1} + f_{n-2} + f_{n-4}, \\ f_0 = 1, f_1 = 0, f_2 = 1, f_3 = 1 \end{cases}$	Thm 3.2	
UD	A215004	1, 1, 3, 5, 10, 17, 30, 50, 84, ... $\begin{cases} g_n = 2g_{n-1} + g_{n-2} - 3g_{n-3} + g_{n-5}, \\ g_1 = 0, g_0 = g_2 = g_3 = 1, g_4 = 3 \end{cases}$	Thm 3.4	
UC	New	1, 1, 2, 4, 5, 9, 15, 24, 40, ... $\begin{cases} i_n = 2i_{n-1} - i_{n-2} + i_{n-3} - i_{n-4} + i_{n-5} - i_{n-6} + i_{n-7}, \\ i_0 = 1, i_1 = 0, i_2 = 1, i_3 = 1, i_4 = 2, i_5 = 4, i_6 = 5 \end{cases}$	Thm 3.6	
DC	New	1, 1, 1, 2, 2, 4, 6, 9, 14, ... $\begin{cases} j_n = 2j_{n-1} - j_{n-2} + j_{n-5} - j_{n-6} + j_{n-7}, \\ j_0 = 1, j_1 = 0, j_2 = j_3 = j_4 = 1, j_5 = 2, j_6 = 2 \end{cases}$	Thm 3.8	
CU	Shift of A000045	1, 1, 1, 2, 3, 5, 8, 13, 21, ... $\begin{cases} k_n = k_{n-1} + k_{n-2}, \\ k_0 = 1, k_1 = 0, k_2 = k_3 = k_4 = 1 \end{cases}$	Thm 3.10	
DU	A212804	1, 1, 2, 3, 5, 8, 13, 21, 34, ... $\begin{cases} \ell_n = \ell_{n-1} + \ell_{n-2}, \\ \ell_0 = 1, \ell_1 = 0, \ell_2 = 1 \end{cases}$	Thm 3.12	
DD	New	1, 1, 2, 1, 4, 5, 11, 11, 27, ... General formula remains to be found	Thm 3.15	Algebraic

Table 1: Summary of results; generating functions are given in the corresponding theorems.

2 Patterns of length one

Our goal in this section is to provide generating functions that count p-equivalence classes with respect to the path length, whenever the pattern p is U, D, or C. For

each pattern, the method consists in constructing a set of representative elements of p-equivalence classes and enumerating them with respect to the length by giving a generating function. From this, we deduce a closed-form expression (that depends on the length) or a recurrence relation for the number of corresponding p-equivalence classes.

2.1 Pattern U

Let us define the set $\mathcal{A}$ consisting of the union of the set $\mathcal{D}$ of Dyck paths with the set $\mathcal{D}'$ of paths in $\mathcal{E}$ having only one catastrophe C_k, $k \geqslant 2$, located at the end of the path. We will prove that there is a one-to-one correspondence between $\mathcal{A}$ and the U-equivalence classes of $\mathcal{E}$, and we enumerate them by providing a generating function that counts $\mathcal{A}$ with respect to the length.

Theorem 2.1 *There is a bijection between $\mathcal{A}$ and the set of U-equivalence classes of $\mathcal{E}$.*

Proof. First, we prove that for every $P \in \mathcal{E}$ there is $Q \in \mathcal{A}$ such that P and Q belong to the same U-equivalence class. If we have $P \in \mathcal{D}$, then we choose $Q = P \in \mathcal{A}$. Otherwise, we decompose $P = \alpha_1 C_{k_1} \alpha_2 C_{k_2} \dots \alpha_r C_{k_r} \alpha_{r+1}$ where $r \geqslant 1$, $k_i \geqslant 2$ for $1 \leqslant i \leqslant r$ and all α_i do not contain any catastrophe steps. If α_{r+1} is empty, then we set $Q = \alpha_1 D \alpha_2 D \dots \alpha_r C_k$ where $k = (k_1 - 1) + (k_2 - 1) + \dots + (k_{r-1} - 1) + k_r = \sum_{i=1}^{r} k_i - (r-1) \geqslant 2r - (r-1) \geqslant 2$. If α_{r+1} is not empty, it can be written $\alpha' D$, and we set $Q = \alpha_1 D \alpha_2 D \dots \alpha_r D \alpha' C_k$ where $k = k_1 - 1 + k_2 - 1 + k_{r-1} - 1 + k_r - 1 + 1 = \sum_{i=1}^{r} k_i - r + 1 \geqslant 2r - r + 1 \geqslant 2$. In these last two cases, we have $Q \in \mathcal{A}$ so that P and Q belong to the same U-equivalence class.

Now, let us prove that if P and Q are two paths of the same length in $\mathcal{A}$ lying in the same U-equivalence class, then $P = Q$. According to the decomposition of a Dyck path with catastrophes (see the Introduction), we write either $P = \alpha$ or $P = \alpha U \alpha_1 U \alpha_2 \dots U \alpha_k C_k$ (respectively $Q = \alpha'$ or $Q = \alpha' U \alpha'_1 U \alpha'_2 \dots U \alpha'_\ell C_\ell$) for some $k, \ell \geqslant 2$ and where α, α', α_i, α'_i are some Dyck paths. Since a Dyck path is characterized by the positions of its up steps, and P and Q are in the same U-equivalence class, we necessarily have $\alpha = \alpha'$, $\alpha'_i = \alpha'_i$ for $i \leqslant \min\{k, \ell\}$, which implies that $P = Q$. □

Theorem 2.2 *The o.g.f. with respect to the length for the set $\mathcal{A}$ is given by*

$$A(x) = \frac{\left(1 - \sqrt{1 - 4x^2}\right)(1 - x)}{x\left(2x - 1 + \sqrt{1 - 4x^2}\right)}.$$

The series expansion of $A(x)$ is

$$1 + x^2 + x^3 + 3x^4 + 4x^5 + 10x^6 + 15x^7 + 35x^8 + 56x^9 + 126x^{10} + 210x^{11} + O\left(x^{12}\right).$$

Proof. A path in $\mathcal{A}$ is either a Dyck path in $\mathcal{D}$, or a path $\alpha U\alpha_1 U\alpha_2 U\alpha_3 \dots U\alpha_k C_k$ where $k \geqslant 2$ and $\alpha, \alpha_1, \dots, \alpha_k$ are some Dyck paths. We deduce that $A(x) = D(x) + D(x) \cdot \frac{x^2 D(x)^2}{1-xD(x)} \cdot x$ where $D(x) = \frac{1-\sqrt{1-4x^2}}{2x^2}$ is the o.g.f. that counts Dyck paths with respect to the length (see [7]). □

Remark 2.1 With Theorem 2.2, it is easy to check that $x^2 - 2x + 1 + (-2x^2 + 3x - 1)A(x) + (2x^2 - x)A(x)^2 = 0$, and the coefficient a_n of x^n in $A(x)$ is $\binom{n-1}{\lfloor \frac{n-2}{2} \rfloor}$ which generates a shift of the sequence A037952 in OEIS [12]. Using a classical analysis [8, 11] of dominant singularity of $A(x)$, a_n has the asymptotic approximation $\dfrac{2^n}{\sqrt{2\pi n}}$.

2.2 Pattern D

Let us define the set $\mathcal{B}$ consisting of the union of the set $\mathcal{D}$ of Dyck paths with the set $\mathcal{D}''$ of paths $P \in \mathcal{E}$ having only one catastrophe.

Theorem 2.3 *There is a bijection between $\mathcal{B}$ and the set of D-equivalence classes of $\mathcal{E}$.*

Proof. First, we prove that for every $P \in \mathcal{E}$ there is $Q \in \mathcal{B}$ such that P and Q belong to the same D-equivalence class. If we have $P \in \mathcal{D}$, then we choose $Q = P \in \mathcal{B}$; otherwise, we decompose $P = \alpha_1 C_{k_1} \alpha_2 C_{k_2} \dots \alpha_r C_{k_r} \alpha_{r+1}$ where $r \geqslant 1$, $k_i \geqslant 2$ for $1 \leqslant i \leqslant r$ and all α_i do not contain any catastrophe steps. If α_{r+1} is empty, then we set $Q = \alpha_1 U\alpha_2 U \dots \alpha_r C_k$ where $k = (k_1 + 1) + (k_2 + 1) + \dots + (k_{r-1} + 1) + k_r = r - 1 + \sum_{i=1}^{r} k_i$. If α_{r+1} is not empty, it is necessarily a Dyck path, and we set $Q = \alpha_1 U\alpha_2 U \dots \alpha_r C_k \alpha_{r+1}$ where $k = k_1 + 1 + k_2 + 1 + k_{r-1} + 1 + k_r = r - 1 + \sum_{i=1}^{r} k_i$. In all these cases, we have $Q \in \mathcal{B}$ so that P and Q belong to the same D-equivalence class.

With a similar argument as for the proof of Theorem 2.1, it is easy to prove that if P and Q are two paths in $\mathcal{B}$ lying in the same class, then $P = Q$.

□

Theorem 2.4 *The o.g.f. with respect to the length for the set $\mathcal{B}$ is given by*

$$B(x) = \frac{\left(1 - \sqrt{1-4x^2}\right)\left(2x^3 - 4x^2 + 1 + (2x^2 - 1)\sqrt{1-4x^2}\right)}{2x^4\left(2x - 1 + \sqrt{1-4x^2}\right)}.$$

The series expansion of $B(x)$ is

$$1 + x^2 + x^3 + 3x^4 + 5x^5 + 11x^6 + 21x^7 + 42x^8 + 84x^9 + 162x^{10} + 330x^{11} + O\left(x^{12}\right).$$

Proof. A path in $\mathcal{B}$ is either a Dyck path in $\mathcal{D}$, or a path of the form $\alpha U\alpha_1 U\alpha_2 U\alpha_3 \dots U\alpha_k C_k \beta$ where $k \geqslant 2$ and $\alpha, \alpha_1, \dots, \alpha_k, \beta$ are some Dyck paths. We deduce that

$B(x)$ satisfies the functional equation $B(x) = D(x) + \frac{x^3 D(x)^4}{1-xD(x)}$ where $D(x) = \frac{1-\sqrt{1-4x^2}}{2x^2}$ is the o.g.f. that counts Dyck paths with respect to the length (see [7]). □

Remark 2.2 We have

$$5x^3 - 4x^2 - x + 1 + (-2x^4 - 5x^3 + 5x^2 + x - 1)B(x) + (2x^5 - x^4)B(x)^2 = 0,$$

and the coefficient b_n of x^n in $B(x)$ is

$$b_n = \begin{cases} \binom{n}{\frac{n-3}{2}} & \text{if } n \text{ is odd,} \\ \binom{n}{\frac{n-4}{2}} + \binom{n}{\frac{n}{2}} / \left(\frac{n}{2}+1\right) & \text{otherwise.} \end{cases}$$

This last result can be easily obtained by combining already known formulæ (see A002054 and A344191 in [12]), or directly using Maple. The sequence b_n has an asymptotic approximation $\frac{2^{n+\frac{1}{2}}}{\sqrt{\pi n}}$.

2.3 Pattern C

In this section, two paths of the same length are *C-equivalent* whenever, for any $k \geqslant 2$ they have the same positions of the occurrences of C_k. For instance,

$$UUDDUUC_2UUUC_3UUC_2UD$$

is C-equivalent to

$$UDUDUUC_2UUUC_3UUC_2UD,$$

but not to

$$UUUDUUC_4UUUC_3UUC_2UD.$$

Let $\mathcal{C}$ be the set of paths $P \in \mathcal{E}$ such that:

(i) $P = (UD)^k$ for $k \geqslant 0$, or
(ii) $P = (UD)^{\ell_1} U^{k_1} C_{k_1} (UD)^{\ell_2} U^{k_2} C_{k_2} \dots (UD)^{\ell_r} U^{k_r} C_{k_r} (UD)^{\ell_{r+1}}$ with $r \geqslant 1$, $\ell_i \geqslant 0$ for $1 \leqslant i \leqslant r+1$, and $k_i \geqslant 2$ for $1 \leqslant i \leqslant r$.

Theorem 2.5 *There is a bijection between $\mathcal{C}$ and the set of C-equivalence classes of $\mathcal{E}$.*

Proof. First, we prove that for every $P \in \mathcal{E}$ there is $Q \in \mathcal{C}$ such that P and Q belong to the same C-equivalence class. If we have $P \in \mathcal{D}$, then we choose $Q = (UD)^k \in \mathcal{C}$ with $k = |P|/2$. Otherwise, we decompose $P = \alpha_1 C_{k_1} \alpha_2 C_{k_2} \dots \alpha_r C_{k_r} \alpha_{r+1}$ where $r \geqslant 1$, $k_i \geqslant 2$ for $1 \leqslant i \leqslant r$ and all α_i do not contain any catastrophe steps. We set $Q = (UD)^{j_1} U^{k_1} C_{k_1} (UD)^{j_2} U^{k_2} C_{k_2} \dots (UD)^{j_r} U^{k_r} C_{k_r} (UD)^{j_{r+1}}$ where $j_i + k_i = |\alpha_i|$, $1 \leqslant i \leqslant r$ and $j_{r+1} = |\alpha_{r+1}|$. We have $Q \in \mathcal{C}$ so that P and Q belong to the same C-equivalence class. Due to the form of Q, it is straightforward to see that if P and Q are two paths in $\mathcal{C}$ lying in the same class, then $P = Q$. □

Theorem 2.6 *The o.g.f. with respect to the length for the set $\mathcal{C}$ is given by*

$$C(x) = \frac{1-x}{1-x-x^2}.$$

The series expansion of $C(x)$ is

$$1 + x^2 + x^3 + 2x^4 + 3x^5 + 5x^6 + 8x^7 + 13x^8 + 21x^9 + 34x^{10} + 55x^{11} + O\left(x^{12}\right).$$

Proof. Due to the definition of the set $\mathcal{C}$, we obtain directly the functional equation $C(x) = \frac{1}{1-x^2} + \frac{1}{1-x^2} \cdot \frac{C'(x)}{1-C'(x)}$ where $C'(x) = \frac{x^3}{1-x} \cdot \frac{1}{1-x^2}$ is the o.g.f. that counts the paths of the form $(UD)^\ell U^k C_k$ for $\ell \geqslant 0$ and $k \geqslant 2$. □

Remark 2.3 From Theorem 2.6, we deduce that the coefficient c_n of x^n in $C(x)$ satisfies $c_n = c_{n-1} + c_{n-2}$ for $n \geqslant 2$ with $c_0 = 1$ and $c_1 = 0$, which generates the sequence A212804 in [12] (it is a variant of the well known Fibonacci sequence A000045). Using the classical method for asymptotic approximation (see [8, 11]), we have

$$c_n \sim \frac{3-\sqrt{5}}{5-\sqrt{5}} \cdot \left(\frac{\sqrt{5}-1}{2}\right)^n.$$

3 Patterns of length two

In this section, we provide generating functions that count p-equivalence classes with respect to the length whenever the pattern p is UU, UD, UC, DC, CU, DU or DD. The techniques used are similar, although more elaborate, to those used in the previous section.

3.1 Pattern UU

Let $\mathcal{F}$ be the set of paths $P \in \mathcal{E}$ such that:

(i) $P = (UD)^k$ for $k \geqslant 0$, or

(ii) $P = (UD)^{\ell_0} U^{k_1} \alpha_1 U^{k_2} \alpha_2 \ldots U^{k_r} \alpha_r$ where $r \geqslant 1$, $\ell_0 \geqslant 0$, $k_i \geqslant 2$ for $1 \leqslant i \leqslant r$, all α_i for $1 \leqslant i \leqslant r-1$ are either $(DU)^k D$ or $(DU)^k DD$ for some $k \geqslant 0$, and α_r is either $(DU)^k C_s$ or $(DU)^k DC_s$ for some $k \geqslant 0$ and with $s \geqslant 1$ is so that the path ends on the x-axis (note that s can be 1, and in this case $C_1 = D$).

Theorem 3.1 *There is a bijection between $\mathcal{F}$ and the set of UU-equivalence classes of $\mathcal{E}$.*

Proof. First, we prove that for every $P \in \mathcal{E}$ there is $Q \in \mathcal{F}$ such that P and Q belong to the same UU-equivalence class. If P does not contain occurrences of UU, then we choose $Q = (UD)^k \in \mathcal{F}$ where $k = |P|/2$. Otherwise, we decompose

$P = \alpha_0 U^{k_1} \alpha_1 U^{k_2} \dots \alpha_{r-1} U^{k_r} \alpha_r$ where $r \geqslant 1$, $k_i \geqslant 2$ for $1 \leqslant i \leqslant r$, and such that all occurrences of UU in P belong necessarily to a run U^{k_i} for some i.

We set $Q = (UD)^{\ell_0} U^{k_1} \beta_1 U^{k_2} \dots \beta_{r-1} U^{k_r} \beta_r$ where $\ell_0 = |\alpha_0|$, and for $1 \leqslant i \leqslant r-1$, $\beta_i = (DU)^{\frac{t_i-1}{2}} D$ if $t_i = |\alpha_i|$ is odd, $\beta_i = (DU)^{\frac{t_i-2}{2}} DD$ otherwise; finally, we set $\beta_r = (DU)^{\frac{t_r-1}{2}} C_s$ if $t_r = |\alpha_r|$ is odd, and otherwise, $\beta_r = (DU)^{\frac{t_r-2}{2}} DC_s$ with $s \geqslant 1$ is chosen so that Q ends on the x-axis.

We have $Q \in \mathcal{F}$ so that P and Q belong to the same UU-equivalence class. It is straightforward to see that if P and Q are two paths in $\mathcal{F}$ lying in the same class, then $P = Q$. □

Theorem 3.2 *The o.g.f. with respect to the length for the set $\mathcal{F}$ is given by*

$$F(x) = \frac{x-1}{(x+1)\,(x^3 - x^2 + 2\,x - 1)}.$$

The series expansion of $F(x)$ is

$$1 + x^2 + x^3 + 3x^4 + 4x^5 + 8x^6 + 13x^7 + 24x^8 + 41x^9 + 73x^{10} + 127x^{11} + O\left(x^{12}\right).$$

Proof. Due to the definition of $\mathcal{F}$, we have $F(x) = \frac{1}{1-x^2} + \frac{1}{1-x^2} \cdot \frac{1}{1-\frac{x^3}{(1-x)^2}} \frac{x^3}{(1-x)^2}$. A simple calculation completes the proof. □

Remark 3.1 The coefficient f_n of x^n in $F(x)$ satisfies $f_n = f_{n-1} + f_{n-2} + f_{n-4}$ with $f_0 = 1, f_1 = 0, f_2 = 1, f_3 = 1$, which generates the sequence A347493 in [12]. An asymptotic approximation of f_n is

$$f_n \sim \frac{(a-1)(a^2-a+2)^n}{a(a+1)(-3a^2+2a-2)} \approx 0.26212 \cdot 1.75487^n,$$

where $a = \dfrac{(44+12\sqrt{69})^{2/3} + 2(44+12\sqrt{69})^{1/3} - 20}{6(44+12\sqrt{69})^{1/3}}$.

3.2 Pattern UD

Let $\mathcal{G}$ be the set of paths $P \in \mathcal{E}$ such that either :

(i) $P = (UD)^k$ for some $k \geqslant 0$, or
(ii) $P = (UD)^\ell U^k C_k (UD)^m$ with $\ell, m \geqslant 0$ and $k \geqslant 2$, or
(iii) $P = (UD)^{\ell_0} U^{k_1} (UD)^{\ell_1} U^{k_2} (UD)^{\ell_2} \dots U^{k_r} (UD)^{\ell_r} U^{k_{r+1}} C_s (UD)^{\ell_{r+1}}$ with $r \geqslant 1$, $\ell_0, \ell_{r+1} \geqslant 0$, $k_i \geqslant 1$ for $1 \leqslant i \leqslant r$, $k_{r+1} \geqslant 0$, $\ell_i \geqslant 1$ for $1 \leqslant i \leqslant r$, and $s \geqslant 1$ is so that the path ends on the x-axis (note that s can be one).

Theorem 3.3 *There is a bijection between $\mathcal{G}$ and the set of UD-equivalence classes of $\mathcal{E}$.*

Proof. First, we prove that for every $P \in \mathcal{E}$ there is $Q \in \mathcal{G}$ such that P and Q belong to the same UD-equivalence class. If P satisfies the case (i), then we obviously set $Q = P$. If $P = (UD)^\ell \alpha (UD)^m$ with $l, m \geqslant 0$ and α is a nonempty path in $\mathcal{E}$ avoiding UD, then we set $Q = (UD)^\ell U^k C_k (UD)^m$. Otherwise, we decompose $P = (UD)^{\ell_0}\alpha_1(UD)^{\ell_1}\alpha_2(UD)^{\ell_2}\dots\alpha_r(UD)^{\ell_r}\alpha_{r+1}$ with all α_i being non-empty partial paths avoiding UD (except α_{r+1} that can be empty). If $|\alpha_{r+1}| > 1$ then we set $Q = (UD)^{\ell_0}U^{k_1}(UD)^{\ell_1}U^{k_2}(UD)^{\ell_2}\dots U^{k_r}(UD)^{\ell_r}U^{k_{r+1}}C_s$ where $k_i = |\alpha_i|$ and $k_{r+1} + 1 = |\alpha_{r+1}|$; otherwise, if $|\alpha_{r+1}| = 0$ then we set $Q = (UD)^{\ell_0}U^{k_1}(UD)^{\ell_1}U^{k_2}(UD)^{\ell_2}\dots U^{k_r-1}C_s(UD)^{\ell_r}$ where $k_i = |\alpha_i|$ and s is so that Q ends on the x-axis. For all these cases, we have $Q \in \mathcal{G}$ so that P and Q belong to the same UD-equivalence class. Due to the form of a path in $\mathcal{G}$, if P and Q are two paths in $\mathcal{G}$ lying in the same UD-equivalence class, the $P = Q$. □

Theorem 3.4 *The o.g.f. with respect to the length for the set $\mathcal{G}$ is given by*

$$G(x) = \frac{2x^3 - 2x + 1}{(1 - x^2 - x)\,(1 - x)^2\,(x + 1)}.$$

The series expansion of $G(x)$ is

$$1 + x^2 + x^3 + 3x^4 + 5x^5 + 10x^6 + 17x^7 + 30x^8 + 50x^9 + 84x^{10} + 138x^{11} + O\left(x^{12}\right).$$

Proof. According to the different cases in the definition of $\mathcal{G}$, $G(x)$ is given by:

$$1 + \frac{x^2}{1-x^2} + \frac{1}{1-x^2}\frac{x^3}{1-x}\frac{1}{1-x^2} + \frac{1}{1-x^2}\frac{R(x)}{1-R(x)}\frac{x}{1-x}\frac{1}{1-x^2}$$

where $R(x) = \frac{x^3}{(1-x)(1-x^2)}$ is the o.g.f. that counts the paths of the form $U^k(UD)^\ell$ with $k, \ell \geqslant 1$. □

Remark 3.2 The coefficient g_n of x^n in $G(x)$ satisfies $g_n = 2g_{n-1} + g_{n-2} - 3g_{n-3} + g_{n-5}$ for $n > 4$ with $g_1 = 0$, $g_0 = g_2 = g_3 = 1$, $g_4 = 3$, which generates the sequence A215004 in [12]. An asymptotic approximation of g_n is

$$g_n \sim \frac{4(5 - 2\sqrt{5})}{5(\sqrt{5} - 3)^2} \cdot \left(\frac{\sqrt{5} + 1}{2}\right)^n \approx 0.72360 \cdot 1.61803^n.$$

3.3 Pattern UC

In this section, two paths are UC-equivalent whenever they coincide on all their occurrences UC_k for $k \geqslant 2$. For instance, $UUDUC_2$ and $UDUUC_2$ are UC-equivalent, while $UDUUC_2$ and $UUUUC_4$ are not.

Let $\mathcal{I}_1$ be the set of paths of length $n \geqslant 0$, $n \notin \{1, 3\}$, defined by either

(i) $(UD)^{\frac{n}{2}}$ if n is even, or

(ii) $(UD)^{\frac{n-5}{2}}UUUDC_2$ if is n is odd.

Let $\mathcal{I}$ be the set consisting of the union of $\mathcal{I}_1$ with the set of paths of the form $\alpha_1 U^{k_1} C_{k_1} \alpha_2 U^{k_2} C_{k_2} \dots \alpha_r U^{k_r} C_{k_r} \alpha_{r+1}$ where $r \geqslant 1$, all values k_i are at least 2, and all α_i are in $\mathcal{I}_1$.

Theorem 3.5 *There is a bijection between $\mathcal{I}$ and the set of UC-equivalence classes of $\mathcal{E}$.*

Proof. First, we prove that for every $P \in \mathcal{E}$ there is $Q \in \mathcal{I}$ such that P and Q belong to the same UC-equivalence class. If P does not contain any pattern UC, then we set $Q = (UD)^{\frac{n}{2}}$ if n is even, and $Q = (UD)^{\frac{n-5}{2}}UUUDC_2$ otherwise. If P contains $r \geqslant 1$ occurrences of UC, then we decompose $P = \alpha_1 UC \alpha_2 UC \dots \alpha_r UC \alpha_{r+1}$ where all α_i are paths avoiding UC. Showing the size of every catastrophes, we obtain the decomposition $P = \alpha_1 UC_{k_1} \alpha_2 UC_{k_2} \dots \alpha_r UC_{k_r} \alpha_{r+1}$ where $k_i \geqslant 2$ for $1 \leqslant i \leqslant r$.

We set $Q = \beta_1 U^{k_1} C_{k_1} \beta_2 U^{k_2} C_{k_2} \dots \beta_r U^{k_r} C_{k_r} \beta_{r+1}$ where all β_i are in $\mathcal{I}_1$ (all β_i are entirely determined by the length of each α_i). We have $Q \in \mathcal{I}$ so that P and Q belong to the same UC-equivalence class. Obviously, due to the definition of $\mathcal{I}$, if P and Q belong to $\mathcal{I}$ in the same UC-equivalence class, then $P = Q$. □

Theorem 3.6 *The o.g.f. with respect to the length for the set $\mathcal{I}$ is given by*

$$I(x) = \frac{x^5 - 2\,x^4 + 2\,x^3 - 2\,x^2 + 2\,x - 1}{x^7 - x^6 + x^5 - x^4 + x^3 - x^2 + 2\,x - 1}.$$

The series expansion of $I(x)$ is

$$1 + x^2 + x^3 + 2x^4 + 4x^5 + 5x^6 + 9x^7 + 15x^8 + 24x^9 + 40x^{10} + 65x^{11} + O\left(x^{12}\right).$$

Proof. The o.g.f. for $\mathcal{I}_1$ is $I_1(x) = \frac{1}{1-x} - x - x^3$ and the o.g.f. for $\mathcal{I}$ is $I_1(x) + I_1(x) \cdot \frac{I_2(x)}{1-I_2(x)}$ where $I_2(x) = \frac{I_1(x)x^3}{1-x}$ is the o.g.f. of a nonempty sequence of terms of the form $\alpha U^k C_k$ with $\alpha \in \mathcal{I}_1$ and $k \geqslant 2$. □

Remark 3.3 The coefficient i_n of x^n in $I(x)$ satisfies $i_n = 2i_{n-1} - i_{n-2} + i_{n-3} - i_{n-4} + i_{n-5} - i_{n-6} + i_{n-7}$ for $n \geqslant 7$ with $i_0 = 1, i_1 = 0, i_2 = 1, i_3 = 1, i_4 = 2, i_5 = 4$, and $i_6 = 5$. This sequence does not appear in [12]. An asymptotic approximation of i_n is

$$i_n \sim 0.28134 \cdot 1.64072^n.$$

3.4 Pattern DC

In this section, two paths are DC-equivalent whenever they coincide on all their occurrences DC_k for $k \geqslant 2$.

Let $\mathcal{J}_1$ be the set of paths of length $n \geqslant 0$, $n \neq 1$, defined by either

(i) $(UD)^{\frac{n}{2}}$ if n is even, or

(ii) $(UD)^{\frac{n-3}{2}}UUC_2$ if is n is odd.

Let $\mathcal{J}$ be the set consisting of the union of $\mathcal{J}_1$ with the set of paths of the form $\alpha_1 U^{k_1+1}DC_{k_1}\alpha_2 U^{k_2+1}DC_{k_2}\dots\alpha_r U^{k_r+1}DC_{k_r}\alpha_{r+1}$ where $r \geqslant 1$, all values k_i are at least 2, and all α_i are in $\mathcal{J}_1$.

Theorem 3.7 *There is a bijection between $\mathcal{J}$ and the set of DC-equivalence classes of $\mathcal{E}$.*

Proof. The proof is obtained *mutatis mutandis* as for Theorem 3.5. □

Theorem 3.8 *The o.g.f. with respect to the length for the set $\mathcal{J}$ is given by*

$$J(x) = \frac{x^3 - 2x^2 + 2x - 1}{x^7 - x^6 + x^5 - x^2 + 2x - 1}.$$

The series expansion of $J(x)$ is

$$1 + x^2 + x^3 + x^4 + 2x^5 + 2x^6 + 4x^7 + 6x^8 + 9x^9 + 14x^{10} + 20x^{11} + O\left(x^{12}\right).$$

Proof. The o.g.f. for $\mathcal{J}_1$ is $J_1(x) = \frac{1}{1-x} - x$ and the o.g.f. for $\mathcal{J}$ is $J_1(x) + J_1(x)\cdot\frac{J_2(x)}{1-J_2(x)}$ where $J_2(x) = \frac{J_1(x)x^5}{1-x}$ is the o.g.f. of a nonempty sequence of terms of the form $\alpha U^{k+1}DC_k$ with $\alpha \in \mathcal{J}_1$ and $k \geqslant 2$. □

Remark 3.4 The coefficient j_n of x^n in $J(x)$ satisfies $j_n = 2j_{n-1} - j_{n-2} + j_{n-5} - j_{n-6} + j_{n-7}$ for $n \geqslant 7$ with $j_0 = 1, j_1 = 0, j_2 = j_3 = j_4 = 1, j_5 = 2, j_6 = 2$ which is not in [12]. An asymptotic approximation of j_n is

$$j_n \sim 0.25317 \cdot 1.48698^n.$$

3.5 Pattern CU

In this section, two paths are CU-equivalent whenever they coincide on all their occurrences C_kU for $k \geqslant 2$.

Let $\mathcal{K}_1$ be the set of paths of even length defined by $(UD)^k$ for $k \geqslant 0$.

Let $\mathcal{K}_2$ be the set of paths of length $n \neq 1$ defined by $(UD)^k$, $k \geqslant 1$, if n is even, and $(UD)^kUUC_2$, $k \geqslant 0$, if n is odd.

Let $\mathcal{K}$ be the set consisting of the union of $\mathcal{K}_2$ with the set of paths of the form $\beta_1 U^{k_1}C_{k_1}\beta_2 U^{k_2}C_{k_2}\dots\beta_r U^{k_r}C_{k_r}\beta_{r+1}$ where $k_i \geqslant 2$ for $1 \leqslant i < r$, and for $1 \leqslant i < r+1$, $\beta_i \in \mathcal{K}_1$, and β_{r+1} is a path from $\mathcal{K}_2$.

Theorem 3.9 *There is a bijection between $\mathcal{K}$ and the set of CU-equivalence classes of $\mathcal{E}$.*

Proof. First, we prove that for every $P \in \mathcal{E}$ there is $Q \in \mathcal{K}$ such that P and Q belong to the same CU-equivalence class. If P does not contain any pattern CU, then we set $Q = (UD)^{\frac{n}{2}}$ whenever $n = |P|$ is even, and we set $Q = (UD)^{\frac{n-3}{2}}UUC_2$ otherwise; in both cases $Q \in \mathcal{K}_2$. If P contains $r \geqslant 1$ occurrences of CU, then we decompose $P = \alpha_1 CU\alpha_2 CU \dots \alpha_r CU\alpha_{r+1}$ where all α_i are paths avoiding CU. Showing the size of every catastrophe, we obtain the decomposition $P = \alpha_1 C_{k_1} U\alpha_2 C_{k_2} U \dots \alpha_r C_{k_r} U\alpha_{r+1}$ where all k_i are at least 2. So, we set $Q = \beta_1 U^{k_1} C_{k_1} \beta_2 U^{k_2} C_{k_2} \dots \beta_r U^{k_r} C_{k_r} \beta_{r+1}$ where $|\alpha_1| = |\beta_1|$, $|\alpha_i| + 1 = |\beta_i| + k_i$ for $1 < i < r+1$, $|\alpha_{r+1}| + 1 = |\beta_{r+1}|$, and

- for $1 \leqslant i < r+1$, $\beta_i \in \mathcal{K}_1$;
- β_{r+1} is a path from $\mathcal{K}_2$.

We have $Q \in \mathcal{K}$ so that P and Q belong to the same CU-equivalence class. Obviously, due to the definition of $\mathcal{K}$, if P and Q belong to $\mathcal{K}$ in the same CU-equivalence class, then $P = Q$. □

Theorem 3.10 *The o.g.f. with respect to the length for the set $\mathcal{K}$ is given by*

$$K(x) = \frac{x^4 + x - 1}{x^2 + x - 1}.$$

The series expansion of $K(x)$ is

$$1 + x^2 + x^3 + x^4 + 2x^5 + 3x^6 + 5x^7 + 8x^8 + 13x^9 + 21x^{10} + 34x^{11} + O\left(x^{12}\right).$$

Proof. Due to the definition of $\mathcal{K}$, the o.g.f. $K(x)$ is given by

$$K(x) = \frac{1}{1-x} - x + \frac{1}{1-x^2} \cdot \frac{1}{1 - \frac{x^3}{(1-x)(1-x^2)}} \cdot \frac{x^3}{1-x} \cdot \frac{x^2}{1-x}.$$

□

Remark 3.5 The coefficient k_n of x^n in $K(x)$ satisfies $k_n = k_{n-1} + k_{n-2}$ for $n \geqslant 5$ with $k_0 = 1, k_1 = 0, k_2 = k_3 = k_4 = 1$ which is a shift of the Fibonacci sequence A000045 in [12]. An asymptotic approximation of k_n is

$$k_n \sim \frac{2(\sqrt{5}-2)}{5-\sqrt{5}} \cdot \left(\frac{2}{\sqrt{5}-1}\right)^n \approx 0.17082 \cdot 1.61803^n.$$

3.6 Pattern DU

Let $\mathcal{L}$ be set of paths $P \in \mathcal{E}$ such that

$$P = U^{\ell_1}(DU)^{k_1} U^{\ell_2}(DU)^{k_2} \dots U^{k_r}(DU)^{k_r} U^{\ell_{r+1}} C_s$$

with $r \geqslant 1$, $\ell_i, k_i \geqslant 1$ for $1 \leqslant i \leqslant r$, $\ell_{r+1} \geqslant 0$, and s is so that the path ends on the x-axis (note that s can be 1, and in this case $C_1 = D$).

Theorem 3.11 *There is a bijection between $\mathcal{L}$ and the set of DU-equivalence classes of $\mathcal{E}$.*

Proof. First, we prove that for every $P \in \mathcal{E}$ there is $Q \in \mathcal{G}$ such that P and Q belong to the same DU-equivalence class. We decompose $P = \alpha_1(DU)^{k_1}\alpha_2(DU)^{k_2}\alpha_3(DU)^{k_3}\dots \alpha_r(DU)^{k_r}\alpha_{r+1}$ with all α_i being non-empty paths avoiding DU, with $r \geqslant 0$, and all k_i are at least one. Then we set

$$Q = U^{\ell_1}(DU)^{k_1}U^{\ell_2}(DU)^{k_2}U^{\ell_3}(DU)^{k_3}\dots U^{\ell_r}(DU)^{k_r}U^{\ell_{r+1}-1}C_s$$

where $\ell_i = |\alpha_i|$, and where s is so that the path ends on the x-axis. For all these cases, we have $Q \in \mathcal{L}$ so that P and Q belong to the same DU-equivalence class. Due to the form of a path in $\mathcal{L}$, if P and Q are two paths in $\mathcal{L}$ lying in the same DU-equivalence class, then $P = Q$. □

Theorem 3.12 *The o.g.f. with respect to the length for the set $\mathcal{L}$ is given by*

$$L(x) = \frac{1-x}{1-x-x^2}.$$

The series expansion of $L(x)$ is

$$1 + x^2 + x^3 + 2x^4 + 3x^5 + 5x^6 + 8x^7 + 13x^8 + 21x^9 + 34x^{10} + 55x^{11} + O\left(x^{12}\right).$$

Proof. According to the definition of $\mathcal{L}$ and handling separately the cases where $r = 1$ and $r \geqslant 2$, the o.g.f. $L(x)$ is given by:

$$1 + \frac{x^2}{1-x^2} + \frac{R(x)}{1-R(x)} \cdot \frac{x}{1-x}$$

where $R(x) = \frac{x^3}{(1-x)(1-x^2)}$ is the o.g.f. that counts paths of the form $U^\ell(DU)^k$ with $k, \ell \geqslant 1$. □

Remark 3.6 The coefficient ℓ_n of x^n in $G(x)$ satisfies $\ell_n = \ell_{n-1} + \ell_{n-2}$ for $n \geqslant 3$ with $\ell_0 = 1, \ell_1 = 0, \ell_2 = 1$, which generates the sequence A212804 in [12] (a variant of the well known Fibonacci sequence A000045). An asymptotic approximation of ℓ_n is

$$\ell_n \sim \frac{(\sqrt{5}-3)}{5-\sqrt{5}} \cdot \left(\frac{2}{\sqrt{5}-1}\right)^n \approx 0.27639 \cdot 1.61803^n.$$

3.7 Pattern DD

In [2], the authors exhibited a set $\overline{\mathcal{A}}$ (denoted $\mathcal{A}$ in [2]) of representatives for the DD-equivalence classes of Dyck paths: $\overline{\mathcal{A}}$ consists of Dyck paths P satisfying

Condition (C)**:** *P avoids $UUDU$ and the minimal ordinate reached by every occurrence of UDU is at most one (i.e. the height of every occurrence of UDU is at most 1).*

For a Dyck path P, we denote by $\psi(P)$ the unique path in $\overline{\mathcal{A}}$ which is DD-equivalent to P. In [2], it is shown how $\psi(P)$ can be constructed from P after several transformations preserving the positions of all occurrences of DD. The following proposition extends this result for the set $\mathcal{D}'$ of Dyck paths with only one catastrophe C_k at the end.

Proposition 3.1 *For any path $P \in \mathcal{D}'$ ending with catastrophe C_k, $k \geqslant 2$, there is a unique path in $\mathcal{D}'$, denoted $\psi(P)$, which is DD-equivalent to P, ending with catastrophe C_k (the same k as for P), and satisfying Condition (C).*

Proof. Let Q be the Dyck path obtained from P by replacing C_k with D^k. Using [2], there is a unique path $\psi(Q)$ in $\overline{\mathcal{A}}$ in the class of Q. Replacing the last D^k of $\psi(Q)$ with C_k, there is a unique Dyck path with only catastrophe C_k at the end equivalent to P. □

Lemma 3.1 *Let P be a path in $\mathcal{E}$. If P contains at least two catastrophes, then there exists $Q \in \mathcal{E}$ in the class of P, such that Q has only one catastrophe.*

Proof. We obtain Q from P by replacing with U every catastrophe, except the last, and by increasing the size of the last catastrophe so that $Q \in \mathcal{E}$. See Figure 2 for an example with two catastrophes. □

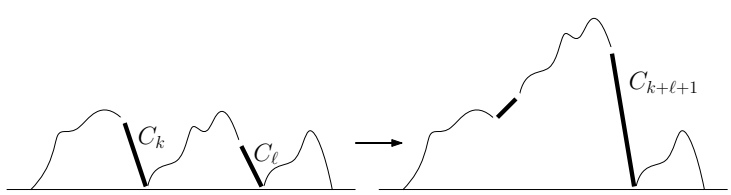


Figure 2: Example for Lemma 3.1 with two catastrophes.

Considering this previous lemma, we can focus our study on the set of Dyck paths with at most one catastrophe. Recall that $\mathcal{D}''$ is the set of paths in $\mathcal{E}$ having only one catastrophe (see Section 2.2).

Lemma 3.2 *Let P be a path of length n in $\mathcal{D}''$ (i.e. P contains only one catastrophe). The size of the catastrophe and n have different parity.*

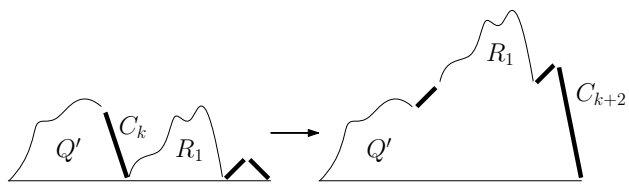
Proof. For any Dyck path in $\mathcal{D}''$ having the catastrophe of size k, the number of U minus the number of D equals k, which implies that the number of U and D (which is equal to $n-1$) has the same parity as k. □

In the following, for a given path P, a D-step will be called *isolated* if and only if it does not lie in an occurrence of DD.

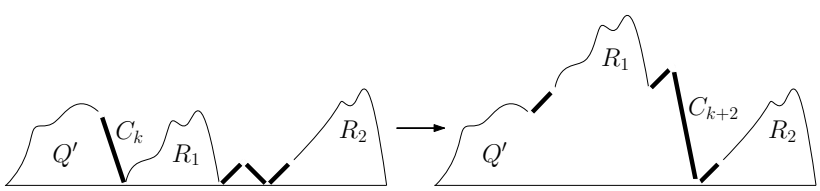
Lemma 3.3 *Let P be a path in $\mathcal{D}''$ having its catastrophe of size $k \geqslant 2$. We have $P = QC_kR$ where R is a Dyck path. If R contains isolated D-steps, then there is a path in $\mathcal{D}''$, DD-equivalent to P, with a catastrophe C_{k+2} of size $k+2$, and such that every D-step on the right of C_{k+2} is not isolated.*

Proof. First, we construct $P' = Q'C_kR'$, where $Q'C_k$ is the result of application of Proposition 3.1 on QC_k and R' is obtained applying ϕ from [2] on R. It implies that P' and P are DD-equivalent and this construction ensures that the height of the rightmost isolated D step in P' is at most one. After this, we distinguish three cases according to the position and the height of the rightmost isolated D-step.

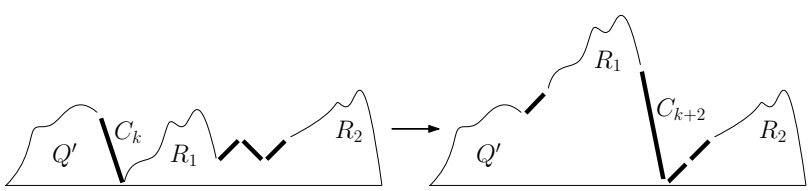
- if $R' = R_1UD$, then the paths $Q'UR_1UC_{k+2}$ and P are equivalent;



- if $R' = R_1UDUR_2$ where UR_2 is a Dyck path, then $Q'UR_1UC_{k+2}UR_2$ and P are equivalent;



- if $R' = R_1UDUR_2$ where UUR_2 is a Dyck path then $Q'UR_1C_{k+2}UUR_2$ and P are equivalent.



□

Lemma 3.4 *Let P be a path in $\mathcal{D}''$ satisfying Condition (C) and having its catastrophe of size $k \geqslant 4$. Then, there is a path $Q \in \mathcal{D}''$, equivalent to P, having its catastrophe of size $k-2$, and such that C_k and C_{k-2} are in the same position in P and Q, respectively.*

Proof. Before describing the construction of Q, it is worth noticing the following fact. There is an occurrence of UUU at height $k-3$ on the left of C_k in P. Indeed, let us consider the rightmost U at height $k-3$ on the left of C_k; clearly it necessarily precedes a step U which constitutes an occurrence of UU at height $k-3$. Moreover, since P avoids $UUDU$, the two steps UU are necessarily followed by another step U, and P can be decomposed $P = P_1UUUP_2C_kP_3$ where P_1 is a prefix of a Dyck path ending at height $k-3$, UP_2D is a Dyck path and P_3 is a Dyck path. We complete the proof by setting $Q = P_1UDUP_2C_{k-2}P_3$. See Figure 3 for an illustration of this construction. □

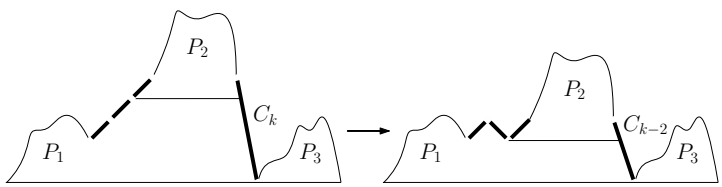


Figure 3: Illustration for Lemma 3.4.

3.7.1 The case of odd length

Let $\mathcal{A}'$ be the subset of paths in $\overline{\mathcal{A}}$ avoiding UDU, and not ending with UD, which also is the set of paths in $\overline{\mathcal{A}}$ without isolated D-steps. Let $\mathcal{A}_2$ be the set of paths satisfying Condition (C) and having only one catastrophe C_2 at the end.

Proposition 3.2 *A set of representatives of the DD-equivalence classes of odd length Dyck paths with catastrophes, is the set $\mathcal{R}_1$ of paths $R = AA'$ where $A \in \mathcal{A}_2$ and $A' \in \mathcal{A}'$.*

Proof. First, let us prove that for any path $P \in \mathcal{E}$, there is a path $Q \in \mathcal{R}_1$ lying in the same class. We obtain Q by applying the following process: (i) we apply Lemma 3.1; (ii) we apply Lemma 3.3; (iii) we apply ψ extended to Dyck paths with only one catastrophe (see Proposition 3.1); (iv) we apply Lemma 3.4 and Proposition 3.1, and we repeat it until the size of the catastrophe reaches 2 (thanks to Lemma 3.2). In the case where UD precedes C_2, i.e. $Q = Q_1UDC_2Q_2$, we replace the path with $Q_1UUC_4Q_2$ (see Fig. 4) and we apply Lemma 3.4 and Proposition 3.1.

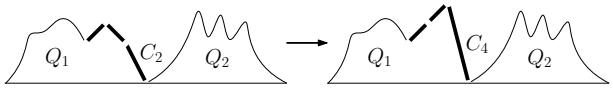


Figure 4: Illustration for Proposition 3.2.

Now, it suffices to prove that two different paths Q and Q' in $\mathcal{R}_1$ with the same length cannot belong to the same class. Let us assume that Q and Q' lie in the same

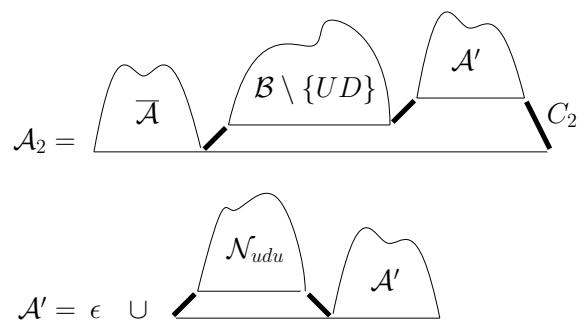
class. We decompose $Q = RC_2S$ and $Q' = R'C_2S'$. Suppose that $|S| \geqslant |S'|$. Since S' does not contain isolated D-steps, S' is entirely determined by the positions of occurrences of DD, which implies that S has the form $S = TS'$. Since S does not contain an isolated D-step, T cannot end with UD, and since Q is DD-equivalent to Q' it cannot end with DD, which means that T is empty. So we have $|S| = |S'|$ and then $S = S'$. Using Proposition 3.1 for RC_2 and $R'C_2$, we obtain $Q = Q'$. □

Theorem 3.13 *(The case of odd length) The o.g.f. for the number of DD-equivalence classes of odd length paths in $\mathcal{E}$ is given by:*

$$N(x) = \frac{8x^3}{(x^2 + \sqrt{1 - 3x^4 - 2x^2} + 1)^2(-x^4 + (1 + x^2)\sqrt{1 - 3x^4 - 2x^2} - 2x^2 + 1)}.$$

We have $N(x) = x^3 + x^5 + 5x^7 + 11x^9 + 33x^{11} + 88x^{13} + 247x^{15} + O(x^{16})$.

Proof. By Proposition 3.2, it suffices to provide the o.g.f. $N(x)$ for $\mathcal{A}_2 \times \mathcal{A}'$ with respect to the length. Sets $\mathcal{A}_2$ and $\mathcal{A}'$ are constructed as follows



where $\mathcal{N}_{udu}$ is the set of non-empty Dyck paths avoiding UDU, $\overline{\mathcal{A}}$ is the set of representatives for the DD-equivalence classes of Dyck paths, e.g. Dyck paths avoiding $UUDU$ and having all UDU at height 0 or 1, and $\mathcal{B}$ is the set of Dyck paths having all UDU on x-axis and not starting with UDU. From [2] it follows that

$$\overline{A}(x) = \frac{1 - x^2 + \sqrt{-3x^4 - 2x^2 + 1}}{1 + x^6 + x^4 - 3x^2 - (x^4 - 1)\sqrt{-3x^4 - 2x^2 + 1}},$$
$$B(x) = \frac{2 - x^2 - x^4 + x^2\sqrt{-3x^4 - 2x^2 + 1}}{1 - x^2 + \sqrt{-3x^4 - 2x^2 + 1}},$$

and from [2, 13] we have $N_{udu}(x) = (x^2 - \sqrt{-3x^4 - 2x^2 + 1} + 1)/(2x^2) - 1$.

Finally, we compute $N(x) = \overline{A}(x)\,(B(x) - x^2)\,A'(x)x^3A'(x)$ where $A'(x)$ satisfies $A'(x) = 1 + x^2N_{udu}(x)A'(x)$.

□

3.7.2 The case of even length

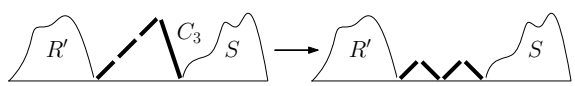
Proposition 3.3 *A set of representatives of the DD-equivalence classes of even length Dyck paths with catastrophes is the set $\mathcal{R}_2$ defined by the union of $\overline{\mathcal{A}}$ with the sets $\mathcal{S}_0$ and $\mathcal{S}_1$ defined below:*

- $\mathcal{S}_0$ is the set of Dyck paths $S = PDUUC_3$ where PDU is a prefix of Dyck path satisfying condition (C), and ending at height 2,

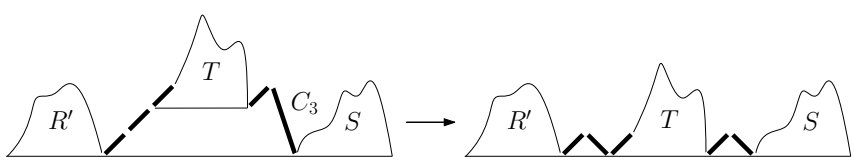
- $\mathcal{S}_1$ is the set of Dyck paths $S = PDDC_3$ where P is a prefix of Dyck path satisfying condition (C), and ending at height 5.

Proof. Let P be a path in $\mathcal{E}$. If P does not contain any catastrophe, then using Lemma 1 from [2], there is a unique path in $\overline{\mathcal{A}}$ equivalent to P. Now, let us assume that P contains at least one catastrophe. We obtain Q by applying the following process: (i) we apply Lemma 3.1; we apply Lemma 3.3; we apply ψ (extended to Dyck paths with only one catastrophe); we apply Lemma 3.4 followed by Proposition 3.1 until the path contains only one catastrophe C_3 (thanks to Lemma 3.2). We decompose the obtained path $Q = RC_3S$ where S is a Dyck path with no isolated D-step. With the same argument used at the beginning of the proof of Lemma 3.4, R is of the form $R = R'UUUR''$ where R' and $UR''D$ are Dyck paths.

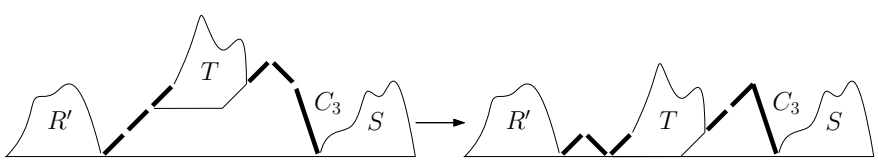
(a) if R'' is empty, i.e. $R = R'UUU$, we replace Q with $R'UDUDS \in \overline{\mathcal{A}}$;



(b) if $R'' = TU$ ends with U, i.e. $R = R'UUUTU$, we replace Q with $R'UDUTUDS \in \overline{\mathcal{A}}$;



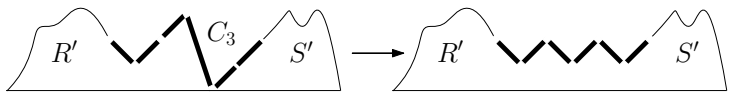
(c) if $R'' = TUD$ ends with UD, i.e. $R = R'UUUTUD$, we replace Q with $R'UDUTUUC_3S$ (see the illustration below); In the case where T ends with U, we apply the transformation (a) and we obtain a path in $\overline{\mathcal{A}}$;



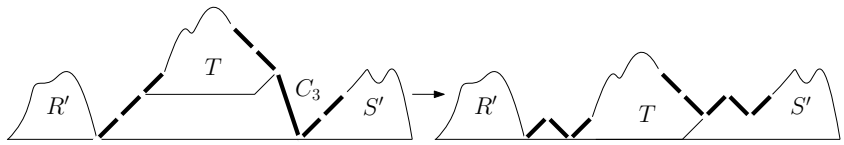
After applying all these transformations, the path is either in $\overline{\mathcal{A}}$, or it satisfies Condition (C) and contains only one catastrophe C_3 that follows an occurrence DUU or DD.

If S is empty then the path lies in $\mathcal{R}_2$. Otherwise, S does not contain isolated D-step and it starts necessarily with UU. In this case we set $S = UUS'$ and make the catastrophe disappear as follows:

(d) if R ends at height 3 with DUU, we set $R = R'DUU$ and replace Q with $R'DUDUDUS' \in \overline{\mathcal{A}}$;



(e) if R ends with $UUUT$, where UT is a prefix of a Dyck path ending at height 1 and with DD, we set $R = R'UUUT$ and replace Q by $R'UDUTUDUS'$.



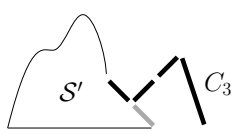
□

Theorem 3.14 *(The case of even length) The o.g.f. $V(x)$ for the number of DD-equivalence classes of even length paths in $\mathcal{E}$ is given by:*

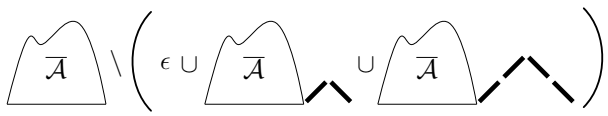
$$V(x) = \frac{4\left(x^4 - x^2 - 1\right)\sqrt{-3x^4 - 2x^2 + 1} + 8x^8 + 20x^6 + 8x^4 - 4}{\left((-x^2-1)\sqrt{-3x^4 - 2x^2 + 1} + x^4 + 2x^2 - 1\right)\left(x^2 + \sqrt{-3x^4 - 2x^2 + 1} + 1\right)^2}.$$

We have $V(x) = 1 + x^2 + 2x^4 + 4x^6 + 11x^8 + 27x^{10} + 73x^{12} + 194x^{14} + 529x^{16} + 1448x^{18} + O(x^{19})$.

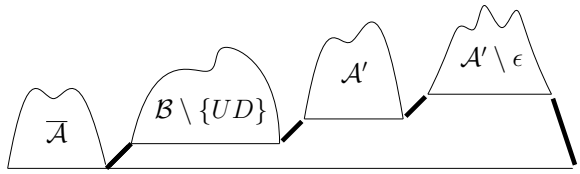
Proof. By Proposition 3.3, it suffices to provide the o.g.f. $M(x)$ for $\overline{\mathcal{A}} \cup \mathcal{S}_0 \cup \mathcal{S}_1$ with respect to the length. Sets $\mathcal{S}_0$ and $\mathcal{S}_1$ are constructed as follows:

$\mathcal{S}_0 =$ 

with

$\mathcal{S}' =$ 

and

$\mathcal{S}_1 =$ 

,

where $\mathcal{A}'$ is the set of non-empty Dyck paths avoiding UDU not ending with UD, $\overline{\mathcal{A}}$ is the set of representatives for the DD-equivalence classes of Dyck paths, e.g. Dyck paths avoiding $UUDU$ and having all UDU at height 0 or 1, and $\mathcal{B}$ is the set of Dyck paths having all UDU on x-axis and not starting with UDU.

Then, we have

$$S_0(x) = x^2 \cdot (\overline{A}(x) - 1 - x^2 \cdot \overline{A}(x) - x^4 \cdot \overline{A}(x))$$

and

$$S_1(x) = x^4 \cdot \overline{A}(x)(B(x) - x^2)A'(x)(A'(x) - 1)$$

which gives the claimed result after a simple calculation. □

3.7.3 The general case

Combining Theorems 3.13 and 3.14, we deduce:

Theorem 3.15 *The o.g.f. $L(x)$ for the number of DD-equivalence in $\mathcal{E}$ is given by*

$$\frac{(4x^4 - 4x^2 - 4)\sqrt{-3x^4 - 2x^2 + 1} + 8x^8 + 20x^6 + 8x^4 - 8x^3 - 4}{\left((-x^2 - 1)\sqrt{-3x^4 - 2x^2 + 1} + x^4 + 2x^2 - 1\right)\left(x^2 + \sqrt{-3x^4 - 2x^2 + 1} + 1\right)^2},$$

and it satisfies the following equation: $x^{10} - x^9 + 4x^8 - 3x^7 + 5x^6 - 5x^5 + 2x^4 - 2x^3 + x^2 - x + 1 + (-x^{10} + x^9 - 6x^8 + 5x^7 - 9x^6 + 6x^5 - 3x^4 + 3x^3 + x - 1)\,L(x) + (x^{10} - x^9 + 4x^8 - 3x^7 + 5x^6 - 3x^5 + 2x^4 - x^3)\,L(x)^2 = 0$.

The first coefficients of x^n, $n \geqslant 2$, of the series expansion of $L(x)$ are $1, 1, 2, 1, 4, 5, 11, 11, 27, 33, 73, 88, 194, 247, \ldots$, and they do not correspond to a part of a sequence listed in [12].

An asymptotic approximation of these coefficients is

$$\frac{41\sqrt{2}}{2\sqrt{\pi}} \cdot \frac{\sqrt{3}^{n+1}}{n^{\frac{3}{2}}} \text{ for } n \text{ even, and}$$

$$\frac{135\sqrt{2}}{4\sqrt{\pi}} \cdot \frac{\sqrt{3}^{n}}{n^{\frac{3}{2}}} \text{ otherwise.}$$

Acknowledgments

The authors would like to thank the anonymous referees for useful remarks and comments. This work was supported in part by the project ANER ARTICO funded by Bourgogne-Franche-Comté region of France and ANR JCJC PICS.

References

[1] C. Banderier and M. Wallner, Lattice paths with catastrophes, *Electron. Notes Discret. Math.* **59** (2017), 131–146.

[2] J.-L. Baril and A. Petrossian, Equivalence of Dyck paths modulo some statistics, *Discrete Math.* **338** (2015), 655–660.

[3] J.-L. Baril and A. Petrossian, Equivalence classes of Motzkin paths modulo a pattern of length at most two, *J. Integer Seq.* **18** (2015), 15.7.1.

[4] J.-L. Baril and S. Kirgizov, Bijections from Dyck and Motzkin meanders with catastrophes to pattern avoiding Dyck paths, *Discrete Math. Lett.* **7** (2021), 5–10.

[5] J.-L. Baril, S. Kirgizov and A. Petrossian, Enumeration of Łukasiewicz paths modulo some patterns, *Discrete Math.* **342**(4) (2019), 997–1005.

[6] J.-L. Baril, J. L. Ramírez and L. M. Simbaqueba, Equivalence classes of skew Dyck paths Modulo some patterns, *Integers* **22** (2022).

[7] E. Deutsch, Dyck path enumeration, *Discrete Math.* **204**(1999), 167–202.

[8] P. Flajolet and R. Sedgewick, *Analytic Combinatorics*, Cambridge University Press, 2009.

[9] A. Krinik, G. Rubino, D. Marcus, R. J. Swift, H. Kasfy and H. Lam. Dual processes to solve single server systems, *J. Stat. Plan. Inference* **135** (1) (2005), 121–147.

[10] K. Manes, A. Sapounakis, I. Tasoulas and P. Tsikouras, Equivalence classes of ballot paths modulo strings of length 2 and 3, *Discrete Math.* **339** (10) (2016), 2557–2572.

[11] A. G. Orlov, On asymptotic behavior of the Taylor coefficients of algebraic functions, *Sib. Math. J.* **25** (5) (1994), 1002–1013.

[12] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, https://oeis.org/.

[13] Y. Sun, The statistic "number of udu's" in Dyck paths, *Discrete Math.* **287** (2004), 177–186.

(Received 11 Mar 2022; revised 6 Sep 2022)

sciendo
PU. M. A. Vol. **30** (2022), No.1, pp. 23 – 30

Asymptotic bit frequency in Fibonacci words

Jean-Luc Baril
LIB
Université de Bourgogne Franche-Comté
B.P. 47 870, 21078 Dijon Cedex France
email: barjl@u-bourgogne.fr

and
Sergey Kirgizov
LIB
Université de Bourgogne Franche-Comté
B.P. 47 870, 21078 Dijon Cedex France
email: sergey.kirgizov@u-bourgogne.fr

and
Vincent Vajnovszki
LIB
Université de Bourgogne Franche-Comté
B.P. 47 870, 21078 Dijon Cedex France
email: vvajnov@u-bourgogne.fr

(Received: March 31, 2022, and in revised form May 15, 2022.)

Abstract. It is known that binary words containing no k consecutive 1s are enumerated by k-step Fibonacci numbers. In this note we discuss the expected value of a random bit in a random word of length n having this property.

Mathematics Subject Classification(2020). 05A16, 05A15, 11B39, 68R15.
Keywords: Fibonacci, word, pattern, frequency, asymptotics.

1 Introduction

For $n \geqslant 0$ and $k \geqslant 2$, we denote by $\mathcal{B}_n(1^k)$ the set of length n binary words avoiding k consecutive 1s. For example, we have

$$\mathcal{B}_4(11) = \{0000, 0001, 0010, 0100, 0101, 1000, 1001, 1010\}, \text{ and}$$
$$\mathcal{B}_4(111) = \{0000, 0001, 0010, 0011, 0100, 0101, 0110, 1000, 1001, 1010, 1011, 1100, 1101\}.$$

It is well known, see Knuth [12, p. 286], that $\mathcal{B}_n(1^k)$ is enumerated by the k-step Fibonacci numbers,

DOI: 10.2478/puma-2022-0005
© 2022 Baril, Kirgizov, Vajnovszki. This is an open access article licensed under the Creative Commons Attribution-NonCommercial-NoDerivs License (http://creativecommons.org/licenses/by-nc-nd/3.0/).

precisely $|\mathcal{B}_n(1^k)| = f_{n+k,k}$, where $f_{n,k}$ is defined, following Miles [14] as

$$f_{n,k} = \begin{cases} 0 & \text{if } 0 \leqslant n \leqslant k-2, \\ 1 & \text{if } n = k-1, \\ \sum_{i=1}^{k} f_{n-i,k} & \text{otherwise.} \end{cases}$$

Denote by $v_{n,k}$ the *frequency* (also called *popularity*) of 1s in $\mathcal{B}_n(1^k)$, i.e. the total number of 1s in all words of $\mathcal{B}_n(1^k)$. For instance, $v_{4,2} = 10$ and $v_{4,3} = 22$. The ratio of frequency of 1s to the overall number of bits in words of $\mathcal{B}_n(1^k)$ is

$$\alpha_{n,k} = \frac{v_{n,k}}{n \cdot |\mathcal{B}_n(1^k)|},$$

and it equals the expected value of a random bit in a random word from $\mathcal{B}_n(1^k)$. In [2], the authors left without proof the fact that, for any $k \geqslant 2$, $\lim_{n\to\infty} \alpha_{n,k}$ converges to a non-zero value as n grows. This note is devoted to proving this fact, which apart from its interest *en soi* has practical counterparts. Indeed, words in $\mathcal{B}_n(1^k)$ play a critical role in some telecommunication frame synchronization protocols, see for example [1, 3, 5], or in particular Fibonacci-like interconnection networks [8].

Our discussion is based on the bivariate generating function

$$F_k(x,y) = \sum_{n=0}^{\infty} \sum_{m=0}^{n-\lfloor \frac{n}{k} \rfloor} a_{n,m} x^n y^m$$

whose coefficient $a_{n,m}$ equals the number of words from $\mathcal{B}_n(1^k)$ containing exactly m 1s. For $k = 2$ and $k = 3$, Table 1 presents some values of $a_{n,m}$ for small n and m.

$m\backslash n$	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
0	1	1	1	1	1	1	1	1	1
1	1	2	3	4	5	6	7	8	9
2			1	3	6	10	15	21	28
3					1	4	10	20	35
4							1	5	15
5									1

$m\backslash n$	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
0	1	1	1	1	1	1	1	1	1
1	1	2	3	4	5	6	7	8	9
2		1	3	6	10	15	21	28	36
3				2	7	16	30	50	77
4					1	6	19	45	90
5							3	16	51

Table 1: First few values of $a_{n,m}$ for $k = 2$ (left) and $k = 3$.

2 Main result

Proposition 2.1 gives the expression of the generating function $F_k(x,y)$. Even though this result is already obtained in [2], in order to make the paper self-contained we give an alternative proof of it. Then we calculate the generating functions for the frequency of 1s and for the overall number of bits in $\mathcal{B}_n(1^k)$ by means of classic generating functions manipulations (Propositions 2.2). Applying

Theorem 4.1 from [16], after ensuring that its conditions are satisfied, we obtain the main result of this note, Theorem 2.6. The evolution of the random bit expectation for $k = 2$ and $k = 3$ is presented on Figure 1 for small values of n. And numerical estimations for the limit value ($n \to \infty$) of the random bit expectation, for small values of k are given in Table 2.

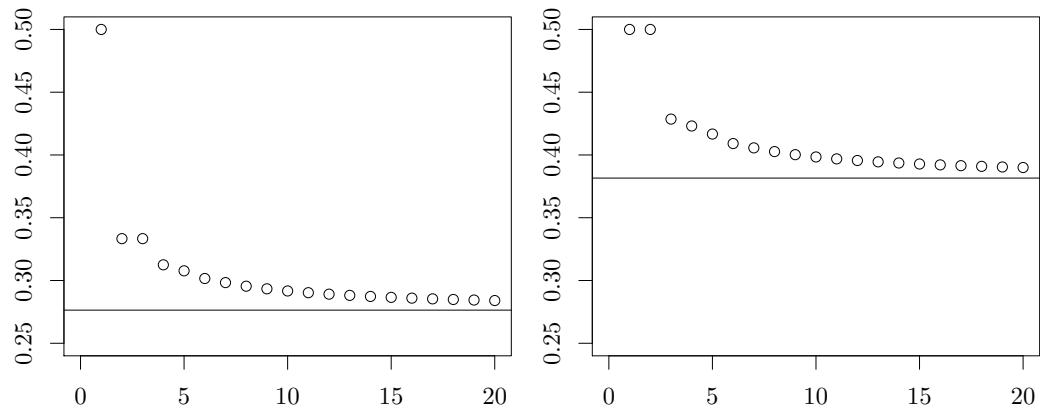


Figure 1: Expected value of a random bit in a random word from $\mathcal{B}_n(1^2)$ (left) and $\mathcal{B}_n(1^3)$ for small values of n.

Proposition 2.1 ([2])

$$F_k(x,y) = \frac{y\left(1-(xy)^k\right)}{y - xy^2 - xy + (xy)^{k+1}}.$$

Proof. The set $\mathcal{B}(1^k) = \bigcup_{n=0}^{\infty} \mathcal{B}_n(1^k)$ respects the following recursive decomposition

$$\mathcal{B}(1^k) = \mathbb{1}_{k-1} \cup \left(\bigcup_{i=0}^{k-1} \left(1^i 0 \cdot \mathcal{B}(1^k) \right) \right)$$

where $\mathbb{1}_{k-1} = \bigcup_{i=0}^{k-1} \{1^i\}$ is the set of words in $\mathcal{B}(1^k)$ containing no 0s, and $\cdot$ denotes the concatenation. Note that the empty word also lies in $\mathbb{1}_{k-1}$. The claimed generating function is the solution of the following functional equation

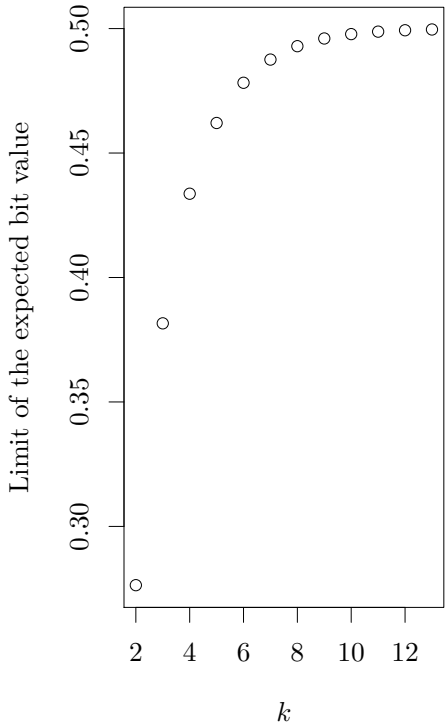
$$F_k(x,y) = \sum_{i=0}^{k-1} x^i y^i + F_k(x,y) \sum_{i=0}^{k-1} x^{i+1} y^i.$$

□

In the proof of Theorem 2.6 we need the following easy to derive results.

Proposition 2.2 • *$P_k(x) = \frac{\partial F_k(x,y)}{\partial y}|_{y=1}$ is the generating function where the coefficient of x^n is the frequency of 1s in $\mathcal{B}_n(1^k)$. We have*

$$P_k(x) = \frac{x \cdot \sum_{i=0}^{k-2} (i+1) x^i}{\left(x^k + x^{k-1} + \cdots + x^2 + x - 1\right)^2}.$$



k	Limit of the expected bit value
2	0.276393202250021
3	0.381580077680607
4	0.433657112297348
5	0.462073883180840
6	0.478227505713290
7	0.487545982771861
8	0.492928265543398
9	0.496019724266083
10	0.497779940783496
11	0.498772398758879
12	0.499326557312936
13	0.499633184444604

Table 2: Numerical estimations for the limit of the expected value of a random bit in a random word from $\mathcal{B}_n(1^k), n \to \infty$.

- $T_k(x) = x\frac{\partial F_k(x,1)}{\partial x}$ *is the generating function where the coefficient of* x^n *equals the total number of all bits in* $\mathcal{B}_n(1^k)$. *We have*

$$T_k(x) = \frac{x\left(\sum_{i=0}^{k-2}(2i+2)x^i + \sum_{i=k-1}^{2k-2}(2k-i-1)x^i\right)}{(x^k + x^{k-1} + \cdots + x^2 + x - 1)^2}.$$

Every root r of a polynomial $h(x)$ of degree n with a non-zero constant term corresponds to the root $1/r$ of its negative reciprocal $-x^n h(1/x)$. The denominator of both $P_k(x)$ and $T_k(x)$ involves $x^k + x^{k-1} + \cdots + x^2 + x - 1$ and its negative reciprocal is $x^k - x^{k-1} - \cdots - x^2 - x - 1$ which is known in the literature as Fibonacci polynomial, see for instance [6, 7, 9, 10, 11, 13, 14, 15, 19] and references therein. In particular, Dubeau proved [7, Theorem 1] that its root of the largest modulus is $\varphi_k = \lim_{n\to\infty} f_{n+1,k}/f_{n,k}$, the generalized golden ratio, and φ_k approaches 2 when $k \to \infty$ [7, Theorem 2]. Wolfram [19, Lemma 3.6] showed that any other root r of the Fibonacci polynomial satisfies

$3^{-1/k} < |r| < 1$. See Figure 2 for an illustration of this fact. Moreover, Corollary 3.8 in [19] proves that Fibonacci polynomial is irreducible over $\mathbb{Q}$. In order to refer later to them we summarize these results in the next proposition.

PROPOSITION 2.3 *The polynomial $g_k(x) = x^k + x^{k-1} + \cdots + x^2 + x - 1$ is irreducible over $\mathbb{Q}$, its root of the smallest modulus is unique and equal to $1/\varphi_k$.*

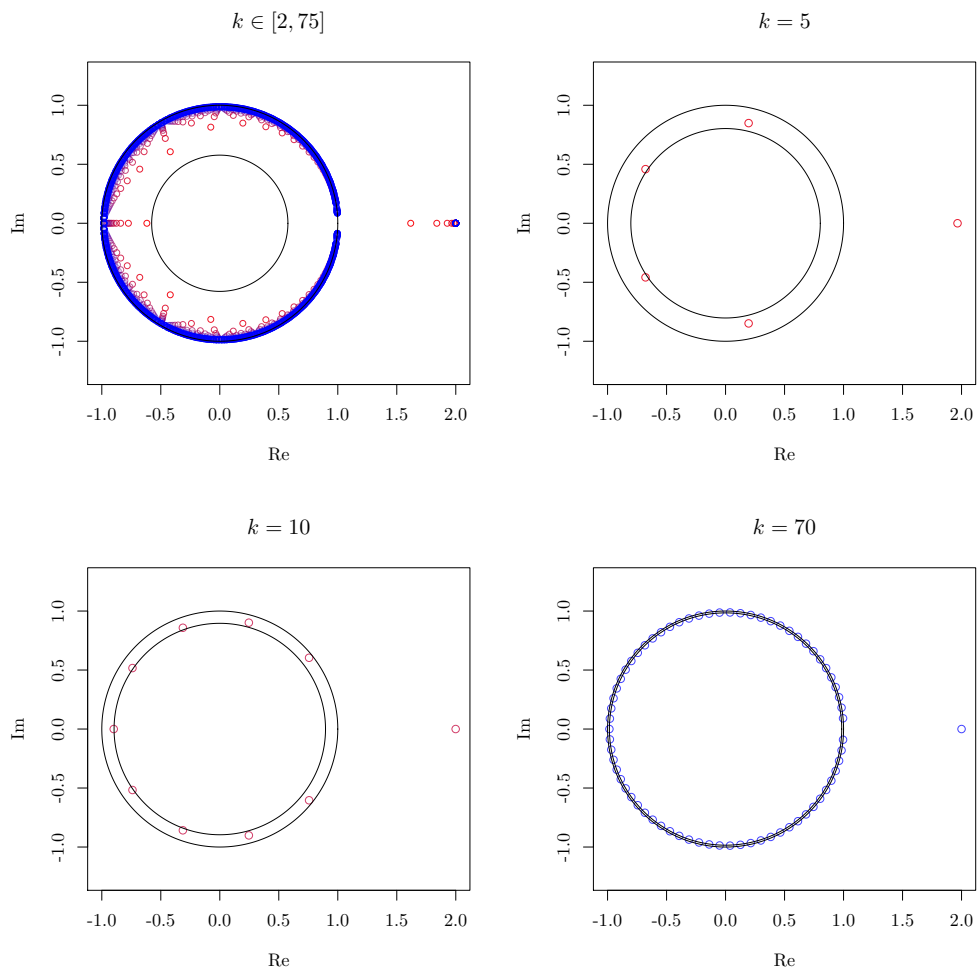


Figure 2: Roots of the polynomial $x^k - x^{k-1} - \cdots - x^2 - x - 1$ (the negative reciprocal of $g_k(x)$) for certain values of k.

The next lemma says that both fractions representing $P_k(x)$ and $T_k(x)$ are irreducible.

LEMMA 2.4 *The polynomials $\sum_{i=0}^{k-2}(i+1)x^i$ and $x^k + x^{k-1} + \cdots + x^2 + x - 1$ are relatively prime; and so are $\sum_{i=0}^{k-2}(2i+2)x^i + \sum_{i=k-1}^{2k-2}(2k-i-1)x^i$ and $x^k + x^{k-1} + \cdots + x^2 + x - 1$.*

Proof. The polynomial $x^k+x^{k-1}+\cdots+x^2+x-1$ is irreducible due to Proposition 2.3. It does not divide $\sum_{i=0}^{k-2}(i+1)x^i$ as it has a greater degree. And it also cannot divide $\sum_{i=0}^{k-2}(2i+2)x^i+\sum_{i=k-1}^{2k-2}(2k-i-1)x^i$ as the latter does not have any positive real roots. □

From Propositions 2.2, 2.3, Dubeau's results [7], and Lemma 2.4 we have:

Lemma 2.5 *Both generating functions $P_k(x)$ and of $T_k(x)$ have the same and unique pole of the smallest modulus with multiplicity 2. The pole equals $1/\varphi_k$, where φ_k is the generalized golden ratio.*

For our main result of this note we need the Theorem 4.1 from [16]:

Theorem 4.1 from [16]. *Assume that a rational generating function $\frac{f(x)}{g(x)}$, with $f(x)$ and $g(x)$ relatively prime and $g(0)\neq 0$, has a unique pole $1/\beta$ of the smallest modulus. Then, if the multiplicity of $1/\beta$ is ν, we have*

$$[x^n]\frac{f(x)}{g(x)} \sim \nu\frac{(-\beta)^\nu f(1/\beta)}{g^{(\nu)}(1/\beta)}\beta^n n^{\nu-1}.$$

Both $P_k(x)$ and $T_k(x)$ are rational generating functions, and by Lemmas 2.4 and 2.5 they fulfill the conditions in the above theorem, so

$$[x^n]P_k(x) \sim 2n\varphi_k^{n+2}\cdot\left.\frac{x\left(\sum_{i=0}^{k-2}(i+1)x^i\right)}{\left((x^k+x^{k-1}+\cdots+x^2+x-1)^2\right)''}\right|_{x=1/\varphi_k}$$

$$[x^n]T_k(x) \sim 2n\varphi_k^{n+2}\cdot\left.\frac{x\left(\sum_{i=0}^{k-2}(2i+2)x^i+\sum_{i=k-1}^{2k-2}(2k-i-1)x^i\right)}{\left((x^k+x^{k-1}+\cdots+x^2+x-1)^2\right)''}\right|_{x=1/\varphi_k}.$$

The expected value of a random bit in a random word from $\mathcal{B}_n(1^k)$ is $\frac{[x^n]P_k(x)}{[x^n]T_k(x)}$. Taking the limit, we obtain:

Theorem 2.6 *The expected value of a random bit in a random word from $\mathcal{B}_n(1^k)$ tends to*

$$\left.\frac{kx^k-kx^{k-1}-x^k+1}{kx^k-kx^{k-1}+x^{2k}-3x^k+2}\right|_{x=1/\varphi_k} \quad \text{when } n\to\infty,$$

where $\varphi_k=\lim_{n\to\infty} f_{n+1,k}/f_{n,k}$ is the generalized golden ratio, in particular φ_2 is the golden ratio.

See Table 2 for some numerical estimations of the result obtained in the previous theorem. This result involves the generalized golden ratio. More than 20 years ago it was conjectured by Wolfram [19] that the Galois group of the polynomial $x^k-x^{k-1}-\cdots-x^2-x-1$ is the symmetric group S_k, and so there is no algebraic expression for φ_k (the root of the largest modulus of this polynomial) when $k\geqslant 5$. In case of even or prime k the conjecture was settled by Martin [13]. Cipu and Luca [6] showed that φ_k cannot be constructed by ruler and compass for $k\geqslant 3$. Nevertheless, good approximations are available, for instance Hare, Prodinger and Shallit [11] expressed φ_k and $1/\varphi_k$ in terms of rapidly converging series.

The generalized golden ratio φ_k tends to 2 as k grows, and we deduce the following.

Corollary 2.7 *The limit of the expected bit value of binary words avoiding k consecutive 1s, whose length tends to infinity, approaches* $1/2$ *as k grows:*

$$\lim_{k\to\infty}\lim_{n\to\infty}\frac{v_{n,k}}{n\cdot|\mathcal{B}_n(1^k)|}=\frac{1}{2}.$$

Finally, note that other sets of restricted binary words are counted by the generalized Fibonacci numbers, for instance q-decreasing words [2] for $q \geqslant 1$. In this case every length maximal factor of the form 0^a1^b satisfies $a = 0$ or $q \cdot a > b$. Theorem 2.6 and Corollary 2.7 apply to this case (with the same limit, see [2, Corollary 5]) by setting $k = q + 1$.

Acknowledgement. We would like to greatly thank Dietrich Burde, Ted Shifrin, Igor Rivin and Sil from Mathematics Stack Exchange[1] for insightful discussions and useful references about the irreducibility of Fibonacci polynomial which is directly related to Proposition 2.3. This work was supported in part by the project ANER ARTICO funded by Bourgogne-Franche-Comté region (France).

References

[1] D. Bajic, *On construction of cross-bifix-free kernel sets*, 2nd MCM COST 2100, TD(07)237, Lisbon, Portugal, 2007.

[2] J.-L Baril, S. Kirgizov and V. Vajnovszki, *Gray codes for Fibonacci q-decreasing words*, arXiv:2010.09505.

[3] A. Bernini, S. Bilotta, R. Pinzani and V. Vajnovszki, *A Gray code for cross-bifix-free sets*, Math. Structures Computer Sci., 27 (2017) 184–196.

[4] A. Brauer, *On algebraic equations with all but one root in the interior of the unit circle. To my teacher and former colleague Erhard Schmidt on his 75th birthday*, Math. Nachr., 4 (1950) 250–257.

[5] Y. M. Chee, H. M. Kiah, P. Purkayastha and C. Wang, *Cross-bifix-free codes within a constant factor of optimality*, IEEE Trans. Inform. Theory, 59 (2013) 4668—4674.

[6] M. Cipu and F. Luca, *On the Galois group of the generalized Fibonacci polynomial*, An. Ştiinţ. Univ. “Ovidius" Constanţa Ser. Mat., 9 (2001) 27—38.

[7] F. Dubeau, *On r-generalized Fibonacci numbers*, Fibonacci Quart., 27 (1989) 221–229.

[8] Ö. Eğecioğlu and V. Iršič, *Fibonacci-run graphs I: Basic properties*, Discrete Appl. Math., 295 (2021) 70–84.

[9] I. Flores, *Direct calculation of k-generalized Fibonacci numbers*, Fibonacci Quart., 5 (1967) 259–266.

[1]See the original discussions here `https://math.stackexchange.com/questions/4120185` and here `https://math.stackexchange.com/questions/4125568`.

[10] G. W. Grossman and S. K. Narayan, *On the characteristic polynomial of the j-th order Fibonacci sequence*, Applications of Fibonacci Numbers, 8, Springer, Dordrecht, 1999, pp. 165–177.

[11] K. Hare, H. Prodinger and J. Shallit, *Three series for the generalized golden mean*, Fibonacci Quart., 52 (2014) 307–314

[12] D. Knuth, *The Art of Computer Programming, Volume 3: Sorting and Searching*, 2nd ed., Addison-Wesley, 1998.

[13] P. A. Martin, *The Galois group of $x^n - x^n - 1 - \ldots - x - 1$*, J. Pure Appl. Algebra, 190 (2004) 213–223.

[14] E. Miles, *Generalized Fibonacci numbers and associated matrices*, Amer. Math. Monthly, 67 (1960) 745–752.

[15] M. D. Miller, *On generalized Fibonacci numbers*, Amer. Math. Monthly, 78 (1971) 1108–1109.

[16] R. Sedgewick and P. Flajolet, *An introduction to the analysis of algorithms*, 2nd ed., Addison-Wesley, 2013.

[17] E. S. Selmer, *On the irreducibility of certain trinomials*, Math. Scand., 4 (1956) 287–302.

[18] E. Rouché, *Mémoire sur la série de Lagrange*, Journal de l'École Polytechnique, 22 (1862) 193–224.

[19] D. A. Wolfram, *Solving generalized Fibonacci recurrences*, Fibonacci Quart., 36 (1998) 129–145.

Math.Comput.Sci. (2024) 18:1
https://doi.org/10.1007/s11786-023-00576-5

Mathematics in Computer Science

Last Symbol Distribution in Pattern Avoiding Catalan Words

Jean-Luc Baril · Javier F. González · José L. Ramírez

Received: 30 November 2022 / Revised: 22 December 2022 / Accepted: 3 March 2023 / Published online: 14 February 2024
© The Author(s) 2024

Abstract We study the distribution of the last symbol statistics on the sets of Catalan words avoiding a pattern of length at most three. For each pattern p, we provide a bivariate rational generating function where the coefficient $\boldsymbol{c}_p(n,k)$ of $x^n y^k$ in its series expansion is the number of length n Catalan words avoiding p and ending with the symbol k. We deduce recurrence relations or closed forms for $\boldsymbol{c}_p(n,k)$ and we provide asymptotic approximations for the expectation of the last symbol on all Catalan words avoiding p. We end this paper by describing a computational approach using computer algebra.

Keywords Catalan word · Generating function · Pattern avoidance

Mathematics Subject Classification 05A15 · 05A19

1 Introduction

A *restricted growth word* $w = w_1 w_2 \cdots w_n$ is a word over the set of non-negative integers such that $w_1 = 0$ and $0 \le w_i \le \mathtt{st}(w_1 \cdots w_{i-1}) + 1$, where $\mathtt{st}$ is an integer statistic. These words have been widely studied in the literature since they simultaneously generalize several classes of combinatorial objects as restricted growth functions [11,14], staircase words [16], ascent sequences [5,9], and Catalan words [1].

In this paper, we focus on Catalan words which are obtained whenever $\mathtt{st}(w)$ returns the last symbol of w. More formally, a word $w = w_1 w_2 \cdots w_n$ is a *Catalan word* if

$$w_1 = 0 \text{ and } 0 \le w_i \le w_{i-1} + 1 \text{ for } i = 2, \ldots, n.$$

J-L. Baril
LIB, Université de Bourgogne, B.P. 47 870, 21078 Dijon-Cedex, France
e-mail: barjl@u-bourgogne.fr

J. F. González · J. L. Ramírez ( )
Departamento de Matemáticas, Universidad Nacional de Colombia, Bogotá, Colombia
e-mail: jlramirezr@unal.edu.co

J. F. González
e-mail: javgonzalezgu@unal.edu.co

For $n \geq 0$, let $\mathcal{C}_n$ denote the set of Catalan words of length n (the Catalan word of length 0 is the empty word ϵ). For example, we have

$$\mathcal{C}_4 = \{\texttt{0000}, \texttt{0001}, \texttt{0010}, \texttt{0011}, \texttt{0012}, \texttt{0100}, \texttt{0101}, \\ \texttt{0110}, \texttt{0111}, \texttt{0112}, \texttt{0120}, \texttt{0121}, \texttt{0122}, \texttt{0123}\}.$$

The cardinality of the set $\mathcal{C}_n$ is given by the Catalan number $c(n) = \frac{1}{n+1}\binom{2n}{n}$, see [17, Exercise 80]. Catalan words have already been studied in the context of exhaustive generation of Gray codes for growth-restricted words [11,14]. More recently, Baril et al. [1,2] study the distribution of descents on restricted Catalan words avoiding a pattern or a pair of patterns of length at most three. Ramírez and Rojas [13] also study the distribution of descents for Catalan words avoiding consecutive patterns of length at most three. Baril and Ramírez [3] give enumerative results for Catalan words avoiding a given pair of relations with respect to the length and the number of descents. Additionally, in [6,10], the authors started a study of combinatorial statistics on the polyominoes associated with words in $\mathcal{C}_n$.

The goal of this work is to complement all these studies by providing enumerative results for Catalan words avoiding a pattern of length at most three with respect to the length and the last symbol. Using classical methods presented in [8,12], we also give the exact value or an asymptotic approximation for the expectation of the last symbol for these words. Notice that this work is, in a way, a counterpart for Catalan words of the study of Bernini et al. [4] which enumerate permutations avoiding generalized patterns with respect to the last symbol.

The remaining of this paper is structured as follows. In Sect. 2, we give some definitions and notations about pattern avoidance and generating functions. As a preliminary result, we yield the bivariate generating function that counts the number of Catalan words (without avoidance of a pattern), with respect to the length and the last symbol, and we deduce the expectation (i.e. the average) of the last symbol on all length n Catalan words. In Sect. 3, we make a similar study for Catalan words avoiding a pattern of length 2. In Sect. 4, we focus on the avoidance of patterns p of length three, and we give asymptotic approximations (see [8,12]) for the expectation of the last symbol. Finally, in Sect. 5 we show how we can obtain recurrence relations and asymptotic approximations for the coefficients of generating functions and for the expectations of the last symbol, by using *Mathematica* and *Maple* libraries.

2 Notations and Preliminaries

For an integer $r \geq 2$, a *pattern* $p = p_1 p_2 \cdots p_r$ is a word (of length r) over the set $\{0, 1, \ldots, r-1\}$ satisfying the condition: if $j > 0$ appears in p, then $j-1$ also appears in p. For instance, there are three patterns of length two (00, 01, 10), and thirteen patterns of length three (000, 001, 010, 011, 012, 021, 100, 101, 102, 110, 120, 201, 210). A Catalan word $w = w_1 w_2 \cdots w_n$ *contains* the pattern $p = p_1 p_2 \cdots p_r$ if there exists a subsequence $w_{i_1} w_{i_2} \cdots w_{i_r}$ of w ($i_1 < i_2 < \cdots < i_r$) which is order-isomorphic to p. We say that w *avoids* p whenever w does not contain the pattern p. For example, the Catalan word $\texttt{0123455543}$ avoids the pattern 001 and contains three subsequences isomorphic to the pattern 210.

For $n \geq 0$, let $\mathcal{C}_n(p)$ denote the set of Catalan words of length n avoiding the pattern p. We denote by $\boldsymbol{c}_p(n)$ the cardinality of $\mathcal{C}_n(p)$, $\mathcal{C}(p) := \bigcup_{n\geq 0} \mathcal{C}_n(p)$, and $\mathcal{C}(p)^+ := \mathcal{C}(p) \setminus \{\epsilon\}$. We denote by $\mathsf{last}(w)$ the last symbol of w. Let $\mathcal{C}_{n,k}(p)$ denote the set of Catalan words $w \in \mathcal{C}_n(p)$ such that $\mathsf{last}(w) = \mathtt{k}$, and let $\boldsymbol{c}_p(n,k) := |\mathcal{C}_{n,k}(p)|$. Obviously, we have $\boldsymbol{c}_p(n) = \sum_{k=0}^{n-1} \boldsymbol{c}_p(n,k)$.

We introduce the bivariate generating function

$$\boldsymbol{H}_p(x,y) := \sum_{w \in \mathcal{C}(p)^+} x^{|w|} y^{\mathsf{last}(w)} = \sum_{n\geq 1,\ k\geq 0} \boldsymbol{c}_p(n,k) x^n y^k,$$

and we set

$$\boldsymbol{H}_p(x) := \sum_{w \in \mathcal{C}(p)^+} x^{|w|} = \boldsymbol{H}_p(x,1).$$

Notice that these generating functions do not consider the empty word. The expectation of the last symbol on all Catalan words in $\mathcal{C}_n(p)$ is given by (see [8])

$$\boldsymbol{a}_p(n) := \frac{[x^n]\partial_y \boldsymbol{H}_p(x,y)|_{y=1}}{\boldsymbol{c}_p(n)} = \frac{[x^n]\partial_y \boldsymbol{H}_p(x,y)|_{y=1}}{[x^n]\boldsymbol{H}_p(x,1)}.$$

Moreover, if we omit the subscript p in all previous notations, then this means we refer to the set of all Catalan words of length n (without avoidance of a pattern).

For a Catalan word $w = w_1 w_2 \cdots w_n$, $n \geq 0$, and an integer $a > 0$, we denote by $(w+a)$ the word $(w_1 + a)(w_2 + a)\cdots(w_n + a)$ obtained from w by increasing by a each of its symbols. In the case where w is the empty word ϵ, we set $(w+a) = \epsilon$. Using this notation, any non-empty Catalan word has a *first return decomposition* as follows: $w = \mathtt{0}(w'+1)w''$ where w', w'' are two Catalan words. From this decomposition, a non-empty Catalan word w is either of the three forms (i) $w = \mathtt{0}$, or (ii) $w = \mathtt{0}(w'+1)$ with $w' \in \mathcal{C}^+$, or (iii) $w = \mathtt{0}(w'+1)w''$ with $w' \in \mathcal{C}$ and $w'' \in \mathcal{C}^+$. So, we deduce the following functional equation

$$\boldsymbol{H}(x,y) = x + xy\boldsymbol{H}(x,y) + x(\boldsymbol{H}(x)+1)\boldsymbol{H}(x,y),$$

which provides

$$\boldsymbol{H}(x,y) = \frac{1-2xy-\sqrt{1-4x}}{2(1-y+xy^2)} \quad \text{and} \quad \boldsymbol{H}(x,1) = \boldsymbol{C}(x),$$

where $\boldsymbol{C}(x) := \frac{1-2x-\sqrt{1-4x}}{2x}$ is the well known generating function for the non-empty Catalan words. Thanks to [6,18], we obtain

$$[x^n y^k]\boldsymbol{H}(x,y) = \frac{k+1}{n}\binom{2n-2-k}{n-1}, \quad n \geq 1, k \geq 0 \text{ and}$$

$$[x^n]\partial_y \boldsymbol{H}(x,y)|_{y=1} = [x^n]\boldsymbol{C}(x)^2 = \frac{4}{n+2}\binom{2n-1}{n-2}.$$

Then, the expectation of the last symbol of a Catalan word in $\mathcal{C}_n$ is

$$\boldsymbol{a}(n) = \frac{4\binom{2n-1}{n-2}(n+1)}{(n+2)\binom{2n}{n}} = \frac{2(n-1)}{n+2},$$

and its limit is 2 when $n \to \infty$.

3 Avoidance of Length 2 Patterns

In this section, we provide the generating functions $\boldsymbol{H}_p(x,y)$ for all patterns $p \in \{00, 01, 10\}$, and we give closed forms for the coefficients of $x^n y^k$ in their series expansions. So, we deduce the expectation of the last symbol on all Catalan words in $\mathcal{C}_p(n)$.

Theorem 3.1 *We have*

$$\boldsymbol{H}_{00}(x,y) = \frac{x}{1-xy}, \quad \boldsymbol{c}_{00}(n,k) = \begin{cases} 1, & \text{if } k = n-1 \\ 0 & \text{otherwise} \end{cases}, \quad \text{and} \quad \boldsymbol{a}_{00}(n) = n-1.$$

Proof The word $\mathtt{012\cdots n\text{-}1}$ is the unique non-empty word avoiding the pattern 00. □

Theorem 3.2 *We have*

$$\boldsymbol{H}_{01}(x,y) = \frac{x}{1-x}, \quad \boldsymbol{c}_{01}(n,k) = \begin{cases} 1, & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}, \quad \text{and} \quad \boldsymbol{a}_{01}(n) = 0.$$

Proof The word $\mathtt{00\cdots 0}$ is the unique non-empty word avoiding the pattern 01. □

Theorem 3.3 *We have*

$$\boldsymbol{H}_{10}(x,y)=\frac{x}{1-x(1+y)},\quad \boldsymbol{c}_{10}(n,k)=\binom{n-1}{k},\quad \textit{and}\quad \boldsymbol{a}_{10}(n)=\frac{n-1}{2}.$$

Proof Let w be a word in $\mathcal{C}_{n,k}(10)$. If $n\geq 2$, then the symbol immediately before the last one has two options $k-1$ or k. Then $\boldsymbol{c}_{10}(n,k)=\boldsymbol{c}_{10}(n-1,k-1)+\boldsymbol{c}_{10}(n-1,k)$ for $n\geq 2$ and $k\geq 1$. Additionally, $\boldsymbol{c}_{10}(n,0)=1$ for all positive integer n. Therefore, the sequence $\boldsymbol{c}_{10}(n,k)$ is equal to the binomial coefficient $\binom{n-1}{k}$ and the generating function can be deduced easily. □

4 Avoidance of Length 3 Patterns

In this section, we provide the generating functions $\boldsymbol{H}_p(x,y)$ for all patterns p of length three. We also yield closed forms or recurrence relations for the number $\boldsymbol{c}_p(n,k)$ of Catalan words $w\in\mathcal{C}_p(n)$ such that $\mathsf{last}(w)=k$, and we deduce exact values or asymptotic approximations for the expectation of the last symbol on all Catalan words in $\mathcal{C}_p(n)$.

4.1 The Pattern 012

Theorem 4.1 *We have*

$$\boldsymbol{H}_{012}(x,y)=\frac{x(1-x+xy)}{1-2x},$$

and for $n\geq 2$,

$$\boldsymbol{c}_{012}(n,k)=\begin{cases}2^{n-2}, & \textit{if } k=0,1\\ 0, & \textit{otherwise}\end{cases},\quad \textit{and}\quad \boldsymbol{a}_{012}(n)=\frac{1}{2}.$$

Proof Let w denote a non-empty Catalan word in $\mathcal{C}(012)$. From the definition of Catalan word, w must be made up exclusively of `0`'s and `1`'s, and the first symbol is `0`. Therefore, the Catalan words avoiding the pattern 012 are given by the regular expression

$$\mathtt{0}\cup\mathtt{0}(\mathtt{0}\cup\mathtt{1})^*\mathtt{0}\cup\mathtt{0}(\mathtt{0}\cup\mathtt{1})^*\mathtt{1}.$$

This regular expression translates to the rational generating function

$$\boldsymbol{H}_{012}(x,y)=x+\frac{x^2}{1-2x}+\frac{x^2y}{1-2x}=\frac{x(1-x+xy)}{1-2x}.$$

The closed formula for the sequence $\boldsymbol{c}_{012}(n,k)$ is obtained by comparing coefficients. □

The series expansion of the generating function $\boldsymbol{H}_{012}(x,y)$ is

$$x+(1+y)x^2+(2+2y)x^3+(\mathbf{4+4y})x^4+(8+8y)x^5+(16+16y)x^6+O(x^7).$$

The Catalan words corresponding to the bold coefficients in the above series are

$\mathcal{C}_{4,0}(012)=\{\mathtt{0000},\mathtt{0010},\mathtt{0100},\mathtt{0110}\}$ and $\mathcal{C}_{4,1}(012)=\{\mathtt{0001},\mathtt{0011},\mathtt{0101},\mathtt{0111}\}$.

4.2 The Pattern 001

Theorem 4.2 *We have*

$$\boldsymbol{H}_{001}(x,y) = \frac{x(1-x)}{(1-2x)(1-xy)},$$

and for $n \geq 2$,

$$\boldsymbol{c}_{001}(n,k) = \begin{cases} 2^{n-k-2}, & \text{if } k < n-1 \\ 1, & \text{if } k = n-1, \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \boldsymbol{a}_{001}(n) = 1 - 2^{1-n}.$$

Proof Let w denote a non-empty Catalan word in $\mathcal{C}(001)$, and let $w = \mathtt{0}(w'+1)w''$ be the first return decomposition, where $w', w'' \in \mathcal{C}(001)$. If $w'' = \epsilon$, then the generating function for this case is $x + xy\boldsymbol{H}_{001}(x,y)$. If w'' is non empty, then $w'' = \mathtt{0}^k$, $k \geq 1$, and the generating function for this case is $x\,(\boldsymbol{H}_{001}(x)+1)\,\frac{x}{1-x}$. Adding both cases we have the functional equation

$$\boldsymbol{H}_{001}(x,y) = x + xy\boldsymbol{H}_{001}(x,y) + (\boldsymbol{H}_{001}(x)+1)\,\frac{x^2}{1-x}.$$

Since $\boldsymbol{H}_{001}(x)+1 = \frac{1-x}{1-2x}$ (see Theorem 5 in [1]), we obtain the result. Classical methods are used for the extraction of the coefficients of $[x^n y^k]\boldsymbol{H}_{001}(x,y)$ and $[x^n]\partial_y \boldsymbol{H}_{001}(x,y)|_{y=1}$. □

The series expansion of the generating function $\boldsymbol{H}_{001}(x,y)$ is

$$x + (1+y)x^2 + (2+y+y^2)x^3 + (\mathbf{4+2y+y^2+y^3})x^4 + (8+4y+2y^2+y^3+y^4)x^5$$
$$+(16+8y+4y^2+2y^3+y^4+y^5)x^6 + O(x^7).$$

The Catalan words corresponding to the bold coefficients in the above series are

$$\mathcal{C}_{4,0}(001) = \{\mathtt{0000}, \mathtt{0100}, \mathtt{0110}, \mathtt{0120}\}, \quad \mathcal{C}_{4,1}(001) = \{\mathtt{0111}, \mathtt{0121}\},$$
$$\mathcal{C}_{4,2}(001) = \{\mathtt{0122}\}, \text{ and } \mathcal{C}_{4,3}(001) = \{\mathtt{0123}\}.$$

The array $[x^n y^k]\boldsymbol{H}_{001}(x,y) = \boldsymbol{c}_{001}(n,k)$ corresponds with the array A155038 in [15] which enumerates the set $\mathcal{P}_{n,k}$ of integer compositions of n with first part k. A constructive bijection ψ between $\mathcal{C}_{n,k}(001)$ and $\mathcal{P}_{n,k}$ can be defined recursively as follows: if $n = 0$ then $\psi(\epsilon) = 0$; if $n \geq 1$, we distinguish two cases: if $w = \mathtt{0}(w'+1)\mathtt{0}^k$, $k \geq 1$ then we set $\psi(w) = 1^k\psi(\mathtt{0}(w'+1))$, and if $w = \mathtt{0}(w'+1)$ then we set $\psi(w) = (c_1+1)c_2\cdots c_r$ with $\psi(w') = c_1c_2\cdots c_r$. For instance, we have $\psi(012200) = 1, 1, 3, 1$.

4.3 The Pattern 010

Notice that a Catalan word w is in set $\mathcal{C}_{n,k}(010)$ if and only if w is in the set $\mathcal{C}_{n,k}(10)$. Therefore, from Theorem 3.3 we conclude the following result.

Theorem 4.3 *We have*

$$\boldsymbol{H}_{010}(x,y) = \frac{x}{1-x-xy},$$

and for $n \geq 1$, $0 \leq k \leq n-1$, $\boldsymbol{c}_{010}(n,k) = \binom{n-1}{k}$, *and* $\boldsymbol{a}_{010}(n) = (n-1)/2$.

The series expansion of the generating function $\boldsymbol{H}_{010}(x, y)$ is

$$x + (1+y)x^2 + (1+2y+y^2)x^3 + (\mathbf{1+3y+3y^2+y^3})x^4 + (1+4y+6y^2+4y^3+y^4)x^5 \\ +(1+5y+10y^2+10y^3+5y^4+y^5)x^6 + O(x^7).$$

The Catalan words corresponding to the bold coefficients in the above series are

$$\mathcal{C}_{4,0}(010) = \{\texttt{0000}\}, \quad \mathcal{C}_{4,1}(010) = \{\texttt{0001}, \texttt{0011}, \texttt{0111}\},$$
$$\mathcal{C}_{4,2}(010) = \{\texttt{0012}, \texttt{0112}, \texttt{0122}\}, \text{ and } \mathcal{C}_{4,3}(001) = \{\texttt{0123}\}.$$

The array $[x^n y^k]\boldsymbol{H}_{010}(x, y) = \boldsymbol{c}_{010}(n, k)$ corresponds with the Pascal matrix A007318 in [15]. A constructive bijection ϕ between $\mathcal{C}_{n,k}(010)$ and the set $\mathcal{S}_{n-1,k}$ of k-subsets of $\{1, \ldots, n-1\}$ can be defined as follows: Let $w \in \mathcal{C}_{n,k}(010)$, we write it $w = \texttt{0}^{i_0}\texttt{1}^{i_1}\cdots\texttt{k}^{i_k}$, with $i_j \geq 1$, $0 \leq j \leq k$, and we set $\phi(w) = \{i_0, i_0+i_1, \ldots, i_0+i_1+\cdots+i_{k-1}\}$. For instance, we have $\phi(\texttt{011233344}) = \{1, 3, 4, 7\} \in \mathcal{S}_{8,4}$.

4.4 The Pattern 021

Theorem 4.4 *We have*

$$\boldsymbol{H}_{021}(x, y) = \frac{x - 5x^2 + 9x^3 - 6x^4 + x^5(1-y)}{(1-2x)^2(1-x)(1-x-xy)}.$$

Moreover, for $n \geq 2$, $\boldsymbol{c}_{021}(n, 0) = (n-2)\cdot 2^{n-3} + 1$, and for $k \geq 1$,

$$\boldsymbol{c}_{021}(n, k) = \sum_{i=k-1}^{n-2} \binom{n-2}{i} \quad \textit{and} \quad \boldsymbol{a}_{021}(n) = \frac{2^{n-5}(n^2+3n-2)}{(n-1)2^{n-2}+1} \sim \frac{n+4}{8}.$$

Proof Let w denote a non-empty Catalan word in $\mathcal{C}(021)$, and let $w = \texttt{0}(w'+1)w''$ be the first return decomposition, where $w', w'' \in \mathcal{C}(021)$. If $w'' = \epsilon$, then $w = \texttt{0}(w'+1)$ where w' is a possibly empty Catalan word in $\mathcal{C}(10)$. From Theorem 3.3 the generating function for this case is

$$x + xy\boldsymbol{H}_{10}(x, y) = x + \frac{x^2 y}{1 - x(1+y)}.$$

If $w' = \epsilon$ and $w'' \neq \epsilon$, then the contribution to the generating function is $x\boldsymbol{H}_{021}(x, y)$. Finally, if w' and w'' are non empty, then we have two cases.

- Case 1: $w' = \texttt{0}^k$ with $k \geq 1$. Since $w'' \in \mathcal{C}(021)$, the generating function is $x\,(x/(1-x))\,\boldsymbol{H}_{021}(x, y)$.
- Case 2: $w' \in \mathcal{C}(10) \setminus \mathcal{C}(01)$. In this case $w'' \in \mathcal{C}(01)$, that is, $w'' = \texttt{0}^k$ $(k \geq 1)$. From Theorems 3.3 and 3.2 the generating function for this case is

$$x\,(\boldsymbol{H}_{01}(x) - \boldsymbol{H}_{10}(x))\,\frac{x}{1-x} = \frac{x^4}{(1-x)^2(1-2x)}.$$

Therefore, we have the functional equation

$$\boldsymbol{H}_{021}(x, y) = x + \frac{x^2 y}{1 - x(1+y)} + x\boldsymbol{H}_{021}(x, y) + x\frac{x}{1-x}\boldsymbol{H}_{021}(x, y) + \frac{x^4}{(1-x)^2(1-2x)}.$$

Solving this equation we obtain the desired result. □

The series expansion of the generating function $\boldsymbol{H}_{021}(x, y)$ is

$$x + (1+y)x^2 + (2+2y+y^2)x^3 + (\mathbf{5+4y+3y^2+y^3})x^4$$
$$+(13+8y+7y^2+4y^3+y^4)x^5 + (33+16y+15y^2+11y^3+5y^4+y^5)x^6 + O(x^7).$$

The Catalan words corresponding to the bold coefficients in the above series are

$\mathcal{C}_{4,0}(021) = \{\mathtt{0000}, \mathtt{0010}, \mathtt{0100}, \mathtt{0110}, \mathtt{0120}\}$, $\mathcal{C}_{4,1}(021) = \{\mathtt{0001}, \mathtt{0011}, \mathtt{0101}, \mathtt{0111}\}$
$\mathcal{C}_{4,2}(021) = \{\mathtt{0012}, \mathtt{0112}, \mathtt{0122}\}$, and $\mathcal{C}_{4,3}(021) = \{\mathtt{0123}\}$.

Notice that the array $[\boldsymbol{c}_{021}(n,k)]_{n,k\geq 1}$ coincides with the array A055248 in [15].

4.5 The Pattern 102

Theorem 4.5 *We have*

$$\boldsymbol{H}_{102}(x, y) = \frac{x(1-4x+4x^2-x^3(1-y))}{(1-x)(1-3x)(1-x(1+y))}.$$

Moreover, $\boldsymbol{c}_{102}(1,0) = 1$, *and for* $n \geq 2$

$$\boldsymbol{c}_{102}(n,k) = \begin{cases} \sum_{k=0}^{n-2} \boldsymbol{c}_{102}(n-1,k), & \text{if } k = 0, 1 \\ \boldsymbol{c}_{102}(n-1,k-1) + \boldsymbol{c}_{102}(n-1,k), & \text{if } 2 \leq k \leq n-1 \text{ , and} \\ 0, & \text{otherwise} \end{cases}$$

$$\boldsymbol{a}_{102}(n) = \frac{4\cdot 3^{n-1} - 2^n}{3^{n-1}+1} \sim 4.$$

Proof Let w denote a non-empty Catalan word in $\mathcal{C}(102)$, and let $w = \mathtt{0}(w'+1)w''$ be the first return decomposition, where $w', w'' \in \mathcal{C}(102)$. If $w'' = \epsilon$, then $w = \mathtt{0}(w'+1)$ with w' possibly empty. The generating function for this case is $x + xy\boldsymbol{H}_{102}(x, y)$. If $w'' \neq \epsilon$, then $w = \mathtt{0}w''$ or $w = \mathtt{0}(w'+1)w''$ with $w'' \in \mathcal{C}(012)^+$. Therefore, the contribution to the generating function is

$$x\boldsymbol{H}_{102}(x, y) + x\boldsymbol{H}_{102}(x)\boldsymbol{H}_{012}(x, y),$$

where (see Theorem 8 of [1])

$$\boldsymbol{H}_{102}(x) = \frac{1-3x+x^2}{(1-x)(1-3x)} - 1.$$

Therefore, we have the functional equation

$$\boldsymbol{H}_{102}(x, y) = x + xy\boldsymbol{H}_{102}(x, y) + x\boldsymbol{H}_{102}(x, y) + x\boldsymbol{H}_{102}(x)\boldsymbol{H}_{012}(x, y).$$

Solving this equation we obtain the desired result. □

The series expansion of the generating function $\boldsymbol{H}_{102}(x, y)$ is

$$x + (1+y)x^2 + (2+2y+y^2)x^3 + (\mathbf{5+5y+3y^2+y^3})x^4$$
$$+(14+14y+8y^2+4y^3+y^4)x^5 + (41+41y+22y^2+12y^3+5y^4+y^5)x^6 + O(x^7).$$

The Catalan words corresponding to the bold coefficients in the above series are

$\mathcal{C}_{4,0}(102) = \{\mathtt{0000}, \mathtt{0010}, \mathtt{0100}, \mathtt{0110}, \mathtt{0120}\}$, $\mathcal{C}_{4,1}(102) = \{\mathtt{0001}, \mathtt{0011}, \mathtt{0101}, \mathtt{0111}, \mathtt{0121}\}$
$\mathcal{C}_{4,2}(102) = \{\mathtt{0012}, \mathtt{0112}, \mathtt{0122}\}$, and $\mathcal{C}_{4,3}(102) = \{\mathtt{0123}\}$.

The array $[\boldsymbol{c}_{102}(n,k)]_{n,k\geq 0}$ does not appear in [15].

4.6 The Pattern 201

Theorem 4.6 *We have*

$$\boldsymbol{H}_{201}(x,y)=\frac{x(1-5x+7x^2-2x^3)}{(1-x)(1-3x)(1+x^2y-x(2+y))},$$

$$\boldsymbol{c}_{201}(n,k)=\begin{cases}\frac{3^{n-2}+1}{2}, & \text{if } k=0\\ \boldsymbol{c}_{201}(n-1,k-1)+\boldsymbol{c}_{201}(n-1,k) & \\ \qquad +\boldsymbol{c}_{201}(n-2,k)+\boldsymbol{c}_{201}(n-2,k+1), & \text{if } 1\le k\le n-1\\ 0, & \text{otherwise}\end{cases}, \text{ and}$$

$$\boldsymbol{a}_{201}(n)=2\frac{3^{n-1}-F(2n-2)}{3^{n-1}+1}\sim 2,$$

where $F(n)$ is the n-th term of the Fibonacci sequence defined by $F(n)=F(n-1)+F(n-2)$ anchored with $F(0)=F(1)=1$.

Proof Let w denote a non-empty Catalan word in $\mathcal{C}(201)$, and let $w=\mathtt{0}(w'+1)w''$ be the first return decomposition, where $w',w''\in\mathcal{C}(201)$. If $w''=\epsilon$, then $w=\mathtt{0}(w'+1)$ with w' possibly empty. The generating function for this case is $x+xy\boldsymbol{H}_{201}(x,y)$. If $w''\neq\epsilon$ and $w'=\epsilon$, then $w=\mathtt{0}w''$ with $w''\in\mathcal{C}(201)$. The contribution to the generating function is $x\boldsymbol{H}_{201}(x,y)$. If w' and w'' are non empty, then we have two cases.

- Case 1: $w'=\mathtt{0}^n$ with $n\ge 1$. Since $w''\in\mathcal{C}(021)^+$, the generating function is $x\frac{x}{1-x}\boldsymbol{H}_{201}(x,y)$.
- Case 2: $w'\in\mathcal{C}(201)^+\setminus\{\mathtt{0}^k,k\ge 1\}$. In this case $w''=\mathtt{0}^n$ for $n\ge 1$. The generating function for this case is

  $$x\left(\boldsymbol{H}_{201}(x)-\frac{x}{1-x}\right)\frac{x}{1-x},$$

  where (see Theorem 8 of [1])

  $$\boldsymbol{H}_{201}(x)=\frac{1-3x+x^2}{(1-x)(1-3x)}-1.$$

Therefore, we have the functional equation

$$\begin{aligned}\boldsymbol{H}_{201}(x,y)&=x+xy\boldsymbol{H}_{201}(x,y)+x\boldsymbol{H}_{201}(x,y)\\&\quad+x\frac{x}{1-x}\boldsymbol{H}_{201}(x,y)+x\left(\boldsymbol{H}_{201}(x)-\frac{x}{1-x}\right)\frac{x}{1-x}.\end{aligned}$$

Solving this equation we obtain the desired result. □

The series expansion of the generating function $\boldsymbol{H}_{201}(x,y)$ is

$$\begin{aligned}&x+(1+y)x^2+(2+2y+y^2)x^3+(\mathbf{5+5y+3y^2+y^3})x^4\\&\quad+(14+13y+9y^2+4y^3+y^4)x^5+(41+35y+26y^2+14y^3+5y^4+y^5)x^6+O(x^7).\end{aligned}$$

The Catalan words corresponding to the bold coefficients in the above series are

$$\mathcal{C}_{4,0}(201)=\{\mathtt{0000},\mathtt{0100},\mathtt{0110},\mathtt{0120}\},\quad \mathcal{C}_{4,1}(201)=\{\mathtt{0001},\mathtt{0011},\mathtt{0101},\mathtt{0111},\mathtt{0121}\}$$
$$\mathcal{C}_{4,2}(201)=\{\mathtt{0012},\mathtt{0112},\mathtt{0122}\},\ \text{ and } \mathcal{C}_{4,3}(201)=\{\mathtt{0123}\}.$$

The array $[\boldsymbol{c}_{201}(n,k)]_{n,k\ge 0}$ does not appear in [15].

4.7 The Pattern 120

Theorem 4.7 *We have*

$$\boldsymbol{H}_{120}(x,y) = \frac{(1-x)x}{1-x(2+y)+yx^2},$$

$$\boldsymbol{c}_{120}(n,k) = \begin{cases} 2^{n-2}, & \text{if } k=0 \\ \boldsymbol{c}_{120}(n-1,k-1)+\boldsymbol{c}_{120}(n-1,k) & \\ \qquad +\boldsymbol{c}_{120}(n-2,k)+\boldsymbol{c}_{120}(n-2,k+1), & \text{if } 1\le k\le n-1 \\ 0, & \text{otherwise} \end{cases}, \text{ and}$$

$$\boldsymbol{a}_{120}(n) = \frac{2(n-1)F(2n-2)-(n-3)F(2n-3)}{5F(2n-2)} \sim \frac{2n}{5+\sqrt{5}},$$

where $F(n)$ is the Fibonacci sequence defined in Theorem 4.6.

Proof Let w denote a non-empty Catalan word in $\mathcal{C}(120)$, and let $w = \mathtt{0}(w'+1)w''$ be the first return decomposition, where $w', w'' \in \mathcal{C}(120)$. If $w'' = \epsilon$, then $w = \mathtt{0}(w'+1)$ with w' possibly empty. The generating function for this case is $x + xy\boldsymbol{H}_{120}(x,y)$. If $w'' \neq \epsilon$, then $w = \mathtt{0}w''$ or $w = \mathtt{0}(w'+1)w''$ with $w' = \mathtt{0}^k, k \ge 1$, and $w'' \in \mathcal{C}(120)^+$. Therefore, the contribution to the generating function is

$$x\boldsymbol{H}_{120}(x,y) + x\left(\frac{x}{1-x}\right)\boldsymbol{H}_{012}(x,y).$$

So we deduce the functional equation

$$\boldsymbol{H}_{120}(x,y) = x + xy\boldsymbol{H}_{120}(x,y) + x\boldsymbol{H}_{120}(x,y) + x\left(\frac{x}{1-x}\right)\boldsymbol{H}_{012}(x,y).$$

Solving this equation we obtain the desired result. □

The series expansion of the generating function $\boldsymbol{H}_{120}(x,y)$ is

$$x + (1+y)x^2 + (2+2y+y^2)x^3 + (\mathbf{4+5y+3y^2+y^3})x^4$$
$$+(8+12y+9y^2+4y^3+y^4)x^5 + (16+28y+25y^2+14y^3+5y^4+y^5)x^6 + O(x^7).$$

The Catalan words corresponding to the bold coefficients in the above series are

$$\mathcal{C}_{4,0}(120) = \{\mathtt{0000}, \mathtt{0010}, \mathtt{0100}, \mathtt{0110}\}, \quad \mathcal{C}_{4,1}(120) = \{\mathtt{0001}, \mathtt{0011}, \mathtt{0101}, \mathtt{0111}, \mathtt{0121}\}$$
$$\mathcal{C}_{4,2}(120) = \{\mathtt{0012}, \mathtt{0112}, \mathtt{0122}\}, \text{ and } \mathcal{C}_{4,3}(120) = \{\mathtt{0123}\}.$$

Notice that the array $[\boldsymbol{c}_{120}(n,k)]_{n\ge1,k\ge0}$ coincides with the array A105306 in [15]. Therefore we have the combinatorial identity

$$\boldsymbol{c}_{120}(n,k) = \sum_{i=0}^{n-k-1}\binom{k+1}{i}\binom{n-1-i}{k}(-1)^i 2^{n-1-k-i}.$$

This array is related to the number of non-decreasing Dyck path of semi-length n with k weakly symmetric peaks [7]. We leave open the question of finding a constructive bijection.

4.8 The Pattern 101

Theorem 4.8 *We have*

$$\boldsymbol{H}_{101}(x,y)=\frac{x(1-3x+2x^2)}{(1-3x+x^2)(1-x(1+y))},$$

$$\boldsymbol{c}_{101}(n,k)=\begin{cases}F(2n-4), & \text{if } k=0\\ \boldsymbol{c}_{101}(n-1,k-1)+\boldsymbol{c}_{101}(n-1,k), & \text{if } 1\le k\le n-1, \text{ and}\\ 0, & \text{otherwise}\end{cases}$$

$$\boldsymbol{a}_{101}(n)=\frac{F(2n-1)-2^{n-1}}{F(2n-2)}\sim\frac{\sqrt{5}+1}{2},$$

where $F(n)$ is the Fibonacci sequence defined in Theorem 4.6.

Proof Let w denote a non-empty Catalan word in $\mathcal{C}(101)$, and let $w=\mathtt{0}(w'+1)w''$ be the first return decomposition, where $w',w''\in\mathcal{C}(101)$. If $w''=\epsilon$, then $w=\mathtt{0}(w'+1)$ with w' possibly empty. The generating function for this case is $x+xy\boldsymbol{H}_{101}(x,y)$. If $w''\neq\epsilon$, then $w=\mathtt{0}w''$ or $w=\mathtt{0}(w'+1)\mathtt{0}^+$ with $w'\in\mathcal{C}(101)^+$. Therefore, the contribution to the generating function is

$$x\boldsymbol{H}_{101}(x,y)+x\boldsymbol{H}_{101}(x)\left(\frac{x}{1-x}\right),$$

where (see Theorem 9 of [1])

$$\boldsymbol{H}_{101}(x)=\frac{1-2x}{1-3x+x^2}-1.$$

Therefore, we have the functional equation

$$\boldsymbol{H}_{101}(x,y)=x+xy\boldsymbol{H}_{101}(x,y)+x\boldsymbol{H}_{101}(x,y)+x\boldsymbol{H}_{101}(x)\left(\frac{x}{1-x}\right).$$

Solving this equation we obtain the desired result. □

The series expansion of the generating function $\boldsymbol{H}_{101}(x,y)$ is

$$x+(1+y)x^2+(2+2y+y^2)x^3+(\mathbf{5+4y+3y^2+y^3})x^4$$
$$+(13+9y+7y^2+4y^3+y^4)x^5+(34+22y+16y^2+11y^3+5y^4+y^5)x^6+O(x^7).$$

The Catalan words corresponding to the bold coefficients in the above series are

$$\mathcal{C}_{4,0}(101)=\{\mathtt{0000},\mathtt{0010},\mathtt{0100},\mathtt{0110},\mathtt{0120}\},\quad \mathcal{C}_{4,1}(101)=\{\mathtt{0001},\mathtt{0011},\mathtt{0111},\mathtt{0121}\}$$
$$\mathcal{C}_{4,2}(101)=\{\mathtt{0012},\mathtt{0112},\mathtt{0122}\},\ \text{ and } \mathcal{C}_{4,3}(101)=\{\mathtt{0123}\}.$$

The array $[\boldsymbol{c}_{101}(n,k)]_{n,k\ge0}$ does not appear in [15].

4.9 The Pattern 011

Theorem 4.9 *We have*

$$\boldsymbol{H}_{011}(x,y)=\frac{x(1-2x+2x^2-x^3y)}{(1-x)^3(1-xy)},$$

$$\boldsymbol{c}_{011}(n,k)=\begin{cases}\frac{(n-1)(n-2)}{2}, & \text{if } k=0\\ 1, & \text{if } 1\le k\le n-1, \text{ and}\\ 0, & \text{otherwise}\end{cases}\quad \boldsymbol{a}_{011}(n)=1+\frac{2}{n(n-1)}\sim 1.$$

Proof Let w denote a non-empty Catalan word in $\mathcal{C}(011)$, and let $w = \mathtt{0}(w'+1)w''$ be the first return decomposition, where $w', w'' \in \mathcal{C}(011)$. If $w'' = \epsilon$, then $w = \mathtt{0}(w'+1)$ with $w' \in \mathcal{C}(00)$ (possibly empty). The generating function for this case is $x + xy\boldsymbol{H}_{00}(x, y)$. If $w'' \neq \epsilon$, then $w = \mathtt{0}w''$ or $w = \mathtt{0}(w'+1)w''$. In the first case $w'' \in \mathcal{C}(011)^+$. In the second case, with $w' \in \mathcal{C}(00)$ and w'' is a word of the form $\mathtt{0}^j\mathtt{123}\cdots$, for $j \geq 1$. Therefore, we have the functional equation

$$\boldsymbol{H}_{011}(x, y) = x + xy\boldsymbol{H}_{00}(x, y) + x\boldsymbol{H}_{011}(x, y) + x\boldsymbol{H}_{00}(x)\frac{x}{1-x}.$$

Solving this equation we obtain the desired result. □

The series expansion of the generating function $\boldsymbol{H}_{011}(x, y)$ is

$$x + (1+y)x^2 + (2+y+y^2)x^3 + (\mathbf{4+y+y^2+y^3})x^4$$
$$+(7+y+y^2+y^3+y^4)x^5 + (11+y+y^2+y^3+y^4+y^5)x^6 + O(x^7).$$

The Catalan words corresponding to the bold coefficients in the above series are

$$\mathcal{C}_{4,0}(011) = \{\mathtt{0000}, \mathtt{0010}, \mathtt{0100}, \mathtt{0120}\}, \quad \mathcal{C}_{4,1}(011) = \{\mathtt{0001}\}$$
$$\mathcal{C}_{4,2}(011) = \{\mathtt{0012}\}, \text{ and } \mathcal{C}_{4,3}(011) = \{\mathtt{0123}\}.$$

The array $[\boldsymbol{c}_{011}(n, k)]_{n,k\geq 0}$ does not appear in [15].

4.10 The Pattern 000

Theorem 4.10 *We have*

$$\boldsymbol{H}_{000}(x, y) = \frac{x(1-3x^2-x^3-(1-x-3x^2+x^3)x^2y)}{(1-x-3x^2+x^3)(1-xy-2x^2y+x^3y^2)},$$

$$\boldsymbol{c}_{000}(n, k) = \begin{cases} 1, & \text{if } k=0,\ n=1,2,3 \\ \boldsymbol{c}_{000}(n-1,k) + 3\boldsymbol{c}_{000}(n-2,k) - \boldsymbol{c}_{000}(n-3,k), & \text{if } k=0,\ n\geq 4 \\ \boldsymbol{c}_{000}(n-1,k-1) + 2\boldsymbol{c}_{000}(n-2,k-1) & \\ \qquad - \boldsymbol{c}_{000}(n-3,k-2), & \text{if } k\geq 1,\ n\geq 4 \\ 0, & \text{otherwise} \end{cases} \quad \text{and}$$

$\boldsymbol{a}_{000}(n) \sim 1$.

Proof Let w denote a non-empty Catalan word in $\mathcal{C}(000)$, and let $w = \mathtt{0}(w'+1)w''$ be the first return decomposition, where $w', w'' \in \mathcal{C}(000)$. If $w'' = \epsilon$, then $w = \mathtt{0}(w'+1)$ with w' possibly empty. The generating function for this case is $x + xy\boldsymbol{H}_{000}(x, y)$. The second case is when $w'' \neq \epsilon$, then we have the following four cases:

- $w = \mathtt{00}$. The generating functions for this case is x^2.
- $w = \mathtt{00}(w'''+1)$, where $w''' \in \mathcal{C}(000)^+$. In this case the generating function is $x^2y\boldsymbol{H}_{000}(x, y)$.
- $w = \mathtt{0}(w'+1)\mathtt{0}$, where $w' \in \mathcal{C}(000)^+$. The generating function for this case is $x^2\boldsymbol{H}_{000}(x)$, where (see Theorem 11 of [1])

  $$\boldsymbol{H}_{000}(x) = \frac{(1-2x^2)}{1-x-3x^2+x^3} - 1.$$
- $w = \mathtt{0}(w'+1)\mathtt{0}(w'''+1)$, where w', w'' are words in $\mathcal{C}(000)^+$ and the concatenation $w'w''$ is again a Catalan word of the second case, that is, the generating function of the words $w'w'''$ is the same of the generating function of the words of the form $\mathtt{0}(w'+1)w''$, where w'' is non-empty. Therefore, the contribution to the generating function is $T(x, y) = x^2y(x^2 + x^2y\boldsymbol{H}_{000}(x, y) + T(x, y))$.

Therefore, we have the functional equation

$$\boldsymbol{H}_{000}(x,y) = x + xy\boldsymbol{H}_{000}(x,y) + x^2 + x^2y\boldsymbol{H}_{000}(x,y) + x^2\boldsymbol{H}_{000}(x) + T(x,y).$$

Solving the system of equations we obtain the desired result. □

The series expansion of the generating function $\boldsymbol{H}_{000}(x,y)$ is

$$x + (1+y)x^2 + (1+2y+y^2)x^3 + (\mathbf{2+3y+3y^2+y^3})x^4$$
$$+(4+4y+6y^2+4y^3+y^4)x^5 + (9+8y+9y^2+10y^3+5y^4+y^5)x^6 + O(x^7).$$

The Catalan words corresponding to the bold coefficients in the above series are

$$\mathcal{C}_{4,0}(000) = \{\texttt{0110}, \texttt{0120}\}, \quad \mathcal{C}_{4,1}(000) = \{\texttt{0011}, \texttt{0101}, \texttt{0121}\}$$
$$\mathcal{C}_{4,2}(000) = \{\texttt{0012}, \texttt{0112}, \texttt{0122}\}, \text{ and } \mathcal{C}_{4,3}(000) = \{\texttt{0123}\}.$$

The array $[\boldsymbol{c}_{000}(n,k)]_{n,k\geq 0}$ does not appear in [15].

4.11 The Pattern 100

Theorem 4.11 *We have*

$$\boldsymbol{H}_{100}(x,y) = \frac{x(1-3x+x^2(1-y)-x^4(1-2y)+x^3(1+2y))}{(1-2x-2x^2)(1+x^2+x^3y^2-x(2+y))},$$

$$\boldsymbol{c}_{100}(n,k) = \begin{cases} 1, & \text{if } k=0,\ n=1,2 \\ 2, & \text{if } k=0,\ n=3 \\ 4, & \text{if } k=0,\ n=4 \\ 4\boldsymbol{c}_{100}(n-1,k) - 3\boldsymbol{c}_{100}(n-2,k) - 2\boldsymbol{c}_{100}(n-3,k) + 2\boldsymbol{c}_{100}(n-4,k), & \text{if } k=0,\ n\geq 5 \\ 2\boldsymbol{c}_{100}(n-1,k) + \boldsymbol{c}_{100}(n-1,k-1) - \boldsymbol{c}_{100}(n-2,k) - \boldsymbol{c}_{100}(n-3,k-2), & \text{if } k\geq 1,\ n\geq 4 \\ 0, & \text{otherwise} \end{cases} \quad \textit{and}$$

$$\boldsymbol{a}_{100}(n) \sim \frac{8\sqrt{3}-12}{9\sqrt{3}-15} \sim 3.154700527.$$

Proof Let w denote a non-empty Catalan word in $\mathcal{C}(100)$, and let $w = \texttt{0}(w'+1)w''$ be the first return decomposition, where $w', w'' \in \mathcal{C}(100)$. If $w'' = \epsilon$, then $w = \texttt{0}(w'+1)$ with w' possibly empty. The generating function for this case is $x + xy\boldsymbol{H}_{100}(x,y)$. Now let us assume that $w'' \neq \epsilon$. Therefore, w contains at least two symbols $\texttt{0}$. We distinguish two cases: (i) w contains $k \geq 3$ symbols $\texttt{0}$, and (ii) w contains exactly two $\texttt{0}$.

- Case (i): w is necessarily of the form $w = \texttt{0}^{k-2}u$ where $u \in \mathcal{C}(100)$ contains exactly two symbols $\texttt{0}$. Thus, the generating function for this case is

  $$A_2(x,y) \cdot \sum_{k\geq 3} x^{k-2},$$

  where $A_2(x,y)$ is the generating function for Catalan words in $\mathcal{C}(100)$ having exactly two symbols $\texttt{0}$.
- Case (ii): w can be written $w = \texttt{0}(w'+1)\texttt{0}(w''+1)$ where $w', w'' \in \mathcal{C}(100)$.
  - If $w' = \epsilon$ then the generating function is $x^2y\boldsymbol{H}_{100}(x,y)$;
  - If $w'' = \epsilon$ then the generating function is $x^2\boldsymbol{H}_{100}(x)$ where (see [1])

    $$\boldsymbol{H}_{100}(x) = \frac{1-2x-x^2+x^3}{1-3x+2x^3} - 1$$

– Otherwise, w' and w'' are non-empty, where the Catalan word $w'w''$ avoids 100 and contains at least two symbols $\mathtt{0}$, that is $w'w'' = \mathtt{0}^{k-2}\mathtt{0}(u+1)\mathtt{0}(v+1)$. So, there are $k-1$ choices possibles for w', namely, $\mathtt{0}, \ldots, \mathtt{0}^{k-2}$ and $\mathtt{0}^{k-2}\mathtt{0}(u+1)$. So, the generating function for these words is $x^2yA_2(x,y)\sum_{k\geq 2}(k-1)x^{k-2}$. So, we obtain

$$A_2(x,y) = x^2y\boldsymbol{H}_{100}(x,y) + x^2\boldsymbol{H}_{100}(x) + x^2yA_2(x,y)\sum_{k\geq 2}(k-1)x^{k-2}.$$

Combining all these cases, we obtain the functional equations:

$$\begin{cases} \boldsymbol{H}_{100}(x,y) = x + xy\boldsymbol{H}_{100}(x,y) + A_2(x,y)\sum_{k\geq 3}x^{k-2} + A_2(x,y) \\ A_2(x,y) = x^2y\boldsymbol{H}_{100}(x,y) + x^2\boldsymbol{H}_{100}(x) + x^2yA_2(x,y)\sum_{k\geq 2}(k-1)x^{k-2}. \end{cases}$$

A simple calculation provides the result. □

The series expansion of the generating function $\boldsymbol{H}_{100}(x,y)$ is

$$\begin{aligned} &x + (1+y)x^2 + (2+2y+y^2)x^3 + (\mathbf{4+5y+3y^2+y^3})x^4 \\ &\quad +(9+12y+9y^2+4y^3+y^4)x^5 + (22+28y+25y^2+14y^3+5y^4+y^5)x^6 + O(x^7). \end{aligned}$$

The Catalan words corresponding to the bold coefficients in the above series are

$$\mathcal{C}_{4,0}(100) = \{\mathtt{0000}, \mathtt{0010}, \mathtt{0110}, \mathtt{0100}\}, \quad \mathcal{C}_{4,1}(100) = \{\mathtt{0001}, \mathtt{0011}, \mathtt{0101}, \mathtt{0111}, \mathtt{0121}\}$$
$$\mathcal{C}_{4,2}(100) = \{\mathtt{0012}, \mathtt{0112}, \mathtt{0122}\}, \text{ and } \mathcal{C}_{4,3}(100) = \{\mathtt{0123}\}.$$

4.12 The Pattern 110

Theorem 4.12 *We have*

$$\boldsymbol{H}_{110}(x,y) = \frac{x(1-2x+2x^2-x^2y+x^3y)}{(1-x)(1-2x+x^2-xy+x^3y^2)}.$$

$$\boldsymbol{a}_{110}(n) \sim \frac{4\sqrt{2}-5}{4\sqrt{2}-4}\cdot n \sim 0.3964466089\cdot n.$$

Proof Let w denote a non-empty Catalan word in $\mathcal{C}(110)$, and let $w = \mathtt{0}(w'+1)w''$ be the first return decomposition, where $w', w'' \in \mathcal{C}(110)$. If $w'' = \epsilon$, then $w = \mathtt{0}(w'+1)$ with w' possibly empty. The generating function for this case is $x + xy\boldsymbol{H}_{110}(x,y)$. If $w'' \neq \epsilon$ and $w' = \epsilon$, then $w = \mathtt{0}w''$. The contribution is $x\boldsymbol{H}_{110}(x,y)$. If $w = 0(w'+1)w''$ with w' and w'' non-empty, then $w' = \mathtt{01}\cdots\mathtt{k}$ with $k \geq 0$, and w'' is of the form either $w'' = \mathtt{0}^{i_1}\mathtt{1}^{i_2}\cdots\ell^{i_\ell}$ with $0 \leq \ell, i_j \geq 1$, that is $w'' \in \mathcal{C}(10)^+$, or $w'' = \mathtt{0}^{i_1}\mathtt{1}^{i_2}\cdots\mathtt{k}^{i_k}(w'''+1)$ with $i_j \geq 1$ and $w''' \in \mathcal{C}(110)\setminus\mathcal{C}(10)$. The contribution for these two cases are respectively $\frac{x}{1-x}\boldsymbol{H}_{10}(x,y)$ and

$$x\left(\sum_{k\geq 1}x^k\cdot\left(\frac{x}{1-x}\right)^k y^k\right)(\boldsymbol{H}_{110}(x,y) - \boldsymbol{H}_{10}(x,y)) = \frac{x^3y}{1-x-x^2y}(\boldsymbol{H}_{110}(x,y) - \boldsymbol{H}_{10}(x,y)).$$

Therefore we have the functional equation

$$\begin{aligned} \boldsymbol{H}_{110}(x,y) &= x + xy\boldsymbol{H}_{110}(x,y) + x\boldsymbol{H}_{110}(x,y) + \frac{x}{1-x}\boldsymbol{H}_{10}(x,y) \\ &\quad + \frac{x^3y}{1-x-x^2y}(\boldsymbol{H}_{110}(x,y) - \boldsymbol{H}_{10}(x,y)). \end{aligned}$$

Solving the system of equations we obtain the desired result. □

The series expansion of the generating function $\boldsymbol{H}_{110}(x, y)$ is

$$x + (1+y)x^2 + (2+2y+y^2)x^3 + (\mathbf{4+5y+3y^2+y^3})x^4$$
$$+(7+12y+9y^2+4y^3+y^4)x^5 + (11+26y+25y^2+14y^3+5y^4+y^5)x^6 + O(x^7).$$

The Catalan words corresponding to the bold coefficients in the above series are

$\mathcal{C}_{4,0}(110) = \{\texttt{0000}, \texttt{0010}, \texttt{0100}, \texttt{0120}\}$, $\mathcal{C}_{4,1}(110) = \{\texttt{0001}, \texttt{0011}, \texttt{0101}, \texttt{0111}, \texttt{0121}\}$
$\mathcal{C}_{4,2}(110) = \{\texttt{0012}, \texttt{0112}, \texttt{0122}\}$, and $\mathcal{C}_{4,3}(110) = \{\texttt{0123}\}$.

The array $[\boldsymbol{c}_{110}(n,k)]_{n,k\geq 0}$ does not appear in [15]. A recurrence relation for the coefficients $\boldsymbol{c}_{110}(n,k)$ can be obtained using *Mathematica* package (see Sect. 5), but the relation is very ugly and too involved. So we do not provide it here.

4.13 The Pattern 210

Theorem 4.13 *We have*

$$\boldsymbol{H}_{210}(x, y) = \frac{x(1-4x-x^3(1-2y)+x^2(5-y))}{(1-2x)(1+x^3y^2+x^2(2+y)-x(3+y))}.$$

$$\boldsymbol{c}_{210}(n,k) = \begin{cases} 1, & \text{if } k=0,\ n=1,2 \\ 2, & \text{if } k=0,\ n=3 \\ 5\boldsymbol{c}_{210}(n-1,k) - 8\boldsymbol{c}_{210}(n-2,k) + 4\boldsymbol{c}_{210}(n-3,k), & \text{if } k=0,\ n\geq 4 \\ 3\boldsymbol{c}_{210}(n-1,k) + \boldsymbol{c}_{210}(n-1,k-1) - 2\boldsymbol{c}_{210}(n-2,k) - \\ \qquad - \boldsymbol{c}_{210}(n-2,k-1) - \boldsymbol{c}_{210}(n-3,k-2), & \text{if } k\geq 1,\ n\geq 4 \\ 0, & \text{otherwise} \end{cases} \quad \textit{and}$$

$$\boldsymbol{a}_{210}(n) \sim \frac{(a+1)(-1+2a)}{3a^2+6a-4} \cdot n \sim 0.2630237717 \cdot n,$$

where

$$a = -1 - \frac{\sqrt{21}}{3}\sin\left(\frac{1}{3}\arctan\left(\frac{\sqrt{3}}{9}\right) + \frac{\pi}{6}\right) + \sqrt{7}\cos\left(\frac{1}{3}\arctan\left(\frac{\sqrt{3}}{9}\right) + \frac{\pi}{6}\right).$$

Proof Let w denote a non-empty Catalan word in $\mathcal{C}(210)$, and let $w = \texttt{0}(w'+1)w''$ be the first return decomposition, where $w', w'' \in \mathcal{C}(210)$. Then, w has one of the following forms:

- $w = \texttt{0}(w'+1)$ where $w' \in \mathcal{C}(210)$; the generating function for these words is $x + xy\boldsymbol{H}_{210}(x, y)$.
- $w = \texttt{0}w''$ where $w'' \in \mathcal{C}(210)^+$; the generating function for these words is $x\boldsymbol{H}_{210}(x, y)$.
- $w = \texttt{0}(w'+1)w''$ where $w' \in \mathcal{C}(01)^+$ and $w'' \in \mathcal{C}(210)^+$; the generating function for these words is $x\boldsymbol{H}_{01}(x)\boldsymbol{H}_{210}(x, y)$.
- $w = \texttt{01}^{a_1}\texttt{2}^{a_2}\cdots\texttt{k}^{a_k}w''$ where $k \geq 2$, $a_i \geq 1$ for $1 \leq i \leq k$, and $w'' \in \mathcal{C}(10)^+$; the generating function for these words is $\boldsymbol{H}_{10}(x, y)\sum_{k\geq 2}\frac{x^{k+1}}{(1-x)^k}$.
- $w = \texttt{01}^{a_1}\texttt{2}^{a_2}\cdots\texttt{k}^{a_k}\texttt{0}^{b_0}\texttt{1}^{b_1}\texttt{2}^{b_2}\cdots\texttt{(k-2)}^{b_{k-2}}(w''+\texttt{k}-\texttt{1})$ where $k \geq 2$, $a_i \geq 1$ for $1 \leq i \leq k$, $b_i \geq 1$ for $0 \leq i \leq k-2$, and $w'' \in \mathcal{C}(210)\backslash\mathcal{C}(10)$; the generating function for these words is

$$(\boldsymbol{H}_{210}(x, y) - \boldsymbol{H}_{10}(x, y))\sum_{k\geq 2}\frac{x^{k+1}}{(1-x)^k}\frac{x^{k-1}}{(1-x)^{k-1}}y^{k-1}.$$

Combining these different cases, we deduce the functional equation:

$$\boldsymbol{H}_{210}(x,y) = x + xy\boldsymbol{H}_{210}(x,y) + x\boldsymbol{H}_{210}(x,y) + x\boldsymbol{H}_{01}(x)\boldsymbol{H}_{210}(x,y)$$
$$+\boldsymbol{H}_{10}(x,y)\sum_{k\geq 2}\frac{x^{k+1}}{(1-x)^k} + (\boldsymbol{H}_{210}(x,y) - \boldsymbol{H}_{10}(x,y))\sum_{k\geq 2}\frac{x^{2k}}{(1-x)^{2k-1}}y^{k-1},$$

which gives the results. □

The series expansion of the generating function $\boldsymbol{H}_{210}(x,y)$ is

$$x + (1+y)x^2 + (2+2y+y^2)x^3 + (\mathbf{5+5y+3y^2+y^3})x^4$$
$$+(13+14y+9y^2+4y^3+y^4)x^5 + (33+40y+28y^2+14y^3+5y^4+y^5)x^6 + O(x^7).$$

The Catalan words corresponding to the bold coefficients in the above series are

$\mathcal{C}_{4,0}(210) = \{0000, 0010, 0100, 0110, 0120\}$, $\mathcal{C}_{4,1}(210) = \{0001, 0011, 0101, 0111, 0121\}$
$\mathcal{C}_{4,2}(210) = \{0012, 0112, 0122\}$, and $\mathcal{C}_{4,3}(210) = \{0123\}$.

5 An Experimental Approach by Using Computer Algebra

In the previous results we give several recurrence relations and asymptotic expansions for the sequences $\boldsymbol{c}_p(n,k)$ and $\boldsymbol{a}_p(n)$, respectively. To find some of these recurrences, we used the package `Guess` (written by Manuel Kauers). It is a *Mathematica* package for guessing multivariate recurrence equations and it is part of the `RISCErgoSum` bundle, developed by the Research Institute for Symbolic Computation (RISC).

For example, consider the sequence $\boldsymbol{c}_{000}(n,k)$. In Theorem 4.10 we prove that the generating function $\boldsymbol{H}_{000}(x,y)$ is rational, then the term $\boldsymbol{c}_{000}(n,k)$ can be expressed as linear combinations of the previous terms. Once `Guess` installed in a *Mathematica* session we find the recurrence relation given in Theorem 4.10 by executing the following commands.

```
In[1]:= data = CoefficientList[Normal[CoefficientList[Series[
        (x (1 - 3 x^2 - x^3 - (x^2 - x^3 - 3 x^4 + x^5) y)) /
        ((1 - x - 3 x^2 + x^3) (1 - x y - 2 x^2 y + x^3 y^2)),
        {x, 0, 20}],x]], y] // PadRight;

In[2]:= sset = Flatten[Table[f[n - u, k - v], {u, 0, 3}, {v, 0, 3}]]

In[3]:= GuessMultRE[Delete[data, 1], sset, {n, k}, 1] // Simplify

Out[3]= {n (f[-3+n,-2+k] - 2 f[-2+n,-1+k] - f[-1+n,-1+k] + f[n,k]),
         k (f[-3+n,-2+k] - 2 f[-2+n,-1+k] - f[-1+n,-1+k] + f[n,k]),
         f[-3+n,-2+k] - 2f[-2+n,-1+k] - f[-1+n, -1+k] + f[n,k],
         n (f[-3+n,-3+k] - 2 f[-2+n,-2+k] - f[-1+n,-2+k] + f[n,-1+k]),
         k (f[-3+n,-3+k] - 2 f[-2+n,-2+k] - f[-1+n,-2+k] + f[n,-1+k]),
         f[-3+n, -3+k] - 2f[-2+n,-2+k] - f[-1+n,-2+k] + f[n,-1+k]}
```

From this result we guess that for $n \geq 4$ and $k \geq 1$

$$\boldsymbol{c}_{000}(n,k) = \boldsymbol{c}_{000}(n-1,k-1) + 2\boldsymbol{c}_{000}(n-2,k-1) - \boldsymbol{c}_{000}(n-3,k-2).$$

From the generating function it is not difficult to prove the above equation. In order to obtain asymptotic approximations for $\boldsymbol{a}_p(n)$, we use two methods: if we can extract the coefficients of $[x^n]\partial_y \boldsymbol{H}_p(x,y)|_{y=1}$ and $[x^n]\boldsymbol{H}_p(x,1)(=\boldsymbol{c}_p(n))$, we do it and we calculate the limit of the quotient; otherwise, we use classical methods from generating functions (see [8,12]). Notice such a routine is implemented in a Maple function `equivalent` of the `Algolib` and `gfun` libraries (see http://algo.inria.fr/libraries/).

Acknowledgements The authors would like to thank the anonymous referee for carefully reading the paper and giving helpful comments and suggestions.

Funding Open Access funding provided by Colombia Consortium

Data Availibility Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.

References

1. Baril, J.-L., Kirgizov, S., Vajnovszki, V.: Descent distribution on Catalan words avoiding a pattern of length at most three. Discr. Math. **341**, 2608–2615 (2018)
2. Baril, J.L., Khalil, C., Vajnovszki, V.: Catalan words avoiding pairs of length three patterns. Discr. Math. Theor. Comput. Sci. **22**(2), 5 (2021)
3. Baril, J.-L., Ramírez, J.L.: Descent distribution on Catalan words avoiding ordered pairs of relations. Adv. Appl. Math. **149**, 102551 (2023)
4. Bernini, A., Bouvel, M., Ferrari, L.: Some statistics on permutations avoiding generalized patterns. Pure Math Appl. (PU.M.A.) **18**, 223–237 (2007)
5. Bousquet-Mélou, M., Claesson, A., Dukes, M., Kitaev, S.: (2+2)-free posets, ascent sequences and pattern avoiding permutations. J. Comb. Theory Ser. A **117**(7), 884–909 (2010)
6. Callan, D., Mansour, T., Ramírez, J.L.: Statistics on bargraphs of Catalan words. J. Autom. Lang. Comb. **26**, 177–196 (2021)
7. Elizalde, S., Flólrez, R., Ramírez, J.L.: Enumerating symmetric peaks in non-decreasing Dyck paths. Ars Math. Contemp. **21** (2021) #P2.04
8. Flajolet, P., Sedgewick, R.: Analytic Combinatorics. Cambridge University Press, Cambridge (2009)
9. Mansour, T., Shattuck, M.: Some enumerative results related to ascent sequences. Discr. Math. **315**, 29–41 (2014)
10. Mansour, T., Ramírez, J.L., Toquica, D.A.: Counting lattice points on bargraphs of Catalan words. Math. Comput. Sci. **15**, 701–713 (2021)
11. Mansour, T., Vajnovszki, V.: Efficient generation of restricted growth words. Inform. Process. Lett. **113**, 613–616 (2013)
12. Orlov, A.G.: On asymptotic behavior of the Taylor coefficients of algebraic functions. Sib. Math. J. **25**(5), 1002–1013 (1994)
13. Ramírez, J.L., Rojas-Osorio, A.: Consecutive patterns in Catalan words and the descent distribution. Bol. Soc. Mat. Mex. **29** (2023)
14. Ruskey, F., Savage, C.: Gray codes for set partitions and restricted growth tails. Australas. J. Combin. **10**, 85–96 (1994)
15. Sloane, N.J.A.: The On-Line Encyclopedia of Integer Sequences. Available at https://oeis.org/
16. Stanley, R.: Enumerative Combinatorics. Vol. 1 Second Ed. Cambridge, (2012)
17. Stanley, R.: Catalan Numbers. Cambridge University Press, Cambridge (2015)
18. Woan, W.-J., Shapiro, L., Rogers, D.G.: The Catalan numbers, the Lebesgue integral, and 4^{n-2}. Am. Math. Month. **104**, 926–931 (1997)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

ecajournal.haifa.ii.il

Enumerative Combinatorics and Applications

ECA **3:1** (2023) Article #S2R2
https://doi.org/10.54550/ECA2023V3S1R2

Enumeration of Partial Łukasiewicz Paths

Jean-Luc Baril† and Helmut Prodinger‡

†*LIB, Université de Bourgogne Franche-Comté, B.P. 47 870, 21078 Dijon Cedex France*
Email: barjl@u-bourgogne.fr

‡*Department of Mathematical Sciences, Stellenbosch University, 7602 Stellenbosch, South Africa*
Email: hproding@sun.ac.za

Received: May 3, 2022, **Accepted**: September 3, 2022, **Published**: September 9, 2022
The authors: Released under the CC BY-ND license (International 4.0)

Abstract: Łukasiewicz paths are lattice paths in $\mathbb{N}^2$ starting at the origin, ending on the x-axis, and consisting of steps in the set $\{(1,k), k \geq -1\}$. We give bivariate generating functions and exact values for the number of n-length prefixes (resp. suffixes) of these paths ending (resp. starting) at height $k \geq 0$ with a given type of step. We make a similar study for paths of bounded height, and we prove that the average height of n-length paths ending at a fixed height behaves as $\sqrt{\pi n}$ when $n \to \infty$. Finally, we study prefixes of alternate Łukasiewicz paths, i.e., Łukasiewicz paths that do not contain two consecutive steps in the same direction.

Keywords: Asymptotics; Enumeration; Partial Łukasiewicz paths
2020 Mathematics Subject Classification: 05A05; 05A15; 05A15

1. Introduction

A *Łukasiewicz path* of length $n \geq 0$ is a lattice path in $\mathbb{N}^2$ starting at the origin $(0,0)$, ending on the x-axis, consisting of n steps lying in $S = \{(1,k), k \geq -1\}$. We denote by ϵ the empty path, i.e., the path of length zero. These paths constitute a natural generalization of Dyck and Motzkin paths (see [3,4]), which are made using steps into the sets $\{(1,1),(1,-1)\}$ and $\{(1,1),(1,0),(1,-1)\}$, respectively. We refer to [1,8,14,17–19] for some combinatorial studies on Łukasiewicz paths. Let $\mathcal{L}_n$, $n \geq 0$, be the set of Łukasiewicz paths of length n, and $\mathcal{L} = \bigcup_{n\geq 0} \mathcal{L}_n$. For convenience, we set $D = (1,-1)$, $F = (1,0)$, $U_k = (1,k)$ for $k \geq 1$. See Figure 1 for an illustration of a Łukasiewicz path of length 18. Note that Łukasiewicz paths can be interpreted as an algebraic language of words $w \in \{x_0, x_1, x_2, \ldots\}^\star$ such that $\delta(w) = -1$ and $\delta(w') \geq 0$ for any proper prefix w' of w where δ is the map from $\{x_0, x_1, x_2, \ldots\}^\star$ to $\mathbb{Z}$ defined by $\delta(w_1 w_2 \ldots w_n) = \sum_{i=1}^{n} \delta(w_i)$ with $\delta(x_i) = i-1$ (see [13,15]).

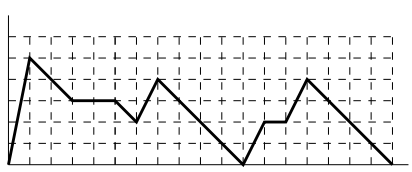


Figure 1: A Łukasiewicz path of length 18: $U_5DDFFDU_2DDDDU_2FU_2DDDD$.

Any non-empty Łukasiewicz path $L \in \mathcal{L}$ can be decomposed (see [6]) into one of the two following forms: (1) $L = FL'$ with $L' \in \mathcal{L}$, or (2) $L = U_k L_1 D L_2 D \ldots L_k D L'$ with $k \geq 1$ and $L_1, L_2, \ldots, L_k, L' \in \mathcal{L}$ (see Figure 2). Due to this decomposition, the generating function $L(z) = \sum_{n\geq 0} a_n z^n$ where a_n is the cardinality of $\mathcal{L}_n$, satisfies the functional equation $L(z) = 1 + zL(z) + \sum_{k\geq 1} z^{k+1} L(z)^{k+1}$, or equivalently, $L(z) = \frac{1}{1-zL(z)}$. Then, $L(z) = \frac{1-\sqrt{1-4z}}{2z}$. Therefore, a_n is the n-th Catalan number $a_n = \frac{1}{n+1}\binom{2n}{n}$ (see sequence A000108 in [16]).

In this paper, we provide enumerating results for several classes of partial Łukasiewicz paths (prefixes and suffixes of Łukasiewicz paths, partial alternate Łukasiewicz paths). More precisely, in Sections 2 and 3, we give bivariate generating functions and exact values for the number of n-length prefixes (resp. suffixes) of these paths ending at height $k \geq 0$ with a given type of step (down, up, or horizontal step). In Sections 4 and 5, we make

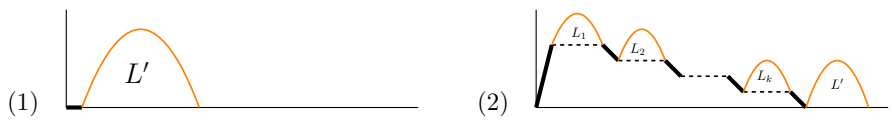


Figure 2: The two forms of a non-empty Łukasiewicz path.

a similar study for paths of bounded height. In Section 6, we prove that the average height of n-length paths ending at a fixed height behaves as $\sqrt{\pi n}$ when $n \to \infty$. In Section 7, we focus on partial alternate Łukasiewicz paths, i.e., Łukasiewicz paths that do contain two consecutive steps with the same direction.

All our *explicit* formulæ follow from the standard identity

$$[z^n]\left(\frac{1-\sqrt{1-4z}}{2z}\right)^k = \binom{2n-1+k}{n} - \binom{2n-1+k}{n}.$$

2. Enumeration of Partial Lukasiewicz paths

Partial Łukasiewicz paths of length n (i.e., n-length prefixes of Łukasiewicz paths) ending at height k can be constructed through the following state diagram (Figure 3). The diagram has three types of states ranging from 0 to infinity on three layers; in the drawing, only the first fifth states of each type are shown. The first type of states (top layer) refers to an up-step leading to a state, the second type (middle layer) refers to a horizontal step leading to a state, and the third type (bottom layer) refers to a down-step leading to a state. Any path from the origin to a state of rank k of a layer represents a partial Łukasiewicz path ending at height k.

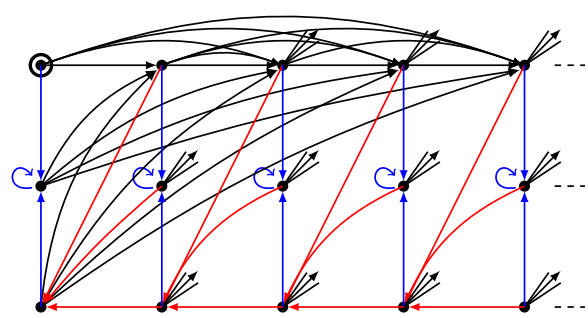


Figure 3: The state diagram for the generation of partial Łukasiewicz paths. Black (resp. red, blue) arrows correspond to up-steps (resp. down-steps, horizontal steps).

For $k \geq 0$, we consider the generating function $f_k = f_k(z)$ (resp. $g_k = g_k(z)$, $h_k = h_k(z)$), where the coefficient of z^n in the series expansion is the number of partial Łukasiewicz paths of length n ending at height k with an up-step U_k, $k \geq 1$, (resp. with a down-step D, resp. with a horizontal step F). Considering the state diagram in Figure 3, f_k (resp. g_k, h_k) is the generating function in the variable z marking the length of the paths ending on the $(k+1)$-th state of the top (resp. middle, bottom) layer. So, we easily obtain the following equations:

$$\begin{aligned} &f_0 = 1, \text{ and } f_k = z\sum_{\ell=0}^{k-1} f_\ell + z\sum_{\ell=0}^{k-1} g_\ell + z\sum_{\ell=0}^{k-1} h_\ell, \quad k \geq 1, \\ &g_k = zf_{k+1} + zg_{k+1} + zh_{k+1}, \quad k \geq 0, \\ &h_k = zf_k + zg_k + zh_k, \quad k \geq 0. \end{aligned} \tag{1}$$

Now, we introduce bivariate generating functions

$$F(u,z) = \sum_{k\geq 0} u^k f_k(z), \quad G(u,z) = \sum_{k\geq 0} u^k g_k(z), \text{ and } H(u,z) = \sum_{k\geq 0} u^k h_k(z).$$

For short, we also use the notation $F(u), G(u)$ and $H(u)$ for these functions. Summing the recursions in (1), we have:

$$F(u) = 1 + z\sum_{k\geq 1} u^k \left(\sum_{\ell=0}^{k-1} f_\ell + \sum_{\ell=0}^{k-1} g_\ell + \sum_{\ell=0}^{k-1} h_\ell\right)$$

$$
\begin{aligned}
&= 1 + z\sum_{k\ge 0}\frac{u^{k+1}}{1-u}f_k + z\sum_{k\ge 0}\frac{u^{k+1}}{1-u}g_k + z\sum_{k\ge 0}\frac{u^{k+1}}{1-u}h_k\\
&= 1 + \frac{uz}{1-u}(F(u)+G(u)+H(u)),\\
G(u) &= z\sum_{k\ge 0}u^k\Big(f_{k+1}+g_{k+1}+h_{k+1}\Big)\\
&= \frac{z}{u}(F(u)+G(u)+H(u)-F(0)-G(0)-H(0)),\\
H(u) &= \frac{z}{1-z}(F(u)+G(u)),
\end{aligned}
$$

where $F(0)+G(0)+H(0)$ is the number of Łukasiewicz paths of length n, i.e.,

$$F(0)+G(0)+H(0) = L(z) = \frac{1-\sqrt{1-4z}}{2z}.$$

Solving these functional equations, we deduce

$$F(u) = 1 - z - \frac{z\left(1+\sqrt{1-4z}\right)}{2u-1-\sqrt{1-4z}}, \quad G(u) = \frac{\sqrt{1-4z}+2z-1}{2u-1-\sqrt{1-4z}}, \text{ and}$$

$$H(u) = z + \frac{z\left(\sqrt{1-4z}-1\right)}{2u-1-\sqrt{1-4z}},$$

which implies that

$$f_k = [u^k]F(u) = \frac{2^k z}{(1+\sqrt{1-4z})^k} = z\left(\frac{1-\sqrt{1-4z}}{2z}\right)^k, \tag{2}$$

$$
\begin{aligned}
g_k = [u^k]G(u) &= \frac{2^k(1-2z-\sqrt{1-4z})}{(1+\sqrt{1-4z})^{k+1}}\\
&= \frac{(1-2z-\sqrt{1-4z})}{2}\left(\frac{1-\sqrt{1-4z}}{2z}\right)^{k+1}\\
&= z\left(\frac{1-\sqrt{1-4z}}{2z}\right)^{k+2} - z\left(\frac{1-\sqrt{1-4z}}{2z}\right)^{k+1},
\end{aligned} \tag{3}
$$

and

$$
\begin{aligned}
h_k = [u^k]H(u) &= \frac{2^k z(1-\sqrt{1-4z})}{(1+\sqrt{1-4z})^{k+1}} = \frac{z(1-\sqrt{1-4z})}{2}\left(\frac{1-\sqrt{1-4z}}{2z}\right)^{k+1}\\
&= z^2\left(\frac{1-\sqrt{1-4z}}{2z}\right)^{k+2}.
\end{aligned} \tag{4}
$$

Theorem 1. *The bivariate generating function for the total number of partial Łukasiewicz paths of length n with respect to the height of the end-point is given by*

$$\mathit{Total}(z,u) = 1 + \frac{-1+\sqrt{1-4z}}{2u-1-\sqrt{1-4z}},$$

and we have

$$[u^k]\,\mathit{Total}(z,u) = [k=0] + zL(z)^{k+2}.$$

Finally, we have for $n \ge 1$,

$$[z^n][u^k]\,\mathit{Total}(z,u) = \frac{k+2}{n+k+1}\binom{2n+k-1}{n-1},$$

$$[z^n][u^k]F(u) = \frac{k}{n+k-1}\binom{2n+k-3}{n-1},$$

$$[z^n][u^k]G(u) = \frac{k+3}{n+k+1}\binom{2n+k-2}{n-2},$$

$$[z^n][u^k]H(u) = \frac{k+2}{n+k}\binom{2n+k-3}{n-2}.$$

Here are examples of the series expansions of $[u^k]\,Total(z,u)$ for $k=0,1,2,3$ (leading terms):

- $1+z+2z^2+5z^3+14z^4+42z^5+132z^6+429z^7+1430z^8+4862z^9$,
- $z+3z^2+9z^3+28z^4+90z^5+297z^6+1001z^7+3432z^8+11934z^9$,
- $z+4z^2+14z^3+48z^4+165z^5+572z^6+2002z^7+7072z^8+25194z^9$,
- $z+5z^2+20z^3+75z^4+275z^5+1001z^6+3640z^7+13260z^8+48450z^9$,

which correspond respectively to A000108, A000245, A002057, and A000344 in [16].

According to Theorem 3.1 and Theorem 3.3 in [9], $[z^n][u^k]\,Total(z,u)$ counts also standard Young tableaux of shape $(n+2,n-k+1)$ (see [11,20] for the definition of a standard Young tableau), and Dyck paths of semilength $n+k$ starting with at least k up-steps and touching the x-axis somewhere between the two end-points.

3. Partial Łukasiewicz paths from right to left

In this section, we count partial Łukasiewicz paths that read from right to left, i.e., paths in $\mathbb{N}^2$ starting at the origin, consisting of steps $(1,k)$, $k\le 1$, and ending at a given height with a given type of step. Of course, this study is completely equivalent to counting suffixes of Łukasiewicz paths starting at a given height with a given type of step. We denote here by f_k, g_k, and h_k the generating functions for the number of these paths (with respect to the length) ending at height k with an up-step, down-step, or a horizontal step, respectively.

Then we have

$$\begin{aligned} &f_0=1, \quad \text{and} \quad f_k=zf_{k-1}+zg_{k-1}+zh_{k-1}, \quad k\ge 1,\\ &g_k=z\sum_{\ell\ge k+1}f_\ell+z\sum_{\ell\ge k+1}g_\ell+z\sum_{\ell\ge k+1}h_\ell, \quad k\ge 0,\\ &h_k=zf_k+zg_k+zh_k, \quad k\ge 0. \end{aligned} \tag{5}$$

Considering the bivariate generating functions

$$F(u)=\sum_{k\ge 0}u^kf_k(z), \quad G(u)=\sum_{k\ge 0}u^kg_k(z), \text{ and } H(u)=\sum_{k\ge 0}u^kh_k(z),$$

and summing the recursions in (5), we obtain:

$$\begin{aligned} F(u)&=1+z\sum_{k\ge 1}u^k\left(f_{k-1}+g_{k-1}+h_{k-1}\right)\\ &=1+zuF(u)+zuG(u)+zuH(u),\\ G(u)&=z\sum_{k\ge 0}u^k\Big(\sum_{\ell\ge k+1}f_\ell+\sum_{\ell\ge k+1}g_\ell+\sum_{\ell\ge k+1}h_\ell\Big)\\ &=z\sum_{k\ge 1}\frac{1-u^k}{1-u}f_k+z\sum_{k\ge 1}\frac{1-u^k}{1-u}g_k+z\sum_{k\ge 1}\frac{1-u^k}{1-u}h_k\\ &=\frac{z}{1-u}(F(1)+G(1)+H(1)-F(u)-G(u)-H(u)),\\ H(u)&=\frac{z}{1-z}(F(u)+G(u)), \end{aligned}$$

with

$$F(0)+G(0)+H(0)=L(z)=\frac{1-\sqrt{1-4z}}{2z}.$$

Moreover, we have

$$F(1)+G(1)+H(1)=\frac{L(z)-1}{z},$$

since there is a bijection between all partial Łukasiewicz paths of length n that read from right to left and Łukasiewicz paths that read from left to right of length $n+1$ (from a Łukasiewicz path, we remove the first step, and we read it from right to left).

Solving these functional equations, we deduce

$$F(u)=-\frac{1+\sqrt{1-4z}}{2zu-\sqrt{1-4z}-1}, \qquad G(u)=\frac{-1+\sqrt{1-4z}+2z}{2zu-\sqrt{1-4z}-1},$$

$$H(u)=-\frac{2z}{2zu-\sqrt{1-4z}-1},$$

which implies that

$$f_k = [u^k]F(u) = \frac{2^k z^k}{(1+\sqrt{1-4z})^k}, \tag{6}$$

$$g_k = [u^k]G(u) = \frac{2^k z^k (1-2z-\sqrt{1-4z})}{(1+\sqrt{1-4z})^{k+1}}, \text{ and} \tag{7}$$

$$h_k = [u^k]H(u) = \frac{2^{k+1} z^{k+1}}{(1+\sqrt{1-4z})^{k+1}}. \tag{8}$$

Theorem 2. *The bivariate generating function for the total number of partial Łukasiewicz paths of length n (from right to left) with respect to the height of the end-point is given by*

$$Total(z,u) = 1 + \frac{2}{1-2zu+\sqrt{1-4z}},$$

and we have

$$[u^k]\,Total(z,u) = z^k L(z)^{k+1}.$$

Finally, for $n \geq 1$, we obtain:

$$[z^n][u^k]Total(z,u) = \frac{k+1}{n+1}\binom{2n-k}{n},$$

$$[z^n][u^k]F(u) = \frac{k}{n}\binom{2n-k-1}{n-1},$$

$$[z^n][u^k]G(u) = \frac{k+3}{n+1}\binom{2n-k-2}{n},$$

$$[z^n][u^k]H(u) = \frac{k+1}{n}\binom{2n-k-2}{n-1}.$$

Here are examples of the series expansions of $[u^k]\,Total(z,u)$ for $k = 0,1,2,3$ (leading terms):

- $1 + z + 2z^2 + 5z^3 + 14z^4 + 42z^5 + 132z^6 + 429z^7 + 1430z^8 + 4862z^9$,
- $z + 2z^2 + 5z^3 + 14z^4 + 42z^5 + 132z^6 + 429z^7 + 1430z^8 + 4862z^9$,
- $z^2 + 3z^3 + 9z^4 + 28z^5 + 90z^6 + 297z^7 + 1001z^8 + 3432z^9$,
- $z^3 + 4z^4 + 14z^5 + 48z^6 + 165z^7 + 572z^8 + 2002z^9$,

which correspond to shifts of A000108, A000245, A002057, and A000344 in [16].

4. Partial Łukasiewicz paths constrained by height

In this section, we count partial Łukasiewicz paths bounded by a given height $t \geq 0$. We introduce the notation f_k^t, g_k^t, h_k^t for $0 \leq k \leq t$, $F^t(u)$, $G^t(u)$ and $H^t(u)$, which are the counterparts of f_k, g_k, h_k, $F(u)$, $G(u)$ and $H(u)$. Considering the state diagram of Figure 3 where each layer consists of only $t+1$ states, we deduce the following system of equations:

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & -1 & 0 & z & z & z & 0 & 0 & 0 & \cdots \\ z & z & z-1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ z & z & z & -1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & -1 & 0 & z & z & z & \cdots \\ 0 & 0 & 0 & z & z & z-1 & 0 & 0 & 0 & \cdots \\ z & z & z & z & z & z & -1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & z & z & z-1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix} \cdot \begin{bmatrix} f_0^t \\ g_0^t \\ h_0^t \\ f_1^t \\ g_1^t \\ h_1^t \\ f_2^t \\ g_2^t \\ h_2^t \\ \vdots \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix}.$$

For a given height $t \geq 0$, the previous matrix (denoted A_t) is square with $3(t+1)$ rows. Using classical properties of the determinant, we can prove that $D_t = \det(A_t)$ satisfies

$$D_{t+2} + D_{t+1} + zD_t = 0,$$

anchored with $D_0 = z - 1$, and $D_1 = 1 - 2z$. Then we deduce,

$$D_t = \frac{z(-1)^{t+2}\left(1-\sqrt{1-4z}\right)^{t+2}}{2^{t+1}\sqrt{1-4z}\left(1+\sqrt{1-4z}\right)} + \frac{z(-1)^{t+1}\left(1+\sqrt{1-4z}\right)^{t+2}}{2^{t+1}\sqrt{1-4z}\left(1-\sqrt{1-4z}\right)},$$

which corresponds to

$$D_t = (-1)^{t+1} \cdot F_t,$$

where F_t is the generalized Fibonacci polynomial (see [7, 12]):

$$F_t = 1 - \binom{t+1}{1} z + \binom{t}{2} z^2 - \binom{t-1}{3} z^3 + \dots.$$

For instance, we have $D_3 = F_3 = 1 - 4z + 3z^2$, and $D_4 = -F_4 = -1 + 5z - 6z^2 + z^3$.

Using Cramer's rule to solve the system, for $0 \le k \le t$, we have

$$f_k^t = \frac{N_{3k+1}^t}{D_t}, \quad g_k^t = \frac{N_{3k+2}^t}{D_t}, \quad h_k^t = \frac{N_{3k+3}^t}{D_t}, \tag{9}$$

where N_k^t is the determinant of the matrix $A_t(k)$ obtained from A_t by replacing the $(k+1)$-th column with the vector $(-1, 0, 0, 0, \dots, 0, 0)^T$.

As we have done for D_t, it is easy to prove that N_k^t satisfies the two recurrence relations

$$N_k^{t+2} + N_k^{t+1} + zN_k^t = 0, \quad 1 \le k,\ 1 + \left\lceil \frac{k}{3} \right\rceil \le t, \text{ and}$$

$$N_{k+3}^{t+1} = -N_k^t, \quad 4 \le k,\ 1 \le t.$$

Calculating N_k^t for $(t,k) \in \{0,1,2\} \times \{1,2,3\}$, and for $(t,k) \in \{1,2,3\} \times \{4,5,6\}$, we can easily obtain a closed form for N_{3k+i}^t, $1 \le i \le 3$. See Table 1 for exact values of N_k^t when $0 \le t \le 4$ and $1 \le k \le 12$.

k/t	0	1	2	3	4
1	$z-1$	$1-2z$	$-(z^2-3z+1)$	$3z^2-4z+1$	z^3-6z^2+5z-1
2	0	z^2	$-z^2$	$z^2(1-z)$	$-z^2(1-2z)$
3	$-z$	$z(1-z)$	$-z(1-2z)$	$z(z^2-3z+1)$	$-z(3z^2-4z+1)$
4		$z(1-z)$	$-z(1-2z)$	$z(z^2-3z+1)$	$-z(3z^2-4z+1)$
5		0	$-z^2$	z^2	$-z^2(1-z)$
6		z^2	$-z^2$	$z^2(1-z)$	$-z^2(1-2z)$
7			$-z(1-z)$	$z(1-2z)$	$-z(z^2-3z+1)$
8			0	z^2	$-z^2$
9			$-z^2$	z^2	$-z^2(1-z)$
10				$z(1-z)$	$-z(1-2z)$
11				0	$-z^2$
12				z^2	$-z^2$
13					...

Table 1: The first values of N_k^t for $0 \le t \le 4$ and $1 \le k \le 12$.

In particular, for $t \ge 0$, we have $N_1^t = D_t$ (see above for a closed form),

$$N_2^t = \frac{z^2\left(-\frac{2z}{1+\sqrt{1-4z}}\right)^t}{\sqrt{1-4z}} - \frac{z^2\left(-\frac{2z}{-\sqrt{1-4z}+1}\right)^t}{\sqrt{1-4z}},$$

$$N_3^t = -\frac{z^2\left(-1+\sqrt{1-4z}\right)\left(-\frac{2z}{1+\sqrt{1-4z}}\right)^t}{\sqrt{1-4z}\left(1+\sqrt{1-4z}\right)} - \frac{z^2\left(1+\sqrt{1-4z}\right)\left(-\frac{2z}{-\sqrt{1-4z}+1}\right)^t}{\sqrt{1-4z}\left(-\sqrt{1-4z}+1\right)},$$

and for $t \ge 1$,

$$N_4^t = N_3^t,\ N_5^t = -N_2^{t-1},\ \text{and } N_6^t = N_2^t.$$

Using (9), we can deduce closed forms for f_k^t, g_k^t, h_k^t, $0 \le k \le 1$, and $k \le t$. Using the above recurrence relations for N_k^t, we deduce closed forms for f_k^t, g_k^t, h_k^t, $2 \le k \le t$.

Theorem 3. *For $2 \le k \le t$, we have*

$$f_k = [u^k]F(u) = \frac{N_3^{t-k+1}}{D_t}(-1)^{k-1}, \tag{10}$$

$$g_k = [u^k]G(u) = \frac{N_2^{t-k}}{D_t}(-1)^{k}, \text{ and} \tag{11}$$

$$h_k = [u^k]H(u) = \frac{N_2^{t-k+1}}{D_t}(-1)^{k-1}. \tag{12}$$

For $t = 2, 3, 4$, the first terms of the series expansion of f_2 are

- $z + 2z^2 + 5z^3 + 13z^4 + 34z^5 + 89z^6 + 233z^7 + 610z^8 + 1597z^9$,
- $z + 2z^2 + 5z^3 + 14z^4 + 41z^5 + 122z^6 + 365z^7 + 1094z^8 + 3281z^9$,
- $z + 2z^2 + 5z^3 + 14z^4 + 42z^5 + 131z^6 + 417z^7 + 1341z^8 + 4334z^9$, which correspond to the sequences A001519, A007051, A080937 in [16], that also count Dyck paths of semilength n of height at most $t + 1$.

Theorem 4. *The generating function $[u^k]\,Total_t(z, u)$ for the number of partial Łukasiewicz paths of height at most $t \ge 0$, ending at height $k \ge 1$, is given by*

$$(-1)^{k-1}\frac{N_3^{t-k+1} - N_2^{t-k} + N_2^{t-k+1}}{D_t}.$$

Moreover, we have

$$[u^0]\,Total_t(z, u) = \frac{D_t + N_2^t + N_3^t}{D_t}.$$

The generating function for the total number of partial Łukasiewicz paths of height at most $t \ge 0$ is given by

$$Total_t(z, 1) = (-1)^{t+1} \cdot D_t^{-1} = F_t^{-1}.$$

For $t = 0, 1, 2, 3, 4$, the first terms of the series expansion of $Total_t(z, 1)$ are

- $1 + z + z^2 + z^3 + z^4 + z^5 + z^6 + z^7 + z^8 + z^9$,
- $1 + 2z + 4z^2 + 8z^3 + 16z^4 + 32z^5 + 64z^6 + 128z^7 + 256z^8 + 512z^9$,
- $1 + 3z + 8z^2 + 21z^3 + 55z^4 + 144z^5 + 377z^6 + 987z^7 + 2584z^8 + 6765z^9$,
- $1 + 4z + 13z^2 + 40z^3 + 121z^4 + 364z^5 + 1093z^6 + 3280z^7 + 9841z^8 + 29524z^9$,
- $1 + 5z + 19z^2 + 66z^3 + 221z^4 + 728z^5 + 2380z^6 + 7753z^7 + 25213z^8 + 81927z^9$, which correspond to A000012, A000079, A001906, A003462, A005021 in [16].

Using [12], $[z^n]\,Total_t(z, 1)$ counts also paths of length $2n + 1 + t$ in $\mathbb{N}^2$ starting at the origin, ending at $(n + t + 1, n)$, consisting of steps $(0, 1)$, and $(1, 0)$, and such that all its points (x, y) satisfy $x - t - 1 \le y \le x$. It would be interesting to exhibit a constructive bijection between these paths and partial Łukasiewicz paths of height at most $t \ge 0$.

5. Partial Łukasiewicz paths constrained by height from right-to-left

In this section, we count partial Łukasiewicz paths from right-to-left bounded by a given height $t \ge 0$. We denote here by f_k^t, g_k^t, h_k^t for $0 \le k \le t$, $F^t(u)$, $G^t(u)$ and $H^t(u)$, the generating functions in the same way as for Section 4. We deduce the following system of equations:

$$\begin{bmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & -1 & 0 & z & z & z & z & z & z & \cdots \\
z & z & z-1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
z & z & z & -1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & -1 & 0 & z & z & z & \cdots \\
0 & 0 & 0 & z & z & z-1 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & z & z & z & -1 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & z & z & z-1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{bmatrix}
\cdot
\begin{bmatrix} f_0^t \\ g_0^t \\ h_0^t \\ f_1^t \\ g_1^t \\ h_1^t \\ f_2^t \\ g_2^t \\ h_2^t \\ \vdots \end{bmatrix}
=
\begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix}.$$

For a given height $t \geq 0$, the previous matrix (denoted A'_t) is square with $3(t+1)$ rows. Using classical properties of the determinant, we can prove that

$$\det(A'_t) = \det(A_t) = D_t, \quad \text{for } t \geq 0.$$

Using Cramer's rule to solve the system, for $0 \leq k \leq t$, we have

$$f_k^t = \frac{N_{3k+1}^t}{D_t}, \quad g_k^t = \frac{N_{3k+2}^t}{D_t}, \quad h_k^t = \frac{N_{3k+3}^t}{D_t}, \tag{13}$$

where N_k^t is the determinant of the matrix $A'_t(k)$ obtained from A'_t by replacing the k-th column with the vector $(-1, 0, 0, 0, \ldots, 0, 0)^T$.

As we have done for D_t, it is easy to prove that N_k^t satisfies the two recurrence relations

$$N_k^{t+2} + N_k^{t+1} + zN_k^t = 0, \quad 1 \leq k \leq 3,\ 2 \leq t, \text{ and}$$

$$N_k^{t+1} = -zN_{k-3}^t, \quad 4 \leq k,\ 0 \leq t,$$

where N_k^t is the same as in Section 4 whenever $(t, k) \in \mathbb{N} \times \{1, 2, 3\}$.

Theorem 5. *For $0 \leq k \leq t$, we have*

$$f_k = [u^k]F(u) = \frac{N_1^{t-k}}{D_t}(-1)^k z^k, \tag{14}$$

$$g_k = [u^k]G(u) = \frac{N_2^{t-k}}{D_t}(-1)^k z^k, \text{ and} \tag{15}$$

$$h_k = [u^k]H(u) = \frac{N_3^{t-k}}{D_t}(-1)^k z^k. \tag{16}$$

Theorem 6. *The generating function $[u^k]\,Total_t(z, u)$ for the number of partial Łukasiewicz paths (from right to left) of height at most $t \geq 0$, ending at height $k \geq 0$, is given by*

$$(-1)^k z^k \frac{N_1^{t-k} + N_2^{t-k} + N_3^{t-k}}{D_t}.$$

The generating function for the total number of partial Łukasiewicz paths (from right to left) of height at most $t \geq 2$ is given by

$$Total_t(z, 1) = \frac{D_{t-2}}{D_t}.$$

Moreover, we have

$$Total_0(z, 1) = \frac{1}{1-z}, \text{ and } Total_1(z, 1) = \frac{1}{1-2z}.$$

For $t = 0, 1, 2, 3$, the first terms of the series expansion of $Total_t(z, 1)$ are

- $1 + z + z^2 + z^3 + z^4 + z^5 + z^6 + z^7 + z^8 + z^9$,
- $1 + 2z + 4z^2 + 8z^3 + 16z^4 + 32z^5 + 64z^6 + 128z^7 + 256z^8 + 512z^9$,
- $1 + 2z + 5z^2 + 13z^3 + 34z^4 + 89z^5 + 233z^6 + 610z^7 + 1597z^8 + 4181z^9$,
- $1 + 2z + 5z^2 + 14z^3 + 41z^4 + 122z^5 + 365z^6 + 1094z^7 + 3281z^8 + 9842z^9$,

  which correspond to A000012, A000079, A001519, A007051 in [16].

Note that the two series in Theorem 4 and Theorem 6 coincide when $t = 0, 1$ since, in these cases, partial Łukasiewicz paths bounded by the height t are identical from left-to-right and from right-to-left.

6. The average height of Łukasiewicz paths

In this section, we prove that the average height of n-length partial Łukasiewicz paths (from left to right, and from right to left) ending at a fixed height behaves as $\sqrt{\pi n}$ when $n \to \infty$

6.1 The left-to-right model

We simplify the expressions given in Section 4 using the substitution $z = \frac{u}{(1+u)^2}$, first used in [2]. Then, we find

$$D_t = (-1)^{t+3}\frac{1-u^{t+3}}{(1-u)(1+u)^{t+2}},$$
$$N_2^t = \frac{(-1)^{t+1}u^2(1-u^t)}{(1+u)^{t+3}(1-u)},$$
$$N_3^t = \frac{(-1)^{t+1}u(1-u^{t+2})}{(1-u)(1+u)^{t+3}}.$$

We start with paths ending on the x-axis, as the formula is (slightly) different. The generating function of these paths bounded by t is

$$\mathscr{L}^{[\le t]} = \frac{D_t + N_2^t + N_3^t}{D_t} = \frac{(1+u) - u^{t+2}(1+u^2)}{1-u^{t+3}}.$$

Taking the limit when $t \to \infty$, we retrieve, as expected,

$$\mathscr{L}^{[\le \infty]} = 1 + u = \frac{1-\sqrt{1-4z}}{2z}.$$

Then, the generating function of paths of height at least $t+1$ is

$$\mathscr{L}^{[>t]} = \mathscr{L}^{[\le\infty]} - \mathscr{L}^{[\le t]} = \frac{u^{t+2}(1-u^2)}{1-u^{t+3}}.$$

We refer to [10] where a similar instance is worked out with an extensive amount of detail. For the average height, we have (before normalizing by the Catalan numbers) to compute

$$\sum_{t\ge 0} t\cdot \mathscr{L}^{[=t]} = \sum_{t\ge 0}\mathscr{L}^{[>t]} = \frac{1-u^2}{u}\sum_{t\ge 3}\frac{u^t}{1-u^t}.$$

The goal is to find the local behavior of $u \sim 1$ since it translates to the local behavior of $z \sim \frac{1}{4}$. To find this, we set $u = e^{-t}$ and we use the Mellin transform. We do not need to do the actual computation, since we just cite the result from [10]. First, the factor is simple since we have

$$\frac{1-u^2}{u} \sim 2(1-u).$$

Since we only compute the leading term of the asymptotic expansion, we use

$$\sum_{t\ge 1}\frac{u^t}{1-u^t} \sim -\frac{\log(1-u)}{1-u},$$

found in [10] for instance. So, we obtain

$$\sum_{t\ge 0} t\cdot\mathscr{L}^{[=t]} \sim -2\log(1-u) \sim -2\log(2\sqrt{1-4z}) \sim -\log(1-4z).$$

Singularity analysis of generating functions (see [5]) allows us to translate this to the coefficients of z^n, with the result $\sim \frac{4^n}{n}$. For Catalan numbers, we have the well-known

$$\frac{1}{n+1}\binom{2n}{n} \sim \frac{4^n}{n^{3/2}\sqrt{\pi}}.$$

Finally, the average height of paths ending on the x-axis behaves as

$$\frac{\frac{4^n}{n}}{\frac{4^n}{n^{3/2}\sqrt{\pi}}} = \sqrt{\pi n}.$$

For path ending at height $k \ge 1$, the generating function of paths bounded by t is

$$\mathscr{L}_k^{[\le t]} = (-1)^{k-1}\frac{N_3^{t-k+1} - N_2^{t-k} + N_2^{t-k+1}}{D_t} = u(1+u)^k\frac{1-u^{t-k+1}}{1-u^{t+3}}.$$

The limit for $t \to \infty$ is $u(1+u)^k$, and $\mathscr{L}_k^{[>t]} = \mathscr{L}_k^{[\le\infty]} - \mathscr{L}_k^{[\le t]}$ is

$$u(1+u)^k - u(1+u)^k \frac{1-u^{t-k+1}}{1-u^{t+3}} = u^{-k-1}(1+u)^k(1-u^{k+2})\frac{u^{t+3}}{1-u^{t+3}}.$$

Locally, we have ($u \sim 1$)

$$u^{-k-1}(1+u)^k(1-u^{k+2}) \sim 2^k(k+2)(1-u).$$

For the average height (the leading term only, before normalization), we compute

$$2^k(k+2)(1-u)\sum_{t\ge1}\frac{u^t}{1-u^t} \sim -2^k(k+2)\log(1-u),$$

and

$$[z^n]\left(-2^k(k+2)\log(1-u)\right) \sim 2^{k-1}(k+2)\frac{4^n}{n}.$$

For the total number of paths ending at height k, we have

$$[z^n]u(1+u)^k = \binom{2n-1+k}{n-1} - \binom{2n-1+k}{n-2} \sim \frac{4^n}{\sqrt{\pi}n^{3/2}}(k+2)2^{k-1},$$

and the average height (k fixed, $n \to \infty$) is asymptotic to

$$\frac{2^{k-1}(k+2)\frac{4^n}{n}}{\frac{4^n}{\sqrt{\pi}n^{3/2}}(k+2)2^{k-1}} = \sqrt{\pi n},$$

as before.

To compute the average height of paths with unspecified endpoints makes no sense in this model since the number of such paths of length n is infinite.

6.2 The right-to left model

We simplify the expressions given in Section 5 using the substitution $z = \frac{u}{(1+u)^2}$. We have to analyze

$$\mathscr{R}_k^{[\le t]} = (-1)^k z^k \frac{N_1^{t-k} + N_2^{t-k} + N_3^{t-k}}{D_t} = \frac{u^k}{(1+u)^{k-1}}\frac{1-u^{t+2-k}}{1-u^{t+3}}.$$

Taking the limit when $t \to \infty$, we obtain

$$\mathscr{R}_k^{[\le\infty]} = \frac{u^k}{(1+u)^{k-1}},$$

and

$$\mathscr{R}_k^{[>t]} = \mathscr{R}_k^{[\le\infty]} - \mathscr{R}_k^{[\le t]} = \frac{u^{t+2}(1-u^{k+1})}{(1+u)^{k-1}(1-u^{t+3})}.$$

For the average, we must compute

$$\frac{(k+1)(1-u)}{2^{k-1}}\sum_{t\ge1}\frac{u^t}{1-u^t},$$

where we took liberties about two missing terms, which do not influence the main term of the average height. As before, we get the asymptotic behavior

$$\frac{(k+1)}{2^k}\frac{4^n}{n}.$$

For the total number of paths ending at height k, we get

$$\frac{k+1}{2^k}\frac{4^n}{\sqrt{\pi}n^{3/2}},$$

and the average height (k fixed, $n \to \infty$) is again asymptotic to $\sqrt{\pi n}$.

Now we move to the Łukasiewicz paths with unspecified end and have to consider

$$\mathscr{R}^{\le t} = \frac{D_{t-2}}{D_t} = \frac{(1+u)^3(1-u)}{u^2}\frac{u^{t+3}}{1-u^{t+3}}.$$

The fact that the formula is different for small values of t and that we start the sum at 1 and not at 3, does not change the main term. We get

$$\mathscr{R}^{>t} \sim \frac{(1+u)^3(1-u)}{u^2} \sum_{t\geq 1} \frac{u^t}{1-u^t} \sim -8\log(1-u),$$

and the coefficient of z^n in it is asymptotic to

$$\frac{4^{n+1}}{n}.$$

For the total number of paths, we get the asymptotic formula

$$\frac{4^{n+1}}{\sqrt{\pi} n^{3/2}},$$

and the average height is again asymptotic to $\sqrt{\pi n}$.

7. Partial alternate Łukasiewicz paths

In this section, we study prefixes of alternate Łukasiewicz paths, i.e., Łukasiewicz paths that do not contain two consecutive steps with the same direction (or equivalently, walks in the state diagram of Figure 3 without two consecutive arrows of the same color). We refer to Figure 4 for the state diagram associated with these paths. We denote here by f_k, g_k, and h_k the generating functions for the number of these paths (with respect to the length) ending at height k with an up-step, down-step, or a horizontal step, respectively.

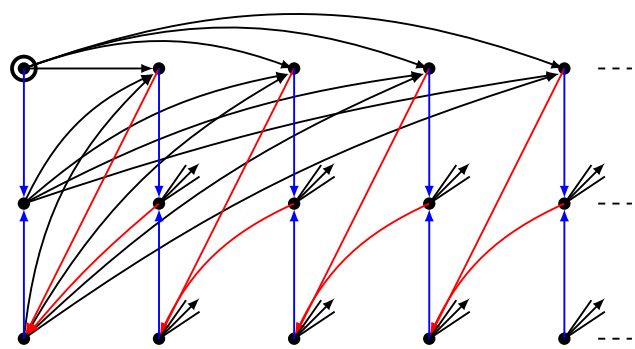


Figure 4: The state diagram for partial alternate Łukasiewicz paths. Black (resp. red, blue) arrows correspond to up-steps (resp. down-steps, horizontal steps).

We have the following equations:

$$\begin{aligned} &f_0 = 1, \quad \text{and} \quad f_k = zf_0 + z\sum_{\ell=0}^{k-1} g_\ell + z\sum_{\ell=0}^{k-1} h_\ell, \quad k \geq 1, \\ &g_k = zf_{k+1} + zh_{k+1}, \quad k \geq 0, \\ &h_k = zf_k + zg_k, \quad k \geq 0. \end{aligned} \tag{17}$$

Considering the bivariate generating functions

$$F(u,z) = \sum_{k\geq 0} u^k f_k(z), \quad G(u,z) = \sum_{k\geq 0} u^k g_k(z), \text{ and } H(u,z) = \sum_{k\geq 0} u^k h_k(z),$$

and summing the recursions in (17), we obtain

$$\begin{aligned} F(u) &= 1 + z\sum_{k\geq 1} u^k \left(1 + \sum_{\ell=0}^{k-1} g_\ell + \sum_{\ell=0}^{k-1} h_\ell\right) \\ &= 1 + \frac{zu}{1-u} + z\sum_{k\geq 0} \frac{u^{k+1}}{1-u} g_k + z\sum_{k\geq 0} \frac{u^{k+1}}{1-u} h_k \end{aligned}$$

$$= 1 + \frac{zu}{1-u}(1 + G(u) + H(u)),$$
$$G(u) = \frac{z}{u}(F(u) + H(u) - 1 - H(0)),$$
$$H(u) = z(F(u) + G(u)).$$

Solving these functional equations, we deduce

$$F(u) = \frac{uz^2(1+z)H(0) + 2uz^3 - u^2z + u^2 + z^2 - u}{u^2z^2 + 2uz^3 + u^2 + z^2 - u},$$
$$G(u) = -\frac{z\left(H(0)(uz^2 + u - 1) + 2uz^2 + z\right)}{u^2z^2 + 2uz^3 + u^2 + z^2 - u},$$
$$H(u) = \frac{z\left(zH(0)(uz - u + 1) - u^2z + u^2 - u\right)}{u^2z^2 + 2uz^3 + u^2 + z^2 - u}.$$

Now we apply the kernel method on $H(u)$. We have

$$H(u) = \frac{z\left(zH(0)(uz - u + 1) - u^2z + u^2 - u\right)}{(1+z^2)(u - s_1)(u - s_2)}, \tag{18}$$

with

$$s_1 = \frac{1 - 2z^3 + \sqrt{4z^6 - 4z^4 - 4z^3 - 4z^2 + 1}}{2z^2 + 2},$$
$$s_2 = \frac{1 - 2z^3 - \sqrt{4z^6 - 4z^4 - 4z^3 - 4z^2 + 1}}{2z^2 + 2}.$$

In order to compute $H(0)$, it suffices to plug $u = s_2$ in the numerator of (18). Then, $H(0)$ satisfies $zH(0)(s_2z - s_2 + 1) - s_2^2z + s_2^2 - s_2 = 0$, which implies that

$$H(0) = \frac{s_2}{z}.$$

After this, and using $s_1s_2(1+z^2) = z^2$, we simplify of both, numerators and denominators, in $F(u)$, $G(u)$, $H(u)$ by factorizing them with $(u - s_2)$.

$$F(u) = \frac{(1-z)u - (1+z^2)s_1}{(1+z^2)(u - s_1)} = -\frac{z-1}{z^2+1} - \frac{s_1z\,(z+1)}{(u - s_1)\,(z^2+1)},$$
$$G(u) = -\frac{z^2s_1^{-1} + 2z^3}{(1+z^2)(u - s_1)} = -\frac{z^2(2zs_1 + 1)}{(1+z^2)s_1(u - s_1)},$$
$$H(u) = \frac{z((1-z)u - 1)}{(1+z^2)(u - s_1)} = \frac{(1-z)\,z}{z^2+1} - \frac{(zs_1 - s_1 + 1)\,z}{(u - s_1)\,(z^2+1)}.$$

Finally, we easily obtain

$$f_k = [u^k]F(u) = \frac{z(z+1)}{(1+z^2)s_1^k}, \tag{19}$$
$$g_k = [u^k]G(u) = \frac{z^2(2zs_1 + 1)}{(1+z^2)s_1^{k+2}}, \quad \text{and} \tag{20}$$
$$h_k = [u^k]H(u) = \frac{z((z-1)s_1 + 1)}{(1+z^2)s_1^{k+1}}. \tag{21}$$

Since the series expansion of s_1 does not have pretty coefficients, we cannot expect this from our final answers.

Theorem 7. *The bivariate generating function for the total number of partial alternate Łukasiewicz paths with respect to the length and the height of the end-point is given by*

$$Total(z,u) = \frac{s_1^2z^2 + s_1uz^2 + 2s_1z^3 + s_1^2 - s_1u + zs_1 + z^2}{(z^2+1)\,(-u + s_1)s_1}.$$

Moreover, we have

$$[u^0]\,Total(z,u) = \frac{s_1^2z^2 + 2s_1z^3 + s_1^2 + s_1z + z^2}{(z^2+1)s_1^2},$$

and for $k \geq 1$,

$$[u^k]\,Total(z,u) = \frac{z(2s_1^2z + 2s_1z^2 + s_1 + z)}{(z^2+1)s_1^{k+2}}.$$

Here are examples of the series expansions of $[u^k]\,Total(z,u)$ for $k=0,1,2,3$ (leading terms):

- $1+z+z^2+3z^3+5z^4+9z^5+19z^6+39z^7+81z^8+173z^9$,
- $z+3z^2+5z^3+11z^4+25z^5+53z^6+115z^7+255z^8+565z^9$;
- $z+3z^2+7z^3+19z^4+45z^5+105z^6+247z^7+575z^8+1333z^9$,
- $z+3z^2+9z^3+27z^4+69z^5+177z^6+443z^7+1087z^8+2645z^9$,

which do not appear in [16]. The first terms of the series expansion of the generating function for the number of alternate Łukasiewicz paths are

$$1+z+z^2+3z^3+5z^4+9z^5+19z^6+39z^7+81z^8+173z^9.$$

A singularity analysis of the generating function $[u^0]\,Total(z,u)$ gives

$$[z^n][u^0]\,Total(z,u) \sim \frac{\sqrt{-6a^6+4a^4+3a^3+2a^2}(a+1)2^n\left(-a^2+a+1\right)^n}{\sqrt{\pi}a^2\left(a^2+1\right)n^{\frac{3}{2}}},$$

with

$$a=\frac{1}{3}-\frac{2\cos\left(\frac{\arctan\left(\frac{15\sqrt{111}}{487}\right)}{6}+\frac{\pi}{6}\right)}{3}+\frac{2\sin\left(\frac{\arctan\left(\frac{15\sqrt{111}}{487}\right)}{6}+\frac{\pi}{6}\right)\sqrt{3}}{3}.$$

The reason that this constant appears, results from the singularity analysis. Indeed, one needs the solution closest to the origin of $4z^6-4z^4-4z^3-4z^2+1=0$, which is $a=0.403031716762\ldots$. Maple provides the curious explicit version if one asks for a *simplification*.

References

[1] J.-L. Baril, S. Kirgizov, and A. Petrossian, *Enumeration of Łukasiewicz paths modulo some patterns* Discrete Math. 342(4) (2019), 997–1005.

[2] N. G. De Bruijn, D. E. Knuth, and S. O. Rice, *The average height of planted plane trees*, In R. C. Read, editor, Graph Theory and Computing, 15–22, Academic Press, 1972.

[3] E. Deutsch, *Dyck path enumeration*, Discrete Math. 204 (1999), 167–202.

[4] R. Donaghey and L. W. Shapiro, *Motzkin numbers*, J. Combin. Theory Ser. A 23 (1977), 291–301.

[5] P. Flajolet and A. M. Odlyzko, *Singularity analysis of generating functions*, SIAM J. Discrete Math. 3 (1990), 216–240.

[6] P. Flajolet and R. Sedgewick, *Analytic Combinatorics*, Cambridge University Press, 2009.

[7] R. Florez and J. C. Saunders, *Irreducibility of generalized Fibonacci polynomials*, Integers 22 (2022), #A69.

[8] I. M. Gessel and S. Ree, *Lattice paths and Faber polynomials*, Advances in Combinatorial Methods and Applications to Probability and Statistics, Birkhauser Verlag, Boston, 1997.

[9] H. H. Gudmundsson, *Dyck paths, standard Young Tableaux, and pattern avoiding permutations*, Pure Math. Appl. (PU.M.A) 21(2) (2010), 265–284.

[10] C. Heuberger, H. Prodinger, and S. Wagner, *The height of multiple edge plane trees*, Aequationes Math. 90 (2016), 625–645.

[11] D. E. Knuth, *The Art of Computer Programming, Vol. III: Sorting and Searching* (2nd ed.), Addison-Wesley, p. 48, "Such arrangements were introduced by Alfred Young in 1900," 1973.

[12] G. Kreweras, *Sur les éventails de segments*, Cahiers du Bureau universitaire de recherche opérationnelle, Série Recherche, 15 (1970), 3–41.

[13] M. Lothaire, *Combinatorics on Words*, Encyclopedia of Mathematics and Its Applications, Vol. 17, Addison-Wesley, Reading, Massachusetts, 1983, Chapter 11, p. 219.

[14] G. N. Raney, *Functional composition patterns and power series reversion*, Trans. Amer. Math. Soc. 94 (1960), 411–451.

[15] M. P. Schützenberger, *Le théorème de Lagrange selon G. N. Raney*, Séminaire IRIA, Rocquencourt, (1971), 199–205.

[16] N. J. A. Sloane, *The On-line Encyclopedia of Integer Sequences*, available electronically at `http://oeis.org`.

[17] R. P. Stanley, *Enumerative Combinatorics*, Vol. 2, Cambridge University Press, 1999.

[18] A. Varvak, *Lattice path encodings in a combinatorial proof of a differential identity*, Discrete Math. 308 (2008), 5834–5840.

[19] G. Viennot, *Une théorie combinatoire des polynômes orthogonaux généraux*, Notes of lectures given at University of Quebec in Montreal, 1983.

[20] A. Young, *On quantitative substitutional analysis*, Proceedings of the London Mathematical Society, Ser. 1, 33(1)(1990), 97–145.

Discrete Mathematics 346 (2023) 113372

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Knight's paths towards Catalan numbers

Jean-Luc Baril [a,*], José L. Ramírez [b]

[a] LIB, Université de Bourgogne Franche-Comté, B.P. 47 870, 21078 Dijon Cedex France
[b] Departamento de Matemáticas, Universidad Nacional de Colombia, Bogotá, Colombia

ARTICLE INFO

Article history:
Received 24 June 2022
Received in revised form 30 January 2023
Accepted 7 February 2023
Available online 20 February 2023

Keywords:
Catalan number
Zigzag knight's path
Generating function

ABSTRACT

We provide enumerating results for partial knight's paths of a given size. We prove algebraically that zigzag knight's paths of a given size ending on the x-axis are enumerated by the generalized Catalan numbers, and we give a constructive bijection with peakless Motzkin paths of a given length. After enumerating partial knight's paths of a given length, we prove that zigzag knight's paths of a given length ending on the x-axis are counted by the Catalan numbers. Finally, we give a constructive bijection with Dyck paths of a given length.

© 2023 Elsevier B.V. All rights reserved.

1. Introduction

For several decades lattice paths have been widely studied in the literature. Their links with many problems of various domains as computer science, biology and physics [24], give them an important place in combinatorics. For instance, they have very tight connections with RNA structures, pattern avoiding permutations, directed animals, and so on [7,12,15,16,24]. In combinatorics, a classical problem consists in enumerating some special kinds of paths with respect to the length and other statistics (see for instance [3,5,6,11,14,18,19,21,22]). Certainly, Dyck paths are the most known. These are lattice paths in $\mathbb{N}^2 = \{0, 1, 2, \ldots\}^2$ starting at the origin $(0, 0)$, ending on the x-axis, consisting of steps lying in $\{(1, 1), (1, -1)\}$. They are enumerated with respect to the number of steps by the famous Catalan numbers (see A000108 in Sloane's On-line Encyclopedia of Integer Sequences [23]).

In this work we deal with several classes of knight's paths (see [17]), i.e., lattice paths in $\mathbb{N}^2$ starting at $(0, 0)$, and consisting of right-moves of a knight in the game of chess (a right-move of a knight is a jump from left to right from one corner of any two-by-three rectangle to the opposite corner on a chessboard). Fig. 1 shows the four possible right-moves of a knight. It is worth noticing that knight's paths always start at the origin and go to the right at each step. Therefore, this work is not related to the well known mathematical problem about the walk of a knight on a chessboard, namely the knight's tour problem [13], which consists in finding a sequence of moves on a chessboard such that the knight visits every square exactly once.

More formally, we define knight's paths as follows.

Definition 1. A *knight's path* is a lattice path in $\mathbb{N}^2$ starting at the origin, ending on the x-axis, and consisting of steps lying in $S = \{(1, 2), (1, -2), (2, 1), (2, -1)\}$. A *partial knight's path* is a prefix of a knight's path.

* Corresponding author.
E-mail addresses: barjl@u-bourgogne.fr (J.-L. Baril), jlramirezr@unal.edu.co (J.L. Ramírez).

https://doi.org/10.1016/j.disc.2023.113372
0012-365X/© 2023 Elsevier B.V. All rights reserved.

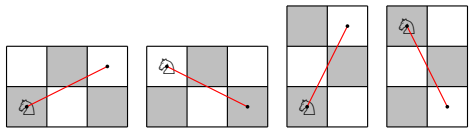


Fig. 1. The four possible right-moves of a knight on a chessboard.

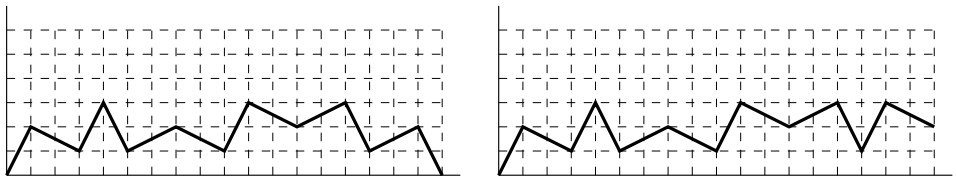


Fig. 2. The left and the right drawing show both a partial knight's path of size 18 and length 12. The left knight's path ends at height 0, whereas the right one ends at height 2.

The *length* of a path is the number of its steps, and the *size* of a path is the abscissa of its last point. We denote by ϵ the empty path, i.e., the path of length 0 (or equivalently of size 0). The *height* of a point (x, y) of a path is the ordinate y. We refer to Fig. 2 for two examples of partial knight's paths where the heights of the last points are 0 and 2, respectively. Defining $E = (2, 1)$, $\bar{E} = (2, -1)$, $N = (1, 2)$ and $\bar{N} = (1, -2)$, these two paths can be written as $N\bar{E}N\bar{N}E\bar{E}N\bar{E}E\bar{N}E\bar{N}$ and $N\bar{E}N\bar{N}E\bar{E}N\bar{E}E\bar{N}N\bar{E}$. Knight's paths of a given size (ending on the x-axis) have been enumerated in [17]. On the other hand, there are enumerative results for knight's paths of a given length which are in one-to-one correspondence with basketball walks in the first quarter plane (see for instance [1,4,8,9]). We will recall some of these results at the beginning of Sections 2 and 3.

In this paper, we will focus on a special kind of knight's paths that are enumerated by Catalan and generalized Catalan numbers (see A004148 in [23] for a definition of the generalized Catalan numbers). More precisely, we will consider zigzag knight's paths defined as follows.

Definition 2. A *zigzag knight's path* (or *zigzag path* for short) is a knight's path with the additional property that two consecutive steps cannot be in the same direction, i.e., two consecutive steps (a, b) and (c, d) lying in S must satisfy $b \cdot d < 0$.

In a zigzag path described as word in N, $\bar{N}$, E, and $\bar{E}$, barred and unbarred letters alternate. Notice that the two paths on Fig. 2 are zigzag paths.

Outline of the paper. In Section 2, we provide enumerating results for partial knight's paths (prefixes of knight's paths) of a given size. We make a similar study for partial zigzag paths of a given size. We prove that zigzag paths of a given size ending on the x-axis are enumerated by the generalized Catalan numbers, and the total number of partial zigzag paths of a given size are enumerated as the symmetric secondary structures of RNA molecules with a given number of nucleotides. Next, we give a constructive one-to-one correspondence between zigzag paths of size n and peakless Motzkin paths of length $n + 1$.

In Section 3, we provide enumerating results for partial zigzag knight's paths of a given length. We prove that zigzag paths of a given length ending on the x-axis are enumerated by the Catalan numbers, and the total number of partial zigzag paths of a given length are enumerated by the central binomial coefficient. Finally, we give a constructive one-to-one correspondence between zigzag paths of length n and Dyck paths of length $2(n + 1)$.

2. Knight's paths of a given size

In [17], Labelle and Yeh provide the generating function for the number of knight's paths of size n (ending on the x-axis):

$$A(z) = \frac{1 + 2z + \sqrt{1 - 4z + 4z^2 - 4z^4} - \sqrt{2}\sqrt{1 - 4z^2 - 2z^4 + (2z + 1)\sqrt{1 - 4z + 4z^2 - 4z^4}}}{4z^2}.$$

The first terms of the series expansion are

$$1 + z^2 + 3z^4 + 2z^5 + 12z^6 + 14z^7 + 54z^8 + 86z^9 + 274z^{10} + 528z^{11} + \dots,$$

and the sequence of coefficients corresponds to A005220 in [23]. As mentioned in [17] (see Theorem 2.13), the generating function $A(z)$ satisfies the functional equation

$$z^4 A(z)^4 - (2z^3 + z^2)A(z)^3 + (z^4 + 2z^2 + 2z)A(z)^2 - (2z + 1)A(z) + 1 = 0.$$

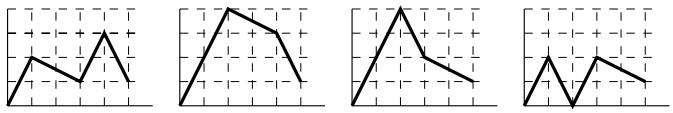


Fig. 3. Partial knight's paths of size 5 ending at height 1.

2.1. Partial knight's paths

In this part, we count partial knight's paths of a given size, i.e., prefixes of knight's paths ending at a given abscissa. For $k \geq 0$, we consider the generating function $f_k = f_k(z)$ (resp. $g_k = g_k(z)$), where the coefficient of z^n in the series expansion is the number of partial knight's paths of size n ending at height k with an up-step N or E, (resp. with a down-step $\bar{N}$ or $\bar{E}$). So, we easily obtain the following equations:

$$\begin{aligned} &f_0 = 1, \text{ and } f_1 = z^2(f_0 + g_0), \\ &f_k = z^2(f_{k-1} + g_{k-1}) + z(f_{k-2} + g_{k-2}), \quad k \geq 2, \\ &g_k = z^2(f_{k+1} + g_{k+1}) + z(f_{k+2} + g_{k+2}), \quad k \geq 0. \end{aligned} \tag{1}$$

Now, we introduce the bivariate generating functions

$$F(u,z) = \sum_{k\geq 0} u^k f_k(z) \quad \text{and} \quad G(u,z) = \sum_{k\geq 0} u^k g_k(z).$$

For short, we use the notation $F(u)$ and $G(u)$ for these functions, respectively. Summing the recursions in (1), we have:

$$\begin{aligned} F(u) &= 1 + z^2 \sum_{k\geq 1} u^k (f_{k-1} + g_{k-1}) + z \sum_{k\geq 2} u^k (f_{k-2} + g_{k-2}) \\ &= 1 + z^2 u (F(u) + G(u)) + z u^2 (F(u) + G(u)) \\ &= 1 + zu(z+u)(F(u) + G(u)), \\ G(u) &= z^2 \sum_{k\geq 0} u^k (f_{k+1} + g_{k+1}) + z \sum_{k\geq 0} u^k (f_{k+2} + g_{k+2}) \\ &= \frac{z^2}{u}(F(u) - 1 + G(u) - g_0) + \frac{z}{u^2}(F(u) - 1 - uz^2(1 + g_0) + G(u) - g_0 - ug_1), \end{aligned}$$

where $g_0 = G(0)$ is the generating function for the number of non-empty knight's paths ending on the x-axis with respect to the size, i.e., $g_0 = A(z) - 1$, and g_1 is the generating function for the number of partial knight's paths ending at height 1 with a down-step $\bar{N}$ or $\bar{E}$. Then, g_1 is the difference between the generating function $A_1(z)$ for the number of partial knight's paths ending at height 1, and the generating function for the number of partial knight's paths ending at height 1 with the step $E = (2,1)$, i.e. $g_1 = A_1(z) - z^2(g_0 + 1)$ where $A_1(z)$ is given in [17] (see Lemma 2.11) as follows:

$$A_1(z) = z^2 A(z)^2 (1 - zA(z))^{-1}.$$

Solving the above functional equations, we deduce

$$\begin{aligned} F(u) &= \frac{uz^2 A(z)(u^2 z + uz^2 + u + z) + u^2 z^2 A_1(z)(u+z) + uz^2 - u^2 + z}{u^4 z + u^3 z^2 + uz^2 - u^2 + z}, \\ G(u) &= -\frac{\big(A(z)(u^3 z^2 + u^2 z^3 + u^2 z + uz^2 - uz - 1) + A_1(z)(u^3 z + u^2 z^2 - u) + zu + 1\big) z}{u^4 z + u^3 z^2 + uz^2 - u^2 + z}, \\ F(u) + G(u) &= \frac{z(uz+1)A(z) + uzA_1(z) - u^2}{u^4 z + u^3 z^2 + uz^2 - u^2 + z}. \end{aligned}$$

The first terms of the series expansion of $F(u) + G(u)$ are

$$\begin{aligned} 1 + u^2 z + (1 + u + u^4) z^2 + (u + 2u^2 + 2u^3 + u^6) z^3 + (3 + 3u + u^2 + 2u^3 + 3u^4 + 3u^5 + u^8) z^4 \\ + (2 + 4u + 9u^2 + 8u^3 + 3u^4 + 3u^5 + 4u^6 + 4u^7 + u^{10}) z^5 + \dots \end{aligned}$$

Notice that the array $[z^n u^k](F(u) + G(u))$ corresponds to A096587. Fig. 3 shows the partial knight's paths of size 5 ending at height 1.

Naturally, we retrieve the generating function for the knight's paths ending on the x-axis $F(0) + G(0) = A(z)$. Moreover, the generating function for the total number of partial knight's paths with respect to the size is

$$F(1)+G(1)=\frac{z(z+1)A(z)+zA_1(z)-1}{2z^2+2z-1}.$$

The first terms of its series expansion are

$$1+z+3z^2+6z^3+16z^4+38z^5+99z^6+248z^7+646z^8+1659z^9+4342z^{10}+\dots,$$

and the sequence of coefficients corresponds to A096588 in [23].

Since it seems difficult to obtain closed forms for f_k and g_k, we only give the first terms of the series expansions of $[u^k](F(u)+G(u))$ for the two values of k ($k=1,2$), since the case $k=0$ is already mentioned above (see the series expansion of $A(x)$). For $k=1,2$ the series expansions are:

- $z^2+z^3+3z^4+4z^5+12z^6+22z^7+61z^8+128z^9+\dots$,
- $z+2z^3+z^4+9z^5+10z^6+42z^7+64z^8+213z^9+\dots$

The first expansion corresponds to the sequence A005221 in [23], whereas the sequence corresponding to the second expansion is not listed in [23]. Notice that the cases $k=0$ and $k=1$ are already known in [17].

2.2. Zigzag knight's paths

In this part, we count partial zigzag knight's paths of a given size, i.e., prefixes of zigzag knight's paths ending at a given abscissa. We keep the notation of the previous section in the context of zigzag paths. For $k\geq 0$, $f_k=f_k(z)$ (resp. $g_k=g_k(z)$) is the generating function where the coefficient of z^n in the series expansion is the number of partial zigzag knight's paths of size n ending at height k with an up-step N or E, (resp. with a down-step $\bar{N}$ or $\bar{E}$).

Then, we easily have the following equations:

$$\begin{aligned}&f_0=1,\quad f_1=z^2(f_0+g_0),\ \text{and}\ f_2=z(f_0+g_0+zg_1),\\&f_k=z^2g_{k-1}+zg_{k-2},\quad k\geq 3,\\&g_k=z^2f_{k+1}+zf_{k+2},\quad k\geq 0.\end{aligned}\tag{2}$$

Summing these recursions, we obtain:

$$F(u)=1+zu(z+u)(1+G(u)),$$

$$G(u)=\frac{z^2}{u}(F(u)-1)+\frac{z}{u^2}(F(u)-1-uz^2(1+g_0)),$$

where $g_0=G(0)$ is the generating function for the number of non-empty zigzag paths ending on the x-axis with respect to the size.

Solving these functional equations, we deduce

$$F(u)=\frac{g_0u^2z^4+g_0uz^5+u^2z^4+uz^5+u^2z^3+uz^4-u^3z-u^2z^2+uz^2+z^3-u}{u^2z^3+uz^4+uz^2+z^3-u}$$

and

$$G(u)=\frac{z^2\left(-u^2z-uz^2+g_0z-u\right)}{u^2z^3+uz^4+uz^2+z^3-u}.$$

In order to compute g_0, we use the kernel method on $G(u)$ (see [2]). We have

$$G(u)=\frac{z^2\left(-u^2z-uz^2+g_0z-u\right)}{z^3(u-s)(u-r)},$$

where

$$r=\frac{1-z^4-z^2-\sqrt{z^8-2z^6-z^4-2z^2+1}}{2z^3},\tag{3}$$

and

$$s=\frac{1-z^4-z^2+\sqrt{z^8-2z^6-z^4-2z^2+1}}{2z^3}.$$

It suffices to plug $u=r$ in the numerator of $G(u)$. Then, g_0 satisfies $-r^2z-rz^2+g_0z-r=0$, which implies that

$$\begin{aligned}g_0&=\frac{r^2z+rz^2+r}{z}=\frac{r^2z^3+rz^4+rz^2}{z^3}=\frac{r^2z^3+rz^4+rz^2+z^3}{z^3}-1\\&=\frac{r}{z^3}-1.\end{aligned}$$

After this, and using $sr=1$, we simplify both numerators and denominators in $F(u)$ and $G(u)$ by extracting the common factor $(u-r)$.

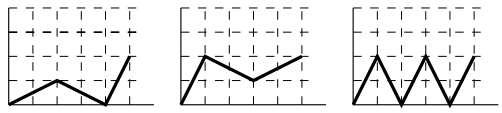


Fig. 4. Partial zigzag knight's paths of size 5 ending at height 2.

Theorem 1. *The bivariate generating functions $F(u)$, $G(u)$, and $F(u)+G(u)$ are given by*

$$F(u)=\frac{ru(u+z)}{z^2(1-ru)}+1,\quad G(u)=\frac{r}{z^3(1-ru)}-1,\ \text{and}\ F(u)+G(u)=\frac{ru^2z+ruz^2+r}{z^3\,(1-ru)},$$

where r is defined by Eq. (3). Moreover, we have

$$f_0=[u^0]F(u)=1, \tag{4}$$

$$f_k=[u^k]F(u)=\frac{rz+1}{z^2}r^{k-1}-[k=1]\frac{1}{z^2},\ k\geq 1, \tag{5}$$

$$g_0=[u^0]G(u)=\frac{r}{z^3}-1, \tag{6}$$

$$g_k=[u^k]G(u)=\frac{r^{k+1}}{z^3},\ k\geq 1, \tag{7}$$

where $[k=1]$ equals 1 whenever $k=1$ and 0 otherwise.

The first terms of the series expansion of $F(u)+G(u)$ are

$$1+u^2z+(1+u)z^2+(u+u^2)z^3+(2+u+u^3)z^4+(2u+3u^2)z^5 \\ +(4+2u+u^2+2u^3)z^6+(5u+6u^2+u^4)z^7\ldots$$

Fig. 4 shows the partial zigzag knight's paths of length 5 ending at height 2.
Here are other examples of the first terms of the series expansions of $[u^k](F(u)+G(u))$ for $k=1,2,3$:

- $z^2+z^3+z^4+2z^5+2z^6+5z^7+4z^8+12z^9+8z^{10}+28z^{11}+17z^{12}+\ldots$,
- $z+z^3+3z^5+z^6+6z^7+3z^8+13z^9+9z^{10}+29z^{11}+25z^{12}+65z^{13}+\ldots$,
- $z^4+2z^6+6z^8+z^9+15z^{10}+4z^{11}+37z^{12}+14z^{13}+91z^{14}+44z^{15}+\ldots$,

which do not appear in [23].

Corollary 1. *The generating function for the number of zigzag knight's paths of a given size ending on the x-axis is*

$$F(0)+G(0)=\frac{r}{z^3},$$

and the sequence of non-zero coefficients in the series expansion corresponds to the generalized Catalan numbers (see A004148 in [23]). The first terms of the series expansion are

$$1+z^2+2z^4+4z^6+8z^8+17z^{10}+37z^{12}+82z^{14}+185z^{16}+423z^{18}+978z^{20}+\ldots$$

Corollary 2. *The generating function for the number of partial zigzag knight's paths of a given size is*

$$F(1)+G(1)=\frac{rz^2+rz+r}{z^3\,(1-r)},$$

and the sequence of coefficients of z^n in the series expansion corresponds to A088518 in [23], which counts also symmetric secondary structures of RNA molecules with n nucleotides. The first terms of the series expansions are

$$1+z+2z^2+2z^3+4z^4+5z^5+9z^6+12z^7+21z^8+29z^9+50z^{10}+\ldots$$

The next theorem provides close forms for the coefficients of z^n in the series expansions of f_k and g_k, $k\geq 0$.

Theorem 2. *We have* $[z^0]f_0 = 1$, *and for* $n \geq 1$, $k \geq 1$,

$$\begin{aligned}
[z^{2n}]f_{2k+1} &= a(n-k, 2k+1) + a(n-k+1, 2k),\\
[z^{2n-1}]f_{2k} &= a(n-k, 2k) + a(n-k+1, 2k-1),\\
[z^{2n}]g_{2k} &= a(n-k+1, 2k+1),\\
[z^{2n-1}]g_{2k-1} &= a(n-k+1, 2k),
\end{aligned}$$

where

$$a(n,k) = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{k}{n-i}\binom{n-i}{i+k}\binom{n-i}{i}.$$

All other coefficients are equal to 0.

Proof. Let $\hat{r}$ be the function $z^{-1/2}r(z^{1/2})$, where $r(z)$ is defined in (3). Then

$$\hat{r} = \frac{1 - z - z^2 - \sqrt{1 - 2z - z^2 - 2z^3 + z^4}}{2z^2}.$$

The function $\hat{r}$ satisfies the functional equation $\hat{r} = z(1 + (1+z)\hat{r} + z\hat{r}^2)$. Consider the auxiliary function $q(z,t)$ defined by

$$q(z,t) = z(1 + (1+t)q(z,t) + tq(z,t)^2) = z\Phi(q(z,t)),$$

where $\Phi(u) = (1+u)(1+tu)$. From Lagrange inversion, (see [20] for instance), we have that

$$\begin{aligned}
[z^n]q(z,t)^k &= \frac{k}{n}[u^{n-k}]\Phi(u)^n = \frac{k}{n}[u^{n-k}](u+1)^n(tu+1)^n\\
&= \frac{k}{n}\sum_{i=0}^{n-k}\binom{n}{n-k-i}\binom{n}{i}t^i, \quad n \geq 1.
\end{aligned}$$

So, we obtain

$$\begin{aligned}
[z^n]\hat{r}^k = [z^n]q(z,z)^k &= [z^n]\sum_{\ell \geq 1}\frac{k}{\ell}\sum_{i=0}^{\ell-k}\binom{\ell}{\ell-k-i}\binom{\ell}{i}z^{\ell+i}\\
&= [z^n]\sum_{i \geq 0}\sum_{\ell \geq 0}\frac{k}{\ell+i+1}\binom{\ell+i+1}{\ell+1-k}\binom{\ell+i+1}{i}z^{\ell+2i+1}.
\end{aligned}$$

Setting $h = \ell + 2i + 1$, this implies

$$\begin{aligned}
[z^n]\hat{r}^k &= [z^n]\sum_{i \geq 0}\sum_{h \geq 2i+1}\frac{k}{h-i}\binom{h-i}{h-2i-k}\binom{h-i}{i}z^h\\
&= \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor}\frac{k}{n-i}\binom{n-i}{i+k}\binom{n-i}{i} = a(n,k).
\end{aligned}$$

Then, for all positive integers n and k, we have

$$[z^{2n}]r^{2k} = a(n-k, 2k) \quad \text{and} \quad [z^{2n-1}]r^{2k-1} = a(n-k, 2k-1).$$

From Theorem 1, we conclude

$$\begin{aligned}
[z^{2n}]f_{2k+1} &= [z^{2n}]\left(\frac{r^{2k+1}}{z} + \frac{r^{2k}}{z^2}\right) = [z^{2n+1}]r^{2k+1} + [z^{2n+2}]r^{2k}\\
&= a(n-k, 2k+1) + a(n-k+1, 2k).
\end{aligned}$$

Analogously, we obtain the remaining identities. □

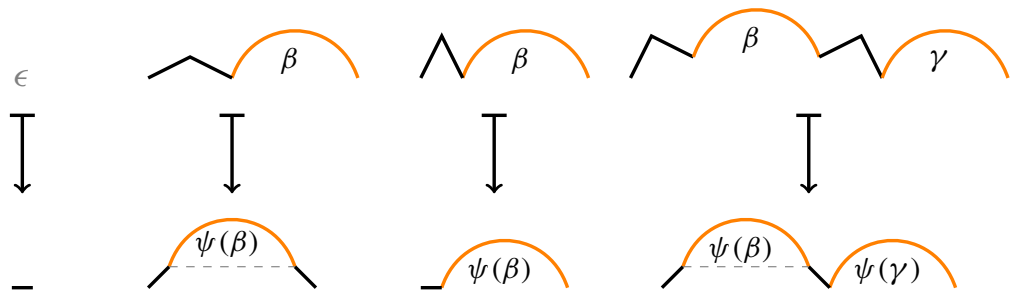


Fig. 5. Illustration of the map ψ according to Definition 3.

2.3. A bijective approach

Corollary 1 proves that the set of zigzag knight's paths of size $2n$ ending on the x-axis is equinumerous to the set of peakless Motzkin paths of length $n+1$, i.e., lattice paths in $\mathbb{N}^2$, starting at the origin, ending at $(n+1,0)$, consisting of steps $U=(1,1)$, $D=(1,-1)$, $F=(1,0)$, and avoiding the pattern UD (see for instance [10]). Any non-empty peakless Motzkin path is either of the form (1) FQ or (2) $URDS$ or (3) URD, where Q (resp. R, S) is a possible empty (resp. non-empty) peakless Motzkin path. Let $\mathcal{M}_n$ be the set of peakless Motzkin paths of length n, and $\mathcal{M}=\bigcup_{n\geq 0}\mathcal{M}_n$.

On the other hand, any non-empty zigzag knight's path is either of the form (1) $N\bar{N}\beta$, (2) $E\bar{E}\beta$ or (3) $N\bar{E}\beta E\bar{N}\gamma$, where β,γ are possibly empty zigzag knight's paths, and $N=(1,2)$, $\bar{N}=(1,-2)$, $E=(2,1)$, $\bar{E}=(2,-1)$. We refer to the top of Fig. 5 for an illustration of these different forms. Let $\mathcal{R}_n$ be the set of zigzag paths of size $2n$ ending on the x-axis, and $\mathcal{R}=\bigcup_{n\geq 0}\mathcal{R}_n$.

Definition 3. We recursively define the map ψ from $\mathcal{R}$ to $\mathcal{M}$ as follows. For $\alpha\in\mathcal{R}$, we set:

$$\psi(\alpha)=\begin{cases} F & \text{if } \alpha=\epsilon, & (i)\\ U\psi(\beta)D & \text{if } \alpha=E\bar{E}\beta \text{ with } \beta\in\mathcal{R}, & (ii)\\ F\psi(\beta) & \text{if } \alpha=N\bar{N}\beta \text{ with } \beta\in\mathcal{R}, & (iii)\\ U\psi(\beta)D\psi(\gamma) & \text{if } \alpha=N\bar{E}\beta E\bar{N}\gamma \text{ with } \beta,\gamma\in\mathcal{R}. & (iv)\end{cases}$$

Due to the recursive definition, the image of a zigzag knight's path of size $2n$ under ψ is a peakless Motzkin path of length $n+1$. For instance, the images of $N\bar{N}$, $E\bar{E}$ and $N\bar{N}N\bar{N}$ are FF, UFD and FFF, respectively. We refer to Fig. 5 for an illustration of this mapping.

Theorem 3. *For all $n\geq 0$, the map ψ induces a bijection between $\mathcal{R}_n$ and $\mathcal{M}_{n+1}$.*

Proof. Due to Corollary 1 in Section 2.2, the cardinality of $\mathcal{R}_n$ is given by the n-th term of generalized Catalan number. So it suffices to prove the injectivity of ψ. We proceed by induction on n. The case $n=0$ is obvious since $\mathcal{R}_0=\{\epsilon\}$ and $\mathcal{M}_1=\{F\}$. For all $k\leq n$, we assume that ψ is an injection from $\mathcal{R}_k$ to $\mathcal{M}_{k+1}$, and we prove the result for $k=n+1$. According to Definition 3, if α and β in $\mathcal{R}_{n+1}$ satisfy $\psi(\alpha)=\psi(\beta)$, then α and β necessarily come from the same case among (i) – (iv). Using the induction hypothesis, we conclude directly that $\alpha=\beta$, which completes the induction. Thus ψ is injective and so bijective. □

For instance, the image of $E\bar{E}N\bar{N}N\bar{E}N\bar{N}E\bar{N}$ under ψ is $UFUFFDFD$.

3. Knight's paths of a given length

Knight's paths of a given length n are clearly in one-to-one correspondence with basketball walks (see [1,4,8,9]), i.e., lattices paths in $\mathbb{N}^2$, starting at the origin, and consisting of n steps lying in $S'=\{(1,1),(1,-1),(1,2),(1,-2)\}$. The correspondence is obtained by replacing $(1,1)$ and $(1,-1)$ with E and $\bar{E}$, respectively.

In [4], Banderier et al. (see Prop. 3.7) prove that the generating function $E(z)=\sum_{n\geq 0}e_nz^n$ for the number of these paths with respect to the length satisfies

$$z^4E^4-(2z^3+z^2)E^3+(3z^2+2z)E^2-(2z+1)E+1=0,$$

and they obtain

$$e_n=\frac{1}{n+1}\sum_{i=0}^{\lfloor n/2\rfloor}\binom{2n+2}{i}\binom{n-i-1}{n-2i}.$$

The first values of e_n, $n\geq 0$, are $1,0,2,2,11,24,93,272,971,3194,11293$ (sequence A187430).

3.1. Partial knight's paths

Partial knight's paths are also handled by Banderier et al. in [4]. In their Prop. 3.2, they prove that the generating function for basketball walks with respect to the number of steps, starting at the origin, ending at height k, is given by

$$\frac{u_1^{k+1}(z)-u_2^{k+1}(z)}{u_1(z)-u_2(z)},$$

where $u_1(z)$ and $u_2(z)$ are the small roots of

$$1-z(u^{-2}+u^{-1}+u+u^2)=0.$$

For instance (see Eq. (3.11) and (3.12) in [4]), the generating functions for basketball walks with respect to the number of steps, starting at the origin, ending at height k, $k=1,2$, are respectively

$$\frac{1}{2}\left(-1+\sqrt{\frac{2-3z-2\sqrt{1-4z}}{z}}\right) \quad \text{and} \quad \frac{3-\sqrt{1-4z}-\sqrt{2+12z+2\sqrt{1-4z}}}{4z}.$$

The first terms of the series expansions are respectively

- $z+z^2+3z^3+7z^4+22z^5+65z^6+213z^7+\dots$ (sequence A166135) and
- $z+z^2+4z^3+9z^4+31z^5+91z^6+309z^7+\dots$ (sequence A111160).

3.2. Zigzag knight's paths

In this part, we count partial zigzag knight's paths of a given length. We adopt the notation used in Section 2.2 for zigzag paths of given length. For $k\geq 0$, $f_k=f_k(z)$ (resp. $g_k=g_k(z)$) is the generating function where the coefficient of z^n in the series expansion is the number of partial zigzag knight's paths of length n. Then, we easily have the following equations:

$$\begin{aligned}&f_0=1,\ f_1=z(f_0+g_0),\ \text{and } f_2=z(f_0+g_0+g_1),\\&f_k=zg_{k-1}+zg_{k-2},\quad k\geq 3,\\&g_k=zf_{k+1}+zf_{k+2},\quad k\geq 0.\end{aligned} \tag{8}$$

Summing the recursions in (8), we have:

$$\begin{aligned}F(u)&=1+zu(1+u)(1+G(u)),\\G(u)&=\frac{z}{u}(F(u)-1)+\frac{z}{u^2}(F(u)-1-uz(1+g_0)),\end{aligned}$$

where $g_0=G(0)$ is the generating function with respect to the length for the number of non-empty zigzag knight's paths ending on the x-axis.

Solving these functional equations, we deduce

$$F(u)=\frac{g_0u^2z^3+g_0uz^3+u^2z^3-u^3z+u^2z^2+uz^3-u^2z+2\,uz^2+z^2-u}{u^2z^2+2\,uz^2+z^2-u},$$

and

$$G(u)=\frac{z^2\left(-u^2+g_0-2u\right)}{u^2z^2+2\,uz^2+z^2-u}.$$

In order to compute g_0, we use the kernel method on $G(u)$. We have

$$G(u)=\frac{z^2\left(-u^2+g_0-2\,u\right)}{z^2(u-r)(u-s)},$$

where

$$r=\frac{1-2\,z^2-\sqrt{1-4\,z^2}}{2z^2} \tag{9}$$

and

$$s=\frac{1-2\,z^2+\sqrt{1-4\,z^2}}{2z^2}.$$

Plugging $u=r$ in the numerator of $G(u)$, we obtain $-r^2+g_0-2r=0$, which implies that $g_0=r(2+r)$. After this, and using $sr=1$, we simplify both numerators and denominators in $F(u)$ and $G(u)$ by extracting the common factor $(u-r)$.

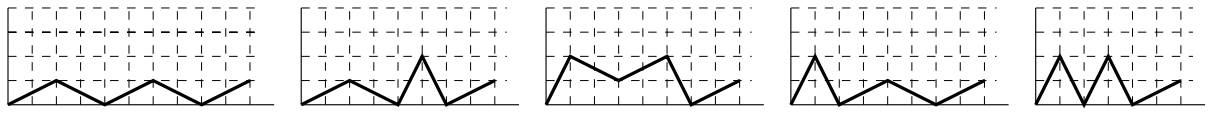


Fig. 6. Partial zigzag knight's path of length 5 ending at height 1.

Theorem 4. *The bivariate generating functions $F(u)$, $G(u)$, and $F(u)+G(u)$ are given by*

$$F(u)=\frac{ru^2+ur(1-z)+z}{z(1-ru)},\quad G(u)=\frac{r(u+r+2)}{1-ru},$$

and

$$F(u)+G(u)=\frac{r^2z+ru^2+ru+2rz+z}{z(1-ru)},$$

where r is defined by Eq. (9). Moreover, we have

$$f_0=[u^0]F(u)=1, \tag{10}$$

$$f_k=[u^k]F(u)=\frac{1+r}{z}r^{k-1}-[k=1]\frac{1}{z},\ k\geq 1, \tag{11}$$

$$g_0=[u^0]G(u)=r(2+r), \tag{12}$$

$$g_k=[u^k]G(u)=\frac{r^{k+1}}{z^2},\ k\geq 1. \tag{13}$$

The first terms of the series expansion of $F(u)+G(u)$ are

$$1+(u+u^2)z+(2+u)z^2+(2u+3u^2+u^3)z^3+(5+4u+u^2)z^4 \\ +(5u+9u^2+5u^3+u^4)z^5+(14+14u+6u^2+u^3)z^6+\dots$$

Fig. 6 shows the partial zigzag knight's paths of length 5 ending at height 1.

Corollary 3. *The generating function for the number of zigzag knight's paths of a given length ending on the x-axis is*

$$F(0)+G(0)=1+r(2+r),$$

and the sequence of non-zero coefficients in the series expansion corresponds to the Catalan numbers (A000108 in [23]). The first terms of the series expansion are

$$1+2z^2+5z^4+14z^6+42z^8+132z^{10}+429z^{12}+1430z^{14}+4862z^{16}+16796z^{18}+58786x^{20}+\dots$$

Corollary 4. *The generating function for the number of partial zigzag knight's paths of a given length is*

$$F(1)+G(1)=\frac{2rz+r}{(1-r)z^2}=\frac{4z+2}{1-4z^2+\sqrt{1-4z^2}},$$

and the sequence of coefficients in the series expansion corresponds to a shift of A001405 in [23]. The first terms of the series expansion are

$$1+2z+3z^2+6z^3+10z^4+20z^5+35z^6+70z^7+126z^8+252z^9+462z^{10}+\dots$$

Corollary 5. *We have $[z^0]f_0=1$ and, for $n\geq 1$,*

$$[z^{2n-1}]f_k=\frac{2k-1}{n+k}\binom{2n}{n-k+1},\ k\geq 1,$$

$$[z^{2n}]g_0=\frac{1}{n+2}\binom{2n+2}{n+1},$$

$$[z^{2n}]g_k=\frac{k+1}{n+1}\binom{2n+2}{n-k},\ k\geq 1.$$

All other coefficients are equal to 0.

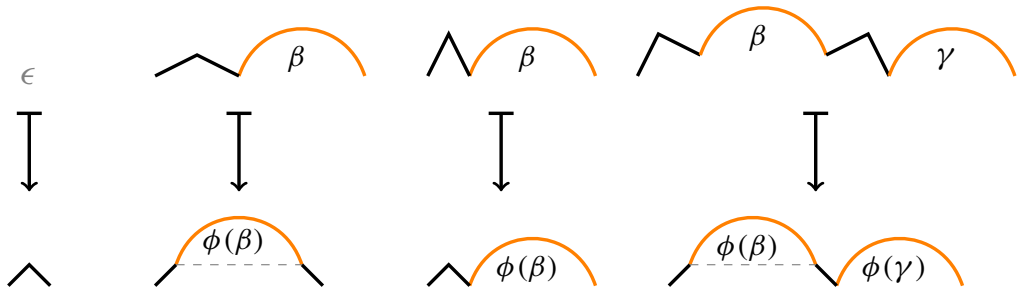


Fig. 7. Illustration of the map ϕ according to Definition 4.

Proof. The generating function $\hat{r} := r(z^{1/2})$ (see Eq. (9)) satisfies the functional equation $\hat{r} = z\Phi(\hat{r})$, where $\Phi(u) = u^2 + 2u + 1$. From the Lagrange inversion we have that

$$[z^{2n}]r^k = [z^n]\hat{r}^k = \frac{k}{n}[u^{n-k}]\Phi(u)^n = \frac{k}{n}[u^{n-k}](u+1)^{2n} = \frac{k}{n}\binom{2n}{n-k}, \quad n \geq 1.$$

From Eq. (11) we have

$$\begin{aligned}[z^{2n-1}]f_k = [z^{2n}](r^{k-1} + r^k) &= \frac{k-1}{n}\binom{2n}{n-k+1} + \frac{k}{n}\binom{2n}{n-k} \\ &= \frac{k-1}{n}\binom{2n}{n-k+1} + \frac{k(n-k+1)}{n(n+k)}\binom{2n}{n-k+1} \\ &= \frac{2k-1}{n+k}\binom{2n}{n-k+1}.\end{aligned}$$

On the other hand,

$$[z^{2n}]g_0 = [z^{2n}]r(2+r) = \frac{2}{n}\binom{2n}{n-1} + \frac{2}{n}\binom{2n}{n-2} = \frac{1}{n+2}\binom{2n+2}{n+1},$$

and, for $k \geq 1$

$$[z^{2n}]g_k = [z^{2n}]\frac{r^{k+1}}{z^2} = [z^{2n+2}]r^{k+1} = \frac{k+1}{n+1}\binom{2n+2}{n-k}. \quad \square$$

Here are other examples of the first terms of the series expansions of $[u^k](F(u) + G(u))$ for $k = 1, 2, 3$:

- $z + z^2 + 2z^3 + 4z^4 + 5z^5 + 14z^6 + 14z^7 + 48z^8 + 42z^9 + 165z^{10} + \ldots$,
- $z + 3z^3 + z^4 + 9z^5 + 6z^6 + 28z^7 + 27z^8 + 90z^9 + 110z^{10} + \ldots$,
- $z^3 + 5z^5 + z^6 + 20z^7 + 8z^8 + 75z^9 + 44z^{10} + 275z^{11} + 208z^{12} + \ldots$,

which do not appear in [23].

3.3. A bijective approach

Corollary 3 proves that there is a bijection between the set of zigzag knight's paths of length $2n$ ending on the x-axis and the set of Dyck paths of length $2(n+1)$. Any non-empty Dyck path is either of the form (1) UD, (2) UQD, (3) UDQ or (4) $UQDR$, where Q and R are non-empty Dyck paths. Let $\mathcal{D}_n$ (resp. $\mathcal{S}_n$) be the set of Dyck paths (resp. zigzag paths) of length $2n$, and $\mathcal{D} = \bigcup_{n\geq 0}\mathcal{D}_n$ (resp. $\mathcal{S} = \bigcup_{n\geq 0}\mathcal{S}_n$).

Definition 4. We recursively define the map ϕ from $\mathcal{S}$ to $\mathcal{D}$ as follows. For $\alpha \in \mathcal{S}$, we set:

$$\phi(\alpha) = \begin{cases} UD & \text{if } \alpha = \epsilon, & (i) \\ U\phi(\beta)D & \text{if } \alpha = E\bar{E}\beta \text{ with } \beta \in \mathcal{S}, & (ii) \\ UD\phi(\beta) & \text{if } \alpha = N\bar{N}\beta \text{ with } \beta \in \mathcal{S}, & (iii) \\ U\phi(\beta)D\phi(\gamma) & \text{if } \alpha = N\bar{E}\beta E\bar{N}\gamma \text{ with } \beta, \gamma \in \mathcal{S}. & (iv) \end{cases}$$

Due to the recursive definition, the image of a zigzag knight's path of length $2n$ under ϕ is a Dyck path of length $2(n+1)$. For instance, the images of $N\bar{N}$, $E\bar{E}$ and $N\bar{N}N\bar{N}$ are $UDUD$, $UUDD$, and $UDUDUD$, respectively. We refer to Fig. 7 for an illustration of this mapping.

The proof of the following theorem is obtained *mutatis mutandis* as for Theorem 2 (see Section 2).

Theorem 5. *For all $n \geq 0$, the map ϕ induces a bijection between $\mathcal{S}_n$ and $\mathcal{D}_{n+1}$.*

For instance, the image of $E\bar{E}N\bar{N}N\bar{E}N\bar{N}E\bar{N}$ under ϕ is $UUDUUDUDDUDD$.

Declaration of competing interest

There is no conflict of interest

Data availability

No data was used for the research described in the article.

Acknowledgements

The authors would like to thank the anonymous referees for useful remarks and comments.

References

[1] A. Ayyer, D. Zeilberger, The number of [old-time] basketball games with final score $n:n$ where the home team was never losing but also never ahead by more than w points, Electron. J. Comb. 14 (1) (2007) #P8.
[2] C. Banderier, M. Bousquet-Mélou, A. Denise, P. Flajolet, D. Gardy, D. Gouyou-Beauchamps, Generating functions for generating trees, Discrete Math. 246 (2002) 29–55.
[3] C. Banderier, M. Wallner, Lattice paths with catastrophes, Discret. Math. Theor. Comput. Sci. 19 (1) (2017) #23.
[4] C. Banderier, C. Krattenthaler, A. Krinik, D. Kruchinin, V. Kruchinin, D. Nguyen, M. Wallner, Explicit formulas for enumeration of lattice paths: basketball and the kernel method, in: Lattice Path Combinatorics and Applications. Developments in Mathematics Series, Springer, 2019, pp. 78–118.
[5] E. Barcucci, A. Del Lungo, S. Fezzi, R. Pinzani, Nondecreasing Dyck paths and q-Fibonacci numbers, Discrete Math. 170 (1997) 211–217.
[6] J.-L. Baril, A. Petrossian, Equivalence classes of Dyck paths modulo some statistics, Discrete Math. 338 (2015) 655–660.
[7] J.-L. Baril, D. Bevan, S. Kirgizov, Bijections between directed animals, multisets and Grand-Dyck paths, Electron. J. Comb. 27 (2020) #P2.10.
[8] J. Bettinelli, E. Fusy, C. Mailler, L. Randazzo, A bijective study of basketball walks, Sémin. Lothar. Comb. 77 (2016) B77a.
[9] M. Bousquet-Mélou, Discrete excursions, Sémin. Lothar. Comb. 57 (2008) B57d.
[10] N. Cameron, E. Sullivan, Peakless Motzkin paths with marked level steps at fixed height, Discrete Math. 344 (2021) 112154.
[11] E. Deutsch, Dyck path enumeration, Discrete Math. 204 (1999) 167–202.
[12] T. Došlić, D. Svrtan, D. Veljan, Enumerative aspects of secondary structure, Discrete Math. 285 (2004) 67–82.
[13] L. Euler, Solution d'une question curieuse qui ne paroit soumise à aucune analyse, Mem. Acad. Sci. Berlin (1959) 310–337.
[14] R. Flórez, J.L. Ramírez, Enumerating symmetric and asymmetric peaks in Dyck paths, Discrete Math. 343 (2020) 112118.
[15] D.E. Knuth, The Art of Computer Programming, vol. III: Sorting and Searching, 2nd ed., Addison-Wesley, 1973, p. 48.
[16] C. Krattenthaler, Permutations with restricted patterns and Dyck paths, Adv. Appl. Math. 27 (2001) 510–530.
[17] J. Labelle, Y.-N. Yeh, Dyck paths of knight moves, Discrete Appl. Math. 24 (1989) 213–221.
[18] T. Mansour, Statistics on Dyck paths, J. Integer Seq. 9 (2006) 06.1.5.
[19] D. Merlini, R. Sprugnoli, M.C. Verri, Some statistics on Dyck paths, J. Stat. Plan. Inference 101 (2002) 211–227.
[20] D. Merlini, R. Sprugnoli, M.C. Verri, Lagrange inversion: when and how, Acta Appl. Math. 94 (2006) 233–249.
[21] A. Sapounakis, I. Tasoulas, P. Tsikouras, Counting strings in Dyck paths, Discrete Math. 307 (2007) 2909–2924.
[22] Y. Sun, The statistic "number of udu's" in Dyck paths, Discrete Math. 287 (2004) 177–186.
[23] N.J.A. Sloane, The on-line encyclopedia of integer sequences, available electronically at http://oeis.org.
[24] R.P. Stanley, Enumerative Combinatorics, vol. 2, Cambridge University Press, 1999.

Journal of Integer Sequences, Vol. 26 (2023),
Article 23.3.2

Enumeration of Dyck Paths with Air Pockets

Jean-Luc Baril, Sergey Kirgizov, Rémi Maréchal, and Vincent Vajnovszki
LIB
Université de Bourgogne
B.P. 47 870
21078 Dijon-Cedex
France
barjl@u-bourgogne.fr
sergey.kirgizov@u-bourgogne.fr
remi_marechal01@etu.u-bourgogne.fr
vvajnov@u-bourgogne.fr

Abstract

We introduce and study the new combinatorial class of Dyck paths with air pockets. We exhibit a bijection with the peakless Motzkin paths, which transports several pattern statistics, and give bivariate generating functions for the distribution of patterns as peaks, returns, and pyramids. Then we deduce the popularities and asymptotic expectations of these patterns, and point out a link between the popularity of pyramids and a special kind of closed smooth self-overlapping curves, a subset of Fibonacci meanders. Finally, we conduct a similar study for non-decreasing Dyck paths with air pockets.

1 Introduction and notation

Lattice paths are widely studied in combinatorics. They have many applications in various domains, such as computer science, biology, and physics [23], and they have very tight links with other combinatorial objects such as directed animals, pattern avoiding permutations, bargraphs, RNA structures, and so on [4, 11, 23]. A classical problem in combinatorics is the enumeration of these paths with respect to their length and other statistics [1, 2, 3, 7, 15, 16, 18, 20, 21]. In the literature, Dyck and Motzkin paths are the ones most often considered.

They are counted by the famous Catalan and Motzkin numbers (see A000108 and A001006 in Sloane's On-line Encyclopedia of Integer Sequences [22]). In 2005, Krinik et al. [13] introduced Dyck paths with catastrophes in the context of queuing theory. They correspond to the evolution of a queue by allowing some resets. The push (resp., pop) operation corresponds to a step $U = (1, 1)$ (resp., $D = (1, -1)$), and the reset operations are modeled by catastrophe steps $D_k = (1, -k)$ ending on x-axis, $k \geq 2$. Banderier and Wallner [1] studied these paths by providing enumerative results and limit laws.

In this paper, we introduce and study paths with air pockets corresponding to a queue evolution with partial reset operations that cannot be consecutive. These paths can also be viewed as airplane flights, sometimes showcasing turbulences that are known as *air pockets*, where consecutive turbulences are considered to be one. More formally, a *Dyck path with air pockets* is a nonempty lattice path in the first quadrant of $\mathbb{Z}^2$ starting at the origin, ending on the x-axis, and consisting of up-steps $U = (1, 1)$ and down-steps $D_k = (1, -k)$, $k \geq 1$, where two down steps cannot be consecutive. For short, we set $D = D_1$. The *length* of a Dyck path with air pockets is the number of its steps. Let $\mathcal{A}_n$ be the set of n-length Dyck paths with air pockets. By definition $\mathcal{A}_0 = \mathcal{A}_1 = \varnothing$ and we set $\mathcal{A} = \bigcup_{n\geq 2} \mathcal{A}_n$.

Figure 1: The Dyck path with air pockets $UUDUD_2UUUD_2UD_2UUD_2$.

A Dyck path with air pockets is called *prime* whenever it ends with D_k, $k \geq 2$, and returns to the x-axis only once. The set of all prime Dyck paths with air pockets of length n is denoted $\mathcal{P}_n$. Notice that UD is not prime so we set $\mathcal{P} = \bigcup_{n\geq 3} \mathcal{P}_n$. If $\alpha = U\beta UD_k \in \mathcal{P}_n$, then $2 \leq k < n$, β is a (possibly empty) prefix of a path in $\mathcal{A}$, and we define the Dyck path with air pockets $\alpha^\flat = \beta UD_{k-1}$, called the 'lowering' of α. For example, the path $\alpha = UUDUUD_3$ is prime, and $\alpha^\flat = UDUUD_2$. The map $\alpha \mapsto \alpha^\flat$ is clearly a bijection from $\mathcal{P}_n$ to $\mathcal{A}_{n-1}$ for all $n \geq 3$, and we let $\gamma^\sharp$ denote the inverse image of $\gamma \in \mathcal{A}_{n-1}$ ($\alpha^\sharp$ is a kind of 'elevation' of α, drawing inspiration for the term from Deutsch's definition of elevated Dyck paths [7]). Any Dyck path with air pockets $\alpha \in \mathcal{A}$ can be decomposed depending on its *second-to-last return to the x-axis*: either

(i) $\alpha = UD$; or

(ii) $\alpha = \beta UD$ with $\beta \in \mathcal{A}$; or

(iii) $\alpha \in \mathcal{P}$; or

(iv) $\alpha = \beta\gamma$ where $\beta \in \mathcal{A}$ and $\gamma \in \mathcal{P}$.

So if $A(x) = \sum_{n\geq 2} a_n x^n$ where a_n is the cardinality of $\mathcal{A}_n$, and $P(x) = \sum_{n\geq 3} p_n x^n$ where p_n is the cardinality of $\mathcal{P}_n$, then we have $P(x) = xA(x)$ and the previous decompositions imply the functional equation $A(x) = x^2 + x^2A(x) + xA(x) + xA(x)^2$, and

$$A(x) = \frac{1 - x - x^2 - \sqrt{x^4 - 2x^3 - x^2 - 2x + 1}}{2x}, \tag{1}$$

which generates the generalized Catalan numbers (see A004148 [22]), which among other things, counts the peakless Motzkin paths. The first values of a_n for $2 \leq n \leq 10$ are $1, 1, 2, 4, 8, 17, 37, 82, 185$. An asymptotic approximation for the coefficient of x^n in the series expansion of $A(x)$ is

$$\frac{\sqrt{14\sqrt{5} - 30}}{2n\sqrt{\pi n}(3 - \sqrt{5})} \left(\frac{\sqrt{5} + 3}{2}\right)^n.$$

If a Dyck path with air pockets $\alpha \in \mathcal{A}_n$ has $k \geq 1$ peaks (a peak is an occurrence UD_i for some $i \geq 1$), then it contains $n - k$ up-steps. If we 'unfurl' all of its down-steps D_i, $i \geq 1$, into runs D^i of i consecutive D-steps, then we obtain a Dyck path of length $2(n - k)$ having k peaks. This gives rise to a bijection between Dyck paths of semilength $n - k$ with k peaks and n-length Dyck paths with air pockets with k peaks. Hence, the number of n-length Dyck paths with air pockets with k peaks is equal to the Narayana number $N(n - k, k) = \frac{1}{n-k}\binom{n-k}{k}\binom{n-k}{k-1}$ [7].

In the following, a *pattern* consists of consecutive steps in a path, and a *statistic* $\mathbf{s}$ is an integer-valued function from a set $\mathcal{S}$ of paths. Given a pattern p, we associate p with the pattern statistic $\mathbf{p} : \mathcal{S} \to \mathbb{N}$ where $\mathbf{p}(a)$ is the number of occurrences of the pattern p in $a \in \mathcal{S}$ (we use the boldface to denote statistics). For example, the statistic giving the number of occurrences of the consecutive pattern UU in a path is denoted by $\mathbf{UU}$. For $n \geq 1$, we let $\hat{\mathbf{n}}$ denote the constant statistic returning the value n. The *popularity* of a pattern p in $\mathcal{S}$ is the total number of occurrences of p over all objects of $\mathcal{S}$, that is $\mathbf{p}(\mathcal{S}) = \sum_{a\in\mathcal{S}} \mathbf{p}(a)$ [6, 9, 10]. Let $\mathcal{S}'$ be another set of combinatorial objects, we say that two statistics, $\mathbf{s}$ on $\mathcal{S}$ and $\mathbf{t}$ on $\mathcal{S}'$, have the *same distribution* if there exists a bijection $f : \mathcal{S} \to \mathcal{S}'$ satisfying $\mathbf{s}(a) = \mathbf{t}(f(a))$ for any $a \in \mathcal{S}$. In this case, with a slight abuse of the notation already used in the literature [5], we write $f(\mathbf{s}) = \mathbf{t}$ or $\mathbf{s} = \mathbf{t}$ whenever f is the identity.

The remainder of this paper is organized as follows. In Section 2, we present a constructive bijection between n-length Dyck paths with air pockets and peakless Motzkin paths of length $n - 1$, and we show how this bijection transports some statistics. In Section 3, we provide bivariate generating functions $A(x, y) = \sum_{n,k\geq 0} a_{n,k} x^n y^k$ for the distributions of some statistics $\mathbf{s}$, i.e., the coefficient $a_{n,k}$ of $x^n y^k$ is the number of paths $\alpha \in \mathcal{A}_n$ satisfying $\mathbf{s}(\alpha) = k$. Then we deduce the popularities of some patterns (U, D, peak, return, catastrophe, pyramid, ...) by calculating $\partial_y(A(x, y))|_{y=1}$, and we provide asymptotic approximations for them using classical methods [8, 17]. We refer to Table 1 for an overview of the results. As a byproduct, we point out a link between the popularity of pyramids and a special kind of closed smooth self overlapping curves in the plane (a subset of Fibonacci meanders defined

by Luschny and Wienand on their web pages [14, 24]). In Section 4, we make a similar study for non-decreasing Dyck paths with air pockets.

Pattern	Pattern popularity in $\mathcal{A}_n$	OEIS
U	$1, 2, 5, 13, 32, 80, 201, 505, 1273, 3217$	A110320
D	$1, 0, 2, 3, 7, 17, 40, 97, 238, 587$	A051291
Peak	$1, 1, 3, 7, 16, 39, 95, 233, 577, 1436$	A203611
Ret	$1, 1, 3, 6, 13, 29, 65, 148, 341, 793$	A093128
Cat	$0, 1, 1, 4, 8, 19, 44, 102, 239, 563$	
Δ_k	$\underbrace{0, \ldots, 0}_{k-1 \text{ zeroes}}, 1, 0, 2, 3, 7, 17, 40, 97, 238, 587$	A051291
$\Delta_{\geq k}$	$\underbrace{0, \ldots, 0}_{k-1 \text{ zeroes}}, 1, 1, 3, 6, 13, 30, 70, 167, 405$	A201631$(= u_n)$
$\Delta_{\leq k}$	$\Delta_{\leq 1}$ $1, 0, 2, 3, 7, 17, 40, 97, 238, 587$	$u_n - u_{n-k}$
	$\Delta_{\leq 2}$ $1, 1, 2, 5, 10, 24, 47, 137, 335, 825, \ldots$	
	$\Delta_{\leq 3}$ $1, 1, 3, 5, 12, 27, 64, 154, 375, 922, \ldots$	
	$\vdots$	

Table 1: Pattern popularity in $\mathcal{A}_n$, for $2 \leq n \leq 11$.

2 Bijection with peakless Motzkin paths

In this section we exhibit a constructive bijection between n-length Dyck paths with air pockets and $(n-1)$-length peakless Motzkin paths, i.e., lattice paths in the first quarter plane starting at the origin, ending at $(n-1, 0)$, made up of U, D and $F = (1, 0)$ and having no occurrence of UD. Moreover, we show how our bijection transports some pattern based statistics. We let $\mathcal{M}_n$ denote the set of peakless Motzkin paths of length n, and $\mathcal{M} = \bigcup_{n \geq 1} \mathcal{M}_n$.

Definition 1. We recursively define the map ψ from $\mathcal{A}$ to $\mathcal{M}$ as follows. For $\alpha \in \mathcal{A}$, we set

$$\psi(\alpha) = \begin{cases} F, & \text{if } \alpha = UD; & \text{(i)} \\ U\psi(\beta)D, & \text{if } \alpha = \beta UD, \text{ with } \beta \in \mathcal{A}; & \text{(ii)} \\ \psi(\alpha^\flat)F, & \text{if } \alpha \in \mathcal{P}; & \text{(iii)} \\ \psi(\gamma^\flat)U\psi(\beta)D, & \text{if } \alpha = \beta\gamma, \text{ with } \beta \in \mathcal{A} \text{ and } \gamma \in \mathcal{P}. & \text{(iv)} \end{cases}$$

Notice that each factor in the above decomposition is nonempty, and that ψ maps nonempty objects to nonempty ones. Due to the recursive definition, the image by ψ of a n-length Dyck path with air pockets is a peakless Motzkin path of length $n-1$. For instance, the images of UD, UUD_2, $UUUD_2UD_2UD$ are respectively F, FF, and $UUFFDFD$. We refer to Figure 2 for an illustration of this mapping.

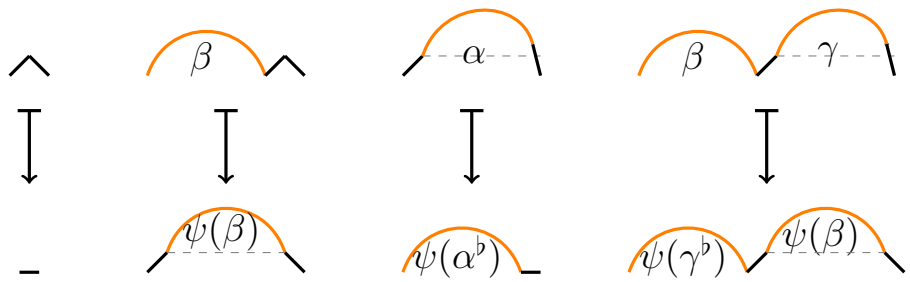


Figure 2: Illustration of the map ψ according to Definition 1.

Theorem 2. *For all $n \geq 2$, the map ψ induces a bijection between $\mathcal{A}_n$ and $\mathcal{M}_{n-1}$.*

Proof. It is well known that the cardinality of $\mathcal{M}_n$ is given by the n-th term of generalized Catalan number (see A004148 [22]). So it suffices (see observation after relation (1)) to prove the injectivity of ψ. We proceed by induction on n. The case $n = 2$ is obvious since $\mathcal{A}_2 = \{UD\}$ and $\mathcal{M}_1 = \{F\}$. For all $k \leq n$, we assume that ψ is an injection from $\mathcal{A}_k$ to $\mathcal{M}_{k-1}$, and we prove the result for $k = n + 1$. According to Definition 1, if α and β in $\mathcal{A}_{n+1}$ satisfy $\psi(\alpha) = \psi(\beta)$, then α and β necessarily come from the same case among (i)–(iv). Using the induction hypothesis, we conclude directly that $\alpha = \beta$, which completes the induction. Thus ψ is injective and so bijective. $\square$

Proposition 3. *For all $n \geq 2$ and $k \geq 1$, and $\psi : \mathcal{A}_n \to \mathcal{M}_{n-1}$, the following hold:*

- $\psi(\mathbf{U}) = \mathbf{F} + \mathbf{U} = \mathbf{F} + \mathbf{D}$
- $\psi(\mathbf{D}) = \psi(\mathbf{UD}) = \mathbb{1}_{\mathbf{F}} + \mathbf{UFD} + \mathbb{1}_{\mathbf{U}\mathcal{M}\mathbf{D}} + \mathbf{U^2}\mathcal{M}\mathbf{D^2}$
- $\psi(\mathbf{DU}) = \mathbf{UFD} + \mathbf{U^2}\mathcal{M}\mathbf{D^2}$
- $\psi(\mathbf{UU}) = \mathbf{F} - \hat{\mathbf{1}}$
- $\psi(\mathbf{\Delta_k}) = \mathbb{1}_{\mathbf{F^k}} + \mathbf{UF^kD} + \mathbb{1}_{\mathbf{F^{k-1}U}\mathcal{M}\mathbf{D}} + \mathbf{UF^{k-1}U}\mathcal{M}\mathbf{D^2}$
- $\psi(\mathbf{Peak}) = \mathbf{U} + \hat{\mathbf{1}}$
- $\psi(\mathbf{Ret}) = \hat{\mathbf{n}} - \mathbf{LastF}$
- $\psi(\mathbf{SLast}) = \mathbf{Ret}$,

where

- $\mathbb{1}_\beta(\alpha) = 1$ *if* $\alpha = \beta$ *and* 0 *otherwise;*
- $\mathbb{1}_{\mathbf{U}\mathcal{M}\mathbf{D}}(\alpha)$ *is equal to* 1 *if there exists* $\beta \in \mathcal{M}$ *such that* $\alpha = U\beta D$ *and* 0 *otherwise;*
- $\mathbf{U^2}\mathcal{M}\mathbf{D^2}(\alpha)$ *is the number of occurrences* $U^2\beta D^2$ *in* α *for* $\beta \in \mathcal{M}$*;*

- $\mathbf{\Delta_k}(\alpha)$ *is the number of occurrences* $U^k D_k$ *in* α*;*
- $\mathbf{Peak}(\alpha) = \sum_{\mathbf{k \geq 1}} \mathbf{UD_k}(\alpha)$*;*
- $\mathbf{Ret}(\alpha)$ *is the number of returns to the* x*-axis of* α*;*
- $\mathbf{LastF}(\alpha)$ *is the position of the rightmost flat-step in* α*; and*
- $\mathbf{SLast}(\alpha)$ *is the size of the the last step of* α *(i.e.,* k *if the last step is* D_k*).*

Proof. We provide the proof for $\psi(\mathbf{U})$ and $\psi(\mathbf{\Delta_k})$ since those for the other relations can be obtained, mutatis mutandis.

We proceed by induction on n. Since $\mathcal{A}_2 = \{UD\}$ and $\mathcal{M}_1 = \{F\}$ the statements trivially hold for $n = 2$. Now assume the statements are true for all $k \leq n$ and let us prove them for $n + 1$.

If $\alpha \in \mathcal{A}_{n+1}$ with $n \geq 2$, then we have either

(i) $\alpha = \beta UD$;

(ii) $\alpha = \gamma^\sharp$; or

(iii) $\alpha = \beta\gamma^\sharp$ where $\beta, \gamma \in \mathcal{A}$.

In case (i), we have $\mathbf{U}(\beta UD) = 1 + \mathbf{U}(\beta)$ and with the induction hypothesis, $\mathbf{U}(\beta UD) = 1 + (\mathbf{U} + \mathbf{F})(\psi(\beta)) = (\mathbf{U} + \mathbf{F})(U\psi(\beta)D) = (\mathbf{U} + \mathbf{F})(\psi(\beta UD))$ as expected.

In case (ii), we have $\mathbf{U}(\gamma^\sharp) = 1 + \mathbf{U}(\gamma)$ and with the induction hypothesis, $\mathbf{U}(\gamma^\sharp) = 1 + (\mathbf{U} + \mathbf{F})(\psi(\gamma)) = (\mathbf{U} + \mathbf{F})(\psi(\gamma)F) = (\mathbf{U} + \mathbf{F})(\psi(\gamma^\sharp))$.

Case (iii) is handled in the same way.

So we have $\psi(\mathbf{U}) = \mathbf{U} + \mathbf{F}$. Using similar reasoning, we can easily prove $\psi(\mathbf{D}) = \mathbb{1}_{\mathbf{F}} + \mathbf{UFD} + \mathbb{1}_{\mathbf{U}\mathcal{M}\mathbf{D}} + \mathbf{U^2}\mathcal{M}\mathbf{D^2}$.

Now let us give details for the slightly less straightforward case of $\psi(\mathbf{\Delta_k})$ for $k \geq 1$. The case $k = 1$ is already handled since we have $\psi(\mathbf{UD}) = \psi(\mathbf{D})$. So we assume $k \geq 2$. We consider the following case analysis: any given Dyck path with air pockets is either of the form

(i) βUD;

(ii) $\beta\Delta_{k-1}^\sharp$;

(iii) $\beta\Delta_k^\sharp$;

(iv) $\beta(\alpha\Delta_k)^\sharp$ with $\alpha \in \mathcal{A}$; or

(v) $\beta\alpha^\sharp$ with $\alpha \in \mathcal{A}$ being neither Δ_{k-1}, nor Δ_k, nor $\alpha'\Delta_k$ ($\alpha' \in \mathcal{A}$), and $\beta \in \mathcal{A} \cup \{\varepsilon\}$.

Reasoning by induction, case (ii) unfolds as follows: if $\beta = \varepsilon$, then we get

$$(\mathbb{1}_{\mathbf{F^k}} + \mathbf{UF^kD} + \mathbb{1}_{\mathbf{F^{k-1}U\mathcal{M}D}} + \mathbf{UF^{k-1}U\mathcal{M}D^2})(F^k) = 1,$$

which is the same as $\mathbf{\Delta_k}(\Delta^\sharp_{k-1})$. Otherwise, we have

$$\begin{aligned}
&(\mathbb{1}_{\mathbf{F^k}} + \mathbf{UF^kD} + \mathbb{1}_{\mathbf{F^{k-1}U\mathcal{M}D}} + \mathbf{UF^{k-1}U\mathcal{M}D^2})(\psi(\beta\Delta^\sharp_{k-1})) \\
&= (\mathbb{1}_{\mathbf{F^k}} + \mathbf{UF^kD} + \mathbb{1}_{\mathbf{F^{k-1}U\mathcal{M}D}} + \mathbf{UF^{k-1}U\mathcal{M}D^2})(F^{k-1}U\psi(\beta)D) \\
&= \mathbb{1}_{\mathbf{F^k}}(\psi(\beta)) + \mathbf{UF^kD}(\psi(\beta)) + 1 + \mathbb{1}_{\mathbf{F^{k-1}U\mathcal{M}D}}(\psi(\beta)) \\
&\quad + \mathbf{UF^{k-1}U\mathcal{M}D^2}(\psi(\beta)) \\
&= 1 + \mathbf{\Delta_k}(\beta) = \mathbf{\Delta_k}(\beta\Delta^\sharp_{k-1}).
\end{aligned}$$

The four remaining cases are obtained in the same way. □

Notice that the mirror of a Dyck path with air pockets is a Łukasiewicz path avoiding flat steps and two consecutive up-steps. Since there is a bijection between Łukasiewicz paths and plane trees [12], we easily deduce that Dyck paths with air pockets are in one-to-one correspondence with plane trees without unary nodes, and such that the first child of any node is always a leaf. We leave open the question of knowing how this bijection transports some pattern-based statistics.

3 Distribution and popularity of patterns

3.1 The numbers of U and D

Theorem 4. *Let $A(x,y,z) = \sum_{n,k,\ell\geq 0} a_{n,k,\ell}x^ny^kz^\ell$ be the generating function (g.f.) where $a_{n,k,\ell}$ is the number of paths in $\mathcal{A}_n$ having k up-steps U and ℓ down-steps $D = D_1$. Then the following holds:*

$$A(x,y,z) = \frac{1 - xy - x^2yz - 2x^3y^2 + 2x^3y^2z - \sqrt{R}}{2xy(1 + x^2y - x^2yz)},$$

with

$$R = x^4y^2z^2 + 2x^3y^2z - 4x^3y^2 + x^2y^2 - 2x^2yz - 2xy + 1.$$

Proof. Due to the first return decomposition, any Dyck path with air pockets has one of the following forms:

(i) $UD\gamma$;

(ii) $U^2D_2\gamma$;

(iii) $(\alpha UD)^\sharp\gamma$ with $\alpha \in \mathcal{A}$; or

(iv) $\alpha^\sharp\gamma$ with $\alpha \in \mathcal{A}$ not being UD nor βUD ($\beta \in \mathcal{A}$), where $\gamma \in \mathcal{A} \cup \{\varepsilon\}$.

These four cases are disjoint and cover $\mathcal{A}$ entirely. Then we deduce the functional equation by taking into account the length, and the numbers of U and D with respect to x, y and z:

$$A = (\underbrace{x^2yz}_{\text{(i)}} + \underbrace{x^3y^2}_{\text{(ii)}} + \underbrace{x^3y^2A}_{\text{(iii)}} + \underbrace{xy(A - x^2yz(1+A))}_{\text{(iv)}})(1+A),$$

where A stands for $A(x, y, z)$. Solving for A, we get the result. □

Corollary 5. *For all $n \geq 1$, the number of Dyck paths with air pockets (of any length) having n up-steps U is the n-th Catalan number $\frac{1}{n+1}\binom{2n}{n}$ (see* A000108 *[22]).*

Proof. We check that $1 + A(1, y, 1)$ is the g.f. of the Catalan numbers. □

Corollary 6. *For all $n \geq 1$, the number of Dyck paths with air pockets having n up-steps U and no down-steps D is the n-th Riordan number $\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}c_k$, where $c_k = \frac{1}{k+1}\binom{2k}{k}$ (see* A005043 *in [22]).*

Proof. We check that $1 + A(1, y, 0)$ is the g.f. of the Riordan numbers. □

Corollary 7. *The g.f. for the popularity of up-steps U in $\mathcal{A}_n$ is*

$$\frac{1 - x - x^2 - \sqrt{x^4 - 2x^3 - x^2 - 2x + 1}}{2x\sqrt{x^4 - 2x^3 - x^2 - 2x + 1}},$$

which generates a shift of the sequence A110320 *[22]. An asymptotic approximation of the n-th term is*

$$\frac{\sqrt{5} - 1}{2\sqrt{\pi n}\sqrt{14\sqrt{5} - 30}}\left(\frac{3 + \sqrt{5}}{2}\right)^n,$$

and an asymptotic for the expectation of the up-step number is

$$\frac{\sqrt{5} + 5}{10}n \sim 0.723606799 \cdot n.$$

Proof. The g.f. is given by $\partial_y(A(x, y, 1))|_{y=1}$. The asymptotic approximation is obtained using classical methods [8, 17]. □

Corollary 8. *The g.f. for the popularity of down-steps $D = D_1$ in $\mathcal{A}_n$ is*

$$\frac{x^2(1 + 2x^2 - x^3 + (1 - x)\sqrt{x^4 - 2x^3 - x^2 - 2x + 1})}{2\sqrt{x^4 - 2x^3 - x^2 - 2x + 1}},$$

which generates a shift of the sequence A051291 *[22]. An asymptotic approximation of the n-th term is*

$$\frac{5\sqrt{5}-11}{2\sqrt{\pi n}\sqrt{14\sqrt{5}-30}}\left(\frac{3+\sqrt{5}}{2}\right)^n,$$

and an asymptotic for the expectation of the D-step number is

$$\frac{5-2\sqrt{5}}{5}n \sim 0.105572797 \cdot n.$$

Proof. The g.f. is given by $\partial_z(A(x,1,z))|_{z=1}$. □

3.2 The number of peaks

In this part, we study the distribution of peaks, i.e., patterns UD_m for $m \geq 1$.

Theorem 9. *Let $P(x,y) = \sum_{n,k\geq 0} p_{n,k}x^n y^k$ be the g.f. where $p_{n,k}$ is the number of n-length Dyck paths with air pockets having k peaks. Then we have*

$$P(x,y) = \frac{1-x-x^2y-\sqrt{(1-x-x^2y)^2-4x^3y}}{2x},$$

which generates a shift of the sequence A089732 *[22].*

Proof. If a Dyck path with air pockets equals αUD with $\alpha \in \mathcal{A} \cup \{\varepsilon\}$, then its contribution to $P(x,y)$ is $(1+P(x,y))x^2y$; if it has the form $\alpha\beta^\sharp$ with $\beta \in \mathcal{A}$, then its contribution is $(1+P(x,y))xP(x,y)$. Hence, the second-to-last return decomposition yields

$$P(x,y) = (1+P(x,y))(x^2y + xP(x,y)),$$

which gives the result after solving for $P(x,y)$. □

Corollary 10. *The g.f. for the popularity of peaks in $\mathcal{A}_n$ is*

$$\frac{x(1+x-x^2-\sqrt{x^4-2x^3-x^2-2x+1})}{2\sqrt{x^4-2x^3-x^2-2x+1}},$$

which generates a shift of the sequence A203611 *[22]. An asymptotic approximation of the n-th term is*

$$\frac{\sqrt{5}-2}{\sqrt{\pi n}\sqrt{14\sqrt{5}-30}}\left(\frac{3+\sqrt{5}}{2}\right)^n,$$

and an asymptotic for the expectation of the peak number is

$$\frac{5-\sqrt{5}}{10}n \sim 0.276393191 \cdot n.$$

Remark 11. Another way of finding the total number of peaks in all Dyck paths with air pockets of length n is the following: since the number of n-length Dyck paths with air pockets with k peaks is $N(n-k,k)$, we have

$$\mathbf{Peak}(\mathcal{A}_n) = \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} kN(n-k,k) = \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \frac{k}{n-k}\binom{n-k}{k}\binom{n-k}{k-1}.$$

Using the formula for the sequence A203611 [22], we get the following identity:

$$\sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k-1}{k-1}\binom{n-k}{k-1} = \sum_{k=0}^{n-1} \binom{k-1}{2k-n}\binom{k}{2k-n+1}.$$

3.3 The number of returns to the x-axis

A return to the x-axis is a step D_m, $m \geq 1$, ending on the x-axis.

Theorem 12. *Let $R(x,y) = \sum_{n,k\geq 0} r_{n,k}x^n y^k$ be the g.f. where $r_{n,k}$ is the number of n-length Dyck paths with air pockets with k returns. Then:*

$$R(x,y) = \frac{2}{2 - y(1 - x + x^2 - \sqrt{x^4 - 2x^3 - x^2 - 2x + 1})} - 1,$$

which generates the triangle A098086 [22] where the row n and column k gives the number of peakless Motzkin paths having its leftmost F-step on the k-th step (see also Proposition 3).

Proof. If a Dyck path with air pockets equals αUD with $\alpha \in \mathcal{A} \cup \{\varepsilon\}$, then its contribution to $R(x,y)$ is $(1 + R(x,y))x^2y$; if it has the form $\alpha\beta^\sharp$ with $\beta \in \mathcal{A}$, then its contribution is $(1 + R(x,y))xyA(x)$. So we deduce that

$$R(x,y) = (1 + R(x,y))(x^2y + xyA(x)),$$

which gives the result using relation (1). □

Corollary 13. *The g.f. for the popularity of returns to the x-axis in $\mathcal{A}_n$ is*

$$2\frac{1 - x + x^2 - \sqrt{x^4 - 2x^3 - x^2 - 2x + 1}}{(1 + x - x^2 + \sqrt{x^4 - 2x^3 - x^2 - 2x + 1})^2},$$

which corresponds to the sequence A093128 [22], where the n-th term counts all possible dissections of a regular $(n+2)$-gon using zero or more strictly disjoint diagonals. An asymptotic approximation of the n-th term is

$$\frac{\sqrt{14\sqrt{5} - 30}\sqrt{5}}{4n\sqrt{\pi n}}\left(\frac{3 + \sqrt{5}}{2}\right)^{n+1},$$

and an asymptotic for the expectation of the return number is $\sqrt{5}$.

3.4 The number of catastrophes

A catastrophe is a step D_m, $m \geq 2$, ending on the x-axis.

Theorem 14. *Let $C(x,y) = \sum_{n,k\geq 0} c_{n,k} x^n y^k$ be the g.f. where $c_{n,k}$ is the number of n-length Dyck paths with air pockets with k catastrophes. Then we have*

$$C(x,y) = \frac{2}{2 - 2x^2 - y(1 - x - x^2 - \sqrt{x^4 - 2x^3 - x^2 - 2x + 1})} - 1.$$

Proof. If a Dyck path with air pockets equals αUD with $\alpha \in \mathcal{A} \cup \{\varepsilon\}$, then its contribution to $C(x,y)$ is $(1 + C(x,y))x^2$; if it has the form $\alpha\beta^\sharp$ with $\beta \in \mathcal{A}$, then its contribution is $(1 + C(x,y))xyA(x)$. So we deduce $C(x,y) = (1 + C(x,y))(x^2 + xyA(x))$. □

Corollary 15. *The g.f. for the popularity of catastrophes in $\mathcal{A}_n$ equals*

$$2\frac{1 - x - x^2 - \sqrt{x^4 - 2x^3 - x^2 - 2x + 1}}{(1 + x - x^2 + \sqrt{x^4 - 2x^3 - x^2 - 2x + 1})^2}.$$

An asymptotic approximation of the n-th term is

$$\frac{\sqrt{14\sqrt{5} - 30}(4 - \sqrt{5})}{4n\sqrt{\pi n}} \left(\frac{3 + \sqrt{5}}{2}\right)^{n+1},$$

and an asymptotic for the expectation of the catastrophe number is $4 - \sqrt{5}$.

Remark 16. As a byproduct of Corollaries 13 and 15, the ratio of the popularity of catastrophes in $\mathcal{A}_n$ to the popularity of returns in $\mathcal{A}_n$ tends to $\frac{4-\sqrt{5}}{\sqrt{5}} = 0.788854\ldots$ when n tends toward ∞.

3.5 The number of pyramids $U^k D_k$

A k-pyramid Δ_k in a path is an occurrence of the pattern $U^k D_k$, $k \geq 1$.

Theorem 17. *For all $k \geq 1$, the g.f. $P_k(x,y) = \sum_{n,m\geq 0} p^k_{n,m} x^n y^m$ where $p^k_{n,m}$ is the number of n-length Dyck paths with air pockets having m k-pyramids is given by*

$$P_k(x,y) = \frac{x^{k+1}(y-1) - 2x^{k+2}(y-1) + x^2 + x - 1 + \sqrt{Q}}{2(x^{k+2}(y-1) - x)},$$

where

$$Q = x^{k+1}(y-1)(x^{k+1}(y-1) + 4x + 2(x^2 - x - 1)) + x^4 - 2x^3 - x^2 - 2x + 1.$$

Proof. We refine the first return decomposition so that any Dyck path with air pockets falls into one of the following cases:

(i) $\Delta_m\gamma$ with $1 \le m \le k-1$;

(ii) $\Delta_k\gamma$;

(iii) $\Delta_{k+1}\gamma$;

(iv) $(\alpha\Delta_k)^\sharp\gamma$ with $\alpha \in \mathcal{A}$;

(v) $\beta^\sharp\gamma$ with $\beta \in \mathcal{A}$ not being Δ_m with $1 \le m \le k$, nor $\alpha\Delta_k$ with $\alpha \in \mathcal{A}$, where $\gamma \in \mathcal{A} \cup \{\varepsilon\}$.

These five cases are disjoint and cover all Dyck paths with air pockets. So we deduce

$$P_k = (\underbrace{\sum_{i=2}^{k} x^i}_{\text{(i)}} + \underbrace{x^{k+1}y}_{\text{(ii)}} + \underbrace{x^{k+2}}_{\text{(iii)}} + \underbrace{x^{k+2}P_k}_{\text{(iv)}} + \underbrace{x \cdot (P_k - \sum_{i=2}^{k} x^i - x^{k+1}y(1+P_k)))}_{\text{(v)}}(1+P_k),$$

where P_k stands for $P_k(x,y)$. Solving for P_k, we get the result. □

Corollary 18. *For $k \ge 1$, the g.f. for the popularity $\mathbf{\Delta_k}(\mathcal{A}_n)$ of k-pyramids in $\mathcal{A}_n$ equals*

$$Y_k(x) = \frac{x^{k+1}(1+2x^2-x^3+(1-x)\sqrt{x^4-2x^3-x^2-2x+1})}{2\sqrt{x^4-2x^3-x^2-2x+1}},$$

which generates the $(n-k-2)$-th term of the sequence A051291 *[22]. In particular, we have $\mathbf{\Delta_1}(\mathcal{A}_n) = \mathbf{\Delta_k}(\mathcal{A}_{n+k-1})$ for all $k \ge 1$ and $n \ge 2$, which means that there are as many 1-pyramids in $\mathcal{A}_n$ as there are k-pyramids in $\mathcal{A}_{n+k-1}$. An asymptotic approximation of the n-th term of this sequence is*

$$\frac{\sqrt{5}-1}{2\sqrt{\pi n}\sqrt{14\sqrt{5}-30}}\left(\frac{3+\sqrt{5}}{2}\right)^{n-k-1},$$

and for the expected number of k-pyramids we have

$$\frac{5-\sqrt{5}}{10}\left(\frac{3-\sqrt{5}}{2}\right)^{k} \cdot n.$$

Notice that $Y_1(x)$ corresponds to the generating function for the popularity of down-steps D (see Corollary 7), since each D is necessarily preceded by an up-step. Moreover, we have $Y_k(x) = x^{k-1}Y_1(x)$ since each pyramid Δ_k in a path of length n comes from a pyramid Δ_1 in a path of length $n-k+1$ by adding $k-1$ up-steps and by increasing the length of the down-step. An immediate consequence of Corollary 18 is the following.

Corollary 19. *For $k \geq 1$, the g.f. for the popularities $\mathbf{\Delta}_{\geq \mathbf{k}}(\mathcal{A}_n)$ and $\mathbf{\Delta}_{\leq \mathbf{k}}(\mathcal{A}_n)$ are respectively given by*

$$Y_{\geq k}(x) = \frac{x^{k-1}}{1-x} Y_1(x) \text{ and } Y_{\leq k}(x) = \frac{1-x^k}{1-x} Y_1(x),$$

which means that $\mathbf{\Delta}_{\leq \mathbf{k}}(\mathcal{A}_{n-k+1}) = \mathbf{\Delta}_{\geq \mathbf{k}}(\mathcal{A}_n) - \mathbf{\Delta}_{\geq \mathbf{k}}(\mathcal{A}_{n-k})$.

For any $k \geq 1$, the popularity of pyramids of size at least k in $\mathcal{A}_n$ (see Corollary 19) seems to correspond to a shift of the sequence A201631 [22], which enumerates *Fibonacci meanders with central angle* 180 *degrees* (see Luschny's [14] and Wienand's [24] posts in OEIS Wiki about meanders). In order to prove this fact, we give the formal definition of such meanders, and provide their g.f. that does not exist in the literature (to our knowledge).

A Fibonacci meander with central angle 180 degrees is a closed smooth self-overlapping curve in the plane, consisting of an even length sequence of two types of arcs of angle 180 degrees, namely $L =$ ↶ and $R =$ ↷, starting at the origin with an L-arc toward the north, having no consecutive L-arcs except at the beginning where a run (of any length) of consecutive L-arcs is authorized. Each arc starts at the end of the previous arc and it preserves the direction of its arrow, i.e., LLR corresponds to ↶↷. Let $\mathcal{F}_{2n}$ be the set of such meanders of length $2n$. For instance, the left part of Figure 3 illustrates a meander in $\mathcal{F}_{20}$.

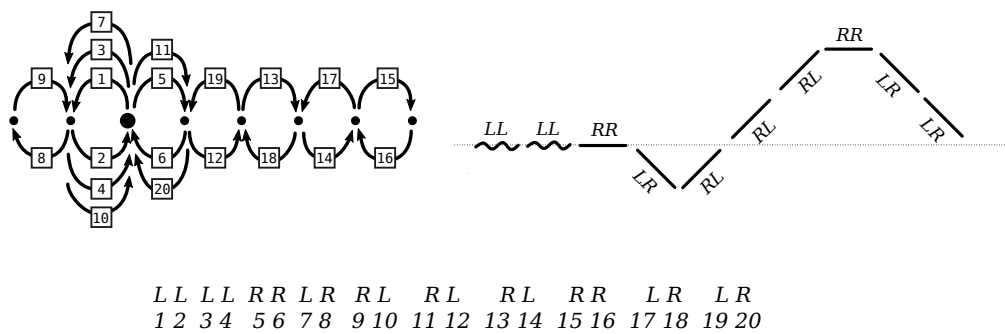


Figure 3: A Fibonacci meander in $\mathcal{F}_{20}$ where the sequence of arcs is given by $LLLLRRLRRLRLRLRRLRLR$, and its associated lattice path.

Now we define a function τ, mapping a two-letter word over the alphabet $\{L, R\}$ into the set $\{U, D, F, \widetilde{F}\}$:

$$\tau(a) = \begin{cases} U, & \text{if } a = RL; \\ D, & \text{if } a = LR; \\ F, & \text{if } a = RR; \\ \widetilde{F}, & \text{if } a = LL. \end{cases}$$

and a function μ, mapping a meander $w = w_1 w_2 \cdots w_{2n} \in \mathcal{F}_{2n}$ into an n-length word over the alphabet $\{U, D, F, \widetilde{F}\}$:

$$\mu(w) = \tau(w_1 w_2)\tau(w_3 w_4) \cdots \tau(w_{2n-1} w_{2n}).$$

Grand Motzkin paths are lattice paths of length n in $\mathbb{N} \times \mathbb{Z}$, consisting of steps $U = (1,1), D = (1,-1)$, and $F = (1,0)$, starting at $(0,0)$, ending at $(n,0)$ (contrarily to classical Motzkin paths, they can go below the x-axis). For instance $DUUDDFDUUUD$ is a grand Motzkin path of length 11. Let $\mathcal{G}_n$ be the set of peakless (i.e., with no occurrence of UD) grand Motzkin paths of length n and $\mathcal{N}_n$ be the subset of paths starting with a D-step in $\mathcal{G}_n$. Let $\mathcal{S}_n$ denote the set of peakless grand Motzkin paths of length n starting with a down step or prefixed by a nonempty sequence of special flats, called *wavy flats*, $\widetilde{F} = (1,0)$. For instance, $\widetilde{F}\widetilde{F}DUUDDFDUUUD \in \mathcal{S}_{13}$.

Proposition 20. *The function μ induces a bijection from $\mathcal{F}_{2n}$ to $\mathcal{S}_n$.*

Proof. Every meander $a \in \mathcal{F}_{2n}$ avoids the pattern LL except if the occurrence of LL is only preceded by letters L, which means that $\mu(a)$ avoids the pattern UD. If the meander a starts with a maximal prefix of the form $(LL)^k$, $k \geq 1$, then $\mu(a)$ starts, with a maximal sequence of k wavy flats, $\widetilde{F}^k$. If the meander a starts with LR, then $\mu(a)$ starts with D. Moreover, the image by μ of a factor RR is $F = \tau(RR)$ in $\mu(a)$, the image of LR is D and the image of RL is U. Thus, the fact that a is a closed curve, implies that $\mu(a)$ starts and ends on x-axis. Due to all these observations, μ is necessarily a bijection from $\mathcal{F}_{2n}$ to $\mathcal{S}_n$. □

Notice that the bijection μ is very close to the bijection of Roitner [19] between 2-watermelons with arbitrary deviation and weighted Motzkin paths, which suggests that there are tight links between 2-watermelons and Wienand-Luschny meanders. It would be interesting to explore this correspondence in future work.

Theorem 21. *The g.f. $S(x) = \sum_{n\geq 0} s_n x^n$, where the coefficient s_n is the number of $2n$-length Fibonacci meanders with a central angle 180 degrees, is*

$$S(x) = \frac{x^2 - x + 1 - \sqrt{R}}{(x-1)(R + (x^2 - x - 1)\sqrt{R})},$$

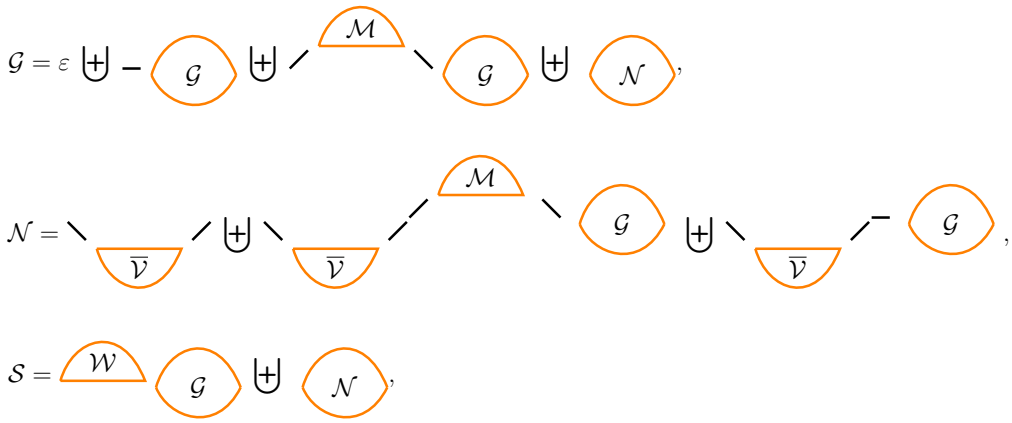
with $R = x^4 - 2x^3 - x^2 - 2x + 1$. Using Corollary 19, we have

$$S(x) = \frac{Y_1(x)}{x^2(1-x)} - 1,$$

which establishes the expected link between Fibonacci meanders and the popularity $\mathbf{\Delta}_{\geq \mathbf{k}}(\mathcal{A}_n)$.

Proof. Considering Proposition 20, it suffices to enumerate $\mathcal{S}_n$. We set $\mathcal{G} = \bigcup_{n\geq 0} \mathcal{G}_n$, $\mathcal{N} = \bigcup_{n\geq 0} \mathcal{N}_n$, and $\mathcal{S} = \bigcup_{n\geq 0} \mathcal{S}_n$. Recall that $\mathcal{M}$ is the set of nonempty peakless Motzkin paths. Let $\mathcal{V}$ denote the set of Motzkin paths without valleys DU and by $\overline{\mathcal{V}}$ the set of paths obtained by symmetry about the x-axis ($U \leftrightarrow D$) of valleyless Motzkin paths, e.g., $DUFDU \in \overline{\mathcal{V}}$ since it is symmetric to $UDFUD \in \mathcal{V}$. Let $\mathcal{W}$ be the set of nonempty sequences of wavy flat steps. We use $M(x)$, $V(x) = \overline{V}(x)$, $G(x)$, $N(x)$, $W(x)$ to denote the corresponding generating functions with respect to the length.

Obviously, we have $W(x) = \frac{x}{1-x}$. From Relation (1) and Theorem 2 we obtain $M(x) = A(x)/x$. Also, there is a one-to-one correspondence ν between $\mathcal{M}_n$ and $\mathcal{V}_{n-1}$ that can be defined recursively by $\nu(F) = \varepsilon$, $\nu(FQ) = F\nu(Q)$, $\nu(UQD) = U\nu(Q)D$, and $\nu(UQDR) = U\nu(Q)DF\nu(R)$ if R is non-empty. So we have $V(x) = M(x)/x = A(x)/x^2$. Finally, we decompose $\mathcal{G}$, $\mathcal{N}$ and $\mathcal{S}$ as illustrated below:



which induces the following system of functional equations:

$$\begin{cases} G(x) = 1 + xG(x) + x^2M(x)G(x) + N(x); \\ N(x) = x^2V(x) + x^4V(x)M(x)G(x) + x^3V(x)G(x); \\ S(x) = W(x)G(x) + N(x). \end{cases}$$

Solving this system, we obtain $S(x)$. □

4 Non-decreasing Dyck paths with air pockets

A Dyck path with air pockets is *non-decreasing* if the sequence of heights of its valleys is non-decreasing, i.e., the sequence of the minimal ordinates of the occurrences D_kU, $k \geq 1$, is non-decreasing from left to right [2]. For example, the Dyck path with air pockets $UUDUDUD_2$ is non-decreasing, since its two valleys both lie at height 1, while the path $UUDUD_2UD$ is not, since its two valleys lie at heights 1 and 0 from left to right. Let $\mathcal{A}_n^{\nearrow}$, $n \geq 2$, be the set of n-length non-decreasing Dyck paths with air pockets and $\mathcal{A}^{\nearrow} = \bigcup_{n\geq 2} \mathcal{A}_n^{\nearrow}$. The subset of n-length prime non-decreasing Dyck paths with air pockets is defined as the intersection $\mathcal{P}_n^{\nearrow} := \mathcal{A}_n^{\nearrow} \cap \mathcal{P}$, and we set $\mathcal{P}^{\nearrow} := \bigcup_{n\geq 2} \mathcal{P}_n^{\nearrow}$. Analogous to generic Dyck paths with air pockets, the map $\alpha \mapsto \alpha^{\flat}$ induces a bijection between $\mathcal{P}_n^{\nearrow}$ and $\mathcal{A}_{n-1}^{\nearrow}$, whose inverse is the map $\alpha \mapsto \alpha^{\sharp}$.

Theorem 22. *For $n \geq 2$, if $a_n^{\nearrow}$ is the number of n-length non-decreasing Dyck paths with air pockets, then $a_2^{\nearrow} = 1$ and $a_n^{\nearrow} = 2^{n-3}$ for $n \geq 3$.*

Proof. Any non-decreasing Dyck path with air pockets α has one of the following two forms: (i) $\alpha \in \mathcal{P}^{\nearrow} \cup \{UD\}$, or (ii) $\alpha = \Delta_k \beta$ where $k \geq 1$ and $\beta \in \mathcal{A}^{\nearrow}$. So if $A^{\nearrow}(x) = \sum_{n \geq 2} a_n^{\nearrow} x^n$ where $a_n^{\nearrow}$ is the cardinality of $\mathcal{A}_n^{\nearrow}$, then the previous decomposition implies the functional equation

$$A^{\nearrow}(x) = x(A^{\nearrow}(x) + x) + \frac{x^2}{1-x} A^{\nearrow}(x).$$

Thus we have $A^{\nearrow}(x) = \frac{x^2(1-x)}{1-2x}$ which completes the proof. □

4.1 The numbers of U and D

Theorem 23. *Let $A^{\nearrow}(x,y,z) = \sum_{n,k,\ell \geq 0} a_{n,k,\ell}^{\nearrow} x^n y^k z^\ell$ be the trivariate g.f. where $a_{n,k,\ell}^{\nearrow}$ is the number of n-length non-decreasing Dyck paths with air pockets having k up-steps U and ℓ down-steps D. Then $A^{\nearrow}(x,y,z)$ equals*

$$\frac{x^2 y(1-xy)(xyz - xy - z)(x^2yz + xy - 1)}{(x^3y^2(z-1) + x^2y(y-z) - 2xy + 1)(x^3y^2(z-1) - x^2yz - xy + 1)}.$$

Proof. Let $Z(x,y,z) = \sum_{n,k,\ell \geq 0} z_{n,k,\ell} x^n y^k z^\ell$, where $z_{n,k,\ell}$ is the number of n-length non-decreasing Dyck paths with air pockets having only valleys at height 0, k up-steps U and ℓ down-steps D. Such a path has the form $UD\alpha$ or $\Delta_k \alpha$ with $k \geq 2$, where α has all its valleys at height 0. Then we have

$$Z(x,y,z) = (1 + Z(x,y,z)) \left(x^2yz + \frac{x^3y^2}{1-xy} \right).$$

Solving for $Z(x,y,z)$, we get

$$Z(x,y,z) = \frac{x^2y(xy(1-z) + z)}{x^3y^2(z-1) - x^2yz - xy + 1}.$$

Now any non-decreasing Dyck path with air pockets belongs to one of the following cases:

(i) βUD;

(ii) $\beta (UD)^\sharp$;

(iii) $\beta (\alpha UD)^\sharp$ (α having all its valleys at height 0);

(iv) $\beta \alpha^\sharp$ (α having all its valleys at height 0, and not ending with UD), where β is either empty or has all its valleys at height 0.

Thus, we have (for short, we use $A^{\nearrow}$ and Z instead of $A^{\nearrow}(x,y,z)$ and $Z(x,y,z)$)

$$A^{\nearrow} = (1+Z)(x^2yz + x^3y^2 + x^3y^2Z + xy(A^{\nearrow} - x^2yz(1+Z))).$$

Solving for $A^{\nearrow}$, we get the result. □

Porism 24. For all $n \geq 1$, the number of n-length non-decreasing Dyck paths with air pockets which have all valleys at height 0 is equal to F_{n-1}, where F_k is the k-th Fibonacci number.

Proof. Plugging $y = z = 1$ into the trivariate g.f. Z in the proof of the previous theorem we obtain $Z(x, 1, 1) = \frac{x^2}{1-x-x^2}$, which is the g.f. for the right shift of the sequence of Fibonacci numbers. □

As we have done in Section 3.1, we deduce the following.

Corollary 25. *For all $k \geq 1$, the number of non-decreasing Dyck paths with air pockets:*

- *having n up-steps U is the k-th term of the sequence* A001519*;*
- *having k up-steps U and no down-steps D is the $(k-1)$-th term of the sequence* A099036*.*

Proof. We calculate $Z(1, y, 1)$ and $Z(1, y, 0)$, respectively. □

Corollary 26. *The popularity of up-steps U in $\mathcal{A}_n^{\nearrow}$ is equal to the $(n-2)$-th term of the sequence* A098156 *[22]. An asymptotic for the expectation of the up-step number is $(3n-2)/4$.*

Proof. We calculate $\partial_y(Z(x, y, 1))|_{y=1}$. □

Corollary 27. *The g.f. for the popularity of down-steps D in $\mathcal{A}_n^{\nearrow}$ equals*

$$\frac{x^2(1-x)(1-4x+5x^2-2x^3+x^5)}{(1-2x)^2(1-x-x^2)}.$$

An asymptotic approximation of the n-th term is $n \cdot 2^{n-6}$. An asymptotic for the expectation of the down-step number is $n/8$.

Proof. We calculate $\partial_z(Z(x, 1, z))|_{z=1}$. □

4.2 The number of peaks

Theorem 28. *For all $n \geq 2$ and $k \geq 1$, the number of n-length non-decreasing Dyck paths with air pockets having k peaks is equal to $\binom{n-2}{2(k-1)}$.*

Proof. Let $B(x, y)$ be the g.f. where the coefficient of $x^n y^k$ is the number of n-length paths in $\mathcal{A}^{\nearrow}$ having k peaks. Any non-decreasing Dyck path with air pockets is either of the form $\Delta_1 = UD$, or $\alpha^\sharp$, or $\Delta_k\beta$ with $k \geq 1$, with $\alpha, \beta \in \mathcal{A}^{\nearrow}$. This yields the following functional equation:

$$B(x, y) = x^2 y + xB(x, y) + \frac{x^2}{1-x} yB(x, y)$$

with the solution $B(x, y) = \frac{(1-x)x^2y}{(1-x)^2-x^2y}$, which generates the sequence A034839 [22]. □

Corollary 29. *The popularity of peaks in $\mathcal{A}_n^{\nearrow}$ is the $(n-2)$-th term of the sequence* A045891 *[22], which is equal to $(n+2) \cdot 2^{n-5}$ for $n \geq 4$. Then the expectation of the peak number is $(n+2)/4$.*

4.3 The number of returns to the x-axis

Theorem 30. *The bivariate g.f. where the coefficient of $x^n y^k$ is the number of n-length non-decreasing Dyck paths with air pockets having k returns is*

$$R(x,y) = \frac{x^2 y(1-x)(1-x-x^2)}{(1-2x)(1-x-x^2y)}.$$

Proof. Using the second-to-last return decomposition of $\mathcal{A}^{\nearrow}$, we easily get the following functional equation:

$$R(x,y) = x^2y + x^2yR(x,y) + xyA^{\nearrow}(x) + \frac{x^3}{1-x}yR(x,y),$$

which gives the result. □

Corollary 31. *The g.f. for the popularity of returns in $\mathcal{A}_n^{\nearrow}$ is*

$$\frac{x^2(1-x)^2}{(1-2x)(1-x-x^2)},$$

and for $n \geq 2$ the coefficient of x^n is $2^{n-2} - F_{n-2}$, where F_n is the n-th Fibonacci number (see A099036 *[22]). Then the expectation of the return number is $2 - F_{n-2}/2^{n-3}$ that tends to 2.*

4.4 The number of catastrophes

Theorem 32. *The bivariate g.f. where the coefficient of $x^n y^k$ is the number of n-length non-decreasing Dyck paths with air pockets having k catastrophes is*

$$C(x,y) = \frac{x^2(1-x)(1+x(y-2)-x^2y)}{(1-2x)(1-x-x^2-x^3(y-1))}.$$

Proof. First, let us determine the bivariate g.f. $U(x,y)$ with respect to the length and number of catastrophes for non-decreasing Dyck paths with air pockets having all their valleys at height 0. It is easy to see that

$$U(x,y) = (1+U(x,y))\left(x^2 + \frac{x^3y}{1-x}\right),$$

which yields

$$U(x,y) = \frac{x^2(1-x+xy)}{1-x-x^2-x^3(y-1)}.$$

Then any non-decreasing Dyck path with air pockets has one of the following forms:

(i) βUD; or

(ii) $\beta U^2 D_2$; or

(iii) $\beta\alpha^\sharp$ with α not belonging to forms (i) or (ii), and where β is either empty or a non-decreasing Dyck path with air pockets which only has valleys that lie at height 0.

Hence, the bivariate generating function $C(x,y)$ satisfies the following equation:

$$C(x,y) = (1+U(x,y))(x^2 + x^3y(1+U(x,1)) + xy(C(x,1) - x^2(1+U(x,1)))),$$

which gives the result. □

Corollary 33. *The g.f. for the popularity of catastrophes in $\mathcal{A}_n^{\nearrow}$ equals*

$$\frac{x^3(1-x)(1-x+x^2)}{(1-2x)(1-x-x^2)},$$

and for $n \geq 4$ the coefficient of x^n is $3 \cdot 2^{n-4} + 2F_{n-3}$, where F_n is the n-th Fibonacci number (see the sequence A175657 [22]). Then the expectation of the catastrophe number is $3/2 - F_{n-3}/2^{n-4}$ that tends to $3/2$.

Remark 34. As a byproduct of Corollary 31 and Corollary 33, the ratio of the popularity of catastrophes in $\mathcal{A}_n^{\nearrow}$ to the popularity of returns in $\mathcal{A}_n^{\nearrow}$ tends to $\frac{3}{4}$ when n tends toward ∞.

4.5 The number of pyramids

Theorem 35. *For $k \geq 1$, let $P_k(x,y) = \sum_{n,m\geq 0} p_{n,m}^k x^n y^m$ be the g.f. where $p_{n,m}^k$ is the number of n-length non-decreasing Dyck paths with air pockets with m occurrences of the pattern $\Delta_k = U^k D_k$. Then the following holds:*

$$P_k(x,y) = \frac{x^2(1 - \frac{x^2}{1-x} + x^{k-1}(1 - x - \frac{x^2(2-x)}{1-x})(y-1) - x^{2k}(y-1)^2)}{(1 - x - \frac{x^2}{1-x} - x^{k+1}(y-1))(1 - \frac{x^2}{1-x} - x^{k+1}(y-1))}.$$

Proof. First, let us determine the expression of the bivariate g.f. $Z_k(x,y)$ with respect to the length and the number of patterns Δ_k for non-decreasing Dyck paths with air pockets having all their valleys at height 0. The second-to-last return decomposition of $\mathcal{A}^{\nearrow}$ yields

$$Z_k(x,y) = (1 + Z_k(x,y))(x^{k+1}y + \left(\frac{x^2}{1-x} - x^{k+1}\right)).$$

Hence, we get $Z_k(x,y) = \frac{1}{1 - \frac{x^2}{1-x} - x^{k+1}(y-1)} - 1$.

Now, assuming that $k \neq 1$, any Dyck path has one of the following forms:

(i) βUD;

(ii) $\beta\Delta_{k-1}^\sharp$;

(iii) $\beta\Delta_k^\sharp$;

(iv) $\beta(\alpha\Delta_k)^\sharp$ with α having all of its valleys at height 0;

(v) $\beta\alpha^\sharp$ with $\alpha \in \mathcal{A}^\nearrow$, $\alpha \neq \Delta_{k-1}, \Delta_k, \gamma\Delta_k$ (γ having all of its valleys at height 0), and where β is either empty or has all its valleys at height 0.

This yields (for short we use P_k and Z_k instead of $P_k(x,y)$ and $Z_k(x,y)$)

$$P_k = (1+Z_k)(x^2 + x^{k+1}y + x^{k+2}(1+Z_k) + x(P_k - x^k - x^{k+1}y(1+Z_k))).$$

Solving for P_k, we get the result for $k \geq 2$.

If $k = 1$, the expression of $P_1(x,y)$ is the same as that of the bivariate g.f. associated with the pattern D in $\mathcal{A}^\nearrow$ (given in Theorem 30), because D occurs exactly as often as $UD = \Delta_1$. □

Corollary 36. *For $k \geq 1$, the g.f. for the popularity $\mathbf{\Delta_k}(\mathcal{A}_n^\nearrow)$ of k-pyramids in $\mathcal{A}_n^\nearrow$ is*

$$W_k(x) = \frac{x^{k+1}(1-x)(1-4x+5x^2-2x^3+x^5)}{(1-2x)^2(1-x-x^2)}.$$

In particular, we can see that $\mathbf{\Delta_1}(\mathcal{A}_n^\nearrow) = \mathbf{\Delta_k}(\mathcal{A}_{n+k-1}^\nearrow)$, which means that there are as many 1-pyramids in $\mathcal{A}_n^\nearrow$ as there are k-pyramids in $\mathcal{A}_{n+k-1}^\nearrow$. An asymptotic approximation of the n-th term is $n \cdot 2^{n-5-k}$, and an asymptotic approximation for the expectation of the k-pyramid number is $n/2^{k+2}$.

An immediate consequence of the previous corollary is the following one, which is the $\mathcal{A}_n^\nearrow$-counterpart of Corollary 19.

Corollary 37. *For $k \geq 1$, the g.f. for the popularities $\mathbf{\Delta_{\geq k}}(\mathcal{A}_n^\nearrow)$ and $\mathbf{\Delta_{\leq k}}(\mathcal{A}_n^\nearrow)$ are respectively given by*

$$W_{\geq k}(x) = \frac{x^{k-1}}{1-x}W_1(x) \text{ and } W_{\leq k}(x) = \frac{1-x^k}{1-x}W_1(x),$$

which means that $\mathbf{\Delta_{\leq k}}(\mathcal{A}_{n-k+1}^\nearrow) = \mathbf{\Delta_{\geq k}}(\mathcal{A}_n^\nearrow) - \mathbf{\Delta_{\geq k}}(\mathcal{A}_{n-k}^\nearrow)$.

Going further. It should be interesting to give natural bijections whenever our enumerative results suggest such bijections. Also, asymptotic investigations of expectations could be extended to a study of the limit distributions. It will also be of interest to investigate the 'Grand' counterpart of Dyck paths with air pockets, that are paths where negative ordinates are allowed, or 'Motzkin' counterpart where flat steps $(1,0)$ are allowed.

5 Acknowledgments

The authors would like to thank the anonymous referees for useful remarks and comments. This work was supported in part by the project ANER ARTICO funded by Bourgogne-Franche-Comté region of France and ANR-22-CE48-0002.

Pattern	Pattern popularity in $\mathcal{A}_n^{\nearrow}$	OEIS
U	$1, 2, 5, 13, 32, 76, 176, 400, 896, 1984$	A098156
D	$1, 0, 2, 3, 7, 15, 33, 72, 157, 341$	
Peak	$1, 1, 3, 7, 16, 36, 80, 176, 384, 832$	A045891
Ret	$1, 1, 3, 6, 13, 27, 56, 115, 235, 478$	A099036
Cat	$0, 1, 1, 4, 8, 18, 38, 80, 166, 342$	A175657
Δ_k	$\underbrace{0, \ldots, 0}_{k-1 \text{ zeroes}}, 1, 0, 2, 3, 7, 15, 33, 72, 157, 341$	
$\Delta_{\geq k}$	$\underbrace{0, \ldots, 0}_{k-1 \text{ zeroes}}, 1, 1, 3, 6, 13, 28, 61, 133, 290, 631$	New ($= v_n$)
$\Delta_{\leq k}$	$\Delta_{\leq 1}$ $1, 0, 2, 3, 7, 15, 33, 72, 157, 341$	$v_n - v_{n-k}$
	$\Delta_{\leq 2}$ $1, 1, 2, 5, 10, 22, 48, 105, 229, 498$	
	$\Delta_{\leq 3}$ $1, 1, 3, 5, 12, 25, 55, 120, 262, 570$	
	$\vdots$	

Table 2: Pattern popularity in $\mathcal{A}_n^{\nearrow}$ for $2 \leq n \leq 11$.

References

[1] C. Banderier and M. Wallner, Lattice paths with catastrophes, *Discrete Math. Theor. Comput. Sci.* **19** (2017), #23.

[2] E. Barcucci, A. Del Lungo, S. Fezzi, and R. Pinzani, Nondecreasing Dyck paths and q-Fibonacci numbers, *Discrete Math.* **170** (1997), 211–217.

[3] J.-L. Baril and A. Petrossian, Equivalence classes of Dyck paths modulo some statistics, *Discrete Math.* **338** (2015), 655–660.

[4] J.-L. Baril, D. Bevan, and S. Kirgizov, Bijections between directed animals, multisets and Grand-Dyck paths, *Electron. J. Combin.* **27** (2020), #P2.10.

[5] J.-L. Baril, R. Genestier, and S. Kirgizov, Pattern distributions in Dyck paths with a first return decomposition constrained by height, *Discrete Math.* **342** (2020), 111995.

[6] M. Bóna, Surprising symmetries in objects counted by Catalan numbers, *Electron. J. Combin.* **19** (2012), #P62.

[7] E. Deutsch, Dyck path enumeration, *Discrete Math.* **204** (1999), 167–202.

[8] P. Flajolet and R. Sedgewick, *Analytic Combinatorics*, Cambridge University Press, 2009.

[9] C. Homberger, Expected patterns in permutation classes, *Electron. J. Combin.* **19** (2012), #P43.

[10] S. Kitaev, *Patterns in Permutations and Words*, Springer Science & Business Media, 2011.

[11] D. E. Knuth, *The Art of Computer Programming*, Volume 1, *Fundamental Algorithms*, Addison-Wesley, 1973.

[12] I. Kortchemski and C. Marzouk, Simply generated non-crossing partitions, *Combin. Probab. Comput.* **26** (2017), 560–592.

[13] A. Krinik, G. Rubino, D. Marcus, R. J. Swift, H. Kasfy, and H. Lam, Dual processes to solve single server systems, *J. Statist. Plann. Inference* **135** (2005), 121–147.

[14] P. Luschny, Fibonacci meanders, http://oeis.org/wiki/User:Peter_Luschny/FibonacciMeanders, 2011.

[15] T. Mansour, Statistics on Dyck paths, *J. Integer Sequences* **9** (2006), Article 06.1.5.

[16] D. Merlini, R. Sprugnoli, and M. C. Verri, Some statistics on Dyck paths, *J. Statist. Plann. Inference* **101** (2002), 211–227.

[17] A. G. Orlov, On asymptotic behavior of the Taylor coefficients of algebraic functions, *Sibirsk. Mat. Zh.* **35** (1994), 1125–1137. Translation in *Siberian Math. J.* **35** (1994), 1002–1013.

[18] A. Panayotopoulos and A. Sapounakis, On the prime decomposition of Dyck paths, *J. Combin. Math. Combin. Comput.* **40** (2002), 33–39.

[19] V. Roitner, Contacts and returns in 2-watermelons without wall, *Bull. Inst. Combin. Appl.* **89** (2020), 75–92.

[20] A. Sapounakis, I. Tasoulas, and P. Tsikouras, Counting strings in Dyck paths, *Discrete Math.* **307** (2007), 2909–2924.

[21] Y. Sun, The statistic "number of *udu*'s" in Dyck paths, *Discrete Math.* **287** (2004), 177–186.

[22] OEIS Foundation Inc., The On-Line Encyclopedia of Integer Sequences, Published electronically at https://oeis.org, 2023.

[23] R. P. Stanley. *Enumerative Combinatorics*, Vol. 2, Cambridge University Press, 1999.

[24] S. Wienand, Personal page on the OEIS Wiki, http://oeis.org/wiki/User:Susanne_Wienand#Definition_of_meanders_concerned_by_these_sequences, 2011.

2020 *Mathematics Subject Classification*: Primary 05A15; Secondary 05A16, 05A19, 68R05.
Keywords: Dyck path, pattern distribution, popularity, Fibonacci meander.

(Concerned with sequences A000108, A001006, A001519, A004148, A005043, A034839, A045891, A051291, A089732, A093128, A098086, A098156, A099036, A110320, A175657, A201631, and A203611.)

Received September 6 2022; revised versions received March 3 2023, March 6 2023, March 8 2023, March 9 2023. Published in *Journal of Integer Sequences*, March 9 2023.

Return to Journal of Integer Sequences home page.

Advances in Applied Mathematics 149 (2023) 102551

Contents lists available at ScienceDirect

Advances in Applied Mathematics

journal homepage: www.elsevier.com/locate/yaama

Descent distribution on Catalan words avoiding ordered pairs of relations

Jean-Luc Baril [a], José L. Ramírez [b,*]

[a] LIB, Université de Bourgogne, B.P. 47 870, 21078 DIJON-Cedex, France
[b] Departamento de Matemáticas, Universidad Nacional de Colombia, Bogotá, Colombia

ARTICLE INFO

Article history:
Received 24 February 2023
Received in revised form 22 April 2023
Accepted 30 April 2023
Available online 12 May 2023

MSC:
05A15
05A05

Keywords:
Catalan word
Consecutive pattern
Pattern avoidance
Descent

ABSTRACT

This work is a continuation of some recent articles presenting enumerative results for Catalan words avoiding one or a pair of consecutive or classical patterns of length 3. More precisely, we provide systematically the bivariate generating function for the number of Catalan words avoiding a given pair of relations with respect to the length and the number of descents. We also present several constructive bijections preserving the number of descents. As a byproduct, we deduce the generating function for the total number of descents on all Catalan words of a given length and avoiding a pair of ordered relations.
© 2023 The Author(s). Published by Elsevier Inc. This is an open access article under the CC BY-NC-ND license (http://creativecommons.org/licenses/by-nc-nd/4.0/).

1. Introduction

A word $w = w_1 w_2 \cdots w_n$ over the set of non-negative integers is called a *Catalan word* if $w_1 = 0$ and $0 \leq w_i \leq w_{i-1} + 1$ for $i = 2, \ldots, n$. Let $\mathcal{C}_n$ denote the set of the

* Corresponding author.
E-mail addresses: barjl@u-bourgogne.fr (J.-L. Baril), jlramirezr@unal.edu.co (J.L. Ramírez).

https://doi.org/10.1016/j.aam.2023.102551
0196-8858/© 2023 The Author(s). Published by Elsevier Inc. This is an open access article under the CC BY-NC-ND license (http://creativecommons.org/licenses/by-nc-nd/4.0/).

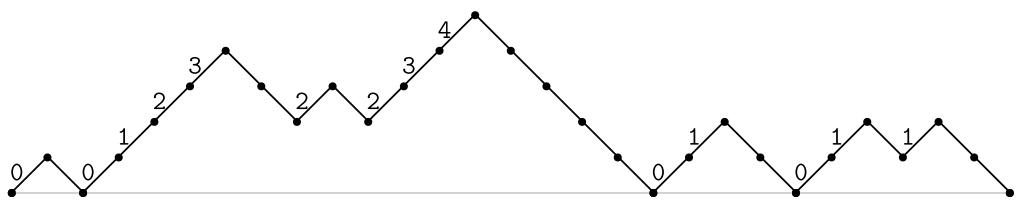


Fig. 1. Dyck path of the Catalan word `00123223401011`.

Catalan words of length n. The cardinality of the set $\mathcal{C}_n$ is given by the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$, see [16, Exercise 80]. A *Dyck path* of semilength n is a lattice path of $\mathbb{Z} \times \mathbb{Z}$ running from $(0,0)$ to $(2n,0)$ that never passes below the x-axis and whose permitted steps are $U = (1,1)$ and $D = (1,-1)$. For a Dyck path of semilength n, we associate a Catalan word in $\mathcal{C}_n$ formed by the y-coordinate of each initial point of the up steps. This construction is a bijection. For example, in Fig. 1 we show the Dyck path associated to the Catalan word $\mathtt{00123223401011} \in \mathcal{C}_{14}$.

Catalan words have already been studied in the context of exhaustive generation of Gray codes for growth-restricted words [11]. More recently, Baril et al. [5,6] study the distribution of descents on restricted Catalan words avoiding a pattern or a pair of patterns of length at most three. Ramírez and Rojas [13] also study the distribution of descents for Catalan words avoiding consecutive patterns of length at most three. Baril, González, and Ramírez [3] enumerate Catalan words avoiding a classical pattern of length at most three according to the length and the value of the last symbol. They also give the exact value or an asymptotic for the expectation of the last symbol. Also, we refer to [7,10], where the authors study several combinatorial statistics on the polyominoes associated with words in $\mathcal{C}_n$. The goal of this work is to complement all these studies by providing enumerative results for Catalan words avoiding a set of consecutive patterns defined from a pair of relations (see below for a formal definition), with respect to the length and the number of descents.

The remaining of this paper is structured as follows. In Section 2, we introduce the notation that will be used in this work. In Section 3, we provide enumerative results for the number of Catalan words avoiding a pattern defined by an ordered pairs of relations. More precisely, in Section 3.1, we focus on the *constant cases*, i.e., pattern avoidances inducing a finite number of Catalan words independently of the length. All other sections handle exhaustively the remaining cases, by providing bivariate generating functions with respect to the length and the number of descents. We also deduce the generating function with respect to the length for the total number of descents in Catalan words avoiding a given pattern. Below, Tables 1 and 2 present an exhaustive list of the sequences counting Catalan words that avoid an ordered pair of relations. Notice that the avoidance of an ordered pairs of relation can be equivalent to the avoidance of a consecutive pattern of length three (for instance, $(<,<)$ is equivalent to $\underline{012}$). In this case, the patterns were already studied in [13].

Table 1
Number of Catalan words avoiding an ordered pair of relations: the constant cases.

(X, Y)	Cardinality of $\mathcal{C}_n(X, Y)$, $n \geq 1$	References
$(\leq, \geq), (\leq, \neq)$	1, 2, 2, 2, ...	Section 3.1
$(\leq, \leq)$	1, 2, 1, 2, 1, 2, ...	Section 3.1
$(\neq, \leq)$	1, 2, 3, 3, 3, ...	Section 3.1

2. Notation

For an integer $r \geq 2$, a *consecutive pattern* $p = \underline{p_1 p_2 \cdots p_r}$ is a word (of length r) over the set $\{0, 1, \ldots, r-1\}$ satisfying the condition: if $j > 0$ appears in p, then $j-1$ also appears in p. A Catalan word $w = w_1 w_2 \cdots w_n$ *contains* the consecutive pattern $p = \underline{p_1 p_2 \cdots p_r}$ if there exists a subsequence $w_i w_{i+1} \cdots w_{i+r-1}$ (for some $i \geq 1$) of w which is order-isomorphic to $p_1 p_2 \ldots p_r$. We say that w *avoids* the consecutive pattern p whenever w does not contain the consecutive pattern p. For example, the Catalan word 0123455543 avoids the consecutive pattern $\underline{001}$ and contains one subsequence isomorphic to the pattern $\underline{210}$. More generally, we consider pattern p as an ordered pair $p = (X, Y)$ of relations X and Y lying into the set $\{<, >, \leq, \geq, =, \neq\}$ (see for instance, Corteel et al. [8], Martinez and Savage [12], and Auli and Elizalde [1]). We will say that a Catalan word w contains the pattern $p = (X, Y)$ if there exists $i \geq 1$ such that $w_i \ X \ w_{i+1}$ and $w_{i+1} \ Y \ w_{i+2}$. As an example, the pattern $(\neq, \geq)$ appears twice in the Catalan word 0123112 on the triplets 231 and 311. Notice that the avoidance of $(\neq, \geq)$ on Catalan words is equivalent to the avoidance of the four consecutive patterns $\underline{010}$, $\underline{011}$, $\underline{100}$, and $\underline{210}$. On the other hand, the avoidance of $(<, <)$ is equivalent to the consecutive pattern $\underline{012}$.

For $n \geq 0$ and for a given consecutive pattern p or an ordered pair of relations $p = (X, Y)$, let $\mathcal{C}_n(p)$ denote the set of Catalan words of length n avoiding the consecutive pattern p. We denote by $\boldsymbol{c}_p(n)$ the cardinality of $\mathcal{C}_n(p)$, and we set $\mathcal{C}(p) := \bigcup_{n \geq 0} \mathcal{C}_n(p)$. We denote by $\mathsf{des}(w)$ the number of descents in w, i.e., the number of indices $i \geq 1$ such that $w_i > w_{i+1}$. Let $\mathcal{C}_{n,k}(p)$ denote the set of Catalan words $w \in \mathcal{C}_n(p)$ such that $\mathsf{des}(w) = k$, and let $\boldsymbol{c}_p(n, k) := |\mathcal{C}_{n,k}(p)|$. Obviously, we have $\boldsymbol{c}_p(n) = \sum_{k=0}^{n-1} \boldsymbol{c}_p(n, k)$. We introduce the bivariate generating function

$$\boldsymbol{C}_p(x, y) := \sum_{w \in \mathcal{C}(p)} x^{|w|} y^{\mathsf{des}(w)} = \sum_{n,k \geq 0} \boldsymbol{c}_p(n, k) x^n y^k,$$

and we set

$$\boldsymbol{C}_p(x) := \sum_{w \in \mathcal{C}(p)} x^{|w|} = \boldsymbol{C}_p(x, 1).$$

The generating function for the total number of descents over all words in $\mathcal{C}_n(p)$ is given by

Table 2
Number of Catalan words avoiding an ordered pair of relations.

(X,Y)	Cardinality of $\mathcal{C}_n(X,Y)$, $n \geq 1$	OEIS	References
$(=,=)$	$\boldsymbol{c}_{\underline{000}}(n) = \sum_{k=1}^{n} \binom{k}{n-k} m_{k-1}$	A247333	Theorem 2.8 of [13], seq. $\boldsymbol{c}_{\underline{000}}(n)$
$(=,\geq)$	1, 2, 4, 10, 26, 72, 206, 606, 1820, 5558, ...	A102407	Section 3.2
$(\geq,=)$	1, 2, 4, 10, 26, 72, 206, 606, 1820, 5558, ...	A102407	Section 3.2
$(=,>)$	$\boldsymbol{C}_{\underline{110}}(x) = \frac{1-2x^2-\sqrt{1-4x+4x^3}}{2x(1-x)}$	A087626	Theorem 2.6 of [13], seq. $\boldsymbol{c}_{\underline{110}}(n)$
$(>,=)$	$\boldsymbol{C}_{\underline{100}}(x) = \frac{1-2x^2-\sqrt{1-4x+4x^3}}{2x(1-x)}$	A087626	Theorem 2.6 of [13], seq. $\boldsymbol{c}_{\underline{100}}(n)$
$(=,\leq)$	1, 2, 3, 7, 17, 43, 114, 310, 861, 2433, ...	A143013	Section 3.3
$(\leq,=)$	1, 2, 3, 7, 17, 43, 114, 310, 861, 2433, ...	A143013	Section 3.3
$(=,<)$	$\boldsymbol{c}_{\underline{001}}(n) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \frac{(-1)^k}{n-k} \binom{n-k}{k} \binom{2n-3k}{n-2k-1}$	A105633	Theorem 2.3 of [13], seq. $\boldsymbol{c}_{\underline{001}}(n)$
$(<,=)$	$\boldsymbol{c}_{\underline{011}}(n) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \frac{(-1)^k}{n-k} \binom{n-k}{k} \binom{2n-3k}{n-2k-1}$	A105633	Theorem 2.4 of [13], seq. $\boldsymbol{c}_{\underline{011}}(n)$
$(<,>)$	$\sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \frac{(-1)^k}{n-k} \binom{n-k}{k} \binom{2n-3k}{n-2k-1}$	A105633	Section 3.4
$(=,\neq)$	1, 2, 4, 8, 17, 38, 89, 216, 539, 1374, ...	A086615	Section 3.5
$(\neq,=)$	1, 2, 4, 8, 17, 38, 89, 216, 539, 1374, ...	A086615	Section 3.5
$(\geq,\geq)$	m_n (Motzkin numbers)	A001006	Section 3.6
$(<,<)$	m_n (Motzkin numbers)	A001006	Theorem 2.1 of [13], seq. $\boldsymbol{c}_n(\underline{012})$
$(\geq,>)$	1, 2, 5, 13, 35, 97, 275, 794, 2327, 6905, ...	A082582	Section 3.7
$(>,\geq)$	1, 2, 5, 13, 35, 97, 275, 794, 2327, 6905, ...	A082582	Section 3.7
$(>,<)$	1, 2, 5, 13, 35, 97, 275, 794, 2327, 6905, ...	A082582	Section 3.7
$(\geq,\leq)$	F_{n+1} (Fibonacci number)	A000045	Section 3.8
$(\leq,<)$	F_{n+1} (Fibonacci number)	A000045	Section 3.8
$(<,\leq)$	F_{n+1} (Fibonacci number)	A000045	Section 3.8
$(\geq,<)$	2^{n-1}	A011782	Section 3.9
$(\leq,>)$	2^{n-1}	A011782	Section 3.9
$(\geq,\neq)$	$\binom{n}{2} + 1$	A000124	Section 3.10
$(>,>)$	$\boldsymbol{c}_{\underline{210}}(n) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{1}{n-k} \binom{n-k}{k} \binom{n-k}{k+1} 2^{n-2k-1}$	A159771	Theorem 2.9 of [13], seq. $\boldsymbol{c}_{\underline{210}}(n)$
$(>,\leq)$	P_{n+1} (Pell numbers)	A000129	Section 3.11.
$(>,\neq)$	1, 2, 5, 13, 34, 90, 242, 660, 1821, 5073, ...	New	Section 3.12
$(<,\geq)$	n	A000027	Section 3.13
$(\neq,\geq)$	n	A000027	Section 3.13
$(<,\neq)$	1, 2, 3, 6, 12, 25, 54, 119, 267, 608, ...	New	Section 3.14
$(\neq,>)$	1, 2, 4, 9, 22, 56, 146, 388, 1048, 2869, ...	A152225	Section 3.15
$(\neq,<)$	1, 2, 4, 8, 17, 37, 82, 185, 423, 978, ...	A292460	Section 3.16
$(\neq,\neq)$	1, 2, 3, 6, 11, 22, 43, 87, 176, 362, ...	A026418	Section 3.17

$$\boldsymbol{D}_p(x) := \left.\frac{\partial \boldsymbol{C}_p(x,y)}{\partial y}\right|_{y=1}.$$

Throughout this work, we will often use the first return decomposition of a Catalan word w, which is $w = \mathtt{0}(w'+1)w''$, where w' and w'' are Catalan words, and where $(w'+1)$ is the word obtained from w' by adding 1 at all these symbols (for instance if $w' = \mathtt{012012}$ then $(w'+1) = \mathtt{123123}$). As an example, the first return decomposition of $w = \mathtt{0122123011201}$ is given by setting $w' = \mathtt{011012}$ and $w'' = \mathtt{011201}$.

3. Enumeration

Section 3.1 handles pattern avoidances inducing a finite number of Catalan words independently of the length. Sections 3.1 to 3.17 handle exhaustively the other patterns.

3.1. Constant cases

The avoidance of $(\leq, \geq)$ on Catalan words is equivalent to the avoidance of $\underline{000}$, $\underline{010}$, $\underline{011}$, $\underline{110}$, and $\underline{120}$. So, Catalan words in $\mathcal{C}_n(\leq, \geq)$ are necessarily of the form $\mathtt{01}\cdots\mathtt{n}$ or $\mathtt{001}\cdots\mathtt{(n-1)}$. Therefore $\boldsymbol{c}_{(\leq,\geq)}(n) = 2$ for all $n \geq 2$.

The avoidance of $(\leq, \neq)$ on Catalan words is equivalent to the avoidance of $\underline{001}$, $\underline{010}$, $\underline{012}$, $\underline{110}$, and $\underline{120}$. So, we necessarily have $\mathcal{C}_1(\leq, \neq) = \{\mathtt{0}\}$ and $\mathcal{C}_n(\leq, \neq) = \{\mathtt{0}^n, \mathtt{01}^{n-1}\}$ for $n \geq 2$. Therefore $\boldsymbol{c}_{(\leq,\neq)}(n) = 2$ for all $n \geq 2$.

The avoidance of $(\leq, \leq)$ on Catalan words is equivalent to the avoidance of $\underline{000}$, $\underline{001}$, $\underline{011}$, and $\underline{012}$. So, Catalan words in $\mathcal{C}_n(\leq, \leq)$ are words made up of alternating zeros and ones except, possibly, for the last symbol, that is

$$\mathcal{C}_n(\leq, \leq) = \begin{cases} (\mathtt{01})^{n/2} \text{ or } (\mathtt{01})^{(n-2)/2}\mathtt{00}, & \text{if } n \text{ is even;} \\ (\mathtt{01})^{(n-1)/2}\mathtt{0}, & \text{if } n \text{ is odd.} \end{cases}$$

Therefore we obtain

$$\boldsymbol{c}_{(\leq,\leq)}(n) = \begin{cases} 2, & \text{if } n \text{ is even;} \\ 1, & \text{if } n \text{ is odd.} \end{cases}$$

Finally, the avoidance of $(\neq, \leq)$ on Catalan words is equivalent to the avoidance of $\underline{011}$, $\underline{012}$, $\underline{100}$, $\underline{101}$, and $\underline{201}$. So, we have $\mathcal{C}_1(\neq, \leq) = \{\mathtt{0}\}$, $\mathcal{C}_2(\neq, \leq) = \{\mathtt{00}, \mathtt{01}\}$, and for $n \geq 3$ $\mathcal{C}_n(\neq, \leq) = \{\mathtt{0}^n, \mathtt{0}^{n-1}\mathtt{1}, \mathtt{0}^{n-2}\mathtt{10}\}$. Therefore,

$$\boldsymbol{c}_{(\neq,\leq)}(n) = \begin{cases} 1, & \text{if } n = 1; \\ 2, & \text{if } n = 2; \\ 3, & \text{if } n \geq 3. \end{cases}$$

We refer to Table 1 for a summary of the results obtained in this part.

3.2. Cases $\mathcal{C}(=,\geq)$ and $\mathcal{C}(\geq,=)$

The avoidance of $(=,\geq)$ (resp. $(\geq,=)$) on Catalan words is equivalent to the avoidance of $\underline{000}$ and $\underline{110}$ (resp. $\underline{000}$ and $\underline{100}$). In [2], the authors established a bijection between Catalan words avoiding $\underline{100}$ and those avoiding $\underline{110}$: from left to right, we replace each maximal factor $k^j(k-\ell)$, $j \geq 2$, $\ell \geq 1$, with the factor $k(k-\ell)^j$. This bijection preserves the number of descents and the avoidance of $\underline{000}$. For example, for $w = \mathtt{0122123300} \in \mathcal{C}_{10}(\geq,=)$, we have the transformation

$$\mathtt{0122123300} \to \mathtt{0121123300} \to \mathtt{0121123000} \in \mathcal{C}_{10}(=,\geq).$$

Therefore, we necessarily have $\boldsymbol{C}_{(\geq,=)}(x,y) = \boldsymbol{C}_{(=,\geq)}(x,y)$, and below, we focus on the pattern $(=,\geq)$.

Theorem 3.1. *We have*

$$\boldsymbol{C}_1(x,y) := \boldsymbol{C}_{(=,\geq)}(x,y) = \frac{1-x-x^2+2xy-\sqrt{(1-x-x^2+2xy)^2-4xy(1+xy)}}{2xy}.$$

Proof. Let w denote a non-empty Catalan word in $\mathcal{C}(=,\geq) = \mathcal{C}(\underline{000},\underline{110})$, and let $w = \mathtt{0}(w'+1)w''$ be the first return decomposition, where $w', w'' \in \mathcal{C}(\underline{000},\underline{110})$. If $w'' = \epsilon$, then $w = \mathtt{0}(w'+1)$ with w' possibly empty. The generating function for this case is $x\boldsymbol{C}_1(x,y)$. If w'' is non-empty and $w' = \epsilon$, then w'' cannot start with $\mathtt{00}$. The corresponding generating function is $xA(x,y)$, where $A(x,y)$ is the bivariate generating function for non-empty Catalan words in $\mathcal{C}(\underline{000},\underline{110})$ that do not start with $\mathtt{00}$. Counting using the complement we have

$$A(x,y) = \boldsymbol{C}_1(x,y) - 1 - \left[x^2\boldsymbol{C}_1(x,y) + x^2y(\boldsymbol{C}_1(x,y) - 1 - B(x,y))(\boldsymbol{C}_1(x,y)-1)\right],$$

where $B(x,y)$ is the bivariate generating function for Catalan words in $\mathcal{C}(=,\geq)$ that end with aa for any $a \geq 0$ (we have to subtract this generating function to ensure the avoidance of $\underline{110}$). Since any word counted by $B(x,y)$ is obtained from a non-empty Catalan word in $\mathcal{C}(=,\geq)$ duplicating the last symbol unless the word ends with two repeated symbols, so this last generating function satisfies $B(x,y) = x(\boldsymbol{C}_1(x,y) - 1 - B(x,y))$.

If w' and w'' are non-empty, then the generating function of this case is

$$E(x,y) := xy(\boldsymbol{C}_1(x,y) - 1 - B(x,y))(\boldsymbol{C}_1(x,y)-1).$$

Therefore, we have the functional equation

$$\boldsymbol{C}_1(x,y) = 1 + x\boldsymbol{C}_1(x,y) + xA(x,y) + E(x,y).$$

Solving this system of equations we obtain the desired result. □

The series expansion of the generating function $\boldsymbol{C}_1(x,y)$ is

$$1+x+2x^2+(3+y)x^3+(\mathbf{5}+\mathbf{5}y)x^4+(8+16y+2y^2)x^5 \\ +(13+43y+16y^2)x^6+O(x^7).$$

The Catalan words corresponding to the bold coefficients in the above series are

$$\mathcal{C}_4(=,\geq)=\mathcal{C}_4(\underline{000},\underline{110})=\{\mathtt{0010},\mathtt{0011},\mathtt{0012},\mathtt{0100},\mathtt{0101},\mathtt{0112},\mathtt{0120},\mathtt{0121},\mathtt{0122},\mathtt{0123}\}.$$

Corollary 3.2. *The g.f. for the cardinality of $\mathcal{C}(=,\geq)$ with respect to the length is*

$$\boldsymbol{C}_{(=,\geq)}(x)=\frac{1+x-x^2-\sqrt{1-2x-5x^2-2x^3+x^4}}{2x}.$$

Using the bijection given in Introduction between Catalan words and Dyck path, it is clear that the sequence $\boldsymbol{c}_{(=,\geq)}(n)=\boldsymbol{c}_{\underline{000},\underline{100}}(n)$ also counts the number of all Dyck paths of semilength n that avoid $DUDU$ (sequence A102407). Then, using [14] we have the combinatorial expression

$$\boldsymbol{c}_{(=,\geq)}(n)=\boldsymbol{c}_{\underline{000},\underline{110}}(n)=\sum_{j=0}^{\lfloor\frac{n}{2}\rfloor}\frac{1}{n-j}\binom{n-j}{j}\sum_{i=0}^{n-2j}\binom{n-2j}{i}\binom{j+i}{n-2j-i+1},\ n\geq 1.$$

Corollary 3.3. *The g.f. for the total number of descents on $\mathcal{C}(=,\geq)$ is*

$$\boldsymbol{D}_{(=,\geq)}(x)=\frac{1-2x-3x^2+x^4-(1-x-x^2)\sqrt{1-2x-5x^2-2x^3+x^4}}{2x\sqrt{1-2x-5x^2-2x^3+x^4}}.$$

The series expansion of $\boldsymbol{D}_{(=,\geq)}(x)$ is

$$x^3+5x^4+20x^5+75x^6+271x^7+964x^8+3397x^9+O(x^{10}),$$

where the coefficient sequence does not appear in [15].

3.3. Cases $\mathcal{C}(=,\leq)$ and $\mathcal{C}(\leq,=)$

The avoidance of $(=,\leq)$ (resp. $(\leq,=)$) on Catalan words is equivalent to the avoidance of $\underline{000}$ and $\underline{001}$ (resp. $\underline{000}$ and $\underline{011}$). In [13] (see Theorem 2.4), the authors established a bijection between Catalan words avoiding $\underline{011}$ and those avoiding $\underline{001}$. From left to right, they replace each factor $k^j(k+1)$ with the factor $k(k+1)^j$ $(j\geq 2)$. This bijection preserves the number of descents and the avoidance of $\underline{000}$. Therefore, we necessarily have $\boldsymbol{C}_2(x,y):=\boldsymbol{C}_{(\leq,=)}(x,y)=\boldsymbol{C}_{(=,\leq)}(x,y)$.

Theorem 3.4. *We have*

$$\boldsymbol{C}_2(x,y) := \boldsymbol{C}_{(=,\leq)}(x,y) = \frac{1-x+2xy-\sqrt{1-2x+x^2-4x^2y-4x^3y}}{2xy}.$$

Proof. Let w denote a non-empty Catalan word in $\mathcal{C}(=,\leq) = \mathcal{C}(\underline{000},\underline{001})$, and let $w = \mathtt{0}(w'+1)w''$ be the first return decomposition, where $w', w'' \in \mathcal{C}(=,\leq)$. If $w'' = \epsilon$, then $w = \mathtt{0}(w'+1)$ with w' possibly empty. The generating function for this case is $x\boldsymbol{C}_2(x,y)$. If w'' is non-empty and $w' = \epsilon$, then w'' cannot start with $\mathtt{00}$ or $\mathtt{01}$, so $w'' = \mathtt{0}$. The corresponding generating function is x^2. If w' and w'' are non-empty, then the generating function is $xy(\boldsymbol{C}_2(x,y)-1)^2$. Therefore, we have the functional equation

$$\boldsymbol{C}_2(x,y) = 1 + x\boldsymbol{C}_2(x,y) + x^2 + xy(\boldsymbol{C}_2(x,y)-1)^2.$$

Solving this equation we obtain the desired result. □

The series expansion of the generating function $\boldsymbol{C}_2(x,y)$ is

$$\begin{aligned}&1 + x + 2x^2 + (2+y)x^3 + (\mathbf{2}+\mathbf{5}y)x^4 + (2+13y+2y^2)x^5\\&\quad + (2+25y+16y^2)x^6 + O(x^7).\end{aligned}$$

The Catalan words corresponding to the bold coefficients in the above series are

$$\mathcal{C}_4(=,\leq) = \{\mathtt{0100}, \mathtt{0101}, \mathtt{0110}, \mathtt{0120}, \mathtt{0121}, \mathtt{0122}, \mathtt{0123}\}.$$

Corollary 3.5. *The g.f. for the cardinality of $\mathcal{C}(=,\leq)$ with respect to the length is*

$$\boldsymbol{C}_{(=,\leq)}(x) = \frac{1+x-\sqrt{1-2x-3x^2-4x^3}}{2x}.$$

This generating function coincides with that of the sequence A143013, that is

$$\boldsymbol{c}_{(=,\leq)}(n) = \boldsymbol{c}_{\underline{000},\underline{001}}(n) = \sum_{i=0}^{n}\sum_{k=1}^{n-i+1}\frac{1}{k}\binom{i-1}{k-1}\binom{k}{n-k-i+1}\binom{k+i-2}{i-1},\ n \geq 1.$$

Notice that sequence A143013 counts also the number of Motzkin paths with two kinds of level steps one of which is a final step.

Corollary 3.6. *The g.f. for the total number of descents on $\mathcal{C}(=,\leq)$ is*

$$\boldsymbol{D}_{(=,\leq)}(x) = \frac{1-2x-x^2-2x^3-(1-x)\sqrt{1-2x-3x^2-4x^3}}{2x\sqrt{1-2x-3x^2-4x^3}}.$$

The series expansion of $\boldsymbol{D}_{(=,\leq)}(x)$ is

$$x^3 + 5x^4 + 17x^5 + 57x^6 + 188x^7 + 610x^8 + 1971x^9 + O(x^{10}),$$

where the coefficient sequence does not appear in [15].

3.4. Cases $(=,<)$, $(<,=)$, and $(<,>)$

Obviously, we have $\mathcal{C}(=,<) = \mathcal{C}(\underline{001})$, $\mathcal{C}(<,=) = \mathcal{C}(\underline{011})$, and $\mathcal{C}(<,>) = \mathcal{C}(\underline{010}, \underline{120})$. With the same bijection used at the beginning of Section 3.3, we have $\boldsymbol{C}_{(=,<)}(x,y) = \boldsymbol{C}_{(<,=)}(x,y)$, and we refer to [13] to see an expression of this generating function. On the other hand, a non-empty Catalan word in $\mathcal{C}(<,>)$ is either of the form (i) $\mathtt{0}\alpha$ with $\alpha \in \mathcal{C}(<,>)$, (ii) $\mathtt{0}(\alpha+1)$ with $\alpha \in \mathcal{C}(<,>)$, $\alpha \neq \epsilon$, or (iii) $\mathtt{0}(\alpha+1)\beta$, where α ends with $a(a+1)$ and $\beta \in \mathcal{C}(<,>)$, $\beta \neq \epsilon$. We deduce the functional equation

$$\begin{aligned}\boldsymbol{C}_{(<,>)}(x) = 1 + x\boldsymbol{C}_{(<,>)}(x) + x(\boldsymbol{C}_{(<,>)}(x) - 1)\\ + x(\boldsymbol{C}_{(<,>)}(x) - 1)(\boldsymbol{C}_{(<,>)}(x) - 1 - x - x(\boldsymbol{C}_{(<,>)}(x) - 1)),\end{aligned}$$

which proves that $\boldsymbol{C}_{(<,>)}(x) = \boldsymbol{C}_{(=,<)}(x) = \boldsymbol{C}_{(<,=)}(x)$. Then, the sets $\mathcal{C}(\underline{011}) = \mathcal{C}(<,=)$ and $\boldsymbol{C}(<,>) = \boldsymbol{C}(\underline{010}, \underline{120})$ are in one-to-one correspondence, but the number of descents cannot be preserved (see for instance the list of Catalan words of length 3).

Below, we focus on the descent distribution over $\mathcal{C}(<,>)$.

Theorem 3.7. *We have*

$$\boldsymbol{C}_3(x,y) := \boldsymbol{C}_{(<,>)}(x,y) = \frac{1 - 2x + 2xy - x^2y - \sqrt{1 - 4x + 4x^2 - 2x^2y + x^4y^2}}{2xy(1-x)}.$$

Proof. Let w denote a non-empty Catalan word in $\mathcal{C}(<,>) = \mathcal{C}(\underline{010}, \underline{120})$. Let $w = \mathtt{0}(w'+1)w''$ be the first return decomposition, where $w', w'' \in \mathcal{C}(<,>)$. If $w'' = \epsilon$, then $w = \mathtt{0}(w'+1)$ with w' possibly empty. The generating function for this case is $x\boldsymbol{C}_3(x,y)$. If w'' is non-empty and $w' = \epsilon$, then w'' is any word of $\mathcal{C}(<,>)$, and the corresponding generating function is $x(\boldsymbol{C}_3(x,y) - 1)$. If w' and w'' are non-empty, then $w' \neq \mathtt{0}$ or w' does not end with an ascent $a(a+1)$, where $a \geq 0$. The generating function is

$$E(x,y) := xy(\boldsymbol{C}_3(x,y) - 1 - x - B(x,y))(\boldsymbol{C}_3(x,y) - 1),$$

where $B(x,y)$ is the generating function for the Catalan words in $\mathcal{C}(<,>)$ ending with an ascent. Since any word counted by $B(x,y)$ is obtained from a non-empty Catalan word duplicating the last symbol plus 1, this generating function is given by $B(x,y) = x(\boldsymbol{C}_3(x,y) - 1)$. Therefore, we have the functional equation

$$\boldsymbol{C}_3(x,y) = 1 + x\boldsymbol{C}_3(x,y) + x(\boldsymbol{C}_3(x,y) - 1) + E(x,y).$$

Solving this system of equations we obtain the desired result. □

The series expansion of the generating function $\boldsymbol{C}_3(x,y)$ is

$$1+x+2x^2+4x^3+(\mathbf{8}+\boldsymbol{y})x^4+(16+6y)x^5+(32+24y+y^2)x^6+O(x^7).$$

The Catalan words corresponding to the bold coefficients in the above series are

$$\mathcal{C}_4(<,>)=\{\mathtt{0000},\mathtt{0001},\mathtt{0011},\mathtt{0012},\mathtt{0110},\mathtt{0111},\mathtt{0112},\mathtt{0122},\mathtt{0123}\}.$$

Corollary 3.8. *The g.f. for the cardinality of $\mathcal{C}(<,>)$ with respect to the length is*

$$\boldsymbol{C}_{(<,>)}(x)=\frac{1-x^2-\sqrt{1-4x+2x^2+x^4}}{2(1-x)x}.$$

This generating function coincides with the generating function of the sequence A105633, and

$$\boldsymbol{c}_{(<,>)}(n)=\boldsymbol{c}_{\underline{010},\underline{120}}(n)=\sum_{k=0}^{\lfloor (n-1)/2\rfloor}\frac{(-1)^k}{n-k}\binom{n-k}{k}\binom{2n-3k}{n-2k-1},\ n\geq 1.$$

Using the bijection given in Introduction between Catalan words and Dyck path, the sequence $\boldsymbol{c}_{(<,=)}(n)=\boldsymbol{c}_{\underline{011}}(n)$ counts the Dyck paths of semilength n avoiding $UUDU$ and $\boldsymbol{c}_{(=,<)}(n)=\boldsymbol{c}_{\underline{001}}(n)$ counts the Dyck paths of semilength n avoiding $UDUU$ (cf. [14]). Notice that the sequence A105633 also counts the number of Dyck paths of semilength $n+1$ with no pairs of consecutive valleys at the same height. More generally, it is proved in [9] that the generating function of the sequence

$$\boldsymbol{c}_{\gamma=\underline{012\cdots k\ell(\ell+1)\cdots(\ell+s)}}(n),\quad 0\leq \ell\leq k \text{ and } s\geq 1,$$

satisfies the functional equation

$$\boldsymbol{C}_\gamma(x)=1+x\boldsymbol{C}_\gamma(x)^2-x^M\boldsymbol{C}_\gamma(x)^{M-(k+1-\ell)}\left(\boldsymbol{C}_\gamma(x)-\frac{1-(x\boldsymbol{C}_\gamma(x))^m}{1-x\boldsymbol{C}_\gamma(x)}\right),$$

where $M=\max\{k+1,s\}$ and $m=\min\{k+1,s\}$ (when $k=0$, $\ell=0$ and $s=1$ we retrieve the generating function $\boldsymbol{C}_{(=,<)}(x)=\boldsymbol{C}_{(<,>)}(x)$).

Corollary 3.9. *The g.f. for the total number of descents on $\mathcal{C}(<,>)$ is*

$$\boldsymbol{D}_{(<,>)}(x)=\frac{1-4x+3x^2-(1-2x)\sqrt{1-4x+2x^2+x^4}}{2(1-x)x\sqrt{1-4x+2x^2+x^4}}.$$

The series expansion of $\boldsymbol{D}_{(<,>)}(x)$ is

$$x^4 + 6x^5 + 26x^6 + 100x^7 + 363x^8 + 1277x^9 + O(x^{10}),$$

where the coefficient sequence does not appear in [15].

3.5. *Cases $\mathcal{C}(=,\neq)$ and $\mathcal{C}(\neq,=)$*

The avoidance of $(=,\neq)$ (resp. $(\neq,=)$) on Catalan words is equivalent to the avoidance of $\underline{001}$ and $\underline{110}$ (resp. $\underline{100}$ and $\underline{011}$). The following map is a bijection from $\mathcal{C}(=,\neq)$ to $\mathcal{C}(\neq,=)$ and it preserves the number of descents: crossing $\sigma \in \mathcal{C}(=,\neq)$ from right to left, we replace each factor $k(k+1)^j$ with the factor $k^j(k+1)$, $j \geq 2$, and we replace each factor $k(k-\ell)^j$ with the factor $k^j(k-\ell)$. Therefore, we necessarily have $\boldsymbol{C}_{(=,\neq)}(x,y) = \boldsymbol{C}_{(\neq,=)}(x,y)$. For example, for the word $\mathtt{01012323412300} \in \mathcal{C}_{14}(=,\neq)$, we have the transformation:

$$\begin{aligned}
&\mathtt{01012323412300} \to \mathtt{01012323412330} \to \mathtt{01012323412230}\\
&\qquad \to \mathtt{01012323411230} \to \mathtt{01012323441230} \to \mathtt{01012323341230}\\
&\qquad \to \mathtt{01012322341230} \to \mathtt{01012332341230} \to \mathtt{01012232341230} \to \mathtt{01011232341230}\\
&\qquad \to \mathtt{01001232341230} \to \mathtt{01101232341230} \to \mathtt{00101232341230} \in \mathcal{C}_{14}(\neq,=).
\end{aligned}$$

Theorem 3.10. *We have*

$$\boldsymbol{C}_4(x,y) := \boldsymbol{C}_{(=,\neq)}(x,y) = \frac{1 - x + 2xy - 2x^2y - \sqrt{1 - 2x + x^2 - 4x^2y}}{2xy(1-x)}.$$

Proof. Let w denote a non-empty Catalan word in $\mathcal{C}(=,\neq) = \mathcal{C}(\underline{001},\underline{110})$, and let $w = \mathtt{0}(w'+1)w''$ be the first return decomposition, where $w', w'' \in \mathcal{C}(=,\neq)$. If $w'' = \epsilon$, then $w = \mathtt{0}(w'+1)$ with w' possibly empty. The generating function for this case is $x\boldsymbol{C}_4(x,y)$. If w'' is non-empty and $w' = \epsilon$, then $w'' = 0^j$ for any $j \geq 1$ (to avoid $\underline{001}$). The corresponding generating function is $x(x/(1-x))$. If w' and w'' are non-empty, then the generating function is

$$E(x,y) := xy(\boldsymbol{C}_4(x,y) - 1 - B(x,y))(\boldsymbol{C}_4(x,y) - 1),$$

where $B(x,y)$ is the bivariate generating function for Catalan words in $\mathcal{C}(=,\neq)$ that do not end with aa $(a \geq 0)$. It is clear that $B(x,y) = x(\boldsymbol{C}_4(x,y) - 1)$. Therefore, we have the functional equation

$$\boldsymbol{C}_4(x,y) = 1 + x\boldsymbol{C}_4(x,y) + \frac{x^2}{1-x} + E(x,y).$$

Solving this system of equations we obtain the desired result. $\square$

The series expansion of the generating function $\boldsymbol{C}_4(x,y)$ is

$$1+x+2x^2+(3+y)x^3+(\mathbf{4}+\mathbf{4}y)x^4+(5+10y+2y^2)x^5 \\ +(6+20y+12y^2)x^6+O(x^7).$$

The Catalan words corresponding to the bold coefficients in the above series are

$$\mathcal{C}_4(=,\neq)=\{\mathtt{0000},\mathtt{0100},\mathtt{0101},\mathtt{0111},\mathtt{0120},\mathtt{0121},\mathtt{0122},\mathtt{0123}\}.$$

Corollary 3.11. *The g.f. for the cardinality of $\mathcal{C}(=,\neq)$ with respect to the length is*

$$\boldsymbol{C}_{(=,\neq)}(x)=\frac{1+x-2x^2-\sqrt{1-2x-3x^2}}{2(1-x)x}.$$

This generating function coincides with the generating function of the sequence of partial sums of the Motzkin numbers m_n (see sequence A086615), that is

$$\boldsymbol{c}_{(=,\neq)}(n)=\boldsymbol{c}_{\underline{001},\underline{110}}(n)=\boldsymbol{c}_{\underline{100},\underline{011}}(n)=\sum_{j=0}^{n-1}m_j,\ n\geq 1,$$

where m_n is the n-th Motzkin number.

Corollary 3.12. *The g.f. for the total number of descents on $\mathcal{C}(=,\neq)$ is*

$$\boldsymbol{D}_{(=,\neq)}(x)=\frac{1-2x-x^2-(1-x)\sqrt{1-2x-3x^2}}{2(1-x)x\sqrt{1-2x-3x^2}}.$$

The series expansion of $\boldsymbol{D}_{(=,\neq)}(x)$ is

$$x^3+4x^4+14x^5+44x^6+134x^7+400x^8+1184x^9+O(x^{10}),$$

where the coefficient sequence corresponds to A097894 in [15], which counts the number of peaks at even height in all Motzkin paths of length $n+1$.

3.6. Cases $\mathcal{C}(\geq,\geq)$ and $\mathcal{C}(<,<)$

The avoidance of $(\geq,\geq)$ (resp. $(<,<)$) on Catalan words is equivalent to the avoidance of $\underline{000}$, $\underline{100}$, $\underline{110}$, and $\underline{210}$ (resp. $\underline{012}$). For the pattern $(<,<)=\underline{012}$, we refer to [13] to see an expression of $\boldsymbol{C}_{(<,<)}(x,y)$. Below, we prove that

$$\boldsymbol{C}_5(x,y):=\boldsymbol{C}_{(\geq,\geq)}(x,y)=\boldsymbol{C}_{(<,<)}(x,y),$$

by exhibiting a bijection between $\mathcal{C}(\geq,\geq)$ and $\mathcal{C}(<,<)$ that preserves the length and the number of descents. Let w be a word in $\mathcal{C}(\geq,\geq)$, we distinguish four cases:

$$\phi(w) = \begin{cases} \epsilon & \text{if } w = \epsilon \\ \mathtt{0}\phi(u) & \text{if } w = \mathtt{0}(u+1) \\ \mathtt{01}(\phi(u)+1) & \text{if } w = \mathtt{00}(u+1) \\ \mathtt{01}(\phi(v)+1)\phi(v') & \text{if } w = \mathtt{0}(va+1)v' \end{cases},$$

where u, va, and v' are Catalan words in $\mathcal{C}(\geq,\geq)$, such that u and v are possibly empty, v' is not empty, v ends with $(a-1)$ when v is not empty, and v' does not start with $\mathtt{00}$. Clearly, this map is a bijection from $\mathcal{C}(\geq,\geq)$ to $\mathcal{C}(<,<)$ that preserves the descent number. For instance, the image by ϕ of $w = \mathtt{0123010122} \in \mathcal{C}_{11}(\geq,\geq)$ is

$$\begin{aligned}\phi(w) &= \mathtt{01}\phi(\mathtt{01}) \cdot \phi(\mathtt{010122}) = \mathtt{0111} \cdot \mathtt{01} \cdot \phi(\mathtt{0122}) \\ &= \mathtt{0111010}\phi(\mathtt{011}) = \mathtt{0111010001} \in \mathcal{C}(<,<).\end{aligned}$$

So, we deduce directly the following.

Theorem 3.13. *We have*

$$\boldsymbol{C}_5(x,y) := \boldsymbol{C}_{(\geq,\geq)}(x,y) = \frac{1-x-x^2+x^2y-\sqrt{(1-x-x^2(1+y))^2-4x^3(1+x)y}}{2x^2y}.$$

The series expansion of the generating function $\boldsymbol{C}_5(x,y)$ is

$$1+x+2x^2+(3+y)x^3+(\mathbf{5}+\mathbf{4}y)x^4+(8+12y+y^2)x^5+(13+31y+7y^2)x^6+O(x^7).$$

The Catalan words corresponding to the bold coefficients in the above series are

$$\mathcal{C}_5(\geq,\geq) = \{\mathtt{0010}, \mathtt{0011}, \mathtt{0012}, \mathtt{0101}, \mathtt{0112}, \mathtt{0120}, \mathtt{0121}, \mathtt{0122}, \mathtt{0123}\}.$$

The coefficients of the bivariate generating function $\boldsymbol{C}_5(x,y)$ coincide with the array A114690, which counts the number of Motzkin paths of length n having k weak ascents.

Corollary 3.14. *The g.f. for the cardinality of $\mathcal{C}(\geq,\geq)$ with respect to the length is*

$$\boldsymbol{C}_{(\geq,\geq)}(x) = \frac{1-x-\sqrt{1-2x-3x^2}}{2x^2}.$$

This generating function coincides with the generating function of the Motzkin numbers (sequence A001006) $m_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{2n}{k} C_k$, where C_k is the k-th Catalan number, that is

$$\boldsymbol{c}_{(\geq,\geq)}(n) = \boldsymbol{c}_{\underline{000},\underline{100},\underline{110},\underline{210}}(n) = m_n, \ n \geq 0.$$

Corollary 3.15. *The g.f. for the total number of descents on $\mathcal{C}(\geq,\geq)$ is*

$$\boldsymbol{D}_{(\geq,\geq)}(x) = \frac{1-2x-2x^2+x^3-(1-x-x^2)\sqrt{1-2x-3x^2}}{2x^2\sqrt{1-2x-3x^2}}.$$

The series expansion of $\boldsymbol{D}_{(\geq,\geq)}(x)$ is

$$x^3 + 4x^4 + 14x^5 + 45x^6 + 140x^7 + 427x^8 + 1288x^9 + O(x^{10}),$$

where the coefficients correspond to the sequence A005775 which counts triangular polyominoes of a given number of cells.

3.7. Cases $\mathcal{C}(\geq,>)$, $\mathcal{C}(>,\geq)$, and $\mathcal{C}(>,<)$

The avoidance of $(\geq,>)$ (resp. $(>,\geq)$, resp. $(>,<)$) on Catalan words is equivalent to the avoidance $\underline{110}$ and $\underline{210}$ (resp. $\underline{100}$ and $\underline{210}$, resp. $\underline{201}$ and $\underline{101}$).

From the bijection described in Section 3.2, the sets $\mathcal{C}(\geq,>) = \mathcal{C}(\underline{110},\underline{210})$ and $\mathcal{C}(>,\geq) = \mathcal{C}(\underline{100},\underline{210})$ are in one-to-one correspondence (the bijection preserves also the descent number), which implies that $\boldsymbol{C}_{(\geq,>)}(x,y) = \boldsymbol{C}_{(>,\geq)}(x,y)$.

On the other hand, let us prove that $\boldsymbol{C}_{(\geq,>)}(x,y) = \boldsymbol{C}_{(>,<)}(x,y)$ by exhibiting a bijection ψ preserving the descent number. Let w be a word in $\mathcal{C}(\geq,>)$, then we distinguish five cases:

$$\psi(w) = \begin{cases} \epsilon & \text{if } w = \epsilon \\ \mathtt{0}\psi(u) & \text{if } w = \mathtt{0}u \\ \mathtt{0}(\psi(u)+1) & \text{if } w = \mathtt{0}(u+1) \\ \mathtt{0}(\psi(ua)+1)\mathtt{0}\psi(v) & \text{if } w = \mathtt{0}(ua(a+1)+1)v \\ \mathtt{0}(\psi(v)+1)\mathtt{0} & \text{if } w = \mathtt{01}v \end{cases},$$

where u, $ua(a+1)$, and v are Catalan words in $\mathcal{C}(\geq,>)$, such that u is possibly empty and v is not empty. Clearly, this map is a bijection from $\mathcal{C}(\geq,>)$ to $\mathcal{C}(>,<)$ that preserves the descent number. For example, for the word $\mathtt{01234012343454} \in \mathcal{C}_{14}(\geq,>)$, we have the transformation:

$$\begin{aligned} \mathtt{0{\cdot}1234{\cdot}012343454} &\to \mathtt{0}(\psi(\mathtt{01{\cdot}2})+1)\mathtt{0{\cdot}}\psi(\mathtt{012343454}) \to \mathtt{01}(\psi(\mathtt{01})+2)\mathtt{0{\cdot}0}(\psi(\mathtt{01232343})+1) \\ &\to \mathtt{01}\cdot\mathtt{22}\cdot\mathtt{00}\cdot\mathtt{1}(\psi(\mathtt{0121232})+2) \to \mathtt{0122001}\cdot\mathtt{2}(\psi(\mathtt{010121})+3) \\ &\to \mathtt{01220012}\cdot\mathtt{3}(\psi(\mathtt{0121})+4)\mathtt{3} \to \mathtt{012200123}\cdot\mathtt{4}(\psi(\mathtt{010})+5)\mathtt{3} \\ &\to \mathtt{0122001234}\cdot\mathtt{5}(\psi(\mathtt{0})+6)\mathtt{5}\cdot\mathtt{3} \to \mathtt{01220012345653} \in \mathcal{C}_{14}(>,<). \end{aligned}$$

Theorem 3.16. *We have*

$$\boldsymbol{C}_6(x,y) := \boldsymbol{C}_{(\geq,>)}(x,y) = \frac{1-2x+x^2y-\sqrt{1-4x+4x^2-2x^2y+x^4y^2}}{2x^2y}.$$

Proof. Let w denote a non-empty Catalan word in $\mathcal{C}(\geq,>) = \mathcal{C}(\underline{110},\underline{210})$, and let $w = \mathtt{0}(w'+1)w''$ be the first return decomposition, where $w', w'' \in \mathcal{C}(\geq,>)$. If $w'' = \epsilon$, then $w = \mathtt{0}(w'+1)$ with w' possibly empty. The generating function for this case is

$x\boldsymbol{C}_6(x,y)$. If w'' is non-empty and $w'=\epsilon$, then w'' is any non-empty word in $\mathcal{C}(\geq,>)$, so the generating function is $x(\boldsymbol{C}_6(x,y)-1)$. If w' and w'' are non-empty, then $w'=\mathtt{0}$ or w' has to finish with an ascent $a(a+1)$ $(a\geq 0)$. Then the generating function is

$$E(x,y):=xy(x(\boldsymbol{C}_6(x,y)-1))+xy(x(\boldsymbol{C}_6(x,y)-1))(\boldsymbol{C}_6(x,y)-1).$$

Therefore, we have the functional equation

$$\boldsymbol{C}_6(x,y)=1+x\boldsymbol{C}_6(x,y)+x(\boldsymbol{C}_6(x,y)-1)+E(x,y).$$

Solving this system of equations we obtain the desired result. □

The series expansion of the generating function $\boldsymbol{C}_6(x,y)$ is

$$\begin{aligned}&1+x+2x^2+(4+y)x^3+(\mathbf{8}+\mathbf{5}y)x^4+(16+18y+y^2)x^5\\&\quad+(32+56y+9y^2)x^6+O(x^7).\end{aligned}$$

The Catalan words corresponding to the bold coefficients in the above series are

$$\mathcal{C}_4(\geq,>)=\{\mathtt{0000},\mathtt{0001},\mathtt{0010},\mathtt{0011},\mathtt{0012},\mathtt{0100},\mathtt{0101},\mathtt{0111},\mathtt{0112},\mathtt{0120},\mathtt{0121},\mathtt{0122},\mathtt{0123}\}.$$

The coefficient sequence of the bivariate generating function $\boldsymbol{C}_6(x,y)$ coincides with A273717, which counts the number of bargraphs of semiperimeter n having k L-shaped corners $(n\geq 2, k\geq 0)$.

Corollary 3.17. *The g.f. for the cardinality of $\mathcal{C}(\geq,>)$ with respect to the length is*

$$\boldsymbol{C}_{(\geq,>)}(x)=\frac{1-2x+x^2-\sqrt{1-4x+2x^2+x^4}}{2x^2}.$$

This generating function coincides with the generating function of the sequence A082582, that is

$$\boldsymbol{c}_{(\geq,>)}(n)=\boldsymbol{c}_{\underline{110},\underline{210}}(n)=\sum_{k=0}^{n}\sum_{j=0}^{n-k}\frac{1}{j+1}\binom{n-k-1}{j}\binom{k}{j}\binom{k+j+2}{j},\ n\geq 1.$$

Notice that this sequence counts also Dyck paths of semilength $n+1$ avoiding $UUDD$.

Corollary 3.18. *The g.f. for the total number of descents on $\mathcal{C}(\geq,>)$ is*

$$\boldsymbol{D}_{(\geq,>)}(x)=\frac{1-4x+3x^2-(1-2x)\sqrt{1-4x+2x^2+x^4}}{2x^2\sqrt{1-4x+2x^2+x^4}}.$$

The series expansion of $\boldsymbol{D}_{(\geq,>)}(x)$ is

$$x^3 + 5x^4 + 20x^5 + 74x^6 + 263x^7 + 914x^8 + 3134x^9 + O(x^{10}),$$

where the coefficients correspond to the sequence A273718, which counts also the total number of descents in all bargraphs of semiperimeter $n-1$.

3.8. Cases $\mathcal{C}(\geq,\leq)$, $\mathcal{C}(\leq,<)$, and $\mathcal{C}(<,\leq)$

The avoidance of $(\geq,\leq)$ (resp. $(\leq,<)$, resp. $(<,\leq)$) on Catalan words is equivalent to the avoidance of $\underline{000}$, $\underline{001}$, $\underline{100}$, $\underline{101}$, and $\underline{201}$ (resp. $\underline{001}$ and $\underline{012}$, resp. $\underline{011}$ and $\underline{012}$).

From the bijection described in Section 3.3, $\mathcal{C}(\leq,<) = \mathcal{C}(\underline{001},\underline{012})$ and $\mathcal{C}(<,\leq) = \mathcal{C}(\underline{011},\underline{012})$ are in one-to-one correspondence by preserving the descent number, which implies that $\boldsymbol{C}_{(\leq,<)}(x,y) = \boldsymbol{C}_{(<,\leq)}(x,y)$.

Theorem 3.19. *We have*

$$\boldsymbol{C}_7(x,y) := \boldsymbol{C}_{(\leq,<)}(x,y) = \frac{1+x^2-x^2y}{1-x-x^2y}.$$

Proof. Let w denote a non-empty Catalan word in $\mathcal{C}(\leq,<) = \mathcal{C}(\underline{001},\underline{012})$. From the conditions we have the decomposition $\mathtt{00}^j$, $\mathtt{011}^j$ or $\mathtt{011}^j w'$, where $j \geq 0$ and w' is a non-empty word in $\mathcal{C}(\leq,<)$. Therefore, we have the functional equation

$$\boldsymbol{C}_7(x,y) = 1 + \frac{x}{1-x} + \frac{x^2}{1-x} + \frac{x^2y}{1-x}\boldsymbol{C}_7(x,y).$$

Solving this equation we obtain the desired result. □

The series expansion of the generating function $\boldsymbol{C}_7(x,y)$ is

$$1 + x + 2x^2 + (2+y)x^3 + (\mathbf{2}+\mathbf{3}y)x^4 + (2+5y+y^2)x^5 + (2+7y+4y^2)x^6 + O(x^7).$$

The Catalan words corresponding to the bold coefficients in the above series are

$$\mathcal{C}_4(\leq,<) = \{\mathtt{0000}, \mathtt{0100}, \mathtt{0101}, \mathtt{0110}, \mathtt{0111}\}.$$

The coefficients of the generating function $\boldsymbol{C}_7(x,y)$ coincide with the table A129710.

Corollary 3.20. *The g.f. for the cardinality of $\mathcal{C}(\leq,<)$ with respect to the length is*

$$\boldsymbol{C}_{(\leq,<)}(x) = \frac{1}{1-x-x^2}.$$

This generating function coincides with that of the Fibonacci sequence F_{n+1}, that is

$$\boldsymbol{c}_{(\leq,<)}(n) = \boldsymbol{c}_{\underline{001},\underline{012}}(n) = F_{n+1},\ n \geq 0.$$

Corollary 3.21. *The g.f. for the total number of descents on $\mathcal{C}(\leq,<)$ is*

$$\boldsymbol{D}_{(\leq,<)}(x) = \frac{x^3(1+x)}{(1-x-x^2)^2}.$$

The series expansion is

$$x^3 + 3x^4 + 7x^5 + 15x^6 + 30x^7 + 58x^8 + 109x^9 + O(x^{10}),$$

where the coefficient sequence corresponds to A023610.

Finally, we can prove easily that $\boldsymbol{C}_{(\geq,\leq)}(x,y) = \boldsymbol{C}_{(\leq,<)}(x,y)$ by exhibiting a bijection ϕ preserving the descent number. Let w be a word in $\mathcal{C}(\leq,<)$, then we distinguish three cases: (i) $\phi(\epsilon) = \epsilon$; (ii) if $w = \mathtt{0}^j$, $j \geq 1$, then we set $\phi(w) = \mathtt{01}^{j-1}$; and (iii) if $w = \mathtt{01}^j$, $j \geq 1$, then $\phi(w) = \mathtt{012}\cdots(j-1)(j-1)$; (iv) if $w = \mathtt{01}^j w'$, $j \geq 1$, then $\phi(w) = \mathtt{01}\cdots(j-1)(\phi(w')+j)\mathtt{0}$. For example, for the word $\mathtt{01101101111011} \in \mathcal{C}_{14}(\leq,<)$, we have the transformation:

$$\begin{aligned}\phi(\mathtt{01}^2\mathtt{01101111011}) &= \mathtt{01}(\phi(\mathtt{01}^2\mathtt{01111011}) + 2)\mathtt{0}\\ &= \mathtt{0123}(\phi(\mathtt{01}^4\mathtt{011}) + 4)\mathtt{20} = \mathtt{01234567}(\phi(\mathtt{01}^2) + 8)\mathtt{420}\\ &= \mathtt{01234557899}(\phi(\epsilon) + 8)\mathtt{420} = \mathtt{01234567899420} \in \mathcal{C}_{14}(\geq,\leq).\end{aligned}$$

3.9. Cases $\mathcal{C}(\geq,<)$ and $\mathcal{C}(\leq,>)$

The avoidance of $(\geq,<)$ (resp. $(\leq,>)$) on Catalan words is equivalent to the avoidance of $\underline{001}$, $\underline{101}$, and $\underline{201}$ (resp. $\underline{010}$, $\underline{110}$. and $\underline{120}$). Below, we will see that for all $n \geq 1$, the cardinalities of $\mathcal{C}_n(\geq,<)$ resp. $\mathcal{C}_n(\leq,>)$ are the same, but the distributions of the number of descents do not coincide.

Theorem 3.22. *We have*

$$\boldsymbol{C}_8(x,y) := \boldsymbol{C}_{(\geq,<)}(x,y) = \frac{1-x+x^2-x^2y}{1-2x+x^2-x^2y}.$$

Proof. Let w denote a non-empty Catalan word in $\mathcal{C}(\geq,<) = \mathcal{C}(\underline{101},\underline{110},\underline{201})$ and let $w = \mathtt{0}(w'+1)w''$ be the first return decomposition, where $w', w'' \in \mathcal{C}(\geq,<)$. If w' is empty, then we have $w = 0^j$, $j \geq 0$. Otherwise, we have $w = \mathtt{0}(w'+1)0^j$, $j \geq 0$. The generating function $\boldsymbol{C}_8(x,y)$ satisfies the functional equation

$$\boldsymbol{C}_8(x,y) = \frac{1}{1-x} + x(\boldsymbol{C}_8(x,y)-1) + \frac{x^2y}{1-x}(\boldsymbol{C}_8(x,y)-1),$$

which gives the result. □

The series expansion of the generating function $\boldsymbol{C}_8(x,y)$ is

$$1+x+2x^2+(3+y)x^3+(\mathbf{4}+\mathbf{4}y)x^4+(5+10y+y^2)x^5+(6+20y+6y^2)x^6+O(x^7).$$

The Catalan words corresponding to the bold coefficients in the above series are

$$\mathcal{C}_4(\geq,<)=\{\mathtt{0000},\mathtt{0110},\mathtt{0111},\mathtt{0100},\mathtt{0120},\mathtt{0121},\mathtt{0122},\mathtt{0123}\}.$$

The coefficient sequence of the bivariate generating function $\boldsymbol{C}_8(x,y)$ coincides with A034867, then

$$[x^ny^k]\boldsymbol{C}_8(x,y)=\binom{n}{2k+1}.$$

Corollary 3.23. *The g.f. for the cardinality of $\mathcal{C}(\geq,<)$ with respect to the length is*

$$\boldsymbol{C}_{(\geq,<)}(x)=\frac{1-x}{1-2x},$$

where the n-th term is 2^{n-1}.

Corollary 3.24. *The g.f. for the total number of descents on $\mathcal{C}(\geq,<)$ is*

$$\boldsymbol{D}_{(\geq,<)}(x)=\frac{x^3}{(1-2x)^2}.$$

The series expansion is

$$x^3+4x^4+12x^5+32x^6+80x^7+192x^8+448x^9+O(x^{10}),$$

where the coefficient sequence corresponds to A001787, i.e., the n-th term is $(n-2)2^{n-3}$.

Theorem 3.25. *We have*

$$\boldsymbol{C}_{(\leq,>)}(x,y)=\frac{1-x}{1-2x}.$$

Proof. Let w denote a non-empty Catalan word in $\mathcal{C}(\leq,>)=\mathcal{C}(\underline{010},\underline{110},\underline{120})$ and let $w=\mathtt{0}(w'+1)w''$ be the first return decomposition, where $w',w''\in\mathcal{C}(\geq,<)$. If w' is empty, then we have $w=0w''$. Otherwise, we have $w=\mathtt{0}(w'+1)$. The generating function $\boldsymbol{C}_{(\leq,>)}(x,y)$ satisfies the functional equation

$$\boldsymbol{C}_{(\leq,>)}(x,y)=1+x\boldsymbol{C}_{(\leq,>)}(x,y)+x(\boldsymbol{C}_{(\leq,>)}(x,y)-1),$$

which gives the result. □

3.10. Case $\mathcal{C}(\geq, \neq)$

The avoidance of $(\geq, \neq)$ on Catalan words is equivalent to the avoidance of 001, 102, 201, 110, and 210. This means that any Catalan word of length n in $\mathcal{C}(\geq, \neq)$ is either of the form $w = 01\dots(k-1)k^{n-k}$, $k \geq 0$, or $w = 01\dots(k-1)m^{n-k}$ where $0 \leq m < k-1$ and $k \geq 1$. Therefore, we can deduce easily that

$$\boldsymbol{C}_9(x,y) := \boldsymbol{C}_{(\geq,\neq)}(x,y) = 1 + \frac{x}{(1-x)^2} + \frac{yx^3}{(1-x)^3},$$

which implies the following.

Theorem 3.26. *We have*

$$\boldsymbol{C}_{(\geq,\neq)}(x,y) = \frac{1 - 2x + 2x^2 - x^3 + x^3y}{(1-x)^3}.$$

The series expansion of the generating function $\boldsymbol{C}_{(\geq,\neq)}(x,y)$ is

$$1 + x + 2x^2 + (3+y)x^3 + (\mathbf{4} + \mathbf{3}y)x^4 + (5+6y)x^5 + (6+10y)x^6 + O(x^7).$$

The Catalan words corresponding to the bold coefficients in the above series are

$$\mathcal{C}_4(\geq, \neq) = \{\mathtt{0000}, \mathtt{0100}, \mathtt{0111}, \mathtt{0120}, \mathtt{0121}, \mathtt{0122}, \mathtt{0123}\}.$$

Corollary 3.27. *The g.f. for the cardinality of $\mathcal{C}(\geq, \neq)$ with respect to the length is*

$$\boldsymbol{C}_{(\geq,\neq)}(x) = \frac{1 - 2x + 2x^2}{(1-x)^3},$$

where the n-th term is $1 + \binom{n}{2}$.

Corollary 3.28. *The g.f. for the total number of descents on $\mathcal{C}(\geq, \neq)$ is*

$$\boldsymbol{D}_{(\geq,\neq)}(x) = \frac{x^3}{(1-x)^3}.$$

The series expansion is

$$x^3 + 3x^4 + 6x^5 + 10x^6 + 15x^7 + 21x^8 + 28x^9 + O(x^{10}),$$

where the coefficient sequence corresponds to A000217, i.e., the n-th term is $\binom{n-1}{2}$.

3.11. Case $\mathcal{C}(>,\leq)$

The avoidance of $(>,\leq)$ on Catalan words is equivalent to the avoidance of $\underline{100}$, $\underline{101}$, and $\underline{201}$.

Theorem 3.29. *We have*

$$\boldsymbol{C}_{10}(x,y) := \boldsymbol{C}_{(>,\leq)}(x,y) = \frac{1-x-x^2y}{1-2x-x^2y}.$$

Proof. Let w denote a non-empty Catalan word in $\mathcal{C}(>,\leq) = \mathcal{C}(\underline{100},\underline{101},\underline{201})$, and let $w = \mathtt{0}(w'+1)w''$ be the first return decomposition, where $w', w'' \in \mathcal{C}(>,\leq)$. If $w'' = \epsilon$, then $w = \mathtt{0}(w'+1)$ with w' possibly empty. The generating function for this case is $x\boldsymbol{C}_{10}(x,y)$. If w'' is non-empty and $w' = \epsilon$, then w'' is any non-empty word in $\mathcal{C}(>,\leq)$, so the generating function is $x(\boldsymbol{C}_{10}(x,y)-1)$. If w' and w'' are non-empty, then w' does not start with the prefix $\mathtt{01}$ or $\mathtt{00}$, otherwise w would contain the pattern $\underline{100}$ or $\underline{201}$, respectively. Therefore, $w'' = \mathtt{0}$ and the generating function is $x^2y(\boldsymbol{C}_{10}(x,y)-1)$. Summarizing, we have the functional equation

$$\boldsymbol{C}_{10}(x,y) = 1 + x\boldsymbol{C}_{10}(x,y) + x(\boldsymbol{C}_{10}(x,y)-1) + x^2y(\boldsymbol{C}_{10}(x,y)-1).$$

Solving this equation we obtain the desired result. □

The series expansion of the generating function $\boldsymbol{C}_{10}(x,y)$ is

$$\begin{aligned}&1 + x + 2x^2 + (4+y)x^3 + (\mathbf{8}+\mathbf{4}y)x^4 + (16+12y+y^2)x^5\\&\quad + (32+32y+6y^2)x^6 + O(x^7).\end{aligned}$$

The Catalan words corresponding to the bold coefficients in the above series are

$$\mathcal{C}_4(>,\leq) = \{\mathtt{0000}, \mathtt{0001}, \mathtt{0010}, \mathtt{0011}, \mathtt{0012}, \mathtt{0110}, \mathtt{0111}, \mathtt{0112}, \mathtt{0120}, \mathtt{0121}, \mathtt{0122}, \mathtt{0123}\}.$$

The coefficients of the above series coincide with the array A207538. Notice that this array coincides with the coefficients of the Pell polynomials.

Corollary 3.30. *The g.f. for the cardinality of $\mathcal{C}(\leq,<)$ with respect to the length is*

$$\boldsymbol{C}_{(>,\leq)}(x) = \frac{1-x-x^2}{1-2x-x^2}.$$

This generating function coincides with the generating function of the Pell numbers P_{n+1}. The Pell sequence is defined by $P_n = 2P_{n-1} + P_{n-2}$ for $n \geq 2$, with the initial values $P_0 = 0$ and $P_1 = 1$ (see sequence A000129).

Corollary 3.31. *The g.f. for the total number of descents on $\mathcal{C}(>,\leq)$ is*

$$\boldsymbol{D}_{(>,\leq)}(x) = \frac{x^3}{(1-2x-x^2)^2}.$$

The series expansion of $\boldsymbol{D}_{(>,\leq)}(x)$ is

$$x^3 + 4x^4 + 14x^5 + 44x^6 + 131x^7 + 376x^8 + 1052x^9 + O(x^{10}),$$

where the coefficients correspond to the sequence A006645.

3.12. Case $\mathcal{C}(>,\neq)$

The avoidance of $(>,\neq)$ on Catalan words is equivalent to the avoidance of 101, 201, and 210, which means that $\mathcal{C}(>,\neq)$ is the set of Catalan words without two consecutive descents and without valleys. Therefore, such a non-empty word is of the form $\mathtt{0}u$, or $\mathtt{0}(u+1)$, or $\mathtt{0}(u+1)u'\mathtt{0}v$ where $u,v \in \mathcal{C}(>,\neq)$ and $u' = k(k+1)\cdots\ell$ with $\ell \geq k \geq 1$ and if $u+1$ ends with a then we set $k=a$, and if u is empty then we set $k=1$. We deduce the following functional equation for $\boldsymbol{C}_{11}(x,y) := \boldsymbol{C}_{(>,\neq)}(x,y)$.

$$\boldsymbol{C}_{11}(x,y) = 1 + x\boldsymbol{C}_{11}(x,y) + x(\boldsymbol{C}_{11}(x,y)-1) + yx^2\frac{x}{1-x}\boldsymbol{C}_{11}(x,y)^2.$$

Then, we obtain the following.

Theorem 3.32. *We have*

$$\boldsymbol{C}_{11}(x,y) := \boldsymbol{C}_{(>,\neq)}(x,y) = \frac{\left(1-2x-\sqrt{1-4x+4x^2-4x^3y}\right)(1-x)}{2x^3y}.$$

The series expansion of the generating function $\boldsymbol{C}_{11}(x,y)$ is

$$1 + x + 2x^2 + (4+y)x^3 + (\mathbf{8}+\mathbf{5}y)x^4 + (16+18y)x^5 + (32+56y+2y^2)x^6 + O(x^7).$$

The Catalan words corresponding to the bold coefficients in the above series are

$$\mathcal{C}_4(>,\neq)=\{\mathtt{0000},\mathtt{0001},\mathtt{0010},\mathtt{0011},\mathtt{0012},\mathtt{0100},\mathtt{0110},\mathtt{0111},\mathtt{0112},\mathtt{0120},\mathtt{0121},\mathtt{0122},\mathtt{0123}\}.$$

Corollary 3.33. *The g.f. for the cardinality of $\mathcal{C}(>,\neq)$ with respect to the length is*

$$\boldsymbol{C}_{(>,\neq)}(x) = \frac{\left(1-2x-\sqrt{1-4x+4x^2-4x^3}\right)(1-x)}{2x^3}.$$

The coefficient sequence of the series expansion does not appear in [15].

Corollary 3.34. *The g.f. for the total number of descents on $\mathcal{C}(>,\neq)$ is*

$$\boldsymbol{D}_{(>,\neq)}(x) = \frac{(1-x)\left(1-4x+4x^2-2x^3-(1-2x)\sqrt{1-4x+4x^2-4x^3}\right)}{2x^3\sqrt{1-4x+4x^2-4x^3}}.$$

The series expansion of $\boldsymbol{D}_{(>,\neq)}(x)$ is

$$x^3+5x^4+18x^5+60x^6+196x^7+632x^8+2015x^9+O(x^{10}),$$

where the coefficient sequence does not appear in [15].

3.13. Cases $\mathcal{C}(<,\geq)$ and $\mathcal{C}(\neq,\geq)$

The avoidance of $(<,\geq)$ (resp. $(\neq,\geq)$) on Catalan words is equivalent to the avoidance of $\underline{010}$, $\underline{120}$, and $\underline{011}$ (resp. $\underline{100}$, $\underline{011}$, $\underline{210}$, $\underline{010}$, and $\underline{120}$). A non-empty Catalan word in $\mathcal{C}(<,\geq)$ is of the form $\mathtt{0}^k(u+1)$, $k\geq 1$ where u is either empty or $u=\mathtt{012}\ldots k$, $k\geq 1$. Moreover, it is easy to check that $\mathcal{C}(<,\geq)=\mathcal{C}(\neq,\geq)$. Then, we deduce the following.

Theorem 3.35. *We have*

$$\boldsymbol{C}_{12}(x,y) := \boldsymbol{C}_{(<,\geq)}(x,y) = \boldsymbol{C}_{(\neq,\geq)}(x,y) = 1+\frac{1}{1-x}\frac{x}{1-x} = \frac{1-x+x^2}{(1-x)^2}.$$

The n-th term of the series expansion is n.

3.14. Case $\mathcal{C}(<,\neq)$

The avoidance of $(<,\neq)$ on Catalan words is equivalent to the avoidance of $\underline{010}$, $\underline{012}$, and $\underline{120}$. We set $\boldsymbol{C}_{13}(x,y) := \boldsymbol{C}_{(<,\neq)}(x,y)$.

Theorem 3.36. *We have*

$$\boldsymbol{C}_{13}(x,y) = \frac{1-2x^2-x^3+2x^2y-(1+x)\sqrt{1-2x-x^2+2x^3+x^4-4x^3y}}{2x^2y}.$$

Proof. Let w denote a non-empty Catalan word in $\mathcal{C}(<,\neq)=\mathcal{C}(\underline{010},\underline{012},\underline{120})$, and let $w=\mathtt{0}(w'+1)w''$ be the first return decomposition, where $w',w''\in\mathcal{C}(<,\neq)$. If w' is empty, then the generating function for these words is $x\boldsymbol{C}_{13}(x,y)$. If w'' is empty, then $w=\mathtt{01}(u+1)$, where $u\in\mathcal{C}(<,\neq)$, and the generating function for these words is $x^2\boldsymbol{C}_{13}(x,y)$. If $w=\mathtt{011}u$, where u is a non-empty word in $\mathcal{C}(<,\neq)$, then the generating function for these words is $x^3y(\boldsymbol{C}_{13}(x,y)-1)$. If $w=\mathtt{01}(u+1)v$, where $u,v\in\mathcal{C}(<,\neq)$, u ending with aa, $a\geq 0$, and v non-empty, then the generating function for these words is $x^3y(\boldsymbol{C}_{13}(x,y)-1)^2$. If $w=\mathtt{01}(u+1)v$, where $u,v\in\mathcal{C}(<,\neq)$, u ending with a descent and v non-empty, then the generating function for these words is $x^2yB(x,y)(\boldsymbol{C}_{13}(x,y)-$

1), where $B(x,y)$ is the generating function for Catalan words in $\mathcal{C}(<,\neq)$ ending by a descent. By considering the complement, we obtain easily that

$$B(x,y) = \boldsymbol{C}_{13}(x,y) - 1 - x - xB(x,y) - x(\boldsymbol{C}_{13}(x,y) - 1) - x^2\boldsymbol{C}_{13}(x,y).$$

Summarizing, we have the following functional equation

$$\begin{aligned}\boldsymbol{C}_{13}(x,y) = 1 + x\boldsymbol{C}_{13}(x,y) + x^2\boldsymbol{C}_{13}(x,y) + x^3y(\boldsymbol{C}_{13}(x,y) - 1)\\ + x^3y(\boldsymbol{C}_{13}(x,y) - 1)^2 + x^2yB(x,y)(\boldsymbol{C}_{13}(x,y) - 1),\end{aligned}$$

which gives the result. □

The series expansion of the generating function $\boldsymbol{C}_{13}(x,y)$ is

$$1 + x + 2x^2 + 3x^3 + (5+y)x^4 + (\mathbf{8+4y})x^5 + (13+12y)x^6 + O(x^7).$$

The Catalan words corresponding to the bold coefficients in the above series are

$$\mathcal{C}_5(<,\neq) = \{\mathtt{00000}, \mathtt{00001}, \mathtt{00011}, \mathtt{00110}, \mathtt{00111}, \mathtt{00112}, \\ \mathtt{01100}, \mathtt{01101}, \mathtt{01110}, \mathtt{01111}, \mathtt{01112}, \mathtt{01122}\}.$$

Corollary 3.37. *The g.f. for the cardinality of $\mathcal{C}(<,\neq)$ with respect to the length is*

$$\boldsymbol{C}_{(<,\neq)}(x) = \frac{1 - x^3 - (1+x)\sqrt{1 - 2x - x^2 - 2x^3 + x^4}}{2x^2}.$$

The coefficient sequence of the series expansion does not appear in [15].

Corollary 3.38. *The g.f. for the total number of descents on $\mathcal{C}(<,\neq)$ is*

$$\boldsymbol{D}_{(<,\neq)}(x) = \frac{1 - 4x^2 - 4x^3 + 2x^5 - x^6 - (1+x)(1 - 2x^2 - x^3)\sqrt{1 - 2x - x^2 - 2x^3 + x^4}}{2x^2(1+x)\sqrt{1 - 2x - x^2 - 2x^3 + x^4}}.$$

The series expansion of $\boldsymbol{D}_{(<,\neq)}(x)$ is

$$x^4 + 4x^5 + 12x^6 + 35x^7 + 97x^8 + 262x^9 + O(x^{10}),$$

where the coefficient sequence does not appear in [15].

3.15. Case $\mathcal{C}(\neq,>)$

The avoidance of $(\neq,>)$ on Catalan words is equivalent to the avoidance of $\underline{010}$, $\underline{210}$, and $\underline{120}$.

Theorem 3.39. *We have*

$$\boldsymbol{C}_{14}(x,y) := \boldsymbol{C}_{(\neq,>)}(x,y) = \frac{1-2x+2x^2y-\sqrt{1-4x+4x^2-4x^3y}}{2x^2y}.$$

Proof. Let w denote a non-empty Catalan word in $\mathcal{C}(\neq,>) = \mathcal{C}(\underline{010},\underline{120},\underline{210})$, and let $w = \mathtt{0}(w'+1)w''$ be the first return decomposition, where $w', w'' \in \mathcal{C}(\neq,>)$. If $w'' = \epsilon$, then $w = \mathtt{0}(w'+1)$ with w' possibly empty. The generating function for this case is $x\boldsymbol{C}_{14}(x,y)$. If w'' is non-empty and $w' = \epsilon$, then w'' is any non-empty word in $\mathcal{C}(\neq,>)$, so the generating function is $x(\boldsymbol{C}_{14}(x,y)-1)$. If w' and w'' are non-empty, then $w' \neq \mathtt{0}$ (to avoid $\underline{010}$), w' does not end with an ascent $a(a+1)$ $(a \geq 0)$ (to avoid $\underline{120}$) or w' does not end with a descent ab $(a > b)$ (to avoid $\underline{210}$). Then, w' ends with aa $(a \geq 0)$ and the generating function is $x^2y(\boldsymbol{C}_{14}(x,y)-1)^2$. Summarizing, we have the functional equation

$$\boldsymbol{C}_{14}(x,y) = 1 + x\boldsymbol{C}_{14}(x,y) + x(\boldsymbol{C}_{14}(x,y)-1) + x^2y(\boldsymbol{C}_{14}(x,y)-1)^2.$$

Solving this equation we obtain the desired result. □

The series expansion of the generating function $\boldsymbol{C}_{14}(x,y)$ is

$$1 + x + 2x^2 + 4x^3 + (\mathbf{8+y})x^4 + (16+6y)x^5 + (32+24y)x^6 + O(x^7).$$

The Catalan words corresponding to the bold coefficients in the above series are

$$\mathcal{C}_4(\neq,>) = \{\mathtt{0000}, \mathtt{0001}, \mathtt{0011}, \mathtt{0012}, \mathtt{0110}, \mathtt{0111}, \mathtt{0112}, \mathtt{0122}, \mathtt{0123}\}.$$

Corollary 3.40. *The g.f. for the cardinality of $\mathcal{C}(\neq,>)$ with respect to the length is*

$$\boldsymbol{C}_{(\neq,>)}(x) = \frac{1-2x+2x^2-\sqrt{1-4x+4x^2-4x^3}}{2x^2}.$$

This generating function coincides with the generating function of the sequence A152225 that counts Dyck paths of a given length with no peaks at height 0 (mod 3) and no valleys at height 2 (mod 3).

Corollary 3.41. *The g.f. for the total number of descents on $\mathcal{C}(\neq,>)$ is*

$$\boldsymbol{D}_{(\neq,>)}(x) = \frac{1-4x+4x^2-2x^3-(1-2x)\sqrt{1-4x+4x^2-4x^3}}{2x^2\sqrt{1-4x+4x^2-4x^3}}.$$

The series expansion of $\boldsymbol{D}_{(\neq,>)}(x)$ is

$$x^4 + 6x^5 + 24x^6 + 84x^7 + 280x^8 + 912x^9 + O(x^{10}),$$

where the coefficient sequence does not appear in [15].

3.16. Case $\mathcal{C}(\neq,<)$

The avoidance of $(\neq,<)$ on Catalan words is equivalent to the avoidance of $\underline{012}$, $\underline{101}$, $\underline{102}$, and $\underline{201}$.

Theorem 3.42. *We have*

$$\boldsymbol{C}_{15}(x,y) := \boldsymbol{C}_{(\neq,<)}(x,y) = \frac{1-x-x^2-\sqrt{(1-x-x^2)^2-4x^3y}}{2x^3y}.$$

Proof. Let w denote a non-empty Catalan word in $\mathcal{C}(\neq,<) = \mathcal{C}(\underline{012},\underline{101},\underline{201})$, and let $w = \mathtt{0}(w'+1)w''$ be the first return decomposition, where $w',w'' \in \mathcal{C}(\neq,<)$. If $w''=\epsilon$, then $w=\mathtt{0}(w'+1)$ with w' possibly empty. For $w'\neq\epsilon$ and to avoid $\underline{012}$, we have $w'=\mathtt{01}(w'''+1)$, where $w'''\in\mathcal{C}(\neq,<)$. The generating function for this case is $x+x^2\boldsymbol{C}_{15}(x,y)$. If w'' is non-empty and $w'=\epsilon$, then w'' is any non-empty word in $\mathcal{C}(\neq,<)$, so the generating function is $x(\boldsymbol{C}_{15}(x,y)-1)$. If w' and w'' are non-empty, then $w'=\mathtt{0}u$ and $w''=\mathtt{0}v$, where $u,v\in\mathcal{C}(\neq,<)$. Then the generating function for this case is $x^3y\boldsymbol{C}_{15}(x,y)$. Summarizing, we have the functional equation

$$\boldsymbol{C}_{15}(x,y) = 1+x+x^2\boldsymbol{C}_{15}(x,y)+x(\boldsymbol{C}_{15}(x,y)-1)+x^3y\boldsymbol{C}_{15}(x,y).$$

Solving this equation we obtain the desired result. □

The series expansion of the generating function $\boldsymbol{C}_{15}(x,y)$ is

$$1+x+2x^2+(3+y)x^3+(\mathbf{5+3y})x^4+(8+9y)x^5+(13+22y+2y^2)x^6+O(x^7).$$

The Catalan words corresponding to the bold coefficients in the above series are

$$\mathcal{C}_4(\neq,>) = \{\mathtt{0000},\mathtt{0001},\mathtt{0010},\mathtt{0011},\mathtt{0100},\mathtt{0110},\mathtt{0111},\mathtt{0112}\}.$$

The coefficients of the generating function $\boldsymbol{C}_{15}(x,y)$ coincide with the array A114711.

Corollary 3.43. *The g.f. for the cardinality of $\mathcal{C}(\neq,>)$ with respect to the length is*

$$\boldsymbol{C}_{(\neq,<)}(x) = \frac{1-x-x^2-\sqrt{1-2x-x^2-2x^3+x^4}}{2x^3}.$$

This generating function coincides with the generating function of the sequence A292460 that gives the number of U_kD-equivalence classes of Łukasiewicz paths (see [4]), which is a shift of the sequences A292460, A004148, and A203019. Then we have

$$\boldsymbol{c}_{(\neq,<)}(n) = \boldsymbol{c}_{\underline{012},\underline{101},\underline{201}}(n) = \sum_{k=0}^{\lfloor\frac{n+1}{2}\rfloor}(-1)^k\binom{n-k-1}{k}m_{n+1-2k},\ n\geq 0.$$

Corollary 3.44. *The g.f. for the total number of descents on $\mathcal{C}(\neq,<)$ is*

$$\boldsymbol{D}_{(\neq,<)}(x)=\frac{1-2x-x^2+x^4-(1-x-x^2)\sqrt{1-2x-x^2-2x^3+x^4}}{2x^3\sqrt{1-2x-x^2-2x^3+x^4}}.$$

The series expansion of $\boldsymbol{D}_{(\neq,<)}(x)$ is

$$x^3+3x^4+9x^5+26x^6+71x^7+191x^8+508x^9+O(x^{10}),$$

where the coefficient sequence does not appear in [15].

3.17. Case $\mathcal{C}(\neq,\neq)$

The avoidance of $(\neq,\neq)$ on Catalan words is equivalent to the avoidance of $\underline{010}$, $\underline{012}$, $\underline{101}$, $\underline{201}$, and $\underline{120}$.

Theorem 3.45. *We have*

$$\boldsymbol{C}_{16}(x,y):=\boldsymbol{C}_{(\neq,\neq)}(x,y)=\frac{1-x-x^2-\sqrt{(1-x-x^2)^2-4x^4y}}{2x^4y}.$$

Proof. Let w denote a non-empty Catalan word in $\mathcal{C}(\neq,\neq)=\mathcal{C}(\underline{010},\underline{012},\underline{101},\underline{120},\underline{201},\underline{210})$, and let $w=\mathtt{0}(w'+1)w''$ be the first return decomposition, where $w',w''\in\mathcal{C}(\neq,\neq)$. If $w''=\epsilon$, then $w=\mathtt{0}(w'+1)$ with w' possibly empty. For $w'\neq\epsilon$ and to avoid $\underline{012}$, we have $w'=\mathtt{01}(w'''+1)$, where $w'''\in\mathcal{C}(\neq,\neq)$. The generating function for this case is $x+x^2\boldsymbol{C}_{16}(x,y)$. If w'' is non-empty and $w'=\epsilon$, then the generating function is $x(\boldsymbol{C}_{16}(x,y)-1)$. If w' and w'' are non-empty, then w' is a non-empty word in $\mathcal{C}(\neq,\neq)$ such that the last two symbols are equal (to avoid $\underline{210}$ and $\underline{120}$) and $w''=\mathtt{0}w'''$, where $w'''\in\mathcal{C}(\neq,\neq)$ (to avoid $\underline{101}$ and $\underline{201}$). Then the generating function for this case is

$$E(x,y):=xy(x^2\boldsymbol{C}_{16}(x,y))(x\boldsymbol{C}_{16}(x,y))).$$

Summarizing, we have the functional equation

$$\boldsymbol{C}_{16}(x,y)=1+x+x^2\boldsymbol{C}_{16}(x,y)+x(\boldsymbol{C}_{16}(x,y)-1)+E(x,y).$$

Solving this equation we obtain the desired result. □

Notice that $\boldsymbol{C}_{16}(x,y)=\boldsymbol{C}_{15}(x,xy)$. The series expansion of the generating function $\boldsymbol{C}_{16}(x,y)$ is

$$1+x+2x^2+3x^3+(\mathbf{5+y})x^4+(8+3y)x^5+(13+9y)x^6+O(x^7).$$

The Catalan words corresponding to the bold coefficients in the above series are

$$\mathcal{C}_4(\neq,\neq) = \{\mathtt{0000}, \mathtt{0001}, \mathtt{0011}, \mathtt{0110}, \mathtt{0111}, \mathtt{0112}\}.$$

Corollary 3.46. *The g.f. for the cardinality of $\mathcal{C}(\neq,\neq)$ with respect to the length is*

$$\boldsymbol{C}_{(\neq,\neq)}(x) = \frac{1 - x - x^2 - \sqrt{1 - 2x - x^2 + 2x^3 - 3x^4}}{2x^4}.$$

This generating function coincides with the generating function of the sequence A026418 that counts ordered trees with a given number of edges and having no branches of length 1. Then, we have

$$\boldsymbol{c}_{(\neq,\neq)}(n) = \boldsymbol{c}_{\underline{010},\underline{012},\underline{101},\underline{120},\underline{201},\underline{210}}(n) = \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n-k}{k} m_k, \ n \geq 1.$$

Corollary 3.47. *The g.f. for the total number of descents on $\mathcal{C}(\neq,\neq)$ is*

$$\boldsymbol{D}_{(\neq,\neq)}(x) = \frac{1 - 2x - x^2 + 2x^3 - x^4 - (1 - x - x^2)\sqrt{1 - 2x - x^2 + 2x^3 - 3x^4}}{2x^4\sqrt{1 - 2x - x^2 + 2x^3 - 3x^4}}.$$

The series expansion of $\boldsymbol{D}_{(\neq,<)}(x)$ is

$$x^4 + 3x^5 + 9x^6 + 22x^7 + 55x^8 + 131x^9 + O(x^{10}),$$

where the coefficient sequence does not appear in [15].

Acknowledgments

The authors are grateful to an anonymous referee for helpful comments.

References

[1] J.S. Auli, S. Elizalde, Consecutive patterns in inversion sequences, Discret. Math. Theor. Comput. Sci. 21 (2) (2019) # 6.
[2] J.-L. Baril, D. Colmenares, J.L. Ramírez, E.D. Silva, L.M. Simbaqueba, D. Toquica, Consecutive pattern-avoidance in Catalan words according to the last symbol, 2023, submitted for publication.
[3] J.-L. Baril, J.F. González, J.L. Ramírez, Last symbol distribution in pattern avoiding Catalan words, 2022, submitted for publication.
[4] J.-L. Baril, S. Kirgizov, A. Petrossian, Enumeration of Łukasiewicz paths modulo some patterns, Discrete Math. 342 (4) (2019) 997–1005.
[5] J.-L. Baril, S. Kirgizov, V. Vajnovszki, Descent distribution on Catalan words avoiding a pattern of length at most three, Discrete Math. 341 (2018) 2608–2615.
[6] J.-L. Baril, C. Khalil, V. Vajnovszki, Catalan words avoiding pairs of length three patterns, Discret. Math. Theor. Comput. Sci. 22 (2) (2021) # 5.
[7] D. Callan, T. Mansour, J.L. Ramírez, Statistics on bargraphs of Catalan words, J. Autom. Lang. Comb. 26 (2021) 177–196.
[8] S. Corteel, M.A. Martinez, C. Savage, M. Weselcouch, Patterns in inversion sequences I, Discrete Math. Theor. Comput. Sci. 18 (2) (2016).

[9] K. Manes, A. Sapounakis, I. Tasoulas, P. Tsikouras, General results on the enumeration of strings in Dyck paths, Electron. J. Comb. 18 (2011), #P74.
[10] T. Mansour, J.L. Ramírez, D.A. Toquica, Counting lattice points on bargraphs of Catalan words, Math. Comput. Sci. 15 (2021) 701–713.
[11] T. Mansour, V. Vajnovszki, Efficient generation of restricted growth words, Inf. Process. Lett. 113 (2013) 613–616.
[12] M.A. Martinez, C.D. Savage, Patterns in inversion sequences II: inversion sequences avoiding triples of relations, J. Integer Seq. 21 (2) (2018) 18.2.2.
[13] J.L. Ramírez, A. Rojas-Osorio, Consecutive patterns in Catalan words and the descent distribution, Bol. Soc. Mat. Mex., in press.
[14] A. Sapounakis, I. Tasoulas, P. Tsikouras, Counting strings in Dyck paths, Discrete Math. 307 (2007) 2909–2924.
[15] N.J.A. Sloane, The on-line encyclopedia of integer sequences, available at https://oeis.org/.
[16] R. Stanley, Catalan Numbers, Cambridge University Press, Cambridge, 2015.

ISSN 2590-9770
The Art of Discrete and Applied Mathematics 7 (2024) #P1.07
https://doi.org/10.26493/2590-9770.1587.b2a
(Also available at http://adam-journal.eu)

Grand Dyck paths with air pockets

Jean-Luc Baril*, Sergey Kirgizov,
Rémi Maréchal, Vincent Vajnovszki
LIB, Université de Bourgogne Franche-Comté,
B.P. 47 870, 21078 Dijon Cedex France

Received 9 November 2022, accepted 24 June 2023, published online 18 September 2023

Abstract

Grand Dyck paths with air pockets (GDAP) are a generalization of Dyck paths with air pockets by allowing them to go below the x-axis. We present enumerative results on GDAP (or their prefixes) subject to various restrictions such as maximal/minimal height, ordinate of the last point and particular first return decomposition. In some special cases we give bijections with other known combinatorial classes.

Keywords: Dyck path, enumeration, composition.

Math. Subj. Class.: 05A15, 05A16

1 Introduction

In a recent paper [2], the authors introduce and study a new class of lattice paths, called *Dyck paths with air pockets* (*DAP* for short). Such a path is a non empty lattice path in the first quadrant of $\mathbb{Z}^2$ starting at the origin, ending on the x-axis, and consisting of up-steps $U = (1, 1)$ and down-steps $D_k = (1, -k)$, $k \geqslant 1$, where two down steps cannot be consecutive. See Figure 1 for an example. These paths can be viewed as ordinary Dyck paths where each maximal run of down-steps is condensed into one large down step. As mentioned in [2], they also correspond to a stack evolution with (partial) reset operations that cannot be consecutive (see for instance [5]). In this paper, we generalize these paths to *grand Dyck path with air pockets* (*GDAP* for short), which have the same definition as DAP, except that they can go below the x-axis, and the empty path ε is considered as a GDAP.

The main goal is to make enumerative studies on GDAP (or prefix of these paths) with various restrictions on the maximal height reached, minimal height reached, height of the last point, ...

*Corresponding author.
E-mail addresses: barjl@u-bourgogne.fr (Jean-Luc Baril), Sergey.Kirgizov@u-bourgogne.fr (Sergey Kirgizov), remi_marechal01@etu.u-bourgogne.fr (Rémi Maréchal), vvajnov@u-bourgogne.fr (Vincent Vajnovszki)

This work is licensed under https://creativecommons.org/licenses/by/4.0/

The remaining of this paper is structured as follows. The next section recalls some useful results from [2] and introduces several notations for particular subsets of GDAP. The main result of Section 3 is Theorem 3.1 which gives the generating function counting the GDAP with respect to the length, which is the 'grand' counterpart of the generating function in (2.1) for DAP. In Section 4 we give similar results for prefixes of GDAP ending at a given ordinate, the corresponding problem for DAP being already solved in [8]. In Section 5, we provide generating functions for the number of partial GDAP that never go below the line $y = m$, with respect to the ordinate of the last point. In Section 6, we count partial GDAP lying between the lines $y = 0$ and $y = t$, which correspond to partial DAP bounded by a given height $t > 0$. We present a constructive bijection between these paths of length n for $t = 2$ and the set of compositions of $n - 2$ such that no two consecutive parts have the same parity. In Section 7, we count partial GDAP lying between the lines $y = -t$ and $y = t$, and we present a constructive bijection between these paths of length n for $t = 1$ and the set of compositions of $n + 3$ such that the first part is odd, the last part is even, and no two consecutive parts have the same parity. Finally, in Section 8 we provide enumerative results for DAP with a special first return decompostion, which proves that there are in one-to-one correspondence with Motzkin paths avoiding the patterns UH, HU and HH. We leave as an open question the problem of finding a constructive bijection between these two sets.

2 Definitions and notations

2.1 DAP

The *length* of a path is the number of its steps, and for $n \geqslant 0$, let $\mathcal{A}_n$ be the set of n-length DAP. By definition $\mathcal{A}_0 = \mathcal{A}_1 = \varnothing$ and we set $\mathcal{A} = \bigcup_{n\geqslant 2} \mathcal{A}_n$, see [2]. A DAP is called *prime* whenever it ends with D_k, $k \geqslant 2$, and returns to the x-axis only once. The set of all prime DAP of length n is denoted $\mathcal{P}_n$. Notice that UD is not prime, where for short we denote D_1 by D, so we set $\mathcal{P} = \bigcup_{n\geqslant 3} \mathcal{P}_n$. If $\alpha = U\beta UD_k \in \mathcal{P}_n$, then $2 \leqslant k < n$ and β is a (possibly empty) prefix of a path in $\mathcal{A}$, and we define the DAP $\alpha^\flat = \beta UD_{k-1}$, called the 'lowering' of α. For example, the path $\alpha = UUDUUD_3$ is prime, and $\alpha^\flat = UDUUD_2$. The map $\alpha \mapsto \alpha^\flat$ is clearly a bijection from $\mathcal{P}_n$ to $\mathcal{A}_{n-1}$ for all $n \geqslant 3$, and we denote by $\gamma^\sharp$ the inverse image of $\gamma \in \mathcal{A}_{n-1}$ ($\alpha^\sharp$ is a kind of 'elevation' of α).

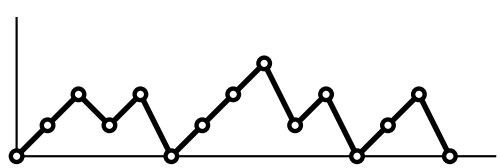


Figure 1: The Dyck path with air pockets $UUDUD_2UUUD_2UD_2UUD_2$.

Any DAP $\alpha \in \mathcal{A}$ can be decomposed depending on its *second-to-last return to the x-axis*: either (i) $\alpha = UD$, or (ii) $\alpha = \beta UD$ with $\beta \in \mathcal{A}$, or (iii) $\alpha \in \mathcal{P}$, or (iv) $\alpha = \beta\gamma$ where $\beta \in \mathcal{A}$ and $\gamma \in \mathcal{P}$. So, if $A(x) = \sum_{n\geqslant 2} a_n x^n$ where a_n is the cardinality of $\mathcal{A}_n$, and $P(x) = \sum_{n\geqslant 3} p_n x^n$ where p_n is the cardinality of $\mathcal{P}_n$, then we have $P(x) = xA(x)$ and the previous decompositions imply the functional equation $A(x) = x^2 + x^2A(x) + xA(x) + xA(x)^2$, and

$$A(x) = \frac{1 - x - x^2 - \sqrt{x^4 - 2\,x^3 - x^2 - 2\,x + 1}}{2x}, \tag{2.1}$$

which generates the generalized Catalan numbers (see A004148 in [10]). The first values of a_n for $2 \leqslant n \leqslant 10$ are $1, 1, 2, 4, 8, 17, 37, 82, 185$. In [2], the authors study the enumeration of these paths according to many parameters, and they give a constructive bijection between these paths and peakless Motzkin paths (i.e. lattice paths in the first quadrant, starting at $(0, 0)$, ending on the x-axis, made of steps $U = (1, 1)$, $D = (1, -1)$ and $H = (1, 0)$, and avoiding peaks of the form UD).

2.2 GDAP

The main object of study in this paper are grand Dyck paths with air pockets (GDAP for short) which generalize DAP by allowing such paths to go below the x-axis; and for convenience the empty path ε is a GDAP. See the first path in Figure 2 for an example. Let $\mathcal{G}_n$ be the set of GDAP of length $n \geqslant 0$, and we set $\mathcal{G} = \bigcup_{n \geqslant 0} \mathcal{G}_n$.

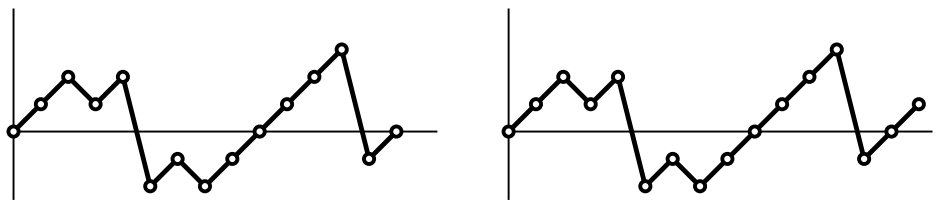


Figure 2: A grand Dyck path with air pockets $UUDUD_4UDUUUUUD_4U$, and a partial GDAP ending at ordinate 1 with an up-step.

We introduce notations for several subsets of $\mathcal{G}$ used in this study:

- $\mathcal{G}^+$ is the set of GDAP starting with U, and the empty path;
  - $\mathcal{G}_1^+$ is the set of elements of $\mathcal{G}^+$ ending with $D_k, k \geqslant 1$;
  - $\mathcal{G}_2^+$ is the set of elements of $\mathcal{G}^+$ ending with U.
- $\mathcal{G}^-$ is the set of GDAP starting with $D_k, k \geqslant 1$;
  - $\mathcal{G}_1^-$ is the set of elements of $\mathcal{G}^-$ ending with $D_k, k \geqslant 1$;
  - $\mathcal{G}_2^-$ is the set of elements of $\mathcal{G}^-$ ending with U.

Obviously, we have $\mathcal{G}^+ = \{\varepsilon\} \cup \mathcal{G}_1^+ \cup \mathcal{G}_2^+$, $\mathcal{G}^- = \mathcal{G}_1^- \cup \mathcal{G}_2^-$, and $\mathcal{G} = \mathcal{G}^+ \cup \mathcal{G}^-$. Also, we denote by $\overleftarrow{\mathcal{P}}$ the set of GDAP obtained from a prime DAP in $\mathcal{P}$ by mirroring it. For instance, the mirror of $U^3D_2UD_2 \in \mathcal{P}$ is $D_2UD_2U^3 \in \overleftarrow{\mathcal{P}}$.

3 Enumeration of GDAP

In this section, we present a generating function that counts GDAP with respect to the length.

Any element of $\mathcal{G}_1^+$ is of the form $\alpha\beta$, where $\alpha \in \mathcal{G}^+$ and $\beta \in \mathcal{P} \cup \{UD\}$. Any element of $\mathcal{G}_2^+$ is either of the form (i) $\alpha\beta$ where $\alpha \in \mathcal{G}_2^+$ and $\beta \in \overleftarrow{\mathcal{P}} \cup \{DU\}$, or of the form (ii) $\overline{\alpha\beta}$ where $\alpha \in \mathcal{G}_1^+$, $\beta \in \overleftarrow{\mathcal{P}} \cup \{DU\}$, and $\overline{\alpha\beta}$ is the path obtained by merging the last step of α together with the first step of β, i.e. if $\alpha = xD_i$ and $\beta = D_jy$, then $\overline{\alpha\beta} = xD_{i+j}y$. Finally, any element of $\mathcal{G}^+$ is either the empty path ε, an element of $\mathcal{G}_1^+$, or an element of $\mathcal{G}_2^+$.

Thus, we deduce the following system of equations:

$$\begin{cases} G_1^+(x) = G^+(x)(x^2 + P(x)) \\ G_2^+(x) = G_2^+(x)(x^2 + P(x)) + \dfrac{1}{x} G_1^+(x)(x^2 + P(x)) \\ G^+(x) = 1 + G_1^+(x) + G_2^+(x), \end{cases}$$

where P, G_1^+, G_2^+, and G^+ are the generating functions with respect to the length for the cardinalities of $\mathcal{P}$, $\mathcal{G}_1^+$, $\mathcal{G}_2^+$, and $\mathcal{G}^+$, respectively. We have $P(x) = xA(x)$, where A is the generating function for the set $\mathcal{A}$ of Dyck paths with air pockets, see (2.1). Solving the system, we get:

$$\begin{aligned} G_1^+(x) &= \frac{x^2}{\sqrt{x^4 - 2x^3 - x^2 - 2x + 1}}, \\ G_2^+(x) &= \frac{(1 - x - x^2)\sqrt{x^4 - 2x^3 - x^2 - 2x + 1} - x^4 + 2x^3 + x^2 + 2x - 1}{2x^4 - 4x^3 - 2x^2 - 4x + 2}, \\ G^+(x) &= \frac{(1 - x + x^2)\sqrt{x^4 - 2x^3 - x^2 - 2x + 1} + x^4 - 2x^3 - x^2 - 2x + 1}{2x^4 - 4x^3 - 2x^2 - 4x + 2}, \end{aligned}$$

and the first terms of the respective series expansions associated with those generating functions are:

- $x^2 + x^3 + 2x^4 + 5x^5 + 11x^6 + 26x^7 + 63x^8 + 153x^9 + 376x^{10} + O(x^{11})$,
- $x^3 + 2x^4 + 5x^5 + 13x^6 + 32x^7 + 80x^8 + 201x^9 + 505x^{10} + O(x^{11})$,
- $1 + x^2 + 2x^3 + 4x^4 + 10x^5 + 24x^6 + 58x^7 + 143x^8 + 354x^9 + 881x^{10} + O(x^{11})$.

They correspond to the OEIS sequences A051286, A110320, and A110236.

On the other hand, any element of $\mathcal{G}^-$ is of the form $\alpha\beta$, where $\alpha \in \overleftarrow{\mathcal{P}} \cup \{DU\}$ and $\beta \in \mathcal{G}$. Any element of $\mathcal{G}$ is either an element of $\mathcal{G}^+$ or an element of $\mathcal{G}^-$.

Thus, we deduce the following system of equations:

$$\begin{cases} G^-(x) = (x^2 + P(x))G(x) \\ G(x) = G^+(x) + (x^2 + P(x))G(x), \end{cases}$$

where G^- and G are the generating functions with respect to the length for the cardinalities of $\mathcal{G}^-$ and $\mathcal{G}$, respectively. Solving the system, we get:

$$G^-(x) = \frac{(1 - x + x^2 - R)(x^4 - 2x^3 - x^2 - 2x + 1 + (1 - x + x^2)R)}{2(1 + x - x^2 + R)R}$$

with $R = \sqrt{x^4 - 2x^3 - x^2 - 2x + 1}$, and we have the following result.

Theorem 3.1. *The o.g.f. that counts the set $\mathcal{G}$ with respect to the length is given by*

$$G(x) = \frac{x^4 - 2x^3 - x^2 - 2x + 1 + (1 - x + x^2)R}{(1 + x - x^2 + R)R}$$

with R defined as above.

Notice that there is a bijection χ between the sets $\mathcal{G}_1^+$ and $\mathcal{G}_2^-$ defined as follows: for $\alpha \in \mathcal{G}_1^+$, $\chi(\alpha)$ is simply the mirror of α, for instance $\chi(UUUD_3) = D_3UUU$.

So, we easily have

$$\begin{cases} G_2^-(x) = G_1^+(x) \\ G_1^-(x) = G^-(x) - G_2^-(x). \end{cases}$$

The first terms of the series expansions of G^-, G, G_1^- and G_2^- are respectively

- $x^2 + x^3 + 3x^4 + 7x^5 + 16x^6 + 39x^7 + 95x^8 + 233x^9 + 577x^{10} + O(x^{11})$,
- $1 + 2x^2 + 3x^3 + 7x^4 + 17x^5 + 40x^6 + 97x^7 + 238x^8 + 587x^9 + 1458x^{10} + O(x^{11})$,
- $x^4 + 2x^5 + 5x^6 + 13x^7 + 32x^8 + 80x^9 + 201x^{10} + O(x^{11})$,
- $x^2 + x^3 + 2x^4 + 5x^5 + 11x^6 + 26x^7 + 63x^8 + 153x^9 + 376x^{10} + O(x^{11})$.

They correspond to the OEIS sequences A203611, A051291, A110320, and A051286.

4 Partial GDAP ending at a given ordinate

Let pre$(\mathcal{G})$ be the set of partial GDAP, i.e. the set of all prefixes of elements of $\mathcal{G}$, see the second path in Figure 2 for an example. In this part, we enumerate partial GDAP ending at a given ordinate with an up-step (resp. with a down-step) with respect to the length. Let f_k (resp. g_k) be the generating function for the number (with respect to the length) of partial GDAP ending at ordinate $k \in \mathbb{Z}$ with an up-step (resp. with a down-step). For short, we will write f_k and g_k instead of $f_k(x)$ and $g_k(x)$.

According to the results in Section 2, for $k = 0$ we obviously have:

$$f_0 = G_2^+(x) + G_2^-(x) = \frac{1 - x + x^2 + \sqrt{x^4 - 2\,x^3 - x^2 - 2\,x + 1}}{2\sqrt{x^4 - 2\,x^3 - x^2 - 2\,x + 1}} - 1,$$

and

$$g_0 = G_1^+(x) + G_1^-(x) = \frac{\left(1 + x - x^2 - \sqrt{x^4 - 2\,x^3 - x^2 - 2\,x + 1}\right) x}{2\sqrt{x^4 - 2\,x^3 - x^2 - 2\,x + 1}}.$$

The first terms of the series expansions of f_0 and g_0 are respectively

- $x^2 + 2x^3 + 4x^4 + 10x^5 + 24x^6 + 58x^7 + 143x^8 + 354x^9 + 881x^{10} + O(x^{11})$,
- $x^2 + x^3 + 3x^4 + 7x^5 + 16x^6 + 39x^7 + 95x^8 + 233x^9 + 577x^{10} + O(x^{11})$.

They correspond to the sequences A110236 and A203611 in OEIS. Clearly, we have $G(x) = 1 + f_0 + g_0$.

For $k > 0$, a partial GDAP ending at ordinate k can be written $\alpha\beta$, where α is either empty or a GDAP ending on the x-axis with an up-step, and β is a partial DAP ending at ordinate k. Then, we obtain

$$f_k + g_k = (1 + f_0) \cdot T_k(x)$$

with

$$T_k(x) = x^k s_2^{k+1}, \text{ and } s_2 = \frac{1 + x - x^2 - \sqrt{-x^2 - 2x^3 - 2x + x^4 + 1}}{2x},$$

where T_k is the o.g.f. that counts DAP ending at ordinate k with respect to length, which is already obtained in [8].

As a consequence, we deduce the following result.

Theorem 4.1. *The o.g.f. that counts the partial GDAP ending at a positive ordinate (with respect to the length) is given by*

$$\frac{\left(x^2 - x - 1 + \sqrt{x^4 - 2\,x^3 - x^2 - 2\,x + 1}\right)^2}{4x\sqrt{x^4 - 2\,x^3 - x^2 - 2\,x + 1}}.$$

The first terms of the series expansion are: $x + x^2 + 4x^3 + 9x^4 + 22x^5$+$55x^6 + 136x^7 + 339x^8 + 849x^9 + 2132x^{10} + O(x^{11})$.

For $k < 0$, a partial GDAP ending at ordinate k can be written β or $\overline{\alpha\beta}$, where α is a GDAP ending with a down-step, and β is the symmetric about the x-axis of a partial DAP ending at ordinate $-k > 0$ in the right-to-left model studied in [8]. Then, we obtain

$$f_k + g_k = R_k(x)\left(1 + \frac{g_0(x)}{x}\right),$$

with

$$R_k(x) = (s_2 - 1) \cdot \frac{s_2^{-k-1}}{x}.$$

For instance, the first terms of the series expansion for $k = -1, -2$ are

- $x + 2x^2 + 4x^3 + 10x^4 + 24x^5 + 58x^6 + 143x^7 + 354x^8 + 881x^9 + 2204x^{10} + O(x^{11})$,
- $x + 2x^2 + 5x^3 + 13x^4 + 32x^5 + 80x^6 + 201x^7 + 505x^8 + 1273x^9 + 3217x^{10} + O(x^{11})$

which correspond to the sequences A110236 and A110320 in OEIS.

Notice that partial GDAP of length $n - 1$ and ending at ordinate $k = -1$ are in one-to-one correspondence with non-empty GDAP of length n and starting with an up-step (see the set $\mathcal{G}^+$ from Section 3). To perceive it, one can add an up-step at the end of a partial GDAP ending at $k = -1$, apply a symmetry about the x-axis, and consider the mirror of this path.

Moreover, $x^2 \cdot G_2^+(x)$ (see Section 3) is the generating function for partial GDAP ending at $k = -2$. Such partial GDAP of length n are in one-to-one correspondence with paths of length $n + 2$ in $\mathcal{G}_2^+$. To see it, from a partial GDAP ending at $k = -2$, one can add an up-step at the beginning of the path and another one at the end.

Since there is an infinite number of partial paths of length n ending at negative height, we cannot provide an ordinary generating function (with respect to the length) for these paths. So, we get around this in the next section by counting partial GDAP lying above the line $y = m$ for a given $m \leqslant 0$.

5 Minorized partial GDAP

Let us denote by $\mathrm{pre}(\mathcal{G})_m$ the set of partial GDAP which never go below the line $y = m$, $m \leqslant 0$, and let us reuse the same notations as in the previous section for the generating functions f_k and g_k in this subset of $\mathrm{pre}(\mathcal{G})$. Obviously, we have $f_k = 0$ for all $k \leqslant m$ and $g_k = 0$ for all $k < m$. By convenience, we count the empty path in f_0. Then, the o.g.f.'s satisfy the following equations:

$$\begin{cases} & f_0 = 1 + xf_{-1} + xg_{-1}, \\ \forall k \geqslant m+1, k \neq 0, & f_k = xf_{k-1} + xg_{k-1}, \\ \forall k \geqslant m, & g_k = \sum_{i=1}^{\infty} xf_{k+i}. \end{cases}$$

As a consequence, we have $f_{m+1} = xg_m$. Now, we introduce the bivariate generating functions

$$f(u,x) = f(u) = \sum_{k=m+1}^{\infty} u^k f_k \quad \text{and} \quad g(u,x) = g(u) = \sum_{k=m}^{\infty} u^k g_k.$$

Making use of the recursions above, we get:

$$\begin{aligned} f(u) &= 1 + xu\left(f(u) + g(u)\right), \\ g(u) &= \frac{x}{1-u}\left(u^m f(1) - f(u)\right). \end{aligned}$$

Plugging the second equation into the first one, we get:

$$f(u) = 1 + xuf(u) + \frac{x^2u}{1-u}\left(u^m f(1) - f(u)\right).$$

Solving for $f(u)$, we finally get:

$$f(u) = \frac{1 - u + x^2u^{m+1}f(1)}{1 - u - xu + xu^2 + x^2u}. \tag{5.1}$$

In order to compute $f(1)$, we use the kernel method (see [1, 7]) on $f(u)$. We can rewrite the denominator—which is a polynomial in u, of degree 2—as $x(u-r_1)(u-r_2)$, where:

$$r_1 = \frac{1 + x - x^2 + \sqrt{x^4 - 2x^3 - x^2 - 2x + 1}}{2x},$$

$$r_2 = \frac{1 + x - x^2 - \sqrt{x^4 - 2x^3 - x^2 - 2x + 1}}{2x},$$

and then, relation (5.1) implies

$$f(u) \cdot (x(u-r_1)(u-r_2)) = 1 - u + x^2u^{m+1}f(1).$$

Plugging $u = r_2$ (which has a Taylor expansion at $x = 0$), we obtain:

$$1 - r_2 + x^2r_2^{m+1}f(1) = 0,$$

which gives an expression for $f(1)$:

$$f(1) = \frac{r_2 - 1}{x^2r_2^{m+1}},$$

and then:

$$\begin{aligned} f(u) &= \frac{1 - u + (r_2 - 1)\left(\frac{u}{r_2}\right)^{m+1}}{1 - u - xu + xu^2 + x^2u}, \\ g(u) &= \frac{x}{1-u}\left(u^m f(1) - f(u)\right). \end{aligned}$$

Finally, we have:

Theorem 5.1. *The o.g.f. that counts the partial GDAP above the line $y = m$ (with respect to the length) is given by*

$$f(1) + g(1) = \frac{r_2^{-m} - r_2^{-1-m} - x^2}{x^3}.$$

For instance, if $m = -1, -2$ the first terms of the series expansions are

- $1+2x+4x^2+8x^3+17x^4+37x^5+82x^6+185x^7+423x^8+978x^9+2283x^{10}+O(x^{11})$,
- $1+3x+6x^2+13x^3+29x^4+65x^5+148x^6+341x^7+793x^8+1860x^9+4395x^{10}+O(x^{11})$.

They correspond to the sequences A004148 and A093128 in OEIS.

6 Partial (G)DAP bounded by $y = 0$ and $y = t$

6.1 Enumerative results

In this section, we count partial GDAP lying between the lines $y = 0$ and $y = t$, which correspond to partial DAP bounded by a given height $t > 0$. We introduce the notation f_k^t, g_k^t for $0 \leqslant k \leqslant t$, $f^t(u)$, and $g^t(u)$, which are the counterparts of f_k, g_k, $f(u)$, and $g(u)$ defined in the previous section. So, we deduce the following system of equations:

$$\left[\begin{array}{cccc|cccc} -1 & & & & 0 & & & \\ x & -1 & & & x & 0 & & \\ & \ddots & \ddots & & & \ddots & \ddots & \\ & & x & -1 & & & x & 0 \\ \hline 0 & x & \dots & x & -1 & & & \\ & 0 & \ddots & \vdots & & \ddots & & \\ & & \ddots & x & & & \ddots & \\ & & & 0 & & & & -1 \end{array}\right] \cdot \left[\begin{array}{c} f_0^t \\ \vdots \\ \vdots \\ f_t^t \\ \hline g_0^t \\ \vdots \\ \vdots \\ g_t^t \end{array}\right] = \left[\begin{array}{c} -1 \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{array}\right].$$

For a given height $t \geqslant 0$, the previous matrix (denoted A_t) is square with $2(t+1)$ rows. Using classical properties of the determinant (in particular $\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(AD - BC)$ whenever D is invertible, and C and D commute [9]), we can easily prove that $D_t = \det(A_t)$ satisfies

$$D_{t+2} + (x^2 - x - 1)D_{t+1} + xD_t = 0,$$

anchored with $D_0 = 1$, and $D_1 = 1 - x^2$. Then we deduce

$$D_t = \frac{2^t x^{t+1}}{W} \left(\frac{W - x^2 + x - 1}{(W - x^2 + x + 1)^{t+1}} + (-1)^{t+1} \frac{W + x^2 - x + 1}{(W + x^2 - x - 1)^{t+1}} \right),$$

where

$$W = \sqrt{x^4 - 2x^3 - x^2 - 2x + 1}.$$

For instance, we have $D_2 = x^4 - x^3 - 2x^2 + 1$, and $D_3 = -x^6 + 2x^5 + 2x^4 - 2x^3 - 3x^2 + 1$.

Using Cramer's rule to solve the system, for $0 \leqslant k \leqslant t$, we have

$$f_k^t = \frac{N_k^t}{D_t}, \quad g_k^t = \frac{N_{t+1+k}^t}{D_t}, \tag{6.1}$$

where N_k^t is the determinant of the matrix $A_t(k)$ obtained from A_t by replacing the $(k+1)$-th column with the vector $(-1, 0, \ldots, 0)^T$.

As we have done for D_t, it is easy to prove that N_k^t satisfies the following recurrence relations, for $0 \leqslant k \leqslant t$:

$$\begin{cases} N_0^t &= D_t \\ N_{2t+1}^t &= 0 \\ N_k^t &= xN_{k-1}^{t-1}1 & \leqslant k \leqslant t \\ N_{t+k}^t &= xN_{t+k-2}^{t-1} & 2 \leqslant k \leqslant t \\ N_{t+1}^t &= x^2N_0^{t-1} + xN_t^{t-1}. \end{cases}$$

See Table 1 for exact values of N_k^t when $0 \leqslant t \leqslant 3$ and $0 \leqslant k \leqslant 7$.

$k\backslash t$	0	1	2	3
0	1	$-x^2+1$	$x^4-x^3-2x^2+1$	$-x^6+2x^5+2x^4-2x^3-3x^2+1$
1	0	x	$-x^3+x$	$x^5-x^4-2x^3+x$
2		x^2	x^2	$-x^4+x^2$
3		0	$-x^4+x^3+x^2$	x^3
4			x^3	$x^6-2x^5-x^4+x^3+x^2$
5			0	$-x^5+x^4+x^3$
6				x^4
7				0

Table 1: The first values of N_k^t for $0 \leqslant t \leqslant 3$ and $0 \leqslant k \leqslant 7$.

Using (6.1) and the above recurrence relations for N_k^t, we can deduce closed forms for f_k^t, g_k^t, $0 \leqslant k \leqslant t$.

So, we can state the following result.

Theorem 6.1. *The o.g.f. that counts the nonempty GDAP bounded by the lines $y = 0$ and $y = t$ (with respect to the length) is*

$$g_0^t = \frac{N_{t+1}^t}{D_t}$$

with

$$N_{t+1}^t = \frac{2^{t+2}x^{t+3}(-1)^t}{W(x^2-x-1)^2 - W^3}\left(\frac{1}{(x^2-x-1+W)^t} - \frac{1}{(x^2-x-1-W)^t}\right).$$

For instance, if $t = 1, 2, 3, 4$, then we have

$$g_0^1 = \frac{x^2}{1-x^2}, g_0^2 = \frac{x^2\left(1+x-x^2\right)}{x^4-x^3-2x^2+1}, g_0^3 = \frac{x^2\left(x^4-2x^3-x^2+x+1\right)}{\left(x^3-2x^2-x+1\right)\left(1+x-x^3\right)}, \text{ and}$$

$$g_0^4 = \frac{-x^8 + 3x^7 - 3x^5 - 2x^4 + x^3 + x^2}{x^8 - 3x^7 - x^6 + 5x^5 + 4x^4 - 3x^3 - 4x^2 + 1},$$

and the first terms of the series expansion of these generating functions are respectively

- $x^2 + x^4 + x^6 + x^8 + x^{10} + O(x^{11})$,
- $x^2 + x^3 + x^4 + 3x^5 + 2x^6 + 6x^7 + 6x^8 + 11x^9 + 16x^{10} + O(x^{11})$,
- $x^2 + x^3 + 2x^4 + 3x^5 + 7x^6 + 9x^7 + 22x^8 + 32x^9 + 66x^{10} + O(x^{11})$,
- $x^2 + x^3 + 2x^4 + 4x^5 + 7x^6 + 16x^7 + 27x^8 + 63x^9 + 112x^{10} + O(x^{11})$.

The first two correspond to shifts of A000035 and A062200. The last two sequences do not appear in [10].

6.2 Bijection with a set of compositions

As stated above, the enumeration of the set $\mathcal{G}_n^{[0,2]}$ of GDAP bounded by $y = 0$ and $y = 2$ is given by $g_0^2 = \frac{x^2(1+x-x^2)}{x^4-x^3-2x^2+1}$ which have a series expansion where the coefficients coincide (up to a shift) with the sequence A062200 in [10]. In this part, for any $n \geqslant 0$, we exhibit a constructive bijection ψ between $\mathcal{G}_n^{[0,2]}$ and the set $\mathcal{C}(n-2)$ of compositions of $n-2$ such that no two consecutive parts have the same parity (see [6] for the enumeration of these objects, and [4] for more results about the enumeration of compositions with regard to several statistics on parts).

Let us define the map ψ. Assuming $n \geqslant 2$, let $\alpha = \alpha_1 \dots \alpha_n \in \mathcal{G}_n^{[0,2]}$ and $\alpha' = \alpha_2\alpha_3 \dots \alpha_{n-1}$. We write $\alpha' = B_1B_2 \dots B_r$ where each B_i is a subpath of α' satisfying the following rules:

- if α' does not contain U^2 and UD_2, then $r = 1$ and $B_1 = \alpha'$;
- otherwise, we split α' into subpaths B_i, $1 \leqslant i \leqslant r$, by cutting it after all up-steps that are followed by another up-step or a D_2-step.

For instance, if $\alpha = UUD_2UUDUD_2UDUDUUD_2 = U\alpha'D_2$, then $\alpha' = B_1B_2B_3B_4B_5$ where $B_1 = U$, $B_2 = D_2U$, $B_3 = UDU$, $B_4 = D_2UDUDU$, $B_5 = U$. We refer to Figure 3 for an illustration of this decompostion.

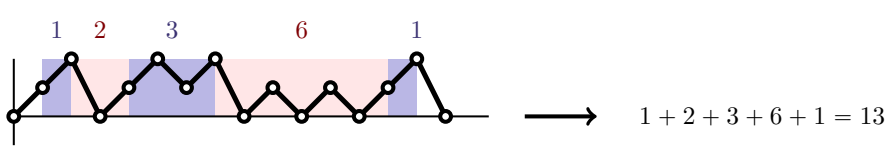


Figure 3: The image by ψ of $\alpha = UUD_2UUDUD_2UDUDUUD_2$ is $\psi(\alpha) = 1, 2, 3, 6, 1$.

Let $b_1, \dots, b_r$ be the lengths of the subpaths $B_1, \dots, B_r$, respectively. It is clear that $b_1 + b_2 + \dots + b_r = n - 2$. Moreover, if the subpath B_i starts with U and ends with U, then B_i is of he form $U(DU)^k$ for some $k \geqslant 0$, and B_{i+1} is necessarily of the form $D_2(UD)^\ell U$ for some ℓ, which implies that b_i is odd and b_{i+1} is even; if the subpath B_i

starts with D_2 and ends with U, then B_i is of the form $D_2(UD)^kU$ for some $k \geqslant 0$, and B_{i+1} is necessarily of the form $U(DU)^\ell$ for some ℓ, which implies that b_i is even and b_{i+1} is odd. Thus, two consecutive b_i and b_{i+1} always have different parities. Then, the above procedure defines a map ψ from $\mathcal{G}_n^{[0,2]}$ to the set $\mathcal{C}(n-2)$, and for $\alpha \in \mathcal{G}_n^{[0,2]}$, we set $\psi(\alpha) = b_1, b_2, \ldots, b_r$.

Theorem 6.2. *The map ψ from $\mathcal{G}_n^{[0,2]}$ to $\mathcal{C}(n-2)$ is a bijection.*

Proof. Since $\mathcal{G}_n^{[0,2]}$ and $\mathcal{C}(n-2)$ have the same cardinality (see the o.g.f. g_0^2 and the sequence A062200 at the end of subsection 6.1, it suffices to prove that ψ is surjective.

Let $c = c_1, \ldots, c_r$ be a composition in $\mathcal{C}(n-2)$, with $r \geqslant 2$ (the case $r = 1$ being trivial since if c_1 is even, then we have $c = \psi(U(DU)^{c_1/2}D)$, and if c_1 is odd we have $c = \psi(U(UD)^{(c_1-1)/2}UD_2)$).

For $r \geqslant 2$, we distinguish four cases:

(i) c_1 and c_r are even,

(ii) c_1 is even and c_r is odd,

(iii) c_1 and c_r are odd,

(iv) c_1 is odd and c_r is even.

According to each case, we define $\alpha \in \mathcal{G}_n^{[0,2]}$ such that $\psi(\alpha) = c$ as follows:

Case (i):

$$\alpha = U\,(DU)^{c_1/2}\,(UD)^{(c_2-1)/2}U\,D_2U(DU)^{(c_3-2)/2}\ldots D_2U(DU)^{(c_r-2)/2}\,D;$$

Case (ii):

$$\alpha = U\,(DU)^{c_1/2}\,(UD)^{(c_2-1)/2}U\,D_2U(DU)^{(c_3-2)/2}\ldots (UD)^{(c_r-1)/2}U\,D_2;$$

Case (iii):

$$\alpha = U\,(UD)^{(c_1-1)/2}U\,D_2U(DU)^{(c_2-2)/2}\,(UD)^{(c_3-1)/2}U\ldots (UD)^{(c_r-1)/2}U\,D_2;$$

Case (iv):

$$\alpha = U\,(UD)^{(c_1-1)/2}U\,D_2U(DU)^{(c_2-2)/2}\,(UD)^{(c_3-1)/2}U\ldots D_2U(DU)^{(c_r-2)/2}\,D.$$

For each case, it is clear that α belongs to $\mathcal{G}_n^{[0,2]}$, which implies that ψ is surjective, and then bijective. □

7 GDAP bounded by $y = -t$ and $y = t$

7.1 Enumerative results

In this section, we count GDAP lying between the lines $y = -t$ and $y = t$. We introduce the notation f_k^t, g_k^t for $-t \leqslant k \leqslant t$, $f^t(u)$, and $g^t(u)$, which are the counterparts of f_k, g_k,

$f(u)$, and $g(u)$. So, we deduce the following system of equations:

$$\left[\begin{array}{cccc|cccc} -1 & & & & 0 & & & \\ x & -1 & & & x & 0 & & \\ & \ddots & \ddots & & & \ddots & \ddots & \\ & & x & -1 & & & x & 0 \\ \hline 0 & x & \dots & x & -1 & & & \\ & 0 & \ddots & \vdots & & \ddots & & \\ & & \ddots & x & & & \ddots & \\ & & & 0 & & & & -1 \end{array}\right] \cdot \left[\begin{array}{c} f_{-t}^t \\ \vdots \\ f_{-1}^t \\ f_0^t \\ f_1^t \\ \vdots \\ f_t^t \\ \hline g_{-t}^t \\ \vdots \\ \vdots \\ g_t^t \end{array}\right] = \left[\begin{array}{c} 0 \\ \vdots \\ 0 \\ -1 \\ 0 \\ \vdots \\ 0 \\ \hline 0 \\ \vdots \\ \vdots \\ 0 \end{array}\right].$$

For a given height $t \geqslant 0$, the previous matrix (denoted $A_t^{'}$) is square with $2(2t+1)$ rows. We notice that for all $t \geqslant 0$, the matrix $A_t^{'}$ is identical to the matrix A_{2t} defined in the previous section. Hence, we have $D_t^{'} := \det(A_t^{'}) = \det(A_{2t}) = D_{2t}$, i.e.

$$D_t^{'} = \frac{4^t x^{2t+1}}{W} \left(\frac{W - x^2 + x - 1}{(W - x^2 + x + 1)^{2t+1}} - \frac{W + x^2 - x + 1}{(W + x^2 - x - 1)^{2t+1}} \right),$$

where

$$W = \sqrt{x^4 - 2x^3 - x^2 - 2x + 1}.$$

Using Cramer's rule to solve the system, for $-t \leqslant k \leqslant t$, we have

$$f_k^t = \frac{\widetilde{N}_k^t}{D_t^{'}}, \quad g_k^t = \frac{\widetilde{N}_{2t+1+k}^t}{D_t^{'}}, \tag{7.1}$$

where $\widetilde{N}_k^t$ is the determinant of the matrix $A_t^{'}(k)$ obtained from $A_t^{'}$ by replacing the $(k+t+1)$-th column with the vector $(0, \dots, 0, -1, 0, \dots, 0)^T$, where the -1 is in the $(t+1)$-th position.

Now, we focus on the calculation of f_k^t and g_k^t for $k = 0$. The other cases can be obtained similarly, but they are much more technical and less interesting to present them here. With the same arguments as in the previous section (in particular, using the mentioned property of the determinant on blocks), it is easy to prove that $\widetilde{N}_0^t$ satisfies:

$$\widetilde{N}_0^t = D_{t-1} \cdot D_t.$$

Moreover, we have

$$\widetilde{N}_{2t+1}^t = D_{t-1} \cdot N_{t+1}^t.$$

Using the results obtained in the previous section, these two relations allow to obtain a closed form for $\widetilde{N}_0^t$ and $\widetilde{N}_{2t+1}^t$. Using (7.1), we deduce closed forms for f_0^t and g_0^t and we can state the following result.

Theorem 7.1. *The o.g.f. that counts the nonempty GDAP bounded by the lines $y = -t$ and $y = t$ (with respect to the length) is*

$$f_0^t + g_0^t = \frac{D_{t-1}}{D_{2t}} \cdot \left(D_t + N_{t+1}^t\right)$$

where D_t and N_{t+1}^t are defined in the previous section.

For instance, if $t = 1, 2, 3$, then we have

$$f_0^1 + g_0^1 = \frac{1}{x^4-x^3-2x^2+1}, f_0^2 + g_0^2 = \frac{(x-1)(x+1)^2}{x^7-2x^6-3x^5+2x^4+6x^3+3x^2-x-1},$$

$$f_0^3 + g_0^3 = \frac{\left(x^4-x^3-2x^2+1\right)^2}{x^{12}-5x^{11}+4x^{10}+10x^9-4x^8-19x^7-4x^6+17x^5+11x^4-5x^3-6x^2+1},$$

and the first terms of the series expansion of these generating functions are respectively

- $1 + 2x^2 + x^3 + 3x^4 + 4x^5 + 5x^6 + 10x^7 + 11x^8 + 21x^9 + 27x^{10} + O(x^{11})$,
- $1 + 2x^2 + 3x^3 + 5x^4 + 13x^5 + 22x^6 + 48x^7 + 93x^8 + 190x^9 + 375x^{10} + O(x^{11})$,
- $1 + 2x^2 + 3x^3 + 7x^4 + 15x^5 + 36x^6 + 75x^7 + 176x^8 + 386x^9 + 869x^{10} + O(x^{11})$.

The first one corresponds to a shift of A122514. The last two sequences do not appear in [10].

7.2 Bijection with a set of compositions

As stated above, the enumeration of the set $\mathcal{G}_n^{[-1,1]}$ of GDAP bounded by $y = -1$ and $y = 1$ is given by $f_0^1 + g_0^1 = \frac{1}{x^4-x^3-2x^2+1}$ which have a series expansion where the coefficients coincide (up to a shift) with the sequence A122514. In this part, for any $n \geqslant 0$, we exhibit a constructive bijection ϕ between $\mathcal{G}_n^{[-1,1]}$ and the set $\mathcal{C}'(n+3)$ of compositions of $n+3$ such that the first part is odd, the last part is even, and no two consecutive parts have the same parity.

Now, let us define the map ϕ. Assuming $n \geqslant 2$, let $\alpha = \alpha_1 \dots \alpha_n \in \mathcal{G}_n^{[-1,1]}$. We write $\alpha = B_1 B_2 \dots B_r$ where each B_i is a subpath of α obtained by applying the same decomposition made on α' in subsection 6.2. Let $b_1, b_2, \dots, b_r$ be the lengths of subpaths $B_1, \dots, B_r$ respectively. In the case $r \geqslant 2$, let L be the reversed composition $b_r, \dots, b_1$. The composition $\phi(\alpha)$ of $n+3$ is obtained from L after going through the following process:

- if b_{r-1} is even, then add 1 to b_r; otherwise, append 1 at the beginning of L;
- if b_1 is even, then add 2 to b_1; otherwise, append 2 at the end of L.

For instance, if $\alpha = UD_2UUDUD_2UDUDUUD$ then we have $B_1 = U$, $B_2 = D_2U$, $B_3 = UDU$, $B_4 = D_2UDUDU$, $B_5 = UD$, $b_1 = 1, b_2 = 2, b_3 = 3, b_4 = 6, b_5 = 2$, $L = 2, 6, 3, 2, 1$, and $\phi(\alpha) = 3, 6, 3, 2, 1, 2$.

In the case $r = 1$, α is either $(UD)^{n/2}$ or $(DU)^{n/2}$. So, we define $\phi((UD)^{n/2}) = n+1, 2$, and $\phi((DU)^{n/2}) = 1, n+2$ (these are valid compositions since n has to be even in these cases).

In the case $r = 0$, α is empty, and we define $\phi(\varepsilon) = 1, 2$.

Due to the definition, it is clear that the composition $\phi(\alpha)$ belongs to $\mathcal{C}'(n+3)$, for any $n \geqslant 0$.

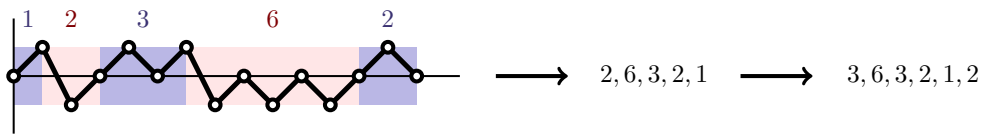


Figure 4: The image by ϕ of $\alpha = UD_2UUDUD_2UDUDUUD$ is $\phi(\alpha) = 3, 6, 3, 2, 1, 2$.

Theorem 7.2. *The map ϕ from $\mathcal{G}_n^{[-1,1]}$ to $\mathcal{C}'(n+3)$ is a bijection.*

Proof. Since $\mathcal{G}_n^{[-1,1]}$ and $\mathcal{C}'(n+3)$ have the same cardinality (see the end of Subsection 7.1), it suffices to prove that ϕ is surjective.

Let $c = c_1, \ldots, c_r$ be an element of $\mathcal{C}'(n+3)$, with $r \geqslant 2$ (the case $r = 1$ does not occur since c_1 is odd and c_r is even implies $r \geqslant 2$). Thus, r is necessarily even. We distinguish four cases:

(i) $c_1 = 1$ and $c_r = 2$,

(ii) $c_1 = 1$ and c_r is even and greater than 2,

(iii) c_1 is odd and greater than 1 and $c_r = 2$,

(iv) c_1 is odd and greater than 1 and c_r is even and greater than 2.

According to each case, we define $\alpha \in \mathcal{G}_n^{[-1,1]}$ such that $\phi(\alpha) = c$:
Case (i): Since c_{r-1} is odd, c_{r-2} is even, and so on, and finally c_2 is even.

$$\alpha = (UD)^{(c_{r-1}-1)/2}U\ D_2U(DU)^{(c_{r-2}-2)/2} \ldots (UD)^{(c_3-1)/2}U\ D_2U(DU)^{(c_2-2)/2}.$$

Case (ii):

$$\alpha = (DU)^{(c_r-2)/2}\ (UD)^{(c_{r-1}-1)/2}U \ldots (UD)^{(c_3-1)/2}U\ D_2U(DU)^{(c_2-2)/2}.$$

Case (iii):

$$\alpha = (UD)^{(c_{r-1}-1)/2}U\ D_2U(DU)^{(c_{r-2}-2)/2} \ldots D_2U(DU)^{(c_2-2)/2}\ (UD)^{(c_1-1)/2}.$$

Case (iv):

$$\alpha = (DU)^{(c_r-2)/2}\ (UD)^{(c_{r-1}-1)/2}U \ldots D_2U(DU)^{(c_2-2)/2}\ (UD)^{(c_1-1)/2}.$$

For each case, it is clear that α belongs to $\mathcal{G}_n^{[-1,1]}$, which implies that ϕ is surjective, and then bijective. □

8 DAP with a special first return decomposition

Recently in [3], the authors introduced and enumerated the subset $\mathcal{D}^{h,\geqslant}$ of restricted Dyck paths defined as follows: the set $\mathcal{D}^{h,\geqslant}$ is the union of the empty Dyck path with all Dyck paths P having a first return decomposition $P = U\alpha D\beta$ satisfying the conditions:

$$\begin{cases} \alpha, \beta \in \mathcal{D}^{h,\geqslant}, \\ h(U\alpha D) \geqslant h(\beta), \end{cases} \tag{8.1}$$

where $h(\alpha)$ is the maximal ordinate reached by the path α. The authors prove algebraically and bijectively that n-length paths in $\mathcal{D}^{h,\geqslant}$ are in one-to-one correspondence with Motzkin paths of length n. Based on this decomposition and in the same way as for Dyck paths, we define a subset of $\mathcal{A} \cup \{\varepsilon\} \subset \mathcal{G}$ as follows. The set $\mathcal{H}$ is the union of the empty path with all DAP $\gamma \in \mathcal{A}$ having a first return decomposition satisfying the following condition:

(C) $\gamma = \alpha\beta$ with $\alpha \in \mathcal{P} \cup \{UD\}$, and $\alpha^\flat \in \mathcal{H}$ whenever $\alpha \neq UD, \beta \in \mathcal{H}$ and $h(\alpha) \geqslant h(\beta)$.

For $n \geqslant 0$, we denote by $\mathcal{H}_n$ the set of DAP of length n in $\mathcal{H}$. For instance, we have $\mathcal{H}_0 = \{\varepsilon\}$, $\mathcal{H}_1 = \emptyset$, $\mathcal{H}_2 = \{UD\}$, $\mathcal{H}_3 = \{UUD_2\}$, $\mathcal{H}_4 = \{UUUD_3, UDUD\}$, $\mathcal{H}_5 = \{UUUUD_4, UUDUD_2, UUD_2UD\}$.

In this section, we enumerate the set $\mathcal{H}_n$. For $k \geqslant 0$, let $A_k(x) = \sum_{n\geqslant 0} a_{n,k}x^n$ (resp. $B_k(x) = \sum_{n\geqslant 0} b_{n,k}x^n$) be the generating function where the coefficient $a_{n,k}$ (resp. $b_{n,k}$) is the number of DAP in $\mathcal{H}_n$ having a maximal height equal to k (resp. of at most k). So, we have $B_k(x) = \sum\limits_{i=0}^{k} A_i(x)$ and the generating function for the set $\mathcal{H}$, namely $B(x)$, is given by $B(x) = \lim\limits_{k\to\infty} B_k(x)$.

Due to the definition of $\mathcal{H}$, we have

$$\begin{cases} A_0(x) = B_0(x) = 1, \\ A_1(x) = x^2 A_0(x) B_1(x), \\ A_k(x) = xA_{k-1}(x)B_k(x). \end{cases}.$$

Lemma 8.1. *For $k \geqslant 1$, we have*

$$B_{k-1}(x) = \frac{(1 - x^3 + x)B_k(x) - 1}{x^2 B_k(x) + x}.$$

Proof. We proceed by induction on k. Since $B_0(x) = 1$ and $B_1(x) = \frac{x^2}{1-x^2}$ it is easy to check that $B_0(x) = \frac{(1-x^3+x)B_1(x)-1}{x^2B_1(x)+x}$.

Now, assume that $B_{i-1}(x) = \frac{(1-x^3+x)B_i(x)-1}{x^2B_i(x)+x}$ for $1 \leqslant i \leqslant k-2$, we prove the result for $i = k-1$. From the above equations, and the recurrence hypothesis on $B_{k-2}(x)$, we obtain

$$\begin{aligned} B_k(x) &= A_k(x) + B_{k-1}(x) \\ &= xA_{k-1}(x)B_k(x) + B_{k-1}(x) \\ &= x(B_{k-1} - B_{k-2})B_k(x) + B_{k-1}(x) \\ &= x(B_{k-1} - \frac{(1 - x^3 + x)B_{k-1}(x) - 1}{x^2 B_{k-1}(x) + x})B_k(x) + B_{k-1}(x). \end{aligned}$$

Isolating $B_{k-1}(x)$, we obtain $B_{k-1}(x) = \frac{(1-x^3+x)B_k(x)-1}{x^2B_k(x)+x}$, which completes the induction. □

Taking the limit in the relation of Lemma 1 whenever k tends to ∞, we obtain

$$B(x) = \frac{(1 + x - x^3)B(x) - 1}{x^2 B(x) + x},$$

which induces the following result.

Theorem 8.2. *The o.g.f. that counts the set $\mathcal{H}$ with respect to the length is given by*

$$B(x) = \frac{1 - x^3 - \sqrt{x^6 - 2\,x^3 - 4\,x^2 + 1}}{2x^2}.$$

The above generating function $B(x)$ counts also Motzkin paths of length n avoiding the patterns UH, HU and HH, see sequence A329699. The first terms of its series expansion are: $1 + x^2 + x^3 + 2x^4 + 3x^5 + 6x^6 + 10x^7 + 20x^8 + 36x^9 + 72x^{10} + 136x^{11} + 273x^{12}$.

We finish this part with a natural question.

Open question: The sets $\mathcal{H}$ and that of $\{UH, HU, HH\}$-avoiding Motzkin paths are thus in bijection and it would be interesting to exhibit a constructive bijection between them.

References

[1] C. Banderier, M. Bousquet-Mélou, A. Denise, P. Flajolet, D. Gardy and D. Gouyou-Beauchamps, Generating functions for generating trees, *Discrete Math.* **246** (2002), 29–55, doi: 10.1016/S0012-365X(01)00250-3, https://doi.org/10.1016/S0012-365X(01)00250-3.

[2] J.-L. Baril, S. Kirgizov, R. Maréchal and V. Vajnovszki, Enumeration of Dyck paths with air pockets, *J. Integer Seq.* **26** (2023), article 23.3.2, 23, https://cs.uwaterloo.ca/journals/JIS/VOL26/Kirgizov/kirg5.pdf.

[3] J.-L. Baril, S. Kirgizov and A. Petrossian, Dyck paths with a first return decomposition constrained by height, *Discrete Math.* **341** (2018), 1620–1628, doi:10.1016/j.disc.2018.03.002, https://doi.org/10.1016/j.disc.2018.03.002.

[4] S. Heubach and T. Mansour, *Combinatorics of Compositions and Words*, Discrete Math. Appl. (Boca Raton), CRC Press, Boca Raton, FL, 2009, doi:10.1201/9781420072686, https://doi.org/10.1201/9781420072686.

[5] A. Krinik, G. Rubino, D. Marcus, R. J. Swift, H. Kasfy and H. Lam, Dual processes to solve single server systems, *J. Stat. Plann. Inference* **135** (2005), 121–147, doi:10.1016/j.jspi.2005.02.010, https://doi.org/10.1016/j.jspi.2005.02.010.

[6] T. Mansour, M. Shattuck and M. C. Wilson, Congruence successions in compositions, *Discrete Math. Theor. Comput. Sci.* **16** (2014), 327–338, http://www.dmtcs.org/dmtcs-ojs/index.php/dmtcs/issue/view/133.1.html.

[7] H. Prodinger, The kernel method: a collection of examples, *Sémin. Lothar. Comb.* **50** (2003), b50f, 19 pp.

[8] H. Prodinger, Partial Dyck paths with air pockets, *Integers* **22** (2022), paper a94, 8 pp., http://math.colgate.edu/~integers/w94/w94.pdf.

[9] J. R. Silvester, Determinants of block matrices, *Math. Gaz.* **84** (2000), 460–467, doi:10.2307/3620776, https://doi.org/10.2307/3620776.

[10] N. J. A. Sloane, *The On-line Encyclopedia of Integer Sequences*, http://oeis.org.

ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA
https://doi.org/10.26493/1855-3974.3035.6ac
(Also available at http://amc-journal.eu)

Two kinds of partial Motzkin paths with air pockets*

Jean-Luc Baril †
LIB, Université de Bourgogne, B.P. 47 870, 21078 Dijon Cedex, France

Paul Barry
School of Science, South East Technological University (SETU), Ireland

Received 23 December 2022, accepted 23 October 2023

Abstract

Motzkin paths with air pockets (MAP) are defined as a generalization of Dyck paths with air pockets by allowing some horizontal steps with certain conditions. In this paper, we introduce two generalizations. The first one consists of lattice paths in $\mathbb{N}^2$ starting at the origin, made of steps $U = (1, 1)$, $D_k = (1, -k)$, $k \geqslant 1$ and $H = (1, 0)$, where two down steps cannot be consecutive, while the second one are lattice paths in $\mathbb{N}^2$ starting at the origin, made of steps U, D_k and H, where each step D_k and H is necessarily followed by an up step, except for the last step of the path. We provide enumerative results for these paths according to the length, the type of the last step, and the height of its end-point. A similar study is made for these paths read from right to left. As a byproduct, we obtain new classes of paths counted by the Motzkin numbers. Finally, we express our results using Riordan arrays.

Keywords: Enumeration, Motzkin paths, kernel method, Riordan array.

Math. Subj. Class. (2020): 05A05, 05A15, 05A15

1 Introduction

In a recent paper [2], the authors introduce, study and enumerate special classes of lattice paths, called *Dyck paths with air pockets* (DAP for short). Such paths are non empty lattice paths in the first quadrant of $\mathbb{Z}^2$ starting at the origin, and consisting of up-steps $U = (1, 1)$ and down-steps $D_k = (1, -k)$, $k \geqslant 1$, where two down steps cannot be consecutive.

*The authors would like to thank the anonymous referees for useful remarks and comments.
†Corresponding author.
E-mail addresses: barjl@u-bourgogne.fr (Jean-Luc Baril), pbarry@wit.ie (Paul Barry)

This work is licensed under https://creativecommons.org/licenses/by/4.0/

Accepted manuscript

The *length* of a path is the number of its steps. These paths can be viewed as ordinary Dyck paths where each maximal run of down-steps is condensed into one large down step. As mentioned in [2], they also correspond to a stack evolution with (partial) reset operations that cannot be consecutive (see for instance [7]). Whenever the last point is on the x-axis, they prove that DAP of length n are in one-to-one correspondence with the peakless Motzkin paths of length $n-1$. They also investigate the popularity of many patterns in these paths and they give asymptotic approximations. We also refer to [10] for the enumeration of DAP with respect to the length, the type (up or down) of the last step and the height of the end-point (i.e., the ordinate of the end-point). In a second work [3], the authors make a study for a generalization of these paths by allowing them to go below the x-axis. They call these paths Grand Dyck paths with air pockets (GDAP), and they also yield enumerative results for these paths according to the length and several restrictions on the height.

In this paper, we introduce two generalizations of partial Dyck paths of air pockets by allowing some possible horizontal steps $H = (1, 0)$ with some conditions. These two kinds of paths can be viewed as special partial Motzkin paths (lattice paths in $\mathbb{N}^2$ starting at the origin and made of $U = (1, 1)$, $D = (1, -1)$, and $H = (1, 0)$), where each maximal run of down-steps is condensed into one large down step.

Definition 1.1. A partial Motzkin path with air pocket of the first kind is a lattice path in $\mathbb{N}^2$ starting at the origin, consisting of steps U, D_k and H, where two down steps cannot be consecutive. Let $\mathcal{M}_1$ be the set of these paths.

Definition 1.2. A partial Motzkin path with air pocket of second kind is a lattice path in $\mathbb{N}^2$ starting at the origin, consisting of steps U, D_k and H, where any step H and D_k (except the last step of the path) is immediately followed by an up step U. Let $\mathcal{M}_2$ be the set of these paths.

Whenever these paths end on the x-axis, we call them *Motzkin paths with air pockets* of the first and second kinds, respectively. For short, we denote by PMAP all paths in $\mathcal{M}_i$, $i \in \{1, 2\}$, and we denote by MAP all paths ending on the x-axis.

For instance, $UUUDHUD_2 \in \mathcal{M}_1$ and $UUUDUHUD_2 \in \mathcal{M}_2$ are two PMAP of the first and second kinds, respectively. The paths $UUUDHUD_3UD$ and $UUUDUHUD_4$ UD are MAP of the first and second kinds, respectively. See also Figure 1 and Figure 5 for other examples of PMAP and MAP.

We also consider lattice paths obtained from MAP in $\mathcal{M}_1$ and $\mathcal{M}_2$ by reading them from right to left and by replacing down-steps with up-steps and *vice versa*. More precisely, we define the two sets $\mathcal{M}'_1$ and $\mathcal{M}'_2$ as follows.

Definition 1.3. Let $\mathcal{M}'_1$ be the set of paths in $\mathbb{N}^2$ starting at the origin, consisting of up steps $U_k = (1, k)$, $k \geqslant 1$, horizontal step H and down step D, where two up steps cannot be consecutive.

Definition 1.4. Let $\mathcal{M}'_2$ be the set of paths in $\mathbb{N}^2$ starting at the origin, consisting of up steps $U_k = (1, k)$, $k \geqslant 1$, horizontal step H and down step D, where any H and U_k, $k \geqslant 1$, (except the first step of the path) is preceded by a down step D.

All paths in $\mathcal{M}'_1$ and $\mathcal{M}'_2$ will be be called *PMAP from right to left*, and in the case where they end on the x-axis, we call them *MAP from right to left*. For instance, we have $UDU_3DHUD \in \mathcal{M}'_1$, $UDU_4DHDU \in \mathcal{M}'_2$, and the path $UDU_3DHUDDD$ (resp.

Accepted manuscript

$UDU_4DHDUDDD$) is a MAP of the first (resp. second) kind from right to left. See also Figure 3 and Figure 7 for other examples of PMAP and MAP from right to left. Notice that $D_1 = D$ and $U_1 = U$, and we will use both notations.

Throughout the paper, and for each class of paths $\mathcal{M}_i$ and $\mathcal{M}'_i$, $i \in \{1, 2\}$, described above, we will use the following notations. For $k \geqslant 0$, we consider the generating function $f_k = f_k(z)$ (resp. $g_k = g_k(z)$, resp. $h_k = h_k(z)$), where the coefficient of z^n in the series expansion is the number of partial Motzkin paths with air pockets of length n ending at height k with an up-step, (resp. with a down-step, resp. with a horizontal step H).

We introduce the bivariate generating functions

$$F(z,u) = \sum_{k\geqslant 0} u^k f_k(z), \quad G(z,u) = \sum_{k\geqslant 0} u^k g_k(z), \text{ and } H(z,u) = \sum_{k\geqslant 0} u^k h_k(z).$$

For short, we also use the notation $F(u), G(u)$ and $H(u)$ for these functions, and we introduce the generating function

$$\texttt{Total}(z,u) = F(u) + G(u) + H(u).$$

Finally, for any bivariate generating function $H(z,u) = H(u)$, we will use the notation $[u^k]H(u)$ for the coefficient of u^k in $H(u)$.

The outline of this paper is the following. In Section 2, we present enumerative results for partial Motzkin paths with air pockets of the first kind, and for these paths when we read them from right to left. We provide bivariate generating functions that count these paths with respect to the length, the type of the last step (up, down or horizontal step) and the height of the end-point. In Section 3, we make a similar study for PMAP of second kind, and we present new classes of lattice paths counted by the well known Motzkin numbers. All these results are obtained algebraically by using the famous kernel method for solving several systems of functional equations. More precisely, Sections 2.1, 2.2, 3.1 and 3.2 have the same structure: we show how to obtain a system of equations involving f_k, g_k and h_k, and we apply the kernel method in order to provide some expressions of $F(u)$, $G(u)$ and $H(u)$. Finally, in Section 4 we express our results using Riordan arrays and we deduce closed forms for PMAP of length n ending at height k.

2 PMAP of the first kind

In this section, we focus on PMAP of the first kind, i.e. lattices paths in $\mathbb{N}^2$ starting at the origin, made of steps U, D_k and H, such that two down steps cannot be consecutive. The first subsection considers the paths in $\mathcal{M}_1$, while the second subsection handles the paths in $\mathcal{M}'_1$ (see Introduction for the definition of these two sets). We yield enumerative results for these paths according to the length, the type of the last step, and the height of its end-point.

2.1 PMAP in $\mathcal{M}_1$ - From left to right

In this part, we consider PMAP in $\mathcal{M}_1$. Figure 1 shows two examples of such paths.

Let P be a length n PMAP in $\mathcal{M}_1$ ending at height $k \geqslant 0$. If the last step of P is U, then $k \geqslant 1$ and P can be written $P = QU$ where Q is a length $(n-1)$ PMAP ending at height $k-1$. So, we obtain the first relation $f_k = zf_{k-1} + zg_{k-1} + zh_{k-1}$ for $k \geqslant 1$, anchored with $f_0 = 1$ by considering the empty path. If the last step of P is a down step D_a, $a \geqslant 1$, then we have $P = QD_a$ where Q is a length $(n-1)$ PMAP ending at height

Accepted manuscript

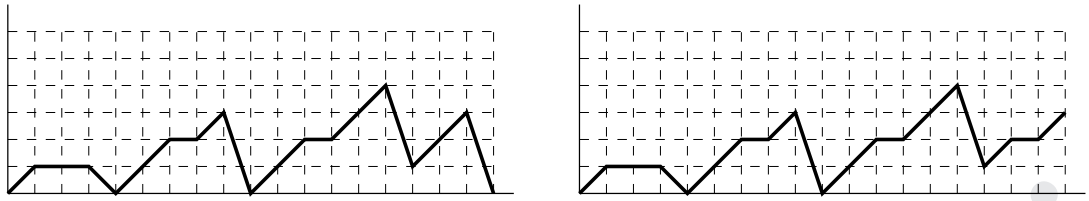


Figure 1: The left drawing shows a Motzkin path with air pockets of length 18 in $\mathcal{M}_1$. The right drawing shows a partial Motzkin path with air pockets of length 18 ending at height 3 in $\mathcal{M}_1$.

$\ell = a + k \geqslant k + 1$ with no down step at its end. So, we obtain the second relation $g_k = z\sum_{\ell\geqslant k+1} f_\ell + z\sum_{\ell\geqslant k+1} h_\ell$. If the last step of P is a horizontal step H, then we have $P = QH$ where Q is a length $(n-1)$ PMAP ending at height k, which implies $h_k = zf_k + zg_k + zh_k$.

Therefore, we have to solve the following system of equations.

$$\begin{cases} f_0 = 1, \text{ and } f_k = zf_{k-1} + zg_{k-1} + zh_{k-1}, \quad k \geqslant 1, \\ g_k = z\sum\limits_{\ell\geqslant k+1} f_\ell + z\sum\limits_{\ell\geqslant k+1} h_\ell, \quad k \geqslant 0, \\ h_k = zf_k + zg_k + zh_k, \quad k \geqslant 0. \end{cases} \tag{2.1}$$

Summing the recursions in (2.1), we have:

$$\begin{aligned} F(u) &= 1 + z\sum_{k\geqslant 1} u^k f_{k-1} + z\sum_{k\geqslant 1} u^k g_{k-1} + z\sum_{k\geqslant 1} u^k h_{k-1} \\ &= 1 + zuF(u) + zuG(u) + zuH(u), \\ G(u) &= z\sum_{k\geqslant 0} u^k\Big(\sum_{\ell\geqslant k+1} f_\ell\Big) + z\sum_{k\geqslant 0} u^k\Big(\sum_{\ell\geqslant k+1} h_\ell\Big) \\ &= z\sum_{k\geqslant 1} f_k(1 + u + \ldots + u^{k-1}) + z\sum_{k\geqslant 1} h_k(1 + u + \ldots + u^{k-1}) \\ &= z\sum_{k\geqslant 1} \frac{u^k - 1}{u - 1} f_k + z\sum_{k\geqslant 1} \frac{u^k - 1}{u - 1} h_k \\ &= \frac{z}{u-1}(F(u) - F(1) + H(u) - H(1)), \\ H(u) &= zF(u) + zG(u) + zH(u). \end{aligned}$$

Notice that we have $F(1) - H(1) = 1$ by considering the difference of the first and third equations. Now, setting $a := F(1)$ and solving these functional equations, we obtain

$$\begin{aligned} F(u) &= \frac{2au\,z^2 - u\,z^2 + uz + z^2 - u - z + 1}{u^2z + u\,z^2 + z^2 - u - z + 1}, \\ G(u) &= -\frac{z\,(2auz + 2az - uz - 2a - z + 2)}{u^2z + u\,z^2 + z^2 - u - z + 1}, \\ H(u) &= \frac{z\,(2az - u - 2z + 1)}{u^2z + u\,z^2 + z^2 - u - z + 1}. \end{aligned}$$

Accepted manuscript

In order to compute $a = F(1)$, we use the kernel method (see [1, 9]) on $F(u)$. This method consists in cancelling the denominator by finding u as an algebraic function of z, $s(z)$. So, if we substitute u with $s(z)$ in the numerator, then it necessarily equals zero (in order to counterbalance the cancellation of the denominator), which allows to find $a = F(1)$. Finally, we can deduce the generating function $F(u)$.

We can write the denominator (which is a polynomial in u of degree 2), as $z(u-r)(u-s)$ with

$$r = \frac{1 - z^2 + \sqrt{z^4 - 4z^3 + 2z^2 - 4z + 1}}{2z},$$
$$s = \frac{1 - z^2 - \sqrt{z^4 - 4z^3 + 2z^2 - 4z + 1}}{2z}.$$

Replacing u with s (which have a Taylor expansion at $z = 0$) in order to cancel the numerator of $F(u)$, we obtain the equation

$$2as\,z^2 - s\,z^2 + sz + z^2 - s - z + 1 = 0.$$

Using $zrs = z^2 - z + 1$, a straightforward calculation provides

$$a = F(1) = H(1) + 1 = \frac{r(s-1)}{2z}.$$

Finally, after simplifying by the factor $(u - s)$ in the numerators and denominators, we obtain

$$F(u) = \frac{r}{r-u}, \quad G(u) = \frac{r(s-1) - z}{r-u}, \quad \text{and} \quad H(u) = \frac{1}{r-u}.$$

Extracting the coefficient of u^k in the series expansion of each generating function, we obtain

$$[u^k]F(u) := f_k = \frac{1}{r^k},$$
$$[u^k]G(u) := g_k = \frac{s-1}{r^k} - \frac{z}{r^{k+1}},$$
$$[u^k]H(u) := h_k = \frac{1}{r^{k+1}}.$$

Theorem 2.1. *The bivariate generating function for the total number of PMAP in $\mathcal{M}_1$ with respect to the length and the height of the end-point is given by*

$$\texttt{Total}(z,u) = \frac{1}{z(r-u)},$$

and we have

$$[u^k]\texttt{Total}(z,u) = \frac{1}{zr^{k+1}}.$$

Finally, setting $t(n,k) := [z^n][u^k]\texttt{Total}(z,u)$, we have for $n \geqslant 2$, $k \geqslant 1$,

$$t(n,k) = t(n,k-1) + t(n-1,k) - t(n-1,k-2) - t(n-2,k) - t(n-2,k-1),$$

and setting $t_n := t(n,0)$, then we have $t_0 = t_1 = 1$, and for $n \geqslant 2$,

$$t_n = t_{n-1} + t_{n-2} + \sum_{k=0}^{n-3} t_k t_{n-k-3} + \sum_{k=2}^{n-1} (t_k - t_{k-1})\, t_{n-k-1}.$$

Accepted manuscript

Proof. Since $\texttt{Total}(z,u) = F(u)+G(u)+H(u)$, the first two equalities are immediately deduced from the previous results. The third equality is obtained using the Mathematica package `Guess.m` (written by Manuel Kauers [6]) for guessing recurrence relations. After this, it suffices to check algebraically:

$$\texttt{Total}(z,u) = (u+z-u^2z-z^2-uz^2)\texttt{Total}(z,u) - u + \frac{(z^2-z+1)(1+rs-z)}{r}.$$

Now, let us prove the last equality. Any length n MAP of the first kind is of the form (i) HP, or (ii) UDP, or (iii) $UPHDQ$ where P, Q are some MAP so that the length k of P lies into $[0, n-3]$, and the length of Q is $n-k-3$, or (iv) $P^\sharp Q$ where $P^\sharp = UP'D_\ell$, $\ell \geqslant 1$, and P' is a PMAP such that $P'D_{\ell-1}$ is a MAP of length k lying into $[2, n-1]$, and the length of Q is $n-k-1$. Taking into account all these cases, we obtain the result. □

Let $\mathcal{T}$ be the infinite matrix $\mathcal{T} := [t(n,k)]_{n\geqslant 0, k\geqslant 0}$. The first few rows of the matrix $\mathcal{T}$ are

$$\mathcal{T} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 5 & 5 & 3 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 13 & 14 & 9 & 4 & 1 & 0 & 0 & 0 & \cdots \\ 36 & 40 & 28 & 14 & 5 & 1 & 0 & 0 & \cdots \\ 105 & 118 & 87 & 48 & 20 & 6 & 1 & 0 & \cdots \\ 317 & 359 & 273 & 161 & 75 & 27 & 7 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Corollary 2.2. *The generating function that counts the PMAP in $\mathcal{M}_1$ with respect to the length is given by*

$$\texttt{Total}(z,1) = \frac{1}{z(r-1)}.$$

The first few terms of the series expansion of $\texttt{Total}(z,1)$ are $1+2z+5z^2+14z^3+41z^4+124z^5+385z^6+1220z^7+3929z^8+12822z^9+O(z^{10})$, which correspond to the sequence A159771 in [13] counting the n-leaf binary trees that do not contain $(()(()())(()())()))$ as a subtree (see [11]). See Figure 2 for an illustration of the 14 PMAP of length 3.

Corollary 2.3. *The generating function that counts the MAP in $\mathcal{M}_1$ with respect to the length is given by*

$$\texttt{Total}(z,0) = \frac{1}{zr}.$$

The first few terms of the series expansion of $\texttt{Total}(z,0)$ are $1+z+2z^2+5z^3+13z^4+36z^5+105z^6+317z^7+982z^8+3105z^9+O(z^{10})$ which correspond to the sequence A114465 in [13] counting Dyck paths of length $2n$ having no ascents of length 2 that start at an odd level. We leave open the question of finding a constructive bijection between these sets.

Accepted manuscript

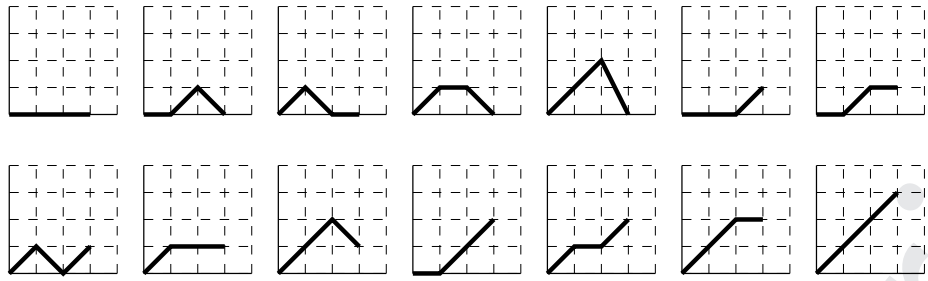


Figure 2: The 14 PMAP of length three in $\mathcal{M}_1$. Notice that five paths end on the x-axis, five paths end at height 1, three paths end at height 2, and one path ends at height 3, which correspond to the fourth row of $\mathcal{T}$.

2.2 PMAP in $\mathcal{M}_1'$ - From right to left

Here, we consider the paths of the previous section, but we read them from right to left. This means that down steps become up steps and *vice versa*, and horizontal steps are unchanged, which implies that two up steps cannot be consecutive now. See Definition 1.3 and Figure 3 for two examples of such paths.

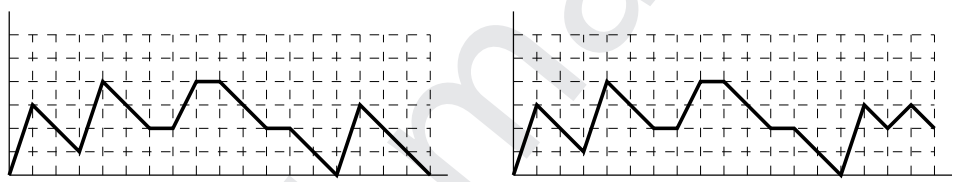


Figure 3: The left drawing shows a Motzkin path with air pockets of length 18 in $\mathcal{M}_1'$. The right drawing shows a partial Motzkin path with air pockets of length 18 ending at height 2 in $\mathcal{M}_1'$.

Let P be a length n PMAP in $\mathcal{M}_1'$ ending at height $k \geqslant 0$. If the last step of P is U_a, $a \geqslant 1$, then $k \geqslant a$ and we have $P = QU_a$ where Q is a length $(n-1)$ PMAP ending at height $\ell = k - a$ with a horizontal step or a down step. So, we obtain the first relation $f_k = z + z(g_0 + g_1 + \ldots + g_{k-1}) + z(h_0 + h_1 + \ldots + h_{k-1})$ for $k \geqslant 1$, anchored with $f_0 = 1$ by considering the empty path. If the last step of P is a down step D, then we have $P = QD$ where Q is a length $(n-1)$ PMAP ending at height $k+1$. So, we obtain the second relation $g_k = zf_{k+1} + zg_{k+1} + zh_{k+1}$. If the last step of P is a horizontal step H, then we have $P = QH$ where Q is a length $(n-1)$ PMAP ending at height k, which implies $h_k = zf_k + zg_k + zh_k$.

Therefore, we have to solve the following system of equations.

$$\begin{cases} f_0 = 1, \\ f_k = z + z(g_0 + g_1 + \ldots + g_{k-1}) + z(h_0 + h_1 + \ldots + h_{k-1}), \quad k \geqslant 1, \\ g_k = zf_{k+1} + zg_{k+1} + zh_{k+1}, \quad k \geqslant 0, \\ h_k = zf_k + zg_k + zh_k, \quad k \geqslant 0. \end{cases} \tag{2.2}$$

Accepted manuscript

Summing the recursions in (2.2), we have:

$$\begin{aligned}
F(u) &= 1 + \frac{zu}{1-u} + z\sum_{k\geqslant 1}(g_0 + \ldots + g_{k-1})u^k + z\sum_{k\geqslant 1}(h_0 + \ldots + h_{k-1})u^k \\
&= 1 + \frac{zu}{1-u} + z\sum_{k\geqslant 0} g_k \frac{u^{k+1}}{1-u} + z\sum_{k\geqslant 0} h_k \frac{u^{k+1}}{1-u} \\
&= 1 + \frac{zu}{1-u}(1 + G(u) + H(u)), \\
G(u) &= z\sum_{k\geqslant 0} f_{k+1}u^k + z\sum_{k\geqslant 0} g_{k+1}u^k + z\sum_{k\geqslant 0} h_{k+1}u^k \\
&= \frac{z}{u}(F(u) - F(0) + G(u) - G(0) + H(u) - H(0)), \\
H(u) &= zF(u) + zG(u) + zH(u).
\end{aligned}$$

Notice that we have $H(0) = \frac{z(1+G(0))}{1-z}$ by considering the third equation and $F(0) = 1$. Solving these functional equations using $a := G(0)$, we obtain

$$\begin{aligned}
F(u) &= \frac{-u^2z^3 + a\,uz^2 + 3\,u^2z^2 - uz^3 - 3\,u^2z + 2\,uz^2 + u^2 + uz - z^2 - u + z}{(1-z)\,(u^2z^2 - u^2z + uz^2 + u^2 - u + z)}, \\
G(u) &= \frac{z\left(a\,uz^2 - a\,uz + a\,u + a\,z + uz - a\right)}{(-1+z)\,(u^2z^2 - u^2z + uz^2 + u^2 - u + z)}, \\
H(u) &= \frac{\left(a\,uz^2 + u^2z^2 - a\,uz - 2\,u^2z + uz^2 + a\,z + u^2 - u + z\right)z}{(1-z)\,(u^2z^2 - u^2z + uz^2 + u^2 - u + z)}.
\end{aligned}$$

In order to compute $a = G(0)$, we use the kernel method on $G(u)$. We can write the denominator (which is a polynomial in u of degree 2), as $(z-1)(z^2-z+1)(u-r)(u-s)$ with

$$r = \frac{1 - z^2 + \sqrt{z^4 - 4\,z^3 + 2\,z^2 - 4\,z + 1}}{2(z^2 - z + 1)},$$

$$s = \frac{1 - z^2 - \sqrt{z^4 - 4\,z^3 + 2\,z^2 - 4\,z + 1}}{2(z^2 - z + 1)}.$$

Replacing u with s (which have a Taylor expansion at $z = 0$) in order to cancel the numerator of $G(u)$, we obtain the equation

$$a(sz^2 - sz + s + z - 1) + sz = 0.$$

Using $sr(z^2 - z + 1) = z$, we deduce

$$a = G(0) = \frac{1-r}{r} - sz.$$

Finally, after simplifying by the factor $(u - s)$ in the numerators and denominators, we obtain

$$F(u) = 1 + \frac{sru}{r-u}, \quad G(u) = \frac{s(1 - r + rz)}{r-u}, \quad \text{and} \quad H(u) = \frac{sr - sru(1-z)}{r-u},$$

Accepted manuscript

which implies that

$$[u^k]F(u) := f_k = [k=0] + [k \neq 0] \cdot \frac{s}{r^{k-1}},$$
$$[u^k]G(u) := g_k = \frac{s(1-r+rz)}{r^{k+1}},$$
$$[u^k]H(u) := h_k = \frac{s}{r^k} - [k \neq 0] \cdot \frac{(1-z)s}{r^{k-1}},$$

where $[k=0]$ (resp. $[k \neq 0]$) equals 1 whenever $k=0$ (resp. $k \neq 0$), and 0 otherwise.

Theorem 2.4. *The bivariate generating function for the total number of PMAP (read from right to left) with respect to the length and the height of the end-point is given by*

$$\texttt{Total}(z,u) = 1 + \frac{s(1+rz+ruz)}{r-u},$$

and we have

$$[u^k]\texttt{Total}(z,u) = [k=0] + \frac{s(rz+1)}{r^{k+1}} + [k \neq 0] \cdot \frac{sz}{r^{k-1}}.$$

Finally, setting $t(n,k) := [z^n][u^k]\texttt{Total}(z,u)$*, we have for* $n \geqslant 2$*,* $k \geqslant 1$*,*

$$t(n,k) = t(n-2,k-1) + t(n-2,k) - t(n-1,k-1) + t(n-1,k+1) + t(n,k-1),$$

and setting $t_n := t(n,0)$*, we have* $t_0 = t_1 = 1$*, and for* $n \geqslant 2$*,*

$$t_n = t_{n-1} + t_{n-2} + \sum_{k=0}^{n-3} t_k t_{n-k-3} + \sum_{k=2}^{n-1} (t_k - t_{k-1})\, t_{n-k-1}.$$

Proof. The first two equalities are directly deduced from the previous results. Since we have

$$(u - u^2z^2 - uz^2 + zu^2 - z - u^2)\texttt{Total}(z,u) + u^2(1-z) - (1+s-sz)u + s = 0,$$

we deduce the third relation. The last equality is already given in Theorem 2.1, since the number of MAP in $\mathcal{M}_1'$ is obviously equal to the number of MAP in $\mathcal{M}_1$. □

Let $\mathcal{T}$ be the infinite matrix $\mathcal{T} := [t(n,k)]_{n \geqslant 0, k \geqslant 0}$. The first few rows of the matrix $\mathcal{T}$ are

$$\mathcal{T} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 2 & 3 & 3 & 3 & 3 & 3 & \cdots \\ 5 & 8 & 10 & 12 & 14 & 16 & \cdots \\ 13 & 23 & 33 & 43 & 53 & 63 & \cdots \\ 36 & 69 & 107 & 149 & 195 & 245 & \cdots \\ 105 & 212 & 348 & 512 & 704 & 924 & \cdots \\ 317 & 665 & 1141 & 1753 & 2509 & 3417 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Accepted manuscript

Since there is an infinite number of PMAP of length n, we do not provide an ordinary generating function (with respect to the length) for these paths. So, we get around this by counting PMAP ending on a point $(x, n-x)$ for a given $n \geqslant 0$.

Corollary 2.5. *The generating function that counts the partial PMAP ending on the line $y = n - x$ is given by*

$$\texttt{Total}(z, z) = 1 + \frac{s(1 + rz + rz^2)}{r - z}.$$

The first few terms of the series expansion of $\texttt{Total}(z, z)$ are $1 + z + 3z^2 + 9z^3 + 25z^4 + 73z^5 + 223z^6 + 697z^7 + 2217z^8 + 7161z^9 + O(z^{10})$, which correspond to the sequence A101499 in [13], which is a Chebyshev transform of the Catalan number that counts peakless Motzkin paths of length n where horizontal steps at level at least one come in 2 colors. See Figure 4 for an illustration of the 9 PMAP in $\mathcal{M}_1'$ ending on the line $y = 3 - x$.

Notice that, from Theorem 2.4 we have

$$\texttt{Total}(z, 0) = 1 + \frac{s(1 + rz)}{r},$$

which is obviously equal to the expression derived in Corollary 2.3 that counts MAP of the first kind with respect to the length.

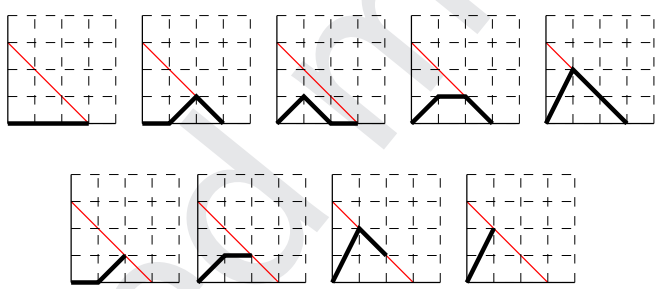


Figure 4: The 9 PMAP ending on the line $y = 3 - x$ in $\mathcal{M}_1'$. Notice that five paths end on the x-axis, three paths end at height 1, and one path ends at height 2, which correspond to the fourth diagonal of $\mathcal{T}$.

3 PMAP of the second kind

In this section, we focus on PMAP of the second kind. See Figure 5 for an illustration of such paths. The first subsection considers paths in $\mathcal{M}_2$, while the second handles paths in $\mathcal{M}_2'$. We yield enumerative results for these paths according to the length, the type of the last step, and the height of the end-point.

3.1 PMAP in $\mathcal{M}_2$ - From left to right

In this part, we consider PMAP in $\mathcal{M}_2$, i.e. lattice paths in $\mathbb{N}^2$ starting at the origin, consisting of steps U, D_k and H, and where any down step or horizontal step (except for the last step of the path) is immediately followed by an up step. Figure 5 shows two examples of such paths.

Accepted manuscript

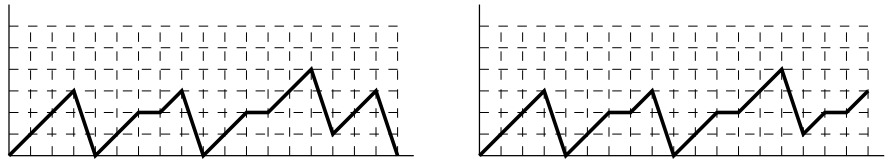


Figure 5: The left drawing shows a MAP of length 18 in $\mathcal{M}_2$. The right drawing shows a PMAP of length 18 ending at height 3 in $\mathcal{M}_2$.

Let P be a length n PMAP in $\mathcal{M}_2$ ending at height $k \geqslant 0$. If the last step of P is U, then $k \geqslant 1$ and we have $P = QU$ where Q is a length $(n-1)$ PMAP ending at height $k-1$. So, we obtain the first relation $f_k = zf_{k-1} + zg_{k-1} + zh_{k-1}$ for $k \geqslant 1$, anchored with $f_0 = 1$ by considering the empty path. If the last step of P is a down step D_a, $a \geqslant 1$, then we have $P = QD_a$ where Q is a length $(n-1)$ PMAP ending at height $\ell = a + k \geqslant k+1$ with an up step. So, we obtain the second relation $g_k = z\sum_{\ell \geqslant k+1} f_\ell$. If the last step of P is a horizontal step H, then we have $P = QH$ where Q is a length $(n-1)$ PMAP ending at height k with an up step, which implies $h_k = zf_k$.

So, we have to solve the following system of equations.

$$\begin{cases} f_0 = 1, \text{ and } f_k = zf_{k-1} + zg_{k-1} + zh_{k-1}, \quad k \geqslant 1, \\ g_k = z \sum\limits_{\ell \geqslant k+1} f_\ell, \quad k \geqslant 0, \\ h_k = zf_k, \quad k \geqslant 0. \end{cases} \tag{3.1}$$

Summing the recursions in (3.1), we have:

$$\begin{aligned} F(u) &= 1 + z\sum_{k\geqslant 1} u^k f_{k-1} + z\sum_{k\geqslant 1} u^k g_{k-1} + z\sum_{k\geqslant 1} u^k h_{k-1} \\ &= 1 + zuF(u) + zuG(u) + zuH(u), \\ G(u) &= z\sum_{k\geqslant 0} u^k \Big(\sum_{\ell\geqslant k+1} f_\ell\Big) = z\sum_{k\geqslant 1} \frac{u^k - 1}{u-1} f_k \\ &= \frac{z}{u-1}(F(u) - F(1)), \\ H(u) &= zF(u). \end{aligned}$$

Now, setting $a := F(1)$ and solving these functional equations, we deduce

$$F(u) = \frac{au\,z^2 - u + 1}{u^2z^2 + u^2z - uz - u + 1},$$

$$G(u) = -\frac{z\left(au\,z^2 + auz - a + 1\right)}{u^2z^2 + u^2z - uz - u + 1}, H(u) = \frac{z\left(au\,z^2 - u + 1\right)}{u^2z^2 + u^2z - uz - u + 1}.$$

In order to compute $a = F(1)$, we use the kernel method on $F(u)$. We can write the denominator (which is a polynomial in u of degree 2), as $(z^2 + z)(u - r)(u - s)$ with

$$r = \frac{1 + z + \sqrt{-3z^2 - 2z + 1}}{2z\,(z+1)}, \text{ and } s = \frac{1 + z - \sqrt{-3z^2 - 2z + 1}}{2z\,(z+1)}.$$

Accepted manuscript

Replacing u with s (which have a Taylor expansion at $z = 0$) in order to cancel the numerator of $F(u)$, we obtain the equation

$$as\,z^2 - s + 1 = 0,$$

and thus

$$a = F(1) = \frac{s-1}{sz^2}.$$

Finally using $z(1+z)rs = 1$ and simplifying by the factor $(u - s)$ in the numerators and denominators, we obtain

$$F(u) = \frac{r}{r-u}, \quad G(u) = \frac{s-1}{sz(r-u)}, \quad \text{and} \quad H(u) = \frac{zr}{r-u},$$

which implies that

$$\begin{aligned}
[u^k]F(u) &:= f_k = \frac{1}{r^k},\\
[u^k]G(u) &:= g_k = (1+z)\cdot\frac{s-1}{r^k},\\
[u^k]H(u) &:= h_k = \frac{z}{r^k}.
\end{aligned}$$

Theorem 3.1. *The bivariate generating function for the total number of PMAP with respect to the length and the height of the end-point is given by*

$$\texttt{Total}(z,u) = \frac{1}{z(r-u)},$$

and we have

$$[u^k]\texttt{Total}(z,u) = \frac{1}{zr^{k+1}}.$$

Finally, setting $t(n,k) = [z^n][u^k]\texttt{Total}(z,u)$*, we have for* $n \geqslant 2$*,* $k \geqslant 1$*,*

$$t(n,k) = t(n,k-1) + t(n-1,k-1) - t(n-1,k-2) - t(n-2,k-2),$$

and setting $t_n := t(n,0)$*, we have* $t_0 = 1$*, and for* $n \geqslant 1$*,*

$$t_n = t_{n-1} + \sum_{k=1}^{n-2} t_k t_{n-1-k}.$$

Proof. The first two equalities are immediately deduced from the previous results. The third equality is obtained using the Mathematica package `Guess.m` ([6]) for guessing recurrence relations. After this, it suffices to check algebraically:

$$\texttt{Total}(z,u) = (u + uz - u^2z - u^2z^2)\texttt{Total}(z,u) - u(1+z) + \frac{1}{zr}.$$

For the last equality, it suffices to remark that the generating function of the sequence $(t_n)_{n\geqslant 0}$, that is $1/(zr)$, generates a shift of the well known Motzkin sequence A001006 in in [13]. □

Accepted manuscript

Let $\mathcal{T}$ be the infinite matrix $\mathcal{T} := [t(n,k)]_{n\geqslant 0, k\geqslant 0}$. The first few rows of the matrix $\mathcal{T}$ are

$$\mathcal{T} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 3 & 3 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 4 & 6 & 6 & 4 & 1 & 0 & 0 & 0 & \cdots \\ 9 & 13 & 13 & 10 & 5 & 1 & 0 & 0 & \cdots \\ 21 & 30 & 30 & 24 & 15 & 6 & 1 & 0 & \cdots \\ 51 & 72 & 72 & 59 & 40 & 21 & 7 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Corollary 3.2. *The generating function that counts the PMAP with respect to the length is given by*

$$\texttt{Total}(z,1) = \frac{1}{z(r-1)}.$$

The first few terms of the series expansion of $\texttt{Total}(z,1)$ are $1+2z+4z^2+9z^3+21z^4+51z^5+127z^6+323z^7+835z^8+2188z^9+O(z^{10})$, which correspond to a shift of the sequence A001006 in [13] that counts the Motzkin paths of a given length. See Figure 6 for an illustration of the 9 paths of length 3.

Corollary 3.3. *The generating function that counts the MAP with respect to the length is given by*

$$\texttt{Total}(z,0) = \frac{1}{zr}.$$

The first few terms of the series expansion of $\texttt{Total}(z,0)$ are $1+z+z^2+2z^3+4z^4+9z^5+21z^6+51z^7+127z^8+323z^9+O(z^{10})$ which correspond to a shift of the sequence A001006 in [13] that counts Motzkin paths of a given length.

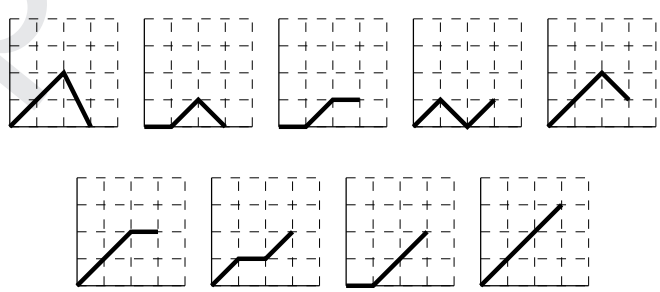


Figure 6: The 9 PMAP of length three in $\mathcal{M}_2$. Notice that two paths end on the x-axis, three paths end at height 1, three paths end at height 2, and one path ends at height 3, which correspond to the fourth row of $\mathcal{T}$.

Accepted manuscript

3.2 PMAP in $\mathcal{M}_2'$ - From right to left

Here, we consider the paths of the previous section, but we read them from right to left. This means that down steps become up steps and *vice versa*, and horizontal steps are unchanged, which implies that any up step or horizontal step (except the first step of the path) is preceded by a down step. See Definition 1.4 and Figure 7 for two examples of such paths.

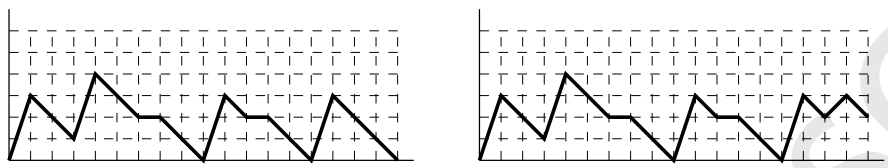


Figure 7: The left drawing shows a Motzkin path with air pockets of length 18 in $\mathcal{M}_2'$ (read from right to left). The right drawing shows a partial Motzkin path with air pockets of length 18 ending at height 2 in $\mathcal{M}_2'$.

Let P be a length n PMAP in $\mathcal{M}_2'$ ending at height $k \geqslant 0$. If the last step of P is U_a, $a \geqslant 1$, then $k \geqslant a$ and we have $P = QU_a$ where Q is a length $(n-1)$ PMAP ending at height $\ell = k-a$ with a down step. So, we obtain the first relation $f_k = z+z(g_0+g_1+\ldots+g_{k-1})$ for $k \geqslant 1$, anchored with $f_0 = 1$ by considering the empty path. If the last step of P is a down step D, then we have $P = QD$ where Q is a length $(n-1)$ PMAP ending at height $k+1$. So, we obtain the second relation $g_k = zf_{k+1} + zg_{k+1} + zh_{k+1}$. If the last step of P is a horizontal step H, then we have $P = QH$ where Q is a length $(n-1)$ PMAP ending at height k with a down step whenever it is nonempty. If $k \geqslant 1$, then we have $h_k = zg_k$; if $k = 0$ then we have $h_0 = z + zg_0$ where the monomial z corresponds to the path $P = H$.

So, we have to solve the following system of equations.

$$\begin{cases} f_0 = 1, \text{ and } f_k = z(1 + g_0 + g_1 + \ldots + g_{k-1}), \quad k \geqslant 1, \\ g_k = zf_{k+1} + zg_{k+1} + zh_{k+1}, \quad k \geqslant 0, \\ h_0 = z + zg_0, \text{ and } h_k = zg_k, \quad k \geqslant 1. \end{cases} \tag{3.2}$$

Using the same notations as in the previous sections, and summing the recursions in (3.2), we have:

$$\begin{aligned} F(u) &= 1 + \sum_{k\geqslant 1} u^k f_k = 1 + z\sum_{k\geqslant 1}(1 + g_0 + \ldots + g_{k-1})u^k \\ &= 1 + \frac{zu}{1-u}(1 + G(u)), \\ G(u) &= z\sum_{k\geqslant 0}(f_{k+1} + g_{k+1} + h_{k+1})u^k \\ &= \frac{z}{u}(F(u) - F(0) + G(u) - G(0) + H(u) - H(0)), \\ H(u) &= z + zG(u). \end{aligned}$$

Notice that $F(0) = 1$ and $H(0) = z + zG(0)$ by the third relation. Now, setting

Accepted manuscript

$a := G(0)$ and solving these functional equations, we deduce

$$F(u) = \frac{a\,uz^3 + a\,uz^2 + uz^3 - u^2z + uz^2 + u^2 - uz + z^2 - u + z}{u^2 - uz + z^2 - u + z},$$
$$G(u) = -\frac{z\,(a\,uz + a\,u - a\,z + uz - a)}{u^2 - uz + z^2 - u + z},$$
$$H(u) = -\frac{z\left(a\,uz^2 + a\,uz - a\,z^2 + uz^2 - a\,z - u^2 + uz - z^2 + u - z\right)}{u^2 - uz + z^2 - u + z}.$$

In order to compute $a = G(0)$, we use the kernel method on $F(u)$. We can write the denominator (which is a polynomial in u of degree 2), as $(u-r)(u-s)$ with

$$r = \frac{1 + z + \sqrt{-3\,z^2 - 2\,z + 1}}{2}, \text{ and } s = \frac{1 + z - \sqrt{-3\,z^2 - 2\,z + 1}}{2}.$$

Replacing u with s (which have a Taylor expansion at $z = 0$) in oder to cancel the numerator of $F(u)$, we obtain the equation

$$a\,sz^3 + a\,sz^2 + sz^3 - s^2z + sz^2 + s^2 - sz + z^2 - s + z = 0.$$

Using $rs = z(1+z)$, we deduce

$$a = G(0) = \frac{1-r}{r}.$$

Finally, after simplifying by the factor $(u-s)$ in the numerators and denominators, we obtain

$$F(u) = \frac{u(z-1)+r}{r-u}, \quad G(u) = \frac{1-r}{r-u}, \quad \text{and} \quad H(u) = \frac{z(1-u)}{r-u},$$

which implies that

$$[u^k]F(u) := f_k = \frac{1}{r^k} + [k \neq 0] \cdot \frac{z-1}{r^k},$$
$$[u^k]G(u) := g_k = \frac{1-r}{r^{k+1}},$$
$$[u^k]H(u) := h_k = \frac{z}{r^{k+1}} - [k \neq 0] \cdot \frac{z}{r^k}.$$

Theorem 3.4. *The bivariate generating function for the total number of PMAP with respect to the length and the height of the end-point is given by*

$$\texttt{Total}(z,u) = \frac{1-u+z}{r-u}.$$

$$[u^k]\texttt{Total}(z,u) = \frac{z+1}{r^{k+1}} - [k \neq 0] \cdot \frac{1}{r^k}.$$

Finally, setting $t(n,k) = [z^n][u^k]\texttt{Total}(z,u)$, we have for $n \geqslant 1$, $k \geqslant 1$,

$$t(n,k) = t(n,k-1) - t(n-1,k) + t(n-2,k+1) + t(n-1,k+1),$$

and setting $t_n := t(n,0)$, we have $t_0 = 1$, and for $n \geqslant 2$,

$$t_n = t_{n-1} + \sum_{k=1}^{n-2} t_k t_{n-1-k}.$$

Accepted manuscript

Proof. The proof are obtained *mutatis mutandis* as for the previous theorems. □

Let $\mathcal{T}$ be the infinite matrix $\mathcal{T} := [t(n,k)]_{n\geqslant 0, k\geqslant 0}$. The first few rows of the matrix $\mathcal{T}$ are

$$\mathcal{T} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 4 & 6 & 8 & 10 & 12 & 14 & 16 & \cdots \\ 9 & 15 & 22 & 30 & 39 & 49 & 60 & \cdots \\ 21 & 36 & 54 & 75 & 99 & 126 & 156 & \cdots \\ 51 & 91 & 142 & 205 & 281 & 371 & 476 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Since there is an infinite number of PMAP of length n, we do not provide an ordinary generating function (with respect to the length) for these paths. So, we get around this by counting PMAP ending on a point $(x, n-x)$ for a given $n \geqslant 0$.

Corollary 3.5. *The generating function that counts the partial PMAP ending on the line $y = n - x$ is given by*

$$\texttt{Total}(z,z) = \frac{1}{r-z}.$$

The first few terms of the series expansion of $\texttt{Total}(z,z)$ are $1 + z + 2z^2 + 4z^3 + 9z^4 + 21z^5 + 51z^6 + 127z^7 + 323z^8 + 835z^9 + O(z^{10})$, which correspond to the sequence A001006 in [13] that counts the Motzkin paths with respect to the length. See Figure 8 for the illustration of the 9 PMAP in $\mathcal{M}_2'$ ending on the line $y = 4 - x$.

Notice that we obviously retrieve the results of Corollary 3.3, i.e., the generating function $\texttt{Total}(z,0)$ that counts the MAP with respect to the length is also a shift of the Motzkin sequence A001006 in [13].

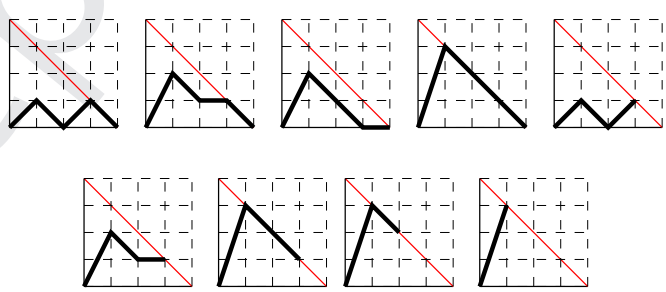


Figure 8: The 9 PMAP ending on the line $y = 4 - x$ in $\mathcal{M}_2'$. Notice that four paths end on the x-axis, three paths end at height 1, one path ends at height 2, and one path ends at height 3, which correspond to the fifth diagonal of $\mathcal{T}$.

Accepted manuscript

4 A Riordan array point of view

In this section, we make links between the previous matrices $\mathcal{T} = [t_{n,k}]_{n\geqslant 0, k\geqslant 0}$ and some Riordan arrays or almost Riordan arrays. We first give a short background on Riordan arrays [4, 5, 12].

An infinite column vector $(a_0, a_1, \dots)^T$ has generating function $f(z)$ if $f(z) = \sum_{n\geqslant 0} a_n z^n$. A *Riordan array* is an infinite lower triangular matrix whose k-th column has generating function $g(z)f(z)^k$ for all $k \geqslant 0$, for some formal power series $g(z)$ and $f(z)$, with $g(0) \neq 0$, $f(0) = 0$, and $f'(0) \neq 0$. Such a Riordan array is denoted by $(g(z), f(z))$. If we multiply this matrix by a column vector $(b_0, b_1, \dots)^T$ having generating function $h(z) = \sum_{n\geqslant 0} b_n z^n$, then the resulting column vector has generating function $g(z)h(f(z))$. This property is known as the *fundamental theorem of Riordan arrays*.

The product of two Riordan arrays $(g(z), f(z))$ and $(h(z), l(z))$ is defined by

$$(g(z), f(z)) * (h(z), l(z)) = (g(z)h(f(z)), l(f(z)))\,.$$

Under the operation "$*$", the set of all Riordan arrays is a group [12]. The identity element is $I = (1, z)$, and the inverse of $(g(z), f(z))$ is

$$(g(z), f(z))^{-1} = \left(1/\left(g \circ \overline{f}\right)(z), \overline{f}(z)\right),$$

where $\overline{f}(z)$ denotes the compositional inverse of $f(z)$.

Moreover, if a matrix $\mathcal{T} = [t_{n,k}]_{n\geqslant 0, k\geqslant 0}$ is a *Riordan array* $(g(z), f(z))$ then $t_{n,k}$ equals the coefficient of $z^n u^k$ in the series expansion of the bivariate generating function

$$\frac{g(z)}{1 - uf(z)},$$

and we say that this is the bivariate generating function of the matrix $\mathcal{T}$.

An *almost Riordan array* $\mathcal{T}'$ is a matrix that consists of an initial column vector $(d_0, d_1, \dots)^T$ with generating function $g_0(z) = \sum_{n\geqslant 0} d_n z^n$, followed by a vertically shifted Riordan array $(g(z), f(z))$ as illustrated below.

$$\mathcal{T}' = \left(\begin{array}{c|ccccc}
d_0 & 0 & \cdots & & 0 & 0 & \cdots \\
\hline
d_1 & & & & & & \\
d_2 & & & & & & \\
d_3 & & & (g(z), f(z)) & & & \\
d_4 & & & & & & \\
d_5 & & & & & & \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots
\end{array}\right)$$

Therefore, the bivariate generating function for this matrix is given by

$$g_0(z) + zu\frac{g(z)}{1 - uf(z)}.$$

Finally, we will say that a matrix $\mathcal{M} = [m_{n,k}]_{n\geqslant 0, k\geqslant 0}$ is the *rectification of the Riordan array* $(g(z), f(z))$ whenever $m_{n,k}$ equals the coefficient of $z^n u^k$ in the series expansion of

Accepted manuscript

the bivariate generating function

$$\frac{g(z)}{1-u\frac{f(z)}{z}}.$$

In this section, $C(z)=\frac{1-\sqrt{1-4z}}{2z}$ will be the generating function where the coefficient of z^n in its series expansion is the Catalan number $c_n=\frac{1}{n+1}\binom{2n}{n}$.

4.1 Comment on Section 2.1

Let $\mathcal{T}=[t(n,k)]_{n\geqslant 0,k\geqslant 0}$ be the matrix given in Section 2.1 where $t(n,k)$ is the number of length n PMAP in $\mathcal{M}_1$ ending at height k.

Proposition 4.1. *The matrix $\mathcal{T}=[t(n,k)]_{n\geqslant 0,k\geqslant 0}$ is a Riordan array defined by*

$$\left(\frac{1}{1-z^2}C\left(\frac{z(1-z+z^2)}{(1-z^2)^2}\right),\frac{z}{1-z^2}C\left(\frac{z(1-z+z^2)}{(1-z^2)^2}\right)\right).$$

Proof. Considering r defined in Section 2.1, we have

$$\begin{aligned}[u^k]\texttt{Total}(z,u)&=\frac{1}{zr^{k+1}}=\left(\frac{1}{zr}\right)\left(\frac{1}{r}\right)^k\\&=\frac{2}{1-z^2+\sqrt{1-4z+2z^2-4z^3+z^4}}\left(\frac{2z}{1-z^2+\sqrt{1-4z+2z^2-4z^3+z^4}}\right)^k.\end{aligned}$$

Therefore, the array $\mathcal{T}$ satisfies

$$\begin{aligned}\mathcal{T}&=\left(\frac{2}{1-z^2+\sqrt{1-4z+2z^2-4z^3+z^4}},\frac{2z}{1-z^2+\sqrt{1-4z+2z^2-4z^3+z^4}}\right)\\&=\left(\frac{1-z^2-\sqrt{1-4z+2z^2-4z^3+z^4}}{2z(1-z+z^2)},\frac{1-z^2-\sqrt{1-4z+2z^2-4z^3+z^4}}{2(1-z+z^2)}\right)\\&=\left(\frac{1}{1-z^2}C\left(\frac{z(1-z+z^2)}{(1-z^2)^2}\right),\frac{z}{1-z^2}C\left(\frac{z(1-z+z^2)}{(1-z^2)^2}\right)\right).\end{aligned}$$

□

Proposition 4.2. *The general term $t(n,k)$ equals*

$$\sum_{i=0}^{n-k}\binom{k+\frac{n-k-i}{2}}{\frac{n-k-i}{2}}\frac{1+(-1)^{n-k-i}}{2}\sum_{j=0}^{i}C_{k+j,k}\sum_{m=0}^{j}\binom{j}{m}(-1)^m\sum_{p=0}^{m}\binom{m}{p}(-1)^p\binom{2j-1+\frac{i-j-m-p}{2}}{\frac{i-j-m-p}{2}}\frac{1+(-1)^{i-j-m-p}}{2},$$

where $C_{n,k}=\frac{k+1}{n+1}\binom{2n-k}{n-k}$ is the general term of the Catalan matrix (A033184 in [13]).

Accepted manuscript

Proof. Setting $Z = \frac{z(1-z+z^2)}{(1-z^2)^2}$, we have

$$\begin{aligned}
t(n,k) &= [z^n]z^k \frac{1}{(1-z^2)^{k+1}} C(Z)^{k+1} \\
&= [z^{n-k}] \frac{1}{(1-z^2)^{k+1}} C(Z)^{k+1} \\
&= \sum_{i=0}^{n-k} [z^{n-k-i}] \frac{1}{(1-z^2)^{k+1}} [z^i] C(Z)^{k+1} \qquad \text{(Product rule [8])} \\
&= \sum_{i=0}^{n-k} [z^{n-k-i}] \frac{1}{(1-z^2)^{k+1}} \sum_{j=0}^{i} [z^j] C(z)^{k+1} [z^i] Z^j \qquad \text{(Composition rule [8])} \\
&= \sum_{i=0}^{n-k} [z^{n-k-i}] \frac{1}{(1-z^2)^{k+1}} \sum_{j=0}^{i} [z^j] \frac{1}{z^k} C(z)(zC(z))^k [z^i] Z^j \\
&= \sum_{i=0}^{n-k} \binom{k + \frac{n-k-i}{2}}{\frac{n-k-i}{2}} \frac{1+(-1)^{n-k-i}}{2} \sum_{j=0}^{i} C_{k+j,k} [z^i] Z^j \qquad \text{(See [8])}.
\end{aligned}$$

Since we have

$$\begin{aligned}
[z^i]Z^j &= [z^i] \left(\frac{z(1-z+z^2)}{(1-z^2)^2} \right)^j \\
&= \sum_{m=0}^{j} \binom{j}{m} (-1)^m \sum_{p=0}^{m} \binom{m}{p} (-1)^p \binom{2j-1+\frac{i-j-m-p}{2}}{\frac{i-j-m-p}{2}} \frac{1+(-1)^{i-j-m-p}}{2},
\end{aligned}$$

the result follows. □

4.2 Comment on Section 2.2

Let $\mathcal{T} = [t(n,k)]_{n \geqslant 0, k \geqslant 0}$ be the matrix given in Section 2.2 where $t(n,k)$ is the number of length n PMAP in $\mathcal{M}_1'$ ending at height k.

Proposition 4.3. *The matrix $\mathcal{T} = [t(n,k)]_{n \geqslant 0, k \geqslant 0}$ can be written*

$$\mathcal{T} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
1 & 1 & 1 & 1 & 1 & 1 & \cdots \\
2 & 3 & 3 & 3 & 3 & 3 & \cdots \\
5 & 8 & 10 & 12 & 14 & 16 & \cdots \\
13 & 23 & 33 & 43 & 53 & 63 & \cdots \\
36 & 69 & 107 & 149 & 195 & 245 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix} = A \cdot B$$

Accepted manuscript

where

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 3 & 0 & 0 & 0 & 0 & \cdots \\ 5 & 8 & 2 & 0 & 0 & 0 & \cdots \\ 13 & 23 & 10 & 0 & 0 & 0 & \cdots \\ 36 & 69 & 38 & 4 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \textit{and} \quad B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 0 & 0 & 1 & 2 & 3 & 4 & \cdots \\ 0 & 0 & 0 & 1 & 3 & 6 & \cdots \\ 0 & 0 & 0 & 0 & 1 & 4 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

are defined as follows:

- *The matrix* $B = [b_{n,k}]_{n,k\geqslant 0}$ *is defined by* $b_{0,0} = 1$*, and* $b_{n,0} = b_{0,n} = 0$ *if* $n \geqslant 1$*, and* $b_{n,k} = \binom{k-1}{n-1}$ *otherwise, which is a kind of Pascal matrix.*
- *The matrix* $A = [a_{n,k}]_{n,k\geqslant 0}$ *is the almost Riordan array with initial column of generating function*

$$g_0(z) = \frac{1 - z^2 - \sqrt{1 - 4z + 2z^2 - 4z^3 + z^4}}{2z(1 - z + z^2)},$$

followed by the shifted Riordan array

$$\left(\frac{1 - 3z + z^2 - z^3 - (1 - z)\sqrt{1 - 4z + 2z^2 - 4z^3 + z^4}}{2z^3(1 - z + z^2)}, \frac{1 - 2z - z^2 - \sqrt{1 - 4z + 2z^2 - 4z^3 + z^4}}{2z}\right).$$

The second column of $\mathcal{T}$ $1, 3, 8, 23, 69, \ldots$ *with generating function*

$$\frac{1 - 3z + z^2 - z^3 - (1 - z)\sqrt{1 - 4z + 2z^2 - 4z^3 + z^4}}{2z^3(1 - z + z^2)}$$

is the convolution of the first column of $\mathcal{T}$ $1, 1, 2, 5, 13, 36, \ldots$ *(*A114465 *in* [13]*) and the sequence* $1, 2, 4, 10, 28, \ldots$ *(*A187256 *in* [13]*).*

Proof. An almost Riordan array is represented by an initial column vector with generating function $g_0(z)$, followed by a vertically shifted Riordan array $(g(z), f(z))$. The bivariate generating function of this matrix is then given by $g_0(z) + zu\frac{g(z)}{1-uf(z)}$. In our case, for the almost Riordan array A, we have

$$g_0(z) = \frac{1 - z^2 - \sqrt{1 - 4z + 2z^2 - 4z^3 + z^4}}{2z(1 - z + z^2)},$$

$$g(z) = \frac{1 - 3z + z^2 - z^3 - (1 - z)\sqrt{1 - 4z + 2z^2 - 4z^3 + z^4}}{2z^3(1 - z + z^2)},$$

$$f(z) = \frac{1 - 2z - z^2 - \sqrt{1 - 4z + 2z^2 - 4z^3 + z^4}}{2z}.$$

Accepted manuscript

We let $G(z,u) = g_0(z) + zu\frac{g(z)}{1-uf(z)}$, the bivariate generating function of the almost Riordan array. Now, it suffices to check that the generating function corresponding to the matrix $A \cdot B$, that is

$$G(z, \frac{u}{1-u}) = \frac{(1-u+uz)(1-2z+2uz-(1+2u)z^2-\sqrt{1-4z+2z^2-4z^3+z^4}}{2(1-z+z^2)(u(u-1)+(1-u^2)z+u(u+1)z^2)},$$

coincides with the generating function $\texttt{Total}(z,u)$ of the matrix $\mathcal{T}$. □

Proposition 4.4. *The matrix*

$$[t(n,k)]_{n\geqslant 1, k\geqslant 1} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & \cdots \\ 3 & 3 & 3 & 3 & 3 & \cdots \\ 8 & 10 & 12 & 14 & 16 & \cdots \\ 23 & 33 & 43 & 53 & 63 & \cdots \\ 69 & 107 & 149 & 195 & 245 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is the rectification of the Riordan array $(g(z), h(z))$ *with*

$$h(z) = \frac{1-z^2-\sqrt{1-4z+2z^2-4z^3+z^4}}{2}, \quad \textit{and}$$
$$g(z) = \frac{1-3z+z^2-z^3-(1-z)\sqrt{1-4z+2z^2-4z^3+z^4}}{2z^3(1-z+z^2)}.$$

Proof. It suffices to check that the generating function of $\mathcal{T}$, i.e. $\texttt{Total}(z,u)$, given in Theorem 2.4, equals to

$$\texttt{Total}(z,0) + zu\frac{g(z)}{1-u\frac{h(z)}{z}}.$$

□

We can express $h(z)$ and $g(z)$, respectively, in the following form

$$h(z) = \frac{z(1-z+z^2)}{1-z^2} C\left(\frac{z(1-z+z^2)}{(1-z^2)^2}\right),$$

$$g(z) = \frac{1}{1-3z+z^2-z^3} C\left(\frac{z^3(1-z+z^2)}{(1-3z+z^2-z^3)^2}\right).$$

Then $g(z)$ expands to give the first column $1, 3, 8, 23, \ldots$, whose n-th term v_n can be expressed

$$v_n = \sum_{k=0}^{n}\sum_{j=0}^{k}\binom{k}{j}(-1)^j\sum_{i=0}^{j}\binom{j}{i}(-1)^i\sum_{\ell=0}^{n-3k-j-i}\binom{2k+\ell}{\ell}\sum_{m=0}^{\ell}\binom{\ell}{m}3^{\ell-m}(-1)^m\binom{m}{n-3k-j-i-\ell-m}(-1)^{n-3k-j-i-\ell-m}c_k.$$

Using v_n, we can deduce the following.

Accepted manuscript

Proposition 4.5. *The general term $t(n,k)$ equals*

$$\sum_{i=0}^{n+k} v_{n+k-i} \sum_{j=0}^{i} \sum_{m=0}^{j} M_{m,k} \sum_{p=0}^{m} \binom{m}{p} (-1)^p \sum_{q=0}^{p} \binom{p}{q} (-1)^q \binom{2m-1+\frac{j-m-p-q}{2}}{\frac{j-m-p-q}{2}} \frac{1+(-1)^{j-m-p-q}}{2} \binom{k}{\frac{i-j}{2}} (-1)^{\frac{i-j}{2}} \frac{1+(-1)^{i-j}}{2},$$

where

$$M_{n,k} = \begin{cases} [k=0] & \text{if } n=0, \\ \frac{n}{k}\binom{2n-k-1}{n-k} & \text{otherwise,} \end{cases}$$

is the general term of Riordan array $(1, zC(z))$ (see A106566 *in* [13]*).*

4.3 Comment on Section 3.1

Let $\mathcal{T} = [t(n,k)]_{n\geqslant 0, k\geqslant 0}$ be the matrix given in Section 3.1 where $t(n,k)$ is the number of length n PMAP in $\mathcal{M}_2$ ending at height k.

Proposition 4.6. *The matrix $\mathcal{T} = [t(n,k)]_{n\geqslant 0, k\geqslant 0}$ is the Riordan array*

$$\begin{aligned} \mathcal{T} &= (1+zM(z), z(1+zM(z))) \\ &= \left(C\left(\frac{z}{1+z}\right), zC\left(\frac{z}{1+z}\right)\right), \end{aligned}$$

where $M(z) = \frac{1-z-\sqrt{1-2z-3z^2}}{2z^2}$ is the generating function of the Motzkin numbers (see A001006 *in* [13]*).*

Proof. It suffices to check that $\texttt{Total}(z,u)$ given in Theorem 3.1 satisfies

$$\texttt{Total}(z,u) = \frac{C\left(\frac{z}{1+z}\right)}{1-uzC\left(\frac{z}{1+z}\right)},$$

and that $1+zM(z) = C\left(\frac{z}{1+z}\right)$. □

This triangle $\mathcal{T}$ corresponds to A091836 in [13] where the coefficient of row $n-1$ and column k is the number of Motzkin paths of length n having k points on the horizontal axis (besides the first and last point). As mentioned in [13], we obtain

$$t(n,k) = \begin{cases} 1, & \text{if } n=k, \\ \frac{k+1}{n+1} \sum_{j=1}^{n-k} j(-1)^{n-k-j} \binom{n+j}{j} \sum_{i=0}^{n-k} \frac{1}{n-k} \binom{i}{n-k-i+j} \binom{n-k}{i}, & \text{otherwise.} \end{cases}$$

A second expression for $t(n,k)$ is given by the following proposition.

Proposition 4.7. *The general term $t(n,k)$ of the Riordan array $(1+zM(z), z(1+zM(z)))$ is given by*

$$t(n,k) = \begin{cases} 1 & \text{if } n=k, \\ \frac{k+1}{n-k} \sum_{j=0}^{k} \binom{k}{j} \sum_{i=0}^{n-k} \binom{n-k}{i} \binom{i}{n-k-i-j-1} & \text{otherwise.} \end{cases}$$

Accepted manuscript

Proof. We prove this using Lagrange inversion, using the fact that

$$\frac{z}{1+z+z^2}M\left(\frac{z}{1+z+z^2}\right)=z,$$

which means that the compositional inverse $(zM(z))^{-1}$ of $zM(z)$ is

$$(zM(z))^{-1}=\frac{z}{1+z+z^2}.$$

Thus we have

$$\begin{aligned}
t(n,k) &= [z^n](1+zM(z))(z(1+zM(z)))^k \\
&= [z^{n-k}](1+zM(z))^{k+1} \\
&= [z^{n-k}]G(zM(z)), \quad \text{with} \quad G(z)=(1+z)^{k+1} \\
&= \frac{1}{n-k}[z^{n-k-1}]G'(z)\left(\frac{z}{(zM(z))^{-1}}\right)^{n-k} \quad \text{(Lagrange inversion)} \\
&= \frac{1}{n-k}[z^{n-k-1}](k+1)(1+z)^k(1+z+z^2)^{n-k} \\
&= \frac{k+1}{n-k}[z^{n-k}]\sum_{j=0}^{k}\binom{k}{j}z^j\sum_{i=0}^{n-k}\binom{n-k}{i}z^i(1+z)^i \\
&= \frac{k+1}{n-k}[z^{n-k}]\sum_{j=0}^{k}\binom{k}{j}z^j\sum_{i=0}^{n-k}\binom{n-k}{i}z^i\sum_{\ell=0}^{i}\binom{i}{\ell}z^\ell \\
&= \frac{k+1}{n-k}\sum_{j=0}^{k}\binom{k}{j}\sum_{i=0}^{n-k}\binom{n-k}{i}\binom{i}{n-k-i-j-1}.
\end{aligned}$$

□

Remark 4.8. The Riordan array $(1+zM(z), z(1+zM(z)))$ is a pseudo-involution in the Riordan group (see [5, Example 8]), that is, the matrix $[(-1)^k t_{n,k}]_{n,k\geqslant 0}$ is idempotent. Thus, this work yields a significant lattice path interpretation of this array.

4.4 Comment on Section 3.2

Let $\mathcal{T}=[t(n,k)]_{n\geqslant 0,k\geqslant 0}$ be the matrix given in Section 3.2 where $t(n,k)$ is the number of length n PMAP in $\mathcal{M}_2'$ ending at height k.

Proposition 4.9. *The matrix $\mathcal{T}=[t(n,k)]_{n\geqslant 0,k\geqslant 0}$ can be written*

$$\mathcal{T}=\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
1 & 1 & 1 & 1 & 1 & 1 & \cdots \\
1 & 1 & 1 & 1 & 1 & 1 & \cdots \\
2 & 3 & 4 & 5 & 6 & 7 & \cdots \\
4 & 6 & 8 & 10 & 12 & 14 & \cdots \\
9 & 15 & 22 & 30 & 39 & 49 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}=A\cdot B$$

Accepted manuscript

where

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 3 & 1 & 0 & 0 & 0 & \cdots \\ 4 & 6 & 2 & 0 & 0 & 0 & \cdots \\ 9 & 15 & 7 & 1 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \textit{and} \quad B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 0 & 0 & 1 & 2 & 3 & 4 & \cdots \\ 0 & 0 & 0 & 1 & 3 & 6 & \cdots \\ 0 & 0 & 0 & 0 & 1 & 4 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

are defined as follows:

- *The matrix $B = [b_{n,k}]_{n,k\geqslant 0}$ is defined by $b_{0,0} = 1$, and $b_{n,0} = b_{0,n} = 0$ if $n \geqslant 1$, and $b_{n,k} = \binom{k-1}{n-1}$ otherwise, which is the same as in* Proposition 4.3.
- *The matrix $A = [a_{n,k}]_{n,k\geqslant 0}$ is the almost Riordan array with initial column whose generating function is*

$$g_0(z) = 1 + zM(z) = \frac{1 + z - \sqrt{1 - 2z - 3z^2}}{2z},$$

which is followed by the shifted Riordan array $(g(z), z^2 g(z))$ where

$$g(z) = \frac{1 - z - 2z^2 - \sqrt{1 - 2z - 3z^2}}{2z^3(1 + z)}.$$

Proof. The almost Riordan array A has generating function

$$g_0(z) + zu\frac{g(z)}{1 - z^2 ug(z)}.$$

Setting $G(z, u) = g_0(z) + zu\frac{g(z)}{1-z^2ug(z)}$, it suffices to check that the generating function corresponding to the matrix $A \cdot B$, that is $G(z, \frac{u}{1-u})$, coincides with the generating function $\texttt{Total}(z, u)$ of the matrix $\mathcal{T}$ given in Theorem 3.4. □

Proposition 4.10. *The matrix*

$$[t(n,k)]_{n\geqslant 1,k\geqslant 1} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 1 & 1 & 1 & 1 & \cdots \\ 3 & 4 & 5 & 6 & 7 & \cdots \\ 6 & 8 & 10 & 12 & 14 & \cdots \\ 15 & 22 & 30 & 39 & 49 & \cdots \\ 36 & 54 & 75 & 99 & 126 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

*is the rectification of the Riordan array $(M(z), zR(z))$ where $M(z) = \frac{1-z-\sqrt{1-2z-3z^2}}{2z^2}$ is the generating function of the Motzkin numbers, and $R(z) = \frac{1+z-\sqrt{1-2z-3z^2}}{2z(1+z)}$ is the generating function of the Riordan numbers (*A005043 *in* [13]*).*

Accepted manuscript

Proof. It suffices to check that the generating function of $\mathcal{T}$, i.e. $\texttt{Total}(z,u)$, given in Theorem 3.4, equals to $g_0(z) + zu\frac{M(z)}{1-u\frac{zR(z)}{z}}$. □

We let m_n denote the n-th Motzkin number $m_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} c_k$ where c_k is the k-th Catalan defined above.

Corollary 4.11. *We have*

$$t(n,k) = \begin{cases} [k=0] & \text{if } n=0, \\ r(n-1,k) & \text{otherwise,} \end{cases}$$

where

$$r(n,k) = \sum_{i=0}^{n} \frac{m_i \cdot (k+[n=i])}{n+k-i+[n=i]} \sum_{j=0}^{n-i} (-1)^{n-i-j} \binom{n+k-i+j-1}{j} \sum_{\ell=0}^{n+k-i} \binom{n+k-i}{\ell} \binom{\ell}{n-i-j-\ell}.$$

Proof. We have $(M(z), zR(z))^{-1} = \left(\frac{(1-x)^2}{1-x+x^2}, \frac{x(1-x)}{1-x+x^2}\right)$. If we denote by $(v(z), u(z))$ this inverse Riordan array, then we obtain

$$(M(z), zR(z)) = \left(\frac{1}{v(\bar{u}(z))}, \bar{u}(z)\right),$$

where $\bar{u}$ is the compositional inverse [8] of u. Using the definition of a Riordan array, and Lagrange inversion, we find that the Riordan array $(M(z), zR(z))$ has general term $\tilde{r}(n,k)$ given by

$$\sum_{i=0}^{n} \frac{m_i \cdot (k+[n=k+i])}{n-i+[n=k+i]} \sum_{j=0}^{n-k-i} (-1)^{n-k-i-j} \binom{n-i+j-1}{j} \sum_{\ell=0}^{n-i} \binom{n-i}{\ell} \binom{\ell}{n-k-i-j-\ell}.$$

This array begins as follows:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 1 & 1 & 0 & 0 & 0 & \cdots \\ 4 & 3 & 1 & 1 & 0 & 0 & \cdots \\ 9 & 6 & 4 & 1 & 1 & 0 & \cdots \\ 21 & 15 & 8 & 5 & 1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

To rectify this array and thus to obtain the array $(M(z), R(z))$, we change n to $n+k$, and we obtain $r(n,k)$ above. To this array we must now prepend the row $(1,0,0,0,\ldots)$, and the result follows. □

Accepted manuscript

References

[1] C. Banderier, M. Bousquet-Mélou, A. Denise, P. Flajolet, D. Gardy and D. Gouyou-Beauchamps, Generating functions for generating trees, *Discrete Math.* **246** (2002), 29–55, doi: 10.1016/S0012-365X(01)00250-3, https://doi.org/10.1016/S0012-365X(01)00250-3.

[2] J.-L. Baril, S. Kirgizov, R. Maréchal and V. Vajnovszki, Enumeration of Dyck paths with air pockets, *J. Integer Seq.* **26** (2023), Article 23.3.2, 23 pp., https://cs.uwaterloo.ca/journals/JIS/VOL26/Kirgizov/kirg5.html.

[3] J.-L. Baril, S. Kirgizov, R. Maréchal and V. Vajnovszki, Grand Dyck paths with air pockets, *Art Discrete Appl. Math.* **7** (2024), Article #P1.07, doi:10.26493/2590-9770.1587.b2a, https://doi.org/10.26493/2590-9770.1587.b2a.

[4] P. Barry, *Riordan Arrays: A Primer*, Logic Press, Kilcock, Co. Kildare, 2017.

[5] A. Burstein and L. W. Shapiro, Pseudo-involutions in the Riordan group, *J. Integer Seq.* **25** (2022), Article 22.3.6, 54 pp., https://cs.uwaterloo.ca/journals/JIS/VOL25/Burstein/burstein14.html.

[6] M. Kauers, *Guessing Handbook, Version 0.32*, 2009, Technical Report 09-07, RISC-Linz, https://www3.risc.jku.at/research/combinat/software/ergosum/RISC/Guess.html.

[7] A. Krinik, G. Rubino, D. Marcus, R. J. Swift, H. Kasfy and H. Lam, Dual processes to solve single server systems, *J. Stat. Plann. Inference* **135** (2005), 121–147, doi:10.1016/j.jspi.2005.02.010, https://doi.org/10.1016/j.jspi.2005.02.010.

[8] D. Merlini, R. Sprugnoli and M. C. Verri, The method of coefficients, *Am. Math. Monthly* **114** (2007), 40–57, doi:10.1080/00029890.2007.11920390, https://doi.org/10.1080/00029890.2007.11920390.

[9] H. Prodinger, The kernel method: a collection of examples, *Sémin. Lothar. Comb.* **50** (2004), Article B50f, 19 pp., https://www.mat.univie.ac.at/%7Eslc/wpapers/s50proding.html.

[10] H. Prodinger, Partial Dyck paths with air pockets, *Integers* **22** (2022), Paper A94, 8 pp., http://math.colgate.edu/~integers/vol22.html.

[11] E. Rowland and R. Yassawi, Automatic congruences for diagonals of rational functions, *J. Théor. Nombres Bordeaux* **27** (2015), 245–288, doi:10.5802/jtnb.901, https://doi.org/10.5802/jtnb.901.

[12] L. W. Shapiro, S. Getu, W. J. Woan and L. C. Woodson, The Riordan group, *Discrete Appl. Math.* **34** (1991), 229–239, doi:10.1016/0166-218X(91)90088-E, https://doi.org/10.1016/0166-218X(91)90088-E.

[13] N. J. A. Sloane et al., *The On-Line Encyclopedia of Integer Sequences*, https://oeis.org.

Accepted manuscript

Aequat. Math.
© The Author(s), under exclusive licence to Springer Nature Switzerland AG 2024
https://doi.org/10.1007/s00010-024-01043-7

Aequationes Mathematicae

Symmetries in Dyck paths with air pockets

Jean-Luc Baril, Rigoberto Flórez, and José L. Ramírez

Abstract. The main objective of this paper is to analyze symmetric and asymmetric peaks in *Dyck paths with air pockets* (DAPs). These paths are formed by combining each maximal run of down-steps in ordinary Dyck paths into a larger, single down-step. To achieve this, we present a trivariate generating function that counts the number of DAPs based on their length and the number of symmetric and asymmetric peaks they contain. We determine the total numbers of symmetric and asymmetric peaks across all DAPs, providing an asymptotic for the ratio of these two quantities. Recursive relations and closed formulas are provided for the number of DAPs of length n, as well as for the total number of symmetric peaks, weight of symmetric peaks, and height of symmetric peaks. Furthermore, a recursive relation is established for the overall number of DAPs, similar to that for classic Dyck paths. A DAP is said to be *non-decreasing* if the sequence of ordinates of all local minima forms a non-decreasing sequence. In the last section, we focus on the sets of non-decreasing DAPs and examine their symmetric and asymmetric peaks.

Mathematics Subject Classification. 05A15, 05A19.

Keywords. Dyck path with air pockets, Non-decreasing Dyck path, Symmetric peak, Popularity.

1. Introduction

In their paper on lattice paths, Baril et al. [3] introduced a new type of lattice paths in the first quadrant of $\mathbb{Z}^2$, called *Dyck paths with air pockets* (DAPs). These paths start at the origin, end on the x-axis, and consist of up-steps $U = (1, 1)$ and down-steps $D_k = (1, -k)$, where $k \geq 1$, and no two down-steps can be consecutive. The *length* of a path P is the number of its steps, denoted by $|P|$. DAPs can be seen as a variation of ordinary Dyck paths where maximal runs of down-steps are replaced by one large down-step. (As remarked in [3], DAPs also correspond to a stack evolution with (partial) reset operations that cannot be consecutive, see [20].) The authors enumerate these paths and their prefixes with respect to the length, the type (up or down) of

Published online: 06 March 2024

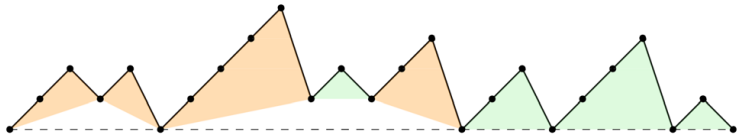


Figure 1. Symmetric and asymmetric peaks of a Dyck path with air pockets

the last step, and the ordinate of the endpoint. Furthermore, they establish a one-to-one correspondence between DAPs of length n and peak-less Motzkin paths of length $n-1$. In a subsequent paper, Baril et al. [4] generalized these paths by allowing them to go below the x-axis, calling them *grand Dyck paths with air pockets* (GDAP). They also provided enumerative results for these paths based on their length and various restrictions on their minimum and maximum ordinates. More recently, the definition of DAPs was extended to include horizontal steps under certain conditions, as described in [5].

Let $\mathcal{D}$ be the set of all DAPs and $\mathcal{D}_n$ be the set of DAPs of length n. Additionally, a special subset of these paths, namely those that are non-decreasing, is considered. A DAP is *non-decreasing* if, as the path is read from left to right, the sequence of the minimal ordinates of the valleys (where a *valley* is an occurrence of D_kU, $k \geq 1$) is non-decreasing. This concept has been previously studied in the literature (see, for example, [2,8,9]). The set of non-decreasing DAPs is denoted by $\mathcal{ND}$.

In this paper, the focus is on analyzing the distribution of symmetric and asymmetric peaks in DAPs and non-decreasing DAPs. A *maximal peak* is defined as an occurrence of U^kD_ℓ, where $k, \ell \geq 1$, that cannot be extended to $U^{k+1}D_\ell$. For the sake of simplicity, the term "peak" is used instead of "maximal peak". (In literature, this type of concept may also be referred to as a *maximal pyramid*.)

On the other hand, a *symmetric peak* is a peak (again, maximal) of the form $\Delta_k := U^kD_k$. An *asymmetric peak* is simply a peak that is not symmetric. We use $\mathsf{peak}(P)$, $\mathsf{sp}(P)$, and $\mathsf{ap}(P)$ to denote the total number of peaks, symmetric peaks, and asymmetric peaks, respectively, in a given path P.

In addition to these parameters, the height and weight of each peak are considered. The *height* of a peak is the maximum ordinate of its points, and the *weight* of a peak is the difference between the maximum and minimum ordinates. For a peak of the form U^kD_ℓ, the weight is simply the $\max\{k, \ell\}$. We use $\mathsf{sumh}(P)$ and $\mathsf{symw}(P)$ to denote the sum of heights and weights of all symmetric peaks in P, respectively.

For example, the path shown in Fig. 1, which has a length of 24, contains four symmetric peaks and four asymmetric peaks. In this particular instance, the height and weight of the symmetric peaks are 2, 3, and 1, corresponding to the height and weight of the last three peaks. It is worth noting that in this

example, both the height and weight are equal for each peak. However, in a more general case, the height and weight can be distinct.

In recent years, there has been a significant amount of research on the concept of symmetric peaks. In 2018, Asakly [1] introduced this concept for words. Later, Flórez and Ramírez [18] extended the concept of symmetric and asymmetric peaks to Dyck paths. The concept was further extended to non-decreasing Dyck paths by Elizalde et al. [14] and Flórez et al. [17], to Motzkin paths by Flórez and Ramírez [16], and to partial Dyck paths by Sun et al. [26]. Elizalde also provided other symmetric results on Dyck paths [13]. In some of these papers, the authors refer to these objects as symmetric pyramids instead of symmetric peaks.

This paper explores various characteristics of symmetric and asymmetric peaks in both DAPs and non-decreasing DAPs. To provide statistics and examine the asymptotic behavior of the different features, generating functions (g.f.) are used. These functions are defined using parameters such as path length, the number of symmetric and asymmetric peaks, and the height of symmetric peaks. Specifically, the total number of symmetric and asymmetric peaks, as well as the height of symmetric peaks, are presented. Additionally, we analyze the asymptotic behavior of the ratio between the number of symmetric peaks and the total number of peaks, and the ratio between the number of asymmetric peaks and the total number of peaks.

Recursive relations for the features examined in this paper are provided, and constructive proofs that rely on combinatorial arguments are offered. These proofs aim to provide intuitive insights into the behavior of symmetric peaks and their weight. For example, we present a recursive relation for the total number of DAPs that is reminiscent of the recursive relation for the total number of classic Dyck paths.

Using the generating functions and recursive relations, we derive closed formulas for the statistics presented in this paper. Most of these formulas use binomial coefficients or Fibonacci numbers. Notably, the total number of DAPs is counted by the generalized Catalan number (see A004148 in [23]), which among other things, counts also the peakless Motzkin paths.

Finally, a combinatorial interpretation of DAPs in terms of binary trees is provided. Thus, we establish a constructive bijection between the trees and DAPs.

2. Symmetric and asymmetric peaks in DAP

In this section, a generating function with three variables is introduced to represent the length of the path, the number of symmetric peaks, and the

number of asymmetric peaks, as defined in the introduction. Using this generating function, we obtain the number of paths in $\mathcal{D}_n$ that avoid symmetric peaks as a corollary. Additionally, the total number of paths in $\mathcal{D}_n$ that avoid asymmetric peaks—peaks located in paths with pyramids at the ground level— is determined, aligning with the Fibonacci numbers F_n. Furthermore, another corollary of the generating function provides insights into the asymptotic behavior of the ratio between the number of symmetric peaks and the total number of peaks, as well as the ratio between the number of asymmetric peaks and the total number of peaks.

We provide a recursive function that simultaneously counts the total number of paths in $\mathcal{D}_n$ and coincides with the generalized Catalan number. Furthermore, we present recursive relations for the number of symmetric peaks and the total number of peaks. We also derive a closed formula using binomial coefficients for the total number of peaks.

Finally, we establish a combinatorial interpretation of DAPs in terms of binary trees. We define a constructive bijection between the trees and DAPs and provide an example to illustrate how the bijection works.

Considering the trivariate generating function:

$$F_{\mathsf{sp},\mathsf{ap}}(x,y,z) = \sum_{P\in\mathcal{D}} x^{|P|} y^{\mathsf{sp}(P)} z^{\mathsf{ap}(P)},$$

which represents the number of Dyck paths with air pockets of length n having k symmetric peaks and ℓ asymmetric peaks. Specifically, the coefficient of $x^n y^k z^\ell$ counts the number of such paths.

The following theorem presents the generating function $F_{\mathsf{sp},\mathsf{ap}}(x,y,z)$ in terms of the length and the numbers of symmetric and asymmetric peaks:

Theorem 2.1. *The generating function $F_{\mathsf{sp},\mathsf{ap}}(x,y,z)$ for the number of DAPs with respect to the length and the numbers of symmetric and asymmetric peaks is*

$$\frac{(1-x)\left(1-x^2-x^2y-x^3y+2x^3z-\sqrt{(1-x^2(1+y+xy-2xz))^2-4xp(x,y,z)}\right)}{2xp(x,y,z)},$$

where $p(x,y,z)=(1-x-x^2(y-z))^2$.

Proof. For short, we set $F = F_{\mathsf{sp},\mathsf{ap}}(x,y,z)$. Let P be a nonempty DAP. Then, we distinguish the following cases.

(1) If $P = U^a D_a Q$, $a \geq 1$, where Q is a DAP, then the g.f. for such paths is $\frac{x^2}{1-x} yF$.

(2) If $P = UU^a D_a Q U^b D_{b+1} R$, $a, b \geq 1$, where Q, R are some DAP, then the g.f. for these paths is

$$x\frac{x^2}{1-x}zF\frac{x^2}{1-x}zF = \frac{x^5}{(1-x)^2}z^2F^2.$$

(3) If $P = UU^aD_a\bar{Q}R$, $a \geq 1$, where Q, R are DAPs (Q non-empty and R possibly empty) and $\bar{Q}$ is obtained from Q after increasing by one the size of the last down-step, and Q does not end with a symmetric peak, then the g.f. for these paths is

$$x\frac{x^2}{1-x}zBF,$$

where B is the g.f. for the number of nonempty DAPs not ending with a symmetric peak. Considering the complement, we obtain easily $B = F - 1 - \frac{x^2}{1-x}yF$.

(4) If $P = UQU^aD_{a+1}R$, $a \geq 1$, where Q, R are some DAPs (Q non-empty and R possibly empty), and Q does not start with a symmetric peak, then the g.f. for these paths is

$$xB'\frac{x^2}{1-x}zF,$$

where B' is the g.f. for the number of nonempty DAPs not starting with a symmetric peak. Using the complement as above, we easily have $B' = B = F - 1 - \frac{x^2}{1-x}yF$.

(5) If $P = U\bar{Q}R$ where Q, R are DAPs (Q non-empty and R possibly empty) such that Q does not start and end with a symmetric peak, and $\bar{Q}$ is obtained from Q after increasing by one the size of the last down-step. The g.f. for this case is xCF where C is the g.f. for the number of nonempty DAPs that do not start and end with a symmetric peak. Considering the complement, we deduce easily $C = B - \frac{x^2y}{1-x}B$.

Summing up all these cases, we obtain the following functional equation:

$$F = 1 + \frac{x^2}{1-x}yF + \frac{x^5}{(1-x)^2}z^2F^2 + 2\frac{x^3}{1-x}zF\left(F - 1 - \frac{x^2}{1-x}yF\right)$$
$$+xF\left(F - 1 - \frac{x^2}{1-x}yF\right)(1 - \frac{x^2y}{1-x}),$$

which gives us the desired result. □

The first terms of the Taylor expansion are as follows:

$$1 + x^2y + x^3y + (y^2 + y)x^4 + (2y^2 + z^2 + y)x^5$$
$$+\boldsymbol{(y^3 + 3y^2 + 3z^2 + y)x^6} + O(x^7),$$

where the weights of the DAPs of length 6 are shown in boldface in the expansion. Figure 5 displays these eight DAPs and their corresponding contributions to $F_{\mathsf{sp},\mathsf{ap}}(x, y, z)$.

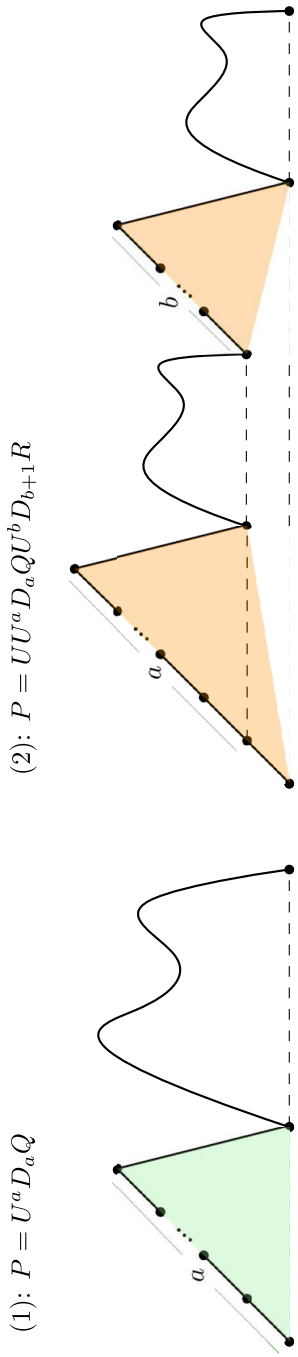


Figure 2. Decomposition of cases (1) and (2)

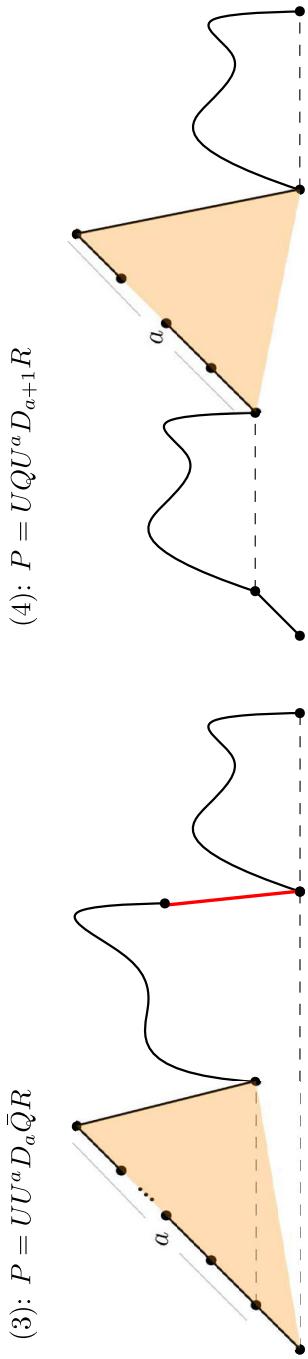


FIGURE 3. Decomposition of cases (3) and (4)

(5): $P = UQR$

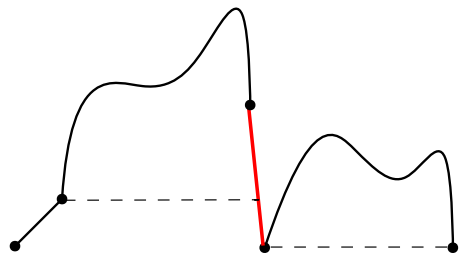


Figure 4. Decomposition of case (5)

Corollary 2.2. *The generating function for the number of DAPs avoiding symmetric peaks is given by:*

$$F(x,0,1) = \frac{(1-x)\left(1-x^2+2x^3-\sqrt{(1-x)(1-3x+3x^2-5x^3+4x^4-4x^5)}\right)}{2x(1-x+x^2)^2}.$$

The Taylor expansion of this generating function is

$$1 + x^5 + \mathbf{3x^6} + 6x^7 + 12x^8 + 25x^9 + 53x^{10} + 115x^{11} + O(x^{12}),$$

where the sequence of coefficients does not appear in [23]. The bold coefficient in the above series can be verified in Fig. 5.

Here, we use the standard notation F_n to denote the n-th Fibonacci number. Specifically, we have $F_n = F_{n-1}+F_{n-2}$ $(n \geq 2)$, with the initial conditions $F_0 = 0$ and $F_1 = 1$.

Corollary 2.3. *The generating function for the number of DAPs avoiding asymmetric peaks is given by:*

$$F(x,1,0) = \frac{1-x}{1-x-x^2}.$$

That is, the number of DAPs of length n without asymmetric peaks is given by the F_{n-1}.

It is worth noting that the Fibonacci sequence counts the compositions of n into parts of size at least 2. In fact, a non-empty DAP of length n without asymmetric peaks necessarily takes the form

$$U^{a_1}D_{a_1}U^{a_2}D_{a_2}\dots U^{a_k}D_{a_k}, \quad \text{with } k \geq 1,\ a_i \geq 1, \quad \text{and}$$
$$n = k + a_1 + a_2 + \cdots + a_k,$$

and it can be associated with the composition of n: $(a_1+1), (a_2+1), \dots, (a_k+1)$, see for instance Fig. 6.

We refer to [3] for the expression of the bivariate generating function for DAPs with respect to the length and the number of peaks (symmetric and asymmetric), which is denoted as $F(x,1,1)$.

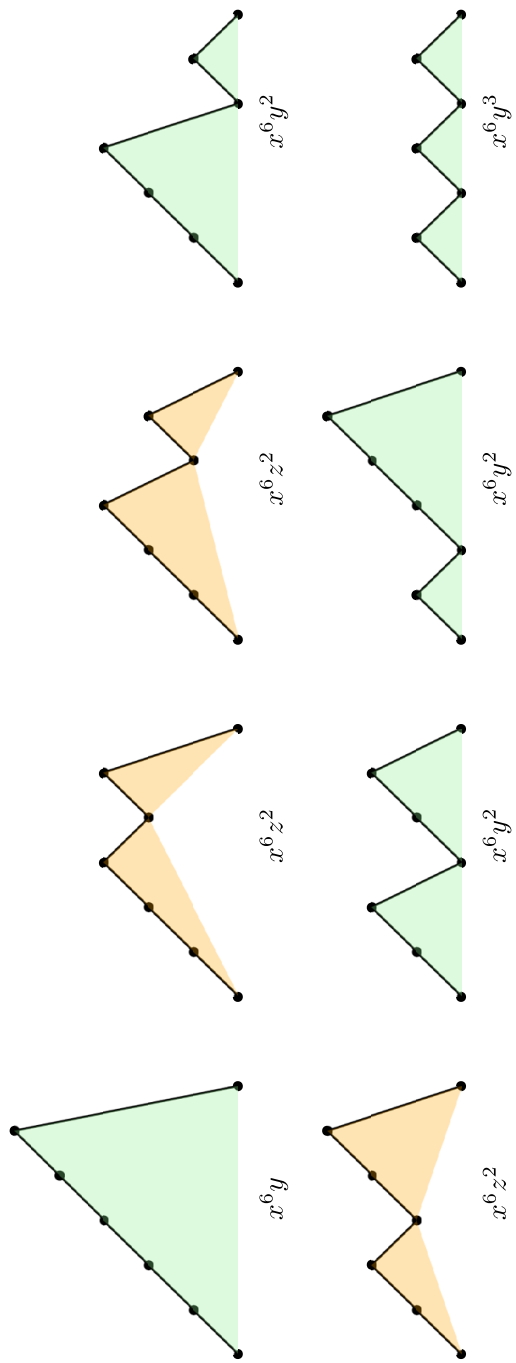


FIGURE 5. The eight DAPs of length 6 and their contribution in $F_{\mathsf{sp,ap}}(x, y, z)$

Figure 6. Illustration of the bijection between length n DAPs without asymmetric peaks and compositions of n

By calculating $\partial_y(F(x,y,1))|_{y=1}$ and $\partial_z(F(x,1,z))|_{y=1}$, we obtain the following two corollaries. The asymptotic approximations of the coefficient of z^n is obtained using classical methods presented in [15,22].

Corollary 2.4. *The generating function for the total number of symmetric peaks in all DAPs is given by:*

$$\frac{x\left(-1+2x+2x^2+3x^3-2x^4+(1+2x)(1-x)\sqrt{(1-3x+x^2)(1+x+x^2)}\right)}{2(1-x)\sqrt{(1-3x+x^2)(1+x+x^2)}},$$

and the asymptotic for the n-th coefficient is

$$\frac{9-4\sqrt{5}}{\sqrt{\pi n}\sqrt{14\sqrt{5}-30}}\left(\frac{1+\sqrt{5}}{2}\right)^{2n}.$$

The Taylor expansion is $x^2+x^3+3x^4+5x^5+\mathbf{10x^6}+21x^7+45x^8+101x^9+O(x^{10})$, where the sequence of coefficients does not appear in [23]. In Fig. 5 it is possible to verify that there are ten symmetric peaks in all DAPs of length 6. Notice that the asymptotic depends on the golden ratio $\frac{1+\sqrt{5}}{2}$ which is a root of the polynomial x^2-x-1. In Theorem 2.9 we give a recurrence relation to calculate the number of symmetric peaks in all DAP.

Corollary 2.5. *The g.f. for the total number of asymmetric peaks in all DAPs is*

$$\frac{x\left(1-x-2x^2-x^3+x^4-(1-x^2)\sqrt{(1-3x+x^2)(1+x+x^2)}\right)}{(1-x)\sqrt{(1-3x+x^2)(1+x+x^2)}},$$

and an asymptotic for the n-th coefficient is

$$\frac{5\sqrt{5}-11}{\sqrt{\pi n}\sqrt{14\sqrt{5}-30}}\left(\frac{1+\sqrt{5}}{2}\right)^{2n}.$$

The Taylor expansion of the generating function for the total number of asymmetric peaks in all DAPs is $2x^5+\mathbf{6x^6}+18x^7+50x^8+132x^9+O(x^{10})$. Note that the sequence of coefficients does not appear in [23]. Figure 5 confirms that there are 6 asymmetric peaks in all DAPs of length 6.

We denote the total number of peaks, symmetric peaks, and asymmetric peaks in all DAPs by $p(n)$, $s(n)$, and $t(n)$, respectively. It is worth noting that from Corollary 10 of [3], we have

$$p(n) \sim \frac{\sqrt{5}-2}{\sqrt{\pi n}\sqrt{14\sqrt{5}-30}}\left(\frac{1+\sqrt{5}}{2}\right)^{2n}.$$

In this corollary we present asymptotic ratios for various peak counts in all DAPs.

Corollary 2.6. *An asymptotic expression for the ratio between the number of symmetric peaks and the total number of peaks in all DAPs:*

$$\lim_{n\to\infty}\frac{s(n)}{p(n)} = \sqrt{5}-2 \sim 0.236067977.$$

An asymptotic expression for the ratio between the number of asymmetric peaks and the total number of all peaks in all DAPs:

$$\lim_{n\to\infty}\frac{t(n)}{p(n)} = 3-\sqrt{5} \sim 0.763932023.$$

An asymptotic expression for the ratio between the numbers of asymmetric and the total number symmetric peaks in all DAPs:

$$\lim_{n\to\infty}\frac{t(n)}{s(n)} = \sqrt{5}+1 \sim 3.236068475.$$

2.1. A recursive relation for the number of DAPs

We use $g(n)$ to denoted the number of DAPs in $\mathcal{D}_n$. The classic proof for the number of Dyck paths can be adapted to obtain a recurrence relation for $g(n)$. It is formally given in the Theorem 2.7.

Let $\mathcal{B}_n \subset \mathcal{D}_n$ the set of all DAPs without valleys at a ground level, i.e., DAPs with no occurrence D_kU, $k \geq 1$, touching the x-axis. There is a bijection between $\mathcal{B}_n$ and $\mathcal{D}_{n-1}$ by deleting the first North-East step (U-step) and replacing the last South-East step of length a (D_a-step) with a step D_{a-1} in all paths in $\mathcal{B}_n$. Notice that $\mathcal{B}_3 = \{U^2D_2\}$ maps bijectively to $\mathcal{D}_2 = \{UD_1\}$, while $\mathcal{B}_2 = \{UD_1\}$ is in bijection with $\mathcal{D}_1 := \{\}$ (the empty set).

Theorem 2.7. *For $n > 3$, we have*

$$g(n) = g(n-1) + g(n-2) + \sum_{k=2}^{n-3} g(k)g(n-k-1),$$

anchored with the initial values $g(1) = 0$ *and* $g(2) = g(3) = 1$. *Furthermore, we have the closed form formula:*

$$g(n) = \sum_{k=1}^{n-1} \frac{1}{n-k}\binom{n-k}{k}\binom{n-k}{k-1}, \quad n \geq 2. \tag{1}$$

Proof. Let us consider $P \in \mathcal{D}_n$. Then P can be decomposed as either $Q \in \mathcal{B}_n$, or UD_1R with $R \in \mathcal{D}_{n-2}$, or QR with $Q \in \mathcal{B}_{k+1}$ and $R \in \mathcal{D}_{n-k-1}$ with $2 \leq k \leq n-3$ (this last decomposition is as in Fig. 4). Due to the above bijection between $\mathcal{D}_{n-1}$ and $\mathcal{B}_n$, the DAPs satisfying the first two cases are enumerated by $g(n-1)+g(n-2)$, and for a fixed k, $2 \leq k \leq n-3$, the DAPs satisfying the third case are enumerated by $g(k)g(n-k-1)$. Varying k in the set $\{2, 3, \ldots, n-3\}$, we obtain the desired result.

On the other hand, let $G(x)$ be the generating function of the sequence $g(n)$. We have

$$G(x) = F_{\mathsf{sp},\mathsf{ap}}(x,1,1) = \frac{1-x-x^2-\sqrt{1-2x-x^2-2x^3+x^4}}{2x}.$$

If $U(x) = G(x)/x$, then $U(x) = x(1+(1+x)U(x)+xU^2(x))$. Consider the auxiliary function $f(x,t)$ defined by $f(x,t) = x(1+(1+t)f(x,t)+tf^2(x,t)) = x(\Phi(x,t))$, where $\Phi(u) = (1+u)(1+tu)$. From the Lagrange inversion, (see [21] for instance), we have that

$$\begin{aligned}[x^n]f(x,t) &= \frac{1}{n}[u^{n-1}]\Phi(u)^n = \frac{1}{n}[u^{n-1}](1+u)^n(1+tu)^n \\ &= \frac{1}{n}\sum_{i=0}^{n-1}\binom{n}{n-1-i}\binom{n}{i}t^i, \quad n \geq 1.\end{aligned}$$

Then $g(n) = [x^n]G(x) = [x^{n-1}]U(x) = [x^{n-1}]f(x,x)$. Comparing coefficients we obtain that

$$\begin{aligned}g(n) &= [x^{n-1}]\sum_{n\geq 1}\frac{1}{n}\sum_{i=0}^{n-1}\binom{n}{n-1-i}\binom{n}{i}x^{n+i} \\ &= [x^{n-1}]\sum_{i\geq 0}\sum_{n\geq 0}\frac{1}{n+i+1}\binom{n+i+1}{n}\binom{n+i+1}{i}x^{n+2i+1}.\end{aligned}$$

Setting $h = n+2i+1$, this implies

$$\begin{aligned}g(n) &= [x^{n-1}]\sum_{i\geq 0}\sum_{h\geq 2i+1}\frac{1}{h-i}\binom{h-i}{h-2i-1}\binom{h-i}{i}x^h \\ &= \sum_{i=0}^{\lfloor\frac{n-2}{2}\rfloor}\frac{1}{n-1-i}\binom{n-1-i}{i+1}\binom{n-1-i}{i}.\end{aligned}$$

□

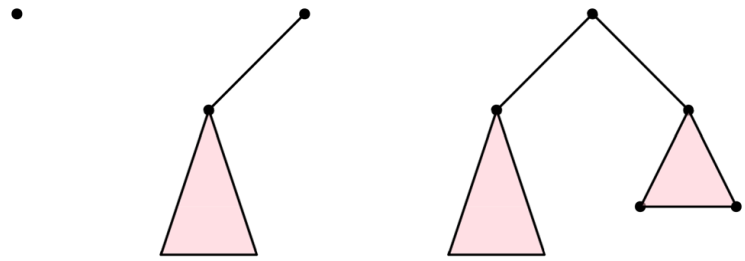


FIGURE 7. Decomposition of the trees in $\mathcal{T}$

The first ten values of the sequence $g(n)$ for $n = 2, \ldots, 11$ are as follows:

$$1, \quad 1, \quad 2, \quad 4, \quad 8, \quad 17, \quad 37, \quad 82, \quad 185, \quad 423.$$

The sequence $g(n)$ has connections to various combinatorial structures, such as generalized bracketings [12], peakless Motzkin paths [7], and zigzag knight's paths [6]. Additionally, recurrence relations of this type have been studied and generalized by Stein and Waterman [25].

We give a probably new combinatorial interpretation in terms of binary trees. An *ordered binary tree* is a rooted tree where each node is either a leaf (with no children), or an internal node (with one or two left/right children). For instance, there are five ordered binary trees with three nodes, and it is well known that these trees with n nodes are counted by the n-th Catalan number (see A000108). Let $\mathcal{T}_n$ be the set of ordered binary trees with n nodes where (i) any right node has a left child, and (ii) any right node is sibling to a left node. Let $\mathcal{T}$ be the set $\bigcup_{n\geq 1} \mathcal{T}_n$. Any tree $T \in \mathcal{T}$ is either a simple root, or a root with a left subtree $T_\ell \in \mathcal{T}$, or a root with a left subtree $T_\ell \in \mathcal{T}$ paired with a right subtree T_r with at least two nodes, see Fig. 7.

From this description of $\mathcal{T}$, we easily deduce that $\mathcal{T}_n$ has the same cardinality as $\mathcal{D}_n$. Indeed, if $T := T(x)$ is the generating function for the cardinality of $\mathcal{T}$ with respect to the number of nodes, then the description induces $T = x + xT + xT(T - x)$.

Let us now exhibit the bijection between $\mathcal{D}_n$ and $\mathcal{T}_n$. Given a path $P \in \mathcal{D}_n$, we can express it as a concatenation of subpaths of the form $U^t D_d$, $t, d \geq 1$. Initiating the process, let us create a root node r. Starting at r as the current node, for the first U-step in P (reading from left to right), we add a new left-child to the current node and move to the newly created leaf, which becomes the new current node. We repeat this procedure for the t consecutive U-steps. For each D_d-step, we create a right-child to the $(d + k)$-th ancestor (i.e., we go back $(d + k)$ times towards the root) where k is the number of ancestors of the current node with a right child, or equivalently the number of ancestors of out-degree 2). We then move to the newly created leaf (which becomes the new current node). We repeat this procedure until we traverse entirely the path P, except the last down-step.

The fact that a DAP does not contain two consecutive down-steps implies that any right node has a left child. Due to the construction, any right node is sibling of a left node, which ensures that the generated tree belongs to $\mathcal{T}$. With the above cardinality argument, this map is a bijection. See Fig. 8 for an illustration of the bijection applied to all DAPs of length 6.

2.2. The total number of symmetric peaks

In this section, a recursive relation for the total number of symmetric peaks in $\mathcal{D}_n$ is given. Let us denote by $s^*(n)$ the total number of all first peaks that are symmetric, excluding the symmetric peak $U^{n-1}D_{n-1}$ of length n (i.e, the path formed by a single peak). Additionally, let us use Δ_k to represent the symmetric peak U^kD_k.

Using the symmetry construction of paths in $\mathcal{D}_n$, we can show that the total number of last symmetric peaks at ground level in $\mathcal{D}_n$ is equal to the total number of first symmetric peaks in $\mathcal{D}_n$, except for the peak $U^{n-1}D_{n-1}$ of length n. If we include the path formed by a single peak, the total count is given by $s^*(n)+1$, for $n \geq 2$. We state this fact formally in the following lemma.

Lemma 2.8. *If $n > 1$, then $s^*(n)$ is given by*

$$s^*(n) = \sum_{i=2}^{n-2}\sum_{k=1}^{i-1} \frac{1}{i-k}\binom{i-k}{k}\binom{i-k}{k-1}.$$

Proof. The total number of the first (resp., last) peaks that are symmetric is counted by the total number of first (resp., last) peaks at a ground level. Thus, this is counted by the total number of paths of the form $\Delta_i P$ (resp., $P\Delta_i$) where P is a DAP of length $n-i-1$ for $1 \leq i \leq n-3$. Clearly, for i fixed it is counted by the total number of paths in $\mathcal{D}_{n-i-1}$; this is given by $g(n-i-1)$ (see Theorem 2.7). Varying i from 1 to $n-3$, we obtain the recurrence relation $s^*(n) = \sum_{i=2}^{n-2} g(i)$ for $n > 4$, with the initial values $s^*(4) = 1$ and $s^*(n) = 0$ for $n < 4$. This, together with Theorem 2.7, implies the desired formula. □

As stated in the introduction, our goal is to derive recurrence relations using combinatorial arguments. Thus, we now present a recurrence relation and its proof, which help us achieve this objective to some extent.

Theorem 2.9. *The sequence $s(n)$ satisfies the following recurrence relation for $n > 4$*

$$s(n) = s(n-1) + s(n-2) - 2s^*(n-1) + g(n-2) + \sum_{k=2}^{n-3} 2g(n-k-1)s'(k),$$

where $s'(k) = s(k) - s^*(k)$, and with initial values $s(2) = s(3) = 1$ and $s(4) = 3$.

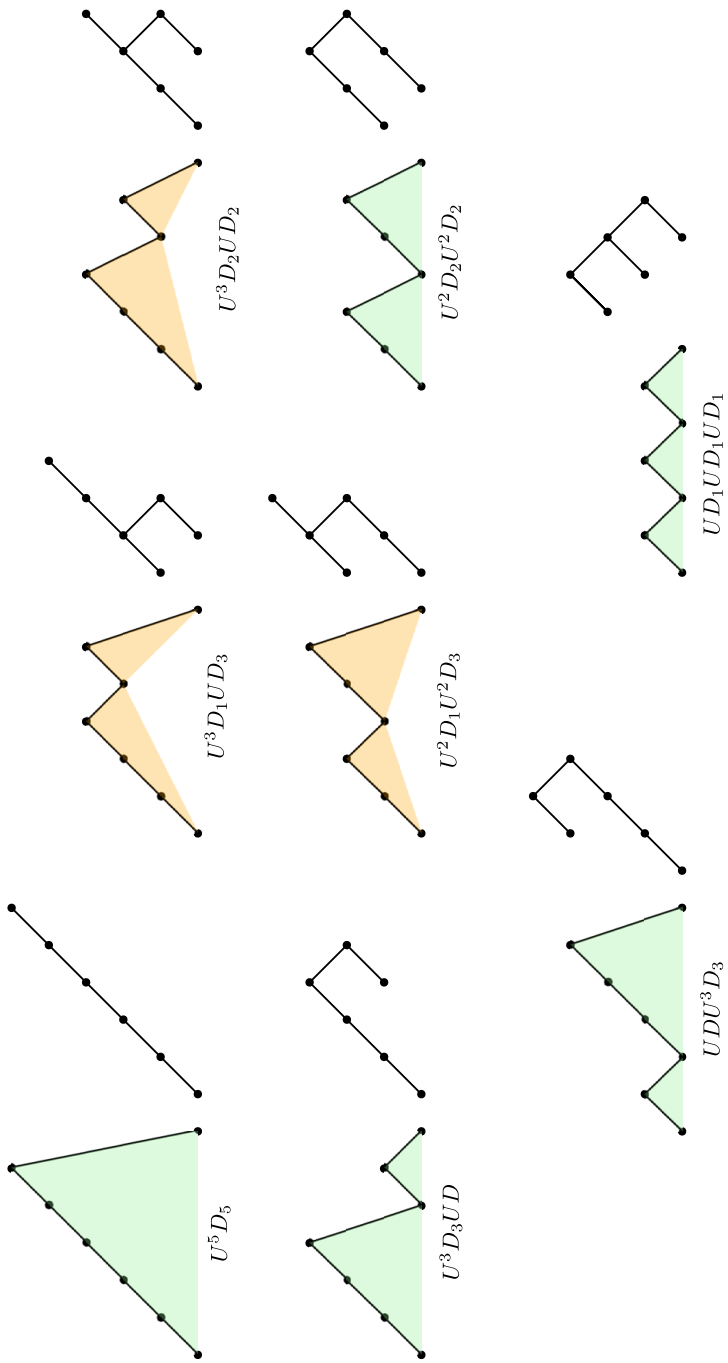


FIGURE 8. Bijection between $\mathcal{D}_6$ and $\mathcal{T}_6$

Proof. Notice that any non-empty path in $\mathcal{D}_n$ can be decomposed as QR where $Q \in \mathcal{B}_{k+1}$ and $R \in \mathcal{D}_{n-k-1}$ for $k \geq 1$ (see Fig. 4).

To count the number of symmetric peaks in a path of the form QR, we observe that it is given by the sum of the number of symmetric peaks in Q and the number of symmetric peaks in R. The first is found multiplying the number of symmetric peaks in $\mathcal{B}_{k+1}$ by the total number of path in $\mathcal{D}_{n-k-1}$. We distinguish four cases.

(1) If $k = 1$, then we have $Q = \Delta_1 \in \mathcal{B}_2$. The total number of peaks derived from $Q = \Delta_1$ is $g(n-2)$ (equal to the number of paths in $\mathcal{D}_{n-2}$). Therefore, adding $g(n-2)$ with the total number of symmetric peaks counted over all paths $R \in \mathcal{D}_{n-2}$, we obtain that there are $s(n-2) + g(n-2)$ symmetric peaks in the paths lying in this case.
(2) If $k = 2$, then using a similar argument as for $k = 1$, there are $s(n-3) + g(n-3)$ symmetric peaks counted over all paths lying in this case.
(3) If $k = n-1$, then the number of symmetric peaks derived from $Q \in \mathcal{B}_{n-1+1}$ can be counted using $s(n-1)$ (by the bijection given above). However the first and the last peaks that are symmetric in all paths in $\mathcal{D}_{n-1}$ are not symmetric peaks in $\mathcal{B}_n$, so we have to subtract them from $s(n-1)$. We can use $s^*(n-1)$ for this purpose. Thus, the total number of symmetric peaks counted over all paths in this case is $s(n-1)-2s^*(n-1)$.
(4) If $2 < k < n-2$, then the number of symmetric peaks derived from $Q \in \mathcal{B}_{k+1}$ can be counted using $s(k) - 2s^*(k)$. (The shaded peaks in Fig. 3 shows the part that we need to subtract by $2s^*(n-1)$.) Therefore, the total number of symmetric peaks counted over all paths of the form QR, $Q \in \mathcal{B}_{k+1}$, $R \in \mathcal{D}_{n-k-1}$, is given by $(s(k) - 2s^*(k))g(n-k-1) + s(n-k-1)g(k)$. By varying k from 3 to $n-3$ and adding the special three cases, we obtain the desired result.

This completes the proof. □

Let us define $p(n)$ as the total number of peaks in $\mathcal{D}_n$. Using a binomial expression, we can write this sequence as:

$$p(n) = \sum_{k=0}^{n} \binom{k-1}{2k-n} \binom{k}{2k-n+1}.$$

A proof of this result, using generating functions, can be found in [3, p. 10]. Alternatively, by modifying the proof of Theorem 2.9 —by removing the use of $s^*(k)$— we can derive a recurrence relation that counts the total number of peaks in $\mathcal{D}_n$, that is, for $n \geq 5$

$$p(n) = p(n-1) + p(n-2) + g(n-2) + 2\sum_{k=2}^{n-3} p(k)g(n-k-1),$$

with the initial values $p(2) = p(3) = 1$ and $p(4) = 3$. It is important to note that a recurrence relation for the number of asymmetric peaks in $\mathcal{D}_n$ can be obtained by subtracting $s(n)$ from $p(n)$.

3. Counting symmetric weight

Recall that the *weight* of a peak is the difference between the maximum and minimum ordinates, which is also equal to the $\max\{k, \ell\}$ whenever the peak is $U^k D_\ell$. For a symmetric peak $U^k D_k$, its weight is k. Let $\mathsf{symw}(P)$, called the *sum of symmetric weights*, be the sum of weights of all symmetric peaks in P. As an illustration, the symmetric peaks of the path depicted in Fig. 1 have weights $1, 2, 3$, and 1, respectively. Thus, $\mathsf{symw}(P) = 7$.

In this section, we present several results related to the sum of weights of symmetric peaks in $\mathcal{D}_n$. Firstly, we derive a generating function in two variables that counts the sum of weights of all symmetric peaks in $\mathcal{D}_n$. Secondly, we establish a recursive relation for the sum of weights of all symmetric peaks in $\mathcal{D}_n$. We start by introducing the generating function $W(x, q)$ of all DAPs with respect to their length and sum of symmetric weights, defined as follows:

$$W(x, q) = \sum_{P \in \mathcal{D}} x^{|P|} q^{\mathsf{symw}(P)}.$$

Using the same decomposition as in the proof of Theorem 2.1, we derive the following result.

Theorem 3.1. *The generating function $W(x, q)$ for the number of DAPs with respect to the length and the sum of symmetric weights is*

$$W(x, q) = \frac{(1-x)(1-qx)\left(1 - qx - (1+q)x^2 + (2+q)x^3 - qx^4 - \sqrt{w(x,q)}\right)}{2x(1 - x(1+q) + x^2)^2},$$

where $w(x, q)$ is defined as:

$$w(x, q) = (1-x)(1 - (3+2q)x + (1+q)(3+q)x^2$$
$$-(5+q^2)x^3 + 2(2+2q-q^2)x^4 - 2(2+q+q^2)x^5 + q(4+q)x^6 - q^2x^7).$$

The first terms of the Taylor expansion are as follows:

$$1 + qx^2 + q^2x^3 + (q^2+q^3)x^4 + (1+2q^3+q^4)x^5$$
$$+\boldsymbol{(3 + q^3 + 3q^4 + q^5)x^6} + O(x^7).$$

All DAPs of length 6 are displayed in Fig. 9, with their corresponding symmetric weights highlighted in boldface in the previous expansion.

The *total symmetric weight* of $\mathcal{D}_n$ is defined as the sum of the symmetric weights over all paths in $\mathcal{D}_n$, and it is denoted by $w(n)$. By calculating $\partial_q(W(x, q))|_{q=1}$, we can obtain the following two corollaries.

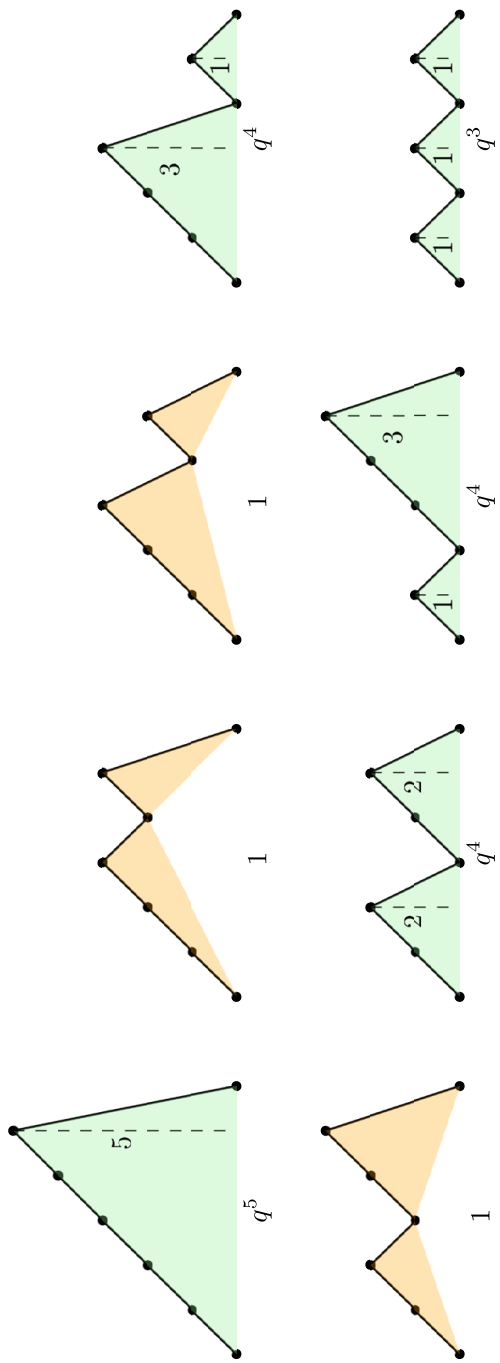


Figure 9. Symmetric weight of all DAPs of length 6

Corollary 3.2. *The generating function for the total symmetric weight in all DAPs is given by*

$$\frac{x\left(-1+2x+2x^2+3x^3-2x^4+(1-x)(1+2x)\sqrt{1-2x-x^2-2x^3+x^4}\right)}{2(1-x)^2\sqrt{1-2x-x^2-2x^3+x^4}},$$

and an asymptotic expression for the n-th coefficient is

$$\frac{5\sqrt{10}-11\sqrt{2}}{4\sqrt{\pi n}\,\sqrt{-15+7\sqrt{5}}}\cdot\left(\frac{1+\sqrt{5}}{2}\right)^{2n}.$$

The Taylor expansion of the generating function is

$$x^2+2x^3+5x^4+10x^5+20x^6+41x^7+86x^8+187x^9+O(x^{10}).$$

It worth to mention that the sequence of coefficients does not appear yet in [23].

Let us use $w^*(n)$ to denote the total sum of symmetric weights of all first peaks that are symmetric over all paths in $\mathcal{D}_n$, except for the path U^nD_n. The weight sum of all first peaks that are symmetric, including the weight of the single peak of length n, is given by $w^*(n)+(n-1)$. Using the symmetry construction of paths in $\mathcal{D}_n$, we can observe that the total weight of the first peaks that are symmetric in $\mathcal{D}_n$ is equal to the total weight of the last peak at a ground level that is symmetric in $\mathcal{D}_n$. It is worth noting that $w^*(n)$ also counts the total height of all first peaks that are symmetric in $\mathcal{D}_n$, except for the path formed by the single peak. Recall that the height of an occurrence U^kD_ℓ in a path is the maximal ordinate of its points.

Lemma 3.3. *If $n>1$, then $w^*(n)$ is given by*

$$w^*(n)=\sum_{i=2}^{n-2}\sum_{k=1}^{n-i-1}\frac{i-1}{n-i-k}\binom{n-i-k}{k}\binom{n-i-k}{k-1}.$$

Proof. For a given $i\geq 1$, the sum of the weights (over all DAPs in $\mathcal{D}_n$) of all first (resp., last) peaks Δ_i is i multiplied by the number of DAPs of length $n-i-1$, which is $i\cdot g(n-i-1)$. Varying i in the set $\{1,2,\ldots,n-3\}$, we obtain the recurrence relation $w^*(n)=\sum_{i=2}^{n-2}(i-1)g(n-i)$ for $n>4$, with the initial values $w^*(4)=1$ and $w^*(n)=0$ for $n<4$.

Using Theorem 2.7, we have the binomial expression for $w^*(n)$ given in the statement of this lemma. □

Theorem 3.4. *For $n\geq 4$, we have*

$$w(n)=w(n-1)+w(n-2)-2w^*(n-1)+1+\sum_{k=1}^{n-3}2w'(k)g(n-k-1),$$

where $w'(k)=w(k)-w^*(k)+\frac{1}{2}$, and the initial values are $w(1)=0$, $w(2)=1$, and $w(3)=2$.

Proof. Consider any non-empty path in $\mathcal{D}_n$, which can be decomposed as QR where $Q \in \mathcal{B}_{k+1}$ and $R \in \mathcal{D}_{n-k-1}$ for $k \geq 1$ (it can be seen in Fig. 4). We consider four cases.

(1) If $k = 1$, then $Q = \Delta_1$ and the total symmetric weight of all symmetric peaks (over $\mathcal{D}_n$) derived from $Q = \Delta_1$ is $g(n-2)$. Adding this to the total symmetric weight of all paths of the form $R \in \mathcal{D}_{n-2}$, we obtain that $w(n-2)+g(n-2)$ is the total symmetric weight of all paths in this case.

(2) If $k = 2$, then, with the same argument as for the previous case, the total symmetric weight of all paths in this case is $w(n-3) + 2g(n-3)$.

(3) If $k = n-1$, then the total symmetric weight of $Q \in \mathcal{B}_{n-1+1}$ is given by the total symmetric weight of $\mathcal{D}_{n-1}$ which is $w(n-1)$. However, there are two sub-cases to consider in this counting: firstly, the weight of the single peak in $\mathcal{D}_{n-1}$ is $n-1$ but in $\mathcal{B}_n$ it is n, so we must add one to the counting; secondly, the first and the last peaks that are symmetric in all paths in $\mathcal{D}_{n-1}$ cannot be counted, as they are not symmetric peaks in $\mathcal{B}_n$, so we have to subtract them from $w(n-1)$. We use $w^*(n-1)$ to adjust for these differences. Thus, the total symmetric weight of $\mathcal{B}_n$ is $w(n-1) - 2w^*(n-1) + 1$.

(4) If $3 \leq k \leq n-3$, then the total symmetric weight derived from $Q \in \mathcal{B}_{k+1}$ can be counted using $w(k) - 2w^*(k) + 1$. Therefore, the total symmetric weight for QR (see Fig. 3) is given by $(w(k) - 2w^*(k) + 1)g(n-k-1) + w(n-k-1)g(k)$. By varying k from 3 to $n-3$ and adding the special three cases, we obtain the desired result.

This completes the proof. □

4. Counting symmetric height

In this section, we present a method for counting the sum of heights of peaks in $\mathcal{D}_n$, using a generating function expressed as a continuous fraction. Additionally, we provide recursive relations to compute both the sum of heights of all peaks and the sum of heights of symmetric peaks in $\mathcal{D}_n$.

Let $h(n)$ and $h_s(n)$ denote the sums of heights of all peaks and all symmetric peaks, respectively, in the set $\mathcal{D}_n$ of DAPs of length n. The height of a peak is defined as the y-coordinate of its highest point measured from the ground level. We use $\mathsf{sumh}(P)$ to denote the sum of heights of all peaks in the path P, and $\mathsf{peak}(P)$ to denote the number of peaks in the path P.

Let $\mathcal{R}$ be the set of all classical Dyck paths. We define the generating function $R(x,p,q)$ over all classical Dyck paths

$$R(x,p,q) = \sum_{P \in \mathcal{R}} x^{|P|/2} p^{\mathsf{peak}(P)} q^{\mathsf{sumh}(P)}.$$

Deutsch [11] proved that this generating function satisfies the functional equation

$$R(x,p,q) = 1 + x(R(x,qp,q) - 1 + pq)R(x,p,q). \tag{2}$$

Let $Q(x,q)$ be the generating functions of all DAPs with respect to the length and the sum of the heights. That is,

$$Q(x,q) = \sum_{P\in\mathcal{D}} x^{|P|} q^{\mathsf{sumh}(P)}.$$

Theorem 4.1. *An expression for generating function of $Q(x,q)$ is given by the continued fraction*

$$Q(x,q) = \cfrac{1}{1 + x - qx^2 - \cfrac{x}{1 + x - q^2x^2 - \cfrac{x}{1 + x - q^3x^2 - \cfrac{x}{\ddots}}}}.$$

Proof. Notice that each peak in a Dyck path has to be counted as a down step in a DAP. Therefore, we have the relation $Q(x,q) = R(x,x,q)$. From the functional equation (2), we obtain

$$Q(x,q) = R(x,x,q) = 1 + x(R(x,qx,q) - 1 + qx)R(x,x,q).$$

Therefore, we have

$$Q(x,q) = \frac{1}{1 - x(R(x,qx,q) - 1 + qx)} = \frac{1}{1 + x - qx^2 - xR(x,qx,q)}.$$

Iterating this expression yields the desired result. □

The first terms of the continued fraction are as follows:

$$1 + qx^2 + q^2x^3 + (q^2+q^3)x^4 + (2q^3+2q^4)x^5$$
$$+(q^3+3q^4+3q^5+q^6)x^6 + (3q^4+6q^5+5q^6+2q^7+q^8)x^7 + O(x^8).$$

Theorem 4.2. *The generating function for the sum of the heights of all peaks in $\mathcal{D}_n$ is*

$$H(x) = \sum_{n\geq 0} h(n)x^n = \frac{x^2}{(1-3x+x^2)(1+x+x^2)}.$$

Moreover,

$$h(n) = \sum_{k=0}^{n} \frac{1}{2}\binom{2n-2k}{2k-1},$$

and an asymptotic for the n-th coefficient is

$$\frac{\sqrt{5}}{20} \cdot \left(\frac{1+\sqrt{5}}{2}\right)^{2n}.$$

Proof. We can use Theorem 4.1 to derive an expression for the generating function $H(x)$ as follows:

$$H(x) = \partial_q(Q(x,q))|_{q=1} = \sum_{\ell \geq 1} \frac{\ell x^{\ell+1}}{M^{2\ell}(x)} = \frac{x^2 M^2(x)}{(x - M^2(x))^2}, \tag{3}$$

where

$$M(x) = 1 + x - x^2 - \cfrac{x}{1 + x - x^2 - \cfrac{x}{1 + x - x^2 - \cfrac{x}{\ddots}}} = 1 + x - x^2 - \frac{x}{M(x)}.$$

Note that we can express $M(x)$ as

$$M(x) = 1/2\left(1 + x - x^2 - \sqrt{1 - 2x - x^2 - 2x^3 + x^4}\right).$$

Substituting this expression for $M(x)$ into (3), we obtain the desired expression for $H(x)$.

To obtain the combinatorial sum, we can manipulate the generating function using standard techniques. □

Upon comparing this generating function with the sequence defined in A182890, it is evident that they are identical.

Theorem 4.3. *The sequence $h(n)$ satisfies the following recurrence relation for $n \geq 4$:*

$$h(n) = h(n-2) + h(n-1) + p(n-1) + g(n-2) + \sum_{k=2}^{n-3} (2h(k) + p(k))\, g(n-k-1),$$

with initial values $h(2) = 1$ and $h(3) = 2$.

Proof. Consider any non-empty path P in $\mathcal{D}_n$, which can be decomposed as $P = QR$ where $Q \in \mathcal{B}_{k+1}$ and $R \in \mathcal{D}_{n-k-1}$ for some $k \geq 1$. We distinguish three cases.

(1) If $k = 1$, then $Q = \Delta_1$, and the sum of the heights of these peaks (over all paths in $\mathcal{D}_n$) is the cardinality of $\mathcal{D}_{n-2}$, which is $g(n-2)$. Adding this to the sum of the heights of all peaks in paths of the form $R \in \mathcal{D}_{n-2}$ gives us that $h(n-2) + g(n-2)$ is the sum of the heights of peaks of all paths in this case.

(2) If $k = n-1$, then the sum of the heights of all peaks in $\mathcal{B}_{n-1+1} = \mathcal{B}_n$ is given by the sum of the heights of all peaks in $\mathcal{D}_{n-1}$ plus the total number of its peaks. This gives us that the sum of the heights of all peaks of all paths in this case is $h(n-1) + p(n-1)$.

(3) If $2 \leq k \leq n-3$, then the sum of the heights of all peaks in $\mathcal{B}_{k+1}$ can be counted using $g(n-k-1)(h(k)+p(k))$. Therefore, the sum of the heights of all peaks of all paths in this case (with k fixed) is given by $h(n-k)g(k) + g(n-k-1)(h(k)+p(k))$. By varying k from 2 to $n-3$, we obtain the recurrence relation in the statement of the theorem.

This completes the proof. □

The first ten values of the sequence $h(n)$ for $n = 2, \ldots, 11$ are as follows:

$$1, \quad 2, \quad 5, \quad 14, \quad 36, \quad 94, \quad 247, \quad 646, \quad 1691, \quad 4428.$$

We use $h_s(n)$ to represent the sum of the heights of all symmetric peaks of $\mathcal{D}_n$.

Theorem 4.4. *The sequence $h_s(n)$, which represents the sum of the heights of all symmetric peaks in $\mathcal{D}_n$, can be described by the following recurrence relation:*

$$h_s(n) = h_s(n-2) - h_s(n-1) + g(n-2) + 2h'_s(n-1) + 2\sum_{k=2}^{n-3} h'(k)g(n-k-1),$$

where $h'_s(k) = h_s(k) + \frac{1}{2}s(k) - s^*(k) - w^*(k)$ for $n \geq 4$, with initial values $h_s(2) = 1$ and $h_s(3) = 2$.

Proof. The proof follows similar steps as the proof of Theorem 3.4, but we need to take into account the height of the peak measured from the ground.

Consider any non-empty path in $\mathcal{D}_n$, which can be decomposed as QR where $Q \in \mathcal{B}_{k+1}$ and $R \in \mathcal{D}_{n-k-1}$ for $k \geq 1$. We distinguish three cases.

(1) If $k = 1$, then the sum of the heights of all symmetric peaks in all these paths is $h_s(n-2) + g(n-2)$.

(2) If $k = n-1$, then the sum of the heights of all symmetric peaks in all these paths is given by $h_s(n-1) - 2w^*(n-1) + s(n-1) - 2s^*(n-1)$.

(3) If $3 \leq k \leq n-3$, then the sum of the heights of all symmetric peaks in all paths in $\mathcal{B}_{k+1}$ is $(h_s(k) - 2w^*(k) + s(k) - 2s^*(k))g(n-k-1)$. Therefore, the sum of the heights of all symmetric peaks in all paths in this case (with k fixed) is given by $(h_s(k) - 2w^*(k) + s(k) - 2s^*(k))g(n-k-1) + w(n-k-1)g(k)$. By varying k from 3 to $n-3$ and adding the special two cases, we obtain the desired result.

This completes the proof. □

The first ten values of the sequence $h_s(n)$ for $n = 2, \ldots, 11$ are as follows:

$$1, \quad 2, \quad 5, \quad 10, \quad 20, \quad 42, \quad 91, \quad 206, \quad 485, \quad 1174.$$

5. Symmetric and asymmetric peaks in non-decreasing DAP

In 1997, Barcucci et al. [2] introduced the concept of non-decreasing Dyck paths. Later in 2001, Prodinger [24] studied non-decreasing Dyck paths in relation to Elena trees. In 2003, Deutsch and Prodinger [10] provided bijections between non-decreasing Dyck paths, directed column-convex polyominoes, Elena trees, and ordered trees of height at most three. In 2015, Flórez et al. [8] used generating functions to count the number of peaks, pyramid weight, and number of valleys in all non-decreasing Dyck paths of a given length. Recently, there has been significant research on the concept of non-decreasing paths, see for example, [3,9,14,17].

In this section, we first focus on the set $\mathcal{ND}$ of non-decreasing DAPs, that is, DAPs where the sequence of the minimal ordinates of the valleys D_kU, $k \geq 1$, (taken from left to right) is non-decreasing. We consider the trivariate generating function

$$G_{\mathsf{sp},\mathsf{ap}}(x,y,z) = \sum_{P \in \mathcal{ND}} x^{|P|} y^{\mathsf{sp}(P)} z^{\mathsf{ap}(P)},$$

where the coefficient of $x^n y^k z^\ell$ is the number of non-decreasing Dyck paths with air pockets of length n with k symmetric peaks and ℓ asymmetric peaks. Corollary 2.3 counts the total number of DAPs formed by symmetric peaks (with no asymmetric peaks). The same result holds here. That is, the total number of non-decreasing DAPs that consist solely of paths with peaks located at a ground level F_{n-1} for $n > 1$.

We summarize our results in a table at the end of this section.

Theorem 5.1. *The generating function $G_{\mathsf{sp},\mathsf{ap}}(x,y,z)$ for the number of non-decreasing DAPs with respect to the length and the numbers of symmetric and asymmetric peaks is*

$$\frac{1-3x+(3-y)x^2-(1-2y+z)x^3-(y-z)x^4+x^5z^2}{(1-x-x^2y)(1-2x-x^2(-1+y)+x^3(y-z))}.$$

Proof. To simplify notation, we set $G = G_{\mathsf{sp},\mathsf{ap}}(x,y,z)$. Let P be a nonempty non-decreasing DAPs. We distinguish the following cases.

(1) If $P = U^a D_a Q$, $a \geq 1$, where Q is a DAP (possibly empty), then the g.f. is $\frac{x^2}{1-x} yG$.

(2) If $P = UU^a D_a Q U^b D_{b+1}$, for $a, b \geq 1$, where Q is a DAP, then necessarily Q is empty or of the form $Q = U^{a_1} D_{a_1} U^{a_2} D_{a_2} \cdots U^{a_k} D_{a_k}$, for some $k \geq 1$, and $a_i \geq 1$ for $i \leq k$. See Fig. 10 for an illustration. The generating function for this case is given by

$$x\frac{x^2}{1-x} z \frac{1}{1-\frac{x^2}{1-x}y} \frac{x^2}{1-x} z = \frac{z^2x^5}{(1-x-x^2y)\,(1-x)}.$$

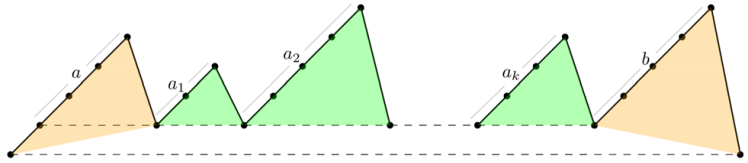


FIGURE 10. Decomposition of case (2)

(3) If $P = UU^aD_a\bar{Q}$, for $a \geq 1$, where Q is not empty and does not start or end with a symmetric peak, and $\bar{Q}$ is obtained from Q by increasing by one the size of its last down-step. The g.f. for this case is $xz\frac{x^2}{1-x}B$, where B is the generating function for non-empty non-decreasing DAPs that do not end with a symmetric peak. Considering the complement, we easily have $B = G - \frac{1}{1-\frac{x^2}{1-x}y}$.

(4) If $P = U\bar{Q}$ where Q is not empty and does not end and start with a symmetric peak, and $\bar{Q}$ is obtained from Q by increasing by one the size of its last down-step. The g.f. for this case is xC, where C is the generating function for non-empty non-decreasing DAPs that do not start or end with a symmetric peak. We easily have $C = B \cdot (1 - \frac{x^2}{1-x}y)$.

Summarizing all these cases, we obtain the following functional equation:

$$G = 1 + \frac{x^2}{1-x}yG + \frac{z^2x^5}{(1-x-x^2y)(1-x)}$$
$$+\frac{x^3z}{1-x}\left(G - \frac{1}{1-\frac{x^2y}{1-x}}\right) + x\left(G - \frac{1}{1-\frac{x^2y}{1-x}}\right)\left(1 - \frac{x^2y}{1-x}\right),$$

which induces the result. □

5.1. Increasing DAPs

Let $\mathcal{I}$ be the set of strictly increasing DAPs, which are non-decreasing DAPs where consecutive valleys cannot have the same ordinate. Notice that any path in $\mathcal{I}$ has one of the following three forms: (i) a non-decreasing DAP without symmetric peaks, (ii) a symmetric peak followed by a non-decreasing DAP without symmetric peaks, or (iii) a symmetric peak followed by another symmetric peak.

Now, let us consider the trivariate generating function

$$H_{\mathsf{sp},\mathsf{ap}}(x,y,z) = \sum_{P\in\mathcal{I}} x^{|P|}y^{\mathsf{sp}(P)}z^{\mathsf{ap}(P)},$$

where the coefficient of $x^ny^kz^\ell$ is the number of increasing Dyck paths with air pockets of length n having k symmetric peaks and ℓ asymmetric peaks.

Table 1. Some statistics for non-decreasing DAPs

	g.f.	Formula
Non-decreasing DAPs avoiding symmetric peaks	$\frac{1-3x+3x^2-2x^3+x^4+x^5}{(1-x)(1-2x+x^2-x^3)}$	A077855
Non-decreasing DAPs avoiding asymmetric peaks	$\frac{1-x}{1-x-x^2}$	F_{n-1}
Total number of symmetric peaks	$\frac{x^2(1-4x+5x^2-3x^3+2x^4+x^5-x^6)}{(1-2x)^2(1-x-x^2)}$	$(n-1)2^{n-6}+F_{n-1}$ $(n\geq 5)$
Total number of asymmetric peaks	$\frac{x^5\left(2-4x+x^3\right)}{(1-x-x^2)(1-2x)^2}$	$(n+5)2^{n-6}+F_{n-1}$ $(n\geq 5)$

TABLE 2. Some statistics for increasing DAPs

	g.f.	Formula/asymptotic
Increasing DAPs avoiding symmetric peaks	$\frac{1-3x+3x^2-2x^3+x^4+x^5}{(1-x)(1-2x+x^2-x^3)}$	A077855
Increasing DAPs avoiding asymmetric peaks	$\frac{1-2x+2x^2-x^3+x^4}{(1-x)^2}$	Sequence of natural numbers (modulo a shift)
Total number of symmetric peaks	$\frac{x^2\left(1-3x+5x^2-6x^3+3x^4-x^5\right)}{(1-x)^2(1-2x+x^2-x^3)}$	Asymptotic for the n-th term is $-\frac{a\left(a^5-3a^4+6a^3-5a^2+3a-1\right)(a^2-a+2)^n}{(3a^2-2a+2)(a-1)^2}$, where $a \sim 0.5698402912$
Total number of asymmetric peaks	$\frac{x^5(1-x+x^2)(2-4x+2x^2-x^3)}{(1-x)^2(1-2x+x^2-x^3)^2}$,	$-\frac{a^3\left(a^3-2a^2+4a-2\right)\left(a^2-a+1\right)(a^2-a+2)^n}{(3a^2-2a+2)^2(a-1)^2}n$, where a is defined above

Based on the three previous forms of an increasing DAP, we can deduce the following.

Theorem 5.2. *The generating function $H_{sp,ap}(x,y,z)$ for the number of increasing DAPs with respect to the length and the numbers of symmetric and asymmetric peaks is*

$$H_{sp,ap}(x,y,z) = G(x,0,z)\left(1+\frac{x^2y}{1-x}\right)+\left(\frac{x^2y}{1-x}\right)^2.$$

The first terms of the Taylor expansion are

$$1+x^2y+x^3y+(y^2+y)x^4+(2y^2+z^2+y)x^5+(3y^2+3z^2+y)x^6+O(x^7).$$

The g.f. for the number of increasing DAPs is

$$H(x,1,1)=\frac{(1-x)^2(1+x^2)}{1-2x+x^2-x^3}.$$

The Taylor expansion of the given expression is

$$1+x^2+x^3+2x^4+4x^5+7x^6+12x^7+21x^8+37x^9+O(x^{10}),$$

where the sequence of coefficients of x^n corresponds to A005251 in [23]. Notice that the increasing DAPs of length n are in bijection with the Dyck path of semi-length $n-1$, subject to the condition that the vector of valleys $(\nu_1,\nu_2,\ldots,\nu_k)$ satisfies $\nu_{i+1}-\nu_i\geq 2$, as shown in [19].

We can adapt the same proofs used in the previous sections to non-decreasing DAPs. Therefore, we will only provide a summary of the results without including the proofs.

Acknowledgements

The authors are grateful to the anonymous referees for helpful comments. The first author was partially supported by University of Burgundy, France. The second author was partially supported by the Citadel Foundation, Charleston SC. The third author was partially supported by Universidad Nacional de Colombia.

Author contributions All authors wrote and reviewed the manuscript.

Declarations

Conflict of interest The authors declare no competing interests.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other

rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.

References

[1] Asakly, W.: Enumerating symmetric and non-symmetric peaks in words. Online J. Anal. Comb. **13** (2018). https://api.semanticscholar.org/CorpusID:53611358

[2] Barcucci, E., Del Lungo, A., Fezzi, A., Pinzani, R.: Nondecreasing Dyck paths and q-Fibonacci numbers. Discrete Math. **170**, 211–217 (1997). https://doi.org/10.1016/S0012-365X(97)82778-1

[3] Baril, J.-L., Kirgizov, S., Maréchal, R., Vajnovszki, V.: Enumeration of Dyck paths with air pockets. J. Integer Seq. **26** (2023), Article 23.3.2. https://cs.uwaterloo.ca/journals/JIS/VOL26/Kirgizov/kirg5.pdf

[4] Baril, J.-L., Kirgizov, S., Maréchal, R., Vajnovszki, V.: Grand Dyck paths with air pockets. Art Discrete Appl. Math. **7** (2024) #P1.07. https://doi.org/10.26493/2590-9770.1587.b2a

[5] Baril, J.-L., Barry, P.: Two kinds of partial Motzkin paths with air pockets. Ars Math. Contemp. (2023). https://doi.org/10.26493/1855-3974.3035.6ac

[6] Baril, J.-L., Ramírez, J.L.: Knight's paths towards Catalan numbers. Discrete Math. **346**, 113372 (2023). https://doi.org/10.1016/j.disc.2023.113372

[7] Cameron, N., Sullivan, E.: Peakless Motzkin paths with marked level steps at fixed height. Discrete Math. **344**, 112154 (2021). https://doi.org/10.1016/j.disc.2020.112154

[8] Czabarka, E., Flórez, R., Junes, L.: Some enumerations on non-decreasing Dyck paths. Electron. J. Combin. **22** (2015), # P1.3, 1–22. https://doi.org/10.37236/3941

[9] Czabarka, E., Flórez, R., Junes, L., Ramírez, J.L.: Enumerations of peaks and valleys on non-decreasing Dyck paths. Discrete Math. **341**, 2789–2807 (2018). https://doi.org/10.1016/j.disc.2018.06.032

[10] Deutsch, E., Prodinger, H.: A bijection between directed column-convex polyominoes and ordered trees of height at most three. Theoret. Comput. Sci. **307**, 319–325 (2003). https://doi.org/10.1016/S0304-3975(03)00222-6

[11] Deutsch, E.: Dyck path enumeration. Discrete Math. **204**, 167–202 (1999). https://doi.org/10.1016/S0012-365X(98)00371-9

[12] Donaghey, R.: Automorphisms on Catalan trees and bracketings. J. Combin. Theory Ser. B **29**, 75–90 (1980). https://doi.org/10.1016/0095-8956(80)90045-3

[13] Elizalde, S.: Symmetric peaks and symmetric valleys in Dyck paths. Discrete Math. **34**, 112364 (2021). https://doi.org/10.1016/j.disc.2021.112364

[14] Elizalde, S., Flórez, R., Ramírez, J.L.: Enumerating symmetric peaks in nondecreasing Dyck paths. Ars Math. Contemp. **21**, 1–23 (2021)

[15] Flajolet, P., Sedgewick, R.: Analytic Combinatorics. Cambridge University Press, Cambridge (2009)

[16] Flórez, R., Ramírez, J.L.: Enumerating symmetric pyramids in Motzkin paths. Ars Math. Contemp. **23** (2023), # 4.06. https://doi.org/10.26493/1855-3974.3061.5bf

[17] : Flórez, R., Junes, L., Ramírez, J. L.: Counting asymmetric weighted pyramids in non-decreasing Dyck paths. Australas. J. Combin. **79**, 123–140 (2021). https://ajc.maths.uq.edu.au/pdf/79/ajc_v79_p123.pdf

[18] Flórez, R., Ramírez, J.L.: Enumerating symmetric and asymmetric peaks in Dyck paths. Discrete Math. **343**, 112118 (2020). https://doi.org/10.1016/j.disc.2020.112118

[19] Flórez, R., Ramírez, J.L., Velandia, F.A., Villamizar, D.: A refinement of Dyck paths: a combinatorial approach. Discrete Math. Algorithms Appl. **14**, 2250026 (2022). https://doi.org/10.1142/S1793830922500264

[20] Krinik, A., Rubino, G., Marcus, D., Swift, R.J., Kasfy, H., Lam, H.: Dual processes to solve single server systems. J. Statist. Plann. Inference **135**, 121–147 (2005). https://doi.org/10.1016/j.jspi.2005.02.010

[21] Merlini, D., Sprugnoli, R., Verri, M.C.: Lagrange inversion: when and how. Acta Appl. Math. **94**, 233–249 (2006). https://doi.org/10.1007/s10440-006-9077-7

[22] Orlov, A.G.: On asymptotic behavior of the Taylor coefficients of algebraic functions. Sib. Math. J. **25**(5), 1002–1013 (1994). https://doi.org/10.1007/BF02104578

[23] Sloane, N.J.A.: The On-line Encyclopedia of Integer Sequences. Available electronically at http://oeis.org

[24] Prodinger, H.: Words, dyck paths, trees, and bijections. In: Words, Semigroups, and Transductions, pp. 369–379. World Scientific (2001). https://doi.org/10.1142/9789812810908_0028

[25] Stein, P.R., Waterman, M.S.: On some new sequences generalizing the Catalan and Motzkin numbers. Discrete Math. **26**, 261–272 (1979). https://doi.org/10.1016/0012-365X(79)90033-5

[26] Sun, Y., Shi, W., Zhao, D.: Symmetric and asymmetric peaks or valleys in (partial) Dyck paths. Enumer. Combin. Appl. **2** (2022). article #S2R24. https://doi.org/10.54550/ECA2022V2S3R24

Jean-Luc Baril
LIB
Université de Bourgogne Franche-Comté
B.P. 47 870 21078 Dijon Cedex
France
e-mail: barjl@u-bourgogne.fr

Rigoberto Flórez
Department of Mathematical Sciences
The Citadel
Charleston SC
USA
e-mail: rigo.florez@citadel.edu

José L. Ramírez
Departamento de Matemáticas
Universidad Nacional de Colombia
Bogotá
Colombia
e-mail: jlramirezr@unal.edu.co

Received: July 11, 2023
Revised: February 5, 2024
Accepted: February 8, 2024

RAIRO-Theor. Inf. Appl. 58 (2024) 1
https://doi.org/10.1051/ita/2024001

RAIRO - Theoretical Informatics and Applications
www.rairo-ita.org

CONSECUTIVE PATTERN-AVOIDANCE IN CATALAN WORDS ACCORDING TO THE LAST SYMBOL

Jean-Luc Baril[1], Daniela Colmenares[2], José L. Ramírez[2,*], Emmanuel D. Silva[2], Lina M. Simbaqueba[2] and Diana A. Toquica[2]

Abstract. We study the distribution of the last symbol statistics on the sets of Catalan words avoiding a consecutive pattern of length at most three. For each pattern p, we provide a bivariate generating function, where the coefficient $\boldsymbol{c}_p(n,k)$ of $x^n y^k$ in its series expansion is the number of length n Catalan words avoiding p and ending with the symbol k. We deduce recurrence relations or closed forms for $\boldsymbol{c}_p(n,k)$ and we provide asymptotic approximations for the expectation of the last symbol on all Catalan words avoiding p. Finally, we characterize the sequence $\boldsymbol{c}_p(n,k)$ using Riordan arrays.

Mathematics Subject Classification. 05A15, 05A19.

Received February 3, 2023. Accepted January 11, 2024.

1. Introduction

Restricted growth words $w = w_1 w_2 \cdots w_n$ are words defined over the set of non-negative integers by $w_1 = \mathtt{0}$ and $0 \le w_i \le \mathtt{st}(w_1 \cdots w_{i-1}) + 1$, where $\mathtt{st}$ is an integer statistic. Whenever the statistic $\mathtt{st}$ returns the last symbol of a word, *i.e.*, $\mathtt{st}(w_1 \cdots w_{i-1}) = w_{i-1}$, the restricted growth words are called *Catalan words*, that is, the word $w = w_1 w_2 \cdots w_n$ is a Catalan word if $w_1 = \mathtt{0}$ and $0 \le w_i \le w_{i-1} + 1$ for $i = 2, \ldots, n$. For $n \ge 0$, let $\mathcal{C}_n$ denote the set of Catalan words of length n. For instance, $\mathcal{C}_3 = \{\mathtt{000}, \mathtt{001}, \mathtt{010}, \mathtt{011}, \mathtt{012}\}$. The cardinality of $\mathcal{C}_n$ is given by the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$, see [1], Exercise 80. Catalan words have already been studied in the context of exhaustive generation of Gray codes for growth-restricted words [2]. More recently, Baril *et al.* [3, 4] study the distribution of descents on restricted Catalan words avoiding a pattern or a pair of patterns of length at most three. Ramírez and Rojas [5] also study the distribution of descents for Catalan words avoiding consecutive patterns of length at most three. Baril, González, and Ramírez [6] enumerate Catalan words avoiding a classical pattern of length at most three according to the length and the value of the last symbol. They also give the exact value or an asymptotic for the expectation of the last symbol. Also, we refer to [7, 8], where the authors study several combinatorial statistics on the polyominoes associated with words in $\mathcal{C}_n$.

Keywords and phrases: Catalan word, consecutive pattern, generating function.

[1] LIB, Université de Bourgogne, BP 47 870, 21078 Dijon-Cedex France.
[2] Departamento de Matemáticas, Universidad Nacional de Colombia, Bogotá, Colombia.

* Corresponding author: jlramirezr@unal.edu.co

© The authors. Published by EDP Sciences, 2024

This is an Open Access article distributed under the terms of the Creative Commons Attribution License (https://creativecommons.org/licenses/by/4.0), which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

The goal of this work is to complement all these studies by providing enumerative results for Catalan words avoiding a consecutive pattern of length at most three with respect to the length and the value of the last symbol. Using classical methods presented in [9, 10], we also give the exact value or an asymptotic approximation for the expectation of the last symbol for these words. The study of consecutive patterns was introduced by Elizalde and Noy [11] in the context of permutations. Since then, several results have appeared in the literature, see for example [12–14] and references therein.

The remainder of this paper is structured as follows. In Section 2, we introduce the notation that will be used in this work. In Section 3, we study the distribution of the last symbol in the set of Catalan words avoiding consecutive patterns of length 2 providing the bivariate generating function that counts Catalan words avoiding a given pattern with respect to the length and the last symbol and a matrix with the total number of words for each length and last symbol. Also, we present a recurrence and an asymptotic approximation for the expectation of the last symbol for these patterns. In Section 4, we present the same for the consecutive patterns of length 3 including bijections between some of them. Finally, in Section 5 we show how the matrices presented above are Riordan arrays.

2. Notations

For an integer $r \geq 2$, a *consecutive pattern* $p = \underline{p_1 p_2 \cdots p_r}$ is a word (of length r) over the set $\{0, 1, \ldots, r-1\}$ satisfying the condition: if $j > 0$ appears in p, then $j-1$ also appears in p. A Catalan word $w = w_1 w_2 \cdots w_n$ *contains* the consecutive pattern $p = \underline{p_1 p_2 \cdots p_r}$ if there exists a subsequence $w_i w_{i+1} \cdots w_{i+r-1}$ (for some $i \geq 1$) of w which is order-isomorphic to $p_1 p_2 \ldots p_r$. We say that w *avoids* the consecutive pattern p whenever w does not contain the consecutive pattern p. For example, the Catalan word $\mathtt{0123455543}$ avoids the consecutive pattern $\underline{001}$ and contains one subsequence isomorphic to the pattern $\underline{210}$.

For $n \geq 0$, let $\mathcal{C}_n(p)$ denote the set of Catalan words of length n avoiding the consecutive pattern p. We denote by $\boldsymbol{c}_p(n)$ the cardinality of $\mathcal{C}_n(p)$, $\mathcal{C}(p) := \bigcup_{n\geq 0} \mathcal{C}_n(p)$, and $\mathcal{C}(p)^+ := \mathcal{C}(p) \setminus \{\epsilon\}$. We denote by $\mathsf{last}(w)$ the last symbol of w. Let $\mathcal{C}_{n,k}(p)$ denote the set of Catalan words $w \in \mathcal{C}_n(p)$ such that $\mathsf{last}(w) = k$, and let $\boldsymbol{c}_p(n,k) := |\mathcal{C}_{n,k}(p)|$. Obviously, we have $\boldsymbol{c}_p(n) = \sum_{k=0}^{n-1} \boldsymbol{c}_p(n,k)$.

We introduce the bivariate generating function

$$\boldsymbol{H}_p(x,y) := \sum_{w \in \mathcal{C}(p)^+} x^{|w|} y^{\mathsf{last}(w)} = \sum_{n\geq 1,\ k\geq 0} \boldsymbol{c}_p(n,k) x^n y^k,$$

and we set

$$\boldsymbol{H}_p(x) := \sum_{w \in \mathcal{C}(p)^+} x^{|w|} = \boldsymbol{H}_p(x,1).$$

Notice that these generating functions do not consider the empty word. Let $\mathcal{T}_p$ be the infinite matrix $\mathcal{T}_p := (\boldsymbol{c}_p(n,k))_{n\geq 1, k\geq 0}$.

The expectation of the last symbol on all Catalan words in $\mathcal{C}_n(p)$ is given by (see [9])

$$\boldsymbol{a}_p(n) := \frac{[x^n]\partial_y \boldsymbol{H}_p(x,y)|_{y=1}}{\boldsymbol{c}_p(n)} = \frac{[x^n]\partial_y \boldsymbol{H}_p(x,y)|_{y=1}}{[x^n]\boldsymbol{H}_p(x,1)}.$$

In order to obtain an asymptotic approximation for $\boldsymbol{a}_p(n)$, we will use classical methods presented in [9, 10] on the two generating functions $[x^n]\partial_y \boldsymbol{H}_p(x,y)|_{y=1}$ and $[x^n]\boldsymbol{H}_p(x,1)$.

Throughout this work, we will often use the first return decomposition of a Catalan word w, which is $w = \mathtt{0}(w'+1)w''$, where w' and w'' are Catalan words, and where $(w'+1)$ is the word obtained from w' by adding 1 at all these symbols (for instance if $w' = \mathtt{012012}$ then $(w'+1) = \mathtt{123123}$). As an example, the first return decomposition of $w = \mathtt{0122123011201}$ is given by setting $w' = \mathtt{011012}$ and $w'' = \mathtt{011201}$.

3. Consecutive patterns of length 2

The goal of this section is to study the distribution of the last symbol in the set of Catalan words avoiding a consecutive pattern of length 2. For each pattern $p \in \{\underline{00}, \underline{01}, \underline{10}\}$, we provide the bivariate generating function $\boldsymbol{H}_p(x,y)$ that counts Catalan words with respect to the length and the last symbol.

Theorem 3.1. *We have*

$$\boldsymbol{H}_{\underline{00}}(x,y) = \frac{2x}{1+x-2xy+\sqrt{1-2x-3x^2}}.$$

Proof. Let w denote a non-empty Catalan word in $\mathcal{C}(\underline{00})$, and let $w = \mathtt{0}(w'+1)w''$ be the first return decomposition, where $w', w'' \in \mathcal{C}(\underline{00})$. If $w'' = \epsilon$, then $w = \mathtt{0}(w'+1)$, where w' is possibly empty. The generating function for this case is $x + xy\boldsymbol{H}_{\underline{00}}(x,y)$. If w'' is non-empty, then w' is necessarily non-empty and this produces the generating function $x\boldsymbol{H}_{\underline{00}}(x)\boldsymbol{H}_{\underline{00}}(x,y)$. Therefore, we have the functional equation

$$\boldsymbol{H}_{\underline{00}}(x,y) = x + xy\boldsymbol{H}_{\underline{00}}(x,y) + x\boldsymbol{H}_{\underline{00}}(x)\boldsymbol{H}_{\underline{00}}(x,y),$$

where $\boldsymbol{H}_{\underline{00}}(x) = \boldsymbol{H}_{\underline{00}}(x,1)$, satisfies the functional equation $\boldsymbol{H}_{\underline{00}}(x) = x + x\boldsymbol{H}_{\underline{00}}(x) + x\boldsymbol{H}_{\underline{00}}(x)^2$. Solving these equations, we obtain the desired result. □

Theorem 3.2. *For $n \geq 2$ and $0 \leq k < n$, we have*

$$\boldsymbol{c}_{\underline{00}}(n,k) = \sum_{i=k-1}^{n-1} \boldsymbol{c}_{\underline{00}}(n-1,i) - \boldsymbol{c}_{\underline{00}}(n-1,k),$$

anchored with $\boldsymbol{c}_{\underline{00}}(1,0) = 1$ and $\boldsymbol{c}_{\underline{00}}(n,k) = 0$ for $k < 0$ or $1 \leq n \leq k$. Moreover, for $n \geq 2$, $2 \leq k < n$, we have

$$\boldsymbol{c}_{\underline{00}}(n,k) = \boldsymbol{c}_{\underline{00}}(n,k-1) - \boldsymbol{c}_{\underline{00}}(n-1,k) + \boldsymbol{c}_{\underline{00}}(n-1,k-1) - \boldsymbol{c}_{\underline{00}}(n-1,k-2).$$

Proof. Let $w = w_1 \cdots w_n$ denote a non-empty Catalan word in $\mathcal{C}_{n,k}(\underline{00})$, with $n \geq 2$. So, the subword $w_1 \cdots w_{n-1}$ necessarily belongs to $\mathcal{C}_{n-1,i}(\underline{00})$ for some $i \geq k-1$, $i \neq k$ (otherwise the word w would end with kk). Consequently, we obtain the first equality. In addition, if we consider the difference $\boldsymbol{c}_{\underline{00}}(n,k) - \boldsymbol{c}_{\underline{00}}(n,k-1)$, then we deduce the second equality. □

The first few rows of the matrix $\mathcal{T}_{\underline{00}}$ are

$$\mathcal{T}_{\underline{00}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{1} & \mathbf{2} & \mathbf{0} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ 3 & 2 & 3 & 0 & 1 & 0 & 0 & 0 & 0 \\ 6 & 7 & 3 & 4 & 0 & 1 & 0 & 0 & 0 \\ 15 & 14 & 12 & 4 & 5 & 0 & 1 & 0 & 0 \\ 36 & 37 & 24 & 18 & 5 & 6 & 0 & 1 & 0 \\ 91 & 90 & 67 & 36 & 25 & 6 & 7 & 0 & 1 \end{pmatrix}$$

The Catalan words corresponding to the bold coefficients in the above array are

$$\mathcal{C}_{4,0}(\underline{00}) = \{\mathtt{0120}\}, \quad \mathcal{C}_{4,1}(\underline{00}) = \{\mathtt{0101}, \mathtt{0121}\}, \quad \mathcal{C}_{4,2}(\underline{00}) = \{\}, \quad \text{and} \quad \mathcal{C}_{4,3}(\underline{00}) = \{\mathtt{0123}\}.$$

$\phi(w) = UFDFFUUDUDFDF$

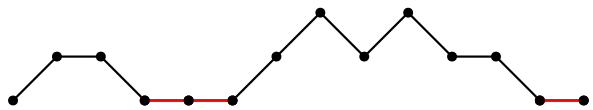


FIGURE 1. Motzkin path corresponding to the word $w = \mathtt{01201234343423} \in \mathcal{C}_{14,3}(\underline{00})$.

The matrix $\mathcal{T}_{\underline{00}}$ coincides with the array A097609, in the OEIS [15], that counts the number of Motzkin paths with respect to the length and the number of horizontal steps at level 0. Recall that a length n Motzkin path is a lattice path in $\mathbb{N}^2$, starting at the origin, ending at $(n, 0)$, consisting of steps $U = (1, 1)$, $D = (1, -1)$, and $F = (1, 0)$. A constructive bijection ϕ between $\mathcal{C}(\underline{00})$ and the set of Motzkin paths can be defined recursively as follows: $\phi(\mathtt{0}) = \epsilon$, $\phi(\mathtt{0}(w'+1)) = F\phi(w')$, and $\phi(\mathtt{0}(w'+1)w'') = U\phi(w')D\phi(w'')$. Clearly, if $w \in \mathcal{C}_{n,k}(\underline{00})$ then $\phi(w)$ is a Motzkin path of length $n-1$ having k horizontal steps at level 0. For example, if $w = \mathtt{01201234343423} \in \mathcal{C}_{14,3}(\underline{00})$, then we have

$$\begin{aligned}\phi(w) &= U\phi(\mathtt{01})D\phi(\mathtt{01234343423}) = UF\phi(\mathtt{0})DF\phi(\mathtt{0123232312}) = UFDFF\phi(\mathtt{012121201})\\ &= UFDFFU\phi(\mathtt{010101})D\phi(\mathtt{01}) = UFDFFUU\phi(\mathtt{0})D\phi(\mathtt{0101})DF\phi(\mathtt{0})\\ &= UFDFFUUDU\phi(\mathtt{0})D\phi(\mathtt{01})DF = UFDFFUUDUDFDF.\end{aligned}$$

Figure 1 shows the corresponding Motzkin path with 3 horizontal steps at level 0.

Corollary 3.3. *An asymptotic approximation for the expectation $\boldsymbol{a}_{\underline{00}}(n)$ of the last symbol over $\mathcal{C}_n(\underline{00})$ is* 2.

The last two theorems of this section have been already presented in [6], since Catalan words avoiding a consecutive pattern $p \in \{\underline{01}, \underline{10}\}$ are those avoiding the corresponding classical pattern.

Theorem 3.4. *We have*

$$\boldsymbol{H}_{\underline{01}}(x, y) = \frac{x}{1-x}, \qquad \boldsymbol{c}_{\underline{01}}(n, k) = \begin{cases} 1, & \text{if } k = 0;\\ 0, & \text{otherwise;}\end{cases} \qquad \text{and} \qquad \boldsymbol{a}_{\underline{01}}(n) = 0.$$

Theorem 3.5. *We have*

$$\boldsymbol{H}_{\underline{10}}(x, y) = \frac{x}{1-x(1+y)}, \qquad \boldsymbol{c}_{\underline{10}}(n, k) = \binom{n-1}{k} \qquad \text{and} \qquad \boldsymbol{a}_{\underline{10}}(n) = \frac{n-1}{2}.$$

4. CONSECUTIVE PATTERNS OF LENGTH 3

In this section we investigate the distribution of the last symbol in the set of Catalan words avoiding a consecutive pattern of length 3.

4.1. The consecutive pattern $\underline{012}$

Theorem 4.1. *We have*

$$\boldsymbol{H}_{\underline{012}}(x, y) = \frac{1 - x + xy - 3x^2y - 2x^3y^2 - (1+xy)\sqrt{1-2x-3x^2}}{2x(1-y+xy+x^2y^2)}.$$

Proof. Let w denote a non-empty Catalan word in $\mathcal{C}(\underline{012})$, and let $w = \mathtt{0}(w'+1)w''$ be the first return decomposition, where $w', w'' \in \mathcal{C}(\underline{012})$. If $w'' = \epsilon$, then $w = \mathtt{0}$, or $w = \mathtt{01}(w'+1)$ with $w' \in \mathcal{C}(\underline{012})$. The corresponding generating function for these words is $x + x^2y(1 + \boldsymbol{H}_{\underline{012}}(x, y))$. If w'' is non-empty, then $w = \mathtt{0}w''$, or

$w = \mathtt{01}(w'+1)w''$, where w' is possibly empty. This case produces the generating function

$$x\boldsymbol{H}_{\underline{012}}(x,y) + x^2(1+\boldsymbol{H}_{\underline{012}}(x))\boldsymbol{H}_{\underline{012}}(x,y),$$

where (see Thm. 2.1 of [5])

$$\boldsymbol{H}_{\underline{012}}(x) = \frac{1-x-\sqrt{1-2x-3x^2}}{2x^2} - 1$$

is the generating function for the number of non-empty Catalan words avoiding $\underline{012}$. Therefore, we have the functional equation

$$\boldsymbol{H}_{\underline{012}}(x,y) = x + x^2y(1+\boldsymbol{H}_{\underline{012}}(x,y)) + x\boldsymbol{H}_{\underline{012}}(x,y) + x^2(1+\boldsymbol{H}_{\underline{012}}(x))\boldsymbol{H}_{\underline{012}}(x,y).$$

Solving this equation, we obtain the desired result. □

Theorem 4.2. *For $n \geq 2$, $3 \leq k \leq n-1$, we have*

$$\boldsymbol{c}_{\underline{012}}(n,k) = \sum_{i=k-1}^{n-1} \boldsymbol{c}_{\underline{012}}(n-1,i) - \sum_{i=k-2}^{n-1} \boldsymbol{c}_{\underline{012}}(n-3,i),$$

and for $n \geq 2$, $k = 0,1$, we have

$$\boldsymbol{c}_{\underline{012}}(n,k) = \sum_{i=k-1}^{n-1} \boldsymbol{c}_{\underline{012}}(n-1,i),$$

anchored with $\boldsymbol{c}_{\underline{012}}(1,0) = 1$ and $\boldsymbol{c}_{\underline{012}}(n,k) = 0$ for $k < 0$ or $1 \leq n \leq k$. Moreover, for $n \geq 4$ and $3 \leq k \leq n-1$, we have

$$\boldsymbol{c}_{\underline{012}}(n,k) = \boldsymbol{c}_{\underline{012}}(n,k-1) - \boldsymbol{c}_{\underline{012}}(n-1,k-2) + \boldsymbol{c}_{\underline{012}}(n-3,k-3).$$

Proof. Let $w = w_1 \cdots w_n$ denote a non-empty Catalan word in $\mathcal{C}_{n,k}(\underline{012})$, with $n \geq 2$. So, the subword $w_1 \cdots w_{n-1}$ necessarily belongs to $\mathcal{C}_{n-1,i}(\underline{012})$ for $i \geq k-1$, but we have not to consider the cases where $w_{n-3} \geq w_{n-2} = k-2$ with $w_{n-1} = k-1$ whenever they can occur, *i.e.* when $k \geq 2$. Therefore, by summing over all possible values of i, we obtain the first equality; whenever $k = 0,1$, all $i \geq k-1$ have to be considered which induces the second equality. Finally, by considering the difference $\boldsymbol{c}_{\underline{012}}(n,k) - \boldsymbol{c}_{\underline{012}}(n,k-1)$ whenever $n \geq 4$ and $k \geq 3$, the third equality follows. □

The first few rows of the matrix $\mathcal{T}_{\underline{012}}$ are

$$\mathcal{T}_{\underline{012}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{4} & \mathbf{4} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 9 & 9 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 21 & 21 & 8 & 1 & 0 & 0 & 0 & 0 & 0 \\ 51 & 51 & 21 & 4 & 0 & 0 & 0 & 0 & 0 \\ 127 & 127 & 55 & 13 & 1 & 0 & 0 & 0 & 0 \\ 323 & 323 & 145 & 39 & 5 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\psi(w) = FUUFUDDUDD$

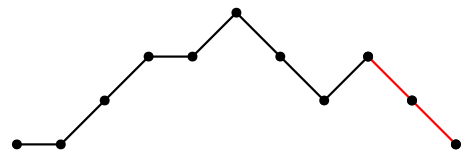


FIGURE 2. Motzkin path corresponding to the word $w = \mathtt{0011222312} \in \mathcal{C}_{10,2}(\underline{012})$.

The Catalan words corresponding to the bold coefficients in the above array are

$$\mathcal{C}_{4,0}(\underline{012}) = \{\mathtt{0000}, \mathtt{0010}, \mathtt{0100}, \mathtt{0110}\}, \quad \mathcal{C}_{4,1}(\underline{012}) = \{\mathtt{0001}, \mathtt{0011}, \mathtt{0101}, \mathtt{0111}\}, \quad \text{and}$$
$$\mathcal{C}_{4,2}(\underline{012}) = \{\mathtt{0112}\}.$$

The matrix $\mathcal{T}_{\underline{012}}$ coincides with the array A098979 that counts the number of Motzkin paths with respect to the length and the length of the final run of down steps. Moreover, the row sums of $\mathcal{T}_{\underline{012}}$ corresponds to the sequence of Motzkin numbers A001006. A constructive bijection ψ can be defined recursively as follows: $\psi(\epsilon) = \epsilon$, $\psi(\mathtt{0}w') = F\psi(w')$, and $\psi(\mathtt{01}(w'+1)w'') = U\psi(w')D\psi(w'')$. Clearly, if $w \in \mathcal{C}_{n,k}(\underline{012})$, then $\psi(w)$ is a Motzkin path of length n where the length of its final descent is k. For example, if $w = \mathtt{0011222312} \in \mathcal{C}_{10,2}(\underline{012})$, then we have

$$\begin{aligned}\psi(w) &= F\psi(\mathtt{011222312})F = FU\psi(\mathtt{0111201})D\psi(\epsilon) = FUU\psi(\mathtt{001})D\psi(\mathtt{01})D \\ &= FUUFUDDUDD.\end{aligned}$$

Figure 2 shows the corresponding Motzkin path with a final descent of length 2.

Corollary 4.3. *An asymptotic approximation for the expectation* $\boldsymbol{a}_{\underline{012}}(n)$ *of the last symbol over* $\mathcal{C}_n(\underline{012})$ *is* $5/4$.

4.2. The consecutive pattern $\underline{001}$ and $\underline{011}$

Theorem 4.4. *We have*

$$\boldsymbol{H}_{\underline{001}}(x,y) = \frac{2x}{1 - x^2 - (2x - 2x^2)y + \sqrt{1 - 4x + 2x^2 + x^4}}.$$

Proof. Let w denote a non-empty Catalan word in $\mathcal{C}(\underline{001})$, and let $w = \mathtt{0}(w'+1)w''$ be the first return decomposition, where $w', w'' \in \mathcal{C}(\underline{001})$. If $w'' = \epsilon$, then $w = \mathtt{0}(w'+1)$ with w' possibly empty. The generating function for this case is $x + xy\boldsymbol{H}_{\underline{001}}(x,y)$. If w'' is non-empty, then $w = \mathtt{00}\cdots\mathtt{0}$ (with $w' = \epsilon$) or $w = \mathtt{0}(w'+1)w''$ (with $w' \neq \epsilon$). This case produces the generating function

$$\frac{x^2}{1-x} + x\boldsymbol{H}_{\underline{001}}(x)\boldsymbol{H}_{\underline{001}}(x,y),$$

where (see Thm. 2.3 of [5])

$$\boldsymbol{H}_{\underline{001}}(x) = \frac{1 - x^2 - \sqrt{(1-x)(1-3x-x^2-x^3)}}{2(1-x)x} - 1$$

is the generating function for the number of non-empty Catalan words avoiding the consecutive pattern $\underline{001}$. Therefore, we have the functional equation

$$\boldsymbol{H}_{\underline{001}}(x,y) = x + xy\boldsymbol{H}_{\underline{001}}(x,y) + \frac{x^2}{1-x} + x\boldsymbol{H}_{\underline{001}}(x)\boldsymbol{H}_{\underline{001}}(x,y).$$

Solving this equation, we obtain the desired result. □

Theorem 4.5. *We have* $\boldsymbol{H}_{\underline{011}}(x,y) = \boldsymbol{H}_{\underline{001}}(x,y)$.

Proof. There exists a bijection between the Catalan words avoiding $\underline{011}$ and those avoiding $\underline{001}$ preserving the last symbol (see Thm. 2.4 of [5]). Indeed, the bijection comes from an algorithm explained in [5], which consists in replacing, from left to right, each factor $k^j(k+1)$ with the factor $k(k+1)^j$ $(j \geq 2)$. For instance, the bijection transforms $w = \texttt{000123222321112001012} \in \mathcal{C}(\underline{011})$ into $\texttt{012333233321222011012} \in \mathcal{C}(\underline{001})$. So, the two bivariate generating functions are equal. □

Theorem 4.6. *Let* $p \in \{\underline{011}, \underline{001}\}$. *Then, for* $n \geq 2$, $0 \leq k \leq n-1$, *we have*

$$\boldsymbol{c}_p(n,k) = \sum_{i=k-1}^{n-2} \boldsymbol{c}_p(n-1,i) - \boldsymbol{c}_p(n-2,k-1),$$

anchored with $\boldsymbol{c}_p(1,0) = 1$ *and* $\boldsymbol{c}_p(n,k) = 0$, *otherwise. Moreover, for* $n \geq 2$ *and* $1 \leq k \leq n$, *we have*

$$\boldsymbol{c}_p(n,k) = \boldsymbol{c}_p(n,k-1) - \boldsymbol{c}_p(n-1,k-2) - \boldsymbol{c}_p(n-2,k-1) + \boldsymbol{c}_p(n-2,k-2).$$

Proof. Let $w = w_1 \cdots w_n$ denote a non-empty Catalan word in $\mathcal{C}_{n,k}(\underline{001})$, with $n \geq 2$. So, the subword $w_1 \cdots w_{n-1}$ necessarily belongs to $\mathcal{C}_{n-1,i}(\underline{001})$ for $i \geq k-1$, but we have not to consider the words where $w_{n-2} = w_{n-1} = k-1$. Therefore, by summing over all possible values of i, we obtain the first equality. The second equality follows from the difference $\boldsymbol{c}_{\underline{001}}(n,k) - \boldsymbol{c}_{\underline{001}}(n,k-1)$. □

The first few rows of the matrix $\mathcal{T}_p$, for $p \in \{\underline{011}, \underline{001}\}$ are

$$\mathcal{T}_p = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{4} & \mathbf{3} & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ 9 & 7 & 4 & 1 & 1 & 0 & 0 & 0 & 0 \\ 22 & 18 & 10 & 5 & 1 & 1 & 0 & 0 & 0 \\ 57 & 48 & 28 & 13 & 6 & 1 & 1 & 0 & 0 \\ 154 & 132 & 79 & 39 & 16 & 7 & 1 & 1 & 0 \\ 429 & 372 & 227 & 115 & 51 & 19 & 8 & 1 & 1 \end{pmatrix}.$$

This array does not appear in the OEIS [15], however the first column (and the row sums) corresponds to the sequence A105633 (this sequence also counts the number of Dyck paths of semilength $n+1$ avoiding $UUDU$). The Catalan words corresponding to the bold coefficients in the above array are

$$\mathcal{C}_{4,0}(\underline{001}) = \{\texttt{0000}, \texttt{0100}, \texttt{0110}, \texttt{0120}\}, \quad \mathcal{C}_{4,1}(\underline{001}) = \{\texttt{0101}, \texttt{0121}, \texttt{0111}\},$$
$$\mathcal{C}_{4,2}(\underline{001}) = \{\texttt{0122}\}, \quad \text{and} \quad \mathcal{C}_{4,3}(\underline{001}) = \{\texttt{0123}\}.$$

An asymptotic analysis of the generating functions $[x^n]\partial_y \boldsymbol{H}_p(x,y)|_{y=1}$ and $[x^n]\boldsymbol{H}_p(x,1)$, $p \in \{\underline{011}, \underline{001}\}$, gives the following result. The constant a is the solution closest to the origin of the polynomial z^3+z^2+3z-1. Maple provides an explicit version if one asks for a *simplification*.

Corollary 4.7. *For $p \in \{\underline{011}, \underline{001}\}$, an asymptotic approximation for the expectation $\boldsymbol{a}_p(n)$ of the last symbol over $\mathcal{C}_n(p)$ is*

$$\frac{a\,(1-a)^3\left(17a^2+22a+57\right)^2}{4\left(2a^2+3a+7\right)^2} \sim 1.6785735141,$$

where $a = \frac{1}{3}\left(26+6\sqrt{33}\right)^{\frac{1}{3}} - \frac{8}{3}\left(26+6\sqrt{33}\right)^{-\frac{1}{3}} - \frac{1}{3}$.

4.3. The consecutive pattern $\underline{010}$

Theorem 4.8. *We have*

$$\boldsymbol{H}_{\underline{010}}(x,y) = \frac{1-2xy+x^2-\sqrt{1-4x+2x^2-4x^3+x^4}}{2(1+x^2-y-x^2y+xy^2)}.$$

Proof. Let w denote a non-empty Catalan word in $\mathcal{C}(\underline{010})$, and let $w = \mathtt{0}(w'+1)w''$ be the first return decomposition, where $w', w'' \in \mathcal{C}(\underline{010})$. If $w'' = \epsilon$, then $w = \mathtt{0}(w'+1)$ with w' possibly empty. The generating function for this case is $x + xy\boldsymbol{H}_{\underline{010}}(x,y)$. If w'' is non-empty, then $w' \neq \mathtt{0}$. This case produces the generating function

$$x(\boldsymbol{H}_{\underline{010}}(x) - x + 1)\boldsymbol{H}_{\underline{010}}(x,y),$$

where (see Thm. 2.3 of [5])

$$\boldsymbol{H}_{\underline{010}}(x) = \frac{1+x^2-\sqrt{(1+x^2)(1-4x+x^2)}}{2x} - 1$$

is the generating function of the non-empty Catalan words avoiding the consecutive pattern $\underline{010}$. Therefore, we have the functional equation

$$\boldsymbol{H}_{\underline{010}}(x,y) = x + xy\boldsymbol{H}_{\underline{010}}(x,y) + x(\boldsymbol{H}_{\underline{010}}(x) - x + 1)\boldsymbol{H}_{\underline{010}}(x,y).$$

Solving this equation, we obtain the desired result. □

Theorem 4.9. *For $n \geq 2$, $0 \leq k \leq n-1$, we have*

$$\boldsymbol{c}_{\underline{010}}(n,k) = \sum_{i=k-1}^{n-2} \boldsymbol{c}_{\underline{010}}(n-1,i) - \boldsymbol{c}_{\underline{010}}(n-2,k),$$

anchored with $\boldsymbol{c}_{\underline{010}}(1,0) = 1$ and $\boldsymbol{c}_{\underline{010}}(n,k) = 0$, otherwise. Moreover, for $n \geq 2$ and $2 \leq k \leq n-1$, we have

$$\boldsymbol{c}_{\underline{010}}(n,k) = \boldsymbol{c}_{\underline{010}}(n,k-1) - \boldsymbol{c}_{\underline{010}}(n-1,k-2) - \boldsymbol{c}_{\underline{010}}(n-2,k) + \boldsymbol{c}_{\underline{010}}(n-2,k-1).$$

Proof. Let $w = w_1 \cdots w_n$ denote a non-empty Catalan word in $\mathcal{C}_{n,k}(\underline{010})$, with $n \geq 2$. So, the subword $w_1 \cdots w_{n-1}$ necessarily belongs to $\mathcal{C}_{n-1,i}(\underline{010})$ for $i \geq k-1$, but we do not have to consider the words where $w_{n-2} = k$ and $w_{n-1} = k+1$. Therefore, by summing over all possible values of i we obtain the first equality. Moreover, if we consider the difference $\boldsymbol{c}_{\underline{010}}(n,k) - \boldsymbol{c}_{\underline{010}}(n,k-1)$, then we have the second equality. □

The first few rows of the matrix $\mathcal{T}_{\underline{010}}$ are

$$\mathcal{T}_{\underline{010}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{3} & \mathbf{3} & \mathbf{3} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ 9 & 8 & 6 & 4 & 1 & 0 & 0 & 0 & 0 \\ 25 & 25 & 16 & 10 & 5 & 1 & 0 & 0 & 0 \\ 73 & 74 & 51 & 28 & 15 & 6 & 1 & 0 & 0 \\ 223 & 223 & 159 & 91 & 45 & 21 & 7 & 1 & 0 \\ 697 & 696 & 496 & 296 & 150 & 68 & 28 & 8 & 1 \end{pmatrix}$$

This array does not appear in the OEIS [15], however, the first column of this matrix is the sequence A101499 (this sequence also counts the number of peakless Motzkin paths of length n in which the horizontal steps at level greater than or equal to 1 come in 2 colors). The row sums of $\mathcal{T}_{\underline{010}}$ corresponds to the sequence A187256. The Catalan words corresponding to the bold coefficients in the above array are

$$\mathcal{C}_{4,0}(\underline{010}) = \{\mathtt{0000}, \mathtt{0110}, \mathtt{0120}\}, \quad \mathcal{C}_{4,1}(\underline{010}) = \{\mathtt{0001}, \mathtt{0011}, \mathtt{0111}\},$$
$$\mathcal{C}_{4,2}(\underline{010}) = \{\mathtt{0012}, \mathtt{0112}, \mathtt{0122}\}, \quad \text{and} \quad \mathcal{C}_{4,3}(\underline{010}) = \{\mathtt{0123}\}.$$

Corollary 4.10. *An asymptotic approximation for the expectation $\boldsymbol{a}_{\underline{010}}(n)$ of the last symbol over $\mathcal{C}_n(\underline{010})$ is* 2.

4.4. The consecutive pattern $\underline{201}$

Theorem 4.11. *We have*

$$\boldsymbol{H}_{\underline{201}}(x,y) = \frac{(1-x)\left(1+2x-\sqrt{1-4x+4x^3}\right)}{(2-x)\left(1-2xy+\sqrt{1-4x+4x^3}\right)}.$$

Proof. Let w denote a non-empty Catalan word in $\mathcal{C}(\underline{201})$ and let $w = \mathtt{0}(w'+1)w''$ be first return decomposition, where $w', w'' \in \mathcal{C}(\underline{201})$. If $w'' = \epsilon$, then $w = \mathtt{0}(w'+1)$ with w' possibly empty. The generating function for this case is $x + xy\boldsymbol{H}_{\underline{201}}(x,y)$. If $w'' \neq \epsilon$ and $w' = \epsilon$, then $w = \mathtt{0}w''$ and the generating function for this case is $x\boldsymbol{H}_{\underline{201}}(x,y)$. If $w'' \neq \epsilon$ and $w' \neq \epsilon$, then $w = \mathtt{0}(w'+1)w''$ and we consider two subcases:

(i) If w' ends with $\mathtt{0}$, then $w' = w'''\mathtt{0}$ with $w''', w'' \in \mathcal{C}(\underline{201})$. The generating function for this case is $x^2(1+\boldsymbol{H}_{\underline{201}}(x))\boldsymbol{H}_{\underline{201}}(x,y)$, where

$$\boldsymbol{H}_{\underline{201}}(x) = \frac{1-2x+2x^2-\sqrt{1-4x+4x^3}}{2(2-x)x^2} - 1,$$

is the generating function for the total number of non-empty Catalan words avoiding $\underline{201}$ (see Thm. 2.5 of [5]).

(ii) If w' does not end with $\mathtt{0}$, then the last symbol of $(w'+1)$ is at least $\mathtt{2}$, which implies that w'' necessarily is $\mathtt{0}$ or $\mathtt{0}w'''$ with $w''' \in \mathcal{C}(\underline{201})$. So, in this case the generating function is

$$x(\boldsymbol{H}_{\underline{201}}(x) - x\boldsymbol{H}_{\underline{201}}(x) - x)x(\boldsymbol{H}_{\underline{201}}(x,y)+1).$$

Therefore, bringing together all the above subcases, we have the functional equation

$$\boldsymbol{H}_{\underline{201}}(x,y) = x + xy\boldsymbol{H}_{\underline{201}}(x,y) + x\boldsymbol{H}_{\underline{201}}(x,y) + x^2(1+\boldsymbol{H}_{\underline{201}}(x))\boldsymbol{H}_{\underline{201}}(x,y) + x(\boldsymbol{H}_{\underline{201}}(x) - x\boldsymbol{H}_{\underline{201}}(x) - x)x(\boldsymbol{H}_{\underline{201}}(x,y)+1).$$

Solving this equation, we obtain the desired result. □

Theorem 4.12. *Let $n \geq 2$, $1 \leq k \leq n-1$, we have*

$$\boldsymbol{c}_{\underline{201}}(n,k) = \sum_{i=k-1}^{n-2} \boldsymbol{c}_{\underline{201}}(n-1,i) - \sum_{i=k+1}^{n-3} \boldsymbol{c}_{\underline{201}}(n-2,i),$$

and for $n \geq 2$, $k = 0$,

$$\boldsymbol{c}_{\underline{201}}(n,0) = \sum_{i=0}^{n-2} \boldsymbol{c}_{\underline{201}}(n-1,i),$$

anchored with $\boldsymbol{c}_{\underline{201}}(1,0) = 1$, and $\boldsymbol{c}_{\underline{201}}(n,k) = 0$ otherwise. Moreover, for $n \geq 3$ and $3 \leq k < n$, we have

$$\boldsymbol{c}_{\underline{201}}(n,k) = \boldsymbol{c}_{\underline{201}}(n,k-1) - \boldsymbol{c}_{\underline{201}}(n-1,k-2) + \boldsymbol{c}_{\underline{210}}(n-2,k).$$

Proof. Let $w = w_1 \cdots w_n$ denote a non-empty Catalan word in $\mathcal{C}_{n,k}(\underline{201})$, with $n \geq 2$. So, the subword $w_1 \cdots w_{n-1}$ necessarily belongs to $\mathcal{C}_{n-1,i}(\underline{201})$ for $i \geq k-1$, but we do not have to consider the words where $w_{n-2} > k$ and $w_{n-1} = k-1$. Therefore, by summing over all possible values of i we obtain the first equality whenever $k \geq 1$, and the second equality for $k = 0$. Finally, by considering the difference $\boldsymbol{c}_{\underline{201}}(n,k) - \boldsymbol{c}_{\underline{201}}(n,k-1)$, we obtain the third equality. □

The first few rows of the matrix $\mathcal{T}_{\underline{201}}$ are

$$\mathcal{T}_{\underline{201}} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
5 & 5 & 3 & 1 & 0 & 0 & 0 & 0 & 0 \\
14 & \mathbf{13} & 9 & 4 & 1 & 0 & 0 & 0 & 0 \\
41 & 37 & 26 & 14 & 5 & 1 & 0 & 0 & 0 \\
124 & 110 & 78 & 45 & 20 & 6 & 1 & 0 & 0 \\
384 & 338 & 240 & 144 & 71 & 27 & 7 & 1 & 0 \\
1212 & 1062 & 756 & 463 & 243 & 105 & 35 & 8 & 1
\end{pmatrix}$$

This array does not appear in OEIS [15], however, the first column of this matrix is the sequence A159769 (this sequence also counts the number of Dyck paths of semilength $n-1$ avoiding $DDUUU$).The row sums of $\mathcal{T}_{\underline{201}}$ corresponds to the sequence A159773. The Catalan words corresponding to the bold coefficient in the above array are

$$\begin{aligned}\mathcal{C}_{5,1}(\underline{201}) = \{&\texttt{00001}, \texttt{00011}, \texttt{00101}, \texttt{00111}, \texttt{00121}, \\ &\texttt{01001}, \texttt{01011}, \texttt{01101}, \texttt{01111}, \texttt{01121}, \texttt{01211}, \texttt{01221}, \texttt{01231}\}\end{aligned}$$

Corollary 4.13. *An asymptotic approximation for the expectation $\boldsymbol{a}_{\underline{201}}(n)$ of the last symbol over $\mathcal{C}_n(\underline{201})$ is*

$$\frac{-128a^7 + 352a^5 - 284a^3 + 55a}{16a^5 + 24a^4 - 12a^3 - 24a^2 + 2a + 6} \sim 2.070578537,$$

where $a = -\frac{\sqrt{3}}{3}\cos\left(\frac{1}{6}\arctan\left(\frac{3\sqrt{111}}{5}\right) + \frac{\pi}{6}\right) + \sin\left(\frac{1}{6}\arctan\left(\frac{3\sqrt{111}}{5}\right) + \frac{\pi}{6}\right)$ is the solution closest to the origin of $4z^3 - 4z + 1$.

4.5. The consecutive pattern $\underline{101}$

Theorem 4.14. *We have*

$$\boldsymbol{H}_{\underline{101}}(x,y) = \frac{x\left(2-x+2x^2-x^3-x\sqrt{1-4x+2x^2-4x^3+x^4}\right)}{(1-x+x^2)(1-2xy+x^2+\sqrt{1-4x+2x^2-4x^3+x^4})}.$$

Proof. Let w denote a non-empty Catalan word in $\mathcal{C}(\underline{101})$, and let $w = \mathtt{0}(w'+1)w''$ be the first return decomposition, where $w', w'' \in \mathcal{C}(\underline{101})$. If $w'' = \epsilon$, then $w = \mathtt{0}(w'+1)$ with w' possibly empty. The generating function for this case is $x + xy\boldsymbol{H}_{\underline{101}}(x,y)$. If w'' is non-empty and $w' = \epsilon$, then the corresponding generating functions is $x\boldsymbol{H}_{\underline{101}}(x,y)$. If w' and w'' are non-empty, we distinguish two cases.

(i) If w' ends with $\mathtt{0}$ then $w' = v\mathtt{0}$ with $v \in \mathcal{C}(\underline{101})$. This implies that w'' cannot start with 01, which means that $w'' = 0w'''$ with $w''' \in \mathcal{C}(\underline{101})$. The generating function for this case is $x^2(1+\boldsymbol{H}_{\underline{101}}(x))x(1+\boldsymbol{H}_{\underline{101}}(x,y))$, where

$$\boldsymbol{H}_{\underline{101}}(x) = \frac{1-x^2-\sqrt{(1+x^2)(1-4x+x^2)}}{2x(1-x+x^2)} - 1$$

is the generating function for the number of all non-empty Catalan words avoiding $\underline{101}$ (see Thm. 2.3 of [5]).

(ii) If w' does not end with $\mathtt{0}$ then w'' has no more restriction, and the generating function for this case is

$$x(\boldsymbol{H}_{\underline{101}}(x) - x\boldsymbol{H}_{\underline{101}}(x) - x)\boldsymbol{H}_{\underline{101}}(x,y).$$

Therefore we have the functional equation

$$\boldsymbol{H}_{\underline{101}}(x,y) = x + xy\boldsymbol{H}_{\underline{101}}(x,y) + x\boldsymbol{H}_{\underline{101}}(x,y) + x^2(1+\boldsymbol{H}_{\underline{101}}(x))x(1+\boldsymbol{H}_{\underline{101}}(x,y)) + x(\boldsymbol{H}_{\underline{101}}(x) - x\boldsymbol{H}_{\underline{101}}(x) - x)\boldsymbol{H}_{\underline{101}}(x,y).$$

Solving this equation, we obtain the desired result. □

Theorem 4.15. *For $n \geq 2$, $1 \leq k \leq n$, we have*

$$\boldsymbol{c}_{\underline{101}}(n,k) = \sum_{i=k-1}^{n-2} \boldsymbol{c}_{\underline{101}}(n-1,i) - \boldsymbol{c}_{\underline{101}}(n-2,k),$$

and for $n \geq 2$,

$$\boldsymbol{c}_{\underline{101}}(n,0) = \sum_{i=0}^{n-2} \boldsymbol{c}_{\underline{101}}(n-1,i),$$

anchored with $\boldsymbol{c}_{\underline{101}}(1,0) = 1$, and $\boldsymbol{c}_{\underline{101}}(n,k) = 0$ otherwise. Moreover, for $n \geq 2$ and $2 \leq k \leq n$, we have

$$\boldsymbol{c}_{\underline{101}}(n,k) = \boldsymbol{c}_{\underline{101}}(n,k-1) - \boldsymbol{c}_{\underline{101}}(n-1,k-2) - \boldsymbol{c}_{\underline{101}}(n-2,k) + \boldsymbol{c}_{\underline{101}}(n-2,k-1).$$

Proof. Let $w = w_1 \cdots w_n$ denote a non-empty Catalan word in $\mathcal{C}_{n,k}(\underline{101})$, with $n \geq 2$. So, the subword $w_1 \cdots w_{n-1}$ necessarily belongs to $\mathcal{C}_{n-1,i}(\underline{101})$ for $i \geq k-1$, but we do not have to consider the words where where $w_{n-2} = k$ and $w_{n-1} = k-1$. By summing over all possible values of i we obtain the first equality for $k \geq 1$, and the second for $k = 0$. Finally, by considering the difference $\boldsymbol{c}_{\underline{101}}(n,k) - \boldsymbol{c}_{\underline{101}}(n,k-1)$, we have the third equality. □

The first few rows of the matrix $\mathcal{T}_{\underline{101}}$ are

$$\mathcal{T}_{\underline{101}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{5} & \mathbf{4} & \mathbf{3} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ 13 & 11 & 7 & 4 & 1 & 0 & 0 & 0 & 0 \\ 36 & 32 & 20 & 11 & 5 & 1 & 0 & 0 & 0 \\ 105 & 94 & 62 & 33 & 16 & 6 & 1 & 0 & 0 \\ 317 & 285 & 192 & 107 & 51 & 22 & 7 & 1 & 0 \\ 982 & 888 & 603 & 347 & 172 & 75 & 29 & 8 & 1 \end{pmatrix}$$

This array does not appear in the OEIS [15], however, the first column (and the row sums) of this matrix corresponds to the sequence A114465 (this sequence also counts the number of Dyck paths semilength n having no ascents of length 2 that start at an odd level). The Catalan words corresponding to the bold coefficients in the above array are

$$\mathcal{C}_{4,0}(\underline{101}) = \{\mathtt{0000}, \mathtt{0010}, \mathtt{0100}, \mathtt{0110}, \mathtt{0120}\}, \quad \mathcal{C}_{4,1}(\underline{101}) = \{\mathtt{0001}, \mathtt{0011}, \mathtt{0111}, \mathtt{0121}\},$$
$$\mathcal{C}_{4,2}(\underline{101}) = \{\mathtt{0012}, \mathtt{0112}, \mathtt{0122}\}, \quad \text{and} \quad \mathcal{C}_{4,3}(\underline{101}) = \{\mathtt{0123}\}.$$

Corollary 4.16. *An asymptotic approximation for the expectation $\boldsymbol{a}_{\underline{101}}(n)$ of the last symbol over $\mathcal{C}_n(\underline{101})$ is*

$$\frac{13 + 8\sqrt{3}}{7 + 4\sqrt{3}} \sim 1.928203230.$$

4.6. The consecutive pattern $\underline{000}$

Theorem 4.17. *We have*

$$\boldsymbol{H}_{\underline{000}}(x, y) = \frac{2x(1+x)}{1 + x + x^2(1-2y) - 2xy + \sqrt{1 - 2x - 5x^2 - 6x^3 - 3x^4}}.$$

Proof. Let w denote a non-empty Catalan word in $\mathcal{C}(\underline{000})$, and let $w = \mathtt{0}(w'+1)w''$ be the first return decomposition, where $w', w'' \in \mathcal{C}(\underline{000})$. If $w'' = \epsilon$, then $w = \mathtt{0}(w'+1)$ with w' possibly empty. The generating function for this case is $x + xy\boldsymbol{H}_{\underline{000}}(x, y)$. If w'' is non-empty and $w' = \epsilon$, then w'' cannot start with $\mathtt{00}$. The corresponding generating function is $xH_{\underline{000},\mathtt{00}}(x, y)$, where $H_{\underline{000},\mathtt{00}}(x, y)$ is the bivariate generating function for Catalan words in $\mathcal{C}(\underline{000})$ that do not start with $\mathtt{00}$. By counting by the complement we have

$$H_{\underline{000},\mathtt{00}}(x, y) = x + xy\boldsymbol{H}_{\underline{000}}(x, y) + x\boldsymbol{H}_{\underline{000}}(x)\boldsymbol{H}_{\underline{000}}(x, y),$$

where (see Thm. 2.8 of [5])

$$\boldsymbol{H}_{\underline{000}}(x) = \frac{1 + x + x^2 - \sqrt{(1 - x - x^2)^2 - 4x^2(1+x)^2}}{2x(1+x)} - 1.$$

If w' and w'' are non-empty, then the generating function is $x\boldsymbol{H}_{\underline{000}}(x)\boldsymbol{H}_{\underline{000}}(x, y)$. Therefore, we have the functional equation

$$\boldsymbol{H}_{\underline{000}}(x, y) \;=\; x \;+\; xy\boldsymbol{H}_{\underline{000}}(x, y) \;+\; x(x \;+\; xy\boldsymbol{H}_{\underline{000}}(x, y) \;+\; x\boldsymbol{H}_{\underline{000}}(x)\boldsymbol{H}_{\underline{000}}(x, y)) \;+\; x\boldsymbol{H}_{\underline{000}}(x)\boldsymbol{H}_{\underline{000}}(x, y).$$

Solving this equation, we obtain the desired result. □

Theorem 4.18. *For $n \geq 2$, $1 \leq k \leq n$, we have*

$$\boldsymbol{c}_{\underline{000}}(n,k) = \sum_{i \geq k-1} \boldsymbol{c}_{\underline{000}}(n-1,i) - \boldsymbol{c}_{\underline{000}}(n-3,k-1) - \sum_{i \geq k+1} \boldsymbol{c}_{\underline{000}}(n-3,i),$$

and for $n \geq 2$,

$$\boldsymbol{c}_{\underline{000}}(n,0) = \sum_{m=0}^{n-1} \frac{1}{m+1} \sum_{i=\lceil n/2 \rceil - m - 1}^{n-m-1} (-1)^i \binom{2m}{m} \binom{m+i}{i} \binom{m+i+1}{n-m-i-1},$$

anchored with $\boldsymbol{c}_{\underline{000}}(1,0) = 1$, and $\boldsymbol{c}_{\underline{000}}(n,k) = 0$, otherwise.

Proof. Let $w = w_1 \cdots w_n$ denote a non-empty Catalan word in $\mathcal{C}_{n,k}(\underline{000})$, with $n \geq 2$. So, the subword $w_1 \cdots w_{n-1}$ necessarily belongs to $\mathcal{C}_{n-1,i}(\underline{000})$ for $i \geq k-1$, but we do not have to consider the words where $w_{n-2} = k$ and $w_{n-1} = k$. Since a word w' ending with kk in $\mathcal{C}_{n-1,k}(\underline{000})$ is of the form $w' = w''k-1kk$, or $w' = w''ikk, i \geq k+1$, the number of such words is given by $\boldsymbol{c}_{\underline{000}}(n-3,k-1) + \sum_{i \geq k+1} \boldsymbol{c}_{\underline{000}}(n-3,i)$. Therefore, we have

$$\boldsymbol{c}_{\underline{000}}(n,k) = \sum_{i \geq k-1} \boldsymbol{c}_{\underline{000}}(n-1,i) - \boldsymbol{c}_{\underline{000}}(n-3,k-1) - \sum_{i \geq k+1} \boldsymbol{c}_{\underline{000}}(n-3,i).$$

The second relation is given in the OEIS [15].

□

Notice that the sequence $\boldsymbol{c}_{\underline{000}}(n,0)$ corresponds to A061639. This sequence counts the number of planar planted trees with n non-root nodes and every 2-valent node isolated.

The first few rows of the matrix $\mathcal{T}_{\underline{000}}$ are

$$\mathcal{T}_{\underline{000}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{4} & \mathbf{3} & \mathbf{3} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ 10 & 10 & 6 & 4 & 1 & 0 & 0 & 0 & 0 \\ 28 & 29 & 19 & 10 & 5 & 1 & 0 & 0 & 0 \\ 85 & 84 & 60 & 32 & 15 & 6 & 1 & 0 & 0 \\ 262 & 262 & 183 & 107 & 50 & 21 & 7 & 1 & 0 \\ 829 & 830 & 586 & 344 & 175 & 74 & 28 & 8 & 1 \end{pmatrix}$$

This array does not appear in the OEIS [15], however the row sums of $\mathcal{T}_{\underline{000}}$ corresponds to the sequence A247333. The Catalan words corresponding to the bold coefficients in the above array are

$$\mathcal{C}_{4,0}(\underline{000}) = \{\texttt{0010}, \texttt{0100}, \texttt{0110}, \texttt{0120}\}, \quad \mathcal{C}_{4,1}(\underline{000}) = \{\texttt{0011}, \texttt{0101}, \texttt{0121}\},$$
$$\mathcal{C}_{4,2}(\underline{000}) = \{\texttt{0012}, \texttt{0112}, \texttt{0122}\}, \quad \text{and} \quad \mathcal{C}_{4,3}(\underline{000}) = \{\texttt{0123}\}.$$

Corollary 4.19. *An asymptotic approximation for the expectation $\boldsymbol{a}_{\underline{000}}(n)$ of the last symbol over $\mathcal{C}_n(\underline{000})$ is 2.*

4.7. The consecutive pattern $\underline{210}$

Theorem 4.20. *We have*

$$\boldsymbol{H}_{\underline{210}}(x,y) = \frac{(2-x)x}{1-x-x^3-2xy+x^2(1+y)+\sqrt{1-4x+4x^3}}.$$

Proof. Let w denote a non-empty Catalan word in $\mathcal{C}(\underline{210})$, and let $w = \mathtt{0}(w'+1)w''$ be the first return decomposition, where $w', w'' \in \mathcal{C}(\underline{210})$. If $w'' = \epsilon$, then $w = \mathtt{0}(w'+1)$ with w' possibly empty. The generating function for this case is $x + xy\boldsymbol{H}_{\underline{210}}(x,y)$. If w'' is non-empty and $w' = \epsilon$, then the corresponding generating function is $x\boldsymbol{H}_{\underline{210}}(x,y)$. If w' and w'' are not empty, then w' cannot end with a descent, ab $(a > b)$, that is, $w' = \mathtt{0}$ or $w' = vb$ where $v = v_1 \dots v_r \in \mathcal{C}(\underline{210})$ with $r \geq 1$ and $b = v_r$ or $b = v_r + 1$. Therefore, the generating function for this case is

$$x^2\boldsymbol{H}_{\underline{210}}(x,y) + 2x^2\boldsymbol{H}_{\underline{210}}(x)\boldsymbol{H}_{\underline{210}}(x,y),$$

where (see Thm. 2.9 in [5])

$$\boldsymbol{H}_{\underline{210}}(x) = \frac{1-2x+2x^2-\sqrt{1-4x+4x^3}}{2(2-x)x^2} - 1.$$

Therefore, we have the functional equation

$$\boldsymbol{H}_{\underline{210}}(x,y) = x + xy\boldsymbol{H}_{\underline{210}}(x,y) + x\boldsymbol{H}_{\underline{210}}(x,y) + x^2\boldsymbol{H}_{\underline{210}}(x,y) + 2x^2\boldsymbol{H}_{\underline{210}}(x)\boldsymbol{H}_{\underline{210}}(x,y)$$

Solving this equation, we obtain the desired result. □

Theorem 4.21. *For $n \geq 2$, $1 \leq k \leq n$, we have*

$$\boldsymbol{c}_{\underline{210}}(n,k) = \sum_{i \geq k-1} \boldsymbol{c}_{\underline{210}}(n-1,i) - \boldsymbol{c}_{\underline{210}}(n-3,k+1) - \sum_{i=1}^{n-2-k} (2i+1) \cdot \boldsymbol{c}_{\underline{210}}(n-3,k+i+1),$$

and for $n \geq 2$,

$$\boldsymbol{c}_{\underline{210}}(n,0) = \sum_{i \geq 0} \boldsymbol{c}_{\underline{210}}(n-1,i) - \boldsymbol{c}_{\underline{210}}(n-3,1) - \sum_{i=1}^{n-2} (2i+1) \cdot \boldsymbol{c}_{\underline{210}}(n-3,i+1),$$

anchored with $\boldsymbol{c}_{\underline{210}}(1,0) = 1$, and $\boldsymbol{c}_{\underline{210}}(n,k) = 0$ otherwise.

Proof. Let $w = w_1 \cdots w_n$ denote a non-empty Catalan word in $\mathcal{C}_{n,k}(\underline{210})$, with $n \geq 2$. So, the subword $w_1 \cdots w_{n-1}$ necessarily belongs to $\mathcal{C}_{n-1,i}(\underline{210})$ for $i \geq k-1$, but we do not have to consider the words where $w_{n-2} > w_{n-1} > k$. Since a word w' ending with ji, $j > i > k$ in $\mathcal{C}_{n-1,i}(\underline{210})$ is of the form $w' = w''ji$, where $w'' \in \mathcal{C}_{n-3,j-1}(\underline{210})$ or $w'' \in \mathcal{C}_{n-3,j}(\underline{210})$ the number of such words is given by

$$\begin{aligned}
&\sum_{i \geq k+1} \sum_{j \geq i+1} \boldsymbol{c}_{\underline{210}}(n-3,j-1) + \sum_{i \geq k+1} \sum_{j \geq i+1} \boldsymbol{c}_{\underline{210}}(n-3,j) \\
&= \sum_{i=1}^{n-2-k} i \cdot \boldsymbol{c}_{\underline{210}}(n-3,k+i) + \sum_{i=0}^{n-2-k} (i+1) \cdot \boldsymbol{c}_{\underline{210}}(n-3,k+i+1) \\
&= \boldsymbol{c}_{\underline{210}}(n-3,k+1) + \sum_{i=1}^{n-2-k} (2i+1) \cdot \boldsymbol{c}_{\underline{210}}(n-3,k+i+1).
\end{aligned}$$

Finally, we obtain the recursion

$$\boldsymbol{c}_{\underline{210}}(n,k) = \sum_{i\geq k-1} \boldsymbol{c}_{\underline{210}}(n-1,i) - \boldsymbol{c}_{\underline{210}}(n-3,k+1) - \sum_{i=1}^{n-2-k} (2i+1)\cdot \boldsymbol{c}_{\underline{210}}(n-3,k+i+1).$$

The case where $k=0$ can be obtained *mutatis mutandis*. □

Notice that the sequence $\boldsymbol{c}_{\underline{210}}(n,0)$ corresponds to A114465. This sequence counts the number of Dyck paths of length $2n$ having no ascents of length 2 that start at an odd level.

The first few rows of the matrix $\mathcal{T}_{\underline{210}}$ are

$$\mathcal{T}_{\underline{210}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{5} & \mathbf{5} & \mathbf{3} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ 13 & 14 & 9 & 4 & 1 & 0 & 0 & 0 & 0 \\ 36 & 40 & 28 & 14 & 5 & 1 & 0 & 0 & 0 \\ 105 & 118 & 87 & 48 & 20 & 6 & 1 & 0 & 0 \\ 317 & 359 & 273 & 161 & 75 & 27 & 7 & 1 & 0 \\ 982 & 1118 & 869 & 536 & 270 & 110 & 35 & 8 & 1 \end{pmatrix}$$

This array does not appear in the OEIS [15], however the row sums of $\mathcal{T}_{\underline{210}}$ corresponds to the sequence A159771. The Catalan words corresponding to the bold coefficients in the above array are

$$\mathcal{C}_{4,0}(\underline{210}) = \{\mathtt{0000}, \mathtt{0010}, \mathtt{0100}, \mathtt{0110}, \mathtt{0120}\}, \quad \mathcal{C}_{4,1}(\underline{210}) = \{\mathtt{0001}, \mathtt{0011}, \mathtt{0101}, \mathtt{0111}, \mathtt{0121}\},$$
$$\mathcal{C}_{4,2}(\underline{210}) = \{\mathtt{0012}, \mathtt{0112}, \mathtt{0122}\}, \quad \text{and} \quad \mathcal{C}_{4,3}(\underline{210}) = \{\mathtt{0123}\}.$$

Corollary 4.22. *An asymptotic approximation for the expectation $\boldsymbol{a}_{\underline{210}}(n)$ of the last symbol over $\mathcal{C}_n(\underline{210})$ is*

$$\frac{32a\left(a^3-2a^2+3a-1\right)\left(8976a^2+1552a-9297\right)^2(-2+a)}{13225\left(152a^2+64a-119\right)^2} \sim 2.943409552,$$

where $a = -\frac{\sqrt{3}}{3}\cos\left(\frac{1}{6}\arctan\left(\frac{3\sqrt{111}}{5}\right)+\frac{\pi}{6}\right) + \sin\left(\frac{1}{6}\arctan\left(\frac{3\sqrt{111}}{5}\right)+\frac{\pi}{6}\right)$ *is the solution closest to the origin of* $4z^3-4z+1$.

4.8. The consecutive patterns $\underline{120}$, $\underline{100}$, and $\underline{110}$

Theorem 4.23. *We have* $\boldsymbol{H}_{\underline{100}}(x,y) = \boldsymbol{H}_{\underline{110}}(x,y) = \boldsymbol{H}_{\underline{120}}(x,y)$.

Proof. There exists a bijection between Catalan words avoiding $\underline{100}$ and those avoiding $\underline{110}$ preserving the last symbol: from left to right, we replace each maximal factor $k^j(k-\ell)$, $j\geq 2$, $\ell\geq 1$, with the factor $k(k-\ell)^j$. For instance, the bijection transforms $w = \mathtt{00012220122331} \in \mathcal{C}(100)$ into $\mathtt{00012000122311} \in \mathcal{C}(110)$. Since this bijection preserves the last symbol, the two bivariate generating functions for the these sets are equal.

Also, there is a bijection between Catalan words avoiding $\underline{120}$ and those avoiding $\underline{110}$ preserving the last symbol: from left to right, we replace each factor $kk(k-\ell)$, $\ell\geq 1$, with the factor $k(k+1)(k-\ell)$. Then, the two bivariate generating functions for the these sets are equal. □

Theorem 4.24. *For $p \in \{\underline{120}, \underline{100}, \underline{110}\}$ we have*

$$\boldsymbol{H}_p(x,y) = \frac{1-2xy-\sqrt{1-4x+4x^3}}{2-2x^2-2y+2xy^2}.$$

Proof. Let w a non-empty Catalan word in $\mathcal{C}(\underline{120})$ whose first return decomposition is $w = \mathtt{0}(w'+1)w''$ such that $w', w'' \in \mathcal{C}(\underline{120})$. If $w'' = \epsilon$ then $w = \mathtt{0}(w'+1)$ with w' possibly empty. The generating function for this case is $x + xy\boldsymbol{H}_{\underline{120}}(x,y)$. If $w'' \neq \epsilon$ and $w' = \epsilon$ the corresponding generating function is $x\boldsymbol{H}_{\underline{120}}(x,y)$. If w' and w'' are non-empty, then $w'+1$ does not finish with an ascent $a(a+1)$, where $a \geq 1$. The corresponding generating function is

$$x(\boldsymbol{H}_{\underline{120}}(x) - \boldsymbol{H}'_{\underline{120}}(x))\boldsymbol{H}_{\underline{120}}(x,y),$$

where (see Thm. 2.6 in [5])

$$\boldsymbol{H}_{\underline{120}}(x) = \frac{1-2x^2-\sqrt{1-4x+4x^3}}{2x(1-x)} - 1$$

and $\boldsymbol{H}'_{\underline{120}}(x)$ is the generating function for non-empty Catalan words avoiding $\underline{120}$ and ending with an ascent, which induces that $\boldsymbol{H}'_{\underline{120}}(x) = x\boldsymbol{H}_{\underline{120}}(x)$.

Consequently, we have the functional equation

$$\boldsymbol{H}_{\underline{120}}(x,y) = x + xy\boldsymbol{H}_{\underline{120}}(x,y) + x\boldsymbol{H}_{\underline{120}}(x,y) + x(\boldsymbol{H}_{\underline{120}}(x) - x\boldsymbol{H}_{\underline{120}}(x))\boldsymbol{H}_{\underline{120}}(x,y)$$

and solving this, we obtain the desired result. □

Theorem 4.25. *Let $p \in \{\underline{120}, \underline{100}, \underline{110}\}$. For $n \geq 2$ and $1 \leq k \leq n-1$, we have*

$$\boldsymbol{c}_p(n,k) = \boldsymbol{c}_p(n-2,k) - \boldsymbol{c}_p(n-1,k-2) + \boldsymbol{c}_p(n,k-1)$$

and for $n \geq 2$,

$$\boldsymbol{c}_p(n,0) = \frac{(-1)^{\frac{n-1-k}{2}}}{2}(1+(-1)^{n-1-k})\binom{k}{\lfloor \frac{n-1-k}{2} \rfloor}C_k,$$

where C_k is the k-th Catalan number.

Proof. For the first recurrence relation, it suffices to check that $\boldsymbol{H}_{\underline{120}}(x,y) - (x^2 - xy^2 + y)\boldsymbol{H}_{\underline{120}}(x,y) + xy$ does not depend on y. For the second relation, the sequence $\boldsymbol{c}_p(n,0)$ ($p \in \{\underline{120}, \underline{100}, \underline{110}\}$) is already known in the OEIS [15] as the sequence A157003. □

Notice that the sequence A157003 also counts the number of Dyck paths of semilength n avoiding any one of the words $UDUDD$, $UUDDD$, $UUDUD$, or $UUUDD$. The first few rows of the matrix $\mathcal{T}_p$ for $p \in \{\underline{120}, \underline{100}, \underline{110}\}$

are

$$\mathcal{T}_p = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{4} & \mathbf{5} & \mathbf{3} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ 10 & 12 & 9 & 4 & 1 & 0 & 0 & 0 & 0 \\ 27 & 32 & 25 & 14 & 5 & 1 & 0 & 0 & 0 \\ 78 & 90 & 72 & 44 & 20 & 6 & 1 & 0 & 0 \\ 234 & 266 & 213 & 137 & 70 & 27 & 7 & 1 & 0 \\ 722 & 812 & 650 & 428 & 235 & 104 & 35 & 8 & 1 \end{pmatrix}$$

This array does not appear in the OEIS [15], however the row sums of $\mathcal{T}_p$ corresponds to the sequence A087626. The Catalan words corresponding to the bold coefficients for $p = \underline{120}$ in the above array are

$$\mathcal{C}_{4,0}(\underline{120}) = \{\mathtt{0000}, \mathtt{0010}, \mathtt{0100}, \mathtt{0110}\}, \quad \mathcal{C}_{4,1}(\underline{120}) = \{\mathtt{0001}, \mathtt{0011}, \mathtt{0101}, \mathtt{0111}, \mathtt{0121}\}$$
$$\mathcal{C}_{4,2}(\underline{120}) = \{\mathtt{0012}, \mathtt{0122}, \mathtt{0112}\}, \quad \text{and} \quad \mathcal{C}_{4,3}(\underline{120}) = \{\mathtt{0123}\},$$

Corollary 4.26. *For $p \in \{\underline{120}, \underline{100}, \underline{110}\}$, an asymptotic approximation for the expectation $\boldsymbol{a}_p(n)$ of the last symbol over $\mathcal{C}_n(p)$ is*

$$-\frac{4a\left(4a^2-5\right)^2(2a-1)}{\left(4a^2+2a-3\right)^2} \sim 2.340172972,$$

with $a = -\frac{\sqrt{3}}{3}\cos\left(\frac{1}{6}\arctan\left(\frac{3\sqrt{111}}{5}\right)+\frac{\pi}{6}\right)+\sin\left(\frac{1}{6}\arctan\left(\frac{3\sqrt{111}}{5}\right)+\frac{\pi}{6}\right).$

5. Catalan matrices as Riordan arrays

In this section, we use Riordan arrays to describe the matrices introduced in the previous sections. We start by giving some background on Riordan arrays [16]. An infinite column vector $(a_0, a_1, \dots)^T$ has generating function $f(x)$ if $f(x) = \sum_{n\geq 0} a_n x^n$, and we index rows and columns starting at 0. A *Riordan array* is an infinite lower triangular matrix whose k-th column has generating function $g(x)f(x)^k$ for all $k \geq 0$, for some formal power series $g(x)$ and $f(x)$ with $g(0) \neq 0$, $f(0) = 0$, and $f'(0) \neq 0$. Such a Riordan array is denoted by $(g(x), f(x))$. If we multiply this matrix by a column vector $(c_0, c_1, \dots)^T$ having generating function $h(x)$, then the resulting column vector has generating function $g(x)h(f(x))$. The product of two Riordan arrays $(g(x), f(x))$ and $(h(x), l(x))$ is defined by

$$(g(x), f(x)) * (h(x), l(x)) = (g(x)h(f(x)), l(f(x)))\,. \tag{5.1}$$

Under this operation, the set of all Riordan arrays is a group [16]. The identity element is $I = (1, x)$, and the inverse of $(g(x), f(x))$ is

$$(g(x), f(x))^{-1} = \left(1/\left(g \circ f^{<-1>}\right)(x), f^{<-1>}(x)\right), \tag{5.2}$$

where $f^{<-1>}(x)$ denotes the compositional inverse of $f(x)$.

For a consecutive pattern p and a formal power series $f(x) = \sum_{i\geq 0} f_i x^i$, such that $f_0 = 1$, we introduce the infinite matrix

$$\mathcal{M}_p^{f(x)} := \begin{pmatrix} 1 & \mathbf{0} \\ \boldsymbol{f} & \mathcal{T}_p \end{pmatrix},$$

where $\mathcal{T}_p = (\boldsymbol{c}_p(n,k))_{n\geq 1, k\geq 0}$, $\boldsymbol{f} = (f_1, f_2, \dots)^T$, and $\mathbf{0} = (0, 0, \dots)$. In other words, let $\mathcal{T}_p[i]$ denote the i-th column of $\mathcal{T}_p$, then $\mathcal{M}_p^{f(x)} = (f(x), \mathcal{T}_p[0], \mathcal{T}_p[1], \dots)$.

Theorem 5.1. *For $p \in \{\underline{010}, \underline{000}, \underline{210}, \underline{120}, \underline{100}, \underline{110}\}$ and $f(x) = 1$, the matrix $\mathcal{M}_p^{f(x)}$ is a Riordan array given by $(1, \boldsymbol{H}_p(x, 0))$.*

Proof. Let us deal with the case $p = \underline{010}$ and the remaining cases can be obtained *mutatis mutandis*. A Catalan word $w \in \mathcal{C}(\underline{010})$, whose last symbol is k, can be decomposed as $w_0(w_1+1)\cdots(w_k+k)$, where $w_i \in \cup_{n\geq 1}\mathcal{C}_{n,0}(\underline{010})$ for $0 \leq i \leq k$. Therefore, the generating function of $\mathcal{T}_p[k]$ is given by $\sum_{n\geq 0} \boldsymbol{c}_{\underline{010}}(n,k)x^n = \boldsymbol{H}_{\underline{010}}(x,0)^k$. Consequently, $\mathcal{M}_{\underline{010}}^{f(x)=1} = (1, \boldsymbol{H}_{\underline{010}}(x,0), \boldsymbol{H}_{\underline{010}}(x,0)^2, \dots)$ is a Riordan array. □

For example,

$$\mathcal{M}_{\underline{010}}^{f(x)=1} = \left(1, \frac{1+x^2-\sqrt{1-4x+2x^2-4x^3+x^4}}{2(1+x^2)}\right) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 3 & 3 & 1 & 0 & 0 & 0 \\ 0 & 9 & 8 & 6 & 4 & 1 & 0 & 0 \\ 0 & 25 & 25 & 16 & 10 & 5 & 1 & 0 \\ 0 & 73 & 74 & 51 & 28 & 15 & 6 & 1 \end{pmatrix}.$$

Theorem 5.2. *For $p \in \{\underline{001}, \underline{011}\}$ and $f(x) = 1/(1-x)$, the matrix $\mathcal{M}_p^{f(x)}$ is a Riordan array given by*

$$\left(f(x), \frac{\boldsymbol{H}_p(x,0)}{f(x)}\right) = \left(\frac{1}{1-x}, (1-x)\boldsymbol{H}_p(x,0)\right).$$

Proof. Let us deal with case $p = \underline{001}$, for the other case we can follow a similar argument. A Catalan word $w \in \mathcal{C}(\underline{001})$, whose last symbol is k, can be decomposed as $w_0(w_1+1)\cdots(w_k+k)$, where $w_i \in \cup_{n\geq 1}\mathcal{C}_{n,0}(\underline{001})$ for $0 \leq i \leq k$. The pattern $\underline{001}$ can be observed in between the blocks $(w_i+i)(w_{i+1}+i+1)$ if it has the form $(w_i'+i)\mathtt{0}^j(w_{i+1}'+i+1)$, for some $j \geq 1$. Therefore, the generating function of the column $\mathcal{T}_{\underline{001}}[k]$ is given by $\boldsymbol{H}_{\underline{001}}(x,0)^k/f(x)^{k-1}$. From the definition of Riordan array we obtain the desired result. □

For example,

$$\mathcal{M}_{\underline{001}}^{f(x)=1/(1-x)} = \left(\frac{1}{1-x}, \frac{2(1-x)x}{1-x^2+\sqrt{1-4x+2x^2+x^4}}\right) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 4 & 3 & 1 & 1 & 0 & 0 & 0 \\ 1 & 9 & 7 & 4 & 1 & 1 & 0 & 0 \\ 1 & 22 & 18 & 10 & 5 & 1 & 1 & 0 \\ 1 & 57 & 48 & 28 & 13 & 6 & 1 & 1 \end{pmatrix}.$$

Theorem 5.3. *The matrix $\mathcal{T}_{\underline{101}}$ is a Riordan array given by $\big(\boldsymbol{H}_{\underline{101}}(x,0)/x, \boldsymbol{H}_{\underline{010}}(x,0)\big)$.*

Proof. Multiplying the right-hand side of the equality by the vector $(1, y, y^2, \dots)^T$, which has generating function $1/(1-xy)$, the resulting vector has bivariate generating function

$$\begin{aligned}\left(\boldsymbol{H}_{\underline{101}}(x,0)/x, \boldsymbol{H}_{\underline{010}}(x,0)\right)\frac{1}{1-xy} &= \frac{\boldsymbol{H}_{\underline{101}}(x,0)}{x}\frac{1}{1-\boldsymbol{H}_{\underline{010}}(x,0)y}\\ &= \frac{2-x+2x^2-x^3-x\sqrt{1-4x+2x^2-4x^3+x^4}}{(1-x+x^2)(1-2xy+x^2+\sqrt{1-4x+2x^2-4x^3+x^4})}\\ &= \frac{\boldsymbol{H}_{\underline{101}}(x,y)}{x},\end{aligned}$$

by Theorem 4.14. □

From a similar argument as in the previous theorem we obtain the following theorem.

Theorem 5.4. *The matrix $\mathcal{T}_{\underline{201}}$ is a Riordan array given by $\left(\boldsymbol{H}_{\underline{201}}(x,0)/x, \boldsymbol{H}_p(x,0)\right)$, where $p \in \{\underline{120}, \underline{100}, \underline{110}\}$.*

Notice that the matrix related to the pattern $\underline{012}$ can not be a Riordan array.

From Theorems 5.3 and 5.4 we obtain the following combinatorial identities:

$$\boldsymbol{c}_{\underline{101}}(n,k) = \sum_{\ell+\ell_1+\cdots+\ell_k=n} \boldsymbol{c}_{\underline{101}}(\ell,0)\boldsymbol{c}_{\underline{010}}(\ell_1,0)\cdots\boldsymbol{c}_{\underline{010}}(\ell_k,0)$$

and

$$\boldsymbol{c}_{\underline{201}}(n,k) = \sum_{\ell+\ell_1+\cdots+\ell_k=n} \boldsymbol{c}_{\underline{201}}(\ell,0)\boldsymbol{c}_p(\ell_1,0)\cdots\boldsymbol{c}_p(\ell_k,0),$$

where $p \in \{\underline{120}, \underline{100}, \underline{110}\}$.

Finally, we will use a characterization of the Riordan arrays given by Rogers in [17]. That is, every element not belonging to row 0 or column 0 in a Riordan array can be expressed as a fixed linear combination of the elements in the preceding row. The A-sequence is defined to be the sequence coefficients of this linear combination. Similarly, Merlini *et al.* [18] introduced the Z-sequence, that characterizes the elements in column 0, except for the top one.

Theorem 5.5 ([18])**.** *An infinite lower triangular array $\mathcal{F} = (d_{n,k})_{n,k\geq 0}$ is a Riordan array if and only if $d_{0,0} \neq 0$ and there exist two sequences $(a_0, a_1, a_2, \dots)$, with $a_0 \neq 0$, and $(z_0, z_1, z_2, \dots)$ (called the A-sequence and the Z-sequence, respectively), such that*

$$\begin{aligned} d_{n+1,k+1} &= a_0 d_{n,k} + a_1 d_{n,k+1} + a_2 d_{n,k+2} + \cdots && \text{for } n,k \geq 0,\\ d_{n+1,0} &= z_0 d_{n,0} + z_1 d_{n,1} + z_2 d_{n,2} + \cdots && \text{for } n \geq 0.\end{aligned}$$

Note that the A-sequence, the Z-sequence, and the upper-left element completely characterize a Riordan array.

Theorem 5.6 ([18, 19])**.** *Let $\mathcal{F} = (g(x), f(x))$ be a Riordan array with inverse $\mathcal{F}^{-1} = (d(x), h(x))$. Then the A-sequence and the Z-sequence of $\mathcal{F}$ have generating functions*

$$A(x) = \frac{x}{h(x)}, \qquad Z(x) = \frac{1}{h(x)}\left(1 - d_{0,0}d(x)\right),$$

respectively.

From the definition of the A-sequence and Z-sequence for the Riordan arrays we can give additional recurrence relations for the sequences $\boldsymbol{c}_p(n,k)$.

Theorem 5.7. *If C_n denotes the n-th Catalan number, then for $n \geq 2$ and $k \geq 0$,*

$$\boldsymbol{c}_{\underline{010}}(n,k) = \sum_{\ell=0}^{n-1} \boldsymbol{c}_{\underline{010}}(n-1, k-1-\ell)\, a_\ell,$$

where

$$a_n := 1 + \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^{i+1} \binom{n-i-1}{i} \overline{C}_{n-i-1} \quad \textit{and} \quad \overline{C}_n := \begin{cases} C_{\frac{n-1}{2}}, & \textit{if } n \textit{ is odd;} \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. By equation (5.2), the inverse of the matrix $\mathcal{M}_{\underline{010}}^{f(x)=1}$ is given by

$$(\mathcal{M}_{\underline{010}}^{f(x)=1})^{-1} = \left(1, \frac{1-\sqrt{1-4x^2+8x^3-4x^4}}{2(1-x)x}\right).$$

Therefore, by Theorem 5.6, the A-sequence and Z-sequence of the Riordan array $\mathcal{M}_{\underline{010}}^{f(x)=1}$ have generating functions given by

$$A(x) = \sum_{n\geq 0} a_n x^n = \frac{1+\sqrt{1-4((1-x)x)^2}}{2(1-x)} \quad \text{and} \quad Z(x) = 0.$$

Notice that $A(x) = xF(x(1-x))$, where $F(x) := \sum_{n\geq 0} f_n x^n = (1+\sqrt{1-4x^2})/(2x)$. It is possible to prove that $f_n = \overline{C}_n$, where $\overline{C}_n$ is as in the statement of the theorem. By comparing coefficients in $A(x)$ and the recurrences from Theorem 5.5 we now obtain the desired result. □

The first few values of the sequence a_n for $n \geq 0$ are

$$1, \quad 1, \quad 0, \quad 2, \quad 0, \quad 4, \quad -4, \quad 12, \quad -24, \quad 56, \quad -128, \ldots.$$

We can obtain similar identities for the remaining sequences.

6. Final remarks

We use generating functions to enumerate Catalan words avoiding any consecutive pattern of length two and three with respect to the length and the value of the last symbol. We note that much of the combinatorial interpretations given by us are (probably) new and they do not correspond to those that appear in the OEIS. In this context, it can be of interest to obtain bijections that show the links between the different combinatorial objects.

Finally, we now give a potential open problem. What is the combinatorial interpretation of the inverse Riordan arrays introduced in the previous theorem? For example, the absolute value of the second and third columns of the inverse matrix $(\mathcal{M}_{\underline{010}}^{f(x)=1})^{-1}$ are the sequences A104545 (number of Motzkin paths of length n having no

consecutive $(1,0)$ steps) and A256169, respectively.

$$(\mathcal{M}_{\underline{010}}^{f(x)=1})^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & -3 & 3 & -3 & 1 & 0 & 0 & 0 \\ 0 & 5 & -8 & 6 & -4 & 1 & 0 & 0 \\ 0 & -11 & 17 & -16 & 10 & -5 & 1 & 0 \\ 0 & 25 & -38 & 39 & -28 & 15 & -6 & 1 \end{pmatrix}.$$

Acknowledgements. We thanks the referees for the detailed comments and corrections that help us to improve the paper.

References

[1] R. Stanley, Catalan Numbers. Cambridge University Press, Cambridge (2015).
[2] T. Mansour and V. Vajnovszki, Efficient generation of restricted growth words. *Inform. Process. Lett.* **113** (2013) 613–616.
[3] J.-L. Baril, S. Kirgizov and V. Vajnovszki, Descent distribution on Catalan words avoiding a pattern of length at most three. *Discrete Math.* **341** (2018) 2608–2615.
[4] J.-L. Baril, C. Khalil and V. Vajnovszki, Catalan words avoiding pairs of length three patterns. *Discret. Math. Theor. Comput. Sci.* **22** (2021) 5.
[5] J.L. Ramírez and A. Rojas-Osorio, Consecutive patterns in Catalan words and the descent distribution. *Bol. Soc. Mat. Mex.* **29** (2023) Article 60.
[6] J.-L. Baril, J.F. González and J.L. Ramírez, Last symbol distribution in pattern avoiding Catalan words. Submitted (2023).
[7] D. Callan, T. Mansour and J.L. Ramírez, Statistics on bargraphs of Catalan words. *J. Autom. Lang. Comb.* **26** (2021) 177–196.
[8] T. Mansour, J.L. Ramírez and D.A. Toquica, Counting lattice points on bargraphs of Catalan words. *Math. Comput. Sci.* **15** (2021) 701–713.
[9] P. Flajolet and R. Sedgewick, Analytic Combinatorics. Cambridge University Press, Cambridge (2009).
[10] A.G. Orlov, On asymptotic behavior of the Taylor coefficients of algebraic functions. *Sib. Math. J.* **25** (1994) 1002–1013.
[11] S. Elizalde and M. Noy, Consecutive patterns in permutations. *Adv. Appl. Math.* **30** (2003) 110–125.
[12] J.S. Auli and S. Elizalde, Consecutive patterns in inversion sequences. *Discret. Math. Theor. Comput. Sci.* **21** (2019) 6.
[13] A. Mendes and J. Remmel, Permutations and words counted by consecutive patterns. *Adv. Appl. Math.* **37** (2006) 443–480.
[14] A. Panholze, Consecutive permutation patterns in trees and mappings. *J. Comb.* **12** (2021) 17–54.
[15] N.J.A. Sloane, The On-Line Encyclopedia of Integer Sequences. Available at https://oeis.org/.
[16] L.W. Shapiro, S. Getu, W. Woan and L. Woodson, The Riordan group. *Discrete Appl. Math.* **34** (1991) 229–239.
[17] D.G. Rogers, Pascal triangles, Catalan numbers and renewal arrays. *Discrete Math.* **22** (1978) 301–310.
[18] D. Merlini, D.G. Rogers, R. Sprugnoli and M.C. Verri, On some alternative characterizations of Riordan arrays. *Can. J. Math.* **49** (1997) 301–320.
[19] T.-X. He and R. Sprugnoli, Sequence characterization of Riordan arrays. *Discrete Math.* **309** (2009) 3962–3974.

Please help to maintain this journal in open access!

This journal is currently published in open access under the Subscribe to Open model (S2O). We are thankful to our subscribers and supporters for making it possible to publish this journal in open access in the current year, free of charge for authors and readers.

Check with your library that it subscribes to the journal, or consider making a personal donation to the S2O programme by contacting subscribers@edpsciences.org.

More information, including a list of supporters and financial transparency reports, is available at https://edpsciences.org/en/subscribe-to-open-s2o.

Aequat. Math.
© The Author(s), under exclusive licence to Springer Nature Switzerland AG 2024
https://doi.org/10.1007/s00010-024-01065-1

Aequationes Mathematicae

Skew Dyck paths with air pockets

Jean-Luc Baril, Rémi Maréchal, and Helmut Prodinger

Abstract. We yield bivariate generating function for the number of n-length partial skew Dyck paths with air pockets (DAPs) ending at a given ordinate. We also give an asymptotic approximation for the average ordinate of the endpoint in all partial skew DAPs of a given length. Similar studies are made for two subclasses of skew DAPs, namely valley-avoiding and zigzagging, valley-avoiding skew DAPs. We express these results as Riordan arrays. Finally, we present two one-to-one correspondences with binary words avoiding the patterns 00 and 0110, and palindromic compositions with parts in $\{2, 1, 3, 5, 7, \ldots\}$.

Mathematics Subject Classification. 05A05, 05A15, 05A15.

1. Introduction

Dyck paths with air pockets (DAPs) are introduced in a recent paper [2]. These paths consist in lattice paths in $\mathbb{N}^2$ starting at the origin, ending on the x-axis, and made of up-steps $U = (1, 1)$ and down-steps $D_k = (1, -k)$ where $k \geqslant 1$, and so that no two down-steps can be consecutive. The *length* of a path P is the number of steps in P. DAPs can be viewed as ordinary Dyck paths where maximal runs of down-steps are replaced by one large down-step. In sorting theory, DAPs also correspond to a stack evolution with (partial) reset operations that cannot be consecutive, see [9]. In [2], the authors enumerate these paths and their prefixes with respect to the length, the type (up or down) of the last step, and the ordinate of the endpoint. Moreover, they establish a one-to-one correspondence between DAPs of length n and peak-less Motzkin paths of length $n - 1$. In a second paper, the authors [3] generalized DAPs by allowing them to go below the x-axis, calling them *grand Dyck paths with air pockets* (GDAPs). They yield enumerative results for these paths with respect to their length and various restrictions on their minimum and maximum ordinates. More recently, the definition of DAPs was extended to include horizontal steps under certain conditions, as described in [1,4].

Published online: 03 May 2024

Birkhäuser

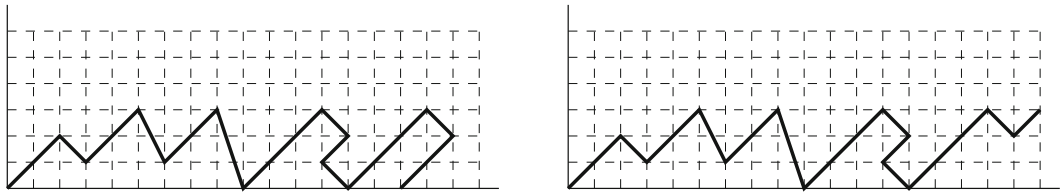


Figure 1. From left to right, a skew Dyck path with air pockets and a partial skew Dyck path with air pockets ending at ordinate 3, respectively

Let $\mathcal{D}$ be the set of all DAPs and $\mathcal{D}_n$ be the set of DAPs of length n. In this paper, we generalize DAPs by allowing back-down steps as follows.

Definition 1. A *skew Dyck path with air pockets* (abbreviated as "skew DAP") is a lattice path in $\mathbb{N}^2$, consisting of steps in the set $\{U, L, D_1, D_2, \ldots\}$, where $U = (1,1)$, $L = (-1,-1)$, and $D_k = (1,-k)$ for every positive integer k (we abbreviate D_1 to D for convenience), starting at the origin $(0,0)$, ending somewhere on the x-axis, and such that any occurrence of the following consecutive patterns is forbidden: UL, LU, and D_iD_j for any $i, j > 0$.

We say a skew DAP has *length* n (where n is a nonnegative integer) if it consists of n steps (the empty path ε counts as a 0-length skew DAP). For all $n \geqslant 0$, we let $\mathcal{S}_n$ denote the set of n-length skew DAPs, and we set $\mathcal{S} = \bigcup_{n\geqslant 0} \mathcal{S}_n$.

Definition 2. A skew DAP is *valley-avoiding* (v.a. skew DAP for short) if it contains no occurrence of the consecutive pattern D_kU, for all $k > 0$.

For all $n \geqslant 0$, we let $\mathcal{V}_n$ denote the set of n-length v.a. skew DAPs, and we set $\mathcal{V} = \bigcup_{n\geqslant 0} \mathcal{V}_n$.

Definition 3. A valley-avoiding skew DAP is *zigzagging* (z.v.a. skew DAP for short) if it contains no occurrence of the consecutive pattern LL.

For all $n \geqslant 0$, we let $\mathcal{Z}_n$ denote the set of n-length z.v.a skew DAPs, and we set $\mathcal{Z} = \bigcup_{n\geqslant 0} \mathcal{Z}_n$.

Finally, for each type of skew Dyck paths defined above, we will use the terminology of *partial skew Dyck paths* to refer to a prefix of a skew DAP (i.e. the path does not necessarily end on the x-axis). For all $n \geqslant 0$, we let $\mathcal{PD}_n$ (resp. $\mathcal{PV}_n$, resp. $\mathcal{PZ}_n$) denote the set of n-length partial skew DAPs (resp. v.a. skew DAPs, resp. z.v.a. skew DAPs), and we use the notation $\mathcal{PD}$, $\mathcal{PV}$, $\mathcal{PZ}$ for the sets of all partial skew DAPs of each type. We refer to Fig. 1 for an illustration of a skew DAP and a partial skew DAP.

For $k \geqslant 0$, we consider the generating function $f_k = f_k(z)$ (resp. $g_k = g_k(z)$, resp. $h_k = h_k(z)$), where the coefficient $[z^n]f_k$ (resp. $[z^n]g_k$, resp. $[z^n]h_k$) of z^n in its series expansion is the number of partial skew Dyck paths of length n,

ending at ordinate k with an up-step U (resp., with a down-step D_k, $k \geqslant 1$, resp., with a back-down step L). Also, we introduce the bivariate generating functions

$$F(u,z) = \sum_{k\geqslant 0} u^k f_k(z), \quad G(u,z) = \sum_{k\geqslant 0} u^k g_k(z), \text{ and } \quad H(u,z) = \sum_{k\geqslant 0} u^k h_k(z).$$

For short, we use the notation $F(u)$, $G(u)$, and $H(u)$ for these functions. In Sects. 3 and 4 respectively, the same notations are preserved for partial v.a. skew DAPs and partial z.v.a. skew DAPs, respectively.

Motivation and outline of the paper The main goal of this work is to present enumerative results for three classes of partial skew Dyck paths, and to exhibit some links between these paths and restricted classes of binary words and compositions (incidentally, we express these results as Riordan arrays). More precisely, in Sect. 2, we give bivariate generating functions for the number of n-length partial skew DAPs ending at ordinate $k \geqslant 0$ with a given type of step (up, down or back-down step). We deduce the generating function for the total number of skew DAPs (resp. of partial skew DAPs). We also give asymptotic approximations for the corresponding cardinalities, and for the average ordinate of the endpoint in all partial skew DAPs of a given length. In Sect. 3, we make a similar study for partial v.a. skew DAPs. In addition, we present a bijection between these paths and binary words avoiding the patterns 00 and 0110. Section 4 presents the counterpart for z.v.a skew DAPs, and we provide a bijection between these paths and palindromic compositions with parts in $\{2, 1, 3, 5, 7, \ldots\}$. Some of the obtained results are summarized in the following table.

Type of paths	First terms	OEIS
Skew DAPs	$1, 0, 1, 1, 3, 5, 13, 26, 64, 143, \ldots$	New
Partial skew DAPs	$1, 1, 2, 4, 9, 19, 44, 100, 236, 558 \ldots$	New
v.a. skew DAPs	$1, 0, 1, 1, 2, 2, 4, 5, 9, 12, \ldots$	A124280
Partial v.a. skew DAPs	$1, 1, 2, 3, 5, 7, 11, 16, 25, 37, \ldots$	A130137
z.v.a skew DAPs	$1, 0, 1, 1, 2, 2, 3, 4, 6, 8, \ldots$	A103632
Partial z.v.a skew DAPs	$1, 1, 2, 3, 5, 7, 10, 14, 20, 28, \ldots$	New

2. Enumerating skew DAPs

In this part, we count partial skew DAPs of a given length, i.e., ending at a given abscissa, according to the type of the last step, and the ordinate of the endpoint.

Theorem 1. *We have*

$$F(u) = \frac{s_1}{s_1 - u}, \qquad G(u) = \frac{1 - s_1 z}{z(s_1 - u)}, \qquad H(u) = \frac{1 - s_1 z}{(s_1 - u)(s_1 - z)},$$

and thus,

$$F(u) + G(u) + H(u) = \frac{s_1(1 - z^2)}{z(s_1 - z)(s_1 - u)},$$

where $s_1 = \frac{A}{6z} + \frac{4z^4 - 2z^3 - \frac{4}{3}z^2 - \frac{2}{3}z + \frac{2}{3}}{zA} + \frac{z+1}{3z}$, *with*

$$A = \left(72z^5 - 72z^4 + 44z^3 + 12B\,z - 48z^2 - 12z + 8\right)^{\frac{1}{3}},$$

and

$$B = \sqrt{-96z^{10} + 144z^9 + 60z^8 - 108z^7 - 24z^6 - 48z^5 + 81z^4 - 18z^2 + 12z - 3}.$$

Proof. By convention, we fix $f_0 = 1$ to take into account the empty path consisting of the origin $(0, 0)$ only. A nonempty skew DAP of length n ending at ordinate $k \geqslant 1$ with an up-step U is uniquely obtained from a skew DAP of length $n - 1$ ending at ordinate $k - 1$ with either an up-step or a down-step, which implies the recurrence relation $f_k(z) = z(f_{k-1}(z) + g_{k-1}(z))$ for $k \geqslant 1$. A skew DAP of length n ending at ordinate $k \geqslant 0$ with a down-step D_i, $i \geqslant 1$, is uniquely obtained from a skew DAP of length $n - 1$ ending at ordinate $k + i$, $i \geqslant 1$, with either an up-step U or a back-down-step L, which implies the recurrence relation $g_k(z) = z \sum_{i \geqslant 1} \left(f_{k+i}(z) + h_{k+i}(z)\right)$, for $k \geqslant 0$. A skew DAP of length n ending at ordinate $k \geqslant 0$ with an L-step is uniquely obtained from a skew DAP of length $n - 1$ ending at ordinate $k + 1$ with either a down-step or an L-step, which implies the recurrence relation $h_k(z) = z(g_{k+1}(z) + h_{k+1}(z))$ for $k \geqslant 0$. Hence, we obtain the following equations:

$$\begin{cases} & f_0(z) = 1, \\ \forall k > 0, & f_k(z) = z\left(f_{k-1}(z) + g_{k-1}(z)\right), \\ \forall k \geqslant 0, & g_k(z) = z \sum_{i \geqslant 1} \left(f_{k+i}(z) + h_{k+i}(z)\right), \\ \forall k \geqslant 0, & h_k(z) = z\left(g_{k+1}(z) + h_{k+1}(z)\right). \end{cases}$$

From the previous system, we multiply both sides of all four equations by u^k, then we sum over k, and using basic algebraic methods on generating functions. For instance, let us examine the case of $G(u)$, which is the least straightforward, we get:

$$G(u) = \sum_{k \geqslant 0} g_k u^k = \sum_{k \geqslant 0} z \sum_{i \geqslant 1} \left(f_{k+i}(z) + h_{k+i}(z)\right) u^k.$$

Interchanging the order of the double summation, the formula becomes:

$$G(u) = z \sum_{i \geqslant 1} (f_i + h_i) \sum_{k=0}^{i-1} u^k = z \sum_{i \geqslant 1} (f_i + h_i) \frac{1-u^i}{1-u}.$$

Then, bringing the factor $\frac{1}{1-u}$ to the front of the expression, and expanding the product inside of the summation, we deduce:

$$G(u) = \frac{z}{1-u} \left(\sum_{i \geqslant 1} f_i - \sum_{i \geqslant 1} f_i u^i + \sum_{i \geqslant 1} h_i - \sum_{i \geqslant 1} h_i u^i \right),$$

which can be rewritten as:

$$G(u) = \frac{z}{1-u} \left(F(1) - F(0) - F(u) + F(0) + H(1) - H(0) - H(u) + H(0) \right).$$

Ultimately, we derive the following system for $F(u)$, $G(u)$, and $H(u)$:

$$\begin{cases} F(u) = 1 + zu(F(u) + G(u)), \\ G(u) = \frac{z}{1-u} \left(F(1) - F(u) + H(1) - H(u) \right), \\ H(u) = \frac{z}{u} \left(H(u) - H(0) + G(u) - G(0) \right). \end{cases}$$

Solving the previous linear system for $F(u)$, $G(u)$, and $H(u)$, we get:

$$F(u) = \frac{u\, z^2 \,(u-z)\,(F(1)+H(1)) + u\, z^3 (G(0)+H(0)) - u^2 + (z+1)\, u + z^2 - z}{u^3 z - 2u\, z^3 - u^2 z + z^2 u - u^2 + zu + z^2 + u - z},$$

$$G(u) = -\frac{z\,((zu-1)\,(u-z)\,(F(1)+H(1)) + (z^2 u - z)\,(G(0)+H(0)) + u - z)}{u^3 z - 2u\, z^3 - u^2 z + z^2 u - u^2 + zu + z^2 + u - z},$$

$$H(u) = -\frac{((z^2 u + (u^2 - u)\, z - u + 1)\,(G(0)+H(0)) + z\,(zu-1)\,(F(1)+H(1)) + z)\, z}{u^3 z - 2u\, z^3 - u^2 z + z^2 u - u^2 + zu + z^2 + u - z}.$$

All three fractions share the same denominator, which we can be rewritten as $z(u-s_1)(u-s_2)(u-s_3)$, where s_1, s_2, s_3 are the roots of $u^3 z - 2u\, z^3 - u^2 z + z^2 u - u^2 + zu + z^2 + u - z$. We observe that only two roots have a Taylor expansion around $z = 0$ (without loss of generality, we assume that these roots are s_2 and s_3). According to the kernel method [11], $u - s_2$ and $u - s_3$ are bad factors that can be cancelled both numerator and denominator. Thus, the numerator is simplified into the coefficient of u^2, and the denominator becomes $z(u - s_1)$. Finally, we obtain:

$$\begin{cases} F(u) = \frac{1 - z^2 (F(1)+H(1))}{z(s_1 - u)}, \\ G(u) = \frac{z(F(1)+H(1))}{s_1 - u}, \\ H(u) = \frac{z(G(0)+H(0))}{s_1 - u}, \end{cases}$$

where the third root s_1 is equal to $s_1 = \frac{A}{6z} + \frac{4z^4 - 2z^3 - \frac{4}{3}z^2 - \frac{2}{3}z + \frac{2}{3}}{zA} + \frac{z+1}{3z}$, with

$$A = \left(72z^5 - 72z^4 + 44z^3 + 12B\, z - 48z^2 - 12z + 8\right)^{\frac{1}{3}},$$

and

$$B=\sqrt{-96z^{10}+144z^9+60z^8-108z^7-24z^6-48z^5+81z^4-18z^2+12z-3}.$$

Fixing $u=0$ and $u=1$ in the above equations, we obtain the values of $G(0)$, $H(0)$, $F(1)$ and $H(1)$:

$$\begin{cases} G(0)=\frac{1-zs_1}{zs_1}, \\ H(0)=\frac{zs_1-1}{(z-s_1)s_1}, \\ F(1)=\frac{(z-1)\left(-s_1^2z+2s_1z^2+z^2+s_1-z\right)}{z(s_1-1)(z-s_1)}, \\ H(1)=\frac{(-2s_1-1)z^4+\left(s_1^2+2s_1+2\right)z^3+\left(-2s_1^2-1\right)z^2+\left(s_1^3-s_1^2+2s_1-1\right)z-s_1^2+s_1}{z^2(s_1-1)(z-s_1)}. \end{cases}$$

Plugging those expressions back into $F(u)$, $G(u)$, and $H(u)$, we obtain the expected result. □

The first terms of the series expansion of $F(u)+G(u)+H(u)$ are

$$\begin{aligned} 1+uz+(u^2+1)z^2+(u^3+2u+1)z^3+(u^4+3u^2+2u+3)z^4 \\ +(u^5+4u^3+3u^2+6u+5)z^5+(u^6+5u^4+4u^3+10u^2+11u+13)z^6 \\ +(u^7+6u^5+5u^4+15u^3+19u^2+28u+26)z^7+O(z^8). \end{aligned}$$

Now, we deduce the coefficient $[u^k](F(u)+G(u)+H(u))$ of u^k in the series expansion of $F(u)+G(u)+H(u)$ by using the well known series expansion $\frac{1}{s_1-u}=\frac{1}{s_1}\sum_{k\geqslant 0}{s_1}^{-k}\cdot u^k$.

Corollary 1. *We have*

$$[u^k]\left(F(u)+G(u)+H(u)\right)=\frac{1-z^2}{z(s_1-z)}s_1^{-k}.$$

The next remark makes a link with Riordan arrays theory. We refer to [5,6,12] for a background on Riordan arrays.

Remark 1. Let $\mathcal{P}$ be the matrix $[p_{n,k}]_{n,k\geqslant 0}$, where $p_{n,k}$ is the number of skew DAPs of length n ending at ordinate k, i.e. the coefficient of z^n in the series expansion of $[u^k]\left(F(u)+G(u)+H(u)\right)$. The first values of $\mathcal{P}$ are

$$\mathcal{P}=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \\ 1 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & \\ 3 & 2 & 3 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 5 & 6 & 3 & 4 & 0 & 1 & 0 & 0 & \\ 13 & 11 & 10 & 4 & 5 & 0 & 1 & 0 & \\ 26 & 28 & 19 & 15 & 5 & 6 & 0 & 1 & \\ & & & & \vdots & & & & \ddots \end{pmatrix}.$$

Since $F(u)+G(u)+H(u)=\frac{g(z)}{1-uf(z)}$ with $f(z)=s_1^{-1}$ and $g(z)=\frac{1-z^2}{z(s_1-z)}$, the matrix $\mathcal{P}$ corresponds to the Riordan array

$$\left(\frac{1-z^2}{z(s_1-z)},\frac{1}{s_1}\right).$$

Now we plug in $u=0$ and $u=1$ to get the generating functions for skew DAPs and partial skew DAPs, respectively, and using classical methods [7,10], we provide an asymptotic approximation of the coefficient of z^n.

Theorem 2. *The generating function for the total number of skew DAPs with respect to the length is given by*

$$F(0,z)+G(0,z)+H(0,z)=\frac{z^2-1}{z\,(z-s_1)},$$

and an asymptotic approximation of the n-th term is

$$0.5292\cdot 2.7309^n\cdot n^{-3/2}.$$

The leading terms of the series expansion of $F(0,z)+G(0,z)+H(0,z)$ are

$$1+z^2+z^3+3z^4+5z^5+13z^6+26z^7+64z^8+143z^9+\mathrm{O}\big(z^{10}\big)\,.$$

Theorem 3. *The generating function for the total number of partial skew DAPs with respect to the length is given by*

$$F(1,z)+G(1,z)+H(1,z)=\frac{s_1(1-z^2)}{z(s_1-z)(s_1-1)},$$

and an asymptotic approximation of the n-th term is

$$2.4909\cdot 2.7309^n\cdot n^{-3/2}.$$

The leading terms of the series expansion of $F(1,z)+G(1,z)+H(1,z)$ are

$$1+z+2z^2+4z^3+9z^4+19z^5+44z^6+100z^7+236z^8+558z^9+\mathrm{O}\big(z^{10}\big)\,.$$

By calculating $\partial_u(F(u,z)+G(u,z)+H(u,z))|_{u=1}$, we obtain the following.

Corollary 2. *An asymptotic approximation for the average of the ordinate of the endpoint in all partial skew DAPs of a given length is* 2.4859.

3. Enumerating v.a. skew DAPs

In this part, we use the same methodology and notation as in Sect. 2 in order to enumerate (partial) valley-avoiding skew Dyck paths with air pockets (v.a. skew DAPs for short), i.e. skew DAPs that do not contain a down-step followed by an up-step.

Theorem 4. *We have*

$$F(u) = \frac{1}{1-zu}, \qquad G(u) = \frac{z^2 - z^4}{(1-zu)(1-z-z^2+z^3-z^4)},$$
$$H(u) = \frac{z^4}{(1-zu)(1-z-z^2+z^3-z^4)},$$

and thus,

$$F(u) + G(u) + H(u) = \frac{1-z+z^3-z^4}{(1-zu)(1-z-z^2+z^3-z^4)}.$$

Proof. By convention, we fix $f_0 = 1$ to take into account the empty path consisting of the origin $(0,0)$ only. A nonempty v.a. skew DAP of length n ending at ordinate $k \geqslant 1$ with an up-step U is uniquely obtained from a skew v.a. DAP of length $n-1$ ending at ordinate $k-1$ with an up-step, which implies the recurrence relation $f_k(z) = zf_{k-1}(z)$ for $k \geqslant 1$. The two recurrence relations for $g_k(z)$ and $h_k(z)$ are the same as in the previous section. So, we refer to Sect. 2 for an explanation on how they are derived, and we have the following system of equations:

$$\begin{cases} & f_0(z) = 1, \\ \forall k > 0, & f_k(z) = zf_{k-1}(z), \\ \forall k \geqslant 0, & g_k(z) = z\sum_{i\geqslant 1}(f_{k+i}(z) + h_{k+i}(z)), \\ \forall k \geqslant 0, & h_k(z) = z(g_{k+1}(z) + h_{k+1}(z)). \end{cases}$$

Using the same method as for the proof of Theorem 1, the previous system induces the following equations for $F(u)$, $G(u)$, and $H(u)$:

$$\begin{cases} F(u) = \frac{1}{1-zu}, \\ G(u) = \frac{z}{1-u}(F(1) - F(u) + H(1) - H(u)), \\ H(u) = \frac{z}{u}(G(u) - G(0) + H(u) - H(0)). \end{cases}$$

Solving the previous linear system for $F(u)$, $G(u)$, and $H(u)$, we get:

$$\begin{cases} F(u) = \frac{1}{1-zu}, \\ G(u) = \frac{C}{u^3z^2 - u^2z^3 - u\,z^4 - u^3z + 2u\,z^3 + z^3 + u^2 - 2z^2 - u + z}, \\ H(u) = \frac{D}{u^3z^2 - u^2z^3 - u\,z^4 - u^3z + 2u\,z^3 + z^3 + u^2 - 2z^2 - u + z}, \end{cases}$$

with

$$\begin{aligned} C = &- z\Big(u^2z^2H(1) - u^2z(1+H(1)) + uz^3(G(0) + H(0) - H(1)) \\ &+ uz^2(1 - G(0) - H(0) + H(1)) + uz(1 - H(1)) + uH(1) \\ &- z^2(1 + G(0) + H(0) - H(1)) + z(G(0) + H(0) - H(1))\Big), \end{aligned}$$

and

$$\begin{aligned} D = &- z\Big(u^2z^2(G(0)+H(0)) - u^2z(G(0)+H(0)) + uz^3H(1) \\ &- uz^2(1+G(0)+H(0)+H(1)) + u(G(0)+H(0)) \\ &+ z^2(1-H(1)) + z(G(0)+H(0)+H(1)) - G(0) - H(0)\Big). \end{aligned}$$

Here, $G(u)$ and $H(u)$ share the same denominator, which we write as $(z^2-z)(u-r_1)(u-r_2)(u-r_3)$, where r_1, r_2, and r_3 are the roots of $u^3z^2 - u^2z^3 - u\,z^4 - u^3z + 2u\,z^3 + z^3 + u^2 - 2z^2 - u + z$. Since two of them have a Taylor expansion around $z = 0$ (without loss of generality, we assume that these roots are r_2 and r_3), the kernel method [11] once again tells us that $u - r_2$ and $u - r_3$ are both bad factors and can be thus cancelled both numerator and denominator in $G(u)$ and $H(u)$. This leaves us with the numerators being simplified into the coefficient of u^2, and the denominators both being simplified into $(z^2-z)(u-r_1)$. Moreover, r_1 happens to be equal to $\frac{1}{z}$, which conveniently leads to simplifications further on. The previous system is then simplified into:

$$\begin{cases} F(u) = \frac{1}{1-zu}, \\ G(u) = \frac{z((z-1)H(1)-1)}{(1-z)(u-1/z)}, \\ H(u) = \frac{-z(G(0)+H(0))}{u-1/z}. \end{cases}$$

Evaluating the second equation at $u = 0$, and the third one at $u = 0$ and $u = 1$ respectively, we derive the values of $G(0)$, $H(0)$, and $H(1)$:

$$\begin{cases} G(0) = \frac{(z^2-1)z}{z^4-z^3+z^2+z-1}, \\ H(0) = \frac{-z^4}{z^4-z^3+z^2+z-1}, \\ H(1) = \frac{z^4}{z^5-2z^4+2z^3-2z+1}. \end{cases}$$

Plugging those expressions back into $F(u)$, $G(u)$, and $H(u)$, we obtain the expected result. □

The first terms of the series expansion of $F(u) + G(u) + H(u)$ are

$$\begin{aligned} 1 + uz &+ \left(u^2+1\right)z^2 + \left(u^3+u+1\right)z^3 + \left(u^4+u^2+u+2\right)z^4 \\ &+ \left(u^5+u^3+u^2+2u+2\right)z^5 + \left(u^6+u^4+u^3+2u^2+2u+4\right)z^6 \\ &+ \left(u^7+u^5+u^4+2u^3+2u^2+4u+5\right)z^7 + \mathrm{O}\left(z^8\right). \end{aligned}$$

Now, we deduce the coefficient $[u^k](F(u) + G(u) + H(u))$ of u^k in the series expansion of $F(u) + G(u) + H(u)$.

Corollary 3. *We have*

$$[u^k](F(u) + G(u) + H(u)) = \frac{1 - z + z^3 - z^4}{1 - z - z^2 + z^3 - z^4} z^k.$$

Remark 2. Let $\mathcal{P}^{\mathcal{V}}$ be the matrix $\mathcal{P}^{\mathcal{V}} = [p^{\mathcal{V}}_{n,k}]_{n,k\geqslant 0}$, where $p^{\mathcal{V}}_{n,k}$ is the number of v.a. skew DAPs of length n ending at ordinate k, i.e. the coefficient of z^n in the series expansion of $[u^k](F(u)+G(u)+H(u))$. The first values of $\mathcal{P}^{\mathcal{V}}$ are

$$\mathcal{P}^{\mathcal{V}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & \\ 2 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 2 & 2 & 1 & 1 & 0 & 1 & 0 & 0 & \\ 4 & 2 & 2 & 1 & 1 & 0 & 1 & 0 & \\ 5 & 4 & 2 & 2 & 1 & 1 & 0 & 1 & \\ & & & & \vdots & & & & \ddots \end{pmatrix}.$$

Since $F(u)+G(u)+H(u) = \frac{g(z)}{1-uf(z)}$, with $f(z)=z$ and $g(z) = \frac{1-z+z^3-z^4}{1-z-z^2+z^3-z^4}$, the matrix $\mathcal{P}^{\mathcal{V}}$ corresponds to the Riordan array

$$\left(\frac{1-z+z^3-z^4}{1-z-z^2+z^3-z^4}, z\right).$$

Now we plug in $u=0$ and $u=1$ to get the generating function for v.a. skew DAPs and partial v.a. skew DAPs, respectively.

Theorem 5. *The generating function for the total number of v.a. skew DAPs with respect to the length is given by:*

$$F(0,z)+G(0,z)+H(0,z) = \frac{1-z+z^3-z^4}{1-z-z^2+z^3-z^4},$$

and an asymptotic of the n-th term is

$$\frac{-a^4+a^3-a+1}{4a^4-3a^3+2a^2+a}\cdot\left(a^3-a^2+a+1\right)^n \approx 0.3051\cdot 1.5129^n,$$

where $a \approx 0.6609925319$ is the smallest root (modulus-wise) of $z^4-z^3+z^2+z-1$.

A simple calculation on generating functions allows to prove that the number a_n of n-length v.a. skew DAPs satisfies $a_0=a_2=a_3=1$, $a_1=0$, $a_4=2$ and $a_n = 2a_{n-2}+a_{n-5}$ for $n\geqslant 5$. The leading terms of the series expansion of $F(0,z)+G(0,z)+H(0,z)$ are:

$$1+z^2+z^3+2z^4+2z^5+4z^6+5z^7+9z^8+12z^9+\mathrm{O}\left(z^{10}\right).$$

The coefficients spell out sequence A124280 in [13], and the n-th term a_n satisfies

$$a_n = \sum_{k=0}^{\lfloor\frac{n-2}{2}\rfloor}\sum_{j=0}^{n-2k-2}\binom{j}{n-2k-j-2}\binom{k}{n-2k-j-2}.$$

Theorem 6. *The generating function for the total number of partial v.a. skew DAPs with respect to the length is given by:*

$$F(1,z)+G(1,z)+H(1,z)=\frac{1+z^3}{1-z-z^2+z^3-z^4},$$

and an asymptotic of the n-th term is

$$\frac{a^3+1}{4a^4-3a^3+2a^2+a}\cdot\left(a^3-a^2+a+1\right)^n\approx 0.9000\cdot 1.5129^n,$$

where $a\approx 0.6609925319$ is the smallest root (modulus-wise) of $z^4-z^3+z^2+z-1$.

A simple calculation on generating functions allows to prove that the number b_n of n-length partial v.a. skew DAPs satisfies $b_0=b_1=1$, $b_2=2$, $b_3=3$, $b_4=5$ and $b_n=2b_{n-2}+b_{n-5}$ for $n\geqslant 5$. The leading terms of the series expansion of $F(1,z)+G(1,z)+H(1,z)$ are:

$$1+z+2z^2+3z^3+5z^4+7z^5+11z^6+16z^7+25z^8+37z^9+\mathrm{O}\left(z^{10}\right).$$

The coefficients spell out sequence A130137 in [13].

In the same way as we did at the end of Sect. 2, we obtain the following.

Corollary 4. *An asymptotic approximation for the average of the ordinate of the endpoint in all partial v.a. skew DAPs of a given length is* 1.9497.

We end this section by exhibiting a constructive bijection between the set $\mathcal{PV}_n$ of n-length partial v.a. skew v.a. DAPs, and the set $\mathcal{B}_{n-1}$ of $(n-1)$-length binary words avoiding the patterns 00 and 0110. Let $\mathcal{B}=\bigcup_{n\geqslant 0}\mathcal{B}_n$.

Definition 4. We recursively define the map χ from the set $\mathcal{PV}\setminus\{\varepsilon\}$ to $\mathcal{B}$ as follows. For $\beta\in\mathcal{PV}\setminus\{\varepsilon\}$, we set:

$$\chi(\beta)=\begin{cases}\epsilon & \text{if } \beta=U & (i)\\ 1^{k-1}0 & \text{if } \beta=U^kD_k,\ k\geqslant 1 & (ii)\\ \chi(\alpha)1 & \text{if } \beta=U\alpha,\ \alpha\in\mathcal{PV}\setminus\{\varepsilon\} & (iii)\\ \underline{\chi(\alpha)}10 & \text{if } \beta=U\alpha L,\ \alpha\in\mathcal{V}\setminus\{\varepsilon\} & (iv)\\ \overline{\chi(\alpha)}1^k0 & \text{if } \beta=U^k\alpha D_k,\ k\geqslant 1,\ \alpha\in\mathcal{V}\setminus\{\varepsilon\} \text{ ending with } L & (v)\end{cases}$$

where the operator $\overline{\bullet}$ acts on a binary word ending with zero, by replacing this last zero with a one, i.e. $\overline{w_1w_2\dots w_{n-1}0}=w_1w_2\dots w_{n-1}1$.

Notice that χ is defined so that the image of any nonempty partial v.a. skew DAP is a binary word that avoids both patterns 00 and 0110. Furthermore, the image of any nonempty v.a. skew DAP ends with a 0. For instance, we have

$$\chi(U^4DLD_2)=\overline{\chi(U^2DL)}110=\overline{\chi(UD)10}110=\overline{010}110=011110.$$

We refer to Fig. 2 for an illustration of the cases in the definition of the bijection χ.

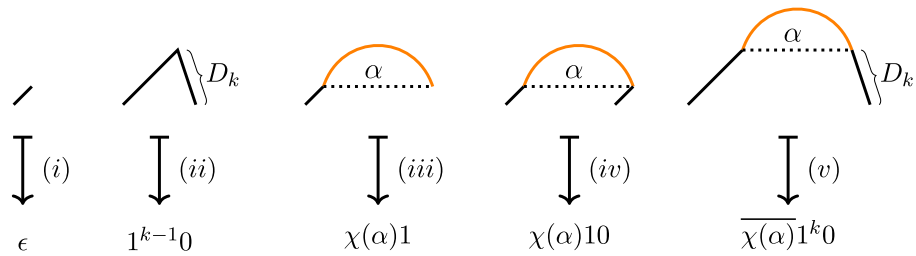


Figure 2. Illustration of the map χ

Theorem 7. *The map χ induces a bijection between $\mathcal{PV}_n \setminus \{\varepsilon\}$ and $\mathcal{B}(n-1)$ for all $n \geqslant 1$.*

Proof. It follows from Theorem 6 and [13] (entry A130137) that $\mathcal{PV}_n \setminus \{\varepsilon\}$ and $\mathcal{B}(n-1)$ have the same cardinality for all $n \geqslant 1$. Thus, it is enough to prove the injectivity of χ. We proceed by induction on n. The statement is trivial for $n = 1, 2$. Now, let $n \geqslant 3$, and let $\alpha, \beta \in \mathcal{PV}_n \setminus \{\varepsilon\}$ such that $\chi(\alpha) = \chi(\beta)$.

If $\chi(\alpha) = 1^k0$, then both $\chi(\alpha)$ and $\chi(\beta)$ belong to case (ii) in the definition of χ. If not, say for example $\alpha = UAL$ with $A \in \mathcal{V} \setminus \{\varepsilon\}$, then $1^k0 = \chi(\alpha) = \chi(UAL) = \chi(A)10$, which implies $\chi(A) = 1^{k-1}$, and in turn $A = U^k$, which contradicts the fact that A is an element of $\mathcal{V} \setminus \{\varepsilon\}$. The hypothesis that α is of the form $U^\ell AD_\ell$ with $\ell \geqslant 1$ and $A \in \mathcal{V} \setminus \{\varepsilon\}$ can be ruled out with a similar reasoning. It now follows from the definition of χ that $\alpha = U^{k+1}D_{k+1} = \beta$.

Otherwise, depending on their ending letters, $\chi(\alpha)$ and $\chi(\beta)$ both either belong to case (iii) ($\chi(\alpha)$ ends with 1), (iv) ($\chi(\alpha)$ ends with 010), or (v) ($\chi(\alpha)$ ends with 110) in the definition of χ. Say they both belong to case (v), for instance. Then, from the definition of χ, it follows that $\alpha = U^kAD_k$ and $\beta = U^\ell BD_\ell$ for some $k, \ell \geqslant 1$ and $A, B \in \mathcal{V} \setminus \{\varepsilon\}$ ending with L. Thus, we have $\chi(\alpha) = \overline{\chi(A)}1^k0$ and $\chi(\beta) = \overline{\chi(B)}1^\ell0$. Suppose for a contradiction that $k \neq \ell$, and without loss of generality, let us assume $k > \ell$. Then, $\chi(\alpha) = \chi(\beta)$ implies $\overline{\chi(A)}1^{k-\ell} = \overline{\chi(B)}$. Since $\chi(A)$ avoids 00 and ends with 0 ($A \in \mathcal{V} \setminus \{\varepsilon\}$), it ends with 10, which implies $\overline{\chi(B)}$ ends with 111. Now, B cannot be of the form UXL with $X \in \mathcal{V} \setminus \{\varepsilon\}$, otherwise $\chi(B)$ would end with 010, and in turn $\overline{\chi(B)}$ would not end with 111 anymore; moreover, B cannot end with a down-step since $\beta = U^\ell BD_\ell$, which yields a contradiction. Hence, we have $k = \ell$, which implies $\overline{\chi(A)} = \overline{\chi(B)}$. Since $A, B \in \mathcal{V} \setminus \{\varepsilon\}$, we have $\chi(A) = \chi(B)$, and, by induction, $A = B$; thus, $\alpha = \beta$.

Cases (iii) and (iv) are handled *mutatis mutandis*, which completes the induction. The cardinality argument then proves the bijectivity. $\square$

4. Enumerating z.v.a. skew DAP

Once again, we use the same methodology and notation as in Sects. 2 and 3 in order to enumerate z.v.a. skew DAPs and partial z.v.a. skew DAPs. Then, we provide here all main results without the details of the proofs. The first system of equations is:

$$\begin{cases} & f_0(z) = 1, \\ \forall k > 0, & f_k(z) = z f_{k-1}(z), \\ \forall k \geqslant 0, & g_k(z) = z \sum_{i \geqslant 1} (f_{k+i}(z) + h_{k+i}(z)), \\ \forall k \geqslant 0, & h_k(z) = z g_{k+1}(z). \end{cases}$$

Redoing the same work as in Sects. 2 and 3, we are led to the following results.

Theorem 8. *We have*

$$F(u) = \frac{1}{1 - zu}, \qquad G(u) = \frac{z^2}{(1 - zu)(1 - z - z^4)},$$
$$H(u) = \frac{z^4}{(1 - zu)(1 - z - z^4)},$$

and thus,

$$F(u) + G(u) + H(u) = \frac{1 - z + z^2}{(1 - zu)(1 - z - z^4)}.$$

The first terms of the series expansion of $F(u) + G(u) + H(u)$ are

$$\begin{aligned} 1 + uz &+ \left(u^2 + 1\right) z^2 + \left(u^3 + u + 1\right) z^3 + \left(u^4 + u^2 + u + 2\right) z^4 \\ &+ \left(u^5 + u^3 + u^2 + 2u + 2\right) z^5 + \left(u^6 + u^4 + u^3 + 2u^2 + 2u + 3\right) z^6 \\ &+ \left(u^7 + u^5 + u^4 + 2u^3 + 2u^2 + 3u + 4\right) z^7 + \mathrm{O}\left(z^8\right). \end{aligned}$$

Therefore, we can obtain the coefficient $[u^k](F(u) + G(u) + H(u))$ of u^k in the series expansion of $F(u) + G(u) + H(u)$.

Corollary 5. *We have*

$$[u^k](F(u) + G(u) + H(u)) = \frac{1 - z + z^2}{1 - z - z^4} z^k.$$

Remark 3. Let $\mathcal{P}^{\mathcal{Z}}$ be the matrix $\mathcal{P}^{\mathcal{Z}} = [p^{\mathcal{Z}}_{n,k}]_{n,k \geqslant 0}$, where $p^{\mathcal{Z}}_{n,k}$ is the number of z.v.a. skew DAPs of length n ending at ordinate k, i.e. the coefficient of z^n in the series expansion of $[u^k](F(u) + G(u) + H(u))$. The first values of $\mathcal{P}^{\mathcal{Z}}$

are

$$\mathcal{P}^{\mathcal{Z}} = \begin{pmatrix} 1\,0\,0\,0\,0\,0\,0\,0 & \\ 0\,1\,0\,0\,0\,0\,0\,0 & \\ 1\,0\,1\,0\,0\,0\,0\,0 & \\ 1\,1\,0\,1\,0\,0\,0\,0 & \\ 2\,1\,1\,0\,1\,0\,0\,0 & \dots \\ 2\,2\,1\,1\,0\,1\,0\,0 & \\ 3\,2\,2\,1\,1\,0\,1\,0 & \\ 4\,3\,2\,2\,1\,1\,0\,1 & \\ \vdots & \ddots \end{pmatrix}.$$

Since $F(u)+G(u)+H(u)=\frac{g(z)}{1-uf(z)}$, with $f(z)=z$ and $g(z)=\frac{1-z+z^2}{1-z-z^4}$, the matrix $\mathcal{P}^{\mathcal{Z}}$ corresponds to the Riordan array

$$\left(\frac{1-z+z^2}{1-z-z^4}, z\right).$$

Now we plug in $u=0$ and $u=1$ to get the generating function for v.a. skew DAPs and partial v.a. skew DAPs, respectively.

Theorem 9. *The generating function for the total number of z.v.a. skew DAPs with respect to the length is given by:*

$$F(0,z)+G(0,z)+H(0,z)=\frac{1-z+z^2}{1-z-z^4},$$

and an asymptotic of the n-th term is

$$\frac{a^2-a+1}{4a^4+a}\cdot\left((a+1)\left(a^2-a+1\right)\right)^n \approx 0.4382\cdot 1.3803^n,$$

where $a\approx 0.7244919590$ is the smallest root (modulus-wise) of z^4+z-1.

A simple calculation on generating functions allows to prove that the number c_n of n-length z.v.a. skew DAPs satisfies $c_0=c_2=c_3=1$, $c_1=0$, and $c_n=c_{n-1}+c_{n-4}$ for $n\geqslant 4$. The leading terms of the series expansion of $F(0,z)+G(0,z)+H(0,z)$ are:

$$1+z^2+z^3+2z^4+2z^5+3z^6+4z^7+6z^8+8z^9+\mathrm{O}\left(z^{10}\right).$$

The coefficients spell out sequence A103632 in [13], and the n-th term c_n satisfies

$$c_n=\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{\lfloor\frac{2n-3k-1}{2}\rfloor}{n-2k}.$$

Theorem 10. *The generating function for the total number of partial z.v.a. skew DAPs with respect to the length is given by:*

$$F(1,z)+G(1,z)+H(1,z)=\frac{z^2-z+1}{(-1+z)\left(z^4+z-1\right)},$$

and an asymptotic of the n-th term is

$$\frac{a^2-a+1}{a\left(4a^3+1\right)(1-a)}\cdot\left((a+1)\left(a^2-a+1\right)\right)^n\approx 1.5905\cdot 1.3803^n,$$

where $a\approx 0.7244919590$ is the smallest root (modulus-wise) of z^4+z-1.

A simple calculation on generating functions allows to prove that the number d_n of n-length partial z.v.a. skew DAPs satisfies $d_0=d_1=1$, $d_2=2$, $d_3=3$, $d_4=5$ and $d_n=2d_{n-1}-d_{n-2}+d_{n-4}-d_{n-5}$ for $n\geqslant 5$. The leading terms of the series expansion of $F(1,z)+G(1,z)+H(1,z)$ are:

$$1+z+2z^2+3z^3+5z^4+7z^5+10z^6+14z^7+20z^8+28z^9+\mathrm{O}\left(z^{10}\right).$$

Finally, we obtain the following.

Corollary 6. *An asymptotic approximation for the average of the ordinate of the endpoint in all partial z.v.a. skew DAPs of a given length is* 2.6296.

It follows from Theorem 9 and [13] (entry A103632) that n-length z.v.a. skew DAPs are in bijection with palindromic compositions of $n-2$ that have parts in $\{2,1,3,5,7,9,\ldots\}$, where a palindromic composition of n is a composition $(c_1,c_2,\ldots,c_k)$, $c_1+c_2+\ldots+c_k=n$, that reads the same backwards as forwards; for instance, $(3,1,5,2,5,1,3)$ is a palindromic composition of 20 (see [8]). Let $\mathcal{C}(n-2)$ be the set of such compositions, and let $\mathcal{C}=\bigcup_{n\geqslant 0}\mathcal{C}(n)$. We shall now provide an explicit bijection.

Definition 5. Let us recursively define the map $\psi:\mathcal{Z}\longrightarrow\mathcal{C}$ as follows. For $\beta\in\mathcal{Z}$, we set:

$$\psi(\beta)=\begin{cases}\epsilon & \text{if } \beta=UD & (i)\\ (1) & \text{if } \beta=U^2D_2 & (ii)\\ (2) & \text{if } \beta=U^2DL & (iii)\\ (3) & \text{if } \beta=U^3D_2L & (iv)\\ (1,\psi(U^aD_{k-2}B),1) & \text{if } \beta=U^{a+2}D_kB,\ a\geqslant 1,k\geqslant 3 & (v)\\ (2,\psi(U^aB),2) & \text{if } \beta=U^{a+2}DLB,\ a\geqslant 1,B\neq\varepsilon & (vi)\\ {}_{+2}\psi(U^aD_{k+1}B)_{+2} & \text{if } \beta=U^{a+2}D_2LD_kB,\ a\geqslant 2,k\geqslant 1 & (vii)\end{cases}$$

where B is a suffix of a z.v.a. skew DAP, and where ${}_{+2}(x_1,x_2,\ldots,x_{n-1},x_n)_{+2}:=((x_1+2),x_2,\ldots,x_{n-1},(x_n+2))$ for $n\geqslant 2$; for the case $n=1$, we define ${}_{+2}(x_1)_{+2}:=(x_1+4)$.

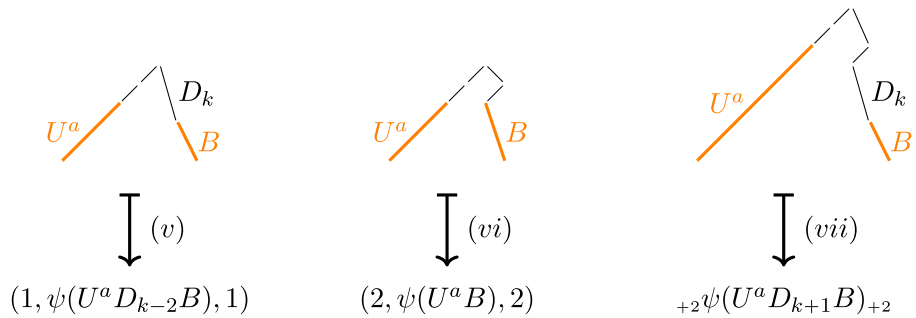


Figure 3. Illustration of the map ψ (the three nontrivial cases)

For instance, we have

$$\begin{aligned}\psi(U^7 D_2 L D_2 L D) &= {}_{+2}\psi(U^5 D_3 L D)_{+2} = {}_{+2}(1, \psi(U^3 D L D), 1)_{+2} \\ &= {}_{+2}(1, 2, \psi(UD), 2, 1)_{+2} = (3, 2, 2, 3).\end{aligned}$$

We refer to Fig. 3 for an illustration of the nontrivial cases in the definition of the bijection ψ.

Theorem 11. *The map ψ induces a bijection between $\mathcal{Z}_n$ and $\mathcal{C}(n-2)$ for all $n \geqslant 2$.*

Proof. Since $\mathcal{Z}_n$ and $\mathcal{C}(n-2)$ have the same cardinality for all $n \geqslant 2$, it is enough to prove that ψ induces an injection from $\mathcal{Z}_n$ to $\mathcal{C}(n-2)$. We proceed by induction on n. The statement is trivial for $n = 2, 3$. Now, let $n \geqslant 4$, and let $\alpha, \beta \in \mathcal{Z}_n$ such that $\psi(\alpha) = \psi(\beta)$. If $\psi(\alpha) \in \{(2), (3)\}$, then we immediately get $\alpha = \beta$. Otherwise, depending on their starting letter, $\psi(\alpha)$ and $\psi(\beta)$ both either belong to case (v), (vi), or (vii) in the definition of ψ. Say they both belong to case (v), for instance. Then, from the definition of ψ, it follows that $\alpha = U^{a_1+2} D_{k_1} B_1$ and $\beta = U^{a_2+2} D_{k_2} B_2$ for some $a_1, a_2, k_1, k_2, B_1, B_2$. Thus, we have $\psi(\alpha) = (1, \psi(U^{a_1} D_{k_1-2} B_1), 1)$ and $\psi(\beta) = (1, \psi(U^{a_2} D_{k_2-2} B_2), 1)$, and in turn, $\psi(U^{a_1} D_{k_1-2} B_1) = \psi(U^{a_2} D_{k_2-2} B_2)$. Since $U^{a_1} D_{k_1-2} B_1$ and $U^{a_2} D_{k_2-2} B_2$ are both elements of $\mathcal{Z}_{n-2}$, and ψ is (by induction) injective from $\mathcal{Z}_{n-2}$ to $\mathcal{C}(n-4)$, we deduce $U^{a_1} D_{k_1-2} B_1 = U^{a_2} D_{k_2-2} B_2$, which implies $\alpha = \beta$. Cases (vi) and (vii) are handled *mutatis mutandis*, which completes the induction. The cardinality argument then proves the bijectivity. $\square$

Author contributions All authors wrote and reviewed the manuscript.

Declarations

Conflict of interests The authors declare no competing interests.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.

References

[1] Baril, J.-L., Barry, P.: Two kinds of partial Motzkin paths with air pockets. Ars Math. Contemp. (2024). https://doi.org/10.26493/1855-3974.3035.6ac

[2] Baril, J.-L., Kirgizov, S., Maréchal, R., Vajnovszki, V.: Enumeration of Dyck paths with air pockets. J. Integer Seq. **26**, Article 23.3.2 (2023)

[3] Baril, J.-L., Kirgizov, S., Maréchal, R., Vajnovszki, V.: Grand Dyck paths with air pockets. Art Discrete Appl. Math. **7**, Article #P1.07 (2024)

[4] Baril, J.-L., Ramirez, J.L.: Partial Motzkin paths with air pockets of the first kind avoiding peaks, valleys or double rises, Submitted (2023)

[5] Barry, P.: Riordan Arrays: A Primer. Logic Press (2017)

[6] Burstein, A., Shapiro, L.W.: Pseudo-involutions in the Riordan group. J. Integer Seq. **25**, Article 22.3.6 (2022)

[7] Flajolet, P., Sedgewick, R.: Analytic Combinatorics. Cambridge University Press, Cambridge (2009)

[8] Hoggat, V.E., Jr., Bicknell, M.: Palindromic compositions. Fibonacci Q. **13**(4), 350–356 (1975)

[9] Krinik, A., Rubino, G., Marcus, D., Swift, R.J., Kasfy, H., Lam, H.: Dual processes to solve single server systems. J. Statist. Plann. Inference **135**, 121–147 (2005)

[10] Orlov, A.G.: On asymptotic behavior of the Taylor coefficients of algebraic functions. Sib. Math. J. **25**, 1002–1013 (1994)

[11] Prodinger, H.: The Kernel method: A collection of examples, *Sém. Lothar. Combin.*, B50f (2004)

[12] Shapiro, L.W., Getu, S., Woan, W., Woodson, L.: The Riordan group. Discrete Appl. Math. **34**, 229–239 (1991)

[13] Sloane, N.J.A.: The On-line Encyclopedia of Integer Sequences, available electronically at http://oeis.org

Jean-Luc Baril and Rémi Maréchal
LIB, Université de Bourgogne
B.P. 47 870
21078 Dijon Cedex
France
e-mail: barjl@u-bourgogne.fr

Rémi Maréchal
e-mail: remi.marechal01@etu.u-bourgogne.fr

Helmut Prodinger
Department of Mathematics
University of Stellenbosch
Stellenbosch 7602
South Africa
e-mail: hproding@sun.ac.za

and

NITheCS (National Institute for Theoretical and Computational Sciences)
Stellenbosch
South Africa

Received: November 9, 2023
Revised: April 4, 2024
Accepted: April 5, 2024

Theoretical Computer Science 1003 (2024) 114626

Contents lists available at ScienceDirect

Theoretical Computer Science

journal homepage: www.elsevier.com/locate/tcs

Bijections between directed-column convex polyominoes and restricted compositions ☆

Jean-Luc Baril [a], José L. Ramírez [b,*], Fabio A. Velandia [b]

[a] LIB, Université de Bourgogne, B.P. 47 870, 21078 Dijon-Cedex, France
[b] Departamento de Matemáticas, Universidad Nacional de Colombia, Bogotá, Colombia

ARTICLE INFO

Communicated by A. Fink

Keywords:
Polyomino
Generating function
Compositions

ABSTRACT

A bijection is given between the set of directed column-convex polyominoes on triangular and honeycomb lattices of area n and some families of restricted compositions. This is an analogous result to one given by Deutsch and Prodinger for polyominoes over square lattices. As a byproduct, we deduce new close forms for the number of hexagonal and triangular directed column-convex polyominoes of area n with k columns.

1. Introduction

A *polyomino* is a connected set of n unit cells on a lattice structure. In the literature, polyominoes are widely studied in the domain of combinatorics. Generally, the studies consist in the enumeration of some special classes of polyominoes with respect to the type of lattice and some given values of parameters (area, height, number of columns, perimeter, ...). We refer to the survey of Viennot [23], the book edited by Guttmann [16], and the papers [2,3,6–9,19–21]. In this paper, we will consider polyominoes in the square (resp. triangular, resp. honeycomb) lattice, where the unit cell is a square (resp. hexagon, resp. triangle). See Fig. 1 for an illustration of these lattices and the associated unit cells. Notice that the unit cell for the honeycomb lattice is an equilateral triangle that can be oriented in two ways (triangle pointing upwards and downwards).

For each lattice, we consider a set of directions (North/East for the square lattice, North/North-East/East for the triangular and honeycomb lattices). A polyomino P is *directed* if there exists a cell S, called the *source* of P, such that any cell C of P can be obtained by repeatedly joining cells from C using the predetermined set of directions. A polyomino P is said *column convex* when any column of P is a connected set, where a *column* of P is defined as the set of cells of P whose centers intersect a fixed line L (vertical line for the square and triangular lattices, and oblique lines of slope $\frac{\pi}{3}$ for the honeycomb lattice).

Definition 1.1. A *dcc-polyomino* consists of a set of unit cells satisfying the three key properties: the set of cells is connected, directed, and column convex.

We refer to Fig. 2 for three examples of dcc-polyominoes in the three kinds of lattices. The source cell is located at the bottom left corner and each dcc-polyomino is constructed by attaching unit cells in the allowed directions of the lattice, by taking into account the property of *directed column convexity*.

☆ This article belongs to Section A: Algorithms, automata, complexity and games, Edited by Paul Spirakis.

* Corresponding author.
E-mail addresses: barjl@u-bourgogne.fr (J.-L. Baril), jlramirezr@unal.edu.co (J.L. Ramírez), fvelandias@unal.edu.co (F.A. Velandia).

https://doi.org/10.1016/j.tcs.2024.114626
Received 29 September 2023; Received in revised form 9 March 2024; Accepted 7 May 2024

Available online 10 May 2024
0304-3975/© 2024 The Author(s). Published by Elsevier B.V. This is an open access article under the CC BY-NC-ND license (http://creativecommons.org/licenses/by-nc-nd/4.0/).

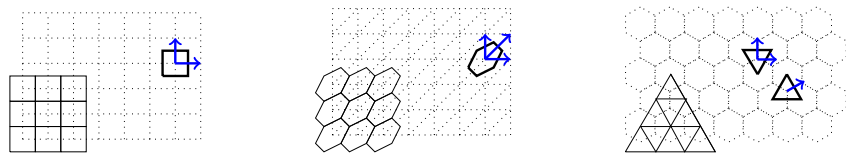


Fig. 1. Square, triangular and honeycomb lattices, and the associated unit cells (square, hexagon, and triangle). For the honeycomb lattice, there are two kinds of cells: triangles pointing upwards and downwards.

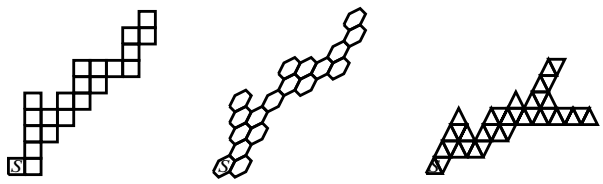


Fig. 2. From left to right, a square, a hexagonal, and a triangular dcc-polyominoes of areas 21, 18 and 37, respectively. All these polyominoes have 9 columns.

Let P be a dcc-polyomino. The *area* of P, denoted by $a(P)$, is the number of unit cells of P. We denote the number of columns of P by $c(P)$. The *height* $h(P)$ of P is the length (number of cells) of a longest path from the source of P to any of the cells in P.

For each kind of lattice described above, Barcucci et al. [3] gave multivariate generating functions for the number of dcc-polyominoes with respect to the area, the number of columns, and the height. The method used consists in giving a recursive description of the set of dcc-polyominoes which induces a functional equation for the multivariate generating function. They also deduce (for each lattice) the average height of dcc-polyominoes and its asymptotic behaviors when the area tends to infinity. In a second study [2], Barcucci, Pinzani, and Sprugnoli use a traditional recurrence relation approach in order to count the number of dcc-polyominoes in the square lattice with area n and with k columns. They prove that this number is given by the binomial coefficient

$$\binom{n+k-2}{n-k},$$

and they deduce that the number of square dcc-polyominoes of area n is the Fibonacci number F_{2n+1}, where $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Moreover, Deutsch and Prodinger [7] exhibit a constructive bijection between these polyominoes of area n and ordered trees of height at most three with n edges, that transports the number of columns into one plus the number of nodes at level 2. They also give a one-to-one correspondence with nondecreasing Dyck paths that transports the number of columns into the number of peaks, knowing that a nondecreasing Dyck path is a Dyck path having a nondecreasing sequence of the heights of its valleys (see [1] for an introduction of nondecreasing paths and [11–14] for some generalizations of these paths).

Motivation: To our knowledge, the literature does not mention any one-to-one correspondence between hexagonal (resp. triangular) dcc-polyominoes of a given area with other classical combinatorial objects so that the number of columns is transported into a natural statistic. The objective of this note is to remedy this shortcoming by exhibiting a unified combinatorial class of objects which is in bijection with the other two kinds of dcc-polyominoes (hexagonal and triangular). As a byproduct, we will deduce new close forms for the number of these dcc-polyominoes of area n with k columns.

Outline of the paper: In Section 2, we exhibit a one-to-one correspondence between hexagonal dcc-polyominoes of area n with k columns and compositions of the integer $n-1$ in which three different types of ones are allowed 1_N, 1_D, and 1_E, and such that $k-1$ parts are different from 1_N. By counting these compositions and using this bijection, we deduce a new close form for the number $H_{n,k}$ of hexagonal dcc-polyominoes of area n with k column. We also give a one-to-one correspondence between these polyominoes and the set of order-consecutive partitions of $\{1, 2, \ldots, n\}$ that transports the number of columns into the number of parts in the partition. Section 3 presents a similar study for triangular dcc-polyominoes in the honeycomb lattice. We exhibit a bijection between these polyominoes of area n with k columns and compositions of the integer $n-1$ in which only parts of the form 2^i, $i \geq 0$, are allowed, and such that $k-1$ parts are different from 1. As previous, this bijection allows us to deduce a new close form for the number $t_{n,k}$ of triangular dcc-polyominoes of area n with k column. We also give a one-to-one correspondence between these polyominoes and the set of consecutive partitions of $\{1, 2, \ldots, n\}$ that transports the number of columns into the number of parts with at least two elements.

We end this section by fixing some definitions about compositions of an integer n. Also, we give some notations used in this note. A *composition* of a positive integer n is a sequence of positive integers $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_\ell)$ such that $\sigma_1 + \sigma_2 + \cdots + \sigma_\ell = n$. The summands σ_i are called *parts* of the composition and n is referred to the *weight* of σ. For example, the compositions of 4 are

$$(4), \quad (3,1), \quad (1,3), \quad (2,2), \quad (2,1,1), \quad (1,2,1), \quad (1,1,2), \quad (1,1,1,1).$$

It is well known [17,26] that the number of compositions of n with k parts is $\binom{n-1}{k-1}$, and the total number of compositions of n is 2^{n-1}. Throughout this note, we will use the following notations. The composition of the integer 0 will be denoted $()$, and if $c = (m_1, m_2, \ldots, m_k)$ is a composition of $n \geq 0$ with k parts, then $\overline{c}$ corresponds to the sequence $m_1, m_2, \ldots, m_k$, and for an integer $a \geq 1$, the notation $(a, \bar{c})$ corresponds to the composition $(a, m_1, m_2, \ldots, m_k)$ of the integer $n + a$. In particular, if $c = ()$ then we have $(a, \overline{c}) = (a)$.

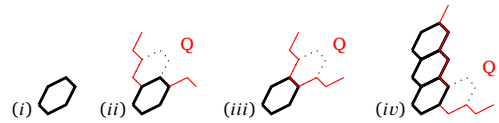


Fig. 3. Recursive decomposition of a hexagonal dcc-polyomino P.

2. Hexagonal dcc-polyominoes

Barcucci et al. [3] proved that the number of hexagonal dcc-polyominoes having area n, denoted by H_n, is equal to

$$H_n = \frac{1}{4}(\theta_1^n + \theta_2^n) = \sum_{k=0}^{n} \binom{n}{2k} 2^{n-k-1} \quad (n \geq 1),$$

where $\theta_1 = 2 + \sqrt{2}$ and $\theta_2 = 2 - \sqrt{2}$. Moreover, the authors give the generating function of the sequence

$$H(x) := \sum_{n\geq 1} H_n x^n = \frac{x(1-x)}{1-4x+2x^2}.$$

The first few values for $n \geq 1$ of H_n are

$$1, \quad 3, \quad 10, \quad 34, \quad 116, \quad 396, \quad 1352, \quad 4616, \quad 15760, \ldots$$

Notice that H_n corresponds with the sequence A007052 in [25]. Among the objects counted by this sequence are the compositions of an integer in which there are three different types of ones, denoted by $1_N, 1_D$, and 1_E, respectively. Let a_n be the number of these compositions of weight n. For example, $a_2 = 10$ and the corresponding compositions are

$$(1_N, 1_N), \quad (1_N, 1_D), \quad (1_N, 1_E), \quad (1_D, 1_N), \quad (1_D, 1_D), \quad (1_D, 1_E),$$
$$(1_E, 1_N), \quad (1_E, 1_D), \quad (1_E, 1_E), \quad (2).$$

Theorem 2.1. *For all $n \geq 0$, we have the equality $a_n = H_{n+1}$.*

Proof. Let C denote the family (combinatorial class) of compositions in which three different types of ones are allowed, then we can write the symbolic equation:

$$C = \mathrm{SEQ}(\{1_N, 1_D, 1_E, 2, 3, 4, \ldots\}),$$

where SEQ denotes the sequence combinatorial class (the previous equation simply rephrases that every element of C is a sequence whose terms belong to $\{1_N, 1_D, 1_E, 2, 3, 4, \ldots\}$). For a general background about the symbolic method see the book [10]. In terms of generating functions, the last equation translates into

$$A(x) := \sum_{n\geq 0} a_n x^n = \frac{1}{1-\left(3x + \sum_{\ell\geq 2} x^\ell\right)} = \frac{1}{1-\frac{3x-2x^2}{1-x}} = \frac{1-x}{1-4x+2x^2},$$

and we obtain that $H(x) = xA(x)$, which means that $a_n = H_{n+1}$. □

As already mentioned in [3], any hexagonal dcc-polyomino P of area $n \geq 1$ can be uniquely decomposed in one of the following forms (see Fig. 3):

(i) P consists of one hexagonal cell;
(ii) P is obtained by attaching a dcc-polyomino Q of area $n-1$ to the north side of a hexagonal cell which becomes the source of P;
(iii) P is obtained by attaching a dcc-polyomino Q of area $n-1$ to the north-east side of a hexagonal cell which becomes the source of P;
(iv) P is obtained by attaching a column C of $k \geq 1$ unit cells so that the most southern cell of C is attached (by its east side) to a dcc-polyomino Q of area $n-k$.

According to this decomposition, we define recursively a map ϕ from the set of hexagonal dcc-polyominoes of area $n+1$ and the set C_n^3 of compositions of n having parts in $\{1_N, 1_D, 1_E, 2, 3, 4, \ldots\}$ (the part 1 can take three different colors).

- If P belongs to the case (i), then we set $\phi(P) = ()$ (empty composition);
- If P belongs to the case (ii), then we set $\phi(P) = (1_N, \overline{\phi(Q)})$;
- If P belongs to the case (iii), then we set $\phi(P) = (1_D, \overline{\phi(Q)})$;
- If P belongs to the case (iv), then we distinguish two cases:

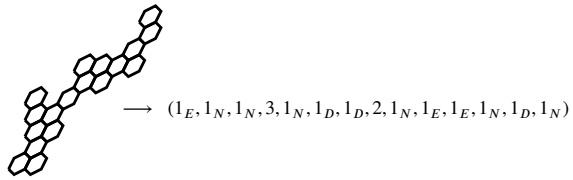


Fig. 4. A hexagonal dcc-polyomino of area 18 with 9 columns and its image by ϕ, which is a composition of 17 with parts in $\{1_N, 1_D, 1_E, 2, 3, 4, \dots\}$. The number of parts different from 1_N equals 8, which also is the number of columns minus one.

- If $k = 1$ (k is the number of cells in the first column of P), then we set $\phi(P) = (1_E, \overline{\phi(Q)})$;
- Otherwise we have $k \geq 2$, and we set $\phi(P) = (k, \overline{\phi(Q)})$.

See Fig. 4 for an illustration of the map ϕ on a hexagonal dcc-polyomino.

Theorem 2.2. *For all $n \geq 0$, ϕ is a bijection between the set of dcc-polyominoes of area $n+1$ and the set of compositions of n where the parts belong to $\{1_N, 1_D, 1_E, 2, 3, 4, \dots\}$. Moreover, ϕ transports the number of columns minus one into the number of parts different from 1_N in the composition.*

Proof. We can easily observe that the image by ϕ of a hexagonal dcc-polymomino of area $n+1$ is a composition in C_n^3. Moreover, if this polyomino P has $k+1$ columns, then $\phi(P)$ has exactly k parts lying in $\{1_D, 1_E, 2, 3, 4, \dots\}$. Conversely, any composition in C_n^3 with k parts different from 1_N can be uniquely decomposed into one of the following forms:

(i) the empty composition () whenever $n = 0$;
(ii) $(1_N, c_1, \dots, c_\ell)$, $\ell \geq 0$, where $(c_1, \dots, c_\ell) \in C_{n-1}^3$ with k parts different from 1_N;
(iii) $(1_D, c_1, \dots, c_\ell)$, $\ell \geq 0$, where $(c_1, \dots, c_\ell) \in C_{n-1}^3$ with $k-1$ parts different from 1_N;
(iv_a) $(1_E, c_1, \dots, c_\ell)$, $\ell \geq 0$, where $(c_1, \dots, c_\ell) \in C_{n-1}^3$ with $k-1$ parts different from 1_N;
(iv_b) $(a, c_1, \dots, c_\ell)$, $\ell \geq 0$, where $a \geq 2$ and $(c_1, \dots, c_\ell) \in C_{n-a}^3$ with $k-1$ parts different from 1_N.

Therefore, the set of hexagonal dcc-polyominoes of area $n+1$ and the set C_n^3 have the same recursive description, which ensures that ϕ is a bijection that transports the number of columns minus one into the number of parts different from 1_N. $\square$

Let $(L_n)_{n\geq 0}$ be the sequence defined by the first difference $L_n = H_{n+1} - H_n$. The first 10 values of this sequence are

1, 2, 7, 24, 82, 280, 956, 3264, 11144, 38048. (A003480)

It is interesting to notice that L_n enumerates the number of L-convex polyominoes with n cells in the square lattice, which are in one-to-one correspondence with 2-compositions of n, i.e. matrices with two rows whose entries are non-negative integers, summing up to n, containing no rows all made of 0 s (see [4,5]).

Let $H_{n,k}$ be the number of hexagonal dcc-polyominoes of area n with exactly k columns. Notice that an immediate consequence of the recursive decomposition of a dcc-polyomino is the recursive formula $H_{n,1} = 1$ for $n \geq 1$, and for $n \geq 2$, $k \geq 2$,

$$H_{n,k} = H_{n-1,k} + H_{n-1,k-1} + \sum_{\ell=1}^{n-1} H_{n-\ell,k-1}.$$

As a byproduct of the bijection ϕ given in Theorem 2.2 we deduce a close form for $H_{n,k}$.

Theorem 2.3. *If $n \geq k \geq 2$, then*

$$H_{n,k} = \sum_{i=0}^{k-1} \binom{k-1}{i} \binom{n+i-1}{n-k}.$$

Proof. Due to the bijection ϕ defined above, $H_{n,k}$ corresponds to the number of compositions of $n-1$ with parts in $\{1_N, 1_D, 1_E, 2, 3, 4, \dots\}$, and where exactly $k-1$ parts are different from 1_N. Such a composition c can be uniquely obtained from a composition of r, $k-1 \leq r \leq n-1$, with $k-1$ parts and where all parts lie in $\{1_D, 1_E, 2, 3, 4, \dots\}$, by adding $(n-1-r)$ parts 1_N in the right places. Since there are $\binom{n-r-1+k-1}{k-1} = \binom{n-r+k-2}{k-1}$ ways for adding these parts into k places (this is the number of ways to choose $k-1$ parts among $n-1-r+k-1$ parts), we obtain

$$H_{n,k} = \sum_{r=k-1}^{n-1} \binom{n-2-r+k}{k-1} a_{r,k-1},$$

where $a_{r,k-1}$ is the number of compositions of r with $(k-1)$ parts lying into $\{1_D, 1_E, 2, 3, 4, \dots\}$.

From the definition of the sequence $a_{r,s}$ and for a given s, we obtain the following expression for its generating function:

$$\sum_{r\geq 0} a_{r,s}x^r = (2x + x^2 + x^3 + \cdots)^s = x^s \sum_{i=0}^{s} \binom{s}{i} \frac{1}{(1-x)^i}.$$

From the equality $1/(1-x)^{m+1} = \sum_{\ell=0}^{\infty} \binom{m+\ell}{\ell} x^\ell$, we have

$$\sum_{r\geq 0} a_{r,s}x^r = x^s + \sum_{i=1}^{s} \sum_{\ell=0}^{\infty} \binom{s}{i} \binom{i+\ell-1}{\ell} x^{\ell+s}.$$

By setting $\ell = 0$, the coefficient of x^s in this expression is $a_{s,s} = 2^s$. For $r > s$, setting $\ell = r - s$ yields

$$a_{r,s} = \sum_{i=1}^{s} \binom{s}{i} \binom{i+r-s-1}{r-s} = \sum_{i=1}^{s} \binom{s}{i} \binom{r-s+i-1}{r-s}.$$

Therefore, by considering the previous value of $a_{r,s}$ for $s = k-1$, we obtain

$$\begin{aligned} H_{n,k} &= a_{k-1,k-1}\binom{n-1}{k-1} + \sum_{j=k}^{n-1} \binom{n-2-j+k}{k-1} a_{j,k-1} \\ &= 2^{k-1}\binom{n-1}{k-1} + \sum_{i=1}^{k-1} \binom{k-1}{i} \sum_{j=1}^{n-k} \binom{(i-1)+j}{j} \binom{n-j-1}{n-j-k} \\ &= 2^{k-1}\binom{n-1}{k-1} + \sum_{i=1}^{k-1} \binom{k-1}{i} \sum_{j=1}^{n-k} \binom{(i-1)+j}{j} \binom{(k-1)+(n-k-j)}{n-j-k}. \end{aligned}$$

The last sum can be simplified by means of the identity (3.2) in [15] by setting $x = i-1$, $y = k-1$, and $m = n-k$. Therefore,

$$\begin{aligned} H_{n,k} &= 2^{k-1}\binom{n-1}{k-1} + \sum_{i=1}^{k-1} \binom{k-1}{i} \left[\binom{n+i-1}{n-k} - \binom{n-1}{n-k}\right] \\ &= \sum_{i=0}^{k-1} \binom{k-1}{i} \binom{n+i-1}{n-k}. \quad \square \end{aligned}$$

Using the same decomposition as previously for defining the bijection with compositions having parts into $\{1_N, 1_D, 1_E, 2, 3, 4, \dots\}$, we can easily exhibit another bijection between dcc-polyominoes of area n with k columns and order-consecutive partitions of $\{1, 2, \dots, n\}$ with k parts, knowing that an *ordered partition* of $\{1, 2, \dots, n\}$ with p parts is a p-uplet $(S_1, S_2, \dots, S_p)$ of subsets such that $S_i \cap S_j = \emptyset$ if $i \neq j$, and $\bigcup_{i=1}^{p} S_i = \{1, 2, \dots, n\}$. An *order-consecutive partition* of $\{1, 2, \dots, n\}$ is an ordered partition satisfying the property: for $j = 1, \dots, p$, $\bigcup_{i=1}^{j} S_i$ is an interval.

So, we define recursively a map ψ from the set of hexagonal polyominoes of area $n+1$ and the set $\mathcal{OCP}_n$ of order-consecutive partitions of $\{1, 2, \dots, n\}$.

- If P belongs to the case (i), then we set $\psi(P) = \{1\}$;
- If P belongs to the case (ii), then $\psi(P)$ is obtained from $\psi(Q)$ by inserting n in the last part; for instance, if $\psi(Q) = \{3,4\}\{2\}\{1\}$, then $\psi(P) = \{3,4\}\{2\}\{1,5\}$;
- If P belongs to the case (iii), then $\psi(P)$ is obtained from $\psi(Q)$ by adding the part $\{n\}$ on the right; for instance, if $\psi(Q) = \{3,4\}\{2\}\{1\}$, then $\psi(P) = \{3,4\}\{2\}\{1\}\{5\}$;
- If P belongs to the case (iv), then $\psi(P)$ is obtained from $\psi(Q)$ by increasing by $k \geq 1$ all values in $\psi(Q)$, and by adding the part $\{1, 2, \dots, k\}$ on the right; for instance, if $\psi(Q) = \{3,4\}\{2\}\{1\}$ and $k = 4$, then $\psi(P) = \{7,8\}\{6\}\{5\}\{1,2,3,4\}$.

With a same argument as the proof of Theorem 2.2, we can easily prove that ψ is a bijection that transports the number of columns into the number of parts. The image of the polyomino represented in Fig. 4 is

$$\{9\}\{8,10,11\}\{5,6,7,12\}\{13\}\{14\}\{3,4,15\}\{2\}\{1,16\}\{17,18\}.$$

As a consequence of this bijection and using Theorem 2.3 and Theorem 6 in [18], we deduce another close form for $H_{n,k}$.

Corollary 2.4. *The number of hexagonal dcc-polyominoes of area n with k columns is*

$$H_{n,k} = \sum_{i=0}^{k-1} (-1)^{k-1-i} \binom{k-1}{i} \binom{2k-i-2}{i}.$$

From the above results we have the bivariate generating function

$$\sum_{n,k\geq 1} H_{n,k} x^n y^k = \frac{(1-x)xy}{1-2x(1+y)+x^2(1+y)}.$$

Let $\mathcal{P}$ be the matrix defined by $\mathcal{P} = [H_{n,k}]_{n,k\geq 1}$. The first few rows of the matrix $\mathcal{P}$ are

$$\mathcal{P} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 5 & 4 & 0 & 0 & 0 & 0 & 0 \\ 1 & 9 & 16 & 8 & 0 & 0 & 0 & 0 \\ 1 & 14 & 41 & 44 & 16 & 0 & 0 & 0 \\ 1 & 20 & 85 & 146 & 112 & 32 & 0 & 0 \\ 1 & 27 & 155 & 377 & 456 & 272 & 64 & 0 \\ 1 & 35 & 259 & 833 & 1408 & 1312 & 640 & 128 \\ \vdots & & \vdots & & \vdots & & \vdots & \end{pmatrix}.$$

The matrix $\mathcal{P}$ corresponds to the array A056242 and it corresponds with the Riordan array (cf. [24]) $\mathcal{P} = \left(\frac{1}{1-x}, \frac{x(2-x)}{(1-x)^2}\right)$. From the theory of Riordan arrays (cf. [22]) we obtain the following curious relation. If $n, k \geq 1$, then

$$\begin{aligned} H_{n+1,k+1} &= 2H_{n,k} + \frac{3}{2} H_{n,k+1} + \sum_{\ell\geq 2} \frac{1}{1-2\ell} \binom{2\ell}{\ell} \left(\frac{-1}{4}\right)^{\ell} H_{n,k+\ell} \\ &= 2H_{n,k} + \frac{3}{2} H_{n,k+1} + \sum_{\ell\geq 2} C_{\ell-1} \frac{(-1)^{\ell-1}}{2^{2\ell-1}} H_{n,k+\ell}, \end{aligned}$$

where C_ℓ is the ℓ-th Catalan number.

3. Triangular dcc-polyominoes

Let $\mathcal{T}$ be the set of triangular dcc-polyominoes. Barcucci et al. [3] proved that the generating function for the number of triangular dcc-polyominoes having area n, denoted by T_n, is given by

$$T(x) := \sum_{n\geq 1} T_n x^n = \frac{x(1-x^2)}{1-x-2x^2+x^3}.$$

The first few values for $n \geq 1$ of T_n are

$$1, \quad 1, \quad 2, \quad 3, \quad 6, \quad 10, \quad 19, \quad 33, \quad 61, \quad 108, \ldots$$

Notice that T_n corresponds with the sequence A028495 in [25]. From the expression of $T(x)$ follows that T_n satisfies the recurrence relation

$$T_n = T_{n-1} + 2T_{n-2} - T_{n-3} \quad (n \geq 4),$$

with initial conditions $T_1 = 1, T_2 = 1$, and $T_3 = 2$. This relation can be applied repeatedly in the following manner:

$$\begin{aligned} T_n - T_{n-1} &= 2T_{n-2} - T_{n-3} \\ &= T_{n-2} + 2T_{n-4} - T_{n-5} \\ &= T_{n-2} + T_{n-4} + (T_{n-4} - T_{n-5}) \\ &\ \ \vdots \\ &= \begin{cases} T_{n-2} + T_{n-4} + \cdots + T_2 + T_0, & \text{if } n \text{ is even;} \\ T_{n-2} + T_{n-4} + \cdots + T_3 + T_1, & \text{if } n \text{ is odd;} \end{cases} \end{aligned}$$

This can be rewritten as

$$T_n = T_{n-1} + \sum_{k=1}^{\lfloor n/2 \rfloor} T_{n-2k}.$$

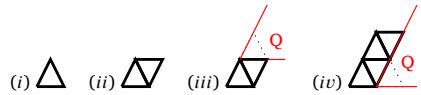


Fig. 5. Recursive decomposition of a triangular dcc-polyomino P.

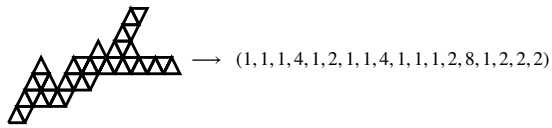


Fig. 6. A triangular dcc-polyomino of area 37 with 9 columns and its image by χ, which is a composition of 36 with parts in $\{1,2,4,6,8,\dots\}$ and so that 8 parts are different from 1.

The sequence $(T_n)_{n\geq 0}$ enumerates a variety of combinatorial objects, such as all paths of length of n on the path graph P_6 and the compositions of n whose parts belong to the set $\{1,2,4,6,8,\dots\}$. Let us establish the relation between triangular dcc-polyominoes and this family of compositions. Let b_n be the number of compositions of n into parts from $\{1,2,4,6,8,\dots\}$.

Proposition 3.1. *For all $n \geq 0$, we have the equality $b_n = T_{n+1}$.*

Proof. We will prove this statement using the symbolic method. Let $\mathcal{B}$ be the family of all compositions whose parts belong to the set $\{1,2,4,6,8,\dots\}$. Thus $\mathcal{B} = \mathrm{SEQ}(\{1,2,4,6,8,\dots\})$. In terms of generating functions, the last equation translates into

$$B(x) := \sum_{n\geq 1} b_n x^n = \frac{1}{1-\left(x+\sum_{\ell\geq 1} x^{2\ell}\right)} = \frac{1}{1-x-\frac{x^2}{1-x^2}} = \frac{1-x^2}{1-x-2x^2+x^3}.$$

Therefore, we obtain $T(x) = xB(x)$, which means that $b_n = T_{n+1}$. $\square$

As already mentioned in [3], any triangular dcc-polyomino P of area $n \geq 1$ can be uniquely decomposed in one of the following forms (see Fig. 5):

(i) P consists of one triangular cell (a triangle pointing upwards);
(ii) P consists of two triangular cells (two triangles pointing upwards and downwards);
(iii) P is obtained by attaching a triangular dcc-polyomino Q of area $n-2$ to the north side of two triangular cells where the leftmost cell becomes the source of P;
(iv) P is obtained by attaching a column C of $k \geq 2$ triangular dcc-polyominoes so that the most southern down-cell of C is attached (by its east side) to a triangular dcc-polyomino Q of area $n-k$.

According to this decomposition, we define recursively a map χ from the set of triangular dcc-polyominoes of area $n+1$ and the set $\mathcal{CP}_n$ of compositions of n having parts in $\{1,2,4,6,8,\dots\}$.

- If P belongs to the case (i), then we set $\chi(P) = ()$ (empty composition);
- If P belongs to the case (ii), then we set $\chi(P) = (1)$;
- If P belongs to the case (iii), then we set $\chi(P) = (1, 1, \overline{\chi(Q)})$;
- If P belongs to the case (iv), then we distinguish two cases:
  - If the number $k \geq 2$ of cells in the first column is odd, then we set $\chi(P) = (1, k-1, \overline{\chi(Q)})$;
  - Otherwise ($k \geq 2$ is even), we set $\chi(P) = (k, \overline{\chi(Q)})$;

See Fig. 6 for an illustration of the map χ on a triangular dcc-polyomino.

Theorem 3.2. *For all $n \geq 0$, χ is a bijection between the set of triangular dcc-polyominoes of area $n+1$ and the set of compositions of n where the parts belong to $\{1,2,4,6,8,\dots\}$. Moreover, χ transports the number of columns minus one into the number of parts different from 1 in the composition.*

Proof. We can easily observe that the image by χ of a triangular dcc-polymomino of area $n+1$ is a composition in $\mathcal{CP}_n$. Moreover, if this polyomino P has $k+1$ columns, then $\phi(P)$ has exactly k parts different to one. Conversely, any composition in $\mathcal{CP}_n$ with k parts different from 1 can be uniquely decomposed into one of the following forms:

(i) the empty composition () whenever $n = 0$;
(ii) the composition (1);
(ii) $(1, 1, c_1, \dots, c_\ell)$, $\ell \geq 0$, where $(c_1, \dots, c_\ell) \in \mathcal{CP}_{n-2}$ with k parts different from 1;
(iv_a) $(1, a, c_1, \dots, c_\ell)$, $\ell \geq 0$, $a \geq 2$, where $(c_1, \dots, c_\ell) \in \mathcal{CP}_{n-a-1}$ with $k-1$ parts different from 1;

(iv_b) $(a, c_1, \dots, c_\ell)$, $\ell \geq 0$, $a \geq 2$, where $(c_1, \dots, c_\ell) \in \mathcal{CP}_{n-a}$ with $k-1$ parts different from 1;

Therefore, the set of triangular dcc-polyominoes of area $n+1$ and the set $\mathcal{CP}_n$ have the same recursive description, which ensures that χ is a bijection that transports the number of columns minus one into the number of parts different from 1. $\square$

Let $t_{n,k}$ be the number of triangular dcc-polyominoes of area n with exactly k columns. Notice that an immediate consequence of the recursive decomposition of a dcc-polyomino is the recursive formula $t_{n,1} = 1$ for $n \geq 1$, and for $n \geq 3$, $k \geq 2$,

$$t_{n,k} = t_{n-2,k} + \sum_{\ell=2}^{n-1} t_{n-\ell,k-1}.$$

As a byproduct of the bijection given in Theorem 3.2 we give a closed form for $t_{n,k}$.

Theorem 3.3. *If $n \geq k \geq 1$, then*

$$t_{n,k} = \sum_{r=\lfloor k/2 \rfloor}^{\lfloor (n-1)/2 \rfloor} \binom{n-2-2r+k}{k-1} \binom{r-1}{r-k+1}.$$

Proof. Due to the bijection χ defined previously, $t_{n,k}$ corresponds to the number of compositions of $n-1$ with parts in $\{1, 2, 4, 6, 8, \dots\}$, and where exactly $k-1$ parts are different from 1. Such a composition c can be uniquely obtained from a composition of r, $k-1 \leq r \leq n-1$, with $k-1$ parts and where all parts lie in $\{2, 4, 6, 8, \dots\}$, by adding $(n-1-r)$ parts 1 in the right places. Since there are $\binom{n-r-1+k-1}{k-1} = \binom{n-r+k-2}{k-1}$ ways for adding these parts into k places (this is the number of ways of choosing $k-1$ parts among $n-1-r+k-1$ parts), we obtain

$$t_{n,k} = \sum_{r=k-1}^{n-1} \binom{n-2-r+k}{k-1} b_{r,k-1},$$

where $b_{r,k-1}$ is the number of compositions of r having all its $(k-1)$ parts lying into $\{2, 4, 6, 8, \dots\}$.

From the definition of the sequence $b_{r,s}$ and for a given s, we obtain the following expression for its generating function:

$$\sum_{r \geq 0} b_{r,s} x^r = (x^2 + x^4 + x^6 + \cdots)^s = x^{2s} \left(\frac{1}{1-x^2} \right)^s.$$

From the equality $1/(1-x)^{m+1} = \sum_{\ell=0}^{\infty} \binom{m+\ell}{\ell} x^\ell$, we have

$$\sum_{r \geq 0} b_{r,s} x^r = x^{2s} \sum_{\ell=0}^{\infty} \binom{s+\ell-1}{\ell} x^{2\ell}.$$

We obtain $b_{r,s} = 0$ when r is odd and $b_{r,s} = \binom{r/2-1}{r/2-s}$ whenever r is even. Therefore, by considering the previous value of $b_{r,s}$ for $s = k-1$, we obtain

$$t_{n,k} = \sum_{r=k-1}^{n-1} \binom{n-2-r+k}{k-1} b_{r,k-1} = \sum_{r=\lfloor k/2 \rfloor}^{\lfloor (n-1)/2 \rfloor} \binom{n-2-2r+k}{k-1} \binom{r-1}{r-k+1}. \quad \square$$

From this bijection, we deduce easily one-to-one correspondence between triangular dcc-polyominoes of area n with k columns and consecutive partitions of n with k parts (i.e. partitions where every subset consists of consecutive elements). Indeed, the consecutive partition p associated to the polyomino P is defined from the composition $\chi(P) = (c_1, c_2, \dots, c_s)$ as follows:

$$p = \{1, \dots, c_1\}\{c_1+1, \dots, c_1+c_2\}\{c_1+c_2+1, \dots, c_1+c_2+c_3\} \cdots$$
$$\cdots \{c_1+c_2+\dots+c_{s-1}+1, \dots, c_1+c_2+\dots+c_s\}.$$

The image of the polyomino represented in Fig. 4 is

$$\{1\}\{2\}\{3\}\{4,5,6,7\}\{8\}\{9,10\}\{11\}\{12\}\{13,14,15,16\}\{17\}\{18\}$$
$$\{19\}\{20,21\}\{22,23,24,25,26,27,28,29\}\{30\}\{31,32\}\{33,34\}\{35,36\}.$$

Let $\mathcal{B} = [b(n,k)]$ be the Riordan array defined by $\mathcal{B} = \left(\frac{1}{1-x}, \frac{x}{(1-x)(1-x^2)} \right)$, and let $\mathcal{T}$ be the matrix defined by $\mathcal{T} = [t_{n,k}]_{n,k \geq 1}$ (A060098). The first few rows of the matrix $\mathcal{T}$ are

$$\mathcal{T} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 & 0 & 0 \\ 1 & 6 & 3 & 0 & 0 & 0 & 0 \\ 1 & 9 & 8 & 1 & 0 & 0 & 0 \\ 1 & 12 & 16 & 4 & 0 & 0 & 0 \\ 1 & 16 & 30 & 13 & 1 & 0 & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \end{pmatrix}.$$

Notice that the anti-diagonals of the matrix $\mathcal{B}$ are the rows of the matrix $\mathcal{T}$, that is, $t_{n,k} = b(n-k,k)$.

CRediT authorship contribution statement

Jean-Luc Baril: Conceptualization, Investigation, Methodology, Writing – original draft, Writing – review & editing. **José L. Ramírez:** Investigation, Writing – original draft, Writing – review & editing, Conceptualization, Formal analysis. **Fabio A. Velandia:** Conceptualization, Formal analysis, Investigation, Writing – original draft, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Acknowledgements

The authors are grateful to the anonymous referees for helpful comments.

References

[1] E. Barcucci, A. Del Lungo, S. Fezzi, R. Pinzani, Nondecreasing Dyck paths and q-Fibonacci numbers, Discrete Math. 170 (1997) 211–217.
[2] E. Barcucci, R. Pinzani, R. Sprugnoli, Directed column-convex polyominoes by recurrence relations, Lecture Notes in Comput. Sci., vol. 668, Springer, Berlin, Heidelberg https://doi.org/10.1007/3-540-56610-471.
[3] E. Barcucci, F. Bertoli, A. Del Lungo, R. Pinzani, The average height of directed column-convex polyominoes having square, hexagonal and triangular cells, Math. Comput. Model. 26 (1997) 27–36.
[4] G. Castiglione, A. Frosini, A. Restivo, S. Rinaldi, Enumeration of L-convex polyominoes by rows and columns, Theor. Comput. Sci. 347 (2005) 336–352.
[5] G. Castiglione, A. Frosini, E. Munarini, A. Restivo, S. Rinaldi, Combinatorial aspects of L-convex polyominoes, Eur. J. Comb. 28 (2007) 1724–1741.
[6] M.-P. Delest, S. Dulucq, Enumeration of directed column-convex animals with given perimeter and area, Croat. Chem. Acta 66 (1993) 59–80.
[7] E. Deutsch, H. Prodinger, A bijection between directed column-convex polyominoes and ordered trees of height at most three, Theor. Comput. Sci. 307 (2003) 319–325.
[8] S. Feretić, A q-enumeration of convex polyominoes by the festoon approach, Theor. Comput. Sci. 319 (2004) 333–356.
[9] S. Feretić, An alternative method for q-counting directed column-convex polyominoes, Discrete Math. 210 (2000) 55–70.
[10] P. Flajolet, R. Sedgewick, Analytic Combinatorics, Cambridge University Press, Cambridge, 2009.
[11] R. Flórez, J.L. Ramírez, Some enumerations on non-decreasing Motzkin paths, Australas. J. Comb. 72 (1) (2018) 138–154.
[12] R. Flórez, J.L. Ramírez, Enumerations of rational non-decreasing Dyck paths with integer slope, Graphs Comb. 37 (2021) 2775–2801.
[13] R. Flórez, T. Mansour, J.L. Ramírez, F. Velandia, D. Villamizar, Restricted Dyck paths on valleys sequence, Sémin. Lothar. Comb. 87B (2023), Special issue for the 9th International Conference on Lattice Path Combinatorics and Applications.
[14] R. Flórez, J.L. Ramírez, F. Velandia, D. Villamizar, A refinement of Dyck paths: a combinatorial approach, Discrete Math. Algorithms Appl. 14 (07) (2022) 2250026.
[15] H.W. Gould, Combinatorial Identities, West Virginia University, 1972.
[16] A.J. Guttmann, Polygons, Polyominoes and Polycubes, Lecture Notes in Physics, vol. 775, Springer, Heidelberg, Germany, 2009.
[17] S. Heubach, T. Mansour, Combinatorics of Compositions and Words, CRC Press, 2009.
[18] F.K. Hwang, C.L. Mallows, Enumerating nested and consecutive partitions, J. Comb. Theory, Ser. A 70 (2) (1995) 323–333.
[19] T. Mansour, R. Rastegar, Enumeration of various animals on triangular lattice, Eur. J. Comb. 394 (2021) 103294.
[20] T. Mansour, R. Rastegar, A.Sh. Shabani, On column-convex and convex Carlitz polyominoes, Math. Comput. Sci. 15 (4) (2021) 889–898.
[21] T. Mansour, A.Sh. Shabani, Smooth column convex polyominoes, Discrete Comput. Geom. 68 (2022) 525–539.
[22] D. Merlini, D.G. Rogers, R. Sprugnoli, M.C. Verri, On some alternative characterizations of Riordan arrays, Can. J. Math. 49 (1997) 301–320.
[23] X.G. Viennot, Survey of polyomino enumeration, in: P. Leroux, C. Reutenauer (Eds.), Séries Formelles et Combinatoire Algébrique, Montréal, in: Publications du LACIM, vol. 11, 1992, pp. 399–420.
[24] L.W. Shapiro, S. Getu, W. Woan, L. Woodson, The Riordan group, Discrete Appl. Math. 34 (1991) 229–239.
[25] N.J.A. Sloane, The on-line encyclopedia of integer sequences, http://oeis.org/.
[26] R.P. Stanley, Enumerative Combinatorics, vol. 1, Cambridge University Press, 1997.

AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **89(2)** (2024), Pages 323–343

Counting symmetric and asymmetric peaks in Motzkin paths with air pockets

Jean-Luc Baril

LIB, Université de Bourgogne Franche-Comté
B.P. 47 870, 21078, Dijon Cedex, France
barjl@u-bourgogne.fr

Rigoberto Flórez

Department of Mathematical Sciences
The Citadel, Charleston, SC, U.S.A.
rigo.florez@citadel.edu

José L. Ramírez

Departamento de Matemáticas
Universidad Nacional de Colombia, Bogotá, Colombia
jlramirezr@unal.edu.co

Abstract

In this paper, we study four subfamilies of Motzkin paths. Two of these subfamilies are well-established in the literature and are known as Motzkin paths with air pockets of the first kind and Motzkin paths with air pockets of the second kind. The remaining two subfamilies extend the concept of non-decreasing paths to the first two families mentioned earlier. Within these four subfamilies, we define two distinct types of subpaths, namely symmetric peaks and asymmetric peaks. Our analysis focuses on understanding the distribution of these symmetric and asymmetric peaks across these subfamilies of paths.

To facilitate this analysis, we present trivariate generating functions. These functions take into consideration parameters such as path length and the numbers of symmetric and asymmetric peaks. These generating functions allow us to calculate, for instance, the total number of symmetric and asymmetric peaks for paths of specific lengths. Furthermore, we conduct an asymptotic analysis of the relationship between these two quantities.

ISSN: 2202-3518 ©The author(s). Released under the CC BY-ND 4.0 International License

1 Introduction

A *Motzkin path* is a lattice path in $\mathbb{Z}^2_{\geq 0}$ that starts at the origin, ends on the x-axis, and consists of steps $U = (1, 1)$, $H = (1, 0)$, and $D_k = (1, -k)$ for any $k \geq 1$. A *Motzkin path with air pockets of the first kind* (MAP1) is a Motzkin path where two consecutive down-steps cannot be adjacent. Similarly, a *Motzkin path with air pockets of the second kind* (MAP2) is a Motzkin path where every step H or D_k is immediately followed by an up-step. We denote the sets of all MAP1 and MAP2 as $\mathcal{M}^1$ and $\mathcal{M}^2$, respectively. These two path families were originally introduced by Baril and Barry (see, for example, [6]).

A *valley* is a subpath of the form D_kU, HU, or D_kH. A subpath in the form of U^kD_k is called a *symmetric peak* if it cannot be extended to a subpath in the form of $U^{k+1}D_k$. An asymmetric peak is a subpath of the form $U^\ell D_k$ whether either $\ell > k$, or $\ell < k$ and the subpath cannot be extended to a subpath of the form $U^\ell D_k$. We use $\mathsf{sp}(P)$ (and $\mathsf{ap}(P)$) to denote the number of symmetric (and asymmetric) peaks in the path P.

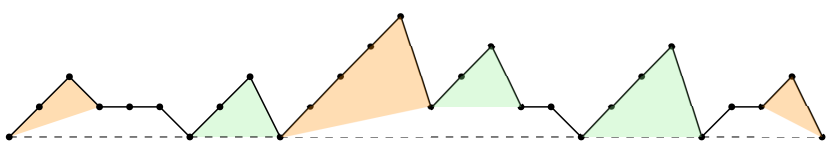


Figure 1: Symmetric and asymmetric peaks of a MAP1.

A path of the form of MAP1 or of the form of MAP2 is called *non-decreasing* if the sequence of ordinates of valleys D_kU, D_kH, HU (considered from left to right) is non-decreasing.

It is worth noting that MAPs without horizontal steps are referred to as *Dyck paths with air pockets*. This term was introduced in a recent paper by Baril et al. [4]. As mentioned in their work, these paths also correspond to a stack evolution with (partial) reset operations, where consecutive resets are not allowed (refer to [14] for more details). Recently, Prodinger [16] used the kernel method to study the partial Dyck paths with air pockets.

In a separate study by Baril et al. [3], the authors explored the prevalence of symmetric and asymmetric peaks in these paths and provided asymptotic approximations for their occurrences.

Furthermore, in a related work [5], these paths were further generalized by allowing them to extend below the x-axis, leading to the concept of *grand Dyck paths with air pockets*. The paper presents enumerative results for these paths, considering parameters such as path length and restrictions on the minimum and maximum ordinates reached.

It is worth mentioning that MAP1 is enumerated by the sequence A114465 in Sloane's On-line Encyclopedia of Integer Sequences [17], while MAP2 corresponds to the sequence of Motzkin numbers (as seen in A001006). These sequences were

originally derived using generating functions. However, in our paper, we employ recurrence relations to achieve the same counting results.

In this paper, we center our focus on analyzing the distribution of symmetric and asymmetric peaks within various classes of Motzkin paths with air pockets, encompassing those of both the first and second kinds, including non-decreasing paths. To accomplish this, we present trivariate generating functions that account for the path's length, the count of symmetric peaks, and the number of asymmetric peaks.

Through the utilization of these generating functions, we are able to determine, for instance, the cumulative count of $\mathsf{sp}(P)$ (as well as $\mathsf{ap}(P)$) for paths of specific lengths. Additionally, we delve into an asymptotic analysis to explore the relationship between these two quantities.

For the sake of simplicity, we adopt the abbreviation 'g.f.' to represent 'generating function'.

The concept of symmetric and asymmetric peaks was first introduced by Asakly in 2018 in the context of words [1]. Since then, several related studies have been published on this subject. For instance, Flórez and Ramírez [12] explored the concept of symmetric and asymmetric peaks for Dyck paths. The concept was further extended to non-decreasing Dyck paths by Elizalde et al. [9] and Flórez et al. [11], to Motzkin paths by Flórez and Ramírez [10], and to partial Dyck paths by Sun et al. [18]. Mansour et al. introduced this concept for integer compositions [15]. Elizalde also contributed other significant findings on Dyck paths [8]. In some of these papers, the authors referred to these objects as 'symmetric pyramids' instead of 'symmetric peaks'.

2 Symmetric and asymmetric peaks in MAP1

In this section, our research is focused on Motzkin paths with air pockets of the first kind, denoted as $\mathcal{M}^1$. We introduce a trivariate generating function that depends on three key parameters: path length, the count of symmetric peaks, and the number of asymmetric peaks. As a corollary to this power series, we derive both a generating function and a closed-form expression for the total number of symmetric peaks. Additionally, we conduct an asymptotic analysis to explore the ratio of the number of symmetric peaks to the total number of peaks for paths of specific lengths. The same analysis is performed for asymmetric peaks. Towards the conclusion of this section, we present a recursive relation for counting the number of paths in $\mathcal{M}^1$ with a given length.

Consider the generating function with the parameters of length, symmetric peaks, and asymmetric peaks:

$$M_{sp,ap}(x,y,z) = \sum_{P\in\mathcal{M}^1} x^{|P|} y^{\mathsf{sp}(P)} z^{\mathsf{ap}(P)}.$$

We establish the following theorem:

Theorem 2.1 *The generating function $M_{sp,ap}(x,y,z)$ for the number of MAP1 with respect to length, number of symmetric peaks, and number of asymmetric peaks is given by:*

$$\frac{2(1-x)}{1-x-x^2y-x^3(y-2z)+\sqrt{(1-x-x^2y-x^3(y-2z))^2-\alpha}},$$

where $\alpha := \alpha(x,y,z) = 4x(1-x-x^2(y-z))(1-2x+x^2(2-y+z)-x^3)$.

Proof. We denote the generating function as $M := M_{sp,ap}(x,y,z)$. Now, let us consider various cases. Except for Case (1) where the paths start with H, we deal with paths UAD_kB where the first return to ground occurs just after D_k for some $k \geq 1$. We consider seven cases $(2)-(8)$ grouped into three types of paths: Case (2) deal with paths where UAD_k is a symmetric peak; Cases (3), (4), and (5) deal with paths where A starts with a symmetric peak; and Cases (6), (7), and (8) deal with the other paths.

Case (1). If the path, denoted as P, is of the form HQ, where Q is another MAP1, then the generating function for this case is simply xM.

Case (2). If the path P is of the form U^aD_aQ (where $a \geq 1$), and Q is a MAP1 (see Figure 2, left-hand side), then the generating function for this case is $\frac{x^2}{1-x}yM$.

Case (3). When P is of the form $P = UU^aD_aQU^bD_{b+1}R$ (with $a,b \geq 1$) and both Q and R are MAP1 (see Figure 2, right-hand side), the generating function becomes

$$x\frac{x^2}{1-x}zM\frac{x^2}{1-x}zM = \frac{x^5}{(1-x)^2}z^2M^2.$$

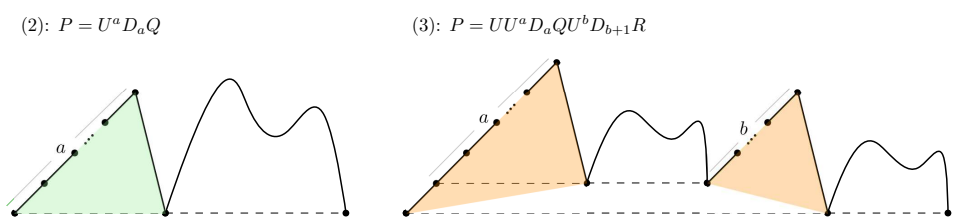


Figure 2: Decomposition of cases (2) and (3).

Case (4). If the path P takes the form of $UU^aD_a\bar{Q}R$ (with $a \geq 1$) and both Q and R are MAP1, with Q ending in a down-step and not concluding with a symmetric peak, and $\bar{Q}$ being an adjusted version of Q after increasing the size of its last down-step by one, the generating function is

$$x\frac{x^2}{1-x}zBM,$$

where $B := B(x,y,z)$ represents the generating function for nonempty MAP1 paths that do not end with a symmetric peak nor with a horizontal step. Considering the complement, we easily obtain $B = M - 1 - \frac{x^2}{1-x}yM - xM$.

Case (5). If the path P is of the form UU^aD_aQHDR (with $a \geq 1$) and both Q and R are MAP1 (see Figure 3 left-hand side), the generating function takes the shape of

$$x\frac{x^2}{1-x}zx^2M^2.$$

Case (6). When P has the form of $UQU^aD_{a+1}R$ (with $a \geq 1$) and both Q and R are MAP1, with Q not starting with a symmetric peak (see Figure 3, right-hand side), the generating function becomes

$$xMB'\frac{x^2}{1-x}z,$$

where $B' := B'(x, y, z)$ denotes the generating function for nonempty MAP1 paths that do not start with a symmetric peak. Considering the complement, we have $B' = M - 1 - \frac{x^2}{1-x}yM$.

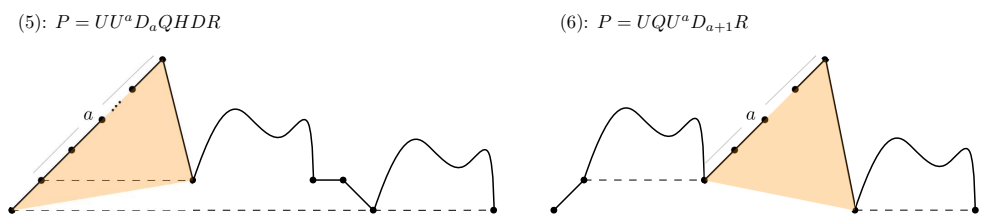


Figure 3: Decomposition of cases (5) and (6).

Case (7). If the path P takes the form of $U\bar{Q}R$, where both Q and R are MAP1, with Q being nonempty, ending with a down-step, not starting or ending with a symmetric peak, and $\bar{Q}$ being an adjusted version of Q, after increasing the size of its last down-step by one, see Figure 4. So, the generating function for this case given by xCM, where $C := C(x, y, z)$ represents the generating function for nonempty MAP1 that end with a down-step and do not start or end with a symmetric peak. Considering the complement, we deduce $C = B' - \frac{x^2y}{1-x}B' - x(B' + 1)$.

Case (8). In the final case, if the path P is of the form $UQHDR$ where both Q and R are MAP1, and Q does not start with a symmetric peak, see Figure 4, the generating function becomes $x^3(B' + 1)M$.

Figure 4: Decomposition of cases (7) and (8).

Summing up these cases, we derive the following functional equation:

$$M = 1 + xM + \frac{x^2}{1-x}yM + \frac{x^5}{(1-x)^2}z^2M^2 + \frac{x^3}{1-x}zM(M - 1 - \frac{x^2}{1-x}yM - xM)$$
$$+ \frac{x^5z}{1-x}M^2 + \frac{x^3z}{1-x}MB' + xM\left(B' - \frac{x^2y}{1-x}B' - x(B'+1)\right) + x^3(B'+1)M,$$

which leads to the desired result. □

The first terms of the Taylor expansion of M are as follows:

$$1 + x + (1+y)x^2 + (3y+2)x^3 + \boldsymbol{(y^2 + 6y + z + 5)x^4}$$
$$+ (5y^2 + z^2 + 12y + 6z + 12)x^5 + O(x^6).$$

In Figure 5, the MAP1 of length 4 are displayed, with their corresponding weights highlighted in boldface in the previous expansion.

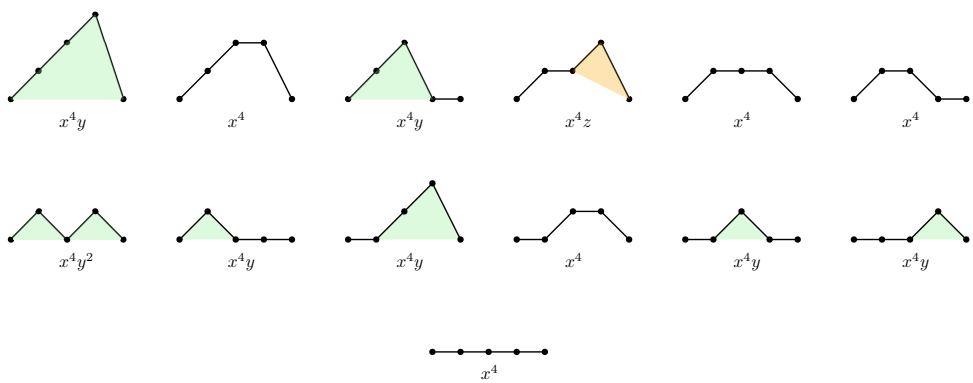


Figure 5: The MAP1 of length 4 and their contribution in $M_{sp,ap}(x, y, z)$.

Corollary 2.2 *The generating function for the number of MAP1 that avoid symmetric and asymmetric peaks is given by:*

$$M(x, 0, 0) = \frac{2}{1 + \sqrt{1 - 4x + 4x^2 - 4x^3}}.$$

The Taylor expansion of this generating function yields:

$$1 + x + x^2 + 2x^3 + 5x^4 + 12x^5 + 29x^6 + 73x^7 + 190x^8 + 505x^9 + O(x^{10}).$$

Remarkably, the coefficients in this sequence correspond to A152171 in [17], which counts Dyck paths of semi-length n without peaks at height 2 (mod 3) and valleys at height 1 (mod 3).

From the formulas provided in A152171 and A025265, we can determine the number of MAP1 of length n that avoid symmetric and asymmetric peaks. Hence, we obtain the expression:

$$\sum_{\ell=0}^{n}\sum_{k=0}^{\ell}\sum_{i=0}^{n-\ell}\sum_{j=0}^{k+1} C_k \binom{j}{\ell-k-j}\binom{k+1}{j}\binom{n-\ell-i}{i}(-1)^{-\ell+k+i},$$

where C_k represents the kth Catalan number.

Corollary 2.3 *The generating function for the number of MAP1 that avoid symmetric peaks is given by:*

$$M(x,0,1) = \frac{2(1-x)}{1-x+2x^3+\sqrt{(1-x)(1-5x+8x^2-12x^3+8x^4-8x^5)}}.$$

The Taylor expansion of this generating function is as follows:

$$1+x+x^2+2x^3+6x^4+19x^5+58x^6+173x^7+519x^8+1585x^9+O(x^{10}).$$

It is worth noting that this sequence of coefficients does not appear in [17].

Corollary 2.4 *The generating function for the number of MAP1 that avoid asymmetric peaks is given by:*

$$M(x,1,0) = \frac{2(1-x)}{1-x-x^2-x^3+\sqrt{(1-x)(1-5x+6x^2-2x^3+x^4+3x^5)}}.$$

The Taylor expansion of this generating function is as follows:

$$1+x+2x^2+5x^3+12x^4+29x^5+73x^6+190x^7+508x^8+1391x^9+O(x^{10}).$$

This sequence of coefficients does not appear in [17].

To find the total number of symmetric peaks and asymmetric peaks in all MAP1, we calculate $\partial_y(M(x,y,1))|_{y=1}$ and $\partial_z(M(x,1,z))|_{z=1}$. This yields two generating functions that we formally state in the following two corollaries.

Corollary 2.5 *The g.f. for the total number of symmetric peaks in all MAP1 is given by:*

$$\frac{2x^2(1-3x+x^2-3x^3+(1+x)\sqrt{\beta})}{(1-x)\sqrt{\beta}(1-x^2+\sqrt{\beta})^2},$$

where $\beta = 1-4x+2x^2-4x^3+x^4$. An asymptotic expression for the n-th coefficient is given by:

$$\frac{(2+\sqrt{3})^n}{3\sqrt{2(12+7\sqrt{3})\pi n}}.$$

The Taylor expansion of this generating function is

$$x^2 + 3x^3 + 8x^4 + 22x^5 + 64x^6 + 196x^7 + 625x^8 + 2053x^9 + O(x^{10}).$$

It is important to note that this sequence of coefficients is not found in [17].

Corollary 2.6 *The g.f. for the total number of asymmetric peaks in all MAP1 is*

$$\frac{4x^3(1 - x + 2x^2 + \sqrt{\beta})}{(1 - x)\sqrt{\beta}(1 - x^2 + \sqrt{\beta})^2},$$

where $\beta = 1 - 4x + 2x^2 - 4x^3 + x^4$. *An asymptotic expression for the* n*-th coefficient is given by:*

$$\frac{(2 + \sqrt{3})^n(-17 + 10\sqrt{3})}{6\sqrt{2(-12 + 7\sqrt{3})\pi n}}.$$

The Taylor expansion of this generating function is

$$x^4 + 8x^5 + 38x^6 + 154x^7 + 590x^8 + 2204x^9 + O(x^{10}).$$

It is important to note that this sequence of coefficients is not found in [17].

Let $p_1(n)$ represent the total number of peaks in all MAP1 of length n. From Corollaries 2.5 and 2.6 we can approximate $p_1(n)$ as follows:

$$p_1(n) \sim \frac{(2 + \sqrt{3})^n \left(2\sqrt{(-12 + 7\sqrt{3})\pi} - (17 - 10\sqrt{3})\sqrt{(12 + 7\sqrt{3})\pi}\right)}{6\sqrt{6n}\pi}.$$

Now, let $s_1(n)$ and $t_1(n)$ denote the numbers of symmetric and asymmetric peaks, respectively, in all MAP1 of length n. We can derive the following asymptotic ratios.

Corollary 2.7 *The asymptotic for the ratio between the number of symmetric peaks and the number of all peaks in all MAP1 is*

$$\lim_{n\to\infty} \frac{s_1(n)}{p_1(n)} = \frac{2\sqrt{(-12 + 7\sqrt{3})\pi}}{2\sqrt{(-12 + 7\sqrt{3})\pi} - (17 - 10\sqrt{3})\sqrt{(12 + 7\sqrt{3})\pi}} \sim 0.309401077.$$

The asymptotic for the ratio between the number of asymmetric peaks and the number of all peaks in all MAP1 is

$$\lim_{n\to\infty} \frac{t_1(n)}{p_1(n)} = \frac{(-17 + 10\sqrt{3})\sqrt{(12 + 7\sqrt{3})\pi}}{2\sqrt{(-12 + 7\sqrt{3})\pi} + (-17 + 10\sqrt{3})\sqrt{(12 + 7\sqrt{3})\pi}} \sim 0.690598923.$$

The asymptotic for the ratio between the numbers of asymmetric and symmetric peaks in all MAP1 is

$$\lim_{n\to\infty} \frac{t_1(n)}{s_1(n)} = \frac{1}{2} + \sqrt{3} \sim 2.232050808.$$

We denote the set of paths of length n in $\mathcal{M}^1$ as $\mathcal{M}^1_n$, and we use $m_1(n)$ to represent the cardinality of $\mathcal{M}^1_n$. Additionally, let $\mathcal{B}_n \subseteq \mathcal{M}^1_n$ denote the subset of paths that do not have valleys at ground level and do not contain sub-paths of the form H at ground level. Essentially, these are MAP1s without sub-paths of the form H and D_kU that touch the x-axis, where $k \geq 1$. It is worth to note that $\mathcal{B}_2 = \{UD_1\}$ and $\mathcal{B}_3 = \{U^2D_1, UHD_1\}$.

Lemma 2.8 *For $n \geq 4$, $|\mathcal{B}_n| = m_1(n-1) - m_1(n-2) + m_1(n-3)$.*

Proof. The set $\mathcal{M}^1_n$ can be partitioned into three disjoint sets: A_n, B_n, and C_n. In A_n we include all paths where the last step D_a satisfies $a \geq 1$. B_n comprises all paths of the form PHH, where $P \in \mathcal{M}^1_{n-2}$. Finally, $C_n = \mathcal{M}^1_n \setminus (A_n \cup B_n)$.

By adding an initial north-east step and replacing the last South-East step of length a (D_a-step) with a step D_{a+1} in all paths in A_n, we obtain a set $W_A \subset \mathcal{B}_{n+1}$. Similarly, by adding an initial north-east step and replacing the last horizontal step H with a step D_1 in all paths in B_n, we obtain a set $W_B \subset \mathcal{B}_{n+1}$.

Notably, $W_A \cup W_B = \mathcal{B}_{n+1}$. Consequently, no path in C_n gives rise to a path in $\mathcal{B}_{n+1}$. This construction establishes the bijection between $\mathcal{B}_{n+1}$ and $\mathcal{M}^1_n \setminus \mathcal{S}$ where $\mathcal{S}$ is the subset of paths from $\mathcal{M}^1_n$ that end with precisely one H step, which induces $|\mathcal{B}_n| = m_1(n-1) - m_1(n-2) + m_1(n-3)$. □

Theorem 2.9 *For $n > 3$, we have*

$$m_1(n) = 3m_1(n-3) + 2m_1(n-1) + \sum_{k=4}^{n-1} \big(m_1(k-1) - m_1(k-2) + m_1(k-3)\big) m_1(n-k),$$

anchored with the initial values $m_1(1) = 1$, $m_1(2) = 2$, and $m_1(3) = 5$.

Proof. Consider a path $P \in \mathcal{M}^1_n$. Such a path P can be decomposed into one of the following forms: HR_{n-1} or Q_kR_{n-k}, where $Q_k \in \mathcal{B}_k$ and $R_{n-k} \in \mathcal{M}^1_{n-k}$, for $2 \leq k < n$, and if $k = n$, we consider R_0 as the empty path.

It is evident that all paths of the form HR_{n-1} are enumerated by $m_1(n-1)$. Since $|\mathcal{B}_2| = 1$ and $|\mathcal{B}_3| = 2$, it follows that all paths of the form Q_2R_{n-2} and Q_3R_{n-3} are enumerated by $m_1(n-2)$ and $2m_1(n-3)$, respectively.

By Lemma 2.8, paths of the form Q_nR_0 are enumerated by $m_1(n-1) - m_1(n-2) + m_1(n-3)$. Furthermore, by Lemma 2.8 paths of the form Q_kR_{n-k}, for a fixed k, $4 \leq k \leq n-1$, are counted as

$$\Big(m_1(k-1) - m_1(k-2) + m_1(k-3)\Big) m_1(n-k).$$

By varying k within the set $\{4, \ldots, n-1\}$, we obtain the desired result. □

The first eleven values of the sequence $m_1(n)$ for $n = 1, \ldots, 11$ are as follows:

$$1, \quad 2, \quad 5, \quad 13, \quad 36, \quad 105, \quad 317, \quad 982, \quad 3105, \quad 9981, \quad 32520.$$

3 Symmetric and asymmetric peaks in MAP2

In this section, our research is focused on Motzkin paths with air pockets of the second kind, denoted as $\mathcal{M}^2$. It is worth recalling that in $P \in \mathcal{M}^2$, two consecutive down-steps cannot occur, and any horizontal-step and down-step (except the last of the path) are immediately followed by an up-step.

We introduce a trivariate generating function that depends on three key parameters: path length, the count of symmetric peaks, and the number of asymmetric peaks. As a corollary to this power series, we derive a generating function for the total number of paths avoiding symmetric peaks, with coefficients corresponding to the Fibonacci numbers. Additionally, we conduct an asymptotic analysis to explore the ratio of the number of symmetric peaks to the total number of peaks for paths of specific lengths. The same analysis is performed for asymmetric peaks. Towards the conclusion of this section, we present a recursive relation for counting the number of paths in $\mathcal{M}^2$ with a given length.

Consider the generating function with the parameters of length, symmetric peaks, and asymmetric peaks:

$$M'_{sp,ap}(x,y,z) = \sum_{P \in \mathcal{M}^2} x^{|P|} y^{\mathsf{sp}(P)} z^{\mathsf{ap}(P)}.$$

For brevity, we set $M' := M'_{sp,ap}(x,y,z)$.

Theorem 3.1 *The generating function $M'_{sp,ap}(x,y,z)$ for the number of MAP2s with respect to length, number of symmetric peaks, and number of asymmetric peaks is as follows:*

$$\frac{(1-x)^2\left(1-x^2y-x^3(1+y-2z)-\sqrt{(1-x)\gamma}\right)}{2x(1-x-x^2y+x^2z)(1-x^2-x^2y+x^2z)},$$

where $\gamma := \gamma(x,y,z)$ is the polynomial:

$$1-3x+x^2(1-2y)-x^4(1-y^2)-x^5(1+y-2z)^2+x^3(3+4y-4z).$$

Proof. We consider a MAP2s which can be categorized into three groups: H, HQ, or Q, where Q is non-empty and starts with U. Now, let $\mathcal{S}$ be the set of non-empty MAP2 that start with U, and we denote $S := S(x,y,z)$ as its trivariate generating function. Clearly, we have:

$$M' = 1 + x + (1+x)S.$$

Let us determine the generating function S for P, where P is a path in $\mathcal{S}$. We distinguish several cases according to the first return decomposition UAD_kB of P. We consider five cases $(1)-(5)$ grouped into three types of paths: Case (1) deal with paths where UAD_k is a symmetric peak; Cases (2) and (3) deal with paths where A starts with a symmetric peak; and Cases (4) and (5) deal with the other paths.

Case (1). When $P = U^a D_a Q$ and $a \geq 1$, where Q can be either empty or belong to $\mathcal{S}$, the generating function for these paths is $\frac{x^2 y}{1-x}(S+1)$. (See Figure 2 (left-hand side).)

Case (2). When $P = UU^a D_a Q U^b D_{b+1} R$ and $a, b \geq 1$, where Q and R are possibly empty or belong to $\mathcal{S}$, see Figure 2 (right-hand side), the generating function for these paths is

$$x\frac{x^2}{1-x}z(S+1)\frac{x^2}{1-x}z(S+1) = \frac{x^5}{(1-x)^2}z^2(S+1)^2.$$

Case (3). When $P = UU^a D_a \bar{Q} R$ and $a \geq 1$, where Q and R are in $\mathcal{S}$, R can be empty, Q does not end with a symmetric peak, and $\bar{Q}$ is obtained from Q by increasing the size of the last down-step by one, the generating function for these paths is

$$x\frac{x^2}{1-x}zV(S+1),$$

where $V := V(x, y, z)$ is the generating function for the paths in $\mathcal{S}$ that do not end with a symmetric peak. Using the complement, it is clear that $V = S - (S+1)\frac{x^2 y}{1-x}$.

Case (4). When $P = UQU^a D_{a+1} R$ and $a \geq 1$, where Q, R are some MAP2, Q does not start with a symmetric peak, see Figure 3 (right-hand side). the contribution is

$$xV'\frac{x^2}{1-x}z(S+1).$$

Here, $V' := V'(x, y, z)$ is the generating function for paths in $\mathcal{S}$ that do not start with a symmetric peak (as in case (3)), plus the g.f. for MAP2 starting with H, that is $V' = V + x(S+1)$.

Case (5). When $P = U\bar{Q}R$, where Q and R are some MAP2, with R in $\mathcal{S}$ and Q does not start or end with symmetric peaks, Q different from H, $\bar{Q}$ is obtained from Q by increasing the size of the last down-step by one, see Figure 4 (left-hand side), the contribution is $xW(S+1)$, where $W := W(x, y, z)$ is the g.f. for paths in $\mathcal{S}$ that do not start or end with a symmetric peak, and different from H. Clearly, we have $W = V' - x - V'\frac{x^2 y}{1-x}$.

Summarizing all these cases, we obtain the following functional equation:

$$S = \frac{x^2 y}{1-x}(S+1) + \frac{x^5}{(1-x)^2}z^2(S+1)^2 + \frac{x^3}{1-x}zV(S+1) + V'\frac{x^3}{1-x}z(S+1) + xW(S+1).$$

This functional equation leads to the desired result. □

The Taylor expansion of this generating function is

$$1 + x + x^2y + 2x^3y + (\boldsymbol{y^2 + 2y + z})x^4 + (3y^2 + z^2 + 2y + 3z)x^5 \\ + (y^3 + 5y^2 + 3yz + 4z^2 + 2y + 6z)x^6 + O(x^7).$$

In Figure 6, the MAP2 of length 4 are displayed, with their corresponding weights highlighted in boldface in the previous expansion.

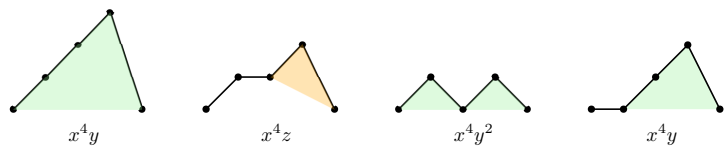


Figure 6: The MAP2 of length 4 and their contribution in $M'_{sp,ap}(x, y, z)$.

Corollary 3.2 *The generating function for the number MAP2 avoiding symmetric peaks is given by:*

$$M'(x, 0, 1) = \frac{(1 - x^2)(1 + x^3 - \sqrt{1 - 4x + 4x^2 - 2x^3 + x^6})}{2x(1 - x + x^2)}.$$

The Taylor expansion is $1 + x + x^4 + 4x^5 + 10x^6 + 23x^7 + 54x^8 + 131x^9 + O(x^{10})$, where the sequence of coefficients does not appear in [17].

Corollary 3.3 *The generating function for the number MAP2 avoiding asymmetric peaks is given by:*

$$M'(x, 1, 0) = \frac{1 - x^2}{1 - x - x^2}.$$

The Taylor expansion is $1+x+x^2+2x^3+3x^4+5x^5+8x^6+13x^7+21x^8+34x^9+O(x^{10})$, where the sequence of coefficients corresponds to the Fibonacci sequence A212804 in [17].

By calculating $\partial_y(M'(x, y, 1))|_{y=1}$ and $\partial_z(M'(x, 1, z))|_{z=1}$ we obtain the following corollaries.

Corollary 3.4 *The g.f. for the total number of symmetric peaks in all MAP2 is*

$$\frac{x(-1 + 3x + 2x^2 + \sqrt{1 - 2x - 3x^2})}{2(1 - x)\sqrt{1 - 2x - 3x^2}},$$

and an asymptotic for the n-th coefficient is

$$\frac{\sqrt{3}\,3^n}{36\sqrt{\pi n}}.$$

The Taylor expansion is $x^2+2x^3+4x^4+8x^5+18x^6+43x^7+109x^8+286x^9+O(x^{10})$, where the sequence of coefficients does not appear in [17].

Corollary 3.5 *The generating function for the total number of symmetric peaks in all MAP2 is given by:*

$$\frac{x(2-3x-3x^2-(2-x)\sqrt{1-2x-3x^2}}{2(1-x)\sqrt{1-2x-3x^2})},$$

and an asymptotic for the n-th coefficient is

$$\frac{\sqrt{3}\,3^n}{12\sqrt{\pi n}}.$$

The Taylor expansion is $x^4+5x^5+17x^6+53x^7+158x^8+464x^9+O(x^{10})$, where the sequence of coefficients does not appear in [17].

Let $p_2(n)$ be the number of peaks in all MAP2 on length n. From Corollaries 3.4 and 3.5 we can deduce that:

$$p_2(n) \sim \frac{3^{n-1}}{\sqrt{3n\pi}}.$$

Now, let us define $s_2(n)$ and $t_2(n)$ as the number of symmetric peaks and asymmetric peaks, respectively, in all MAP2s of length n.

Corollary 3.6 *The asymptotic for the ratio between the number of symmetric peaks and the number of all peaks in all MAP2 is*

$$\lim_{n\to\infty}\frac{s_2(n)}{p_2(n)}=\frac{1}{4}.$$

The asymptotic for the ratio between the number of asymmetric peaks and the number of all peaks in all MAP2 is

$$\lim_{n\to\infty}\frac{t_2(n)}{p_2(n)}=\frac{3}{4}.$$

The asymptotic for the ratio between the numbers of asymmetric and symmetric peaks in all MAP2 is 3.

Corollary 3.7 *The generating function for the total number of peaks in all MAP2 is given by:*

$$\frac{z(1+z-\sqrt{1-2z-3z^2})}{2\sqrt{1-2z-3z^2}}$$

and an asymptotic for the n-th coefficient is

$$\frac{\sqrt{3}\,3^n}{9\sqrt{\pi n}}.$$

The Taylor expansion is $z^2 + 2z^3 + 5z^4 + 13z^6 + 96z^7 + 267z^8 + 750z^9 + O(x^{10})$, where the sequence of coefficients corresponds to A005773 in [17], which also counts the directed animals of size n (see [7] for instance).

We use $\mathcal{M}_n^2$ to denote the set of paths of length n in $\mathcal{M}^2$, and we use $m_2(n)$ to denote the cardinality of $\mathcal{M}_n^2$. Let $\mathcal{S}_n^2 \subseteq \mathcal{M}_n^2$ to represent all paths starting with a U step. Define $\mathcal{B}_n^2 := \mathcal{M}_n^2 \setminus \{HQ_{n-1} | Q_{n-1} \in \mathcal{S}_{n-1}\}$. We use $s(n)$ to denote the cardinality of $\mathcal{S}_n$.

Theorem 3.8 *For $n \geq 3$, we have*

$$m_2(n) = s(n-2) + s(n-1) + m_2(n-1) + \sum_{k=2}^{n-3} m_2(k)s(n-k-1),$$

where $m_2(2) = 1$, $s(2) = 1$, and $s(n) = m_2(n) - s(n-1)$.

Proof. Let us consider a path $P \in \mathcal{M}_n^2$. Such a path P can be decomposed into one of the following forms: HQ_{n-1}, UD_1Q_{n-2}, UM_{n-1}, or $UM_kQ_{n-(k+1)}$, where $Q_i \in \mathcal{S}_i$, and M_i is obtained from a path in $\mathcal{M}_i^2$ by increasing the size of the last down-step by one.

From definition of $\mathcal{B}_n$, we can deduce that $s(i) = m_2(i) - s(i-1)$. Therefore, from the decomposition, we can see that all paths of the form HQ_{n-1} and UD_1Q_{n-2} are counted by $s(n-1)$ and $s(n-2)$, respectively. Paths of the form UM_{n-1} are counted by $m_2(n-1)$.

For a fixed k, $2 \leq k \leq n-3$, once again from the decomposition, we have that all paths of the form $UM_kQ_{n-(k+1)}$ are counted by $m_2(k)s(n-k-1)$. So, by varying k in the set $\{2, \ldots, n-3\}$, and adding $s(n-1)$, $s(n-2)$, and $m_2(n-1)$, we obtain the desired result. □

The first eleven values of the sequence $m_2(n)$ for $n = 2, \ldots, 12$ are as follows:

$$1, \quad 2, \quad 4, \quad 9, \quad 21, \quad 51, \quad 127, \quad 323, \quad 835, \quad 2188, \quad 5798.$$

This sequence is related to the Motzkin numbers, see A001006.

4 The non-decreasing MAPS

In this section we introduce the concept of non-decreasing Motzkin paths with air pockets of both kinds. The concept of non-decreasing path was first introduced by Barcucci et al. [2] in the context of the classical Dyck paths. Recently, Flórez and Ramírez [13] studied this concept for Motzkin paths.

We present a trivariate generating function that depends on three parameters: path length, the number of symmetric peaks, and the number of asymmetric peaks. We then count the number of non-decreasing paths in the form of MAP1 and MAP2, and provide a table with a similar structure to those counted in the previous sections.

4.1 Symmetric and asymmetric peaks in non-decreasing MAP1

In this section, we focus our attention to the set $\mathcal{NM}^1$ of non-decreasing MAP1. These are MAP1 in which the sequence of valley ordinates, specifically D_kU, D_kH for $k \geq 1$, and HU, forms a non-decreasing pattern when read from left to right.

Theorem 4.3 provides the generating function

$$N_{sp,ap}(x,y,z) = \sum_{P \in \mathcal{NM}^1} x^{|P|} y^{\mathsf{sp}(P)} z^{\mathsf{ap}(P)}.$$

For brevity, we will refer to this generating function as N. To facilitate our analysis, we will begin by introducing some essential lemmas.

This lemma explores non-decreasing MAP1, with all its valleys positioned at ground level.

Lemma 4.1 *Let $V := V_{sp,ap}(x,y,z)$ denote the generating function for the number of non-decreasing MAP1 having all its valleys at ordinate 0, with respect to the path's length and the number of symmetric and asymmetric peaks. The generating function is given by:*

$$V = \frac{(1-x)^2}{1-3x+(3-y)x^2-(2-y)x^3}.$$

It is important to note that non-decreasing MAP1 with valleys at ground level do not contain any asymmetric peaks. Consequently, the variable z does not appear in the expression for V.

Proof. To establish the generating function for non-decreasing MAP1 with all its valleys at ground level (or equivalently on the x-axis), we consider non-empty non-decreasing paths of the form $H^{a_0}R_1H^{a_1}\cdots R_kH^{a_k}$. Here, $k \geq 0$, $a_i \geq 0$ for $0 \leq i \leq k$, and R_i can either be a symmetric peak U^aD_a or a truncated symmetric peak $U^aH^bD_a$, where $U^aH^bD_a$, $a,b \geq 1$, for $1 \leq i \leq k$. Therefore, the generating function V comes from the decomposition of paths into a run of initial H steps, followed by a sequence of subpaths of the form $U^kD_kH^\ell$ or $U^kH^mD_kH^\ell$:

$$V = \frac{1}{1-x} \cdot \frac{1}{1-\frac{1}{1-x}\cdot\left(\frac{x^2y}{1-x}+\frac{x^3}{(1-x)^2}\right)}.$$

This completes the proof. □

Lemma 4.2 *The g.f. $B := B_{sp,ap}(x,y,z)$ for the number of non-decreasing MAP1 ending with a down-step, that do not end end with a symmetric peak, with respect to the length and the numbers of symmetric and asymmetric peaks satisfies*

$$B = N - 1 - xV - \frac{x^2yV}{1-x},$$

where V is given in the previous lemma.

Proof. By reasoning with the complement, a non-decreasing MAP1 ending with a down-step, that does not end with a symmetric peak, is a non-decreasing MAP1 different from (i) the empty path, (ii) QH, and (iii) QU^aD_a, where Q has all of its valleys at ordinate 0. Summarizing all these cases, we deduce functional equation. □

Theorem 4.3 *The g.f. $N_{sp,ap}(x,y,z)$ for the number of non-decreasing MAP1 with respect to length, number of symmetric peaks, and number of asymmetric peaks is $\frac{P(x,y,z)}{Q(x,y,z)}$, where $P(x,y,z)$ is*

$$1-6x-x^2(-14+y)-x^3(17-4y+z)-x^4(-11+6y-4z)+x^5(-2+4y-5z+z^2) \\ -x^6(1-3z+2z^2)-x^7(-1+2y-z^2)-x^8(-y+z),$$

and $Q(x,y,z)$ is

$$(1-3x-x^2(-3+y)+x^3(-2+y))(1-4x-x^2(-5+y)-x^4(y-z)-x^3(3-2y+z)).$$

Proof. Let P be a non-decreasing MAP1 (NMAP1 for short). Except for the first case where P starts with H, we distinguish several cases according to the first return decomposition UAD_kB of P. We consider eight cases (2)–(9) grouped into four types of paths: Case (2) deals with paths where UAD_k is a symmetric peak; Case (3) deals with paths where A starts with H; Cases (4), (5) and (6) deal with paths where A starts with a symmetric peak; and Cases (7), (8) and (9) deal with the other paths.

Case (1). If $P = HQ$ where Q also is a NMAP1, then the g.f. for this case is xN.

Case (2). If $P = U^aD_aQ$ for $a \geq 1$ and Q a NMAP1, then the g.f. is $\frac{x^2y}{1-x}N$.

Case (3). If $P = U^aH^bD_aQ$, for $a, b \geq 1$ and Q a non-empty NMAP1, then the g.f. is $\frac{x^3}{(1-x)^2}N$.

Case (4). If $P = UU^aD_aQU^bD_{b+1}$, for $a, b \geq 1$ and Q a NMAP1 having all its valleys on the x-axis, then the g.f. is

$$\frac{x^3z}{1-x}V\frac{x^2z}{1-x},$$

where $V := V(x,y,z)$ is given in Lemma 4.1.

Case (5). If $P = UU^aD_aQHD$ for $a \geq 1$, Q having all its valleys at ordinate 0, then the g.f. is $\frac{x^5z}{1-x}V$.

Case (6). If $P = UU^aD_a\bar{Q}$ for $a \geq 1$, Q a non-empty NMAP1 ending with a down-step and that does not end with a symmetric peak, and $\bar{Q}$ is obtained from Q by increasing by one the size of the last down step, then the g.f. is

$$\frac{x^3z}{1-x}B,$$

where $B := B(x,y,z)$ is the g.f. of Lemma 4.2.

Case (7). If $P = UQU^aD_{a+1}$ for $a \geq 1$, Q a non-empty NMAP1 that does not start with a symmetric peak, and having all its valleys on the x-axis, then the g.f. is

$$\left(V - 1 - \frac{x^2y}{1-x}V\right)\frac{x^3z}{1-x}.$$

Case (8). If $P = U\bar{Q}$, where Q does not start with a symmetric peak, ends with a down step but does not end with a symmetric peak, and $\bar{Q}$ is obtained from Q by increasing the last down-step, then the g.f. is

$$x\left(B - \frac{x^2yB}{1-x} - \frac{x^3}{(1-x)^2}\right),$$

where B satisfies Lemma 4.2. Indeed, $B - \frac{x^2yB}{1-x}$ corresponds to the paths Q that do not start with a symmetric peak, ending with a down step, but not end with a symmetric peak, and we must substract $\frac{x^3}{(1-x)^2}$ in order to eliminate paths of the form $U^aH^bD_a$, $a, b \geq 1$.

Case (9). If $P = UQHD$ where Q does not start with a symmetric peak and having all its valleys at ordinate 0, and different from H^a, $a \geq 1$, then the g.f. is

$$x^3\left(V - 1 - \frac{x^2yV}{1-x} - \frac{x}{1-x}\right).$$

Summarizing all these cases and using the the previous lemmas, we obtain the result. □

The Taylor expansion of this generating function is

$$1+x+(y+1)x^2+(2+3y)x^3+(y^2+6y+z+5)x^4+(5y^2+z^2+12y+5z+12)x^5+O(x^6)$$

Corollary 4.4 *The g.f. for the number of non-decreasing MAP1 is*

$$N(x,1,1) = \frac{1-6x+13x^2-14x^3+9x^4-2x^5}{(1-3x+2x^2-x^3)(1-4x+4x^2-2x^3)}.$$

The Taylor expansion of this generating function is

$$1 + x + 2x^2 + 5x^3 + 13x^4 + 35x^5 + 96x^6 + 265x^7 + 734x^8 + 2040x^9 + O(x^{10}),$$

where the sequence does not appear in [17].

We can adapt the same proofs used in the previous sections to non-decreasing MAPs of both kinds. Therefore, we will only provide in Table 1 a summary of the results without including the proofs.

Notice that the sequences associated to the statistics given in Table 1 do not appear in [17].

Corollary 4.5 *An asymptotic for the ratio between the numbers of asymmetric and symmetric peaks in non-decreasing MAP1 is*

$$\frac{\left(\sqrt{33}+7\right)\left(26+6\sqrt{33}\right)^{\frac{1}{3}} - 2\sqrt{33} - \left(26+6\sqrt{33}\right)^{\frac{2}{3}} + 2}{3\left(26+6\sqrt{33}\right)^{\frac{2}{3}}} \sim 0.5436890133.$$

Statistic population	Generating function
Non-decreasing MAP1 avoiding symmetric peaks	$N(x,0,1) = \frac{1-5x+8x^2-5x^3+2x^4+x^5-x^6}{(1-2x)(1-3x+x^2)(1-x+x^2)}$
Non-decreasing MAP1 avoiding asymmetric peaks	$N(x,1,0) = \frac{1-6x+13x^2-13x^3+5x^4+2x^5-x^6-x^7+x^8}{(1-3x+2x^2-x^3)(1-4x+4x^2-x^3-x^4)}$
Non-decreasing MAP1 avoiding symmetric and asymmetric peaks	$N(x,0,0) = \frac{(1-3x+2x^2-x^3)(1-3x+3x^2-x^3-x^4)}{(1-2x)(1-x+x^2)(1-4x+5x^2-3x^3)}$
Symmetric peaks in all non-decreasing MAP1	$\frac{(1-x)x^2(1-x+x^2)(1-9x+31x^2-52x^3+48x^4-32x^5+22x^6-16x^7+8x^8-2x^9)}{(1-3x+2x^2-x^3)^2(1-4x+4x^2-2x^3)^2}$
Asymmetric peaks in all non-decreasing MAP1	$\frac{(1-x)x^4(1-3x-x^2+8x^3-10x^4+6x^5-2x^6)}{(1-3x+2x^2-x^3)(1-4x+4x^2-2x^3)^2}$

Table 1: Some statistics for non-decreasing MAP1.

4.2 Symmetric and asymmetric peaks in non-decreasing MAP2

Finally, we focus on the set $\mathcal{NM}^2$ of non-decreasing Dyck paths with air pockets of second kind. The following theorem provides the generating function

$$N'_{sp,ap}(x,y,z) = \sum_{P \in \mathcal{NM}^2} x^{|P|} y^{\mathsf{sp}(P)} z^{\mathsf{ap}(P)}.$$

For short, we set $N' := N'_{sp,ap}(x,y,z)$.

Theorem 4.6 *The g.f. $N'_{sp,ap}(x,y,z)$ for the number of non-decreasing MAP2 with respect to the numbers of symmetric and asymmetric peaks is*

$$\frac{(1+x)(1-3x+2x^2+x^3-x^4-x^2y+2x^3y-x^4y-x^3z+2x^4z-x^5z+x^5z^2)}{(1-x-x^2y)(1-2x+x^3-x^2y+x^3y-x^3z)}.$$

Proof. We set $N' = N'_{sp,ap}(x,y,z)$. Any nonempty non-decreasing MAP2 (NMAP2 for short) is either H, HQ, or Q, where Q is non-empty and Q starts with U.

Now, let $\mathcal{S}$ be the set of non-empty NMAP2 that starts with U, and $S := S(x,y,z)$ its associated generating function. Obviously, we have $N' = 1 + x + (1+x)S$. We distinguish several cases according to the beginning and the end of A in the first return decomposition UAD_kB of P.

Case (1). If $P = U^aD_aQ$, $a \geq 1$, where Q is either empty or $Q \in \mathcal{S}$, the g.f. for these paths is $\frac{x^2y}{1-x}(S+1)$.

Case (2). If $P = UU^aD_aQU^bD_{b+1}$, $a, b \geq 1$, where Q is either empty or $Q \in \mathcal{S}$ with all its valleys on the x-axis, then the g.f. is

$$x\frac{x^2}{1-x}zW\frac{x^2}{1-x}z = \frac{x^5}{(1-x)^2}z^2W,$$

where $W := W(x, y, z)$ is the g.f. for NMAP2 in $\mathcal{S} \cup \{\epsilon\}$ having valleys on the x-axis, i.e., W is the solution of $W = 1 + \frac{x^2y}{1-x}W$.

Case (3). If $P = UHQU^aD_{a+1}$, $a \geq 1$, where Q is either empty or $Q \in \mathcal{S}$ having all its valleys on the x-axis, then the g.f. is $\frac{x^4z}{1-x}W$.

Case (4). If $P = UU^aD_a\bar{Q}$, $a \geq 1$, where $Q \in \mathcal{S}$, Q ends with an asymmetric peak, and $\bar{Q}$ is obtained from Q after increasing by one the size of the last down-step, then the g.f. is

$$x\frac{x^2z}{1-x}z(S-(W-1)).$$

Case (5). If $P = UH\bar{Q}$, where $Q \in \mathcal{S}$, Q ends with an asymmetric peak, and $\bar{Q}$ is obtained from Q after increasing by one the size of the last down-step, then the g.f. is

$$x^2(S-W+1).$$

Case (6). If $P = U\bar{Q}$, where Q is a NMAP2 in $\mathcal{S}$, Q starts and ends with an asymmetric peak, $\bar{Q}$ is obtained from Q by increasing by one the last down-step, then the contribution is

$$x\left(S-W+1-\frac{x^2y}{1-x}(S-W+1)\right).$$

Summarizing all these cases, we obtain the following functional equation.

$$S = \frac{x^2y}{1-x}(S+1) + \frac{x^5z^2W}{(1-x)^2} + \frac{x^4zW}{1-x} + \frac{x^3z}{1-x}(S-W+1) + x^2(S-W+1) + x(S-W+1)\left(1-\frac{x^2}{1-x}\right),$$

which induces the result for S, and thus for N'. □

The first terms of the Taylor expansion of N' are

$$1 + x + yx^2 + 2yx^3 + (2y+y^2+z)x^4 + (2y+3y^2+3z+z^2)x^5 + (2y+5y^2+y^3+6z+2yz+4z^2)x^6 + (2y+7y^2+4y^3+11z+8yz+11z^2+2yz^2)x^7 + O(x^8).$$

Corollary 4.7 *The g.f. for the number of non-decreasing MAP2 is*

$$N'(x,1,1) = \frac{(1+x)(1-2x)}{1-2x-x^2+x^3}.$$

The first terms of the Taylor expansion are

$$1 + x + x^2 + 2x^3 + 4x^4 + 9x^5 + 20x^6 + 45x^7 + 101x^8 + 227x^9 + O(x^{10}),$$

where the sequence of coefficients corresponds to the sequence A052534 in [17].

In Table 2 we summarized the results for the MAP2.

Statistic population	Generating function
Non-decreasing MAP2 avoiding symmetric peaks	$N'(x,0,1) = \frac{(1+x)(1-3x+2x^2+x^4)}{(1-x)(1-2x)}$
Non-decreasing MAP2 avoiding asymmetric peaks	$N'(x,1,0) = \frac{1-x^2}{1-x-x^2}$, Fibonacci sequence
Symmetric peaks in all non-decreasing MAP2	$\frac{x^2(1+x)(1-4x+3x^2+3x^3-x^4-x^5)}{(1-x-x^2)(1-2x-x^2+x^3)^2}$
Asymmetric peaks in all non-decreasing MAP2	$\frac{x^4(1+x)(1-x-3x^2+x^3+x^4)}{(1-x-x^2)(1-2x-x^2+x^3)^2}$

Table 2: Some statistics for non-decreasing MAP2.

Corollary 4.8 *An asymptotic for the ratio between the numbers of asymmetric and symmetric peaks in NMAP2 is*

$$\frac{a^2\left(a^4 + a^3 - 3a^2 - a + 1\right)}{-a^5 - a^4 + 3a^3 + 3a^2 - 4a + 1} \sim 0.8019374457,$$

where

$$a = \frac{1}{3} - \frac{\sqrt{7}\,\sin\left(\frac{\arctan\left(3\sqrt{3}\right)}{3} + \frac{\pi}{6}\right)}{3} + \frac{\sqrt{3}\,\sqrt{7}\,\cos\left(\frac{\arctan\left(3\sqrt{3}\right)}{3} + \frac{\pi}{6}\right)}{3} \sim 0.4450418680.$$

Acknowledgements

The authors are grateful to the referees for the detailed comments and corrections that helped us improve the paper.

References

[1] W. Asakly, Enumerating symmetric and non-symmetric peaks in words, *Online J. Anal. Comb.* **13** (2018), 7 pp.

[2] E. Barcucci, A. Del Lungo, A. Fezz, and R. Pinzani, Nondecreasing Dyck paths and q-Fibonacci numbers, *Discrete Math.* **170** (1997), 211–217.

[3] J.-L. Baril, R. Flórez and J. L. Ramírez, Symmetries in Dyck paths with air pockets, *Aequat. Math.* (2024), https://doi.org/10.1007/s00010-024-01043-7.

[4] J.-L. Baril, S. Kirgizov, R. Maréchal and V. Vajnovszki, Enumeration of Dyck paths with air pockets, *J. Integer Seq.* **26** (2023), Article 23.3.2.

[5] J.-L. Baril, S. Kirgizov, R. Maréchal and V. Vajnovszki, Grand Dyck paths with air pockets, *Art Discrete Appl. Math.* **7** (2024), Article #P1.07.

[6] J.-L. Baril and P. Barry, Two kinds of partial Motzkin paths with air pockets, *Ars Math. Contemp.* (2023) (to appear), https://doi.org/10.26493/1855-3974.3035.6ac.

[7] D. Dhar, M. K. Phani and M. Barma, Enumeration of directed site animals on two-dimensional lattices, *J. Phys. A: Math. Gen.* **15** (1982), 279–284.

[8] S. Elizalde, Symmetric peaks and symmetric valleys in Dyck paths, *Discrete Math.* **34** (2021), 112364.

[9] S. Elizalde, R. Flórez and J. L. Ramírez, Enumerating symmetric peaks in non-decreasing Dyck paths, *Ars Math. Contemp.* **21** (2021), 1–23.

[10] R. Flórez and J. L. Ramírez, Enumerating symmetric pyramids in Motzkin paths, *Ars Math. Contemp.* **23** (2023), # 4.06.

[11] R. Flórez, L. Junes and J. L. Ramírez, Counting asymmetric weighted pyramids in non-decreasing Dyck paths, *Australas. J. Combin.* **79** (2021), 123–140.

[12] R. Flórez and J. L. Ramírez, Enumerating symmetric and asymmetric peaks in Dyck paths, *Discrete Math.* **343** (2020), 112118.

[13] R. Flórez and J. L. Ramírez, Some enumeration on non-decreasing Motzkin paths, *Australas. J. Combin.* **72** (2018), 138–154.

[14] A. Krinik, G. Rubino, D. Marcus, R. J. Swift, H. Kasfy and H. Lam. Dual processes to solve single server systems, *J. Statist. Plann. Inference* **135** (2005), 121–147.

[15] T. Mansour, A. Moreno and J. L. Ramírez, Symmetric and asymmetric peaks in compositions, *Online J. Anal. Comb.* **17** (2022), #05.

[16] H. Prodinger, Partial Dyck paths with air pockets, *Integers* **22** (2022), #A24.

[17] N. J. A. Sloane, The On-line Encyclopedia of Integer Sequences, Available electronically at `http://oeis.org`.

[18] Y. Sun, W. Shi and D. Zhao, Symmetric and asymmetric peaks or valleys in (partial) Dyck paths, *Enumer. Combin. Appl.* **2** (2022), article #S2R24.

(Received 26 Oct 2023; revised 26 Mar 2024, 23 May 2024)

Partial Motzkin paths with air pockets of the first kind avoiding peaks, valleys or double rises

Jean-Luc Baril[1] and José L. Ramírez[2]

[1]LIB, Université de Bourgogne
B.P. 47 870, 21078 Dijon Cedex France
E-mail: barjl@u-bourgogne.fr
[2]Departamento de Matemáticas, Universidad Nacional de Colombia
Bogotá, Colombia
E-mail: jlramirezr@unal.edu.co

July 18, 2024

Abstract

Motzkin paths with air pockets (MAP) of the first kind are defined as a generalization of Dyck paths with air pockets. They are lattice paths in $\mathbb{N}^2$ starting at the origin made of steps $U = (1, 1)$, $D_i = (1, -i)$, $i \geqslant 1$ and $H = (1, 0)$, where two down-steps cannot be consecutive. We enumerate MAP and their prefixes avoiding peaks (resp. valleys, resp. double rise) according to the length, the type of the last step, and the height of its end-point. We express our results using Riordan arrays. Finally, we provide constructive bijections between these paths and restricted Dyck and Motzkin paths.

1 Introduction

In a recent paper [6], the authors introduce, study, and enumerate special classes of lattice paths, called *Dyck paths with air pockets* (DAP for short). Such paths are non empty lattice paths in the first quadrant of $\mathbb{Z}^2$ starting at the origin, and consisting of up-steps $U = (1, 1)$ and down-steps $D_i = (1, -i)$, $i \geqslant 1$, where two down-steps cannot be consecutive. These paths can be viewed as ordinary Dyck paths (i.e., paths in $\mathbb{N}^2$ starting at the origin, ending on the x-axis and consisting of U and $D_1(= D)$), where each maximal run of down-steps is condensed into one large down-step. As mentioned in [6], they also correspond to a stack evolution with (partial) reset operations that cannot be consecutive (see for instance [13]). The authors enumerate these paths with respect to the length, the type (up or down) of the

last step and the height of the end-point. Whenever the last point is on the x-axis, they prove that the DAP of length n are in one-to-one correspondence with the peak-less Motzkin paths of length $n-1$. They also investigate the popularity of many patterns in these paths and they give asymptotic approximations. In a second work [7], the authors make a study for a generalization of these paths by allowing them to go below the x-axis. They call these paths Grand Dyck paths with air pockets (GDAP), and they also yield enumerative results for these paths according to the length and several restrictions on the height. In a third paper, Baril and Barry [8] study two generalizations of DAP by allowing some horizontal steps $H = (1, 0)$ with some conditions. They call them *Motzkin paths with air pockets of the first and second kind*. For other generalizations of Motzkin paths, we refer to [17, 15] for example.

In this paper we study Motzkin paths with air pockets of the first kind, which are defined as Motzkin paths [9] (lattice paths in $\mathbb{N}^2$ starting at the origin and made of U, D, and H), where each maximal run of down-steps is condensed into one large down-step. More precisely, we consider lattice paths in $\mathbb{N}^2$ starting at the origin, consisting of steps U, H, and D_i, $i \geqslant 1$, where *two down-steps cannot be consecutive*. We denote by $\mathcal{D}$ (resp. $\mathcal{M}$, resp. $\mathcal{MP}$) the set of Dyck paths (resp. Motzkin paths, resp. Motzkin paths with air pockets of the first kind). Moreover, we denote by $\mathcal{PMP}$ the set of partial Motzkin paths with air pockets (PMAP for short). The MAP of the first kind are enumerated by the sequence A114465. This sequence also counts the Dyck paths having no ascents of length 2 that start at an odd level.

Throughout the paper, we will use the following notations. For $k \geqslant 0$, we consider the generating function $f_k = f_k(z)$ (resp. $g_k = g_k(z)$, resp. $h_k = h_k(z)$), where the coefficient of z^n in the series expansion is the number of partial Motzkin paths with air pockets of length n ending at height k with an up-step, (resp. with a down-step, resp. with a horizontal step H). We introduce the bivariate generating functions

$$F(u,z) = \sum_{k\geqslant 0} u^k f_k(z), \quad G(u,z) = \sum_{k\geqslant 0} u^k g_k(z), \text{ and } H(u,z) = \sum_{k\geqslant 0} u^k h_k(z).$$

For short, we also use the notation $F(u), G(u)$, and $H(u)$ for these functions.

A *Riordan array* is an infinite lower triangular matrix whose k-th column has generating function $g(z)f(z)^k$ for all $k \geqslant 0$, for some formal power series $g(z)$ and $f(z)$, with $g(0) \neq 0$, $f(0) = 0$, and $f'(0) \neq 0$. Such a Riordan array is denoted by $(g(z), f(z))$. We refer to [25, 26] for more details on Riordan arrays. Several authors have used Riordan arrays to study lattice paths; see for example [19, 22, 23, 29, 30, 31].

The outline of this paper is the following. We present enumerative results for partial Motzkin paths with air pockets of the first kind avoiding peaks (resp. avoiding valleys, resp. avoiding double rises), knowing that a *peak* is an occurrence UD_i for some $i \geqslant 1$, a *valley* is an occurrence D_iU for some $i \geqslant 1$, and a *double rise* is an occurrence UU. For each avoidance, we provide bivariate generating functions that count the PMAP with respect to the length, the type of the last step (up, down or horizontal step) and the height of the end-point. All these results are obtained algebraically by using the famous kernel method for solving several systems of functional equations [1, 20]. We express our results using Riordan

arrays and we deduce closed forms for PMAP of length n ending at height k. Finally, we provide constructive bijections between these paths and some restricted Dyck and Motzkin paths.

2 Partial peak-less Motzkin paths with air pockets

In this section, we study partial Motzkin paths with air pockets of the first kind avoiding occurrences of UD_i for all $i \geqslant 1$.

2.1 Enumerative results

Let P be a length n PMAP ending at height $k \geqslant 0$ and avoiding the occurrences of UD_i for $i \geqslant 1$. If the last step of P is U, then $k \geqslant 1$ and we have $P = QU$, where Q is a length $(n-1)$ MAP ending at height $k-1$ and avoiding the peaks (Q can be the empty path). So, we obtain the first relation $f_k = zf_{k-1} + zg_{k-1} + zh_{k-1}$ for $k \geqslant 1$, anchored with $f_0 = 1$ by considering the empty path. If the last step of P is a down-step D_i, $i \geqslant 1$, then we have $P = QD_i$, where Q is a length $(n-1)$ PMAP ending at height $\ell \geqslant k+1$ with no up- and down-steps at its end, and with no peaks. So, we obtain the second relation $g_k = z \sum_{\ell \geqslant k+1} h_\ell$. If the last step of P is a horizontal step H, then we have $P = QH$, where Q is a length $(n-1)$ PMAP ending at height k with no peaks, which implies that $h_k = z(f_k + g_k + h_k)$ for $k \geqslant 0$.

Therefore, we have to solve the following system of equations:

$$\begin{cases} f_0 = 1, \text{ and } f_k = zf_{k-1} + zg_{k-1} + zh_{k-1}, \quad k \geqslant 1, \\ g_k = z \sum_{\ell \geqslant k+1} h_\ell, \quad k \geqslant 0, \\ h_k = zf_k + zg_k + zh_k, \quad k \geqslant 0. \end{cases} \tag{1}$$

Multiplying by u^k the recursions in (1) and summing over k, we have:

$$\begin{aligned} F(u) &= 1 + z\sum_{k\geqslant 1} u^k f_{k-1} + z\sum_{k\geqslant 1} u^k g_{k-1} + z\sum_{k\geqslant 1} u^k h_{k-1} \\ &= 1 + zuF(u) + zuG(u) + zuH(u), \\ G(u) &= z\sum_{k\geqslant 0} u^k \Big(\sum_{\ell\geqslant k+1} h_\ell\Big) = z\sum_{k\geqslant 1} h_k(1 + u + \cdots + u^{k-1}) \\ &= z\sum_{k\geqslant 1} \frac{u^k - 1}{u-1} h_k = \frac{z}{u-1}(H(u) - H(1)), \\ H(u) &= zF(u) + zG(u) + zH(u). \end{aligned}$$

Notice that we have $F(1) - H(1) = 1$ by considering the difference of the first and third equations. Now, setting $h1 := H(1)$ and solving these functional equations, we obtain

$$F(u) = \frac{h1\, uz^2 + zu + z^2 - u - z + 1}{u^2z + z^2 - u - z + 1},$$

$$G(u) = -\frac{z\,(h1\, uz + zh1 - h1 + z)}{u^2z + z^2 - u - z + 1}, \quad H(u) = \frac{z\,(zh1 - u + 1)}{u^2z + z^2 - u - z + 1}.$$

In order to compute $h1$, we use the kernel method (see [2, 20]) on $H(u)$. We can write the denominator (which is a polynomial in u of degree 2), as $z(u-r)(u-s)$ with

$$r = \frac{1 + \sqrt{-4\,z^3 + 4\,z^2 - 4\,z + 1}}{2z} \text{ and } s = \frac{1 - \sqrt{-4\,z^3 + 4\,z^2 - 4\,z + 1}}{2z}.$$

Plugging $u = s$ (which has a Taylor expansion at $z = 0$) in $H(u)z(u-r)(u-s)$, we obtain the equation $zh1 - s + 1 = 0$, which implies that

$$h1 = \frac{s-1}{z}.$$

Finally, after simplifying by the factor $(u-s)$ in the numerators and denominators, we obtain

$$F(u) = \frac{r}{r-u}, \quad G(u) = \frac{s-1}{r-u}, \quad \text{and} \quad H(u) = \frac{1}{r-u},$$

which induces that

$$f_k = [u^k]F(u) = \frac{1}{r^k}, \quad g_k = [u^k]G(u) = \frac{s-1}{r^{k+1}}, \quad \text{and} \quad h_k = [u^k]H(u) = \frac{1}{r^{k+1}}.$$

It is worth noticing that the system of equations for $F(u)$, $G(u)$, and $H(u)$ can be solved directly using the algorithm presented in [12].

Theorem 1 *The bivariate generating function for the total number of peak-less PMAP with respect to the length and the height of the end-point is given by*

$$Total(z,u) = \frac{1}{z(r-u)},$$

and we have

$$[u^k]\,Total(z,u) = \frac{1}{zr^{k+1}}.$$

Finally, setting $t(n,k) = [z^n][u^k]\,Total(z,u)$, we have for $n \geqslant 2$ and $k \geqslant 1$,

$$t(n,k) = t(n,k-1) + t(n-1,k) - t(n-1,k-2) - t(n-2,k),$$

and setting $t_n := t(n,0)$, then we have

$$t_n = t_{n-1} + \sum_{k=0}^{n-3} t_k t_{n-k-3} + \sum_{k=2}^{n-1} (t_k - t_{k-1})\, t_{n-k-1}.$$

Proof. The first three equalities are immediately deduced from the previous results. Now, let us prove the last equality. Any length n peak-less MAP is of the form (i) HP where P is a MAP of length $n-1$, or (ii) $UQHDR$, where Q, R are some MAP such that the length of Q lies into $[0, n-3]$, or (iii) $P^\sharp Q$, where $P^\sharp = UP'D_i$, $i \geqslant 2$, and $P'D_{i-1}$ is a MAP of length lying into $[2, n-1]$. The number of $P'D_{i-1}$ of a given length k is the total number of peak-less MAP of length k minus the number of peak-less MAP of length k and ending with H. Taking into account all these cases, we obtain the result. □

Corollary 1 *The generating function that counts all peak-less PMAP with respect to the length is given by*

$$Total(z, 1) = \frac{1}{z(r-1)}.$$

The first few terms of the series expansion of $Total(z, 1)$ are

$$1 + 2z + 4z^2 + 9z^3 + 22z^4 + 56z^5 + 146z^6 + 388z^7 + 1048z^8 + 2869z^9 + O(x^{10}),$$

which corresponds to the sequence A152225 in [27] counting Dyck paths of semilength $n+1$ with no peaks of height 0 (mod 3) and no valleys of height 2 (mod 3); see [14].

Corollary 2 *The generating function that counts the peak-less MAP with respect to the length is given by*

$$Total(z, 0) = \frac{1}{zr}.$$

The first few terms of the series expansion of $Total(z, 0)$ are

$$1 + z + z^2 + 2z^3 + 5z^4 + 12z^5 + 29z^6 + 73z^7 + 190z^8 + 505z^9 + O(z^{10}),$$

which corresponds to the sequence A152171 in [27] counting Dyck paths of length $2n$ with no peaks of height 2 (mod 3) and no valleys of height 1 (mod 3). In Section 2.2, we will exhibit a constructive bijection between these two classes of paths.

Let $\mathcal{T}$ be the infinite matrix $\mathcal{T} := [t(n,k)]_{n,k\geqslant 0}$, where $t(n,k) = [z^n][u^k]\,Total(z,u)$. The first few rows of the matrix $\mathcal{T}$ are

$$\mathcal{T} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 3 & 3 & 1 & 0 & 0 & 0 & 0 & 0 \\ 5 & 6 & 6 & 4 & 1 & 0 & 0 & 0 & 0 \\ 12 & 15 & 13 & 10 & 5 & 1 & 0 & 0 & 0 \\ 29 & 38 & 33 & 24 & 15 & 6 & 1 & 0 & 0 \\ 73 & 96 & 87 & 63 & 40 & 21 & 7 & 1 & 0 \\ 190 & 248 & 229 & 172 & 110 & 62 & 28 & 8 & 1 \end{pmatrix}.$$

In Figure 1 we show the peak-less PMAP counted by $t(5,1) = 15$.

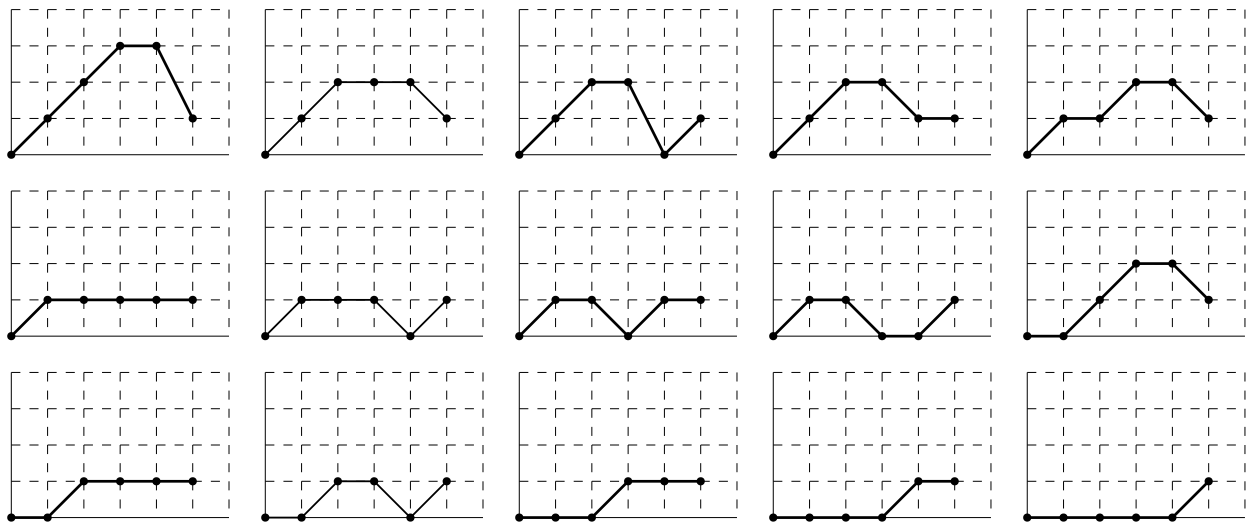


Figure 1: Peak-less PMAP of length 5 ending at height 1.

Corollary 3 *The matrix $\mathcal{T} = [t(n,k)]_{n,k\geqslant 0}$ is a Riordan array defined by*

$$\left(C\left(z(1-z+z^2)\right), zC\left(z(1-z+z^2)\right)\right),$$

where $C(z) = \frac{1-\sqrt{1-4z}}{2z}$ is the generating function of the Catalan numbers $c_n = \frac{1}{n+1}\binom{2n}{n}$.

Proof. Indeed, we directly deduce the result from the following.

$$[u^k]\,Total(z,u) = \frac{1}{zr^{k+1}} = \frac{1}{zr}\cdot\frac{1}{r^k} = C\left(z(1-z+z^2)\right)\cdot\left(zC\left(z(1-z+z^2)\right)\right)^k.$$

□

Corollary 4 *We have*

$$t(n,k) = \sum_{j=0}^{n-k} \frac{k+1}{2(n-j)-k+1}\binom{2(n-j)-k+1}{n-k-j} a(n-k-j,j),$$

where $a(n,k) = (-1)^k \sum_{i=0}^{n} \binom{n}{i}\binom{n-i}{k-2i}$.

Proof. From the Lagrange Inversion Formula (cf. [16, 28]), we know that

$$C(z)^k = \sum_{n\geqslant 0} \frac{k}{2n+k}\binom{2n+k}{n} z^n. \tag{2}$$

From definition of Riordan arrays and Eq. (2) we have

$$\begin{aligned}
t(n,k) &= [z^{n-k}]\left(C\left(z(1-z+z^2)\right)\right)^{k+1} \\
&= [z^{n-k}]\sum_{\ell=0}^{\infty}\frac{k+1}{2\ell+k+1}\binom{2\ell+k+1}{\ell}z^{\ell}(1-z+z^2)^{\ell} \\
&= [z^{n-k}]\sum_{\ell=0}^{\infty}\frac{k+1}{2\ell+k+1}\binom{2\ell+k+1}{\ell}z^{\ell}\sum_{j=0}^{2\ell}a(\ell,j)z^{j} \\
&= [z^{n-k}]\sum_{j=0}^{\infty}\sum_{\ell=0}^{\infty}\frac{k+1}{2(\ell+\lfloor j/2\rfloor)+k+1}\binom{2(\ell+\lfloor j/2\rfloor)+k+1}{\ell+\lfloor j/2\rfloor}a(\ell+\lfloor j/2\rfloor,j)z^{j+\ell+\lfloor j/2\rfloor}.
\end{aligned}$$

If we take $s = j+\ell+\lfloor j/2\rfloor$, then

$$\begin{aligned}
t(n,k) &= [z^{n-k}]\sum_{j=0}^{\infty}\sum_{s=j}^{\infty}\frac{k+1}{2(s-j)+k+1}\binom{2(s-j)+k+1}{s-j}a(s-j,j)z^{s} \\
&= \sum_{j=0}^{n-k}\frac{k+1}{2(n-j)-k+1}\binom{2(n-j)-k+1}{n-k-j}a(n-k-j,j).
\end{aligned}$$

□

In [24], Rogers gave an equivalent characterization of the Riordan arrays. That is, every element not belonging to row 0 or column 0 in a Riordan array can be expressed as a fixed linear combination of the elements in the preceding row. The *A-sequence* is defined to be the sequence coefficients of this linear combination. Analogously, Merlini et al. [18] introduced the *Z-sequence*, that characterizes the elements in column 0, except for the top one.

An infinite lower triangular matrix $[d_{n,k}]_{n,k\geqslant 0}$ is a Riordan array if and only if $d_{0,0}\neq 0$ and there exist two sequences $(a_0,a_1,a_2,\dots)$, with $a_0\neq 0$, and $(z_0,z_1,z_2,\dots)$ (called the A-sequence and the Z-sequence, respectively), such that

$$d_{n+1,k+1} = a_0 d_{n,k} + a_1 d_{n,k+1} + a_2 d_{n,k+2} + \cdots \qquad \text{for } n,k\geqslant 0, \tag{3}$$

$$d_{n+1,0} = z_0 d_{n,0} + z_1 d_{n,1} + z_2 d_{n,2} + \cdots \qquad \text{for } n\geqslant 0. \tag{4}$$

The product of two Riordan arrays $(g(z),f(z))$ and $(h(z),l(z))$ is defined by

$$(g(z),f(z))*(h(z),l(z)) = (g(z)h(f(z)),l(f(z)))\,. \tag{5}$$

Under this operation, the set of all Riordan arrays is a group [25]. The identity element is $I=(1,z)$ and the inverse of $(g(z),f(z))$ is given by

$$(g(z),f(z))^{-1} = \left(1/\left(g\circ f^{<-1>}\right)(z), f^{<-1>}(z)\right), \tag{6}$$

where $f^{<-1>}(z)$ denotes the compositional inverse of $f(z)$.

The generating functions for the A-sequence and Z-sequence of the Riordan array $\mathcal{F} = (g(z), f(z))$, with inverse $\mathcal{F}^{-1} = (d(z), h(z))$, are given by ([18, 11])

$$A(z) = \frac{z}{h(z)} \quad \text{and} \quad Z(z) = \frac{1}{h(z)}\left(1 - d_{0,0}d(z)\right),$$

respectively.

From the definition of the A-sequence and Z-sequence for the Riordan arrays we can give an additional recurrence relation for the sequence $t(n,k)$.

Corollary 5 *We have*

$$t(n+1, k+1) = \sum_{j \geqslant 0} a(j)t(n, k+j),$$

where $a(n) = (-1)^{n+1} \sum_{k=1}^{n} \sum_{j=0}^{k} \frac{1}{k} \binom{j}{n-k-j} \binom{k}{j} \binom{n-k-2}{k-1}$ *for* $n \geqslant 1$ *and* $a(0) = 1$. *Moreover,*

$$t_{n+1} = \sum_{j \geqslant 0} a(j+1)t(n, j).$$

Proof. By Equation (6), the inverse of the matrix $\mathcal{T} = [t(n,k)]_{n,k \geqslant 0}$ is given by $\mathcal{T}^{-1} = (g_2(z), zg_2(z))$, where

$$g_2(z) = \frac{-1 + z^2 + \sqrt{1 - 2z^2 + 4z^3 - 3z^4}}{2z^3}.$$

Therefore, the A-sequence and Z-sequence of the Riordan array $\mathcal{T}$ have generating functions

$$A(z) = \sum_{n \geqslant 0} a(n)z^n = \frac{2z^3}{-1 + z^2 + \sqrt{1 - 2z^2 + 4z^3 - 3z^4}} \quad \text{and} \quad Z(z) = \frac{A(z) - 1}{z}.$$

The generating function $A(z)$ corresponds with the sequence A247162, where the explicit formula for $a(n)$ can be found. From (3) we obtain the result. □

The first few values of the sequence $a(n)$ for $n \geqslant 0$ are

$$1, \quad 1, \quad 0, \quad 1, \quad 0, \quad 1, \quad -1, \quad 2, \quad -3, \quad 6, \quad -10, \ldots.$$

2.2 A bijective approach

Corollary 2 proves that the set of peak-less Motzkin paths with air pockets of length n (ending on the x-axis) is equinumerous to the set $\mathcal{D}_n(2,1)$ of Dyck paths of length $2n$ with no peak at height 2 (mod 3) and no valley at height 1 (mod 3).

Any non-empty peak-less Motzkin path with air pockets is either of the form (1) $H\alpha$ or (2) $U\alpha_1 U\alpha_2 \cdots U\alpha_k H D_k \beta$, where $k \geqslant 1$ and $\alpha, \alpha_1, \ldots, \alpha_k, \beta$ are possibly empty peak-less MAP. We refer to the left part of Figure 2 for an illustration of this form.

Remember that $\mathcal{MP}$ denotes the set of Motzkin paths with air pockets of the first kind.

Definition 1 *We recursively define the map ψ from $\mathcal{MP}$ to $\cup_{n\geqslant 0}\mathcal{D}_n(2,1)$ as follows. For $\alpha \in \mathcal{MP}$, we set:*

$$\psi(P) = \begin{cases} \epsilon & \text{if } P = \epsilon \quad (i) \\ UD\psi(\alpha) & \text{if } P = H\alpha \text{ with } \alpha \in \mathcal{MP} \quad (ii) \\ U^3\psi(\alpha_1)DU\psi(\alpha_2)DU\ldots DU\psi(\alpha_k)D^3\psi(\beta) & \text{if } P = U\alpha_1 U\alpha_2 \ldots U\alpha_k HD_k\beta \text{ with} \\ & k \geqslant 1 \text{ and } \alpha_1,\ldots,\alpha_k,\beta \in \mathcal{MP}. \quad (iii) \end{cases}$$

We refer to Figure 3 for an illustration of the third case of the definition of ψ.

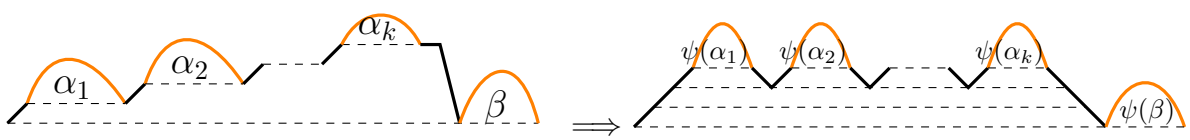


Figure 2: Illustration of the map ψ for the more general case (iii) in Definition 1.

Due to the recursive definition, the image of peak-less MAP of length n under ψ is a Dyck path of length $2n$. Moreover it is clear that the obtained path is a Dyck path in $\cup_{n\geqslant 0}\mathcal{D}_n(2,1)$. For instance (see Figure 3 for an illustration of this example).

$$\begin{aligned} \psi(UUHDHUHUHD_3HH) &= \psi(U \cdot \overbrace{UHDH}^{\alpha_1} \cdot U \cdot \overbrace{H}^{\alpha_2} \cdot U \cdot \overbrace{\epsilon}^{\alpha_3} \cdot HD_3 \cdot \overbrace{HH}^{\beta} \\ &= U^3\psi(UHDH) \cdot DU \cdot \psi(H) \cdot DU \cdot \psi(\epsilon) \cdot D^3 \cdot \psi(HH) \\ &= U^3 \cdot (U^3D^3\psi(H)) \cdot DU \cdot UD \cdot DU \cdot D^3 \cdot UDUD \\ &= U^6D^3UD^2U^2D^2UD^2DUDUD. \end{aligned}$$

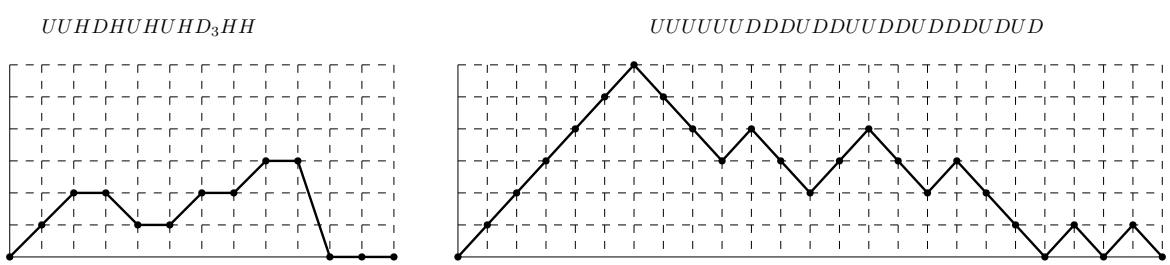


Figure 3: $\psi(UUHDHUHUHD_3HH) = UUUUUUDDDUDDUUDDUDDDUDUD$.

Theorem 2 *For all $n \geqslant 0$, the map ψ induces a bijection between $\mathcal{MP}_n$ and $\mathcal{D}_n(2,1)$.*

Proof. Since $\mathcal{MP}_n$ and $\mathcal{D}_n(2,1)$ have the same cardinality (due to Corollary 2 and A152171 in [27]), it suffices to prove that for $P, Q \in \mathcal{MP}$, $P \neq Q$ implies $\psi(P) \neq \psi(Q)$. A simple induction on n allows to obtain the result. □

3 Partial valley-less Motzkin paths with air pockets

Using the same arguments we used for the system of the previous section, we study partial Motzkin paths with air pockets of the first kind avoiding occurrences of D_iU for all $i \geqslant 1$.

3.1 Enumerative results

In the same way as we done in Section 2.1, we have to solve the following system of equations:

$$\begin{cases} f_0 = 1, \text{ and } f_k = zf_{k-1} + zh_{k-1}, \quad k \geqslant 1, \\ g_k = z \sum\limits_{\ell \geqslant k+1} f_\ell + z \sum\limits_{\ell \geqslant k+1} h_\ell, \quad k \geqslant 0, \\ h_k = zf_k + zg_k + zh_k, \quad k \geqslant 0. \end{cases} \tag{7}$$

Multiplying by u^k the recursions in (7) and summing over k, we have:

$$\begin{aligned} F(u) &= 1 + zuF(u) + zuH(u), \\ G(u) &= \frac{z}{u-1}(F(u) - F(1) + H(u) - H(1)), \\ H(u) &= zF(u) + zG(u) + zH(u). \end{aligned}$$

This system of equations for $F(u)$, $G(u)$, and $H(u)$ can be solved directly using the algorithm presented in [12]. However we provide the main steps for solving it.

Notice that we have $(1-z)F(1) = 1 + zH(1)$ by considering the first equation. Now, setting $f1 := F(1)$ and solving these functional equations, we obtain

$$F(u) = \frac{f1\, uz^2 - uz^2 + zu + z^2 - u - z + 1}{u^2z + z^2 - u - z + 1},$$

$$G(u) = -\frac{f1\, uz + f1\, z - zu - f1 + 1}{u^2z + z^2 - u - z + 1}, \quad H(u) = -\frac{z\,(f1\, uz - zu - f1 + u + z)}{u^2z + z^2 - u - z + 1}.$$

In order to compute $f1$, we use the kernel method (see [2, 20]) on $F(u)$. We can write the denominator (which is a polynomial in u of degree 2), as $z(u-r)(u-s)$ with

$$r = \frac{1 + \sqrt{-4\,z^3 + 4\,z^2 - 4\,z + 1}}{2z} \quad \text{and} \quad s = \frac{1 - \sqrt{-4\,z^3 + 4\,z^2 - 4\,z + 1}}{2z}.$$

Plugging $u = s$ (which has a Taylor expansion at $z = 0$) in $F(u)z(u-r)(u-s)$, we obtain the equation $f1\, sz^2 - sz^2 + zs + z^2 - s - z + 1 = 0$, which implies that

$$f1 = 1 + \frac{s-1}{z}.$$

Finally, after simplifying by the factor $(u-s)$ in the numerators and denominators, we obtain

$$F(u) = \frac{r}{r-u}, \quad G(u) = \frac{s-1}{z(r-u)}, \quad \text{and} \quad H(u) = \frac{s}{r-u},$$

which implies that

$$f_k = [u^k]F(u) = \frac{1}{r^k}, \quad g_k = [u^k]G(u) = \frac{s-1}{zr^{k+1}}, \quad \text{and} \quad h_k = [u^k]H(u) = \frac{s}{r^{k+1}}.$$

Theorem 3 *The bivariate generating function for the total number of partial valley-less MAP with respect to the length and the height of the end-point is given by*

$$Total(z,u) = \frac{s}{z(r-u)},$$

and we have

$$[u^k]\,Total(z,u) = \frac{s}{zr^{k+1}}.$$

Finally, setting $t(n,k) = [z^n][u^k]\,Total(z,u)$*, we have for* $n \geqslant 2$*,* $k \geqslant 1$*,*

$$t(n,k) = t(n,k-1) + t(n-1,k) - t(n-1,k-2) - t(n-2,k),$$

and setting $t_n := t(n,0)$ *and* $t_{-1} = 1$*, then we have for* $n \geqslant 2$*,*

$$t_{n-1} = t_{n-2} + \sum_{k=0}^{n-3} t_{k-1}t_{n-k-4} + \sum_{k=2}^{n-1} (t_{k-1} - t_{k-2})\, t_{n-k-2}.$$

Proof. The first three equalities are immediately deduced from the previous results. For the last equality, the term t_n satisfies the same recurrence relation as in Theorem 1 (modulo shift of n) since the two sequences are equal modulo a shift. □

Corollary 6 *The generating function that counts the partial valley-less MAP with respect to the length is given by*

$$Total(z,1) = \frac{s}{z(r-1)}.$$

The first few terms of the series expansion of $Total(z,1)$ are

$$1 + 2z + 5z^2 + 13z^3 + 34z^4 + 90z^5 + 242z^6 + 660z^7 + 1821z^8 + 5073z^9 + O(x^{10}),$$

which does not appear in [27].

Corollary 7 *The generating function that counts the partial valley-less MAP with respect to the length is given by*

$$Total(z,0) = \frac{s}{zr}.$$

The first few terms of the series expansion of $Total(z,0)$ are

$$1+z+2z^2+5z^3+12z^4+29z^5+73z^6+190z^7+505z^8+1363z^9+O(z^{10}),$$

which corresponds to a shift of the sequence A152171 in [27] counting Dyck paths of length $2n$ with no peaks of height 2 (mod 3) and no valleys of height 1 (mod 3) (see also Section 3.1).

Let $\mathcal{T}$ be the infinite matrix $\mathcal{T} := [t(n,k)]_{n,k\geqslant 0}$. The first few rows of the matrix $\mathcal{T}$ are

$$\mathcal{T} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 5 & 4 & 3 & 1 & 0 & 0 & 0 & 0 & 0 \\ 12 & 10 & 7 & 4 & 1 & 0 & 0 & 0 & 0 \\ 29 & 26 & 18 & 11 & 5 & 1 & 0 & 0 & 0 \\ 73 & 67 & 49 & 30 & 16 & 6 & 1 & 0 & 0 \\ 190 & 175 & 133 & 85 & 47 & 22 & 7 & 1 & 0 \\ 505 & 467 & 361 & 241 & 139 & 70 & 29 & 8 & 1 \end{pmatrix}.$$

In Figure 4 we show the valley-less PMAP counted by $t(4,0)=12$.

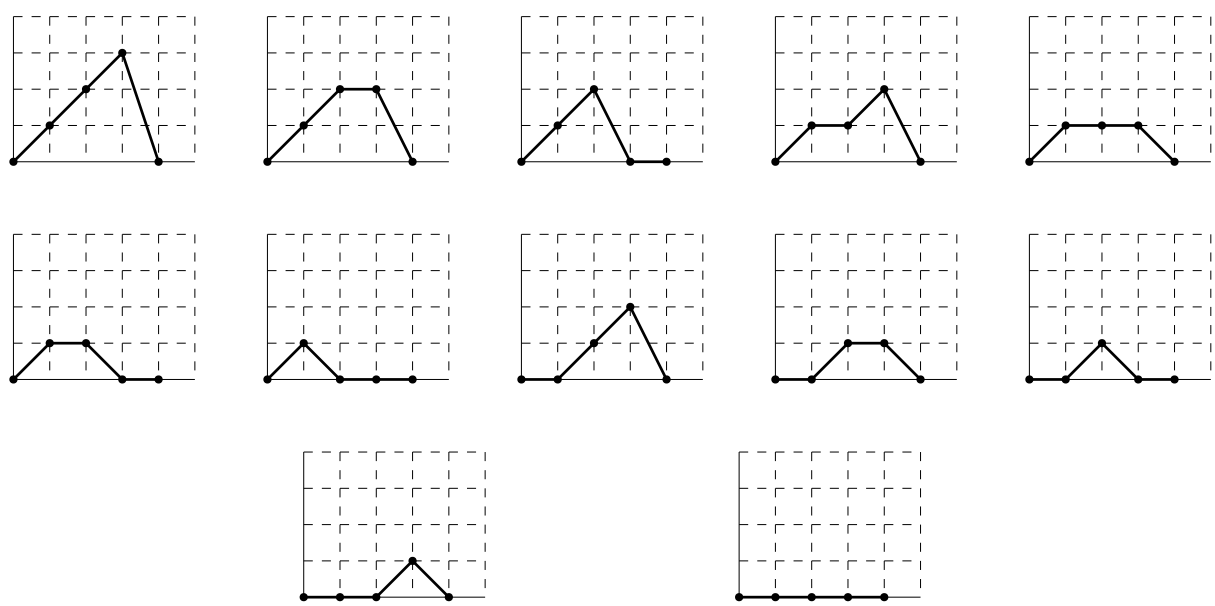


Figure 4: The 12 UU-less PMAP of length 4 ending at height 0.

Corollary 8 *The matrix $\mathcal{T} = [t(n,k)]_{n,k\geqslant 0}$ is a Riordan array defined by*

$$\left((z^2-z+1)C\left(z(1-z+z^2)\right)^2, zC\left(z(1-z+z^2)\right)\right)$$

where $C(z) = \frac{1-\sqrt{1-4z}}{2z}$ *is the generating function of the Catalan numbers* $c_n = \frac{1}{n+1}\binom{2n}{n}$.

Proof. Indeed, we directly deduce the result from the following.

$$[u^k]\,Total(z,u) = \frac{s}{zr^{k+1}} = \frac{s}{zr}\cdot\frac{1}{r^k} = (z^2-z+1)C\left(z(1-z+z^2)\right)^2\cdot\left(zC\left(z(1-z+z^2)\right)\right)^k.$$

□

From a similar argument as in Corollary 4 we obtain the following result.

Corollary 9 *We have*

$$t(n,k) = \sum_{j=0}^{n-k}\frac{k+2}{2(n-j)-k+2}\binom{2(n-j)-k+2}{n-k-j}a(n-k-j+1,j),$$

where $a(n,k) = (-1)^k\sum_{i=0}^{n}\binom{n}{i}\binom{n-i}{k-2i}$.

3.2 A bijective approach

Corollary 7 and Corollary 2 prove that the set of valley-less Motzkin paths with air pockets of length $n-1$ (ending on the x-axis) is equinumerous to the set of peak-less Motzkin paths with air pockets of length n, which is in one-to-one correspondence with the set $\mathcal{D}_n(2,1)$ of Dyck paths of length $2n$ with no peak at height 2 (mod 3) and no valley at height 1 (mod 3) (see a constructive bijection in Section 2.2). Below, we provide a bijection between valley-less MAP of length $n-1$ and peak-less MAP of length n.

Any valley-less Motzkin path with air pockets is either of the form (i) ϵ, (ii) αH, (iii) $U\alpha D$, (iv) $\beta HU\alpha D$, (v) $U\gamma D_k$, or (vi) $\beta HU\gamma D_k$, where α,β are valley-less MAP (possibly empty), and γD_{k-1} is a valley-less MAP. According to all these cases, we define the map ϕ.

Definition 2 *We recursively define the map* ϕ *from valley-less MAP of length* $n-1$ *to peak-less MAP of length* n. *Let* P *be a valley-less MAP, we set:*

$$\phi(P) = \begin{cases} H & \text{if } P=\epsilon, & (i)\\ \phi(\alpha)H & \text{if } P=\alpha H, & (ii)\\ U\phi(\alpha)D & \text{if } P=U\alpha D, & (iii)\\ \phi(\beta)U\phi(\alpha)D & \text{if } P=\beta HU\alpha D, & (iv)\\ \phi(\alpha D_{k-1})^\sharp & \text{if } P=U\alpha D_k, & (v)\\ \phi(\beta)\phi(\alpha D_{k-1})^\sharp & \text{if } P=\beta HU\alpha D_k, & (vi)\end{cases}$$

where the $\sharp$*-operator maps a peak-less MAP of the form* αD_{k-1} *into the peak-less MAP* $(\alpha D_{k-1})^\sharp = U\alpha D_k$.

Due to the recursive definition, the image of valley-less MAP of length $n-1$ under ϕ is a peak-less MAP of length n. The recursive definition naturally induces that ϕ is a bijection. Using the bijection ψ presented above, we can easily obtain a constructive bijection between valley-less MAP of length $n-1$ and Dyck paths of length $2n$ with no peak at height 2 (mod 3) and no valley at height 1 (mod 3).

4 Partial UU-less Motzkin paths with air pockets

In this section, we study partial Motzkin paths with air pockets of the first kind avoiding occurrences of UU.

4.1 Enumerative results

In the same way as we done in the previous section, we have to solve the following system of equations:

$$\begin{cases} f_0 = 1, f_1 = z + zg_0 + zh_0, \text{ and } f_k = zg_{k-1} + zh_{k-1}, \quad k \geqslant 2, \\ g_k = z \sum_{\ell \geqslant k+1} f_\ell + z \sum_{\ell \geqslant k+1} h_\ell, \quad k \geqslant 0, \\ h_k = zf_k + zg_k + zh_k, \quad k \geqslant 0. \end{cases} \tag{8}$$

Multiplying by u^k the recursions in (8) and summing over k, we have:

$$\begin{aligned} F(u) &= 1 + zu + zuG(u) + zuH(u), \\ G(u) &= \frac{z}{u-1}(F(u) - F(1) + H(u) - H(1)), \\ H(u) &= zF(u) + zG(u) + zH(u). \end{aligned}$$

Again, this system of equations for $F(u)$, $G(u)$, and $H(u)$ can be solved directly using the algorithm presented in [12]. However we provide the main steps for solving it.

Notice that we have $H(1) = (1+z)(F(1)-1)$ by considering the first and the last equation. Now, setting $f1 := F(1)$ and solving these functional equations, we obtain

$$F(u) = \frac{f1\,u\,z^3 + 2f1\,u\,z^2 + u^2z^2 - u^2z - 2u\,z^2 + 2uz + z^2 - u - z + 1}{u^2z^2 + u\,z^3 + uz + z^2 - u - z + 1},$$

$$G(u) = -\frac{z\,(f1\,u\,z^3 + 2f1\,u\,z^2 - u\,z^3 + f1\,z^2 - u\,z^2 + f1\,z + uz - z^2 - 2f1 + 2)}{u^2z^2 + u\,z^3 + uz + z^2 - u - z + 1},$$

$$H(u) = \frac{z\,(f1\,u\,z^3 + 2f1\,u\,z^2 - u\,z^3 + f1\,z^2 - u^2z - 2u\,z^2 + 2f1\,z + uz - z^2 - u - 2z + 1)}{u^2z^2 + u\,z^3 + uz + z^2 - u - z + 1}.$$

In order to compute $f1$, we use the kernel method (see [2, 20]) on $F(u)$. We can write the denominator (which is a polynomial in u of degree 2), as $z^2(u-r)(u-s)$, with

$$r = \frac{-z^3 - z + 1 + \sqrt{z^6 - 2\,z^4 + 2\,z^3 - 3\,z^2 - 2\,z + 1}}{2z^2} \quad \text{and}$$

$$s = \frac{1 - z - z^3 - \sqrt{z^6 - 2\,z^4 + 2\,z^3 - 3\,z^2 - 2\,z + 1}}{2z^2}.$$

Plugging $u = s$ (which has a Taylor expansion at $z = 0$) in $F(u)z^2(u-r)(u-s)$, we obtain the equation $f1\,s\,z^3 + 2f1\,s\,z^2 + s^2z^2 - s^2z - 2s\,z^2 + 2sz + z^2 - u - z + 1 = 0$, which implies that

$$f1 = 1 + \frac{s-1}{z(z+2)}.$$

Finally, after simplifying by the factor $(u-s)$ in the numerators and denominators, we obtain

$$F(u) = \frac{z-1}{z} + \frac{r}{z(r-u)}, \quad G(u) = \frac{s+z}{r-u}, \quad \text{and} \quad H(u) = \frac{zr+1}{z(r-u)} - 1,$$

which implies that $f_0 = 1$, $g_0 = \frac{s+z}{r}$, $h_0 = \frac{1}{zr}$ and

$$f_k = [u^k]F(u) = \frac{1}{zr^k}, \quad g_k = [u^k]G(u) = \frac{s+z}{r^{k+1}}, \quad \text{and} \quad h_k = [u^k]H(u) = \frac{1+zr}{zr^{k+1}}.$$

Theorem 4 *The bivariate generating function for the total number of UU-less PMAP with respect to the length and the height of the end-point is given by*

$$Total(z,u) = \frac{1+uz}{(r-u)z^2},$$

and we have

$$[u^0]\,Total(z,u) = \frac{1}{z^2r},$$

and for $k \geqslant 1$

$$[u^k]\,Total(z,u) = \frac{rz+1}{z^2r^{k+1}}.$$

Finally, setting $t(n,k) = [z^n][u^k]\,Total(z,u)$, we have for $n \geqslant 2$, $k \geqslant 1$,

$$t(n,k) = t(n,k-1) + t(n-1,k) - t(n-1,k-1) - t(n-2,k) \\ - t(n-2,k-2) - t(n-3,k-1).$$

and setting $t_n := t(n,0)$, then we have

$$t_n = t_{n-1} + t_{n-2} + t_{n-3} + \sum_{k=2}^{n-2} t_{k-2}t_{n-k-2} + \sum_{k=3}^{n-1}(t_{k-1} - t_{k-2})t_{n-k-1}.$$

Proof. The first three equalities are immediately deduced from the previous results. Now, let us prove the last equality. Any non-empty length n UU-less MAP is of the form (i) HP where P is a UU-less MAP of length $n-1$, or (ii) UDQ where Q is a MAP of length $n-2$ avoiding UU, or (iii) $UQDR$, where Q, R are some MAP avoiding UU such that the length of Q lies into $[1, n-2]$ and Q starts and ends with H, or (iv) PQ, where $P = UP'D_i$, $i \geqslant 2$, and $P'D_{i-1}$ is a MAP of length lying into $[3, n-1]$ and starting with H. The number of $P'D_{i-1}$ of a given length k is the total number of UU-less MAP of length $k-1$ minus the total number of UU-less MAP of length $k-2$. Taking into account all these cases, we obtain the result.

□

Corollary 10 *The generating function that counts the partial UU-less MAP with respect to the length is given by*

$$Total(z,1) = \frac{1+z}{(r-1)z^2}.$$

The first few terms of the series expansion of $Total(z,1)$ are

$$1 + 2z + 4z^2 + 9z^3 + 21z^4 + 50z^5 + 122z^6 + 302z^7 + 759z^8 + 1928x^9 + O(x^{10}),$$

which does not appear in [27].

Corollary 11 *The generating function that counts UU-less MAP with respect to the length is given by*

$$Total(z,0) = \frac{1}{z^2 r}.$$

The first few terms of the series expansion of $Total(z,0)$ are

$$1 + z + 2z^2 + 4z^3 + 9z^4 + 20z^5 + 47z^6 + 112z^7 + 274z^8 + 679x^9 + O(z^{10}),$$

which corresponds to the sequence A095980 in [27] counting Motzkin paths of length n with no occurrences of UHU.

Let $\mathcal{T}$ be the infinite matrix $\mathcal{T} := [t(n,k)]_{n,k\geqslant 0}$. The first few rows of the matrix $\mathcal{T}$ are

$$\mathcal{T} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 9 & 9 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 20 & 21 & 8 & 1 & 0 & 0 & 0 & 0 & 0 \\ 47 & 50 & 21 & 4 & 0 & 0 & 0 & 0 & 0 \\ 112 & 121 & 55 & 13 & 1 & 0 & 0 & 0 & 0 \\ 274 & 298 & 143 & 39 & 5 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

In Figure 5 we show the UU-less PMAP counted by $t(4,0) = 9$.

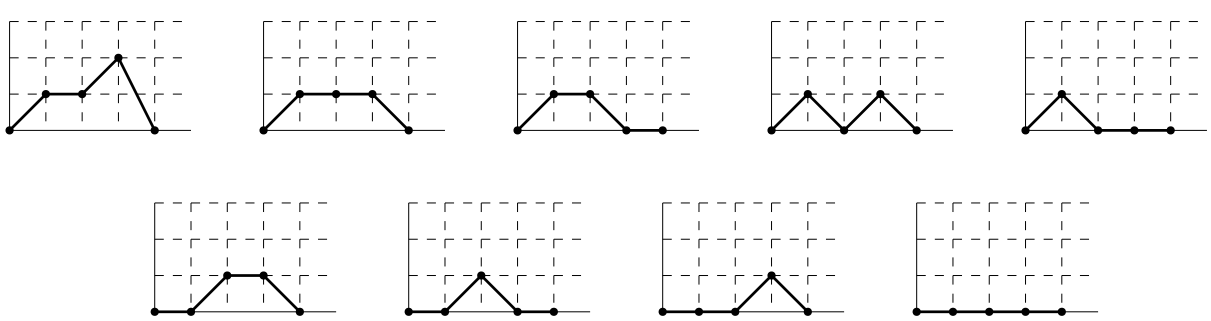


Figure 5: The 9 UU-less PMAP of length 4 ending at height 0.

The matrix $\mathcal{G}$ is not a (proper) Riordan array. For this reason, we consider the matrix $\mathcal{G} := [g(n,k)]_{n\geqslant 0, k\geqslant 0}$, where

$$g(n,k) = \begin{cases} 1, & \text{if } n = k = 0; \\ t(n+k-1,k), & \text{if } n \geqslant 1. \end{cases}$$

The first few rows of the matrix $\mathcal{G}$ are

$$\mathcal{G} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 4 & 3 & 1 & 0 & 0 & 0 & 0 & 0 \\ 4 & 9 & 8 & 4 & 1 & 0 & 0 & 0 & 0 \\ 9 & 21 & 21 & 13 & 5 & 1 & 0 & 0 & 0 \\ 20 & 50 & 55 & 39 & 19 & 6 & 1 & 0 & 0 \\ 47 & 121 & 143 & 113 & 64 & 26 & 7 & 1 & 0 \\ 112 & 298 & 372 & 319 & 203 & 97 & 34 & 8 & 1 \end{pmatrix}.$$

Corollary 12 *The matrix $\mathcal{G} = [g(n,k)]_{n,k\geqslant 0}$ is the Riordan array defined by $(t(z), t(z)-1)$, where*

$$t(z) = \frac{1 + z(1-z)^2 - \sqrt{(1-3z+z^3)(1+z+z^3)}}{2z(1-z+z^2)}.$$

Proof. It follows from the relation $t(z) = \frac{1}{zr} + 1$. □

It seems difficult to obtain a close form for the coefficient $g(n,k)$ using the Lagrange Inversion Formula. Indeed, $t(z)$ can be expressed in terms of $C(u)$, where $u = \frac{1}{4}(-z^6 + 2z^4 - 2z^3 + 3z^2 + 2z)$ is a polynomial of degree 6, which complicates the calculations.

4.2 A bijective approach

Corollary 11 proves that the set of UU-less Motzkin paths with air pockets of length n (ending on the x-axis) is equinumerous to the set of Motzkin paths of length n avoiding UHU. Below, we provide a bijection between these two sets.

Any UU-less Motzkin path with air pockets is either of the form (i) ϵ, (ii) $H\alpha$, (iii) $UD\alpha$, (iv) $UHD\alpha$, (v) $UH\alpha HD\beta$, or (vi) $UH^k\gamma D_i\beta$, where α, β are UU-less MAP (possibly empty), and γD_{i-1} is a UU-less MAP and $k \geqslant 1$. According to all these cases, we define the map χ.

Definition 3 *We recursively define the map χ from UU-less MAP of length n to UHU-less Motzkin paths of length n. Let P be a UU-less MAP, we set:*

$$\chi(P) = \begin{cases} \epsilon & \text{if } P = \epsilon, & (i) \\ H\chi(\alpha) & \text{if } P = H\alpha, & (ii) \\ UD\chi(\alpha) & \text{if } P = UD\alpha, & (iii) \\ UHD\chi(\alpha) & \text{if } P = UHD\alpha, & (iv) \\ UHH\chi(\alpha)D\chi(\beta) & \text{if } P = UH\alpha HD\beta, & (v) \\ U\chi(\gamma D_{i-1})H^{k-1}D\chi(\beta) & \text{if } P = UH^k\gamma D_i\beta, & (vi) \end{cases}$$

Due to the recursive definition, the image of UU-less MAP of length n by χ is a UHU-less Motzkin path of length n. The recursive definition naturally induces that χ is a bijection. For instance, if $P = UHUHD_2UDUHUHHUD_3H$ then we obtain $\chi(P) = \chi(UH^1UHD_2)\chi(UD)\chi(UHUHHUD_3)\chi(H) = UUHDDUDUUUDHDDH$ (see Figure 6 for an illustration of this example).

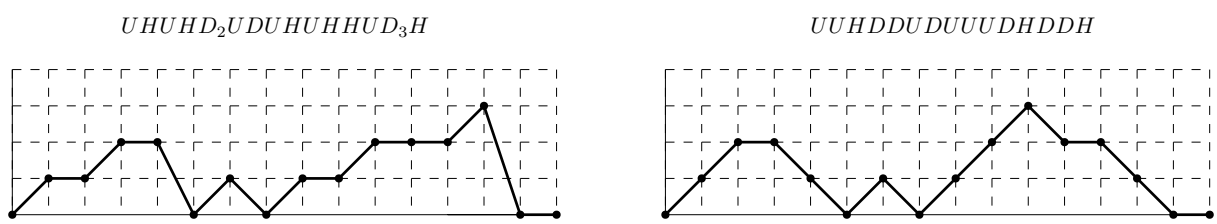


Figure 6: $\chi(P) = UUHDDUDUUUDHDDH$.

5 Going further

In [1] and [9] some interesting results on the generating functions of restricted lattice paths are obtained for a finite set of types of moves. Can we adapt these techniques to Motzkin paths with air pockets (avoiding some patterns? Considering [3, 4], is it possible to obtain an efficient algorithm for the exhaustive generation of partial Motzkin paths with air pockets? Is it possible to list PMAP of a given length in Gray code order using [5, 10]?

6 Statements and Declarations

Competing Interests: Authors have non-financial interests that are directly or indirectly related to the work submitted for publication.
Funding: The authors declare that no funds, grants, or other support were received during the preparation of this manuscript.

References

[1] A. Asinowski, C. Banderier, and V. Roitner. Generating functions for lattice paths with several forbidden patterns. *Sémin. Lothar. Comb.* (2020), 84B.

[2] C. Banderier, M. Bousquet-Mélou, A. Denise, P. Flajolet, and D. Gouyou-Beauchamps. Generating functions for generating trees. *Discrete Math.* **246** (2002), 29–55.

[3] E. Barcucci, A. Bernini, and R. Pinzani. Exhaustive generation of some lattice paths and their prefixes. *Theoret. Comput. Sci.* **878-879** (2021), 47–52.

[4] E. Barcucci, A. Bernini, and R. Pinzani. Exhaustive generation of positive lattice paths. *CEUR Workshop proceedings*, **2113** (2018), 79–86.

[5] E. Barcucci, A. Bernini, and R. Pinzani. A Gray code for regular languages. *CEUR Workshop proceedings*, **2113** (2018), 87–93.

[6] J.-L. Baril, S. Kirgizov, R. Maréchal, and V. Vajnovszki. Enumeration of Dyck paths with air pockets. *J. Integer Seq.* **26** (2023), Article 23.3.2.

[7] J.-L. Baril, S. Kirgizov, R. Maréchal, and V. Vajnovszki. Grand Dyck paths with air pockets. *Art Discrete Appl. Math.* **7** (2024), # P1.07.

[8] J.-L. Baril and P. Barry. Two kinds of partial Motzkin paths with air pockets. *Ars Math. Contemp.* 2024.

[9] C. Bean, A. Bernini, M. Cervetti, and L. Ferrari. On the generating functions of pattern-avoiding Motzkin paths. *J. Symbolic Comput.* **113** (2022), 126-–138.

[10] A. Bernini, S. Bilotta, R. Pinzani, and V. Vajnovszki. A trace partitioned Gray code for q-ary generalized Fibonacci strings. *J. Discret. Math. Sci. Cryptogr.*, **18** (2015), 751-–761.

[11] T.-X. He and R. Sprugnoli. Sequence characterization of Riordan arrays. *Discrete Math.* **309** (2009), 3962–3974.

[12] Q. Hou and T. Mansour. Kernel method and linear recurrence system. *J. Comput. Appl. Math.*, **261**(1) (2008), 227–242.

[13] A. Krinik, G. Rubino, D. Marcus, R.J. Swift, H. Kasfy, and H. Lam. Dual processes to solve single server systems. *Journal of Stat. Planning and Inference*, **135**(1) (2005), 121–147.

[14] S.-C. Liu, J. Ma, and Y.-N. Yeh. Dyck paths with peak- and valley- avoiding sets. *Stud. Appl. Math.* **121** (2008), 263–289.

[15] T. Mansour, M. Schork, and Y. Sun. Motzkin numbers of higher rank: Generating function and explicit expression, *J. Integer Seq.*, **10** (2007), Article 07.7.4.

[16] D. Merlini, R. Sprugnoli, and M. C. Verri. Lagrange inversion: when and how, *Acta Appl. Math.* **94** (2006), 233–249.

[17] T. Mansour and M. Shattuck. Counting humps and peaks in generalized Motzkin paths, *Discrete Appl. Math.*, **161** (2013), 2213–2216.

[18] D. Merlini, D. G. Rogers, R. Sprugnoli, and M. C. Verri. On some alternative characterizations of Riordan arrays. *Canadian J. Math.* **49** (1997), 301–320.

[19] D. Merlini, D. G. Rogers, R. Sprugnoli, and M. C. Verri. Underdiagonal lattice paths with unrestricted steps. *Discrete Appl. Math.* **91** (1999), 197–213.

[20] H. Prodinger. The kernel method: a collection of examples. *Sém. Lothar. Combin.* **50** (2004), B50f.

[21] H. Prodinger. Partial Dyck paths with air pockets. *Integers* **22** (2022), #A24.

[22] J. L. Ramírez and V. F. Sirvent. A generalization of the k-bonacci sequence from Riordan arrays. *Electron. J. Combin.* **22** (2015), # P1.38.

[23] J. L. Ramírez and V. F. Sirvent. Generalized Schröder matrix and its combinatorial interpretation. *Linear Multilinear Algebra* **66**(2) (2018), 418–433.

[24] D. G. Rogers. Pascal triangles, Catalan numbers and renewal arrays. *Discrete Math.* **22** (1978), 301–310.

[25] L. W. Shapiro, S. Getu, W. Woan, and L. Woodson, The Riordan group, *Discrete Appl. Math.* **34** (1991), 229–239.

[26] L. W. Shapiro, R. Sprugnoli, P. Barry, G.-S. Cheon, T.-X. He, D. Merlini, and W. Wang. *The Riordan Group and Applications*, Springer Monographs in Mathematics, Springer, 2022.

[27] N. J. A. Sloane. The On-line Encyclopedia of Integer Sequences. Available electronically at http://oeis.org.

[28] H. Wilf. *Generating functionology*, Academic Press, New York, 1990.

[29] S-.L. Yang, Y.-N. Dong, and T.-X. He. Some matrix identities on colored Motzkin paths. *Discrete Math.* **340** (2017), 3081–3091.

[30] S-.L. Yang and M. Jiang. The m-Schröder paths and m-Schröder numbers. *Discret. Math.* **344**(2) (2021), 112209.

[31] L. Yang, S.-L. Yang, and T.-X. He. Generalized Schröder matrices arising from enumeration of lattice paths. *Czechoslovak Math. J.* **70** (2020), 411–433.

Flattened Catalan Words

Jean-Luc Baril

LIB, Université de Bourgogne, B.P. 47 870, 21078, Dijon Cedex, France
barjl@u-bourgogne.fr

Pamela E. Harris

Department of Mathematical Sciences, University of Wisconsin-Milwaukee, Milwaukee, WI 53211 United States
peharris@uwm.edu

José L. Ramírez

Departamento de Matemáticas, Universidad Nacional de Colombia, Bogotá, Colombia
jlramirezr@unal.edu.co

Abstract

In this work, we define flattened Catalan words as Catalan words whose runs of weak ascents have leading terms that appear in weakly increasing order. We provide generating functions, formulas, and asymptotic expressions for the number of flattened Catalan words based on the number of runs of ascents (descents), runs of weak ascents (descents), ℓ-valleys, valleys, symmetric valleys, ℓ-peaks, peaks, and symmetric peaks.

Key words and phrases: Catalan word; generating function; combinatorial statistic; Dyck path; flattened words

AMS (MOS) Subject Classifications: 05A15, 05A19

1 Introduction

A word $w = w_1 w_2 \cdots w_n$ over the set of nonnegative integers is called a *Catalan word* if $w_1 = \mathtt{0}$ and $\mathtt{0} \leq w_i \leq w_{i-1} + \mathtt{1}$ for $i = 2, \ldots, n$. Throughout this paper, $|w|$ denotes the length of w and ϵ denotes the *empty word*, which is the unique word of length zero. For $n \geq 0$, let $\mathcal{C}_n$ denote the set of Catalan words of length n. We set $\mathcal{C} := \bigcup_{n \geq 0} \mathcal{C}_n$ and $\mathcal{C}^+ := \bigcup_{n \geq 1} \mathcal{C}_n$ be the set of nonempty Catalan words. For example,

$$\mathcal{C}_4 = \left\{ \begin{matrix} \mathtt{0000}, \mathtt{0001}, \mathtt{0010}, \mathtt{0011}, \mathtt{0012}, \mathtt{0100}, \mathtt{0101}, \\ \mathtt{0110}, \mathtt{0111}, \mathtt{0112}, \mathtt{0120}, \mathtt{0121}, \mathtt{0122}, \mathtt{0123} \end{matrix} \right\}.$$

Note that $|\mathcal{C}_n| = C_n = \frac{1}{n+1}\binom{2n}{n}$ is the nth Catalan number. The exploration of Catalan words has begun with the comprehensive generation of Gray codes tailored for growth-constricted words [12]. Baril et al. [2, 4, 5] have delved into analyzing the distribution of descents and the last symbol in Catalan words avoiding one or two classical patterns of length at most three. Similar findings [1, 7, 17] emerge in studies of restricted Catalan words avoiding consecutive patterns of length three or pairs of relations. Callan et al. [10] initiate the enumeration of statistics, including area and perimeter, on the polyominoes associated with Catalan words. Furthermore, assorted combinatorial statistics regarding polyominoes associated with both Catalan and Motzkin terminologies have been scrutinized [6, 13, 14, 15]. Next Shattuck [18] initiated an examination into the frequency of distinct subword occurrences, spanning no more than three characters, nestled within Catalan words, like descents, ascents, and levels. In a recent paper [3], Baril et al. provide generating functions, formulas, and asymptotic expressions for the number of Catalan words based on the number of runs of ascents (descents), runs of weak ascents (descents), ℓ-valleys, valleys, symmetric valleys, ℓ-peaks, peaks, and symmetric peaks.

Given a permutation of $[n] = \{1, \ldots, n\}$ in one-line notation $\pi = \pi_1 \pi_2 \cdots \pi_n$, the *runs* of π are the maximal contiguous increasing subwords of π. If the sequence of leading terms of the runs of π appears in increasing order, then π is called *flattened partition* of length n. Nabawanda et al. give recursive formula for the number of flattened partitions of length n with k runs [16, Theorem 1]. Callan gives the number of flattened partitions of length n avoiding a single 3-letter pattern [9]. Elder et al. extended the work Nabawanda et al. to establish recursive formulas for the number of flattened parking functions built from permutations of $[n]$, with r additional ones inserted that have k runs [11, Theorems 29, 30 and 35]. A further generalization includes the work of Buck et al. [8] who establish that flattened Stirling permutations are enumerated by the Dowling numbers,

which corresponds to the OEIS entry [19, A007405].

In this work, we define *flattened Catalan words*, which are Catalan words whose maximal contiguous nondecreasing subwords have leading terms in weakly increasing order. For example, the Catalan word $\mathtt{0012301222345523343} \in \mathcal{C}_{19}$ is a flattened Catalan word with four maximal contiguous nondecreasing subwords $\mathtt{00123}$, $\mathtt{012223455}$, $\mathtt{2334}$, and $\mathtt{3}$, whose leading terms satisfy $\mathtt{0} \leq \mathtt{0} \leq \mathtt{2} \leq \mathtt{3}$. Conversely, $\mathtt{012321} \in \mathcal{C}_6$ is not a flattened Catalan word as it has maximal contiguous nondecreasing subwords $\mathtt{0123}$, $\mathtt{2}$, and $\mathtt{1}$, and the leading terms $\mathtt{0}$, $\mathtt{2}$, and $\mathtt{1}$ are not in weakly increasing order. We denote the sets of nonempty flattened Catalan words and flattened Catalan words of length n as $\mathsf{Flat}(\mathcal{C}^+)$ and $\mathsf{Flat}(\mathcal{C}_n)$, respectively.

Let $w = w_1 w_2 \cdots w_n \in \mathsf{Flat}(\mathcal{C}_n)$. As usual, we say that w has an *ascent* (*descent*) at position ℓ if $w_\ell < w_{\ell+1}$ ($w_\ell > w_{\ell+1}$), where $\ell \in [n-1]$. Similarly, we define *weak ascent* (resp. *weak descent*) at position ℓ if $w_\ell \leq w_{\ell+1}$ ($w_\ell \geq w_{\ell+1}$), where $\ell \in [n-1]$. A *run* (resp. *weak run*) of ascents (resp. *weak ascents*) in a word w is a maximal subword of consecutive ascents (resp. weak ascents). The number of runs in w is denoted by $\overline{\mathsf{runs}}(w)$, and the number of weak runs in w is denoted by $\overline{\mathsf{wruns}}(w)$. The runs of descents and weak descents are defined similarly, and the statistics will be denoted $\underline{\mathsf{runs}}(w)$ and $\underline{\mathsf{wruns}}(w)$, respectively. An ℓ-*valley* in a flattened Catalan word w is a subword of the form $ab^\ell(b+1)$, where $a > b$ and ℓ is a positive integer and b^ℓ denotes ℓ consecutive copies of the letter b. If $\ell = 1$, we say that it is a *short valley*. The number of ℓ-valleys of w is denoted by $\ell\text{-}\mathsf{val}(w)$ and the number of all ℓ-valleys for $\ell \geq 1$ of w is denoted by $\mathsf{val}(w)$. A *symmetric valley* is a valley of the form $a(a-1)^\ell a$ with $\ell \geq 1$. The number of symmetric valleys of w is denoted by $\mathsf{symv}(w)$. Analogously, we define the peak statistic. Namely, an ℓ-*peak* in w is a subword of the form $a(a+1)^\ell b$, where $a \geq b$ and ℓ is a positive integer. The number of ℓ-peaks of w is denoted by $\ell\text{-}\mathsf{peak}(w)$ and the sum of all ℓ-peaks for $\ell \geq 1$ of w is denoted by $\mathsf{peak}(w)$. If $\ell = 1$, we say that it is a *short peak*; and if $a = b$, it is called a *symmetric peak*. The number of symmetric peaks of w is denoted by $\mathsf{symp}(w)$.

Our contributions include generating functions and combinatorial expressions for the number of flattened Catalan words based on the number of runs of ascents (descents), runs of weak ascents (descents), ℓ-valleys, valleys, symmetric valleys, ℓ-peaks, peaks, and symmetric peaks. We also establish one-to-one correspondences between:

- flattened Catalan words of length n with k runs of ascents and k-part

order-consecutive partitions of n, which have been studied in [21], see Theorem 3.5;

- flattened Catalan words of length n and compositions of all even natural numbers into $n-1$ parts of at most two where the part 0 is allowed, see Theorem 3.4;
- flattened Catalan words of length n with k runs of weak ascents and binary words of length $n-1$ where $2k-2$ symbols are replaced with a dot $\bullet$, see Theorem 3.11;
- flattened Catalan words of length n with k short valleys and Dyck paths of semilength n with k occurrences of `DDUU`, where the height sequence of occurrences `DDU` (from left to right) is nondecreasing, see Remark 4.3.
- flattened Catalan words of length n and ordered trees with n edges and with $k+1$ nodes having only leaves as children and satisfying two additional conditions, see Remark 4.6.

We aggregate our results and the notation used throughout in Table 1.

2 Basic Definitions

Throughout the article, we will use the following decomposition of Catalan words, called *first return decomposition* of a Catalan word w, which is $w = \mathtt{0}(w'+1)w''$, where w' and w'' are Catalan words (w' and w'' could be empty), and where $(w'+1)$ is the word obtained from w' by adding 1 at all these symbols. Note that whenever w' is the empty word, denoted by ϵ, then $(w'+1)$ remains the empty word.

For example, the first return decomposition of $w = \mathtt{0122200122322334544} \in \mathsf{Flat}(\mathcal{C}_{19})$ is given by setting $w' = \mathtt{0111}$ and $w'' = \mathtt{00122322334544}$. For this word w, we have $\overline{\mathsf{runs}}(w) = 11$, $\overline{\mathsf{wruns}}(w) = 4$, $\underline{\mathsf{runs}}(w) = 16$, $\underline{\mathsf{wruns}}(w) = 9$, $1\text{-}\mathsf{val}(w) = 0$, $2\text{-}\mathsf{val}(w) = 2$, $\ell\text{-}\mathsf{val}(w) = 0$ $(\ell > 2)$, $\mathsf{symv}(w) = 1$, $1\text{-}\mathsf{peak}(w) = 2$, $2\text{-}\mathsf{peak}(w) = 0$, $3\text{-}\mathsf{peak}(w) = 1$, $\ell\text{-}\mathsf{peak}(w) = 0$ $(\ell > 3)$, and $\mathsf{symp}(w) = 2$.

Drawing Catalan words as lattice diagrams on the plane proves to be a convenient representation. These diagrams are constructed using unit up steps $(0,1)$, down steps $(0,-1)$, and horizontal steps $(1,0)$. Each symbol

	Statistics					
	runs of asc.	runs of w. asc.	runs of desc.	runs of w. desc.	ℓ-valleys	short valleys
Statistic on w	$\overline{\mathsf{runs}}(w)$	$\overline{\mathsf{wruns}}(w)$	$\underline{\mathsf{runs}}(w)$	$\underline{\mathsf{wruns}}(w)$	ℓ-$\mathsf{val}(w)$	1-$\mathsf{val}(w)$
Bivariate g. function	$R(x,y)$	$W(x,y)$	$\bar{R}(x,y)$	$\bar{W}(x,y)$	$V_\ell(x,y)$	$V_1(x,y)$
Distribution	$r(n,k)$	$w(n,k)$	$\bar{r}(n,k)$	$\bar{w}(n,k)$	$v_\ell(n,k)$	$v_1(n,k)$
Total occurrences over $\mathsf{Flat}(\mathcal{C}_n)$	$r(n)$	$w(n)$	$\bar{r}(n)$	$\bar{w}(n)$	$v_\ell(n)$	$v_1(n)$
	valleys	sym. valleys	ℓ-peaks	short peaks	peaks	sym. peaks
Statistic on w	$\mathsf{val}(w)$	$\mathsf{symv}(w)$	ℓ-$\mathsf{peak}(w)$	1-$\mathsf{peak}(w)$	$\mathsf{peak}(w)$	$\mathsf{symp}(w)$
Bivariate g. function	$V(x,y)$	$S(x,y)$	$P_\ell(x,y)$	$P_1(x,y)$	$P(x,y)$	$T(x,y)$
Distribution	$v(n,k)$	$s(n,k)$	$p_\ell(n,k)$	$p_1(n,k)$	$p(n,k)$	$t(n,k)$
Total occurrences over $\mathsf{Flat}(\mathcal{C}_n)$	$v(n)$	$s(n)$	$p_\ell(n)$	$p_1(n)$	$p(n)$	$t(n)$

Statistic	Bivariate g. f.	Total occurrences over $\mathsf{Flat}(\mathcal{C}_n)$	OEIS
$\overline{\mathsf{runs}}$	$\frac{xy(1-x-xy)}{1-2x+x^2-2xy+x^2y+x^2y^2}$	$\frac{1}{4}(3^{n-1}+1)(n+1)$	
$\overline{\mathsf{wruns}}$	$\frac{(1-2x)xy}{1-4x+4x^2-x^2y}$	$\frac{1}{36}\left(27-9n+(5+n)3^n\right)$	
$\underline{\mathsf{runs}}$	$\frac{xy(1-2xy)}{1-4xy-x^2y+4x^2y^2}$	$\frac{1}{36}\left(27n-9+(5n+1)3^n\right)$	
$\underline{\mathsf{wruns}}$	$\frac{yx(1-xy-x)}{x^2y^2+x^2y+x^2-2\,xy-2\,x+1}$	$\frac{n+1}{4}(1+3^{n-1})$	
ℓ-val	$\frac{x(1-2x+x^{\ell+1}-x^{\ell+1}y)}{(1-x)(1-3x+x^{\ell+1}-x^{\ell+1}y)}$	$\frac{1}{4}\left(1-3^{n-2-\ell}+2\cdot 3^{n-2\ell}(n-2-\ell)\right)$	
val	$\frac{x-3x^2+x^3(3-y)}{(1-x)(1-4x+4x^2-x^2y)}$	$\frac{1}{36}\left(3^n(n-4)+9n\right)$	A212337
symv	$\frac{x(1-2x)(1-2x+2x^2-x^2y)}{(1-x)(1-5x+8x^2-5x^3-x^2y+2x^3y)}$	$\frac{1}{144}\left(3^n(2n-5)-18n^2+54n-27\right)$	
ℓ-peak	$\frac{x(1-2x)}{(1-x)(1-3x+x^{\ell+1}(1-y))}$	$\frac{1}{4}\left((3^{n-\ell-2}(2n+1-2\ell))-1\right)$	
peak	$\frac{x(1-2x)}{1-4x+4x^2-x^2y}$	$\frac{1}{4}(3^{n-2}-1)(n-1)$	A261064
symp	$\frac{x(1-x)(1-2x)}{1-5x+8x^2-5x^3-x^2y+2x^3y}$	$\frac{1}{144}\left(63+3^n+2(-45+3^n)n+18n^2)\right)$	

Table 1: Summary of notation and results for statistics considered.

w_i of a Catalan word is represented by the horizontal segment between the points $(i-1, w_i)$ and (i, w_i), and the vertical steps are inserted to obtain a connected diagram. For example, in Figure 1, we illustrate the lattice diagram associated to the Catalan word w.

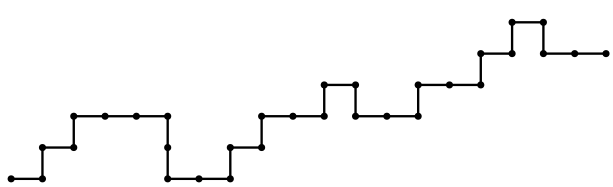


Figure 1: Lattice diagram of the word $w = 0122200122322334544$.

Remark 2.1. *Let $\mathcal{C}_n^{\uparrow}$ denote the set of weakly increasing Catalan words of length n. Notice that $|\mathcal{C}_0^{\uparrow}| = 1$ and for $n \geq 1$ $|\mathcal{C}_n^{\uparrow}| = 2^{n-1}$, then its generating functions is $1 + x/(1-2x)$ if we include the empty word. Note that the set of nonempty weakly increasing Catalan words is precisely the set of flattened Catalan words with a single weak run. Hence, the generating functions for the later set is $x/(1-2x)$.*

3 The Distribution of Runs

3.1 Runs of Ascents

In order to count nonempty flattened Catalan words according to the length and the number runs of ascents, we introduce the following bivariate generating function

$$R(x,y) = \sum_{w \in \mathsf{Flat}(\mathcal{C}^+)} x^{|w|} y^{\overline{\mathsf{runs}}(w)} = \sum_{n \geq 1} x^{|w|} \sum_{w \in \mathsf{Flat}(\mathcal{C}_n)} y^{\overline{\mathsf{runs}}(w)},$$

where the coefficient of $x^n y^k$ is the number of flattened Catalan words of length n with k runs of ascents.

In Theorem 3.2, we give an expression for this generating function, but first we provide an example.

Example 3.1. *Consider the flattened Catalan word* $w = \mathtt{012230123122} \in \mathsf{Flat}(\mathcal{C}_{12})$. *Then* w *has* 5 *runs of ascents:* $\mathtt{012}$, $\mathtt{23}$, $\mathtt{0123}$, $\mathtt{12}$, *and* $\mathtt{2}$.

Theorem 3.2. *The generating function for nonempty flattened Catalan words with respect to the length and the number of runs of ascents is*

$$R(x,y) = \frac{xy(1-x-xy)}{1-2x+x^2-2xy+x^2y+x^2y^2}.$$

Proof. Let w be a nonempty flattened Catalan word and let $w = \mathtt{0}(w'+1)w''$ be the first return decomposition, with $w', w'' \in \mathsf{Flat}(\mathcal{C})$. There are four different types of this word. Figure 2 illustrates this case.

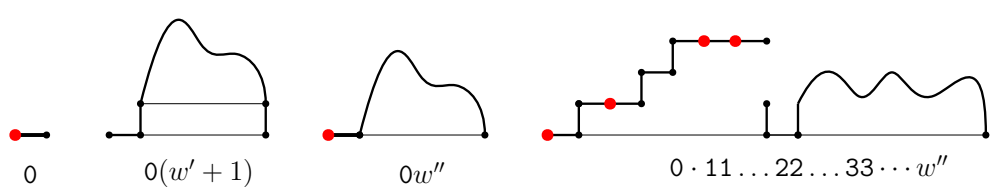


Figure 2: Decomposition of a nonempty flattened Catalan word in $\mathsf{Flat}(\mathcal{C})$.

If $w' = w'' = \epsilon$, then $w = 0$. Then its generating function is xy.

If $w'' = \epsilon$ and $w' \neq \epsilon$, then $w = \mathtt{0}(w'+1)$. Then the generating function is $xR(x,y)$.

If $w' = \epsilon$ and $w'' \neq \epsilon$, then $w = \mathtt{0}w''$. Then the generating function is $xyR(x,y)$ because we have an extra run.

If $w' \neq \epsilon$ and $w'' \neq \epsilon$, then $w = \mathtt{0}(w'+1)w''$. Note w' is a weakly increasing word because $w \in \mathsf{Flat}(\mathcal{C}^+)$. Then the bivariate generating function for such words w' is

$$\sum_{n\geq 1}\sum_{k=1}^{n}\binom{n-1}{k-1}x^n y^k = \sum_{n\geq 0} y(1+y)^{n-1}x^n = \frac{xy}{1-x(1+y)}.$$

Therefore, the generating function for this case is given by

$$\frac{x^2y}{1-x-xy}R(x,y).$$

Therefore, we have the functional equation

$$R(x,y) = xy + x(1+y)R(x,y) + \frac{x^2y}{1-x-xy}R(x,y).$$

Solving this equation, we obtain the desired result. □

Corollary 3.3. *The generating function for nonempty flattened Catalan words is given by*

$$R(x,1) = \sum_{n\geq 1} f(n)x^n = \frac{x(1-2x)}{(1-3x)(1-x)}.$$

Therefore,

$$f(n) = \frac{1}{2}\left(3^{n-1}+1\right).$$

The first few values of the sequence $f(n)$ $(n \geq 1)$ correspond to the OEIS entry [19, A007051]:

$$1, \quad 2, \quad 5, \quad 14, \quad 41, \quad 122, \quad 365, \quad 1094, \quad 3281, \quad 9842, \ldots.$$

This sequence also counts the compositions of all even natural numbers (from 0 to $2(n-1)$) into $n-1$ parts of at most two (the part 0 is allowed).

Theorem 3.4. *Flattened Catalan words of length n and compositions of all even natural numbers (from 0 to $2(n-1)$) into $n-1$ parts of at most two (the part 0 is allowed) are in bijection.*

Proof. A bijection ψ between flattened Catalan words of length n and this combinatorial class is given by $\psi(\mathtt{0}) = \epsilon$; $\psi(\mathtt{0}(w+1)) = 2\psi(w)$; $\psi(\mathtt{0}w) = \mathtt{0}\psi(w)$; and $\psi(\mathtt{0}(w+1)w') = 1\psi(w)1\psi(w')$. □

Let $r(n,k)$ denote the number of flattened Catalan words of length n with exactly k runs of ascents, that is $r(n,k) = [x^n y^k]R(x,y)$, which denotes the coefficient of $x^n y^k$ in $R(x,y)$. The first few rows of this array are

$$\mathcal{R} := [r(n,k)]_{n,k\geq 1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 6 & \boxed{\mathbf{6}} & 1 & 0 & 0 & 0 & 0 \\ 1 & 10 & 19 & 10 & 1 & 0 & 0 & 0 \\ 1 & 15 & 45 & 45 & 15 & 1 & 0 & 0 \\ 1 & 21 & 90 & 141 & 90 & 21 & 1 & 0 \\ 1 & 28 & 161 & 357 & 357 & 161 & 28 & 1 \end{pmatrix}.$$

For example, $r(4,3) = 6$, the entry boxed in $\mathcal{R}$ above, and the corresponding flattened Catalan words (and lattice diagrams) are shown in Figure 3.

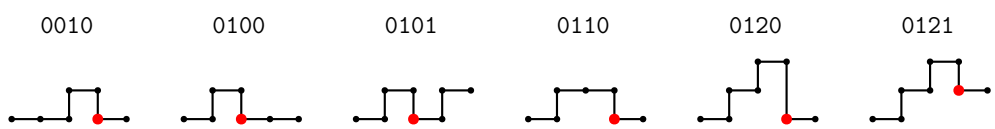


Figure 3: Flattened Catalan words of length 4 with 2 runs of ascents. The red marked vertex denotes the start of the second run of ascents.

The array $\mathcal{R}$ corresponds to the OEIS entry [19, A056241]. Notice that this sequence has a different combinatorial interpretation. It counts the number of k-part order-consecutive partitions of n. An *order-consecutive partition* of $\{1,2,\ldots,n\}$ with k parts is a k-uplet $(S_1, S_2, \ldots, S_k)$ of subsets such that $S_i \cap S_j = \emptyset$ if $i \neq j$, $\bigcup_{i=1}^{k} S_i = \{1,2,\ldots,n\}$, where every subset S_i are in increasing order relatively to their maximum elements, and satisfying the property: for $j = 1,\ldots,k$, $\bigcup_{i=1}^{j} S_i$ is an interval (cf. [21]).

Theorem 3.5. *Flattened Catalan words of length n with exactly k runs of ascents are in bijection with k-part order-consecutive partitions of n.*

Proof. We define recursively a map ψ from the set of words in $\mathsf{Flat}(\mathcal{C}_n)$ and the set $\mathcal{OCP}_n$ of order-consecutive partitions of $\{1,2,\ldots,n\}$. We consider the four cases of Figure 2.

- If w belongs to the case (i), then $w = \mathtt{0}$ and we set $\psi(w) = \{1\}$;

- If w belongs to the case (ii), then $w = \mathtt{0}(w'+1)$ and $\psi(w)$ is obtained from $\psi(w')$ by inserting n in the last part; for instance, if $f(w') = \{2,3\}\{1,4\}$, then $f(w) = \{2,3\}\{1,4,5\}$;

- If w belongs to the case (iii), then $w = \mathtt{0}w'$ and $\psi(w)$ is obtained from $\psi(w')$ by adding the part $\{n\}$ on the right; for instance, if $f(w') = \{2,3\}\{1,4\}$, then $f(w) = \{2,3\}\{1,4\}\{5\}$;

- If w belongs to the case (iv), then $w = w'w''$ where w' consists of one weak run starting with $\mathtt{01}$. Using the previous cases, $\psi(w') = S_1 \dots S_k$ where $S_k = \{a_1, \dots a_\ell, |w'|-1, |w'|\}$ ends with a part containing both $|w'|-1$ and $|w'|$. So, we set $\psi(w) = S_1 \dots S_{k-1}(\psi(w'') + |w'| - 1)\{a_1, \dots, a_\ell, |w'|-1, |w'|+|w''|\}$. For instance if $w = \mathtt{0112\,0120}$, $w' = \mathtt{0112}$, $w'' = \mathtt{0120}$ and $f(w') = \{1,2\}\{3,4\}$ and $f(w'') = \{3\}\{1,2,4\}$ then $f(w) = \{1,2\}\{6\}\{4,5,7\}\{3,8\}$. $\square$

Theorem 3.5 and [21, Theorem 6] imply the following combinatorial expression.

Corollary 3.6. *If $n, k \geq 1$, then*

$$r(n,k) = \sum_{j=0}^{k-1} \binom{n-1}{2k-j-2} \binom{2k-j-2}{j}.$$

Let $r(n)$ be the total number of runs of ascents over all flattened Catalan words of length n.

Corollary 3.7. *We have*

$$\sum_{n \geq 0} r(n) x^n = \frac{x - 5x^2 + 8x^3 - 3x^4}{(1-3x)^2(1-x)^2}.$$

Moreover, for $n \geq 1$, we have

$$r(n) = \frac{1}{4}(3^{n-1}+1)(n+1).$$

The first few values of the sequence $r(n)$ $(n \geq 1)$ are

$$1, \quad 3, \quad 10, \quad 35, \quad 123, \quad 427, \quad 1460, \quad 4923, \quad 16405, \quad 54131, \dots.$$

3.2 Runs of Weak Ascents

In order to count nonempty flattened Catalan words according to the length and the number runs of weak ascents, we introduce the following bivariate generating function

$$W(x,y) = \sum_{w \in \mathsf{Flat}(\mathcal{C}^+)} x^{|w|} y^{\overline{\mathsf{wruns}}(w)} = \sum_{n \geq 1} x^{|w|} \sum_{w \in \mathsf{Flat}(\mathcal{C}_n)} y^{\overline{\mathsf{wruns}}(w)},$$

where the coefficient of $x^n y^k$ is the number of flattened Catalan words of length n with k runs of weak ascents.

Example 3.8. *Consider the flattened Catalan word* $w = \mathtt{012230123122} \in \mathsf{Flat}(\mathcal{C}_{12})$. *Then* w *has* 3 *runs of weak ascents:* $\mathtt{01223}$, $\mathtt{0123}$, $\mathtt{122}$.

In Theorem 3.9, we give an expression for this generating function.

Theorem 3.9. *The generating function for the number of nonempty flattened Catalan words with respect to the length and the number of runs of weak ascents is*

$$W(x,y) = \frac{(1-2x)xy}{1-4x+4x^2-x^2y}.$$

Proof. Let w be a nonempty flattened Catalan word and let $w = \mathtt{0}(w'+1)w''$ be the first return decomposition, with $w', w'' \in \mathsf{Flat}(\mathcal{C})$. There are four different types of this word. If $w' = w'' = \epsilon$, then $w = \mathtt{0}$. Then its generating function is xy. If $w'' = \epsilon$ and $w' \neq \epsilon$, then $w = \mathtt{0}(w'+1)$. Then the generating function is $xW(x,y)$. Similarly, if $w' = \epsilon$ and $w'' \neq \epsilon$, then $w = \mathtt{0}w''$. Then the generating function is $xW(x,y)$. If $w' \neq \epsilon$ and $w'' \neq \epsilon$, then $w = \mathtt{0}(w'+1)w''$. Note w' is a weakly increasing word because $w \in \mathsf{Flat}(\mathcal{C}^+)$. Then the generating function is given by

$$x \sum_{k \geq 1} 2^k x^k y W(x,y) = \frac{x^2 y}{1-2x} W(x,y).$$

Therefore, we have the functional equation

$$W(x,y) = xy + 2xW(x,y) + \frac{x^2 y}{1-2x} W(x,y).$$

Solving this equation, we obtain the desired result. □

Let $w(n,k)$ denote the number of flattened Catalan words of length n with exactly k runs of weak ascents, that is $w(n,k) = [x^n y^k]W(x,y)$, which denotes the coefficient of $x^n y^k$ in $W(x,y)$. The first few values of this array are

$$\mathcal{W} := [w(n,k)]_{n,k\geq 1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 & 0 \\ 8 & \boxed{\mathbf{6}} & 0 & 0 & 0 \\ 16 & 24 & 1 & 0 & 0 \\ 32 & 80 & 10 & 0 & 0 \\ 64 & 240 & 60 & 1 & 0 \\ 128 & 672 & 280 & 14 & 0 \\ 256 & 1792 & 1120 & 112 & 1 \end{pmatrix}.$$

For example, $w(4,2) = 6$, the entry boxed in $\mathcal{W}$ above, and the corresponding flattened Catalan words (and lattice diagrams) are shown in Figure 4.

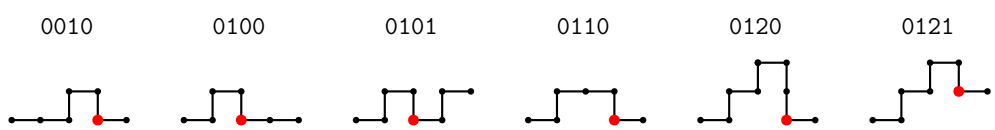


Figure 4: Flattened Catalan words of length 4 with 2 runs of weak ascents. The red marked vertex denotes the start of the second run of weak ascents.

Corollary 3.10. *For $n, k \geq 1$, we have*

$$w(n,k) = 2^{n-2k+1}\binom{n-1}{2k-2}.$$

Proof. From Theorem 3.9, we obtain the recurrence relation

$$w(n,k) - 4w(n-1,k) + 4w(n-2,k) - 4w(n-2,k-1) = 0, \quad n \geq 3, k \geq 1,$$

with the initial values $w(2,1) = 2$, $w(1,1) = 1$, and $w(n,k)$ for $n < k$. It is not difficult to verify that $2^{n-2k+1}\binom{n-1}{2k-2}$ satisfies the same recurrence relation and the same initial values. Therefore, the sequences are the same. □

We give an alternate proof of Corollary 3.10 through a bijective proof. We state the result formally for ease of reference.

Theorem 3.11. *Flattened Catalan words of length n with k runs of weak ascents and binary words of length $n-1$ where $2k-2$ symbols are replaced with a dot $\bullet$ are in bijection.*

Proof. We now give bijection between flattened Catalan words of length n with k runs of weak ascents and binary words of length $n-1$ where $2k-2$ symbols are replaced with a dot $\bullet$ (Corollary 3.10 and a simple combinatorial argument prove that the two classes of objects have the same cardinality). Let $u = u_1 u_2 \cdots u_{n-1}$ be such a binary word with $2k-2$ $\bullet$'s, and let us suppose that the $\bullet$'s are on the positions $\{i_1, i_2, \ldots, i_{2k-2}\}$. Then, we define the flattened Catalan words with k runs of weak ascents as follows:

Let $v = v_0 v_1 \cdots v_{n-1}$ be the word of length n constructed from u by fixing $v_0 = \mathtt{0}$, $v_{i_{2a+1}} := \mathtt{1}$, $v_{i_{2a}} := \mathtt{0}$, $a = 0, 1, \ldots, k-1$, and $v_i := u_i$ for all other positions i. We fix $i_0 = 0$ and $i_{2k-1} = n$. Now, v consists of the juxtaposition of k nonempty factors of the form $r_a = v_{i_{2a}} \cdots v_{i_{2a+2}-1}$, $a = 0, 1, \ldots, k-1$, all of them starting with $\mathtt{0}$. We associate to each factor $s = 0 s_2 \cdots s_p$ the nondecreasing Catalan word $c(s) = \mathtt{0} c_2 \cdots c_{|s|}$, where $c_i = c_{i-1}$ if $s_i = 0$ and $c_i = c_{i-1} + \mathtt{1}$, otherwise (for instance, if $s = 011010110$ then $c(s) = \mathtt{012233455}$).

The bijection f is defined as follows:

$$f(u) = c(r_0)(a_0 + c(r_1))(a_0 + a_1 + c(r_2)) \cdots (a_0 + a_1 + \cdots + a_{k-2} + c(r_{k-1})),$$

where a_j is the number of $\mathtt{1}$'s in the factor $v_{i_{2(j+1)}} \cdots v_{i_{2(j+1)+1}-1}$.

For instance, if $n = 29$ and $k = 4$ and $u = 10100\bullet1010\bullet0110\bullet01\bullet0110\bullet0\bullet00$. We have

$$v = 01010\mathbf{0}11010\ \mathbf{0}0110\mathbf{1}01\ \mathbf{0}0110\mathbf{1}0\ \mathbf{0}00,$$

and

$$f(u) = \mathtt{01122234455\ 22344556\ 4456677\ 666}. \qquad \square$$

Let $w(n)$ be the total number of runs of weak ascents over all flattened Catalan words of length n.

Corollary 3.12. *For $n \geq 1$, we have*

$$\sum_{n \geq 1} w(n) x^n = \frac{x(1-2x)^3}{(1-4x+3x^2)^2}.$$

Moreover, for $n \geq 1$, we have

$$w(n) = \frac{1}{36}\left(27 - 9n + (5+n)3^n\right).$$

The first few values of the sequence $w(n)$ $(n \geq 1)$ are

$$1, \quad 2, \quad 6, \quad 20, \quad 67, \quad 222, \quad 728, \quad 2368, \quad 7653, \quad 24602, \ldots$$

3.3 Runs of Descents

In order to count nonempty flattened Catalan words according to the length and the number runs of descents, we introduce the following bivariate generating function

$$\bar{R}(x,y) = \sum_{w \in \mathsf{Flat}(\mathcal{C}^+)} x^{|w|} y^{\underline{\mathsf{runs}}(w)} = \sum_{n \geq 1} x^{|w|} \sum_{w \in \mathsf{Flat}(\mathcal{C}_n)} y^{\underline{\mathsf{runs}}(w)},$$

where the coefficient of $x^n y^k$ is the number of flattened Catalan words of length n with k runs of descents.

Example 3.13. *Consider the flattened Catalan word* $w = \mathtt{012230123122} \in \mathsf{Flat}(\mathcal{C}_{12})$. *Then* w *has 10 runs of descents:* `0`, `1`, `2`, `2`, `30`, `1`, `2`, `31`, `2`, *and* `2`.

It is worth noticing that in any flattened Catalan word w of length n, we have $\underline{\mathsf{runs}}(w) = n + 1 - \overline{\mathsf{wruns}}(w)$. Therefore, we can directly deduce Theorem 3.14 and Corollary 3.15.

Theorem 3.14. *The generating function for the number of nonempty flattened Catalan words with respect to the length and the number of runs of descents is*

$$\bar{R}(x,y) = yW\left(xy, \frac{1}{y}\right) = \frac{xy(1-2xy)}{1-4xy-x^2y+4x^2y^2}.$$

Let $\bar{r}(n,k)$ denote the number of flattened Catalan words of length n with exactly k runs of descents, that is $\bar{r}(n,k) = [x^n y^k]\bar{R}(x,y)$, which denotes the coefficient of $x^n y^k$ in $\bar{R}(x,y)$. The first few values of this arrays are

$$\bar{\mathcal{R}} := [\bar{r}(n,k)]_{n,k\geq 1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \boxed{\mathbf{6}} & 8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 24 & 16 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 10 & 80 & 32 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 60 & 240 & 64 & 0 & 0 \\ 0 & 0 & 0 & 0 & 14 & 280 & 672 & 128 & 0 \end{pmatrix}.$$

For example, $\bar{r}(4,3) = 6$, the entry boxed in $\bar{\mathcal{R}}$ above, and the corresponding flattened Catalan words (and lattice diagrams) are shown in Figure 5.

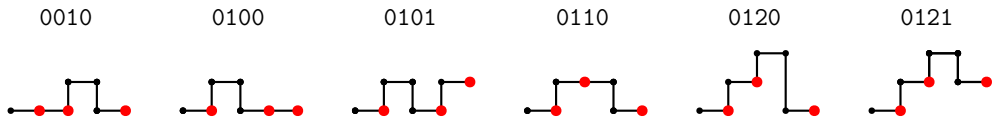


Figure 5: Flattened Catalan words of length 4 with 3 runs of descents. The red marked vertices denote the end of a run of descents.

Corollary 3.15. *For $n, k \geq 1$, we have*

$$\bar{r}(n,k) = 2^{2k-n-1}\binom{n-1}{2(n-k)}.$$

A combinatorial interpretation of this last formula can be obtained from the bijection f (see Section 3.2) between flattened Catalan words of length n with $n+1-k$ runs of weak ascents (or equivalently with k descents) and binary words of length $n-1$ with $(2n-2k)$ dots $\bullet$.

Let $\bar{r}(n)$ be the total number of runs of descents over all flattened Catalan words of length n.

Corollary 3.16. *We have*

$$\sum_{n\geq 0} \bar{r}(n)x^n = \frac{x(1-4x+4x^2+2x^3)}{(1-4x+3x^2)^2}.$$

Moreover, for $n \geq 1$, we have

$$\bar{r}(n) = \frac{1}{36}\left(27n - 9 + (5n+1)3^n\right).$$

The first few values of the sequence $\bar{r}(n)$ $(n \geq 1)$ are

$$1, \quad 4, \quad 14, \quad 50, \quad 179, \quad 632, \quad 2192, \quad 7478, \quad 25157, \quad 83660, \ldots.$$

3.4 Runs of Weak Descents

In a flattened Catalan word of length n, the number of runs of ascents plus the number of runs of weak descents equals $n+1$. Hence, the number $\bar{w}(n,k)$ of flattened Catalan words of length n with k runs of weak descents equals the number $r(n,k)$ of flattened Catalan words of length n with k runs of ascents. Moreover, we can defined a simple involution ϕ on $\mathsf{Flat}(\mathcal{C}_n)$

such that $\phi(w) = w'$ with $\underline{\mathsf{wruns}}(\phi(w)) = \overline{\mathsf{runs}}(w)$, as follows: $\phi(\epsilon) = \epsilon$, $\phi(\mathsf{0}(w+1)) = \mathsf{0}\phi(w)$, $\phi(\mathsf{0}w) = \mathsf{0}(1+\phi(w))$, and $\phi(\mathsf{0}(1+w)w') = \mathsf{0}(1+\phi(w))\phi(w')$ whenever $w, w' \neq \epsilon$. Then, we the results can be restated as those in Section 3.1.

Theorem 3.17. *The generating function for the number of nonempty flattened Catalan words with respect to the length and the number of runs of weak descents is*

$$\bar{W}(x,y) = R(x,y) = \frac{yx\,(1-xy-x)}{x^2y^2+x^2y+x^2-2\,xy-2\,x+1}.$$

Therefore,

$$\bar{w}(n,k) = r(n,k) = \sum_{j=0}^{k-1} \binom{n-1}{2k-j-2}\binom{2k-j-2}{j}.$$

Corollary 3.18. *We have*

$$\sum_{n\geq 0} \bar{w}(n)x^n = \sum_{n\geq 0} r(n)x^n = \frac{x\left(1-3\,x^3+8\,x^2-5\,x\right)}{\left(3\,x^2-4\,x+1\right)^2}.$$

Moreover, for $n \geq 1$, we have

$$\bar{w}(n) = r(n) = \frac{n+1}{4}(1+3^{n-1}).$$

4 The Distribution of Valleys

4.1 Valleys

In order to count nonempty flattened Catalan words according to the length and the number ℓ-valleys, we introduce the following bivariate generating function

$$V_\ell(x,y) = \sum_{w\in\mathsf{Flat}(\mathcal{C}^+)} x^{|w|}y^{\ell\text{-}\mathsf{val}(w)} = \sum_{n\geq 1} x^{|w|} \sum_{w\in\mathsf{Flat}(\mathcal{C}_n)} y^{\ell\text{-}\mathsf{val}(w)},$$

where ℓ-$\mathsf{val}(w)$ denotes the number of occurrences of subwords of the form $ab^\ell(b+1)$, and $a > b$, in w. The coefficient of x^ny^k in $V_\ell(x,y)$ is the number of flattened Catalan words of length n with k ℓ-valleys.

In Theorem 4.1, we give an expression for this generating function.

Theorem 4.1. *The generating function for nonempty flattened Catalan words with respect to the length and the number of ℓ-valleys is*

$$V_\ell(x,y) = \frac{x(1-2x+x^{\ell+1}-x^{\ell+1}y)}{(1-x)(1-3x+x^{\ell+1}-x^{\ell+1}y)}.$$

Proof. Let w be a nonempty flattened Catalan word, and let $w = \mathtt{0}(w'+1)w''$ be the first return decomposition, with $w', w'' \in \mathsf{Flat}(\mathcal{C})$. If $w' = w'' = \epsilon$, then $w = \mathtt{0}$, and its generating function is x. If $w' \neq \epsilon$ and $w'' = \epsilon$, then $w = \mathtt{0}(w'+1)$, and its generating function is $xV_\ell(x,y)$. Similarly, if $w' = \epsilon$ and $w'' \neq \epsilon$, then $w = \mathtt{0}w''$, and its generating function is $xV_\ell(x,y)$. Finally, if $w' \neq \epsilon$ and $w'' \neq \epsilon$, then $w = \mathtt{0}(w'+1)w''$. Because w is a flattened Catalan word, w' must be a weakly increasing word, and we distinguish two cases. If w'' is of the form $\mathtt{0}^{\ell-1}w'''$, where w''' starts with $\mathtt{01}$, then $w = \mathtt{0}(w'+1)\mathtt{0}^{\ell-1}w'''$, and the generating function is

$$\left(\frac{x^{\ell+1}y}{1-2x}\right)\left(V_\ell(x,y) - (x + xV_\ell(x,y))\right).$$

Notice that $T_\ell(x,y) := V_\ell(x,y) - (x + xV_\ell(x,y))$ is obtained using the complement of the generating function for the word $\mathtt{0}$ and the words starting with $\mathtt{00}$.

The second case is the negation, so, w'' does not start with $\mathtt{0}^\ell\mathtt{1}$. Notice that ℓ is fixed because we are interested in the ℓ-valleys, so the generating function is

$$\frac{x^2}{1-2x}(V_\ell(x,y) - x^{\ell-1}T_\ell(x,y)).$$

Therefore, we have the functional equation

$$V_\ell(x,y) = x + 2xV_\ell(x,y) + \left(\frac{x^{\ell+1}y}{1-2x}\right)T_\ell(x,y) + \frac{x^2}{1-2x}(V_\ell(x,y) - x^{\ell-1}T_\ell(x,y)).$$

Solving this equation, we obtain the desired result. □

Let $v_\ell(n,k)$ denote the number of flattened Catalan words of length n with exactly k ℓ-valleys, that is $v_\ell(n,k) = [x^n y^k]V_\ell(x,y)$, which denotes the coefficient of $x^n y^k$ in $V_\ell(x,y)$. For example, the first few values of this

array for $\ell = 2$ are

$$\mathcal{V}_2 := [v_2(n,k)]_{n\geq 4, k\geq 0} = \begin{pmatrix} 14 & 0 & 0 & 0 \\ 40 & 1 & 0 & 0 \\ 115 & \boxed{\mathbf{7}} & 0 & 0 \\ 331 & 34 & 0 & 0 \\ 953 & 140 & 1 & 0 \\ 2744 & 527 & 10 & 0 \\ 7901 & 1877 & 64 & 0 \end{pmatrix}.$$

For example, $v_2(6,1) = 7$, the entry boxed in $\mathcal{V}_2$ above, and the corresponding flattened Catalan words of length 6 with one 2-valley (and lattice diagrams) are shown in Figure 6.

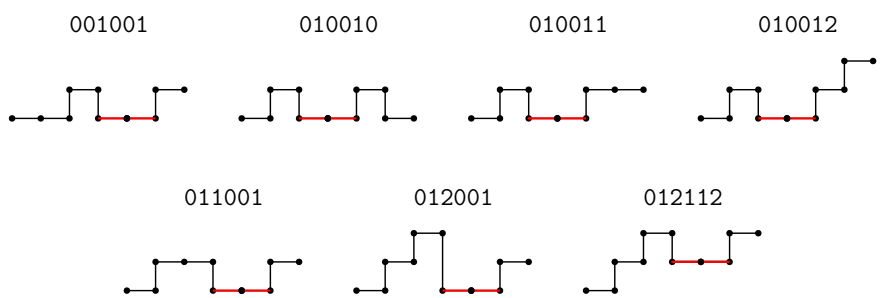


Figure 6: Flattened Catalan words of length 6 with one 2-valley. The red edges indicate the location of the 2-valley.

The first column of the array $\mathcal{V}_2$ corresponds to OEIS entry [19, A052963].

Let $v_\ell(n)$ be the sum of all ℓ-valleys in the set of flattened Catalan words of length n.

Corollary 4.2. *The generating function of the sequence $v_\ell(n)$ is*

$$\sum_{n\geq 1} v_\ell(n) x^n = \frac{x^{\ell+3}}{(1-x)(1-3x)^2}.$$

Moreover, for $n \geq 1$, we have

$$v_\ell(n) = \frac{1}{4}\left(1 - 3^{n-2-\ell} + 2\cdot 3^{n-2\ell}(n-2-\ell)\right).$$

Taking $\ell = 1$ in Theorem 4.1, we obtain the generating function for nonempty flattened Catalan words with respect to the length and the number of short

valleys

$$V_1(x,y) = \sum_{w \in \mathsf{Flat}(\mathcal{C}^+)} x^{|w|} y^{1\text{-}\mathsf{val}(w)} = \frac{x - 2x^2 + x^3(1-y)}{(1-x)(1-3x+x^2(1-y))}.$$

Let $v_1(n,k)$ denote the number of flattened Catalan words of length n with exactly k short valleys, that is $v_1(n,k) = [x^n y^k]V_1(x,y)$, which denotes the coefficient of $x^n y^k$ in $V_1(x,y)$. The first few values of this array are

$$\mathcal{V}_1 = [v_1(n,k)]_{n\geq 1, k\geq 0} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 \\ 13 & 1 & 0 & 0 \\ 34 & \boxed{\mathbf{7}} & 0 & 0 \\ 89 & 32 & 1 & 0 \\ 233 & 122 & 10 & 0 \\ 610 & 422 & 61 & 1 \\ 1597 & 1376 & 295 & 13 \end{pmatrix}.$$

For example, $v_1(5,1) = 7$, the entry boxed in $\mathcal{V}_1$ above, and the corresponding flattened Catalan words of length 5 with exactly one short valley (and lattice diagrams) are shown in Figure 7.

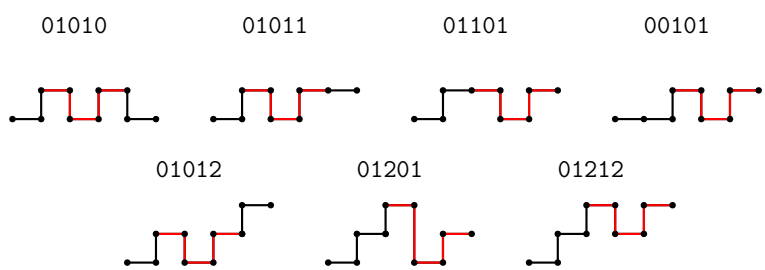


Figure 7: Flattened Catalan words of length 5 with one short valley. The red edges indicates the location of the short valley.

Remark 4.3. *In [3], we proved that Catalan words of length n with k short valleys are in one-to-one correspondence with Dyck paths of semilength n with k occurrences of* `DDUU`*. Taking the restriction on flattened Catalan words of this bijection, we obtain a one-to-one correspondence between flattened Catalan words of length n with k short valleys and Dyck paths of semilength n with k occurrences of* `DDUU`*, where the height sequence of occurrences* `DDU` *(from left to right) is nondecreasing.*

We can also obtain the generating function for the number of flattened Catalan words of length n with respect to the number of valleys (we consider all ℓ-valleys for $\ell \geq 1$).

Theorem 4.4. *The generating function for nonempty flattened Catalan words with respect to the length and the number of valleys is*

$$V(x,y) = \frac{x - 3x^2 + x^3(3-y)}{(1-x)(1-4x+4x^2-x^2y)}.$$

Let $v(n,k)$ denote the number of flattened Catalan words of length n with exactly k valleys, that is $v(n,k) = [x^n y^k]V(x,y)$, which denotes the coefficient of $x^n y^k$ in $V(x,y)$. The first few values of this arrays are

$$\mathcal{V} = [v(n,k)]_{n\geq 1, k\geq 0} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 \\ 13 & 1 & 0 & 0 \\ 33 & 8 & 0 & 0 \\ 81 & 40 & 1 & 0 \\ 193 & 160 & \boxed{\mathbf{12}} & 0 \\ 449 & 560 & 84 & 1 \\ 1025 & 1792 & 448 & 16 \end{pmatrix}.$$

For example, $v(7,2) = 12$, the entry boxed in $\mathcal{V}$ above, and the corresponding flattened Catalan words of length 7 with exactly two valleys are

`0010101`, `0100101`, `0101001`, `0101010`, `0101011`, `0101012`,
`0101101`, `0101201`, `0101212`, `0110101`, `0120101`, `0121212`.

Corollary 4.5. *For $n \geq 0$ we have*

$$v(n,k) = \begin{cases} (n-1)2^{n-2} + 1, & \text{if } k = 0 \\ 2^{n-2k-2}\binom{n-1}{2k+1}, & \text{if } k \geq 1 \end{cases}.$$

Note that $v(n,0)$ corresponds to OEIS entry [19, A005183].

Remark 4.6. *In [3], we proved that Catalan words of length n with k valleys are in one-to-one correspondence with ordered trees with n edges and having exactly $k+1$ non-leaf nodes all of those children are leaves. Taking the restriction on flattened Catalan words of this bijection, we obtain a one-to-one correspondence between flattened Catalan words of length n and ordered trees with n edges and with $k+1$ nodes having only leaves as children and satisfying the following:*

- *if $T_1, T_2, \ldots, T_r$ are the subtrees of the root, then T_i, $i \in [1, r-1]$, is nondecreasing (i.e. for any node, its subtrees, except the rightmost, consist of one node only),*

- *the rightmost subtree of the root again satisfies all these properties.*

Let $v(n)$ be the sum of all valleys in the set of flattened Catalan words of length n.

Corollary 4.7. *The generating function of the sequence $v(n)$ is*

$$\sum_{n\geq 0} v(n)x^n = \frac{x^4}{(1-x)^2(1-3x)^2}.$$

Moreover, for $n \geq 4$, we have

$$v(n) = \frac{1}{36}\left(3^n(n-4)+9n\right).$$

For $n \geq 4$, the first few values of the sequence $v(n)$ are

1, 8, 42, 184, 731, 2736, 9844, 34448, 118101, 398584, ...

This sequence corresponds to OEIS entry [19, A212337].

4.2 Symmetric Valleys

A *symmetric valley* is a valley of the form $a(a-1)^\ell a$ with $\ell \geq 1$. Let $\mathsf{symv}(w)$ denote the number of symmetric valleys in the word w. In order to count flattened Catalan words according to the length and the number of symmetric valleys, we introduce the following bivariate generating function generating function

$$S(x,y) = \sum_{w\in\mathsf{Flat}(\mathcal{C}^+)} x^{|w|}y^{\mathsf{symv}(w)} = \sum_{n\geq 1} x^{|w|} \sum_{w\in\mathsf{Flat}(\mathcal{C}_n)} y^{\mathsf{symv}(w)},$$

where the coefficient of $x^n y^k$ in $S(x,y)$ is the number of nonempty flattened Catalan words of length n with k symmetric ℓ-valleys.

In Theorem 4.8, we give an expression for this generating function.

Theorem 4.8. *The generating function of the nonempty flattened Catalan words with respect to the length and the number of symmetric valleys is*

$$S(x,y) = \frac{x(1-2x)(1-2x+2x^2-x^2y)}{(1-x)(1-5x+8x^2-5x^3-x^2y+2x^3y)}.$$

Proof. Let w be a nonempty flattened Catalan word, and let $w = \mathtt{0}(w'+1)w''$ be the first return decomposition, with $w', w'' \in \mathcal{C}$. If $w' = w'' = \epsilon$, then $w = \mathtt{0}$, and its generating function is x. If $w' \neq \epsilon$ and $w'' = \epsilon$, then $w = \mathtt{0}(w'+1)$, and its generating function is $xS(x,y)$. Similarly, if $w' = \epsilon$ and $w'' \neq \epsilon$, then $w = \mathtt{0}w''$, and its generating function is $xS(x,y)$. Finally, if $w' \neq \epsilon$ and $w'' \neq \epsilon$, then $w = \mathtt{0}(w'+1)w''$, we consider three cases.

1. If $w' = \mathtt{0}^k$ and w'' has a nonzero entry, then its generating function is
$$\left(\frac{x^2}{1-x}\right) y \left(S(x,y) - \frac{x}{1-x}\right).$$
2. If w' is a weakly increasing flattened Catalan word different than $\mathtt{0}^k$, and w'' has a nonzero entry, then its generating function is
$$x\left(\frac{x}{1-2x} - \frac{x}{1-x}\right)\left(S(x,y) - \frac{x}{1-x}\right).$$
3. If w' is a weakly increasing flattened Catalan word and $w'' = \mathtt{0}^k$, then its generating function is
$$\frac{x^3}{(1-x)(1-2x)}.$$

Therefore, we have the functional equation

$$S(x,y) = x + 2xS(x,y) + \left(\frac{x^2}{1-x}\right) y \left(S(x,y) - \frac{x}{1-x}\right) + x\left(\frac{x}{1-2x} - \frac{x}{1-x}\right)\left(S(x,y) - \frac{x}{1-x}\right) + \frac{x^3}{(1-x)(1-2x)}.$$

Solving the obtained functional equation yields the desired result. □

Let $s(n,k)$ denote the number of flattened Catalan words of length n with exactly k symmetric valleys, that is $s(n,k) = [x^n y^k]S(x,y)$, which denotes the coefficient of $x^n y^k$ in $S(x,y)$. The first few values of this arrays are

$$\mathcal{S} = [s(n,k)]_{n\geq 1, k\geq 0} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 & 0 \\ 13 & 1 & 0 & 0 & 0 \\ 34 & \boxed{\mathbf{7}} & 0 & 0 & 0 \\ 90 & 31 & 1 & 0 & 0 \\ 242 & 113 & 10 & 0 & 0 \\ 659 & 375 & 59 & 1 & 0 \end{pmatrix}.$$

For example, $s(5,1) = 7$, the entry boxed in $\mathcal{S}$ above, and the corresponding flattened Catalan words of length 5 with 1 symmetric valley are given in Figure 8.

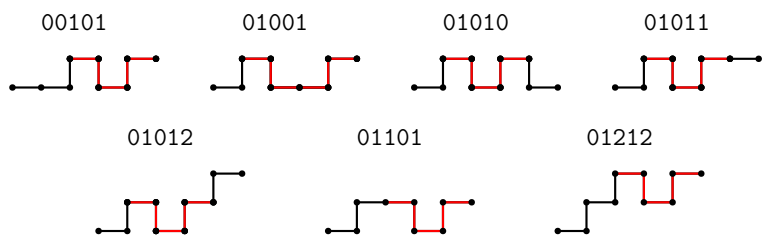


Figure 8: Flattened Catalan words of length 5 with one symmetric valley. In red we mark the location of the symmetric valley.

Let $s(n)$ be the sum of all symmetric valleys in the set of flattened Catalan words of length n.

Corollary 4.9. *The generating function of the sequence $s(n)$ is*

$$\sum_{n\geq 0} s(n)x^n = \frac{x^4(1+2x)}{(1-3x)^2(1-x)^3}.$$

Moreover, for $n \geq 4$, we have

$$s(n) = \frac{1}{144}\left(3^n(2n-5) - 18n^2 + 54n - 27\right).$$

The first few values of the sequence $s(n)$ $(n \geq 4)$ are

$$1, \quad 7, \quad 33, \quad 133, \quad 496, \quad 1770, \quad 6142, \quad 20902, \quad 70107, \quad 232489, \ldots.$$

5 The Distribution of Peaks

5.1 Peaks

In order to count flattened Catalan words according to the length and the number of ℓ-peaks, we introduce the following bivariate generating function

$$P_\ell(x,y) = \sum_{w\in \mathsf{Flat}(\mathcal{C}^+)} x^{|w|} y^{\ell\text{-peak}(w)} = \sum_{n\geq 1} x^{|w|} \sum_{w\in \mathsf{Flat}(\mathcal{C}_n)} y^{\ell\text{-peak}(w)},$$

where ℓ-$\mathsf{peak}(w)$ denotes the number of occurrences of subwords of the form $a(a+1)^\ell b$, and $a \geq b$, in w. The coefficient of $x^n y^k$ in $P_\ell(x,y)$ is the number of flattened Catalan words of length n with k ℓ-peaks.

In Theorem 5.1, we give an expression for this generating function.

Theorem 5.1. *The generating function for nonempty flattened Catalan words with respect to the length and the number of ℓ-peaks is*

$$P_\ell(x,y) = \frac{x(1-2x)}{(1-x)(1-3x+x^{\ell+1}(1-y))}.$$

Proof. Let w be a nonempty flattened Catalan word, and let $w = \mathtt{0}(w'+1)w''$ be the first return decomposition, with $w', w'' \in \mathcal{C}$. If $w' = w'' = \epsilon$, then $w = \mathtt{0}$, and its generating function is x. If $w' \neq \epsilon$ and $w'' = \epsilon$, then $w = \mathtt{0}(w'+1)$, and its generating function is $xP_\ell(x,y)$. Similarly, if $w' = \epsilon$ and $w'' \neq \epsilon$, then $w = \mathtt{0}w''$, and its generating function is $xP_\ell(x,y)$. Finally, if $w' \neq \epsilon$ and $w'' \neq \epsilon$, then $w = \mathtt{0}(w'+1)w''$, its generating function is

$$x\left(\frac{x}{1-2x} - x^\ell - \frac{x^{\ell+1}}{1-2x}\right)P_\ell(x,y) + xy\left(x^\ell + \frac{x^{\ell+1}}{1-2x}\right)P_\ell(x,y).$$

Therefore, we have the functional equation

$$\begin{aligned} P_\ell(x,y) = x + 2xP_\ell(x,y) + x\left(\frac{x}{1-2x} - x^\ell - \frac{x^{\ell+1}}{1-2x}\right)P_\ell(x,y) \\ + xy\left(x^\ell + \frac{x^{\ell+1}}{1-2x}\right)P_\ell(x,y). \end{aligned}$$

Solving the obtained functional equation yields the desired results. □

Let $p_\ell(n)$ be the sum of all ℓ-peaks in the set of flattened Catalan words of length n.

Corollary 5.2. *The generating function of the sequence $p_\ell(n)$ is*

$$\sum_{n\geq 1} p_\ell(n)x^n = \frac{x^{\ell+2}(1-2x)}{(1-3x)^2(1-x)}.$$

Moreover, for $n \geq 1$ we have

$$p_\ell(n) = \frac{1}{4}\left((3^{n-\ell-2}(2n+1-2\ell)) - 1\right).$$

Taking $\ell = 1$ in Theorem 5.1, establishes that the generating function for flattened Catalan words with respect to the length and the number of short peaks is

$$P_1(x,y) = \frac{x(1-2x)}{(1-x)(1-3x+x^2(1-y))}.$$

Let $p_1(n,k)$ denote the number of flattened Catalan words of length n with exactly k short peaks, that is $p_1(n,k) = [x^n y^k]P_1(x,y)$, which denotes the coefficient of $x^n y^k$ in $P_1(x,y)$. The first few values of this array are

$$\mathcal{P}_1 = [p_1(n,k)]_{n\geq 1, k\geq 0} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 & 0 \\ 9 & 5 & 0 & 0 & 0 \\ 22 & 18 & 1 & 0 & 0 \\ 56 & 58 & \boxed{\mathbf{8}} & 0 & 0 \\ 145 & 178 & 41 & 1 & 0 \\ 378 & 532 & 173 & 11 & 0 \\ 988 & 1563 & 656 & 73 & 1 \end{pmatrix}.$$

For example, $p_1(6,2) = 8$, the entry boxed in $\mathcal{S}$ above, and the corresponding flattened Catalan words of length 6 with 2 short peaks are

`001010`, `010100`, `010101`, `010010`, `010120`, `010121`, `012010`, `012121`.

For $n \geq 1$ we have $p_1(n,0) = F_{2(n-1)} + 1$, where F_m is the mth Fibonacci number with initial values $F_1 = F_2 = 1$. For $n \geq 1$, the sequence $p_1(n,0)$ corresponds to the OEIS entry [19, A055588].

Using a similar proof as for Theorem 5.1, we generalize the result in order to obtain the following generating function for the number of flattened Catalan words of length n with respect to the number of peaks (we consider all ℓ-peaks for $\ell \geq 1$).

Theorem 5.3. *The generating function for flattened Catalan words with respect to the length and the number of peaks is*

$$P(x,y) = \frac{x(1-2x)}{1-4x+4x^2-x^2y}.$$

Let $p(n,k)$ denote the number of flattened Catalan words of length n with exactly k peaks, that is $p(n,k) = [x^n y^k]P(x,y)$, which denotes the coeffi-

cient of $x^n y^k$ in $P(x,y)$. The first few values of this arrays are

$$\mathcal{P} = [p(n,k)]_{n\geq 1, k\geq 0} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 & 0 \\ 8 & \boxed{\mathbf{6}} & 0 & 0 & 0 \\ 16 & 24 & 1 & 0 & 0 \\ 32 & 80 & 10 & 0 & 0 \\ 64 & 240 & 60 & 1 & 0 \\ 128 & 672 & 280 & 14 & 0 \\ 256 & 1792 & 1120 & 112 & 1 \end{pmatrix}.$$

For example, $p(4,1) = 6$, the entry boxed in $\mathcal{P}$ above, and the corresponding flattened Catalan words of length 4 with 1 peaks are

$$\texttt{0010}, \quad \texttt{0100}, \quad \texttt{0110}, \quad \texttt{0101}, \quad \texttt{0120}, \quad \texttt{0121}.$$

Let $p(n)$ be the sum of all peaks in the set of flattened Catalan words of length n.

Corollary 5.4. *The generating function of the sequence $p(n)$ is*

$$\sum_{n\geq 0} p(n)x^n = \frac{(1-2x)x^3}{(1-4x+3x^2)^2}.$$

Moreover, for $n \geq 3$, we have

$$p(n) = \frac{1}{4}(3^{n-2}-1)(n-1).$$

The first few values of the sequence $p(n)$ $(n \geq 3)$ are

$$1, \quad 6, \quad 26, \quad 100, \quad 363, \quad 1274, \quad 4372, \quad 14760, \quad 14760, \quad 49205, \ldots.$$

This sequence corresponds to the OEIS entry [19, A261064]. Our combinatorial interpretation is new.

5.2 Symmetric Peaks

A *symmetric peak* is a peak of the form $a(a+1)^\ell a$ with $\ell \geq 1$. Let $\mathsf{symp}(w)$ denote the number of the symmetric peaks of the word w. In order to count

flattened Catalan words according to the length and the number symmetric peaks, we introduce the following bivariate generating function

$$T(x,y) = \sum_{w \in \mathsf{Flat}(\mathcal{C}^+)} x^{|w|} y^{\mathsf{symp}(w)} = \sum_{n \geq 1} x^{|w|} \sum_{w \in \mathsf{Flat}(\mathcal{C}_n)} y^{\mathsf{symp}(w)},$$

where the coefficient of $x^n y^k$ in $T(x,y)$ is the number of flattened Catalan words of length n with k symmetric peaks.

Theorem 5.5, we give an expression for this generating function.

Theorem 5.5. *The generating function of the nonempty flattened Catalan words with respect to the length and the number of symmetric peaks is*

$$T(x,y) = \frac{x(1-x)(1-2x)}{1-5x+8x^2-5x^3-x^2y+2x^3y}.$$

Proof. Let w be a nonempty flattened Catalan word, and let $w = \mathtt{0}(w'+1)w''$ be the first return decomposition, with $w', w'' \in \mathsf{Flat}(\mathcal{C})$. If $w' = w'' = \epsilon$, then $w = \mathtt{0}$, and its generating function is x. If $w' \neq \epsilon$ and $w'' = \epsilon$, then $w = \mathtt{0}(w'+1)$, and its generating function is $xT(x,y)$. Similarly, if $w' = \epsilon$ and $w'' \neq \epsilon$, then $w = \mathtt{0}w''$, and its generating function is $xT(x,y)$.

Finally, if $w' \neq \epsilon$ and $w'' \neq \epsilon$, then $w = \mathtt{0}(w'+1)w''$, and we have two cases to consider.

1. If w' is all $\mathtt{0}$'s, its generating function is

$$\frac{x^2y}{1-x}T(x,y).$$

2. Otherwise, the generating function is

$$x\left(\frac{x}{1-2x} - \frac{x}{1-x}\right)T(x,y).$$

Therefore, we have the functional equation is

$$T(x,y) = x + 2xT(x,y) + \frac{x^2y}{1-x}T(x,y) + x\left(\frac{x}{1-2x} - \frac{x}{1-x}\right)T(x,y).$$

Solving this equation yields the desired result. □

Let $t(n,k)$ denote the number of flattened Catalan words of length n with exactly k symmetric peaks, that is $t(n,k) = [x^n y^k]T(x,y)$, which denotes the coefficient of $x^n y^k$ in $T(x,y)$. The first few values of this arrays are

$$\mathcal{T} = [t(n,k)]_{n\geq 1, k\geq 0} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 & 0 \\ 9 & \boxed{\mathbf{5}} & 0 & 0 & 0 \\ 23 & 17 & 1 & 0 & 0 \\ 63 & 51 & 8 & 0 & 0 \\ 176 & 149 & 39 & 1 & 0 \\ 491 & 439 & 153 & 11 & 0 \end{pmatrix}.$$

For example, $t(4,1) = 5$, the entry boxed in $\mathcal{T}$ above, and the corresponding flattened Catalan words of length 4 with 1 symmetric peak (and lattice diagrams) are shown in Figure 9.

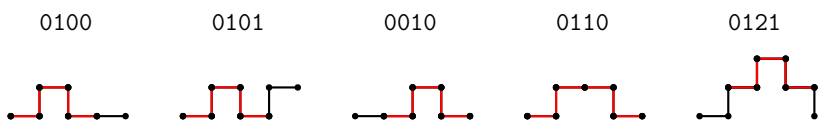


Figure 9: Flattened Catalan words of length 4 with 1 symmetric peak. In red we mark the location of the symmetric peak.

The first and second column of the array $\mathcal{T}$ coincides with OEIS entries [19, A369328, A290900].

Let $t(n)$ be the sum of all symmetric peaks in the set of flattened Catalan words of length n.

Corollary 5.6. *The generating function of the sequence $t(n)$ is*

$$\sum_{n\geq 0} t(n)x^n = \frac{(1-2x)^2 x^3}{(1-3x)^2(1-x)^3}.$$

Moreover, for $n \geq 3$, we have

$$t(n) = \frac{1}{144}\left(63 + 3^n + 2(-45+3^n)n + 18n^2\right).$$

For $n \geq 3$, the first few values of the sequence $t(n)$ are

$$1, \quad 5, \quad 19, \quad 67, \quad 230, \quad 778, \quad 2602, \quad 8618, \quad 28303, \quad 92275, \ldots.$$

Acknowledgements

The authors thank an anonymous referee for some advice and comments which helped to improve the exposition. Jean-Luc Baril was supported by University of Burgundy. Pamela E. Harris was supported in part by a Karen Uhlenbeck EDGE Fellowship. José L. Ramírez was partially supported by Universidad Nacional de Colombia. The authors thank Kimberly J. Harry and Matt McClinton for their helpful discussions during the completion of this manuscript.

References

[1] J.-L. Baril, D. Colmenares, J. L. Ramírez, D. Silva, L. M. Simbaqueba, and D. Toquica, Consecutive pattern-avoidance in Catalan words according to the last symbol, *RAIRO Theor. Inform. Appl.* **58** (2024), Paper No. 1. https://doi.org/10.1051/ita/2024001.

[2] J.-L. Baril, J. F. González, and J. L. Ramírez, Last symbol distribution in pattern avoiding Catalan words, *Math. Comput. Sci.* **18** (1) (2024). https://doi.org/10.1007/s11786-023-00576-5.

[3] J.-L. Baril, P. E. Harris, K. J. Harry, M. McClinton, and J. L. Ramírez, Enumerating runs, valleys, and peaks in Catalan words, arXiv:2404.05672 (2024).

[4] J.-L. Baril, C. Khalil, and V. Vajnovszki, Catalan words avoiding pairs of length three patterns, *Discret. Math. Theor. Comput. Sci.* **22** (2) (2021), # 5. https://doi.org/10.46298/dmtcs.6002

[5] J.-L. Baril, S. Kirgizov, and V. Vajnovszki, Descent distribution on Catalan words avoiding a pattern of length at most three, *Discrete Math.* **341** (2018), 2608–2615. https://doi.org/10.1016/j.disc.2018.06.001

[6] J.-L. Baril, S. Kirgizov, J. L. Ramírez, and D. Villamizar, The combinatorics of Motzkin polyominoes, arXiv:2401.06228 (2024).

[7] J.-L. Baril and J. L. Ramírez, Descent distribution on Catalan words avoiding ordered pairs of relations, *Adv. in Appl. Math.* **149** (2023), 102551. https://doi.org/10.1016/j.aam.2023.102551

[8] A. Buck, J. Elder, A. A. Figueroa, P. E. Harris, K. J. Harry, and A. Simpson, Flattened Stirling permutations, arXiv:2306.13034 (2023).

[9] D. Callan, Pattern avoidance in "flattened" partitions, *Discrete Math.* **309** (12) (2009), 4187–4191. https://doi.org/10.1016/j.disc.2008.11.019

[10] D. Callan, T. Mansour, and J. L. Ramírez, Statistics on bargraphs of Catalan words, *J. Autom. Lang. Comb.* **26** (2021), 177–196. https://doi.org/10.25596/jalc-2021-177.

[11] J. Elder, P. E. Harris, Z. Markman, I. Tahir, and A. Verga, On flattened parking functions, *J. Integer Seq.* **26** (2023), Article 23.5.8. https://cs.uwaterloo.ca/journals/JIS/VOL26/Harris/harris3.pdf

[12] T. Mansour and V. Vajnovszki, Efficient generation of restricted growth words, *Inform. Process. Lett.* **113** (2013), 613–616. https://doi.org/10.1016/j.ipl.2013.05.008.

[13] T. Mansour and J. L. Ramírez, Enumerations on polyominoes determined by Fuss-Catalan words, *Australas. J. Combin.* **81** (3) (2021), 447–457.

[14] T. Mansour and J. L. Ramírez, Exterior corners on bargraphs of Motzkin words. In: F. Hoffman et al. (eds) Combinatorics, Graph Theory and Computing. SEICCGTC 2021. *Springer Proceedings in Mathematics & Statistics*, **448** (2024), 355–367.

[15] T. Mansour, J. L. Ramírez, and D. A. Toquica, Counting lattice points on bargraphs of Catalan words, *Math. Comput. Sci.* **15** (2021), 701–713. https://doi.org/10.1007/s11786-021-00501-8.

[16] O. Nabawanda, F. Rakotondrajao, and A. Bamunoba, Run distribution over flattened partitions, *J. Integer Seq.* **23** (2020), Article 20.9.6.

[17] J. L. Ramírez and A. Rojas-Osorio, Consecutive patterns in Catalan words and the descent distribution, *Bol. Soc. Mat. Mex.* **29** (2023), Article #60. https://doi.org/10.1007/s40590-023-00532-0.

[18] M. Shattuck, Counting subword patterns in Catalan words, *Art Discrete Appl. Math.* Accepted, (2024). https://doi.org/10.26493/2590-9770.1695.4da.

[19] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, `http://oeis.org/`.

[20] R. Stanley, *Catalan Numbers*, Cambridge University Press, Cambridge, 2015.

[21] F. K. Hwang and C. L. Mallows, Enumerating nested and consecutive partitions, *J. Combin. Theory Ser. A* **70** (2) (1995), 323–333.

ISSN 2590-9770
The Art of Discrete and Applied Mathematics
https://doi.org/10.26493/2590-9770.1794.fc5
(Also available at http://adam-journal.eu)

Enumerating runs, valleys, and peaks in Catalan words

Jean-Luc Baril*
LIB, Université de Bourgogne Franche-Comté, B.P. 47 870, 21078, Dijon Cedex, France

Pamela E. Harris†, Kimberly J. Harry, Matt McClinton
Department of Mathematical Sciences, University of Wisconsin-Milwaukee, Milwaukee, WI 53211 United States

José L. Ramírez‡
Departamento de Matemáticas, Universidad Nacional de Colombia, Bogotá, Colombia

Received 9 April 2024, accepted 8 September 2024

Abstract

We provide generating functions, formulas, and asymptotic expressions for the number of Catalan words based on the number of runs of ascents (descents), runs of weak ascents (descents), ℓ-valleys, valleys, symmetric valleys, ℓ-peaks, peaks, and symmetric peaks. We also establish some bijections with restricted Dyck paths and ordered trees that transports some statistics.

Keywords: Catalan word, generating function, combinatorial statistic, Dyck path.

Math. Subj. Class.: 05A15, 05A19

*Corresponding author.
†Pamela E. Harris was supported in part by a Karen Uhlenbeck EDGE Fellowship.
‡José L. Ramírez was partially supported by Universidad Nacional de Colombia. The last author would like to thank the Laboratoire d'Informatique de Bourgogne for the warm hospitality during his visit at Université de Bourgogne where part of this work was done.
E-mail address: barjl@u-bourgogne.fr (Jean-Luc Baril), peharris@uwm.edu (Pamela E. Harris), kjharry@uwm.edu (Kimberly J. Harry), mcclin33@uwm.edu (Matt McClinton), jlramirezr@unal.edu.co (José L. Ramírez)

This work is licensed under https://creativecommons.org/licenses/by/4.0/

Accepted manuscript

1 Introduction

A word $w = w_1 w_2 \cdots w_n$ over the set of nonnegative integers is called a *Catalan word* if $w_1 = 0$ and $0 \leq w_i \leq w_{i-1} + 1$ for $i = 2, \ldots, n$. We let $|w|$ denote the length of w. We denote by ϵ the *empty word*, that is the unique word of length zero. For $n \geq 0$, let $\mathcal{C}_n$ denote the set of Catalan words of length n. We set $\mathcal{C} :- \bigcup_{n\geq 0} \mathcal{C}_n$ and $\mathcal{C}^+ :- \bigcup_{n\geq 1} \mathcal{C}_n$ be the set of nonempty Catalan words. For example,

$$\mathcal{C}_4 = \left\{ \begin{array}{l} \texttt{0000}, \texttt{0001}, \texttt{0010}, \texttt{0011}, \texttt{0012}, \texttt{0100}, \texttt{0101}, \\ \texttt{0110}, \texttt{0111}, \texttt{0112}, \texttt{0120}, \texttt{0121}, \texttt{0122}, \texttt{0123} \end{array} \right\}.$$

As may be expected given their name, the cardinality of the set $\mathcal{C}_n$ is given by the nth Catalan number $c_n = \frac{1}{n+1}\binom{2n}{n}$, see [27, Exercise 80]. Catalan words have been previously studied within the framework of exhaustively generation of Gray codes for growth-restricted words [19]. Baril et al. [3, 4, 6] have investigated the distribution of descents and the last symbol in restricted Catalan words that avoid a classical pattern or a pair of patterns of length at most three. Analogous results have been found in [2, 7, 21] for restricted Catalan words avoiding a consecutive pattern or an ordered pair of relations of length at most three. On the other hand, Callan et al. [8] initiate the enumeration of statistics, including area and perimeter, on the polyominoes associated with Catalan words. Additionally, we refer to [5, 16, 17, 18], where the authors investigate several combinatorial statistics on the polyominoes linked to both Catalan and Motzkin words. Recently, Shattuck [25] began studying the number of occurrences of different subwords of length at most three contained within Catalan words, such as descents, ascents, levels, etc. Our paper continues the study of different statistics or parameters in Catalan words (runs of ascents, run of weak ascents, ℓ-valleys, ℓ peaks,...) and their connections with other combinatorial objects, like trees and restricted Dyck paths. Notice that the joint distribution of patterns 121 and 231 studied in [25] corresponds to our study for ℓ-peaks whenever $\ell = 1$. Similarly, the joint distribution of consecutive patterns 212 and 312 studied in [25] corresponds to our study for ℓ-valleys for $\ell = 1$.

Let $w = w_1 w_2 \cdots w_n \in \mathcal{C}_n$. As usual, we say that w has an *ascent* (*descent*) at position ℓ if $w_\ell < w_{\ell+1}$ ($w_\ell > w_{\ell+1}$), where $\ell \in [n-1]$. Similarly, we define *weak ascent* (resp. *weak descent*) whenever the inequality is not strict. A *run* (resp. *weak run*) of ascents (resp. *weak ascents*) in a word w is a maximal subword of consecutive ascents (resp. weak ascents). The number of runs in w is denoted by $\mathsf{runs}(w)$, and the number of weak runs in w is denoted by $\mathsf{wruns}(w)$. The runs of descents and weak descents are defined similarly, and the statistics will be denoted $\overline{\mathsf{runs}}(w)$ and $\overline{\mathsf{wruns}}(w)$, respectively. An *$\ell$-valley* in a Catalan word w is a subword of the form $ab^\ell(b+1)$, where $a > b$ and ℓ is a positive integer and b^ℓ denotes ℓ consecutive copies of the letter b. If $\ell = 1$, we say that it is a *short valley*. The number of ℓ-valleys of w is denoted by ℓ-$\mathsf{val}(w)$ and the number of all ℓ-valleys for $\ell \geq 1$ of w is denoted by $\mathsf{val}(w)$. A *symmetric valley* is a valley of the form $a(a-1)^\ell a$ with $\ell \geq 1$. The number of symmetric valleys of w is denoted by $\mathsf{symv}(w)$. Analogously, we define the peak statistic. An *ℓ-peak* in w is a subword of the form $a(a+1)^\ell b$, where $a \geq b$ and ℓ is a positive integer. The number of ℓ-peaks of w is denoted by ℓ-$\mathsf{peak}(w)$ and the sum of all ℓ-peaks for $\ell \geq 1$ of w is denoted by $\mathsf{peak}(w)$. If $\ell = 1$, we say that it is a *short peak*; and if $a = b$, it is called a *symmetric peak*. The number of symmetric peaks of w is denoted by $\mathsf{symp}(w)$.

Among our contributions, we give generating functions, formulas, and asymptotic expressions for the number of Catalan words based on the number of runs of ascents (de-

Accepted manuscript

	Statistics					
	runs of asc.	runs of w. asc.	runs of desc.	runs of w. desc.	ℓ-valleys	short valleys
statistic on w	$\mathsf{runs}(w)$	$\mathsf{wruns}(w)$	$\overline{\mathsf{runs}}(w)$	$\overline{\mathsf{wruns}}(w)$	ℓ-$\mathsf{val}(w)$	1-$\mathsf{val}(w)$
bivariate g. function	$R(x,y)$	$W(x,y)$	$\bar{R}(x,y)$	$\bar{W}(x,y)$	$V_\ell(x,y)$	$V_1(x,y)$
distribution	$r(n,k)$	$w(n,k)$	$\bar{r}(n,k)$	$\bar{w}(n,k)$	$v_\ell(n,k)$	$v_1(n,k)$
total number over $\mathcal{C}_n$	$r(n)$	$w(n)$	$\bar{r}(n)$	$\bar{w}(n)$	$v_\ell(n)$	$v_1(n)$
	valleys	sym. valleys	ℓ-peaks	short peaks	peaks	sym. peaks
statistic on w	$\mathsf{val}(w)$	$\mathsf{symv}(w)$	ℓ-$\mathsf{peak}(w)$	1-$\mathsf{peak}(w)$	$\mathsf{peak}(w)$	$\mathsf{symp}(w)$
bivariate g. function	$V(x,y)$	$S(x,y)$	$P_\ell(x,y)$	$P_1(x,y)$	$P(x,y)$	$T(x,y)$
distribution	$v(n,k)$	$s(n,k)$	$p_\ell(n,k)$	$p_1(n,k)$	$p(n,k)$	$t(n,k)$
total number over $\mathcal{C}_n$	$v(n)$	$s(n)$	$p_\ell(n)$	$p_1(n)$	$p(n)$	$t(n)$

Table 1: Summary of notation and results for statistics considered.

scents), runs of weak ascents (descent), ℓ-valleys, valleys, symmetric valleys, ℓ-peaks, peaks, and symmetric peaks. We also establish some bijections with restricted Dyck paths and ordered trees. These statistics have been studied in detail for Dyck paths, see for example [9, 13, 14, 22, 23, 24].

We aggregate our results and the notation used in Table 1.

2 Basic definitions

A *Dyck path* of semilength n is a lattice path from $(0,0)$ to $(2n,0)$ consisting of $(1,1)$ up steps, and $(1,-1)$ down steps, in which the path never falls below the x-axis. Let $\mathcal{D}_n$ denote the set of all Dyck paths of semilength n. Given a Dyck path $D \in \mathcal{D}_n$, we can associate a Catalan word in $\mathcal{C}_n$ formed by the y-coordinate of each initial point of the up steps in D. It is known that this construction yields a bijection between $\mathcal{D}_n$ and $\mathcal{C}_n$, see [8, 27]. For example, in Figure 1, we illustrate the Dyck path associated to the Catalan word $\mathtt{012120012312} \in \mathcal{C}_{12}$.

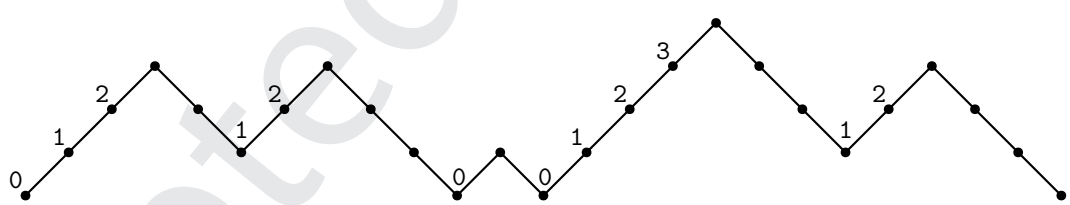


Figure 1: Dyck path of the Catalan word $\mathtt{012120012312}$.

Throughout, we use the *first return decomposition* of a Catalan word w, which is

$$w = \mathtt{0}(w'+1)w'',$$

where w' and w'' are Catalan words (w' and w'' could be empty), and where $(w'+1)$ is the word obtained from w' by adding 1 at all these symbols. Note that whenever w' is the empty word, denoted by ϵ, then $(w'+1)$ remains the empty word. For example, the first return decomposition of $w = \mathtt{012120012312}$ is given by setting $w' = \mathtt{0101}$ and $w'' = \mathtt{0012312}$. For this word w, we have $\mathsf{runs}(w) = 5$, $\mathsf{wruns}(w) = 4$, $\overline{\mathsf{runs}}(w) = 9$, $\overline{\mathsf{wruns}}(w) = 8$, 1-$\mathsf{val}(w) = 2$, 2-$\mathsf{val}(w) = 1$, ℓ-$\mathsf{val}(w) = 0$ ($\ell > 2$), $\mathsf{symv}(w) = 1$, 1-$\mathsf{peak}(w) = 3$, ℓ-$\mathsf{peak}(w) = 0$ ($\ell > 1$), and $\mathsf{symp}(w) = 1$.

Accepted manuscript

Drawing Catalan words as lattice diagrams on the plane proves to be a convenient representation. These diagrams are constructed using unit up steps $(0,1)$, down steps $(0,-1)$, and horizontal steps $(1,0)$. Each symbol w_i of a Catalan word is represented by the segment in between the points $(i-1, w_i)$ and (i, w_i), and the vertical steps are inserted to obtain a connected diagram. For example, in Figure 2, we illustrate the lattice diagram associated to the Catalan word of Figure 1.

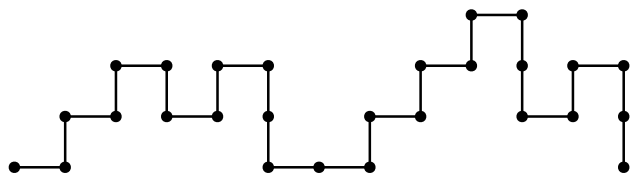


Figure 2: Lattice diagram of the word `012120012312`.

3 The distribution of runs

3.1 Runs of ascents

In order to count nonempty Catalan words according to the length and the number runs of ascents, we introduce the following bivariate generating function

$$R(x,y) = \sum_{w \in \mathcal{C}^+} x^{|w|} y^{\mathsf{runs}(w)} = \sum_{n \geq 1} x^{|w|} \sum_{w \in \mathcal{C}_n} y^{\mathsf{runs}(w)},$$

where the coefficient of $x^n y^k$ is the number of Catalan words of length n with k runs of ascents.

In Theorem 3.1, we give an expression for this generating function.

Theorem 3.1. *The generating function for the number of nonempty Catalan words with respect to the length and the number of runs of ascents is*

$$R(x,y) = \frac{1 - x(1-y) - \sqrt{(1-x-xy)^2 - 4x^2 y}}{2x}.$$

Proof. Let w be a nonempty Catalan word, and let $w = \mathtt{0}(w'+1)w''$ be the first return decomposition of w, with $w', w'' \in \mathcal{C}$. If $w' = w'' = \epsilon$, then $w = \mathtt{0}$, and its generating function is xy. If $w'' = \epsilon$ and $w' \neq \epsilon$, then $w = \mathtt{0}(w'+1)$, and its generating function is $xR(x,y)$. Similarly, if $w' = \epsilon$ and $w'' \neq \epsilon$, then $w = \mathtt{0}w''$, and then its generating function is $xyR(x,y)$ because we have made an extra run. Finally, if w' and w'' are not empty, then $w = \mathtt{0}(w'+1)w''$, and then its generating function is given by $xR^2(x,y)$. Therefore, we have the functional equation

$$R(x,y) = xy + x(1+y)R(x,y) + xR^2(x,y).$$

Solving this equation, we obtain the desired result. □

Figure 3 illustrates the cases given in the decomposition described in the proof of Theorem 3.1. The red point denotes the new run of ascents.

Accepted manuscript

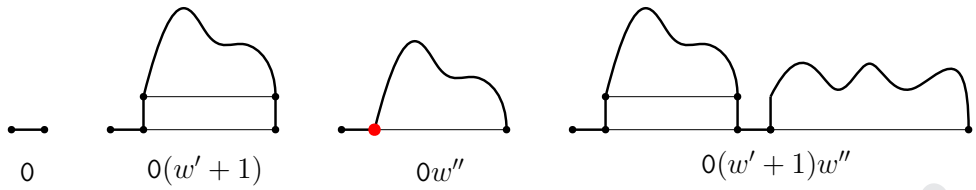


Figure 3: Decomposition of a nonempty Catalan word.

Let $r(n,k)$ denote the number of Catalan words of length n with exactly k runs of ascents, that is $r(n,k) = [x^n y^k]R(x,y)$, which denotes the coefficient of $x^n y^k$ in $R(x,y)$. The first few values of this arrays are

$$\mathcal{R} :- [r(n,k)]_{n,k\geq 1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & \boxed{\mathbf{6}} & 6 & 1 & 0 & 0 & 0 & 0 \\ 1 & 10 & 20 & 10 & 1 & 0 & 0 & 0 \\ 1 & 15 & 50 & 50 & 15 & 1 & 0 & 0 \\ 1 & 21 & 105 & 175 & 105 & 21 & 1 & 0 \\ 1 & 28 & 196 & 490 & 490 & 196 & 28 & 1 \end{pmatrix}.$$

For example, $r(4,2) = 6$, the entry boxed in $\mathcal{R}$ above, and the corresponding Catalan words (and lattice diagrams) are shown in Figure 4.

Figure 4: Catalan words of length 4 with 2 runs of ascents.

The array $\mathcal{R}$ corresponds with the OEIS entry [26, A001263], which is called the Narayana array. Therefore, we have the combinatorial expression

$$r(n,k) = \frac{1}{k}\binom{n-1}{k-1}\binom{n}{k-1}.$$

Notice, that this sequence counts the number of peaks in Dyck paths. Let $r(n)$ be the total number of runs of ascents over all Catalan words of length n.

Corollary 3.2. *For $n \geq 1$, we have $r(n) = \binom{2n-1}{n}$.*

Proof. By differentiating $R(x,y)$ with respect to y and evaluating at $y = 1$, Theorem 3.1 gives

$$\left.\frac{\partial R(x,y)}{\partial y}\right|_{y=1} = \frac{1+\sqrt{1-4x}}{2\sqrt{1-4x}} = \sum_{n\geq 1}\binom{2n-1}{n}x^n.$$

By comparing the nth coefficient, we obtain the desired result. We use the generating function of the central binomial coefficients:

$$\frac{1}{\sqrt{1-4x}} = \sum_{n\geq 0}\binom{2n}{n}x^n. \qquad \square$$

Accepted manuscript

The first few values of the sequence $r(n)$ $(n \geq 1)$ correspond with the OEIS entry [26, A001700]:

$$1, \quad 3, \quad 10, \quad 35, \quad 126, \quad 462, \quad 1716, \quad 6435, \quad 24310, \quad 92378, \ldots.$$

Notice that in the notation of patterns in subwords, see [25], the sequence $r(n)$ counts the total number of maximal patterns of the form $12\cdots(\ell-1)\ell$, with $\ell \geq 1$, in all Catalan words of length n.

3.2 Runs of weak ascents

In order to count nonempty Catalan words according to the length and the number runs of weak ascents, we introduce the following bivariate generating function

$$W(x,y) = \sum_{w\in\mathcal{C}^+} x^{|w|} y^{\mathsf{wruns}(w)} = \sum_{n\geq 1} x^{|w|} \sum_{w\in\mathcal{C}_n} y^{\mathsf{wruns}(w)},$$

where the coefficient of $x^n y^k$ is the number of Catalan words of length n with k runs of weak ascents.

In Theorem 3.3, we give an expression for this generating function.

Theorem 3.3. *The generating function for the number of nonempty Catalan words with respect to the length and the number of runs of weak ascents is*

$$W(x,y) = \frac{1-2x-\sqrt{1-4x+4x^2-4x^2y}}{2x}.$$

Proof. From a similar argument as in the proof of Theorem 3.1, we obtain the functional equation

$$W(x,y) = xy + 2xW(x,y) + xW^2(x,y).$$

Solving this equation, we obtain the desired result. □

Let $w(n,k)$ denote the number of Catalan words of length n with exactly k runs of weak ascents, that is $w(n,k) = [x^n y^k]W(x,y)$, which denotes the coefficient of $x^n y^k$ in $W(x,y)$. The first few values of this array are

$$\mathcal{W} :- [w(n,k)]_{n,k\geq 1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 & 0 \\ 8 & \boxed{\mathbf{6}} & 0 & 0 & 0 \\ 16 & 24 & 2 & 0 & 0 \\ 32 & 80 & 20 & 0 & 0 \\ 64 & 240 & 120 & 5 & 0 \\ 128 & 672 & 560 & 70 & 0 \\ 256 & 1792 & 2240 & 560 & 14 \end{pmatrix}.$$

For example, $w(4,2) = 6$, the entry boxed in $\mathcal{W}$ above, and the corresponding Catalan words (and lattice diagrams) are shown in Figure 5.

Accepted manuscript

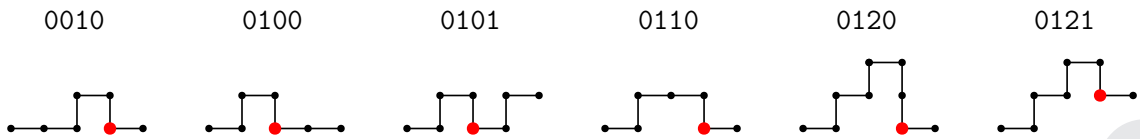


Figure 5: Catalan words of length 4 with 2 runs of weak ascents.

Theorem 3.4. *For $n, k \geq 1$, we have*

$$w(n,k) = \frac{1}{n}\binom{n}{k}\binom{n-k}{k-1}2^{n-2k+1}.$$

Proof. Rewriting the functional equation in the proof of Theorem 3.3, we obtain

$$W(x,y) = xy + 2xW(x,y) + xW^2(x,y) = x\Phi(W(x,y)),$$

where $\Phi(z) = y + 2z + z^2$. From the Lagrange inversion theorem (see for instance [20]), we have

$$\begin{aligned} w(n,k) = [x^n y^k]W(x,y) &= \frac{1}{n}[z^{n-1}y^k](y+2z+z^2)^n \\ &= \frac{1}{n}[z^{n-1}y^k]\sum_{i=0}^{n}\binom{n}{i}y^i(2z+z^2)^{n-i} \\ &= \frac{1}{n}[z^{n-1}]\binom{n}{k}(2z+z^2)^{n-k} \\ &= \frac{1}{n}\binom{n}{k}[z^{n-1-(n-k)}]\sum_{i=0}^{n-k}\binom{n-k}{i}2^{n-k-i}z^i \\ &= \frac{1}{n}\binom{n}{k}\binom{n-k}{k-1}2^{n-2k+1}. \end{aligned}$$

□

From Theorem 3.4, we obtain the combinatorial identity for the Catalan numbers

$$c_n = \frac{1}{n}\sum_{k=1}^{n}\binom{n}{k}\binom{n-k}{k-1}2^{n-2k+1}. \tag{3.1}$$

Open Problem 3.1. Find a combinatorial interpretation for the identity in (3.1).

The array $\mathcal{W}$ corresponds with the OEIS entry [26, A091894]. Among the combinatorial objects counted by this sequence, there are Dyck paths of semilength n having exactly $k-1$ UU's with midpoint at even height. Let $\mathcal{U}_n$ be the set of Dyck paths of semilength n and $\mathcal{U} = \bigcup_{n\geq 0}\mathcal{U}_n$. For instance, there are 6 Dyck paths of semilength 4 having one occurrence UU with midpoint at even height:

$$\begin{aligned} &U(UU)DUDDD, \quad U(UU)UDDDD, \quad UDU(UU)DDD, \\ &U(UU)DDDUD, \quad U(UU)DDUDD, \quad UUD(UU)DDD. \end{aligned}$$

We define recursively the bijection ψ from $\mathcal{C}_n$ to $\mathcal{U}_n$ as follows:

$$\psi(w) = \begin{cases} \epsilon & \text{if } w = \epsilon, \\ UD\psi(u) & \text{if } w = \mathtt{0}(u+1), \\ UU\psi(u_1)DU\cdots DU\psi(u_{a-1})DD\psi(u_a) & \text{if } w = \mathtt{0}(u_1+1)\cdots\mathtt{0}(u_a+1),\ a\geq 2,\ u_a \neq \epsilon, \\ UU\psi(u_1)DU\cdots DU\psi(u_{a-1})DU\psi(u_a)DD & \text{if } w = \mathtt{0}(u_1+1)\cdots\mathtt{0}(u_a+1)\mathtt{0},\ a\geq 1, \end{cases}$$

Accepted manuscript

where u_i, $1 \leq i \leq a$, are Catalan words (possibly empty except for the third case where $u_a \neq \epsilon$), and where, as in the first return decomposition, $(u+1)$ corresponds to the word obtained from u by increasing by one all letters of u. For instance, we have

$$\begin{aligned}\psi(\mathtt{0012231010}) &= \psi(\mathtt{0} \cdot \epsilon \cdot \mathtt{012231} \cdot \mathtt{01} \cdot \mathtt{0}) \\ &= UU\psi(\epsilon)DU\psi(\mathtt{01120})DU\psi(\mathtt{0})DD \\ &= UU \cdot \epsilon \cdot DU \cdot UU\psi(\mathtt{001})DD \cdot DU \cdot UD \cdot DD \\ &= UUDUUU \cdot UU\psi(\epsilon)DD\psi(\mathtt{0}) \cdot DDDUUDDD \\ &= UUDUUUUDDUDDDDUUDDD.\end{aligned}$$

Refer to Figure 6 for an illustration of the above example. The red points denote the midpoints at even height.

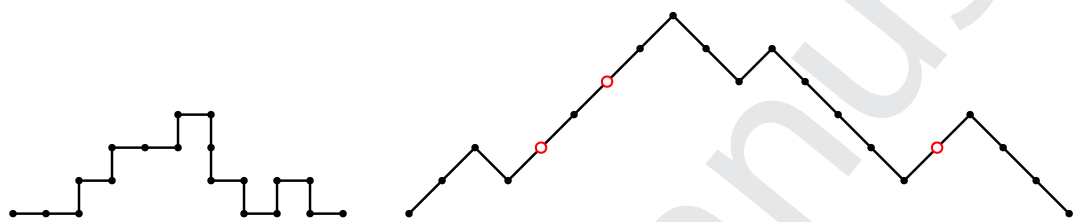


Figure 6: An example for the bijection ψ from $\mathcal{C}_{10}$ to $\mathcal{U}_{10}$.

A simple observation allows us to prove that ψ is a bijection. Now, let us check (by induction on the length) that ψ transports the number of runs of weak ascents minus one into the number of occurrences UU with midpoint at even height. Let $\mathtt{mid}(P)$ be the number of such occurrences in the path P. By convenience, we set $\mathsf{wruns}(\epsilon) = 1$, $\mathtt{mid}(\epsilon) = 0$ and the base case $n = 0$ follows. Let w satisfying $w = \mathtt{0}(1+u_1)\cdots\mathtt{0}(1+u_a)$, $a \geq 2$, $u_a \neq \epsilon$, and let k be the number of nonempty u_i for $1 \leq i \leq a-1$. We have

$$\begin{aligned}\mathtt{mid}(\psi(w)) &= \mathtt{mid}(UU\psi(u_1)DU \cdots DU\psi(u_{a-1})DD\psi(u_a)) \\ &= k + \mathtt{mid}(\psi(u_1)) + \cdots + \mathtt{mid}(\psi(u_{a-1})) + \mathtt{mid}(\psi(u_a)).\end{aligned}$$

Using the recurrence hypothesis, we have $\mathtt{mid}(\psi(u_i)) = \mathsf{wruns}(u_i) - 1$ for $1 \leq i \leq a$. So, we deduce

$$\begin{aligned}\mathtt{mid}(\psi(w)) &= k + \big(\mathsf{wruns}(u_1) - 1\big) + \cdots + \big(\mathsf{wruns}(u_{a-1}) - 1\big) + \big(\mathsf{wruns}(u_a) - 1\big) \\ &= k + \mathsf{wruns}(u_1) + \cdots + \mathsf{wruns}(u_{a-1}) - (a-1) + \mathsf{wruns}(u_a) - 1 \\ &= \mathsf{wruns}(\mathtt{0}(1+u_1)\cdots\mathtt{0}(1+u_a)) - 1 \\ &= \mathsf{wruns}(w) - 1.\end{aligned}$$

The second case where $w = \mathtt{0}(1+u_1)\cdots\mathtt{0}(1+u_a)\mathtt{0}$, $a \geq 1$ is handled *mutatis mutandis*.

Let $w(n)$ be the total number of runs of weak ascents over all Catalan words of length n.

Corollary 3.5. *For $n \geq 1$, we have $w(n) = \binom{2n-2}{n-1}$.*

The first few values of the sequence $w(n)$ corresponds to the OEIS entry [26, A000984]

$$1, \quad 2, \quad 6, \quad 20, \quad 70, \quad 252, \quad 924, \quad 3432, \quad 12870, \ldots.$$

Our combinatorial interpretation does not appear in the OEIS.

Accepted manuscript

3.3 Runs of descents

In order to count nonempty Catalan words according to the length and the number runs of descents, we introduce the following bivariate generating function

$$\bar{R}(x,y) = \sum_{w\in\mathcal{C}^+} x^{|w|} y^{\overline{\mathsf{runs}}(w)} = \sum_{n\geq 1} x^{|w|} \sum_{w\in\mathcal{C}_n} y^{\overline{\mathsf{runs}}(w)},$$

where the coefficient of $x^n y^k$ is the number of Catalan words of length n with k runs of descents.

In Theorem 3.6, we give an expression for this generating function.

Theorem 3.6. *The generating function for the number of nonempty Catalan words with respect to the length and the number of runs of descents is*

$$\bar{R}(x,y) = \frac{1-2xy-\sqrt{4x^2y^2-4x^2y-4xy+1}}{2x}$$

Proof. Let w be a nonempty Catalan word, and let $w = \mathtt{0}(w'+1)w''$ be the first return decomposition of w, with $w', w'' \in \mathcal{C}$. If $w' = w'' = \epsilon$, then $w = \mathtt{0}$, and its generating function is xy. If $w'' = \epsilon$ and $w' \neq \epsilon$, then $w = \mathtt{0}(w'+1)$, and then its generating function is $xy\bar{R}(x,y)$. Similarly, if $w' = \epsilon$ and $w'' \neq \epsilon$, then $w = \mathtt{0}w''$, and then its generating function is $xy\bar{R}(x,y)$. Finally, if w' and w'' are not empty, then $w = \mathtt{0}(w'+1)w''$, and then its generating function is given by $x\bar{R}^2(x,y)$. Therefore, we have the functional equation

$$\bar{R}(x,y) = xy + 2xy\bar{R}(x,y) + x\bar{R}^2(x,y).$$

Solving this equation yields the desired result. □

Let $\bar{r}(n,k)$ denote the number of Catalan words of length n with exactly k runs of descents, that is $\bar{r}(n,k) = [x^n y^k]\bar{R}(x,y)$, which denotes the coefficient of $x^n y^k$ in $\bar{R}(x,y)$. The first few values of this arrays are

$$\bar{\mathcal{R}} :- [\bar{r}(n,k)]_{n,k\geq 1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 24 & 16 & 0 & 0 & 0 \\ 0 & 0 & 0 & 20 & 80 & 32 & 0 & 0 \\ 0 & 0 & 0 & 5 & 120 & 240 & 64 & 0 \\ 0 & 0 & 0 & 0 & 70 & 560 & 672 & 128 \end{pmatrix}.$$

The array $\bar{\mathcal{R}}$ does not appear in the OEIS.

Corollary 3.7. *For $n \geq 1$, we have $\bar{r}(n) = \binom{2n}{n} - \binom{2n-2}{n-1}$.*

Proof. By differentiating $\bar{R}(x,y)$ with respect to y and evaluating at $y = 1$, Theorem 3.6 gives

$$\left.\frac{\partial \bar{R}(x,y)}{\partial y}\right|_{y=1} = \frac{1-x-\sqrt{1-4x}}{\sqrt{1-4x}}.$$

Accepted manuscript

By comparing the nth coefficient, we obtain the desired result. We use the generating function of the central binomial coefficients:

$$\frac{1}{\sqrt{1-4x}} = \sum_{n\geq 0} \binom{2n}{n} x^n. \qquad \square$$

The first few values of the sequence $\bar{r}(n)$ $(n \geq 1)$ correspond with the OEIS entry [26, A051924]:

$$1, \quad 4, \quad 14, \quad 50, \quad 182, \quad 672, \quad 2508, \quad 9438, \quad 35750, \quad 136136, \ldots.$$

3.4 Runs of Weak Descents

In a Catalan word of length n, the number of runs of ascents plus the number of runs of weak descents equals $n+1$. Due to the relation $r(n,k) = r(n, n+1-k)$ in the Narayana array (see Section 3.1), the number $\bar{w}(n,k)$ of Catalan words of length n with k runs of weak descents equals the number $r(n,k)$ of Catalan words of length n with k runs of ascents. Then, we refer to Section 3.1.

4 The distribution of valleys

4.1 Valleys

In order to count Catalan words according to the length and the number ℓ-valleys, we introduce the following bivariate generating function

$$V_\ell(x,y) = \sum_{w\in\mathcal{C}^+} x^{|w|} y^{\ell\text{-val}(w)} = \sum_{n\geq 1} x^{|w|} \sum_{w\in\mathcal{C}_n} y^{\ell\text{-val}(w)},$$

where ℓ-val(w) denotes the number of occurrences of subwords of the form $ab^\ell(b+1)$, and $a > b$, in w. The coefficient of $x^n y^k$ in $V_\ell(x,y)$ is the number of Catalan words of length n with k ℓ-valleys.

In Theorem 4.1, we give an expression for this generating function.

Theorem 4.1. *The generating function for nonempty Catalan words with respect to the length and the number of ℓ-valleys is*

$$V_\ell(x,y) = \frac{1-2x-x^{\ell+1}(1-y)-\sqrt{1-4x+2x^{\ell+1}(1-y)+x^{2(\ell+1)}(1-y)^2}}{2x(1-x^{\ell-1}(1-y)+x^\ell(1-y))}.$$

Proof. Let w be a nonempty Catalan word, and let $w = \mathtt{0}(w'+1)w''$ be the first return decomposition, with $w', w'' \in \mathcal{C}$. If $w' = w'' = \epsilon$, then $w = \mathtt{0}$, and its generating function is x. If $w' \neq \epsilon$ and $w'' = \epsilon$, then $w = \mathtt{0}(w'+1)$, and its generating function is $xV_\ell(x,y)$. Similarly, if $w' = \epsilon$ and $w'' \neq \epsilon$, then $w = \mathtt{0}w''$, and its generating function is $xV_\ell(x,y)$. Finally, if $w' \neq \epsilon$ and $w'' \neq \epsilon$, then $w = \mathtt{0}(w'+1)w''$, and we distinguish two cases. If w'' is of the form $\mathtt{0}^{\ell-1}w'''$, where w''' starts with $\mathtt{01}$, then $w = \mathtt{0}(w'+1)\mathtt{0}^{\ell-1}w'''$, and the generating function is $xyV_\ell(x,y)x^{\ell-1}A_\ell(x,y)$, where

$$A_\ell(x,y) = V_\ell(x,y) - x(1+V_\ell(x,y))$$

Accepted manuscript

is obtained using the complement of the generating function for the word 0 and the words starting with 00. Otherwise, the generating function is

$$xV_\ell(x,y)(V_\ell(x,y) - x^{\ell-1}A_\ell(x,y)).$$

Therefore, we have the functional equation

$$V_\ell(x,y) = x + 2xV_\ell(x,y) + xyV_\ell(x,y)x^{\ell-1}\left(V_\ell(x,y) - x(1+V_\ell(x,y)) + \right.$$
$$\left. xV_\ell(x,y)\left(V_\ell(x,y) - x^{\ell-1}\left(V_\ell(x,y) - x(1+V_\ell(x,y))\right)\right)\right).$$

Solving this equation, we obtain the desired result. □

Let $v_\ell(n,k)$ denote the number of Catalan words of length n with exactly k ℓ-valleys, that is $v_\ell(n,k) = [x^n y^k]V_\ell(x,y)$, which denotes the coefficient of $x^n y^k$ in $V_\ell(x,y)$. For example, the first few values of this array for $\ell = 2$ are

$$\mathcal{V}_2 := [v_2(n,k)]_{n\geq 1, k\geq 0} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 \\ 14 & 0 & 0 & 0 \\ 41 & 1 & 0 & 0 \\ 125 & \boxed{\mathbf{7}} & 0 & 0 \\ 393 & 36 & 0 & 0 \\ 1266 & 163 & 1 & 0 \\ 4158 & 693 & 11 & 0 \end{pmatrix}.$$

For example, $v_2(6,1) = 7$, the entry boxed in $\mathcal{V}_2$ above, and the corresponding Catalan words (and lattice diagrams) are shown in Figure 7. The red segments denote the valley.

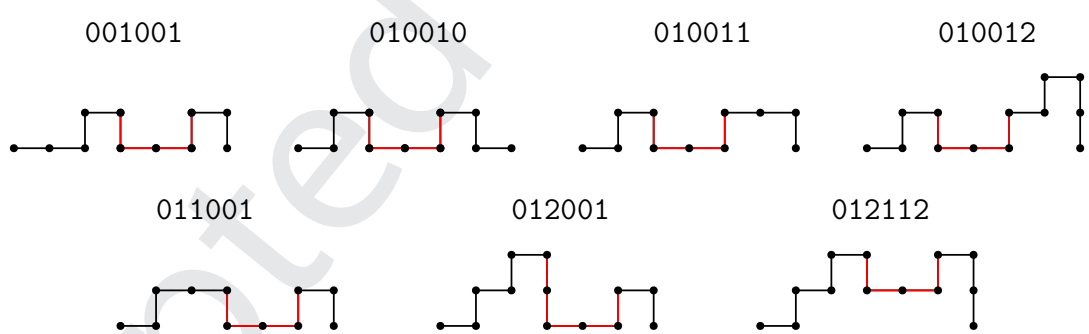


Figure 7: Catalan words of length 6 with one 2-valley.

Let $v_\ell(n)$ be the sum of all ℓ-valleys in the set of Catalan words of length n.

Corollary 4.2. *The generating function of the sequence $v_\ell(n)$ is*

$$\sum_{n\geq 1} v_\ell(n)x^n = \frac{x^{\ell-2}\left(1 - 5x + 5x^2 - (1-3x+x^2)\sqrt{1-4x}\right)}{2\sqrt{1-4x}}.$$

Moreover, for all $n \geq 1$, we have

$$v_\ell(n) = \binom{2(n-\ell)-1}{n-\ell-3}, \tag{4.1}$$

Accepted manuscript

and an asymptotic approximation for $v_\ell(n)$ is

$$\frac{2^{2(n-\ell)-1}}{\sqrt{\pi n}}.$$

We compute asymptotics of the combinatorial sequences using classical singularity analysis (cf. [11]).

Taking $\ell = 1$ in Theorem 4.1, we obtain the generating function for nonempty Catalan words with respect to the length and the number of short valleys

$$V_1(x,y) = \sum_{w\in\mathcal{C}^+} x^{|w|} y^{\text{1-val}(w)} = \frac{1-2x-x^2(1-y)-\sqrt{1-4x+2x^2(1-y)+x^4(1-y)^2}}{2x(y+x(1-y))}.$$

Notice that in the notation of subwords patterns, see [25], the sequence $v_1(n)$ counts the joint distribution of the patterns 212 and 312.

Let $v_1(n,k)$ denote the number of Catalan words of length n with exactly k short valleys, that is $v_1(n,k) = [x^n y^k]V_1(x,y)$, which denotes the coefficient of $x^n y^k$ in $V_1(x,y)$. The first few values of this array are

$$\mathcal{V}_1 = [v_1(n,k)]_{n\geq 1, k\geq 0} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 & 0 \\ 13 & 1 & 0 & 0 & 0 \\ 35 & \boxed{\mathbf{7}} & 0 & 0 & 0 \\ 97 & 34 & 1 & 0 & 0 \\ 275 & 143 & 11 & 0 & 0 \\ 794 & 558 & 77 & 1 & 0 \\ 2327 & 2083 & 436 & 16 & 0 \end{pmatrix}.$$

For example, $v_1(5,1) = 7$, the entry boxed in $\mathcal{V}_1$ above, and the corresponding Catalan words (and lattice diagrams) are shown in Figure 8.

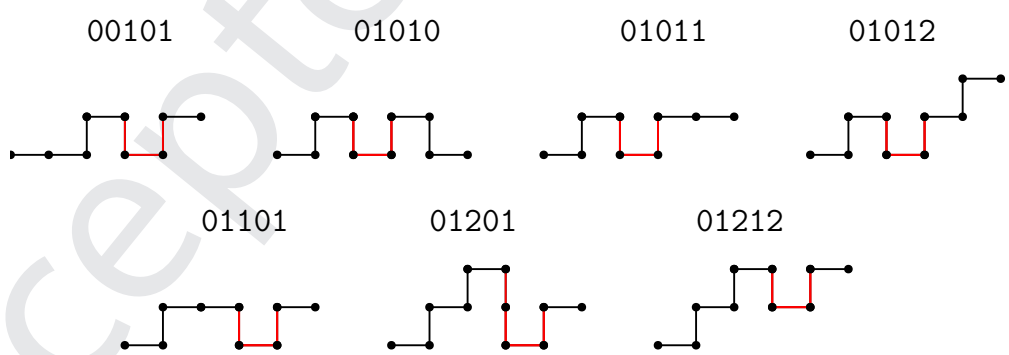


Figure 8: Catalan words of length 5 with one short valley.

The matrix $\mathcal{V}_1$ corresponds to the OEIS sequence [26, A114492]. Among the combinatorial objects counted by this sequence are Dyck paths of semilength n having exactly k occurrences of $DDUU$. For instance, the 7 Dyck paths of semilength 5 having one short valley are

Accepted manuscript

$$\begin{array}{ll} UDUU(DDUU)DD, & UUDU(DDUU)DD, \\ UU(DDUU)DDUD, & UUUD(DDUU)DD, \\ UU(DDUU)UUUDDD, & UU(DDUU)UUDUDD, \\ UUUD(DDUU)DDD. & \end{array}$$

We recursively define the bijection χ from $\mathcal{C}_n$ to the set of Dyck paths of semilength n as follows:

$$\chi(w) = \begin{cases} \epsilon & \text{if } w = \epsilon, \\ U\chi(u)D & \text{if } w = \mathtt{0}(1+u), \\ UD\chi(u) & \text{if } w = \mathtt{0}u, \\ U\chi(u)D\chi(v) & \text{if } w = \mathtt{0}(1+u)v,\ u, v \neq \epsilon, \end{cases}$$

where $(1+u)$ corresponds to the word obtained from u by increasing by one all letters of u. For instance, we have

$$\begin{aligned} \chi(\mathtt{0012231010}) &= \chi(\mathtt{0} \cdot \mathtt{012231010}) \\ &= UD\chi(\mathtt{0} \cdot \mathtt{12231} \cdot \mathtt{010}) \\ &= UDU\chi(\mathtt{0} \cdot \mathtt{112} \cdot \mathtt{0})D\chi(\mathtt{0} \cdot \mathtt{1} \cdot \mathtt{0}) \\ &= UDUUUDUUDDDUDDUUDDUD. \end{aligned}$$

A simple observation allows us to prove that χ is a bijection. Now, let us check (by induction on the length) that χ transports the number of short valleys into the number of occurrences of $DDUU$. Let $\mathtt{dduu}(P)$ be the number of such occurrences in the path P. The case $n = 0$ is trivial. Let us consider the general decomposition of a Catalan word $w = \mathtt{0}(1+u)v$. The cases where u or v are empty is easy to check, so we assume that u and v nonempty. We have $\mathtt{dduu}(\chi(w)) = \mathtt{dduu}(U\chi(u)D\chi(v))$. We distinguish two cases.

Case (i): v starts with $\mathtt{01}$. By definition, $\chi(v)$ starts with UU and we have

$$\mathtt{dduu}(\psi(w)) = \mathtt{dduu}(U\chi(u)D\chi(v)) = \mathtt{dduu}(\chi(u)) + 1 + \mathtt{dduu}(\chi(v)).$$

Using the recurrence hypothesis, we deduce

$$\mathtt{dduu}(\psi(w)) = 1 + \mathsf{sval}(u) + \mathsf{sval}(v) = \mathsf{sval}(w).$$

Case (ii): v does not start with $\mathtt{01}$. In the same way, we have

$$\begin{aligned} \mathtt{dduu}(\psi(w)) &= \mathtt{dduu}(U\chi(u)D\chi(v)) \\ &= \mathtt{dduu}(\chi(u)) + \mathtt{dduu}(\chi(v)) \\ &= \mathsf{sval}(u) + \mathsf{sval}(v) = \mathsf{sval}(w). \end{aligned}$$

Using a similar proof as for Theorem 4.1, we generalize the result in order to obtain the following generating function for the number Catalan words of length n with respect to the number of valleys (we consider all ℓ-valleys for $\ell \geq 1$).

Theorem 4.3. *The generating function for nonempty Catalan words with respect to the length and the number of valleys is*

$$V(x,y) = \frac{1 - 3x + x^2 + x^2y - \sqrt{(1 - 3x + x^2 + x^2y)^2 - 4x^2y(1-x)^2}}{2xy(1-x)}.$$

Accepted manuscript

Let $v(n,k)$ denote the number of Catalan words of length n with exactly k valleys, that is $v(n,k) = [x^n y^k]V(x,y)$, which denotes the coefficient of $x^n y^k$ in $V(x,y)$. The first few values of this arrays are

$$\mathcal{V} = [v(n,k)]_{n\geq 1, k\geq 0} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 & 0 \\ 13 & 1 & 0 & 0 & 0 \\ 34 & \boxed{\mathbf{8}} & 0 & 0 & 0 \\ 89 & 42 & 1 & 0 & 0 \\ 233 & 183 & 13 & 0 & 0 \\ 610 & 717 & 102 & 1 & 0 \\ 1597 & 2622 & 624 & 19 & 0 \end{pmatrix}.$$

Note that $v(n,1) = F_{2n-1}$ $(n \geq 1)$, where F_n is the nth Fibonacci number. The array $\mathcal{V}$ corresponds to the OEIS entry [26, A114502], where the nth term counts the number of ordered trees with n edges and having exactly $k+1$ nodes all of whose children are leaves. For example, $v(5,1) = 8$, the entry boxed in $\mathcal{V}$ above, and the corresponding Catalan words and trees are shown in Figure 9.

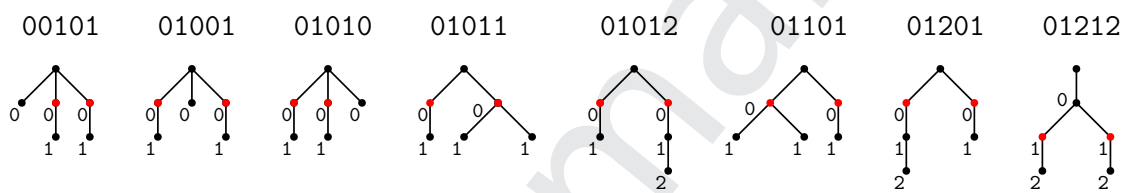


Figure 9: Bijection ψ.

Theorem 4.4. *The set of ordered trees with n edges and having exactly $k+1$ nodes all of whose children are leaves is in bijection with the set of Catalan words of length n having k valleys.*

Proof. Recall that an ordered tree is either a single node, called root and denoted $()$ or equivalently $\bullet$, or a root whose some ordered subtrees $S_1, S_2, \ldots, S_k$, $k \geq 1$, denoted $(S_1, S_2, \ldots, S_k)$. A leaf of an ordered tree is a node with no children. We define recursively the bijection ψ from Catalan words of length n to ordered trees with n edges as follows:

(i) $\psi(w) = \bullet$, if $w = \epsilon$,

(ii) $\psi(w) = (\psi(v_1), \psi(v_2), \ldots, \psi(v_k))$, if $w = \mathtt{0}(1+v_1)\cdots\mathtt{0}(1+v_k)$, $k \geq 1, 1 \leq i \leq k$.

Clearly, ψ is a bijection, and the word w corresponds to the level minus one of the nodes (except the root) read in the preorder traversal of $\psi(w)$ (see Figure 9). Now let us prove by induction on the length that ψ transports the number $\mathsf{val}(w)$ of valleys plus one into the number $\mathtt{nl}(\psi(w))$ of nodes whose all children are leaves in the tree $\psi(w)$. Assuming that $w = \mathtt{0}(1+v_1)\cdots\mathtt{0}(1+v_k)$, where v_i are possibly empty, we have

$$\mathtt{nl}(\psi(w)) = \mathtt{nl}\left((\psi(v_1), \psi(v_2), \ldots, \psi(v_k))\right).$$

We distinguish two cases.

Accepted manuscript

Case (i): For all j, $1 \leq j \leq k$, we have $v_j = \epsilon$. Thus, we have $w = 0^k$, $\psi(w) = (\bullet, \ldots, \bullet)$, and $\mathtt{nl}(\psi(w)) = 1 = \mathsf{val}(w) + 1$.

Case (ii): Otherwise, let $j_1, j_2, \ldots, j_a$, $a \geq 1$, the indices j where $v_j \neq \epsilon$. Thus, we have $\mathtt{nl}(\psi(w)) = \mathtt{nl}(\psi(v_{j_1})) + \cdots + \mathtt{nl}(\psi(v_{j_a}))$. Using the recurrence hypothesis, we obtain $\mathtt{nl}(\psi(w)) = \mathsf{val}(v_{j_1}) + \cdots + \mathsf{val}(v_{j_a}) + a = \mathsf{val}(w) + 1$, which completes the induction. □

Let $v(n)$ be the sum of all valleys in the set of Catalan words of length n.

Corollary 4.5. *The generating function of the sequence $v(n)$ is*

$$\sum_{n \geq 0} v(n) x^n = \frac{1 - 5x + 5x^2 - (1 - 3x + x^2)\sqrt{1 - 4x}}{2(1 - x)x\sqrt{1 - 4x}}.$$

Moreover, for all $n \geq 1$, we have

$$v(n) = \sum_{\ell=1}^{n-1} \binom{2(n - \ell) - 1}{n - \ell - 3}, \tag{4.2}$$

and an asymptotic approximation for $v(n)$ is

$$\frac{2^{2n}}{6\sqrt{\pi n}}.$$

For $n \geq 4$, the first few values of the sequence $v(n)$ are

$$1, \quad 8, \quad 44, \quad 209, \quad 924, \quad 3927, \quad 16303, \quad 66691, \quad 270181, \ldots.$$

This sequence does not appear in the OEIS.

Open Problem 4.1. Find a combinatorial proof for the equality in (4.2).

4.2 Symmetric valleys

A *symmetric valley* is a valley of the form $a(a - 1)^\ell a$ with $\ell \geq 1$. Let $\mathsf{symv}(w)$ denote the number of symmetric valleys in the word w. In order to count Catalan words according to the length and the number of symmetric valleys, we introduce the following bivariate generating function generating function

$$S(x, y) = \sum_{w \in \mathcal{C}^+} x^{|w|} y^{\mathsf{symv}(w)} = \sum_{n \geq 1} x^{|w|} \sum_{w \in \mathcal{C}_n} y^{\mathsf{symv}(w)},$$

where the coefficient of $x^n y^k$ in $S(x, y)$ is the number of nonempty Catalan words of length n with k symmetric ℓ-valleys.

We remark that the symmetric valley statistic has been recently studied in the context of k-ary words [1], Dyck paths [10, 12], and integer compositions [15].

Theorem 4.6. *The generating function of the nonempty Catalan words with respect to the length and the number of symmetric valleys is*

$$S(x, y) = \frac{1 - 3x + x^2(3 - y) - 2x^3(1 - y) - \sqrt{1 - 6x + x^2(11 - 2y) - 2x^3(5 - 3y) + x^4(5 - 6y + y^2)}}{2x(1 - x)(1 - x(1 - y))}.$$

Accepted manuscript

Proof. Let w be a nonempty Catalan word, and let $w = \mathtt{0}(w'+1)w''$ be the first return decomposition, with $w', w'' \in \mathcal{C}$. If $w' = w'' = \epsilon$, then $w = \mathtt{0}$, and its generating function is x. If $w' \neq \epsilon$ and $w'' = \epsilon$, then $w = \mathtt{0}(w'+1)$, and its generating function is $xS(x,y)$. Similarly, if $w' = \epsilon$ and $w'' \neq \epsilon$, then $w = \mathtt{0}w''$ and its generating function is $xS(x,y)$. Finally, if w' and w'' are not empty, then $w = \mathtt{0}(w'+1)w''$ and we have a new symmetric valley in the case where w' ends with $\mathtt{0}$ and $w'' \neq \mathtt{0}^\ell$ ($\ell \geq 1$). The generating function for this case is

$$xS(x,y)^2 + x(xS(x,y)+x)y\left(S(x,y) - \frac{x}{1-x}\right) - x(xS(x,y)+x)\left(S(x,y) - \frac{x}{1-x}\right).$$

Summing all the cases and solving the obtained functional equation yields the desired result. □

Let $s(n,k)$ denote the number of Catalan words of length n with exactly k symmetric valleys, that is $s(n,k) = [x^n y^k]S(x,y)$, which denotes the coefficient of $x^n y^k$ in $S(x,y)$. The first few values of this arrays are

$$\mathcal{S} = [s(n,k)]_{n\geq 1, k\geq 0} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 & 0 \\ 13 & 1 & 0 & 0 & 0 \\ 35 & \boxed{\mathbf{7}} & 0 & 0 & 0 \\ 98 & 33 & 1 & 0 & 0 \\ 284 & 135 & 10 & 0 & 0 \\ 846 & 519 & 64 & 1 & 0 \\ 2576 & 1933 & 340 & 13 & 0 \end{pmatrix}.$$

For example, $s(5,1) = 7$, the entry boxed in $\mathcal{S}$ above, and the corresponding Catalan words of length 5 with 1 symmetric valley are

`00101, 01001, 01010, 01011, 01012, 01101, 01212.`

Let $s(n)$ be the sum of all symmetric valleys in the set of Catalan words of length n.

Corollary 4.7. *The generating function of the sequence $s(n)$ is*

$$\sum_{n\geq 0} s(n)x^n = \frac{1-4x+2x^2-(1-2x)\sqrt{1-4x}}{2(1-x)\sqrt{1-4x}}.$$

Moreover, for all $n \geq 1$, we have

$$s(n) = (3n-2)c_{n-1} - \frac{1}{2}\sum_{k=1}^{n}\binom{2k}{k}, \tag{4.3}$$

and an asymptotic approximation for $s(n)$ is

$$\frac{2^{2n}}{12\sqrt{\pi n}}.$$

Accepted manuscript

For $n \geq 4$, the first few values of the sequence $s(n)$ are

$$1, \quad 7, \quad 35, \quad 155, \quad 650, \quad 2652, \quad 10660, \quad 42484, \quad 168454, \quad 665874, \ldots.$$

This sequence does not appear in the OEIS. An asymptotic approximation for the ratio between $s(n)$ and $v(n)$ is $1/2$.

Open Problem 4.2. Find a combinatorial proof for the expression for $s(n)$ given in equation (4.3), as stated in Corollary 4.7.

5 The distribution of peaks

5.1 Peaks

It is clear that the number of weak valleys of a Catalan word w is exactly the number of weak peaks of w minus one, but there is no natural link between the number of ℓ-valleys and the number of ℓ-peaks. So, in order to count Catalan words according to the length and the number of ℓ-peaks, we introduce the following bivariate generating function

$$P_\ell(x,y) = \sum_{w \in \mathcal{C}^+} x^{|w|} y^{\ell\text{-peak}(w)} = \sum_{n \geq 1} x^{|w|} \sum_{w \in \mathcal{C}_n} y^{\ell\text{-peak}(w)},$$

where ℓ-peak(w) denotes the number of occurrences of subwords of the form $a(a+1)^\ell b$, and $a \geq b$, in w. The coefficient of $x^n y^k$ in $P_\ell(x,y)$ is the number of Catalan words of length n with k ℓ-peaks.

In Theorem 5.1, we give an expression for this generating function.

Theorem 5.1. *The generating function for nonempty Catalan words with respect to the length and the number of ℓ-peaks is*

$$P_\ell(x,y) = \frac{1 - 2x + x^{\ell+1}(1-y) - \sqrt{1 - 4x + 2x^{\ell+1}(1-y) + x^{2(\ell+1)}(1-y)^2}}{2(x^{\ell+1}(y-1) + x)}.$$

Proof. From a similar argument as in the proof of Theorem 4.1, we have the functional equation:

$$\begin{aligned} P_\ell(x,y) = x + 2xP_\ell(x,y) &+ \frac{x}{1-x}(P_\ell(x,y) - 2xP_\ell(x,y) - x)P_\ell(x,y) \\ &+ xP_\ell(x,y)\left(\frac{x}{1-x} - x^\ell\right) + xyx^\ell P_\ell(x,y) \\ &+ x\left(P_\ell(x,y)\frac{x}{1-x} - P_\ell(x,y)x^\ell\right)P_\ell(x,y) + xyP_\ell(x,y)^2 x^\ell. \end{aligned}$$

Solving this equation yields the desired result. □

Let $p_\ell(n)$ be the sum of all ℓ-peaks in the set of Catalan words of length n.

Corollary 5.2. *The generating function of the sequence $p_\ell(n)$ is*

$$\sum_{n \geq 1} p_\ell(n) x^n = \frac{x^{\ell-1}\left(1 - 3x - (1-x)\sqrt{1-4x}\right)}{2\sqrt{1-4x}}.$$

Accepted manuscript

Moreover, for all $n \geq 1$ we have

$$p_\ell(n) = \binom{2(n-\ell)-1}{n-\ell-2}, \tag{5.1}$$

and an asymptotic approximation for $p_\ell(n)$ is

$$\frac{2^{2(n-\ell)-1}}{\sqrt{\pi n}}.$$

Taking $\ell = 1$ in Theorem 5.1, establishes that the generating function for Catalan words with respect to the length and the number of short peaks is

$$P_1(x,y) = \frac{1-2x+x^2(1-y)-\sqrt{1-4x+2x^2(1-y)+x^4(1-y)^2}}{2(x^2(y-1)+x)}.$$

Let $p_1(n,k)$ denote the number of Catalan words of length n with exactly k short peaks, that is $p_1(n,k) = [x^n y^k]P_1(x,y)$, which denotes the coefficient of $x^n y^k$ in $P_1(x,y)$. The first few values of this array are

$$\mathcal{P}_1 = [p_1(n,k)]_{n\geq 1, k\geq 0} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 & 0 \\ 9 & 5 & 0 & 0 & 0 \\ 22 & 19 & 1 & 0 & 0 \\ 57 & 66 & 9 & 0 & 0 \\ 154 & 221 & 53 & 1 & 0 \\ 429 & 729 & 258 & 14 & 0 \\ 1223 & 2391 & 1131 & 116 & 1 \end{pmatrix}.$$

The array $\mathcal{P}_1$ corresponds to the OEIS entry [26, A116424]. Notice that in the notation of subwords patterns, see [25], the sequence $p_1(n)$ counts the joint distribution of the patterns 121 and 231.

Using a similar proof as for Theorem 5.1, we generalize the result in order to obtain the following generating function for the number Catalan words of length n with respect to the number of peaks.

Theorem 5.3. *The generating function for Catalan words with respect to the length and the number of peaks is*

$$P(x,y) = \frac{1-3x-x^2y+3\,x^2-\sqrt{1-6x+11x^2-6x^3+x^4-2x^2y+2x^3y-2x^4y+x^4y^2}}{2x\,(xy-2\,x+1)}.$$

Let $p(n,k)$ denote the number of Catalan words of length n with exactly k peaks, that is $p(n,k) = [x^n y^k]P(x,y)$, which denotes the coefficient of $x^n y^k$ in $P(x,y)$. The first

Accepted manuscript

few values of this arrays are

$$\mathcal{P} = [p(n,k)]_{n\geq 1, k\geq 0} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 & 0 \\ 8 & \boxed{\mathbf{6}} & 0 & 0 & 0 \\ 16 & 25 & 1 & 0 & 0 \\ 32 & 89 & 11 & 0 & 0 \\ 64 & 290 & 74 & 1 & 0 \\ 128 & 893 & 392 & 17 & 0 \\ 256 & 2645 & 1796 & 164 & 1 \end{pmatrix}.$$

This array does not appear in the OEIS. For example, $p(4,1) = 6$, the entry boxed in $\mathcal{P}$ above, and the corresponding Catalan words of length 4 with one peak are shown in Figure 10.

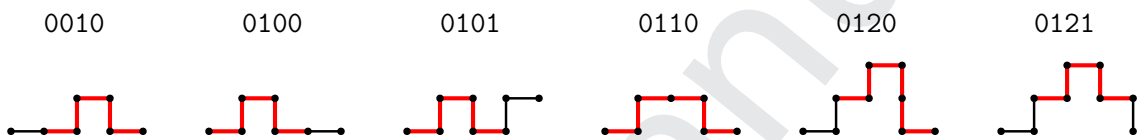


Figure 10: Catalan words of length 4 with one peak.

Let $p(n)$ be the sum of all peaks in the set of Catalan words of length n.

Corollary 5.4. *The generating function of the sequence $p(n)$ is*

$$\sum_{n\geq 0} p(n)x^n = \frac{1-3\,x-(1-x)\sqrt{1-4\,x}}{2(1-x)\sqrt{1-4x}},$$

and an asymptotic approximation for $p(n)$ is

$$\frac{2^{2n}}{6\sqrt{\pi n}}.$$

The first few values of the sequence $p(n)$ ($n \geq 3$) are

$$1, \quad 6, \quad 27, \quad 111, \quad 441, \quad 1728, \quad 6733, \quad 26181, \quad 101763, \quad 395693, \ldots.$$

This sequence does not appear in the OEIS. Since $p(n) = \sum_{\ell\geq 1} p_\ell(n)$, by equation (5.1), we deduce

$$p(n) = \sum_{\ell=1}^{n-1} \binom{2(n-\ell)-1}{n-\ell-2}.$$

5.2 Symmetric peaks

A *symmetric peak* is a peak of the form $a(a+1)^\ell a$ with $\ell \geq 1$. Let $\mathsf{symp}(w)$ denote the number of the symmetric peaks of the word w. In order to count Catalan words according to the length and the number symmetric peaks, we introduce the following bivariate

Accepted manuscript

generating function

$$T(x,y) = \sum_{w\in\mathcal{C}^+} x^{|w|} y^{\mathsf{symp}(w)} = \sum_{n\geq 1} x^{|w|} \sum_{w\in\mathcal{C}_n} y^{\mathsf{symp}(w)},$$

where the coefficient of $x^n y^k$ in $T(x,y)$ is the number of Catalan words of length n with k symmetric peaks.

In Theorem 5.5, we give an expression for this generating function.

Theorem 5.5. *The generating function of the nonempty Catalan words with respect to the length and the number of symmetric peaks is*

$$T(x,y) = \frac{x^2y - 3\,x^2 + 3\,x - 1 + \sqrt{x^4y^2 - 6\,x^4y + 5\,x^4 + 6\,x^3y - 10\,x^3 - 2\,x^2y + 11\,x^2 - 6\,x + 1}}{2x\,(-1+x)}.$$

Proof. Let w be a nonempty Catalan word, and let $w = \mathtt{0}(w'+1)w''$ be the first return decomposition, with $w', w'' \in \mathcal{C}$. If $w' = w'' = \epsilon$, then $w = \mathtt{0}$, and its generating function is x. If $w' \neq \epsilon$ and $w'' = \epsilon$, then $w = \mathtt{0}(w'+1)$, and its generating function is $xT(x,y)$. Similarly, if $w' = \epsilon$ and $w'' \neq \epsilon$, then $w = \mathtt{0}w''$, and its generating function is $xT(x,y)$. Finally, if w' and w'' are not empty, then $w = \mathtt{0}(w'+1)w''$, and we have a new symmetric peak in the case where w' is $\mathtt{0}^k$, $k \geq 1$. The generating for this case is

$$\frac{x^2y}{1-x}T(x,y) + x\left(T(x,y) - \frac{x}{1-x}\right)T(x,y).$$

Summing all the cases and solving the obtained functional equation yields the desired result. □

Let $t(n,k)$ denote the number of Catalan words of length n with exactly k symmetric peaks, that is $t(n,k) = [x^n y^k]T(x,y)$, which denotes the coefficient of $x^n y^k$ in $T(x,y)$. The first few values of this arrays are

$$\mathcal{T} = [t(n,k)]_{n\geq 1, k\geq 0} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 & 0 \\ 9 & \boxed{\mathbf{5}} & 0 & 0 & 0 \\ 23 & 18 & 1 & 0 & 0 \\ 64 & 60 & 8 & 0 & 0 \\ 187 & 199 & 42 & 1 & 0 \\ 563 & 667 & 189 & 11 & 0 \\ 1731 & 2259 & 795 & 76 & 1 \end{pmatrix}.$$

For example, $t(4,1) = 5$, the entry boxed in $\mathcal{T}$ above, and the corresponding Catalan words of length 4 with 1 symmetric peak are

$$\mathtt{0100}, \quad \mathtt{0101}, \quad \mathtt{0010}, \quad \mathtt{0110}, \quad \mathtt{0121}.$$

Let $t(n)$ be the sum of all symmetric peaks in the set of Catalan words of length n.

Accepted manuscript

Corollary 5.6. *The generating function of the sequence $t(n)$ is*

$$\sum_{n\geq 0} t(n)x^n = \frac{x\left(2\,x^2 - 3\,x + 1 - (1-x)\sqrt{1-4x}\right)}{2\,(1-x)^2\,\sqrt{1-4x}}.$$

Moreover, for all $n \geq 1$, we have

$$t(n) = \sum_{k=0}^{n-3} \binom{2k+2}{k},$$

and an asymptotic approximation for $t(n)$ is

$$\frac{2^{2n}}{12\sqrt{\pi n}}.$$

For $n \geq 3$, the first few values of the sequence $t(n)$ are

$$1, \quad 5, \quad 20, \quad 76, \quad 286, \quad 1078, \quad 4081, \quad 15521, \quad 59279, \quad 227239, \ldots.$$

This sequence corresponds to the OEIS entry [26, A057552]. An asymptotic approximation for the ratio between $t(n)$ and $p(n)$ is $1/2$.

References

[1] W. Asakly, Enumerating symmetric and non-symmetric peaks in words, *Online J. Anal. Comb.* **13** (2018), Id/No 2, 7 pp.

[2] J.-L. Baril, D. Colmenares, J. L. Ramírez, E. D. Silva, L. M. Simbaqueba and D. A. Toquica, Consecutive pattern-avoidance in Catalan words according to the last symbol, *RAIRO, Theor. Inform. Appl.* **58** (2024), Id/No 1, 21 pp., doi:10.1051/ita/2024001, https://doi.org/10.1051/ita/2024001.

[3] J.-L. Baril, J. F. González and J. L. Ramírez, Last symbol distribution in pattern avoiding Catalan words, *Math. Comput. Sci.* **18** (2024), Id/No 1, 16 pp., doi:10.1007/s11786-023-00576-5, https://doi.org/10.1007/s11786-023-00576-5.

[4] J.-L. Baril, C. Khalil and V. Vajnovszki, Catalan words avoiding pairs of length three patterns, *Discrete Math. Theor. Comput. Sci.* **22** (2021), Id/No 5, 19 pp.

[5] J.-L. Baril, S. Kirgizov, J. L. Ramírez and D. Villamizar, The combinatorics of Motzkin polyominoes, *Discrete Appl. Math.* **364** (2025), 1–15, doi:10.1016/j.dam.2024.12.002, https://doi.org/10.1016/j.dam.2024.12.002.

[6] J.-L. Baril, S. Kirgizov and V. Vajnovszki, Descent distribution on Catalan words avoiding a pattern of length at most three, *Discrete Math.* **341** (2018), 2608–2615, doi:10.1016/j.disc.2018.06.001, https://doi.org/10.1016/j.disc.2018.06.001.

[7] J.-L. Baril and J. L. Ramírez, Descent distribution on Catalan words avoiding ordered pairs of relations, *Adv. Appl. Math.* **149** (2023), Id/No 102551, 28 pp., doi:10.1016/j.aam.2023.102551, https://doi.org/10.1016/j.aam.2023.102551.

[8] D. Callan, T. Mansour and J. L. Ramírez, Statistics on bargraphs of Catalan words, *J. Autom. Lang. Comb.* **26** (2021), 177–196, doi:10.25596/jalc-2021-177, https://doi.org/10.25596/jalc-2021-177.

Accepted manuscript

[9] E. Deutsch, Dyck path enumeration, *Discrete Math.* **204** (1999), 167–202, doi: 10.1016/S0012-365X(98)00371-9, https://doi.org/10.1016/S0012-365X(98)00371-9.

[10] S. Elizalde, Symmetric peaks and symmetric valleys in Dyck paths, *Discrete Math.* **344** (2021), Id/No 112364, 13 pp., doi:10.1016/j.disc.2021.112364, https://doi.org/10.1016/j.disc.2021.112364.

[11] P. Flajolet and R. Sedgewick, *Analytic Combinatorics*, Cambridge University Press, Cambridge, 2009.

[12] R. Flórez and J. L. Ramírez, Enumerating symmetric and asymmetric peaks in Dyck paths, *Discrete Math.* **343** (2020), Id/No 112118, 9 pp., doi:10.1016/j.disc.2020.112118, https://doi.org/10.1016/j.disc.2020.112118.

[13] K. Manes, A. Sapounakis, I. Tasoulas and P. Tsikouras, General results on the enumeration of strings in Dyck paths, *Electron. J. Comb.* **18** (2011), research paper p74, 22 pp., doi:10.37236/561, https://doi.org/10.37236/561.

[14] T. Mansour, Statistics on Dyck paths, *J. Integer Seq.* **9** (2006), article 06.1.5, 13 pp.

[15] T. Mansour, A. R. Moreno and J. L. Ramírez, Symmetric and asymmetric peaks in compositions, *Online J. Anal. Comb.* **17** (2022), 14 pp.

[16] T. Mansour and J. L. Ramírez, Enumerations on polyominoes determined by Fuss-Catalan words, *Australas. J. Comb.* **81** (2021), 447–457, ajc.maths.uq.edu.au/pdf/81/ajc_v81_p447.pdf.

[17] T. Mansour and J. L. Ramírez, Exterior corners on bargraphs of Motzkin words, in: *Combinatorics, Graph Theory and Computing. Proceedings of the Combinatorics, Graph Theory and Computing 2021*, Springer, Singapore, pp. 355–367, 2024, doi:10.1007/978-3-031-52969-6_30, https://doi.org/10.1007/978-3-031-52969-6_30.

[18] T. Mansour, J. L. Ramírez and D. A. Toquica, Counting lattice points on bargraphs of Catalan words, *Math. Comput. Sci.* **15** (2021), 701–713, doi:10.1007/s11786-021-00501-8, https://doi.org/10.1007/s11786-021-00501-8.

[19] T. Mansour and V. Vajnovszki, Efficient generation of restricted growth words, *Inf. Process. Lett.* **113** (2013), 613–616, doi:10.1016/j.ipl.2013.05.008, https://doi.org/10.1016/j.ipl.2013.05.008.

[20] D. Merlini, R. Sprugnoli and M. C. Verri, Lagrange inversion: when and how, *Acta Appl. Math.* **94** (2006), 233–249, doi:10.1007/s10440-006-9077-7, https://doi.org/10.1007/s10440-006-9077-7.

[21] J. L. Ramírez and A. Rojas-Osorio, Consecutive patterns in Catalan words and the descent distribution, *Bol. Soc. Mat. Mex., III. Ser.* **29** (2023), 14 pp., doi:10.1007/s40590-023-00532-0, https://doi.org/10.1007/s40590-023-00532-0.

[22] A. Sapounakis, I. Tasoulas and P. Tsikouras, Counting strings in Dyck paths, *Discrete Math.* **307** (2007), 2909–2924, doi:10.1016/j.disc.2007.03.005, https://doi.org/10.1016/j.disc.2007.03.005.

[23] A. Sapounakis, P. Tsikouras and I. Tasoulas, Enumeration of strings in Dyck paths: A bijective approach, *Discrete Math.* **309** (2009), 3032–3039, doi:10.1016/j.disc.2008.08.005, https://doi.org/10.1016/j.disc.2008.08.005.

[24] A. Sapounakis, P. Tsikouras, I. Tasoulas and K. Manes, Strings of length 3 in grand-Dyck paths and the Chung-Feller property, *Electron. J. Comb.* **19** (2012), research paper p2, 10 pp., doi:10.37236/2181, https://doi.org/10.37236/2181.

Accepted manuscript

[25] M. Shattuck, Counting subword patterns in Catalan words, *Art Discrete Appl. Math.* **7** (2024), P3.04, doi:10.26493/2590-9770.1695.4da, https://doi.org/10.26493/2590-9770.1695.4da.

[26] N. J. A. Sloane et al., *The On-Line Encyclopedia of Integer Sequences*, https://oeis.org.

[27] R. Stanley, *Catalan Numbers*, Cambridge University Press, Cambridge, 2015.

Accepted manuscript

FIBONACCI AND CATALAN PATHS IN A WALL

JEAN-LUC BARIL AND JOSÉ L. RAMÍREZ

Abstract. We study the distribution of some statistics (width, number of steps, length, area) defined for paths contained in walls. We present the results by giving generating functions, asymptotic approximations, as well as some closed formulas. We prove algebraically that paths in walls of a given width and ending on the x-axis are enumerated by the Catalan numbers, and we provide a bijection between these paths and Dyck paths. We also find that paths in walls with a given number of steps are enumerated by the Fibonacci numbers. Finally, we give a constructive bijection between the paths in walls of a given length and peakless Motzkin paths of the same length.

1. Introduction and notation

Lattice path theory takes an important place in combinatorics. In the literature, there are many articles that study combinatorial problems on lattice paths (see [15]). Most of the time, lattice paths are defined in $\mathbb{Z}^2$ by a starting point (almost always the origin), and a sequence of vectors (also called steps) lying in a given set S. For instance, paths (also called walks) defined with $S = \{N, S, E, W\}$, where $N = (0, 1)$, $S = (0, -1)$, $E = (1, 0)$, and $W = (-1, 0)$, are widely studied in the quarter plane $\mathbb{N}^2$ (see [9, 17, 18] for instance). Such a path may overlap itself (i.e., vertices and edges can be repeated). The problem of the enumeration of these paths is very interesting to solve whenever boundary constraints are imposed. On the other hand, we can consider lattice paths in $\mathbb{N}^2$ with no overlaps by forcing the paths to go to the right; this is the subclass of *directed paths*. For example, if $S = \{U, D\}$, where $U = (1, 1)$, $D = (1, -1)$, then the paths in $\mathbb{N}^2$ starting at the origin and ending on the x-axis are the famous Dyck paths that are counted with respect to the semilength (number of steps divided by 2) by the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$ (see A000108 in Sloane's On-line Encyclopedia of Integer Sequences [27]). Moreover, if we permit steps $H = (1, 0)$, then we obtain the class of Motzkin paths that are enumerated by the sequence A001006 in [27]. We refer to [1, 4, 5, 6, 7, 8, 10, 12, 19, 20, 25, 26] for several works on the enumeration and the generation of such paths (with and without overlaps) with respect to the length and various statistics.

In this work we introduce a new class of paths in $\mathbb{N}^2$ induced by a regular tiling of the first quadrant: the *wall*. More precisely, a *wall* is a tiling of $\mathbb{N}^2$ using tiles (or bricks) of size 1×2 organized as shown in Figure 1. More formally, a wall is a subgrid of $\mathbb{N}^2$ constituted by the segments $(0, b) - (\infty, b)$ for every $b \geq 0$, and $(a, b) - (a, b+1)$ for $a \geq 0$ and $b \geq 0$ of

Date: September 14, 2024.
2010 *Mathematics Subject Classification.* 05A15, 05A19.
Key words and phrases. Catalan, Fibonacci, Dyck path, generating function, kernel method.

the same parity. Note that this tiling can be also viewed as the cell structure of the plant tissues (see [28] for instance), and our work studies the enumeration of paths of the sap [16] according to several parameters defined below.

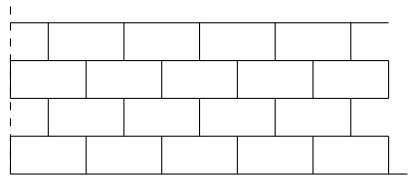


FIGURE 1. The wall tiling of $\mathbb{N}^2$.

A *path* in a wall is a lattice path on the subgrid defined by the wall. More precisely, it is a lattice path in $\mathbb{N}^2$ starting at the origin $(0,0)$ where each step links two adjacent corners of the bricks, by following the sides of the bricks with no overlap and no return to the left (each step touching exactly two corners of some bricks at its beginning and its end). Thus a path consists of steps $N = (0,1)$, $S = (0,-1)$, and $E \in \{E_1 = (1,0), E_2 = (2,0)\}$ and their connections are constrained by the tiling (E_2 is used on the x-axis and E_1 above). Let $\mathcal{P}$ be the set of all paths in a wall. For instance, Figure 2 shows the two paths $NEEENEESESENEES$ and $NEEENEESESENENEN$, which can also written as $NE_1E_1E_1NE_1E_1SE_1SE_2NE_1E_1S$ and $NE_1E_1E_1NE_1E_1SE_1SE_2NE_1NE_1N$. The first path ends on the x-axis and the second path ends at ordinate 3. Note that some works [13, 14] have investigated the connection between paths and tilings of the plane, but this does not correspond to our definition of the paths in a wall.

A *statistic* on the set $\mathcal{P}$ is a function $\mathtt{w}$ from $\mathcal{P}$ to $\mathbb{N}$. Below, we define three important statistics for our study. The *width* of a path P, denoted $\mathtt{width}(P)$, is the abscissa of its last point. For instance Figure 2 shows two paths of width 10. The *length* of a path P, denoted $\mathtt{length}(P)$, is the length of the path considering as a curve in $\mathbb{R}^2$. Figure 2 shows two paths of length 16 and 17, respectively. The *number of steps* of a path P, denoted $\mathtt{nbstep}(P)$, is the number of steps in the path (or equivalently the number of connections of two corners). Figure 2 shows two paths with 15 and 16 steps, respectively.

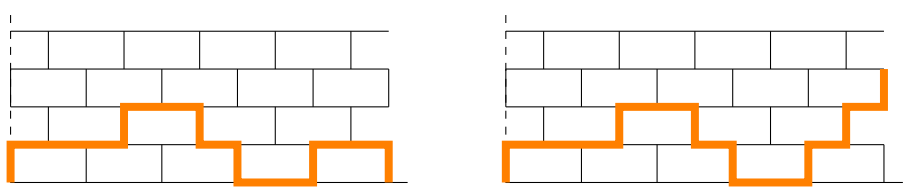


FIGURE 2. Two paths in a wall $NEEENEESESENEES$ and $NEEENEESESENENEN$. The left path ends on the x-axis, its width is 10, it has 15 steps, and its length is 16. The right path ends at ordinate 3, its width is 10, it has 16 steps, and its length is 17.

Now, let us assume that the statistic $\mathtt{w}$ returns either the width, or the length, or the number of steps of a path. For $k \geq 0$, we consider the generating function $f_k = f_k(z)$ (resp. $g_k = g_k(z)$, resp. $h_k = h_k(z)$), where the coefficient of z^n in the series expansion is the number of paths $P \in \mathcal{P}$ such that $\mathtt{w}(P) = n$, ending at ordinate k with an up-step N (resp., with a down-step S, resp., with a horizontal-step E). Let f_k^0 (resp. f_k^1) be the generating function consisting of the terms z^n in f_k such that $n + k \equiv 0 \mod 2$ (resp., $n+k \equiv 1 \mod 2$). Similarly, we define g_k^0, g_k^1, h_k^0, and h_k^1. Obviously, we have $f_k = f_k^0 + f_k^1$, $g_k = g_k^0 + g_k^1$, and $h_k = h_k^0 + h_k^1$ for any $k \geq 0$.

Also, we introduce the bivariate generating functions for $i \in \{0, 1\}$,

$$F^i(u, z) = \sum_{k\geq 0} u^k f_k^i(z), \quad G^i(u, z) = \sum_{k\geq 0} u^k g_k^i(z), \text{ and } \quad H^i(u, z) = \sum_{k\geq 0} u^k h_k^i(z).$$

For short, we use the notation $F^i(u)$, $G^i(u)$, and $H^i(u)$, $i \in \{0, 1\}$, for these functions.

We will use all these notations for the three following sections of this study according to the choice of the statistic $\mathtt{w}$ ($\mathtt{w} = \mathtt{width}$ in Section 3, $\mathtt{w} = \mathtt{nbstep}$ in Section 4, and $\mathtt{w} = \mathtt{length}$ in Section 5).

Outline of the paper. In this paper, we investigate the enumeration problem of the paths defined by the wall with respect to several parameters. In Section 2, we count paths of a given width (ending on a given abscissa) according to the type of the last step and the ordinate of the last point. We provide an asymptotic approximation for the expected ordinate of the last point, and we prove that such paths ending on the x-axis are counted by the well known Catalan numbers. We exhibit a bijection between these paths and Dyck paths. Note that this last result was already found by Odlyzko [22] in the context of the enumeration of fountains with a given number of coins on the basis. *En passant*, Odlyzko also enumerates fountains with n coins, which allows us to say that paths ending on the x-axis in the wall and having a given number of bricks below the path, are counted by the infinite continued fraction

$$\cfrac{1}{1 - \cfrac{z}{1 - \cfrac{z^2}{1 - \cfrac{z^3}{\ddots}}}}.$$

In this section we also enumerate the paths ending on the x-axis with a given area and width. We note that the total area of the paths is related with the path length in binary trees.

In Section 3, we count paths of a given number of steps according to the type of the last step and the ordinate of the last point. We prove that such paths are counted by the Fibonacci numbers. We exhibit a bijection between these paths and binary words avoiding two consecutive ones.

Finally, in Section 4, we make an analogous study for paths having a given length. We prove that such paths are counted by the generalized Catalan number, which are known to also count RNA structures. We exhibit a bijection between these paths and peakless

Motzkin paths. Note that the study made in this last section is equivalent to the study of paths of a given length in the honeycomb (i.e., hexagonal) lattice (it suffices to expand all bricks of the wall into hexagonal cells).

2. Paths of a given width

In this part, we count paths in $\mathcal{P}$ of a given width, i.e., ending at a given abscissa.

By convention, we fix $f_0^0 = 1$ for considering the empty path consisting of the origin $(0,0)$ only. Observing that a step $N = (0,1)$ in a path cannot end on a corner (n,k), $k \geq 1$, with $n + k \equiv 0 \mod 2$, we deduce $f_k^0 = 0$ for $k \geq 1$. Of course, a path ending with N cannot end at ordinate 0, which implies $f_0^1 = 0$. A path ending with N at ordinate $k = 1$ ends necessarily at an even abscissa, and then, it equals either N or QN, where Q is a path ending with a horizontal step $(2,0)$, which implies $f_1^1 = 1 + h_0^0$. Finally, a path ending with N on a corner (n,k), $k > 1$, $n + k \equiv 1 \mod 2$, follows necessarily an horizontal step $(1,0)$ that ends at $(n, k-1)$, which implies $f_k^1 = h_{k-1}^0$, $k > 1$.

Other recurrence relations for g_k^i and h_k^i, $k \geq 0$, $i \in \{0,1\}$, can be obtained *mutatis mutandis*. Thus we obtain the following equations:

$$\begin{cases} f_0^0 = 1 \text{ and } f_k^0 = 0, & k \geq 1, \\ f_0^1 = 0, f_1^1 = 1 + h_0^0, \text{ and } f_k^1 = h_{k-1}^0, & k > 1, \end{cases}, \quad \begin{cases} g_k^0 = h_{k+1}^1, & k \geq 0, \\ g_k^1 = 0, & k \geq 0, \end{cases} \quad \text{and}$$

$$\begin{cases} h_0^0 = z^2 + z^2(h_0^0 + g_0^0) \text{ and } h_k^0 = z(h_k^1 + f_k^1), & k \geq 1, \\ h_0^1 = 0 \text{ and } h_k^1 = z(h_k^0 + g_k^0), & k \geq 1. \end{cases} \tag{1}$$

Summing the recursions in (1), we have:

$$\begin{cases} F^0(u) &= 1, \\ F^1(u) &= u + \sum_{k\geq 1} u^k h_{k-1}^0 = u + uH^0(u), \end{cases} \quad \begin{cases} G^0(u) &= \sum_{k\geq 0} u^k h_{k+1}^1 = \frac{1}{u} H^1(u), \\ G^1(u) &= 0, \end{cases},$$

$$\begin{cases} H^0(u) &= h_0^0 + z \sum_{k\geq 1} u^k (f_k^1 + h_k^1) \\ &= h_0^0 + z(F^1(u) + H^1(u)), \\ H^1(u) &= z \sum_{k\geq 0} u^k (g_k^0 + h_k^0) - z(h_0^0 + g_0^0) \\ &= z(H^0(u) + G^0(u)) - \frac{h_0^0 - z^2}{z}. \end{cases}$$

Solving the above functional equations, we deduce

$$F^0(u) = 1 \quad \text{and} \quad F^1(u) = \frac{u\,(h_0^0\, z - u + z)}{u^2 z - u + z},$$

$$G^1(u) = 0 \quad \text{and} \quad G^0(u) = -\frac{h_0^0\, uz + h_0^0\, z^2 + z^2 - h_0^0}{(u^2 z - u + z)\, z},$$

$$H^0(u) = \frac{z\,(-u^2 + h_0^0)}{u^2 z - u + z} \quad \text{and} \quad H^1(u) = -\frac{u\,(h_0^0\,uz + h_0^0\,z^2 + z^2 - h_0^0)}{(u^2 z - u + z)\,z}.$$

In order to compute h_0^0, we use the kernel method [2, 3, 24, 25] on $F^1(u)$. This method consists in cancelling the denominator of $F^1(u)$ by finding u as an algebraic function r of z. Therefore, if we substitute u with r in the numerator then it necessarily equals zero (in order to counterbalance the cancellation of the denominator), which induces the value of h_0^0.

Thus, we factorize the denominator $u^2 z - u + z = z(u - r)(u - s)$ with

$$r = \frac{1 - \sqrt{1 - 4z^2}}{2z} \quad \text{and} \quad s = \frac{1 + \sqrt{1 - 4z^2}}{2z}.$$

Note that r is the generating function of Catalan numbers, which ensures us that we remain in the ring of formal power series.

Hence we obtain

$$h_0^0 = \frac{r - z}{z}.$$

Now, substituting h_0^0 with its value in the above generating functions, and simplifying by $(u - r)$ in the numerator and the denominator, we obtain the following:

Theorem 1. *We have*

$$F^0(u) = 1, \quad F^1(u) = \frac{u}{z(s-u)}, \quad G^0(u) = \frac{r - z}{z^2(s-u)}, \quad G^1(u) = 0, \text{ and}$$

$$H^0(u) = \frac{r + u}{s - u}, \quad H^1(u) = \frac{u(r - z)}{z^2(s - u)}.$$

Finally, the bivariate generating function $S(z, u)$, where the coefficient of $z^n u^k$ is the number of paths of width n ending at ordinate k, satisfies

$$S(z, u) = \frac{r(1 + u)}{z^2(s - u)}.$$

The first terms of the series expansion of $S(z, u)$ are

$$\begin{aligned} 1 + u + \left(u^2 + u\right) z + \left(u^3 + u^2 + 2\,u + 2\right) z^2 + \left(u^4 + u^3 + 3\,u^2 + 3\,u\right) z^3 + \\ + \left(u^5 + u^4 + 4\,u^3 + 4\,u^2 + 5\,u + 5\right) z^4 + \left(u^6 + u^5 + 5\,u^4 + 5\,u^3 + 9\,u^2 + 9\,u\right) z^5 + \\ + \left(u^7 + u^6 + 6\,u^5 + 6\,u^4 + 14\,u^3 + 14\,u^2 + 14\,u + 14\right) z^6 + \cdots \end{aligned}$$

Using the Vieta relations $rs = 1$ and $r + s = 1/z$, we deduce the following.

Corollary 1. *We have*

$$[u^k]F^0(u) = [k = 0],\ k \geq 0, \quad [u^k]F^1(u) = \frac{r^k}{z},\ k \geq 1,$$

$$[u^k]G^0(u) = \frac{(r - z)r^{k+1}}{z^2},\ k \geq 0,$$

$$[u^k]H^0(u) = \frac{r^{k+1}}{z},\ k \geq 1, \quad [u^k]H^1(u) = \frac{(r-z)r^k}{z^2},\ k \geq 1,$$

$$[u^k]S(z,u) = \frac{r^{k+1}(r+1)}{z^2} - \frac{[k=0]}{z},\ k \geq 0,\ \textit{and}$$

all other coefficients are equal to zero.

In order to provide closed forms for the coefficients of $z^n u^k$ in the previous generating functions, we need to obtain a closed form for the coefficient of z^n in r^k, $k \geq 0$. The quantity r satisfies the functional equation $r = z(1+r^2) = z\phi(r)$ with $\phi(t) = 1+t^2$. From Lagrange inversion, (see [21] for instance), we have

$$[z^n]r^k = \frac{k}{n}[t^{n-k}]\phi(t)^n = \frac{k}{n}[t^{n-k}](1+t^2)^n.$$

Thus we have

$$[z^n]r^k = 0 \text{ if } n-k \not\equiv 0 \mod 2 \quad \text{and} \quad [z^n]r^k = \frac{k}{n}\binom{n}{\frac{n-k}{2}} \text{ otherwise.}$$

Therefore, we obtain the following.

Theorem 2. *The number $f(n,k)$ of paths of width n ending at ordinate k with a step N is given by*

$$f(n,k) = \frac{k}{n+1}\binom{n+1}{\frac{n+1-k}{2}} \quad \textit{if } n+k \not\equiv 0 \mod 2 \ \textit{ and } \ 0 \textit{ otherwise.}$$

The number $g(n,k)$ of paths of width n ending at ordinate k with a step S is given by

$$g(n,k) = \frac{k+2}{n+2}\binom{n+2}{\frac{n-k}{2}} - \frac{k+1}{n+1}\binom{n+1}{\frac{n-k}{2}} \quad \textit{if } n+k \equiv 0 \mod 2 \ \textit{ and } \ 0 \textit{ otherwise.}$$

Theorem 3. *The number $s(n,k)$ of paths of width n ending at ordinate k is given by*

$$s(n,k) = \frac{k+2}{n+2}\binom{n+2}{\frac{n-k}{2}} \quad \textit{if } n+k \equiv 0 \mod 2 \textit{ and}$$

$$s(n,k) = \frac{k+1}{n+2}\binom{n+2}{\frac{n-k+1}{2}} \quad \textit{otherwise.}$$

From Theorem 2 and Theorem 3, we can easily deduce a closed form for the number $h(n,k)$ of paths of width n ending at ordinate k with a horizontal step. As a byproduct of Theorem 3, if we set $k=0$ and $n=2m$ is even, then $s(2m,0) = \frac{1}{m+1}\binom{2m+2}{m} = \frac{1}{m+2}\binom{2m+2}{m+1}$, which corresponds to the $(m+1)$-th Catalan number (see A000108). Figure 3 shows the 14 paths of width 6 ending on the x-axis (i.e., ending at ordinate $k=0$).

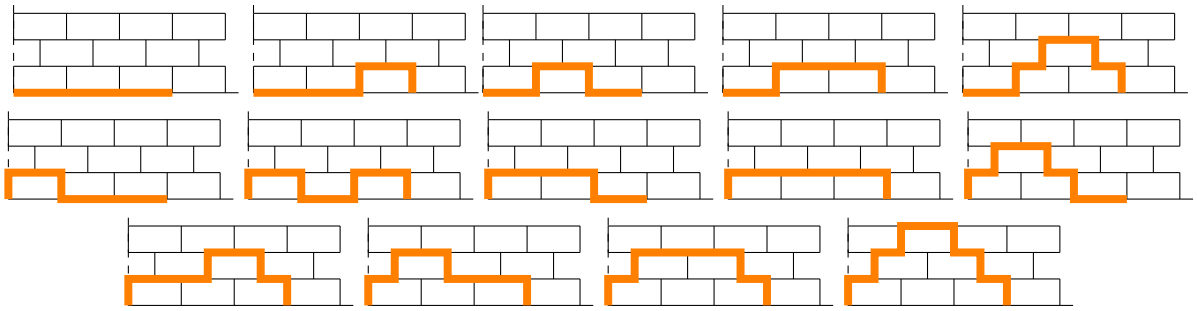


FIGURE 3. The 14 paths of width 6 ending on the x-axis (Catalan number).

Corollary 2. *The generating function for the number of paths of a given width is*

$$S(z,1) = \frac{2r}{z^2(s-1)} = \frac{1-z-2z^2-(1-z)\sqrt{1-4z^2}}{z^3(-1+2z)},$$

and the n-th coefficient in the series expansion is given by

$$2 \cdot \binom{n+1}{\lfloor n/2 \rfloor},$$

which corresponds to twice the coefficients of A037952 in [27].

The first terms of the series expansion are

$$2 + 2z + 6z^2 + 8z^3 + 20z^4 + 30z^5 + 70z^6 + 112z^7 + 252z^8 + 420z^9 + \cdots$$

By calculating $\partial_u(S(z,u))|_{u=1}$, and using classical methods [11, 23] for an asymptotic approximation of the coefficient of z^n, we obtain the following.

Corollary 3. *An asymptotic for the expected ordinate of the last point in all paths of a given width is given by*

$$\sqrt{\frac{\pi n}{2}} \approx 1.253314137\sqrt{n}.$$

We end this section by exhibiting a constructive bijection ϕ between paths of width $2n$ ending on the x-axis in $\mathcal{P}$ and Dyck paths with $n+1$ up steps. Recall that $E_1 = (1,0)$ and $E_2 = (2,0)$. Let $\mathcal{P}^1$ be the set of paths in the wall starting at $(1,1)$, ending at ordinate 1, and never going to the x-axis. If Q is a path in $\mathcal{P}^1$, then we define the path $\bar{Q}$ in $\mathcal{P}$ obtained from Q after a translation by the vector $(-1,-1)$ (note that some occurrences of E_1E_1 in Q can be transformed by the translation into a step E_2). For instance, if $Q = E_1E_1NE_1E_1S$ then $\bar{Q} = E_2NE_1E_1S$.

Proposition 1. *Let us consider the map ϕ recursively defined from $\mathcal{P}$ to the set $\mathcal{D}$ of Dyck paths as follows. For $P \in \mathcal{P}$, we set:*

$$\phi(P) = \begin{cases} UD & \text{if } P = \epsilon, \\ UD\phi(Q) & \text{if } P = E_2\ Q \text{ with } Q \in \mathcal{P}, \\ U\phi(\bar{Q})D & \text{if } P = NE_1QE_1S \in \mathcal{P}, \text{ and } Q \in \mathcal{P}^1, \\ U\phi(\bar{Q})D\phi(R) & \text{if } P = NE_1QE_1SE_2R \in \mathcal{P}, \text{ with } R \in \mathcal{P} \text{ and } Q \in \mathcal{P}^1. \end{cases}$$

Then, the map ϕ is a bijection.

Due to the recursive definition, the image by ϕ of a path ending on the x-axis of width $2n$ in $\mathcal{P}$ is a Dyck path of semilength $n+1$, and it is easy to see that ϕ is a bijection. For instance, the image of $NE_1E_1E_1NE_1E_1SE_1SE_2NE_1E_1S$ (of width 10) is the Dyck path $U\phi(E_2NE_1E_1S)D\phi(NE_1E_1S) = UUD\phi(NE_1E_1S)D\phi(NE_1E_1S) = UUDUUDDDUUDD$ (of length 12), see Figure 4. We also refer to Figure 5 for an illustration of the last three cases used in the definition of ϕ.

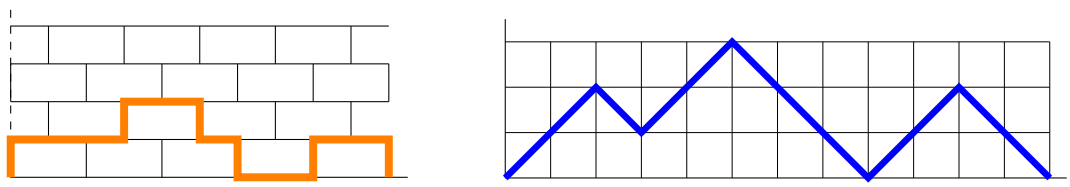


FIGURE 4. The path $NEEENEESESENEES$ of width 10 and its image by ϕ, the Dyck path $UUDUUDDDUUDD$ of length 12,

2.1. **The total area.** The *area* of a path P ending on the x-axis is defined as two times the number of bricks between the path and the x-axis. It is denoted by $\texttt{area}(P)$. For example, the path in Figure 2 has area 10. In this part, we enumerate the paths in $\mathcal{P}$ of a given width, ending on the x-axis, and with a given area.

The following theorem provides a functional equation satisfied by the generating function

$$F(z,q) = \sum_{P \in \mathcal{P}} z^{\texttt{width}(P)} q^{\texttt{area}(P)}.$$

Theorem 4. *The bivariate generating function $F(z,q)$, where the coefficient of $z^n q^k$ is the number of paths of width n ending on the x-axis and area k, satisfies the functional equation*

$$F(z,q) = 1 + z^2F(z,q) + z^2q^2F(zq,q) + z^4q^2F(zq,q)F(z,q). \tag{2}$$

Proof. In the bijection introduced in Proposition 1 we note that any non-empty path can be decomposed as E_2Q, NE_1QE_1S, or $NE_1QE_1SE_2R$, where $Q \in \mathcal{P}^1$ and $R \in \mathcal{P}$, see Figure 5 for a pictorial representation. For the paths P of the form E_2Q, the contribution is $z^2F(z,q)$ since the area of P and Q are the same, and the width of P is equal to that of Q plus two. For the paths P of the form NE_1QE_1S, the contribution is $z^2q^2F(zq,q)$ since the

area of P is $2+\texttt{width}(Q)+\texttt{area}(Q)$, and the width of P is $2+\texttt{width}(Q)$. The contribution of the third case is obtained *mutatis mutandis*. Therefore, fom this decomposition follows the functional equation.

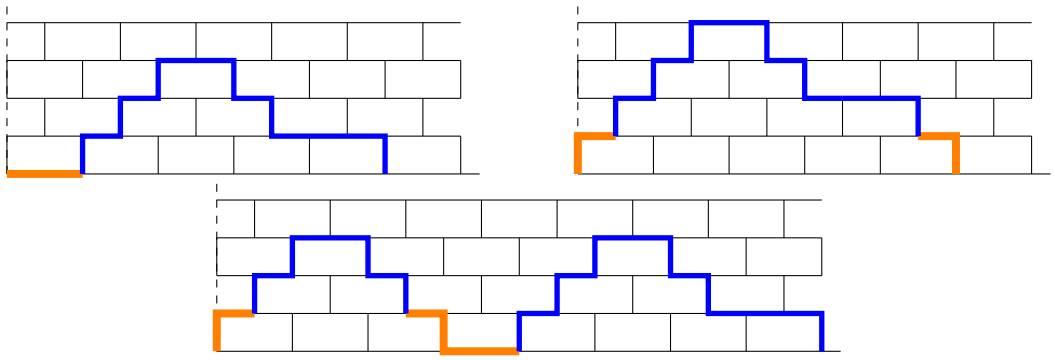


FIGURE 5. Decomposition of a path into E_2Q, or NE_1QE_1S, or $NE_1QE_1SE_2R$.

□

Theorem 5. *An expression for the generating function $F(z,q)$ is given by the continued fraction*

$$F(z,q) = \cfrac{1}{-z^2 + \cfrac{1}{1 + \cfrac{q^2z^2}{-q^2z^2 + \cfrac{1}{1 + \cfrac{q^4z^2}{-q^4z^2 + \cfrac{1}{\ddots}}}}}}.$$

Proof. From Theorem 4 we have

$$F(z,q) = \frac{1 + z^2q^2F(zq,q)}{1 - z^2 - z^4q^2F(zq,q)} = \cfrac{1}{-z^2 + \cfrac{1}{1 + z^2q^2F(zq,q)}}.$$

Iterating this expression yields the desired result. □

The first terms of the continued fraction are as follows:

$$1 + (1+q^2)z^2 + (1+2q^2+q^4+q^6)z^4 + (\mathbf{1+3q^2+3q^4+3q^6+2q^8+q^{10}+q^{12}})z^6 + \\ (1+4q^2+6q^4+7q^6+7q^8+5q^{10}+5q^{12}+3q^{14}+2q^{16}+q^{18}+q^{20})z^8 + \cdots$$

The paths corresponding to the boldface in the expansion are displayed in Figure 3. Note that by considering $1 + q \cdot F(\sqrt{q}, \sqrt{q})$, we obtain the generating function for the paths

without steps on the x-axis having a given number of bricks below the path, and we retrieve the results of Odlyzko [22] that enumerates fountains with n coins with the continued fraction

$$\cfrac{1}{1-\cfrac{z}{1-\cfrac{z^2}{1-\cfrac{z^3}{\ddots}}}}.$$

Corollary 4. *The generating function of the total area in all paths in $\mathcal{P}$ of a given width and ending on the x-axis is*

$$\frac{1-3z^2-(1-z^2)\sqrt{1-4z^2}}{z^4(1-4z^2)}.$$

The n-th coefficient of the series expansion is asymptotically equivalent to

$$(1+(-1)^n)2^{n+1}.$$

Proof. Let $G := \partial_q(F(z,q))|_{q=1}$. Then, by differentiating (2) with respect to q, we obtain

$$G = z^2G + (2z^2 + 2z^4F(z,1) + z^4G)F(z,1) + z^2(1+z^2F(z,1))H, \tag{3}$$

where $H = \partial_q(F(qz,q))|_{q=1}$. From Theorem 1 we have

$$F(z,1) = \frac{1-2z^2-\sqrt{1-4z^2}}{2z^4} = \sum_{n\geq 0} C_{n+1}z^{2n},$$

where C_n is the n-th Catalan number. From the definition of $F(z,q)$ we have

$$H = \partial_q(F(qz,q))|_{q=1} = \sum_{n\geq 0} 2nC_{n+1}z^{2n} + G.$$

Since we have

$$\sum_{n\geq 0} 2nC_{n+1}z^{2n} = 2\frac{1-3z^2-(1-z^2)\sqrt{1-4z^2}}{z^4\sqrt{1-4z^2}},$$

we can substitute this expression in (3). Solving for G, we obtain the desired result. □

The first terms of the series expansion are

$$2z^2 + 14z^4 + 74z^6 + 352z^8 + 1588z^{10} + 6946z^{12} + 29786z^{14} + 126008z^{16} + \cdots,$$

which corresponds to A138156 in [27], that also counts the sum of the path lengths of all incomplete binary trees with n edges. It would be interesting to investigate the link between the area in these paths and the path length in these trees.

3. Paths with a given number of steps

In this part, we count paths in $\mathcal{P}$ with a given number of steps (recall that a step is a move $N=(0,1)$, $S=(0,-1)$, $E\in\{(1,0),(2,0)\}$ connecting two adjacent corners in a wall. With the similar arguments used in the previous section, we easily obtain the following recurrence relations.

$$\begin{cases} f_0^0=1, \text{ and } f_k^0=0, & k\geq 1,\\ f_0^1=0, f_1^1=z+zh_0^0, \text{ and } f_k^1=zh_{k-1}^0, & k\geq 2,\end{cases}, \quad \begin{cases} g_k^0=zh_{k+1}^1, & k\geq 0,\\ g_k^1=0, & k\geq 0,\end{cases} \quad \text{and}$$

$$\begin{cases} h_0^0=z+z(h_0^0+g_0^0), \text{ and } h_k^0=z(h_k^1+f_k^1), & k\geq 1,\\ h_0^1=0, \text{ and } h_k^1=z(h_k^0+g_k^0), & k\geq 1.\end{cases} \tag{4}$$

Summing the recursions in (4), we have:

$$\begin{cases} F^0(u) &= 1,\\ F^1(u) &= zu+z\sum_{k\geq 1}u^kh_{k-1}^0=zu+zuH^0(u),\end{cases} \quad \begin{cases} G^0(u) &= z\sum_{k\geq 0}u^kh_{k+1}^1=\frac{z}{u}H^1(u),\\ G^1(u) &= 0,\end{cases}$$

$$\begin{cases} H^0(u) &= h_0^0+z\sum_{k\geq 1}u^k(f_k^1+h_k^1)\\ &= h_0^0+z(F^1(u)+H^1(u)),\\ H^1(u) &= z\sum_{k\geq 0}u^k(g_k^0+h_k^0)-z(h_0^0+g_0^0)\\ &= z(H^0(u)+G^0(u))-h_0^0+z.\end{cases}$$

Solving these functional equations, we deduce

$$F^0(u)=1 \text{ and } F^1(u)=\frac{uz\,(h_0^0\,uz+h_0^0\,z^2-h_0^0\,u+z^2-u)}{-uz^4+u^2z^2+uz^2+z^2-u},$$

$$G^1(u)=0 \text{ and } G^0(u)=-\frac{z\,(h_0^0\,uz^2+h_0^0\,z-h_0^0+z)}{-uz^4+u^2z^2+uz^2+z^2-u},$$

$$H^0(u)=\frac{uz^4-(u^2-h_0^0+u)z^2+h_0^0uz-h_0^0u}{-uz^4+(u^2+u+1)z^2-u} \text{ and } H^1(u)=-\frac{u\,(h_0^0\,uz^2+h_0^0\,z-h_0^0+z)}{-uz^4+u^2z^2+uz^2+z^2-u}.$$

In order to compute h_0^0, we use the kernel method on $F^1(u)$ (see Section 2 for more details). We factorize the denominator $-uz^4+u^2z^2+uz^2+z^2-u=z^2(u-r)(u-s)$ with

$$r=\frac{1-z^2+z^4-\sqrt{(z^2+z+1)\,(z^2-z+1)\,(z^2+z-1)\,(z^2-z-1)}}{2z^2} \quad \text{and}$$

$$s=\frac{1-z^2+z^4+\sqrt{(z^2+z+1)\,(z^2-z+1)\,(z^2+z-1)\,(z^2-z-1)}}{2z^2}.$$

Cancelling the numerator of $F^1(u)$ by substituting u by r, we obtain

$$h_0^0=\frac{z}{1-z-rz^2}.$$

The first terms of the series expansion of h_0^0 are

$$z + z^2 + z^3 + z^4 + 2\,z^5 + 3\,z^6 + 5\,z^7 + 7\,z^8 + 11\,z^9 + \cdots,$$

and this sequence does not appear in [27]. However, if we set $h(n) := [z^n]h_0^0$ then the sequence of odd powers corresponds to A051286, and we have

$$h(2n+1) = \sum_{k=0}^{n} \binom{n-k}{k}^2.$$

Moreover, the sequence of even powers corresponds to A203611, and we have

$$h(2n) = \sum_{k=0}^{n} \binom{k-1}{2k-1-n}\binom{k}{2k-n}.$$

Corollary 5. *The generating function for the number of paths of a given number of steps in $\mathcal{P}$ and ending on the x-axis is*

$$\frac{h_0^0}{z} = \frac{2}{1 - 2z + z^2 - z^4 + \sqrt{1 - 2z^2 - z^4 - 2z^6 + z^8}},$$

and the n-th coefficient of the series expansion is given by $h(n+1)$.

Now, substituting h_0^0 by its value in the previous generating functions, and simplifying by $(u - r)$ in the numerators and denominators, we obtain the following:

Theorem 6. *We have*

$$F^0(u) = 1, \quad F^1(u) = \frac{u(1 + h_0^0 - h_0^0 z)}{z(s-u)}, \quad G^0(u) = \frac{z h_0^0}{s-u}, \quad G^1(u) = 0, \text{ and}$$

$$H^0(u) = \frac{ru + h_0^0}{r(s-u)}, \quad H^1(u) = \frac{u h_0^0}{s-u},$$

where

$$h_0^0 = \frac{z}{1 - z - rz^2}.$$

Finally, the bivariate generating function $S(z,u)$, where the coefficient of $z^n u^k$ is the number of paths with n steps ending at ordinate k, satisfies

$$S(z,u) = 1 + \frac{ru + zru + h_0^0(ru + z^2 r + z)}{zr(s-u)}.$$

The first terms of the series expansion of $S(z,u)$ are

$$\begin{aligned} 1 + (u+1)z + (1+2u)z^2 + (u^2+3u+1)z^3 + (2u^2+4u+2)z^4 + (u^3+4u^2+5u+3)z^5 + \\ + (2u^3+6u^2+8u+5)z^6 + (u^4+5u^3+9u^2+12u+7)z^7 + \\ + (2u^4+8u^3+14u^2+20u+11)z^8 + \cdots \end{aligned}$$

Theorem 7. *We have*

$$[u^k]F^0(u) = [k=0],\ k \geq 0, \quad [u^k]F^1(u) = \frac{1+h_0^0(1-z)}{z} \cdot r^k,\ k \geq 1,$$

$$[u^k]G^0(u) = zh_0^0 r^{k+1},\ k \geq 0,$$

$$[u^k]H^0(u) = (1+h_0^0)r^k - [k=0],\ k \geq 0, \quad [u^k]H^1(u) = h_0^0 r^k,\ k \geq 1,$$

$$S(z,0) = h_0^0/z, \quad [u^k]S(z,u) = \frac{1+2h_0^0}{z} \cdot r^k,\ k \geq 1,\ \textit{and}$$

all other coefficients are equal to zero.

In order to provide a closed form for the coefficient of z^n of all these previous quantities, we need to provide a closed form of $r(n,k) := [z^n]r^k$. We set $r' = r(\sqrt{z})$. Then r' is the generating function for the generalized Catalan numbers, and using the comment of Barry in [27] (see A004148) giving the general term of r'^k, we can easily obtain:

$$r(2k,k) := [z]^{2k}r^k = 1, \quad k \geq 0$$

$$r(2k+2\ell,k) := [z]^{2k+2\ell}r^k = k \sum_{i=\lceil \frac{\ell+1}{2} \rceil}^{\ell} \frac{1}{i}\binom{i}{\ell-i}\binom{i+k-1}{\ell+k-i}, \quad k \geq 1, \ell \geq 1,$$

$$r(2n+1,k) := [z^{2n+1}]r^k = 0, \quad \text{otherwise.}$$

Corollary 6. *The number $s(n,k)$ of paths with n steps in $\mathcal{P}$ ending at ordinate k is given by*

$$s(n,k) = r(n+1,k) + 2\sum_{i \geq 0}^{n+1} h(i)r(n+1-i,k),$$

where $r(n,k)$ and $h(n)$ are defined previously.

Corollary 7. *The generating function for the number of paths of a given number of steps in $\mathcal{P}$ is*

$$S(z,1) = \frac{1+z}{1-z-z^2},$$

and the n-th coefficient $s(n)$ in the series expansion is given by a shift of the Fibonacci sequence A000045 in [27], defined by $s(n) = s(n-1) + s(n-2)$ anchored with $s(0) = 1$ and $s(1) = 2$.

The first terms of the series expansion are

$$1 + 2z + 3z^2 + 5z^3 + 8z^4 + 13z^5 + 21z^6 + 34z^7 + 55z^8 + 89z^9 + \cdots$$

Figure 6 shows the eight paths with four steps. Note that there is a simple bijection ψ between the set of paths in $\mathcal{P}$ with n steps and the set of binary words of length n that do not contain two adjacent ones: reading the path from left to right, we replace each vertical step by 1, and each horizontal step by 0. For instance, the image of the path $ENENEESEEN$ is 0101001001.

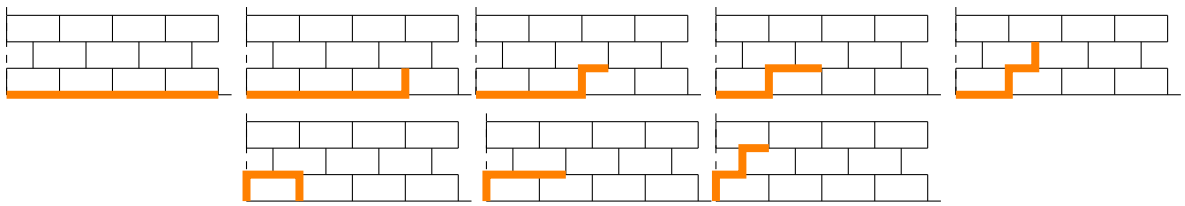


FIGURE 6. The eight paths with four steps (Fibonacci).

By calculating $\partial_u(S(z,u))|_{u=1}$ and using classical methods [11, 23] for an asymptotic approximation of the coefficient of z^n, we obtain the following.

Corollary 8. *The expected ordinate of the last point in all paths in $\mathcal{P}$ with n steps is asymptotically equivalent to*

$$\frac{\sqrt{-15+7\sqrt{5}}\left(5+3\sqrt{5}\right)\sqrt{n}}{10\sqrt{\pi}} \approx 0.5335775634\sqrt{n}.$$

4. Paths with a given length

In this part, we count paths in $\mathcal{P}$ with a given length. We easily obtain the following recurrence relations.

$$\begin{cases} f_0^0 = 1 \text{ and } f_k^0 = 0, & k \geq 1, \\ f_0^1 = 0, f_1^1 = z + zh_0^0, \text{ and } f_k^1 = zh_{k-1}^0, & k \geq 2, \end{cases}, \quad \begin{cases} g_k^0 = zh_{k+1}^1, & k \geq 0, \\ g_k^1 = 0, & k \geq 0, \end{cases} \quad \text{and}$$

$$\begin{cases} h_0^0 = z^2 + z^2(h_0^0 + g_0^0) \text{ and } h_k^0 = z(h_k^1 + f_k^1), & k \geq 1, \\ h_0^1 = 0 \text{ and } h_k^1 = z(h_k^0 + g_k^0), & k \geq 1. \end{cases} \tag{5}$$

Summing the recursions in (5), we have:

$$\begin{cases} F^0(u) &= 1, \\ F^1(u) &= zu + z\sum_{k\geq 1} u^k h_{k-1}^0 = zu + zuH^0(u), \end{cases} \quad \begin{cases} G^0(u) &= z\sum_{k\geq 0} u^k h_{k+1}^1 = \frac{z}{u}H^1(u), \\ G^1(u) &= 0, \end{cases}$$

$$\begin{cases} H^0(u) &= h_0^0 + z\sum_{k\geq 1} u^k (f_k^1 + h_k^1) \\ &= h_0^0 + z(F^1(u) + H^1(u)), \\ H^1(u) &= z\sum_{k\geq 0} u^k (g_k^0 + h_k^0) - z(h_0^0 + g_0^0) \\ &= z(H^0(u) + G^0(u)) - \frac{h_0^0 - z^2}{z}. \end{cases}$$

Solving the above functional equations, we deduce

$$F^0(u) = 1 \text{ and } F^1(u) = \frac{uz\left(h_0^0 z^2 + z^2 - u\right)}{-uz^4 + u^2z^2 + uz^2 + z^2 - u},$$

$$G^1(u) = 0 \text{ and } G^0(u) = -\frac{h_0^0\, uz^2 + h_0^0\, z^2 + z^2 - h_0^0}{-uz^4 + u^2z^2 + uz^2 + z^2 - u},$$

$$H^0(u) = \frac{z^2\,(uz^2 - u^2 + h_0^0 - u)}{-uz^4 + u^2z^2 + uz^2 + z^2 - u} \text{ and } H^1(u) = -\frac{u\,(h_0^0\, uz^2 + h_0^0\, z^2 + z^2 - h_0^0)}{(-uz^4 + u^2z^2 + uz^2 + z^2 - u)\, z}.$$

In order to compute h_0^0, we use the kernel method on $F^1(u)$ (see Section 2 for more details). We factorize the denominator $-uz^4+u^2z^2+uz^2+z^2-u = z^2(u-r)(u-s)$ where r and s are the same as those defined in Section 3.

Cancelling the numerator of $F^1(u)$ by substituting u with r, we obtain

$$h_0^0 = \frac{r - z^2}{z^2}.$$

Now, substituting h_0^0 by its value in the previous generating functions, and simplifying by $(u-r)$ in the numerators and denominators, we obtain the following:

Theorem 8. *We have*

$$F^0(u) = 1, \quad F^1(u) = \frac{u}{z(s-u)}, \quad G^0(u) = \frac{r - z^2}{z^2(s-u)}, \quad G^1(u) = 0, \text{ and}$$

$$H^0(u) = \frac{1}{z^2(s-u)} - 1, \quad H^1(u) = \frac{u(r - z^2)}{z^3(s-u)}.$$

Finally, the bivariate generating function $S(z,u)$, where the coefficient of z^nu^k is the number of paths of length n ending at ordinate k, satisfies

$$S(z,u) = \frac{rz + z + ru - z^3}{z^3(s-u)}.$$

The first terms of the series expansion of $S(z,u)$ are

$$1 + uz + (u+1)z^2 + (u^2+2u)z^3 + (u^2+2u+2)z^4 + (u^3+3u^2+3u)z^5 + \\ + (u^3+3u^2+4u+4)z^6 + (u^4+4u^3+6u^2+6u)z^7 + \\ + (u^4+4u^3+7u^2+9u+8)z^8 + \cdots$$

Theorem 9. *We have*

$$[u^k]F^0(u) = [k=0],\ k \geq 0, \quad [u^k]F^1(u) = \frac{r^k}{z},\ k \geq 1,$$

$$[u^k]G^0(u) = r^{k+2}/z^2 - r^{k+1},\ k \geq 0,$$

$$[u^k]H^0(u) = r^{k+1}/z^2 - [k=0],\ k \geq 0, \quad [u^k]H^1(u) = r^{k+1}/z^3 - r^k/z,\ k \geq 1,$$

$$S(z,0) = \frac{r - z^2}{z^4}, \qquad [u^k]S(z,u) = \frac{r^{k+1}}{z^2}(1 + r + \frac{1}{z} - z^2),\ k \geq 1, \text{ and}$$

all other coefficients are equal to zero.

Using the closed form of $[z^n]r^k$ obtained in the previous section, we deduce the following.

Corollary 9. *The number $s(n,k)$ of paths of length n in $\mathcal{P}$ ending at ordinate k is given by*

$$s(2,0)=1, s(n,0)=r(n+4,1),\ \ n\geq 4,\ \ \textit{and}$$
$$s(n,k)=r(n+2,k+1)+r(n+3,k+1)-r(n,k+1)+r(n+2,k+2),$$

where $r(n,k):=[z^n]r^k$ is given before Corollary 6

Corollary 10. *The generating function for the number of paths of a given length in $\mathcal{P}$ is*

$$S(z,1)=\frac{z^3-rz-z-r}{z^3(1-s)}=\frac{-1+2z^2+2z^5+z^6+(1-z^2)\sqrt{1-2z^2-z^4-2z^6+z^8}}{2z^5(1-z-z^2)}.$$

The sequences of coefficients of z^{2n+1} corresponds to A003440 and the sequences of coefficients of z^{2n} does appear in [27]

Corollary 11. *The generating function for the number of paths of a given length in $\mathcal{P}$ ending on the x-axis is $\frac{r-z^2}{z^4}$ and the n-th coefficient of the series expansion is given by $r(2n,0)$, which corresponds to the n-th generalized Catalan number (see A004148).*

Corollary 12. *An asymptotic for the expected ordinate of the last point in all paths in $\mathcal{P}$ of a given length is*

$$\frac{\left(3+\sqrt{5}\right)\sqrt{-15+7\sqrt{5}}}{2\sqrt{\pi n}}\approx 1.193115703\cdot\frac{1}{\sqrt{n}}.$$

Figure 7 shows the eight paths of length 8 ending on the x-axis.

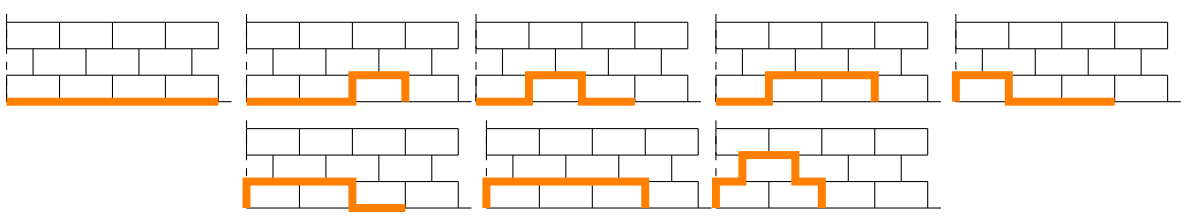


FIGURE 7. The eight paths of length 8 ending on the x-axis (generalized Catalan)

We end the section by exhibiting a constructive bijection ψ between paths of length $2n$ ending on the x-axis in $\mathcal{P}$ and peakless Motzkin paths with $n+1$ steps (i.e., Motzkin paths with no occurrence of UD). Recall that $E_1=(1,0)$ and $E_2=(2,0)$, and that $\mathcal{P}^1$ is the set of paths in the wall starting at $(1,1)$, ending at ordinate 1, and never going to the x-axis. If $Q\in\mathcal{P}^1$, then we recall that $\bar{Q}$ is the path in $\mathcal{P}$ obtained from Q after the translation by the vector $(-1,-1)$. These definitions are already given in Section 2 just before Proposition 1.

Proposition 2. *Let us consider the map ψ recursively defined from $\mathcal{P}$ to the set $\mathcal{PM}$ of peakless Motzkin paths as follows. For $P \in \mathcal{P}$, we set:*

$$\psi(P) = \begin{cases} H & \text{if } P = \epsilon, \\ H\psi(Q) & \text{if } P = E_2\ Q \text{ with } Q \in \mathcal{P}, \\ U\psi(\bar{Q})D & \text{if } P = NE_1QE_1S \in \mathcal{P}, \text{ and } Q \in \mathcal{P}^1, \\ U\psi(\bar{Q})D\psi(R) & \text{if } P = NE_1QE_1SE_2R \in \mathcal{P}, \text{ with } R \in \mathcal{P} \text{ and } Q \in \mathcal{P}^1. \end{cases}$$

Due to the recursive definition, the image by ψ of a path ending on the x-axis of length $2n$ in $\mathcal{P}$ is a peakless Motzkin path with $n+1$ steps, and it is easy to see that ψ is a bijection. For instance, the image of $NE_1E_1E_1NE_1E_1SE_1SE_2NE_1E_1S$ (of length 16) is the peakless Motzkin path $UHUHDDUHD$ (of length 9), see Figure 8. Note that the definition of ψ is basically the same as in Proposition 1 since we use the same decomposition of a path (the only change is that we use horizontal steps instead of peaks UD).

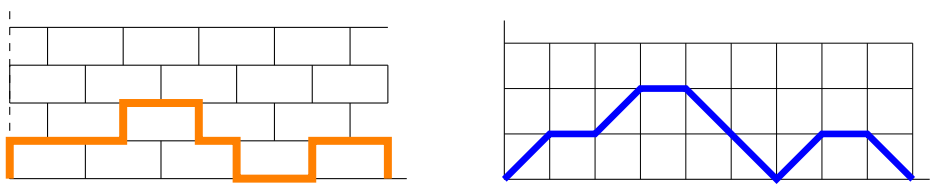


FIGURE 8. The path $NE_1E_1E_1NE_1E_1SE_1SE_2NE_1E_1S$ of length 16 and its image by ψ, the peakless Motzkin path $UHUHDDUHD$ of length 9.

5. Conclusions

In this work, we enumerate lattice paths in the wall with respect to their width, number of steps, length, and area. Can we identify the underlying limit law (see [11])? The main tools are multivariate generating functions and the kernel method. We can extend our results by considering additional statistics, for example the number of *turns* of the path, i.e., the number of occurrences of the subpaths EN, ES, NE, and SE. For example, it is possible to give an expression for the multivariate generating function $S(z,u,w)$, where the coefficient of $z^n u^k w^\ell$ is the number of paths with n steps, ℓ turns, and ending at ordinate k. The expression is too large, however, the first few terms of the Taylor expansion are

$$\begin{aligned} S(z,u,w) = 1 + (1+u)z + (1+2uw)z^2 + (1+2uw+uw^2+u^2w^2)z^3 \\ + (1+2uw+w^2+2uw^2+2u^2w^3)z^4 \\ + (1+2uw+3uw^2+u^2w^2+2w^3+2u^2w^3+u^2w^4+u^3w^4)z^5 + \cdots \end{aligned}$$

Moreover, the bivariate generating function $S(z,1,w)$, where the coefficient of $z^n w^\ell$ is the number of paths with n steps and ℓ turns is given by

$$S(z,1,w) = \frac{(w^2-2w+1)z^2 - z - 1}{w^2z^2 + z - 1}.$$

Therefore, the generating function for the total number of turns in all paths of a given width is

$$\frac{2z^2(1+z)}{(1-z-z^2)^2} = 2z^2 + 6z^3 + 14z^4 + 30z^5 + 60z^6 + 116z^7 + 218z^8 + 402z^9 + \cdots,$$

and the n-th coefficient corresponds to twice the convolution of the Fibonacci sequence (see A023610 in [27]), that is,

$$2 \cdot \sum_{i=2}^{n+1} F_i F_{n+1-i}.$$

Can we obtain enumerative results for paths with other boundary conditions? Can we generalize our study if we drop the non-overlapping constraint and allow the step $W = (-1, 0)$? Finally, the paths studied in the last section are in one-to-one correspondence with paths of a given length in the honeycomb lattice, which suggests to explore in more detail the link between the paths in the wall and the honeycomb lattice.

6. Acknowledgments

The authors are grateful to the anonymous referees for helpful comments.

References

[1] A. Asinowski, C. Banderier, and V. Roitner. Generating functions for lattice paths with several forbidden patterns. *Sémin. Lothar. Comb.*, 84B (2020).

[2] C. Banderier, M. Bousquet-Mélou, A. Denise, P. Flajolet, and D. Gouyou-Beauchamps. Generating functions for generating trees. *Discrete Math.*, 246 (2002), 29–55.

[3] C. Banderier, and Ph. Flajolet. Basic Analytic Combinatorics of Directed Lattice Paths. *Theoretical Computer Science*, 281(1-2) (2002), 37–80.

[4] C. Banderier and M. Wallner. Lattice paths with catastrophes. *Discrete Math Theor. Comput. Sci.*, 19(1) (2017), #23.

[5] E. Barcucci, A. Del Lungo, S. Fezzi, and R. Pinzani. Nondecreasing Dyck paths and q-Fibonacci numbers. *Discrete Math.*, 170 (1997), 211–217.

[6] E. Barcucci, A. Bernini, and R. Pinzani. Exhaustive generation of some lattice paths and their prefixes. *Theoret. Comput. Sci.*, 478 (2021), 47–52.

[7] J.-L. Baril and A. Petrossian. Equivalence classes of Dyck paths modulo some statistics. *Discrete Math.*, 338 (2015), 655–660.

[8] C. Bean, A. Bernini, M. Cervetti, and L. Ferrari. On the generating functions of pattern-avoiding Motzkin paths. *J. Symbolic Comput.*, 113 (2022), 126–138.

[9] M. Bousquet-Mélou and M. Mishna. Walks with small steps in the quarter plane. *In Algorithmic probability and combinatorics, vol. 520 of Contemp. Math.*, (2010), Amer. Math. Soc., 1–39.

[10] E. Deutsch. Dyck path enumeration. *Discrete Math.*, 204 (1999), 167–202.

[11] Ph. Flajolet and R. Sedgewick. *Analytic Combinatorics*, Cambridge University Press, 2009.

[12] R. Flórez and J.L. Ramírez. Enumerating symmetric and asymmetric peaks in Dyck paths. *Discrete Math.*, 343 (2020), 112118.

[13] S. Heubach and T. Mansour. Staircase tilings and lattice paths. *Congr. Numer.*, 182 (2006), 97–109.

[14] S. Heubach, N.Y. Li, and T. Mansour. Staircase tilings and k-Catalan structures. *Discrete Math.*, 308(24) (2008), 5954–5953.

[15] K. Humphreys. A history and a survey of lettice path enumeration. *J. of Stat. Plan. and Inference*, 140(8) (2010), 2237-2254.

[16] W.O. James and H. Baker. Sap pressure and the movements of sap. The New Phytologist, 32(5) (1933), 317–343.
[17] M. Kauers and R. Yatchak. Walks in the quarter plane with multiple steps. *In Proceedings of FPSAC 2015, Discrete Math. Theor. Comput. Sci.*, (2015), 25–36.
[18] I. Kurkova and K. Raschel. On the functions counting walks with small steps in the quarter plane. *Publ. Math. Inst. Hautes Études Sci.*, 116 (2012), 69–114.
[19] T. Mansour. Statistics on Dyck paths. *J. Integer Sequences*, 9 (2006), 06.1.5.
[20] D. Merlini, R. Sprugnoli, and M.C. Verri. Some statistics on Dyck paths, *J. Statis. Plann. Inference*, 101 (2002), 211–227.
[21] D. Merlini, R. Sprugnoli, and M.C. Verri. Lagrange inversion: when and how. *Acta Appl. Math*, 94 (2006), 233–249.
[22] A.M. Odlyzko and H.S. Wilf. The editor's corner: n coins in a fountain. *Amer. Math. Monthly*, 95 (1988), 840–843.
[23] A.G. Orlov. On asymptotic behavior of the Taylor coefficients of algebraic functions. *Sib. Math. J.*, 25(5) (1994), 1002–1013.
[24] H. Prodinger. The kernel method: a collection of examples. *Sém. Lothar. Combin*, 50 (2004), Paper B50f.
[25] H. Prodinger. Partial skew Dyck paths: a kernel method approach. *Graphs Combin.*, 38 (2022), Paper 135.
[26] A. Sapounakis, I. Tasoulas, and P. Tsikouras. Counting strings in Dyck paths. *Discrete Math.*, 307 (2007), 2909–2924.
[27] N. J. A. Sloane. The On-line Encyclopedia of Integer Sequences, available electronically at http://oeis.org.
[28] L.A. Staehelin. Cell wall. Encyclopedia Britannica, https://www.britannica.com/science/cell-wall-plant-anatomy.

LIB, Université de Bourgogne Franche-Comté, B.P. 47 870, 21078, Dijon Cedex, France
Email address: barjl@u-bourgogne.fr

Departamento de Matemáticas, Universidad Nacional de Colombia, Bogotá, Colombia
Email address: jlramirezr@unal.edu.co

ecajournal.haifa.ac.il

Enumerative Combinatorics and Applications

ECA **5:1** (2025) Article #S2R6
https://doi.org/10.54550/ECA2025V5S1R6

Grand Zigzag Knight's Paths

Jean-Luc Baril†, Nathanaël Hassler†, Sergey Kirgizov†, and José L. Ramírez‡

†*LIB, Université de Bourgogne, B.P. 47 870, 21078, Dijon Cedex, France*
Email: barjl@u-bourgogne.fr, nathanael.hassler@ens-rennes.fr, sergey.kirgizov@u-bourgogne.fr

‡*Departamento de Matemáticas, Universidad Nacional de Colombia, Bogotá, Colombia*
Email: jlramirezr@unal.edu.co

Received: April 30, 2024, **Accepted**: October 21, 2024, **Published**: November 1, 2024
The authors: Released under the CC BY-ND license (International 4.0)

Abstract: We study the enumeration of different classes of grand knight's paths in the plane. In particular, we focus on the subsets of the zigzag knight's paths that are subject to constraints. These constraints include ending at y-coordinate 0, bounded by a horizontal line, confined within a tube, among other considerations. We present our results using generating functions or direct closed-form expressions. We derive asymptotic results, finding approximations for quantities such as the probability that a zigzag knight's path stays in some area of the plane, or for the average of the altitude of such a path. Additionally, we exhibit some bijections between grand zigzag knight's paths and some pairs of compositions.

Keywords: Asymptotics; Enumeration; Generating function; Kernel method; Knight's paths
2020 Mathematics Subject Classification: 05A15; 05A19.

1 Introduction

In combinatorics, the enumeration of different classes of lattice paths according to several parameters is often studied. One of the motivations is to exhibit new one-to-one correspondences between these classes and various objects from other domains, such as computer science, biology, and physics [26]. For instance, they have close connections with RNA structures, pattern-avoiding permutations, directed animals, and other related topics [10, 19, 26]. We refer to [7–9, 14, 17, 21, 22, 24, 27, 28], and the references therein, for such studies.

Labelle and Yeh [20] investigate knight's paths, that are lattice paths in $\mathbb{N}^2$ that start at the origin and consist of steps $N = (1, 2)$, $\bar{N} = (1, -2)$, $E = (2, 1)$, and $\bar{E} = (2, -1)$. Note that these steps correspond to the right moves of a knight on a chessboard. Recently, in [11], the authors focus on zigzag knight's paths, i.e., knight's paths where the direction of the steps alternate up and down, or equivalently knight's paths avoiding the consecutive patterns NN, NE, EN, EE, $\bar{N}\bar{N}$, $\bar{N}\bar{E}$, $\bar{E}\bar{N}$, $\bar{E}\bar{E}$. They prove that such paths ending on the x-axis with a given number of steps are enumerated by the well-known Catalan numbers (see sequence A000108 in Sloane's On-line Encyclopedia of Integer Sequences [25]), and they exhibit a bijection between these paths and Dyck paths (lattice paths in $\mathbb{N}^2$ starting at the origin, ending on the x-axis and made of steps $U = (1, 1)$ and $D = (1, -1)$).

In this work, we extend the aforementioned study by allowing knight's paths to go below the x-axis.

Definition 1.1. *A grand knight's path is a lattice path in $\mathbb{Z}^2$ that starts at the origin and consists of steps $N = (1, 2)$, $\bar{N} = (1, -2)$, $E = (2, 1)$, and $\bar{E} = (2, -1)$.*

The *size* of such a path is the x-coordinate of its last point. The *empty path* ϵ is a path of size 0. The *height* is the maximal y-coordinate reached by a point of the path, and the *altitude* is the y-coordinate of the last point of the path. We also define grand zigzag knight's paths as follows.

Definition 1.2. *A grand zigzag knight's path is a grand knight's path with the additional property that the vertical components of two consecutive steps cannot be in the same direction, i.e., two consecutive steps cannot be NN, NE, $\bar{N}\bar{N}$, $\bar{N}\bar{E}$, EE, EN, $\bar{E}\bar{E}$, $\bar{E}\bar{N}$.*

See Figure 1 for an illustration of two grand knight's paths, with the second one also being a grand zigzag knight's path.

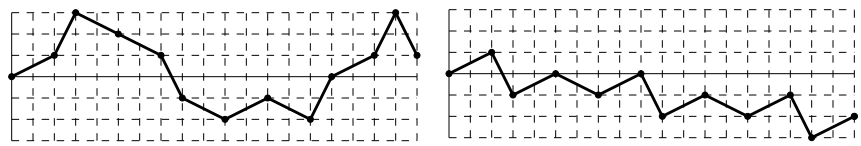


Figure 1: On the left: a grand knight's path of size 19, height 3, and altitude 1. On the right: a grand zigzag knight's path of size 19, height 1, and altitude -2.

The enumeration of grand knight's paths of a given size can be obtained using the work of Banderier and Flajolet [6]. Indeed, it suffices to consider the step polynomial $P(z,u) = zu^2 + \frac{z}{u^2} + \frac{z^2}{u} + z^2u$, where z (resp. u) marks the size (resp. height) of the step, i.e., zu^2, $\frac{z}{u^2}$, $\frac{z^2}{u}$, and z^2u correspond to the steps N, $\bar{N}$, $\bar{E}$, and E, respectively. The Laurent series providing the number of grand knight's paths with respect to the size (marked with z), the number of steps (marked with x), and the altitude (marked with u) is given by

$$\frac{1}{1 - xP(z,u)}.$$

Fixing $x = u = 1$, we deduce the generating function $1/(1 - 2z - 2z^2)$ for the number of grand knight's paths with respect to the size. The coefficient of z^n is $\sum_{k=0}^{n+1} \binom{n+1}{2k+1} 3^k$, which corresponds to the sequence A002605 in the OEIS [25]. The first values of u_n are

$$1,\ 2,\ 6,\ 16,\ 44,\ 120,\ 328,\ 896,\ 2448,\ 6688,\ 18272.$$

An asymptotic approximation is given by $\sqrt{3}/6 \cdot (\sqrt{3}+1)^{n+1}$.

The generating function for the number of grand knight's paths ending on the x-axis is

$$x \left(\frac{\mathbf{u}_3(x)'}{\mathbf{u}_3(x)} + \frac{\mathbf{u}_4(x)'}{\mathbf{u}_4(x)} \right) \Bigg|_{x=1},$$

where $\mathbf{u}_3(x)$ and $\mathbf{u}_4(x)$ are the small roots in u of $1 - xP(u)$ (i.e. the roots $\mathbf{u}_i(x)$ that satisfy $\lim_{x\to 0} \mathbf{u}_i(x) = 0$), and the derivative with respect to x. The bivariate generating function for the number of grand knight's paths of a given size and altitude $k \geq 0$ is then

$$W_k(z,u) = \left(\frac{x}{k} \left(\mathbf{u}_3(x)^{-k} \right)' + \frac{x}{k} \left(\mathbf{u}_4(x)^{-k} \right)' \right) \Bigg|_{x=1}.$$

In the subsequent sections of the paper, among other results, we recover the aforementioned formulae, although in a different statement (without involving any derivatives).

Recently, Asinowski, Bacher, Banderier, Gittenberger, and Roitner [1–5] developed a method for solving enumerative problems that arise out of paths avoiding consecutive patterns. Grand zigzag knight's paths could be modeled as grand knight's paths avoiding eight consecutive patterns of size 2. However, in this paper we present a more specific method, using the point of view of Prodinger [23] based on the kernel method. It consists in studying two generating functions that count the desired paths by distinguishing the direction of the last step (up or down). Then, we provide formulae for different kinds of constraints, give bijections, direct closed-form expressions for the n-th coefficient, and asymptotic results.

Outline of the paper. In Section 2, we apply the kernel method for providing generating function of the number of grand knight's paths with respect to the size and the altitude. We deduce asymptotic approximations for the number of grand knight's paths of a given size and a non-negative altitude, and for the expected altitude of a grand knight's path ending at a positive altitude. In the following sections, we focus on grand zigzag knight's paths. In Section 3, we give the counterpart of Section 2 for grand zigzag knight's paths. Moreover, we exhibit a bijection between pairs of integer compositions and grand zigzag knight's paths ending at (n,k), where n and k have the same parity. As a byproduct, we obtain a closed form for the number of grand zigzag knight's paths ending at (n,k), and the expected value for the number of steps of a grand zigzag knight's path ending at (n,k); asymptotic approximations are also derived. In Section 4, we consider grand zigzag knight's paths staying in a region delimited by horizontal lines. We make a similar study as for the previous sections, which allows us to estimate asymptotic approximations for the probability that a grand zigzag knight's path chosen uniformly at random among all grand zigzag knight's paths stays above a horizontal line. We end by giving an appendix for the general case of grand zigzag knight's paths staying in a general tube.

2 Unrestricted grand knight's paths

In this section, we enumerate grand knight's paths starting at the origin $(0,0)$ and ending at (n,k) for $n \geq 0$, $k \in \mathbb{Z}$. Let $\mathcal{H}_{n,k}$ be the set of such paths and let $h_{n,k}$ be its cardinality. Due to the symmetry with respect to the x-axis, we obviously have $h_{n,k} = h_{n,-k}$. Therefore it is sufficient to focus on non-negative k. For (fixed) $k \geq 0$, let $h_k = \sum_{n\geq 0} h_{n,k} z^n$ be the generating function for the number of grand knight's paths of altitude k with respect to the size. A non-empty grand knight's path of altitude k and size $n \geq 1$ is of one of the following forms (a) $P\bar{N}$ with $P \in \mathcal{H}_{n-1,k+2}$, (b) $P\bar{E}$ with $P \in \mathcal{H}_{n-2,k+1}$, (c) PE with $P \in \mathcal{H}_{n-2,k-1}$, (d) PN with $P \in \mathcal{H}_{n-1,k-2}$. Therefore we easily obtain the following equations after considering the symmetry $h_{-k} = h_k$:

$$h_0 = 1 + 2zh_2 + 2z^2 h_1, \tag{1}$$

$$h_1 = z^2(h_0 + h_2) + z(h_1 + h_3), \tag{2}$$

$$h_k = z^2(h_{k-1} + h_{k+1}) + z(h_{k-2} + h_{k+2}), \text{ for } k \geq 2. \tag{3}$$

Let $H(u,z) = \sum_{k\geq 0} h_k(z)u^k$ be the bivariate generating function for the number of grand knight's paths with respect to the size and the altitude. We will write $H(u)$ for short. Thus, by multiplying (3) by u^k, summing over $k \geq 2$, with $h_0 + h_1 u$ we obtain

$$H(u) = h_0 + h_1 u + z^2 \sum_{k\geq 2}(h_{k-1} + h_{k+1})u^k + z\sum_{k\geq 2}(h_{k-2} + h_{k+2})u^k,$$

and

$$\begin{aligned} H(u) = h_0 + h_1 u + z^2 &\left(u(H(u) - h_0) + \frac{H(u) - h_0 - h_1 u - h_2 u^2}{u} \right) \\ &+ z\left(u^2 H(u) + \frac{H(u) - h_0 - h_1 u - h_2 u^2 - h_3 u^3}{u^2} \right). \end{aligned} \tag{4}$$

From (1) and (2) we obtain $h_2 = -\frac{1}{2z} + \frac{h_0}{2z} - zh_1$ and $h_3 = \frac{1}{2} + (z^2 + \frac{1}{z} - 1)h_1 - (\frac{1}{2} + z)h_0$. By replacing h_2 and h_3 by these expressions in (4), and after simplifying, we eventually have

$$H(u)\left(u^2 - zu^4 - z - z^2 u - z^2 u^3\right) = \frac{u^2}{2} + h_0\left(\frac{u^2}{2} - z - z^2 u\right) + zuh_1(u^2 - 1). \tag{5}$$

We use the kernel method (see [23]) to determine h_0 and h_1. Let $K(u) = u^2 - zu^4 - z - z^2 u - z^2 u^3$ be the *kernel* of Equation (5), and let u_1 and u_2 be two of the roots of $K(u)$:

$$u_1 = \frac{-z^2 + \sqrt{z^4 + 8z^2 + 4z}}{4z} + \frac{1}{2\sqrt{2}}\sqrt{\frac{z^3 - z\sqrt{z^4 + 8z^2 + 4z} - 4z + 2}{z}} \tag{6}$$

and

$$u_2 = \frac{-z^2 - \sqrt{z^4 + 8z^2 + 4z}}{4z} - \frac{1}{2\sqrt{2}}\sqrt{\frac{z^3 + z\sqrt{z^4 + 8z^2 + 4z} - 4z + 2}{z}}. \tag{7}$$

Note that $u_1 = \mathbf{u}_1(1)$ and $u_2 = \mathbf{u}_2(1)$ where $\mathbf{u}_1(x)$ and $\mathbf{u}_2(x)$ are the two other roots of the step polynomial $1 - xP(z,u)$ (see Introduction), which implies that $1/(u - u_1)$ and $1/(u - u_2)$ have no power series expansions around $(u,z) = (0,0)$. This implies that $(u - u_1)(u - u_2)$ must be a factor of the numerator. Then by (5), we have

$$\frac{u_i^2}{2} + h_0\left(\frac{u_i^2}{2} - z - z^2 u_i\right) + zu_i h_1(u_i^2 - 1) = 0, \quad \text{for } i = 1, 2.$$

After solving for h_0 and h_1, we obtain

$$h_0 = \frac{1 + u_1 u_2}{1 + u_1 u_2 - 2z^2(u_1 + u_2) - 2z + 2z(1 - u_1^2 - u_2^2)u_1^{-1}u_2^{-1}} \tag{8}$$

and

$$h_1 = \frac{u_1 + u_2}{1 + u_1 u_2} h_0. \tag{9}$$

Then we can deduce $H(u)$ from (5), which yields the following.

Theorem 2.1. *The bivariate generating function for grand knight's paths with respect to the size and the altitude is*

$$H(u) = \frac{-h_1 u + h_0 u_1 u_2}{(u-u_1)\,(u-u_2)},$$

where u_1, u_2, h_0, h_1 are defined in (6), (7), (8) and (9).

By decomposing $H(u)$ into partial fractions, we deduce a close form for the coefficient $[u^k]H(u)$ of u^k in $H(u)$.

Corollary 2.1. *The generating function for grand knight's paths of altitude k with respect to the size is*

$$[u^k]H(u) = \frac{h_0 u_1 - h_1}{u_1 - u_2} u_2^{-k} - \frac{h_0 u_2 - h_1}{u_1 - u_2} u_1^{-k}.$$

The following table presents the numbers $h_{n,k}$ for $0 \le n \le 9$, $0 \le k \le 9$. It is given in the entry A096608 of [25].

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
2 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 2 & 3 & 2 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\
\mathbf{8} & 6 & 1 & 3 & 4 & 3 & 0 & 0 & 1 & 0 & \cdots \\
6 & 12 & 16 & 12 & 3 & 4 & 5 & 4 & 0 & 0 & \cdots \\
44 & 33 & 18 & 21 & 27 & 20 & 6 & 5 & 6 & 5 & \cdots \\
60 & 76 & 95 & 72 & 40 & 34 & 41 & 30 & 10 & 6 & \cdots \\
256 & 210 & 154 & 155 & 177 & 135 & 75 & 52 & 58 & 42 & \cdots \\
460 & 520 & 581 & 480 & 335 & 288 & 299 & 228 & 126 & 76 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.$$

We refer to Figure 2 for the illustration of the eight grand knight's paths of size 4 ending on the x-axis (see the entry indicated by the boldface in the table).

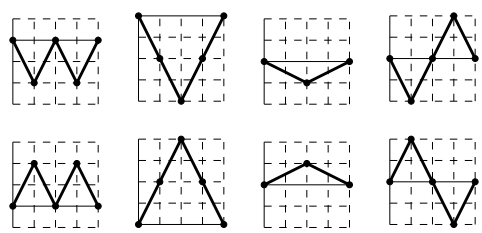


Figure 2: The eight grand knight's paths of size 4 ending on the x-axis.

Corollary 2.2. *The generating function for the total number t_n of grand knight's paths ending at a non-negative altitude with respect to the size is*

$$H(1) = \frac{h_0 u_1 u_2 - h_1}{(1-u_1)(1-u_2)}.$$

An asymptotic approximation of t_n is

$$\frac{\sqrt{3}}{12}\left(1+\sqrt{3}\right)^{n+1}.$$

The first terms of t_n, $0 \le n \le 12$, are

$$1,\ 1,\ 4,\ 8,\ 26,\ 63,\ 186,\ 478,\ 1352,\ 3574,\ 9927,\ 26640,\ 73354,$$

and this sequence does not appear in the OEIS.

Corollary 2.3. *The generating function for the total sum s_n of the altitudes in all grand knight's paths of size n ending at a non-negative altitude is*

$$\frac{\partial}{\partial u} H(u) \mid_{u=1} = \frac{h_0 u_1^2 u_2 + u_1 u_2 (h_0 u_2 - 2h_0 - h_1) + h_1}{(1-u_1)^2 (1-u_2)^2}.$$

An asymptotic approximation of s_n is

$$\frac{(4\sqrt{3}+7)\sqrt{2}\sqrt{137\sqrt{3}-237}}{6}\sqrt{\frac{n}{\pi}}\left(1+\sqrt{3}\right)^{n}.$$

The first terms of s_n, $0 \le n \le 12$, are

$$0,\ 2,\ 5,\ 20,\ 56,\ 180,\ 516,\ 1552,\ 4452,\ 13000,\ 37120,\ 106684,\ 303090,$$

and this sequence does not appear in OEIS.

Corollary 2.4. *The expected altitude of a grand knight's path of size n ending at positive altitude is asymptotically*

$$(5+3\sqrt{3})\sqrt{137\sqrt{3}-237}\sqrt{\frac{2n}{3\pi}}.$$

3 Unrestricted grand zigzag knight's paths

Recall from Definition 1.2 that a grand zigzag knight's path is a grand knight's path avoiding the consecutive patterns NN, NE, $\bar{N}\bar{N}$, $\bar{N}\bar{E}$, EE, EN, $\bar{E}\bar{E}$, $\bar{E}\bar{N}$.

Let $\mathcal{Z}_{n,k}$ be the set of grand zigzag knight's paths of size n and altitude k, and $\mathcal{Z}^+_{n,k}$ (resp. $\mathcal{Z}^-_{n,k}$) be the subset of $\mathcal{Z}_{n,k}$ of paths starting with E or N (resp. $\bar{E}$ or $\bar{N}$). Note that the symmetry with respect to the x-axis provides a direct bijection between $\mathcal{Z}_{n,k}$ and $\mathcal{Z}_{n,-k}$, and also between $\mathcal{Z}^+_{n,k}$ and $\mathcal{Z}^-_{n,-k}$.

3.1 Enumeration using algebraic method

Let $\hat{F}(x,u,z)$ be the generating function for the number of grand zigzag knight's paths ending with E or N with respect to the number of steps (marked by x), the altitude (marked by u, the power of u is negative for paths ending below x-axis) and the size (final x-coordinate, marked by z). Denote by $\hat{G}(x,u,z)$ the generating function for grand zigzag knight's paths ending with $\bar{E}$ or $\bar{N}$ and by $\hat{Z}(x,u,z)$ the generating function for all grand zigzag knight paths.

Then we have readily the following system of equations:

$$\begin{cases} \hat{F}(x,u,z) &= (1+\hat{G}(x,u,z))(xzu^2+xz^2u), \\ \hat{G}(x,u,z) &= (1+\hat{F}(x,u,z))(xzu^{-2}+xz^2u^{-1}), \\ \hat{Z}(x,u,z) &= 1+\hat{F}(x,u,z)+\hat{G}(x,u,z). \end{cases} \tag{10}$$

The solutions of (10) are rational:

$$\begin{aligned} \hat{F}(x,u,z) &= -\frac{xz\,(u+z)\left(u^2+uxz^2+xz\right)}{u^2x^2z^3+ux^2z^4+ux^2z^2-u+x^2z^3}, \\ \hat{G}(x,u,z) &= -\frac{xz\,(uz+1)\left(u^2xz+uxz^2+1\right)}{u\left(u^2x^2z^3+ux^2z^4+ux^2z^2-u+x^2z^3\right)}, \\ \hat{Z}(x,u,z) &= -\frac{\left(u^2+uxz^2+xz\right)\left(u^2xz+uxz^2+1\right)}{u\left(u^2x^2z^3+ux^2z^4+ux^2z^2-u+x^2z^3\right)}. \end{aligned}$$

Fixing $u=1$ and $x=1$ in the third equation, we obtain the generating function for all grand zigzag knight's paths of a given size:

$$\hat{Z}(1,1,z) = \frac{1+z+z^2}{1-z-z^2}. \tag{11}$$

Here are the first coefficients of z^n for $0 \le n \le 16$:

$$1,\ 2,\ 4,\ 6,\ 10,\ 16,\ 26,\ 42,\ 68,\ 110,\ 178,\ 288,\ 466,\ 754,\ 1220,\ 1974,\ 3194.$$

This corresponds to the sequence A128588 in [25], which is a Fibonacci-like sequence with $a_0=1$, $a_1=2$ and $a_2=4$ (the n-th term, except the first, is twice the Fibonacci number F_n defined by $F_n=F_{n-1}+F_{n-2}$ for $n\ge 2$, with $F_0=0$ and $F_1=1$).

By decomposing the rational fraction $\hat{Z}(x,u,z)$ into partial fractions, and after isolating the terms which are analytic at $u=0$, we can obtain an expression for the generating function for paths of positive height. In this paper, we use a different technique inspired by Prodinger [23] and based on the kernel method. We describe it in full details in order to introduce the routine that we will use in Section 4 for bounded paths.

For $k\in\mathbb{Z}$, let f_k (resp. g_k) be the generating function of the number of grand zigzag knight's paths of altitude k with E or N (resp. $\bar{E}$ or $\bar{N}$), with respect to the size. Due to the symmetry with respect to the x-axis, we have $f_{-k}=g_k$ for $k\neq 0$, and $f_0=1+g_0$. So, we will focus on positive altitudes. Let us consider

$F(u,z)=\sum_{k\geq 0} f_k(z)u^k$ and $G(u,z)=\sum_{k\geq 0} g_k(z)u^k$ (for short, we will write $F(u)$ and $G(u)$). We easily obtain the following equations, with the convention that the empty path is counted in f_0:

$$f_1 = z^2 + zg_{-1} + z^2 g_0 = z^2 + zf_1 + z^2 g_0 = zf_1 + z^2 f_0, \tag{12}$$
$$f_2 = z + zg_0 + z^2 g_1,$$
$$f_k = zg_{k-2} + z^2 g_{k-1} \quad \text{for } k \geq 3,$$
$$g_k = zf_{k+2} + z^2 f_{k+1} \quad \text{for } k \geq 0. \tag{13}$$

From equations (12-13) we deduce the following functional equations:

$$F(u) = \left(1 + \frac{z^3 u}{1-z}\right) f_0 + zu(z+u)(G(u)+1),$$
$$G(u) = -\left(\frac{z}{u^2} + \frac{z^2}{u} + \frac{z^3}{u(1-z)}\right) f_0 + \left(\frac{z}{u^2} + \frac{z^2}{u}\right) F(u),$$

which yields

$$F(u) = \frac{f_0(-uz^4 + uz^3 + z^4 - uz^2 - z^3 - uz + u) - u^3 z^2 - u^2 z^3 + u^3 z + u^2 z^2}{(-1+z)\,(u^2 z^3 + uz^4 + z^2 u + z^3 - u)},$$
$$G(u) = \frac{\left(f_0(uz^3 + z^2 - z) - u^2 z^2 - z^3 u + u^2 z + z^2 u - zu - z^2 + u + z\right) z^2}{(-1+z)\,(u^2 z^3 + uz^4 + z^2 u + z^3 - u)}.$$

Let

$$r := \frac{1 - z^4 - z^2 - \sqrt{z^8 - 2\,z^6 - z^4 - 2\,z^2 + 1}}{2z^3},$$
$$s := \frac{1 - z^4 - z^2 + \sqrt{z^8 - 2\,z^6 - z^4 - 2\,z^2 + 1}}{2z^3}$$

be the two roots in u of the kernel $u^2 z^3 + uz^4 + z^2 u + z^3 - u$. Applying the kernel method, we obtain

$$f_0 = \frac{r(z-1)}{z^3(rz^2 + z - 1)} = \frac{(1-z)(-1 + z^2 + z^4 + \sqrt{1 - 2z^2 - z^4 - 2z^6 + z^8})}{z^5(1 - 2z + z^2 - z^4 - \sqrt{1 - 2z^2 - z^4 - 2z^6 + z^8})},$$

and we can state the following cancelling the factor $(u-r)$.

Theorem 3.1. *The generating functions $F(u)$, $G(u)$ are given by:*

$$F(u) = \frac{-u^2 - u(r+z) - z^2 f_0 s}{z^2(u-s)} \quad \textit{and} \quad G(u) = \frac{-z^2 u - z^2 f_0 s + z^2 s}{z^2(u-s)}.$$

The bivariate generating function $H(u) := F(u) + G(u)$ for the number of grand zigzag knight's paths (ending at a non-negative y-coordinate) with respect to the size and the altitude is

$$H(u) = -\frac{u^2 + u(r + z + z^2) + z^2 s(2f_0 - 1)}{z^2(u-s)}.$$

The generating function for the total number of grand zigzag knight's paths of size n ending at a non negative y-coordinate is

$$H(1) = -\frac{1 + r + z + z^2 + z^2 s(2f_0 - 1)}{z^2(1-s)}.$$

Here are the first terms of $H(1)$ for $0 \leq n \leq 16$:

1, 1, 3, 3, 7, 9, 18, 24, 45, 63, 115, 166, 296, 435, 763, 1138, 1973.

Note that the generating function for the total number of grand zigzag knight's paths is thus

$$2H(1) - (2f_0 - 1) = \frac{1 + z + z^2}{1 - z - z^2}.$$

Consistently with the beginning of this section, the coefficient of z^n in $2H(1) - (2f_0 - 1)$ is twice the n-th Fibonacci number (see sequence A128588 in [25]). This can be also seen bijectively as follows. A grand zigzag knight's paths of size n starting with an up-step (so half of the total number of grand zigzag knight's paths of size n) can start either with E, and then followed by a grand zigzag knight's path of size $n-2$ starting with a down-step, or it can start with N, and then followed by a grand zigzag knight's path of size $n-1$ starting with a down-step. See Table 1 for the first values of $|\mathcal{Z}_{n,k}|$. Note that the sequence $(|\mathcal{Z}^+_{2n,0}|)_{n\geq 0}$ corresponds to sequence A051286 in [25].

Corollary 3.1. *The generating function for the number of grand zigzag knight's paths ending at y-coordinate k with respect to the size is given by:*

$$[u^0]H(u) = 2f_0 - 1, \quad [u^1]H(u) = \frac{r + z + 2z^2 f_0}{z^2 s},$$
$$[u^k]H(u) = \frac{r^{k-1}}{z^2}(1 + r(r+z) + 2rz^2 f_0) \quad \text{for } k \geq 2.$$

Corollary 3.2. *An asymptotic approximation for the expected altitude of a grand zigzag knight's path of size n ending on or above the x-axis is*

$$\frac{2(\sqrt{5}-2)}{\sqrt{7\sqrt{5}-15}}\sqrt{\frac{n}{\pi}}.$$

This is also the variance (divided by 2) of the altitude of a grand zigzag knight's path (not necessarily ending on or above the x-axis) of size n (the expected value of the altitude being simply 0 for those paths).

Proof. The generating function for the total number of grand zigzag knight's paths of size n is $H(1)$, and the one for the total sum of altitudes over all grand zigzag knight's paths of size n is $\frac{\partial}{\partial u}H(u)\mid_{u=1}$. We compute asymptotics of the coefficients of both those generating functions using singularity analysis (see [15]), and then we take the ratio to obtain the stated result. □

Remark 3.1. *From [11, Theorem 1], we deduce that the expected altitude of a grand zigzag knight's path of size n and staying above the x-axis is equivalent as $n \to \infty$ to*

$$\frac{(5+\sqrt{5})\sqrt{7\sqrt{5}-15}}{20}\sqrt{\pi n}.$$

Thus, the altitude of a zigzag knight's path (i.e. a grand zigzag knight's path in $\mathbb{N}^2$), is, in average, ≈ 1.57079633 times the altitude of a grand zigzag knight's path in $\mathbb{Z}^2$ ending at non-negative altitude.

k \ n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	1	0	2	0	4	2	10	**6**	22	16	52	44	126	116	306	302
1	0	0	1	2	2	4	4	10	11	26	28	64	71	160	183	402
2	0	1	0	1	0	3	2	7	6	16	18	40	52	100	142	252
3	0	0	0	0	1	0	2	0	6	2	16	8	41	28	107	90
4	0	0	0	0	0	0	0	1	0	3	0	10	2	30	10	85

Table 1: The number of grand zigzag knight's paths from $(0,0)$ to (n,k) for $(n,k) \in [0,15] \times [0,4]$.

We refer to Figure 3 for the illustration of the six grand zigzag knight's paths of size 7 ending on the x-axis (see the entry indicated by the boldface in the table).

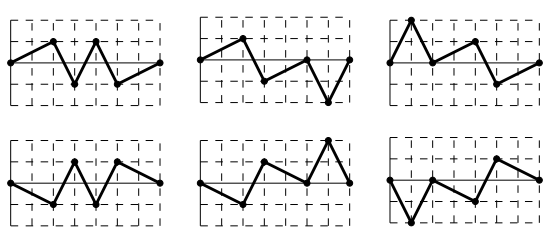


Figure 3: The six grand zigzag knight's paths of size 7 ending on the x-axis.

Now, we focus on grand zigzag knight's paths that never touch the x-axis except for the initial and the final point. Let $\mathcal{B}$ the set of such paths.

Corollary 3.3. *The generating function $B(z)$ for the number b_n of grand zigzag knight's paths in $\mathcal{B}$ with respect to the size is*

$$B(z) = \frac{2z^5 r + 2z^4 - 2z^3 - 3rz + 3r}{r(1-z)}.$$

Proof. Let $\Gamma = 2f_0 - 1$ be the generating function for the grand zigzag knight's paths ending on the x-axis. Each element counted by Γ is either empty or can be uniquely decomposed into a juxtaposition $B_1 \cdots B_k$ of non-empty paths $B_i \in \mathcal{B}$ so that $B_i B_{i+1}$ forms a zigzag, which forces B_{i+1} to start with an up step (resp. down step) if B_i ends with a down step (resp. up step). Therefore, the generating function for B_1 is $B(z) - 1$, and the generating function for B_i, $i \geq 2$, is $(B(z) - 1)/2$. Thus, we have

$$\Gamma = 1 + (B(z) - 1) \sum_{k \geq 0} \left(\frac{B(z) - 1}{2} \right)^k = \frac{1 + B(z)}{3 - B(z)}.$$

We deduce $B(z) = (3\Gamma - 1)/(\Gamma + 1)$ and the expression of $B(z)$ follows from Corollary 3.1. □

The first terms of b_n for $0 \leq n \leq 22$ are

$$1, 0, 2, 0, 2, 2, 4, 2, 4, 2, \mathbf{6}, 2, 10, 2, 18, 2, 36, 2, 76, 2, 166, 2, 372.$$

All the odd terms equal 2 for $2n + 1 \geq 5$, because the only paths of size $2n + 1$ in $\mathcal{B}$ are $E\bar{N}N \cdots N\bar{N}E$ and its symmetric $\bar{E}N\bar{N} \cdots \bar{N}N\bar{E}$. Figure 4 shows the six grand zigzag knight paths of size 10 in $\mathcal{B}$ (see the entry indicated by the boldface in the list above).

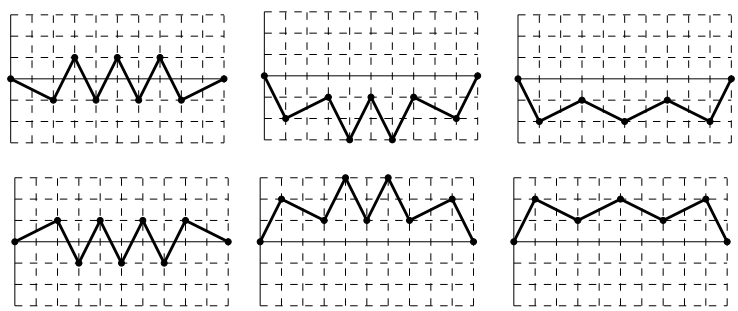


Figure 4: The six grand zigzag knight's paths of size 10 having exactly the starting and the ending point on the x-axis.

3.2 A bijective approach

In this part, we exhibit a bijection between pairs of integer compositions and grand zigzag knight's paths ending at (n, k) where n and k have the same parity.

In [12], Bóna and Knopfmacher provided the following bijection φ between pairs of compositions of n with parts in $\{1, 2\}$ that have the same number of parts, and lattice paths starting at $(0, 0)$ with size n and steps from the set $\{(1, 0), (2, 0), (2, 1), (1, -1)\}$. Let $X = (x_1, \ldots, x_k)$ and $Y = (y_1, \ldots, y_k)$ be compositions of n with k parts. The mapping is defined by $\varphi(X, Y) = \varphi(x_1, y_1) \cdots \varphi(x_k, y_k)$ with

$$\varphi(x_i, y_i) = \begin{cases} (1, 0), & \text{if } x_i = y_i = 1, \\ (2, 0), & \text{if } x_i = y_i = 2, \\ (2, 1), & \text{if } x_i = 2 \text{ and } y_i = 1, \\ (1, -1), & \text{if } x_i = 1 \text{ and } y_i = 2. \end{cases}$$

Here we construct a similar bijection with grand zigzag knight's paths ending at a point whose coordinates have the same parity.

Let $\mathcal{C}_{n,m}$ be the set of ordered pairs (X, Y) of compositions of n and m, respectively, where every part is 1 or 2, such that X and Y have the same number of parts. Note that $\mathcal{C}_{n,m} = \varnothing$ if $n \leq \lceil m/2 \rceil$ or $m \leq \lceil n/2 \rceil$.

Lemma 3.1. *If $n \leq m$, then*

$$|\mathcal{C}_{m,n}| = |\mathcal{C}_{n,m}| = \sum_{i=0}^{n - \lceil m/2 \rceil} \binom{n - i}{i} \binom{n - i}{m - n + i}.$$

Proof. Let us count the compositions of n into parts equal to 1 or 2 with i parts ($n \geq i \geq \lceil n/2 \rceil$). Since there are i parts, there are necessarily $n - i$ parts equal to 2, and $2i - n$ parts equal to 1. Then it remains to choose

the places of the 1's among those i parts, which can be done in $\binom{i}{2i-n} = \binom{i}{n-i}$ ways. Therefore, if we assume $n \le m$,

$$|\mathcal{C}_{n,m}| = \sum_{i=\lceil m/2 \rceil}^{n} \binom{i}{n-i}\binom{i}{m-i} = \sum_{i=0}^{n-\lceil m/2 \rceil} \binom{n-i}{i}\binom{n-i}{m-n+i}. \qquad \square$$

Lemma 3.2. *If $n, k \in \mathbb{N}$ with $n = k \pmod 2$, then there is a bijection ϕ between $\mathcal{C}_{\frac{n-k}{2},\frac{n+k}{2}}$ and $\mathcal{Z}^+_{n,k}$, and a bijection ψ between $\mathcal{C}_{\frac{n-k}{2},\frac{n+k}{2}}$ and $\mathcal{Z}^-_{n,k}$.*

Proof. Let $(X, Y) \in \mathcal{C}_{\frac{n-k}{2},\frac{n+k}{2}}$. Let i be the number of parts of X and Y, so that $X = (x_1, \dots, x_i)$ and $Y = (y_1, \dots, y_i)$, with $x_j, y_j \in \{1, 2\}$. We define $\phi(X, Y)$ as the path $\phi(x_1, y_1) \cdots \phi(x_k, y_k)$, where

$$\phi(x_j, y_j) = \begin{cases} E\bar{E}, & \text{if } x_j = y_j = 2, \\ N\bar{N}, & \text{if } x_j = y_j = 1, \\ N\bar{E}, & \text{if } x_j = 1 \text{ and } y_j = 2, \\ E\bar{N}, & \text{if } x_j = 2 \text{ and } y_j = 1. \end{cases}$$

Note that the size of $\phi(x_j, y_j)$ is $x_j + y_j$, so the size of $\phi(X, Y)$ is $\sum_{j=1}^{i}(x_j + y_j) = n$. The altitude of $\phi(x_j, y_j)$ is $y_j - x_j$, and consequently, the altitude of $\phi(X, Y)$ is $\sum_{j=0}^{i}(y_j - x_j) = k$. Therefore, ϕ maps into $\mathcal{Z}^+_{n,k}$. The inverse of ϕ is easy to obtain, since each path of $\mathcal{Z}^+_{n,k}$ can be seen as a path of steps belonging to $\{E\bar{E}, N\bar{N}, N\bar{E}, E\bar{N}\}$. Thus, ϕ is a bijection between $\mathcal{C}_{n,m}$ and $\mathcal{Z}^+_{n,k}$. The bijection ψ can be constructed similarly since each path of $\mathcal{Z}^-_{n,k}$ can be seen as a path of steps belonging to $\{\bar{E}E, \bar{N}N, \bar{E}N, \bar{N}E\}$. $\square$

Example 3.1. *If we consider the compositions*

$$X = (2, 2, 2, 1, 1, 1, 1, 2, 1) \quad \textit{and} \quad Y = (1, 2, 1, 2, 2, 1, 2, 1, 2),$$

then $(X, Y) \in \mathcal{C}_{13,14}$ is mapped to the path $E\bar{N}E\bar{E}E\bar{N}N\bar{E}N\bar{E}N\bar{N}N\bar{E}E\bar{N}N\bar{E} \in \mathcal{Z}^+_{27,1}$, see Figure 5.

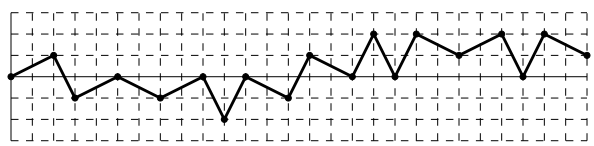


Figure 5: Illustration to Example 3.8. The path $\phi(X, Y)$ is a grand zigzag knight's path of size 27 and altitude 1.

Theorem 3.2. *If $n = k \pmod 2$, with $(n, k) \neq (0, 0)$, then*

$$|\mathcal{Z}_{n,k}| = 2 \sum_{i=0}^{\frac{n-|k|}{2}} \binom{\frac{n-|k|}{2} - i}{i}\binom{\frac{n-|k|}{2} - i}{|k| + i}.$$

If $n \neq k \pmod 2$, then

$$|\mathcal{Z}_{n,k}| = |\mathcal{Z}^+_{n-1,k-2}| + |\mathcal{Z}^+_{n-2,k-1}| + |\mathcal{Z}^-_{n-1,k+2}| + |\mathcal{Z}^-_{n-2,k+1}|,$$

where each of those four terms can be expressed with the above formula (without the factor 2).

Proof. By Lemma 3.2, if $n = k \pmod 2$, then we have $|\mathcal{Z}_{n,k}| = |\mathcal{Z}^+_{n,k}| + |\mathcal{Z}^-_{n,k}| = 2|\mathcal{C}_{\frac{n-k}{2},\frac{n+k}{2}}|$. We conclude using Lemma 3.1. If $n \neq k \pmod 2$, then we have

$$\mathcal{Z}_{n,k} = \mathcal{Z}_{n-1,k-2} \cdot N \cup \mathcal{Z}_{n-2,k-1} \cdot E \cup \mathcal{Z}_{n-1,k+2} \cdot \bar{N} \cup \mathcal{Z}_{n-2,k+1} \cdot \bar{E},$$

which completes the proof. $\square$

Remark 3.2. *When $n = k \pmod 2$, $|\mathcal{Z}_{n,k}|$ is even (except for $(n, k) = (0, 0)$). Indeed, since each path of $\mathcal{Z}_{n,k}$ has an even number of steps, we have a bijection from $\mathcal{Z}^+_{n,k}$ to $\mathcal{Z}^-_{n,k}$, induced by*

$$\begin{array}{ccc} N\bar{N} & \to & \bar{N}N \\ E\bar{E} & \to & \bar{E}E \\ N\bar{E} & \to & \bar{E}N \\ E\bar{N} & \to & \bar{N}E. \end{array}$$

3.3 Step number of a grand zigzag knight's path

For $n/2 \leq i \leq n$, let $\mathcal{Z}^i_{n,k}$ be the subset of $\mathcal{Z}_{n,k}$ consisting of paths with i steps (see Section 3 for a definition of $\mathcal{Z}_{n,k}$). We define similarly $\mathcal{Z}^{i,+}_{n,k}$ and $\mathcal{Z}^{i,-}_{n,k}$. Note that $|\mathcal{Z}^i_{n,k}| = 0$ if $i \neq n-k \pmod 2$.

Theorem 3.3. *For $n = k \pmod 2$ and i even,*

$$|\mathcal{Z}^{i,+}_{n,k}| = |\mathcal{Z}^{i,-}_{n,k}| = \binom{i/2}{\frac{n-i-k}{2}}\binom{i/2}{\frac{n-i+k}{2}}.$$

For $n \neq k \pmod 2$ and i odd,

$$|\mathcal{Z}^{i,+}_{n,k}| = \binom{\frac{i+1}{2}}{\frac{n-k-i}{2}+1}\binom{\frac{i-1}{2}}{\frac{n+k-i}{2}-1},$$

$$|\mathcal{Z}^{i,-}_{n,k}| = \binom{\frac{i-1}{2}}{\frac{n-k-i}{2}-1}\binom{\frac{i+1}{2}}{\frac{n+k-i}{2}+1}.$$

Proof. Let us first assume $n = k \pmod 2$ and i is even. Let $\mathbf{e}, \mathbf{n}, \bar{\mathbf{e}}, \bar{\mathbf{n}}$ be the numbers of steps $E, N, \bar{E}, \bar{N}$ of a path in $\mathcal{Z}^{i,+}_{n,k}$ (or $\mathcal{Z}^{i,-}_{n,k}$). Then we have the following equations:

$$\begin{cases} \mathbf{e} + \bar{\mathbf{e}} &= n - i, \\ \mathbf{n} + \bar{\mathbf{n}} &= 2i - n, \\ \mathbf{e} + \mathbf{n} &= i/2, \\ \bar{\mathbf{e}} + \bar{\mathbf{n}} &= i/2, \\ \mathbf{e} + 2\mathbf{n} - \bar{\mathbf{e}} - 2\bar{\mathbf{n}} &= k. \end{cases}$$

This linear system is non-degenerate, thus we deduce the unique solution

$$(\mathbf{e}, \mathbf{n}, \bar{\mathbf{e}}, \bar{\mathbf{n}}) = \left(\frac{n-i-k}{2}, \frac{2i-n+k}{2}, \frac{n-i+k}{2}, \frac{2i-n-k}{2}\right).$$

Then, it remains to choose the positions of the N's among the up-steps, and the positions of the $\bar{N}$'s among the down-steps. This eventually yields the desired formula. When $n \neq k \pmod 2$ and i is odd, the third and the fourth equations become

$$\begin{cases} \mathbf{e} + \mathbf{n} &= (i+1)/2, \\ \bar{\mathbf{e}} + \bar{\mathbf{n}} &= (i-1)/2, \end{cases}$$

for $\mathcal{Z}^{i,+}_{n,k}$, and

$$\begin{cases} \mathbf{e} + \mathbf{n} &= (i-1)/2, \\ \bar{\mathbf{e}} + \bar{\mathbf{n}} &= (i+1)/2, \end{cases}$$

for $\mathcal{Z}^{i,-}_{n,k}$. We then conclude similarly. □

Corollary 3.4. *If $n = k \pmod 2$ with $(n,k) \neq (0,0)$,*

$$|\mathcal{Z}_{n,k}| = 2\sum_{\substack{i=0 \\ i \text{ even}}}^{n-k} \binom{i/2}{\frac{n-i-k}{2}}\binom{i/2}{\frac{n-i+k}{2}}.$$

If $n \neq k \pmod 2$,

$$|\mathcal{Z}_{n,k}| = \sum_{\substack{i=0 \\ i \text{ odd}}}^{n-k+1} \left[\binom{\frac{i+1}{2}}{\frac{n-k-i}{2}+1}\binom{\frac{i-1}{2}}{\frac{n+k-i}{2}-1} + \binom{\frac{i-1}{2}}{\frac{n-k-i}{2}-1}\binom{\frac{i+1}{2}}{\frac{n+k-i}{2}+1}\right].$$

Remark 3.3. *Note that we obtain the same formula as the one in Theorem 3.2. Also, we now have a formula for $|\mathcal{Z}_{n,k}|$ when $n \neq k \pmod 2$ involving only 2 sums, instead of 4 as stated in Theorem 3.2.*

Corollary 3.5. *The expected value for the number of steps of a grand zigzag knight's path ending at (n,k) is*

$$\frac{2}{|\mathcal{Z}_{n,k}|}\sum_{\substack{i=0 \\ i \text{ even}}}^{n-k} i\binom{i/2}{\frac{n-i-k}{2}}\binom{i/2}{\frac{n-i+k}{2}} \quad \text{if } n = k \pmod 2 \text{ and}$$

$$\frac{1}{|\mathcal{Z}_{n,k}|}\sum_{\substack{i=0 \\ i \text{ odd}}}^{n-k} i\left[\binom{\frac{i+1}{2}}{\frac{n-k-i}{2}+1}\binom{\frac{i-1}{2}}{\frac{n+k-i}{2}-1} + \binom{\frac{i-1}{2}}{\frac{n-k-i}{2}-1}\binom{\frac{i+1}{2}}{\frac{n+k-i}{2}+1}\right],$$

otherwise.

Theorem 3.4. *An asymptotic approximation for the expected number of steps of a grand zigzag knight's path ending on the x-axis of size $2n$ (which is also the expected number of parts of a pair of compositions of n with parts in $\{1,2\}$ and having the same number of parts) is*

$$\frac{1+\sqrt{5}}{2\sqrt{5}} \cdot 2n.$$

Proof. By Corollary 3.5 and Theorem 3.2, it suffices to estimate $\sum_{i=0}^{n} \binom{i}{n-i}^2$ and $\sum_{i=0}^{n} i\binom{i}{n-i}^2$ (respectively A051286 and A182879 in [25]). Their generating functions are respectively

$$\frac{1}{\sqrt{1-2z-z^2-z^3+z^4}}, \quad \text{and} \quad \frac{z(1+2z^2-z^3)}{((1-3z+z^2)(1+z+z^2))^{3/2}}.$$

We conclude using singularity analysis. □

Based on Mathematica calculations, we suspect that the behavior is the same for odd sizes.

Conjecture 3.1. *An asymptotic approximation for the expected number of steps of a grand zigzag knight's path ending on the x-axis of size n is*

$$\frac{1+\sqrt{5}}{2\sqrt{5}} \cdot n.$$

4 Bounded grand zigzag knight's paths

In this section, we focus on grand zigzag knight's paths staying in some regions delimited by horizontal lines.

4.1 Grand zigzag knight's paths staying above a horizontal line

First, we consider paths that stay above the line $y = -m$ for a given integer $m \geq 0$. By symmetry, this is equivalent to count grand zigzag knight's paths staying below the line $y = +m$. Note that the case $m = 0$ has been done in [11], thus we will assume $m \geq 1$.

Remark 4.1. *When $n = k \pmod 2$, there is a bijection between those paths ending at (n,k) and pairs of compositions $(X,Y) \in \mathcal{C}_{\frac{n-k}{2},\frac{n+k}{2}}$ such that for all j,*

$$-m \leq \sum_{i=1}^{j}(y_i - x_i)$$

(see the bijections ϕ and ψ used in Lemma 3.2).

Figure 6 shows an example of a grand zigzag knight's path above the line $y = -3$, with size 27, and altitude -2.

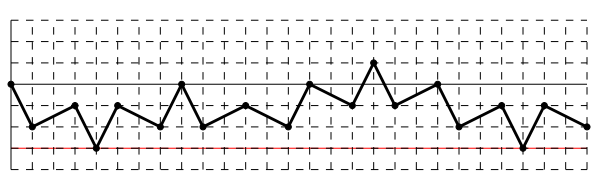


Figure 6: A grand zigzag knight's path staying above the line $y = -3$, with size 27, and altitude -2.

As in the previous section, for $k \geq 0$, let f_{-m+k} (resp. g_{-m+k}) be the generating function of the number of grand zigzag knight's paths staying above the y-coordinate $-m$ and ending at altitude $-m+k$ with an up-step N or E (resp. with a down-step $\bar{N}$ or $\bar{E}$). Then we easily obtain the following equations

$$f_{-m} = 0, \; f_{-m+1} = \mathbb{1}_{[m=1]} + z^2 g_{-m}, \tag{14}$$

$$f_{-m+k} = z g_{-m+k-2} + z^2 g_{-m+k-1} + \mathbb{1}_{[k=m]} + \mathbb{1}_{[k=m+1]} z^2 + \mathbb{1}_{[k=m+2]} z \quad \text{for } k \geq 2$$

and

$$g_{-m+k} = z f_{-m+k+2} + z^2 f_{-m+k+1} \text{ for } k \geq 0,$$

where $\mathbb{1}_{[a=b]}$ is 1 if $a = b$, and 0 otherwise.

Setting $F(u) = \sum_{k\geq 0} f_{-m+k}u^k$ and $G(u) = \sum_{k\geq 0} g_{-m+k}u^k$, we obtain

$$F(u) = zu^{m+1}(z+u) + u^m + zu(z+u)G(u),$$
$$G(u) = -\frac{z}{u}f_{-m+1} + \left(\frac{z}{u^2} + \frac{z^2}{u}\right)F(u).$$

Using the kernel method (the roots r and s of the kernel are the same as for Section 3.1), we find

$$f_{-m+1} = (1+zr)r^{m-1} + \frac{r^{m+1}}{z}.$$

Remark 4.2. *From equation (14), we obtain* $g_{-m} = \frac{1+zr}{z^2}r^{m-1} + \frac{r^{m+1}}{z^3} - \mathbb{1}_{[m=1]}\frac{1}{z^2}$. *This is the same expression as the generating function for the total number of grand zigzag knight's paths of altitude m and staying above the x-axis, see [11, Theorem 1]. Indeed, reading the paths backwards, we get a bijection between those ending at altitude m and staying above $y = 0$, and those ending at altitude $-m$ and staying above $y = -m$, see Figure 7.*

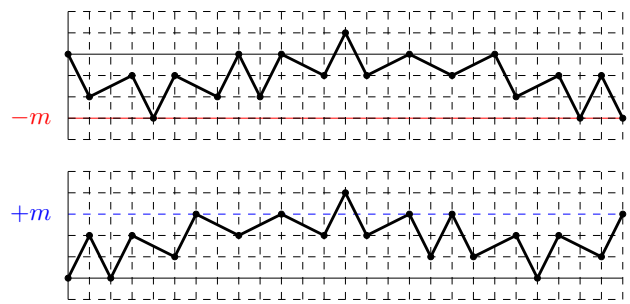


Figure 7: There is a bijection between grand zigzag knight's paths staying above $y = -m$ and ending at altitude $-m$, and grand zigzag knight's paths staying above $y = 0$ and ending at altitude k.

Theorem 4.1. *The bivariate generating functions for the number of grand zigzag knight's paths staying above $y = -m$ and ending with respectively an up-step and a down-step with respect to the size and the altitude are*

$$F(u) = -\frac{u(u^m(1+z^2u+zu^2) - r^{m-1}z(u+z)(z+rz^2+r^2))}{z^3(u-r)(u-s)},$$

$$G(u) = \frac{r^{m-1}(z+r^2+z^2r) - zu^{m-1}(1+zu)(1+z^2u+zu^2)}{z^3(u-r)(u-s)}.$$

Setting $H(u) := F(u)+G(u)$, the generating function for the total number of grand zigzag knight's paths staying above $y = -m$ with respect to the size is

$$H(1) = \frac{zr^{m-1} + z^2r^m + r^{m+1} - z^2 - z - 1}{z^2+z-1}.$$

Here are the first terms of $H(1)$ when $m = 2$ and $0 \leq n \leq 15$:

$$1, 2, 4, 6, 9, 15, 23, 38, 58, 95, 147, 239, 373, 603, 947, 1525.$$

Remark 4.3. *$H(1)$ converges as $m \to \infty$ to $(1+z+z^2)/(1-z-z^2)$, the generating function for the total number of grand zigzag knight's paths (see the beginning of Section 3). Moreover, the valuation of r is 3, thus we can estimate the rate of convergence:*

$$val\left(H(1) - \frac{1+z+z^2}{1-z-z^2}\right) = 3m-2.$$

In particular, every grand zigzag knight's path of size $\leq 3m-3$ stays above $y = -m$, and we know that the path $(\bar{N}E)^{m-1}\bar{N}$ has size $3m-2$ and goes below $y = -m$.

Corollary 4.1. *For $m \geq 0$, the probability that a grand zigzag knight's path chosen uniformly at random among all grand zigzag knight's paths of size n stays above the line $y = -m$ is asymptotically $c_m/\sqrt{n}$, with*

$$c_m = \begin{cases} \frac{2+\sqrt{5}}{2}\sqrt{\frac{7\sqrt{5}-15}{\pi}}, & \text{if } m = 0, \\ \frac{4m+3-\sqrt{5}}{4(\sqrt{5}-2)}\sqrt{\frac{7\sqrt{5}-15}{\pi}}, & \text{if } m \geq 1. \end{cases}$$

Proof. Using basic singularity analysis, since the generating function for the total number of grand zigzag knight's paths with respect to the size is $(1+z+z^2)/(1-z-z^2)$, we deduce that the asymptotic for the total number $a(n)$ of grand zigzag knight's paths of size n is $\frac{4}{\sqrt{5}(\sqrt{5}-1)}\alpha^{-n}$, with $\alpha = \frac{\sqrt{5}-1}{2}$. Now we study $H(1)$ (see Theorem 4.1). It turns out that the main singularity of this function (the one with the smallest absolute value) is also α. Using the following approximation near α:

$$\frac{r^j}{z-\alpha} = \frac{1}{z-\alpha} + \frac{j\sqrt{10-4\sqrt{5}}}{\sqrt{5}-2}(z-\alpha)^{-1/2} + O(1),$$

we deduce

$$H(1) \sim \frac{(2m+1-\alpha)\sqrt{10-4\sqrt{5}}}{\sqrt{5}(\sqrt{5}-2)}(z-\alpha)^{-1/2}$$

as $z \to \alpha$. We thus have the following approximation for the number $a_m(n)$ of grand zigzag knight's paths of size n and staying above $y = -m$ (see for instance [15, Chapter VI]) as $n \to \infty$, when $m \geq 1$:

$$a_m(n) \sim \frac{(2m+1-\alpha)\sqrt{10-4\sqrt{5}}}{\sqrt{5}\alpha(\sqrt{5}-2)}\frac{\alpha^{-n}}{\sqrt{\pi n}}.$$

By taking the quotient, we obtain the asymptotic for the desired probability as $n \to \infty$:

$$\frac{a_m(n)}{a(n)} \sim \frac{4m+3-\sqrt{5}}{4(\sqrt{5}-2)}\sqrt{\frac{7\sqrt{5}-15}{\pi n}}.$$

When $m = 0$, we use similarly the generating function from [11, Corollary 2]. □

Remark 4.4. *We can deduce from the previous proof (going one order further) that the asymptotic probability that a grand zigzag knight's path of size n chosen uniformly at random has minimal height $-m$ is*

$$\frac{a_m(n) - a_{m-1}(n)}{a(n)} \sim \begin{cases} \frac{2+\sqrt{5}}{2}\sqrt{\frac{7\sqrt{5}-15}{\pi n}}, & \text{if } m = 0, \\ \frac{5+3\sqrt{5}}{4}\sqrt{\frac{7\sqrt{5}-15}{\pi n}}, & \text{if } m = 1, \\ (2+\sqrt{5})\sqrt{\frac{7\sqrt{5}-15}{\pi n}}, & \text{if } m \geq 2. \end{cases}$$

In particular, asymptotically, every minimal height ≤ -2 has the same probability. By symmetry, this is exactly the same for maximal heights.

4.2 Grand zigzag knight's paths staying in a symmetric tube

Here we count grand zigzag knight's paths that stay in a symmetric tube, i.e. between two horizontal lines $y = -m$ and $y = +m$, for $m \geq 1$. See [13,16,18] for previous studies of lattices paths staying in a tube. Figure 8 shows an example of a grand zigzag knight's path staying between the lines $y = -2$ and $y = 2$, with size 27 and altitude 0.

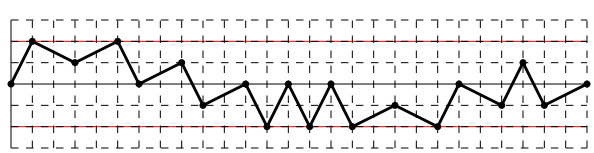


Figure 8: A grand zigzag knight's path staying between the lines $y = -2$ and $y = +2$, with size 27 and altitude 0.

We use the same notation as in the previous subsection. Note that $f_{-k} = g_k$ for all $-m \leq k \leq m$. We obtain the following equations:

$$f_{-m} = g_m = 0,\ f_{-m+1} = \mathbb{1}_{[m=1]} + z^2 g_{-m},\ g_{m-1} = z^2 f_m,$$

$$f_{-m+k} = zg_{-m+k-2} + z^2 g_{-m+k-1} + \mathbb{1}_{[k=m]} + \mathbb{1}_{[k=m+1]} z^2 + \mathbb{1}_{[k=m+2]} z$$

for $2 \le k \le 2m$, and

$$g_{-m+k} = zf_{-m+k+2} + z^2 f_{-m+k+1} \quad \text{for } 0 \le k \le 2m-2.$$

Remark 4.5. *We actually have a finite linear system with unknowns f_k, g_k, with polynomial coefficients. Therefore, we expect rational solutions and in particular, the generating function for the total number of grand zigzag knight's paths staying between $y = -m$ and $y = +m$ is rational.*

Setting $F(u) = \sum_{k=0}^{2m} f_{-m+k} u^k$ and $G(u) = \sum_{k=0}^{2m} g_{-m+k} u^k$ we obtain

$$\begin{aligned} F(u) &= u^m - zu^{2m+1}(f_{-m+1} - \mathbb{1}_{[m=1]}) + zu(z+u)(G(u) + u^m), \\ G(u) &= -\frac{z}{u} f_{-m+1} + \left(\frac{z}{u^2} + \frac{z^2}{u}\right) F(u). \end{aligned}$$

The kernel method then leads to

$$f_{-m+1} = \frac{r^m(1 + rz^2 + r^2 z(1 + \mathbb{1}_{[m=1]}))}{z(rz + z^2 + r^{2m+1})}.$$

Theorem 4.2. *The generating functions $F(u)$ and $G(u)$ counting the number of grand zigzag knight's paths staying between $y = -m$ and $y = +m$ with respect to the size and the altitude are given by:*

$$\begin{aligned} F(u) &= -\frac{(z(\mathbb{1}_{[m=1]} - f_{-m+1})u^{2m+1} - z^2(u+z)f_{-m+1} + u^m(1 + z^2 u + zu^2))u}{z^3(u-r)(u-s)}, \\ G(u) &= -\frac{z(1+zu)(\mathbb{1}_{[m=1]} - f_{-m+1})u^{2m+1} - uf_{-m+1} + u^m(1+zu)(1+z^2u+zu^2)}{z^2(u-r)(u-s)}. \end{aligned}$$

Setting $H(u) = F(u) + G(u)$, the generating function for the total number of grand zigzag knight's paths staying between $y = -m$ and $y = +m$ is

$$H(1) = \frac{(2f_{-m+1} - \mathbb{1}_{[m=1]})z - z^2 - z - 1}{z^2 + z - 1}.$$

Remark 4.6. *As a consequence of Remark 4.5, f_{-m+1} and $H(1)$ are rational. As in Remark 4.3, $H(1)$ converges as $m \to \infty$ to $(1 + z + z^2)/(1 - z - z^2)$, the generating function for the total number of grand zigzag knight's paths (see (11) at the beginning of Section 3).*

Example 4.1. *For $m = 1$, the generating function for the number of grand zigzag knight's paths staying between $y = -1$ and $y = +1$ and ending on the x-axis is $(-z^4 + z - 1)/(z^4 + z - 1)$. Its first terms for $0 \le n \le 18$ are:*

$$1,\ 0,\ 0,\ 0,\ 2,\ 2,\ 2,\ 2,\ 4,\ 6,\ 8,\ 10,\ 14,\ 20,\ 28,\ 38,\ 52,\ 72,\ 100.$$

The term of order $2n+4$ corresponds to 2×sequence A052535(n)= $2\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{2n-3k}{k}$. The term of order $2n+3$ corresponds to 2×sequence A158943(n).

For instance, we can easily exhibit three bijections between such paths from $(0,0)$ to $(2n+4, 0)$ starting with E and

- tilings of a $2 \times 2n$ rectangle with 1×2 and 4×1 tiles,
- pairs $X = (x_1, \ldots, x_k), Y = (y_1, \ldots, y_k)$ of compositions of $n+2$ with same number of parts, with all parts in $\{1, 2\}$ and such that for all $1 \le j \le k$,

$$\left| \sum_{i=1}^{j} (x_i - y_i) \right| \le 1.$$

- compositions of n with parts in $\{2, 1, 3, 5, 9, 11, \ldots\}$. Indeed, a grand zigzag knight's path of size $2n+4$ staying between $y = -1$ and $y = +1$ can be uniquely decomposed as $E\mathbf{P}\bar{E}$ with $\mathbf{P}$ of size $2n$ and having its steps in $\{\bar{E}E\} \cup \bigcup_{k \ge 0} \{\bar{N}(E\bar{E})^k N\}$. If $\mathbf{P} = S_1 \cdots S_j$, then we set $\Phi(E\mathbf{P}\bar{E}) = (\Phi(S_1), \ldots, \Phi(S_j))$ with $\Phi(\bar{E}E) = 2$ and $\Phi(\bar{N}(E\bar{E})^k N) = 2k+1$.

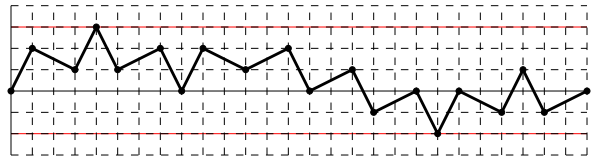


Figure 9: A grand zigzag knight's path staying between the lines $y = -2$ and $y = +3$, with size 27 and altitude 0.

A Grand zigzag knight's paths staying in a general tube

Here we provide, without going into details, additional results that can be obtained by similar methods. We enumerate grand zigzag knight's paths that stay between the lines $y = -m$ and $y = +M$ for $m, M \geq 0$ with $M \geq m$. In particular, we always have $M \geq 1$ (the case $m = M = 0$ being trivial). Figure 9 shows an example of a grand zigzag knight's path staying between the lines $y = -2$ and $y = +3$, with size 27 and altitude 0.

Remark A.1. *When $n = k \pmod 2$, there is a bijection between those paths ending at (n, k) and pairs of compositions $(X, Y) \in \mathcal{C}_{\frac{n-k}{2}, \frac{n+k}{2}}$ such that for all $1 \leq j \leq k$,*

$$-m \leq \sum_{i=1}^{j} (y_i - x_i) \leq M$$

(see bijections ϕ and ψ in Lemma 3.2).

With a similar reasoning as in Section 4, we obtain the following equations:

$$F(u) = u^m - z^3 u^{m+M+1} f_M + zu(z+u)(G(u) + u^m),$$
$$G(u) = -\frac{z}{u} f_{-m+1} + \left(\frac{z}{u^2} + \frac{z^2}{u}\right)(F(u) - \mathbb{1}_{[m=0]}).$$

Remark A.2. *As in Remark 4.5, the solutions are rational.*

Using the kernel method, we find:

$$f_{-m+1} = \frac{(1+sz)\left((\mathbb{1}_{[m=0]} - s^m(1+sz^2+s^2z))r^{m+M+2} + s^{m+M+1}(r^{m+1}(1+rz^2+r^2z) - \mathbb{1}_{[m=0]} z^2(z+r)(1+zr))\right)}{s^{m+M} z^2(r + z(2+sz)) - r^{m+M+1}},$$

$$f_M = -\frac{r^m + z(z+r)\left(r^{m+1} - z\left(s^{m-1}(1+sz)(1+s^2z+sz^2) - \mathbb{1}_{[m=0]}(r-s)\right)\right)}{z^3(z^2(r+z)(1+sz)s^{m+M} - r^{m+M+1})}.$$

Theorem A.1. *The generating functions $F(u)$ and $G(u)$ counting the number of grand zigzag knight's paths staying between $y = -m$ and $y = +M$ with respect to the size and the altitude are given by:*

$$F(u) = \frac{z^3 u^{m+M+2} f_M + uz^2(u+z) f_{-m+1} + \mathbb{1}_{[m=0]} z^2(u+z)(1+uz) - u^{m+1}(1+uz^2+u^2z)}{z^3(u-r)(u-s)},$$
$$G(u) = \frac{z^3 u^{m+M+1}(1+uz) f_M + u f_{-m+1} + \mathbb{1}_{[m=0]}(1+uz) - u^m(1+uz)(1+uz^2+u^2z)}{uz^2(u-r)(u-s)}.$$

We set $H(u) := F(u) + G(u)$.

Corollary A.1. *The generating function for the number of grand zigzag knight's path staying between $y = 0$ and $y = +M$ and ending on the x-axis is $H(0) = \frac{f_1}{z^2}$.*

When $M = 2$, we obtain the first terms, $0 \leq n \leq 18$,

$$1, 0, 1, 0, 2, 0, 4, 0, 7, 0, 14, 0, 26, 0, 50, 0, 95, 0, 181,$$

where the even terms are sequence A052535 (see Example 4.1).

Remark A.3. *We have a convergence to the generating function for all grand zigzag knight's path staying above the x-axis and ending on it $A = \frac{r}{z^3}$ (see [11, Corollary 1]) as $M \to \infty$.*

Corollary A.2. *The generating function for the total number of grand zigzag knight's paths staying between $y = -m$ and $y = +M$ is*

$$H_{m,M}(z) := H(1) = \frac{(f_{-m+1} + z^2 f_M - 1)z + \mathbb{1}_{[m=0]} z(1+z) - 1 - z^2}{z^2 + z - 1}.$$

As a byproduct and using the inclusion–exclusion principle, the generating function for grand zigzag knight's paths staying between lines $y = -m$ and $y = M$, and reaching them, is $H_{m,M}(z) - H_{m-1,M}(z) - H_{m,M-1}(z) + H_{m-1,M-1}(z)$ when $1 \leq m \leq M$, and $H_{0,M}(z) - H_{0,M-1}(z)$ when $0 \leq M$. Then the generating function grand zigzag knight's paths such that the difference between the maximal and minimal y-coordinates of a point of the path is exactly k is

$$2\left(H_{0,k}(z) - H_{0,k-1}(z) + \sum_{m=1}^{\lfloor k/2 \rfloor} \left(H_{m,k-m}(z) - H_{m-1,k-m}(z) - H_{m,k-m-1}(z) + H_{m-1,k-m-1}(z)\right)\right).$$

Table 2 gives the first coefficients for $k = 1, 2, 3$. The sequences do not appear in OEIS.

k \ n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	0	0	2	0	2	0	2	0	2	0	2	0	2	0	2	0	2
2	0	2	2	6	6	12	14	24	30	46	60	88	118	168	228	320	438
3	0	0	0	0	2	4	10	16	32	52	94	148	252	392	648	996	1612

Table 2: The number of grand zigzag knight's paths such that the difference of the maximal and minimal heights is at most k.

Acknowledgments. The authors would like to thank the anonymous referees for their useful remarks and comments. This work was supported in part by ANR-22-CE48-0002 funded by l'Agence Nationale de la Recherche and by the project ANER ARTICO funded by Bourgogne-Franche-Comté region (France). José L. Ramírez was partially supported by Universidad Nacional de Colombia, Project No. 57340.

References

[1] A. Asinowski, A. Bacher, C. Banderier, and B. Gittenberger, *Analytic combinatorics of lattice paths with forbidden patterns: enumerative aspects*, In: International Conference on Language and Automata Theory and Applications, Lecture Notes in Comput. Sci. 10792 (2018), 195–206.

[2] A. Asinowski, A. Bacher, C. Banderier, and B. Gittenberger, *Analytic combinatorics of lattice paths with forbidden patterns: asymptotic aspects and Borges's Theorem*, In: 29th International Conference on Probabilistic, Combinatorial and Asymptotic Methods for the Analysis of Algorithms (AofA 2018), LIPIcs. Leibniz Int. Proc. Inform. 110 (2018), 10.1–10.13.

[3] A. Asinowski, A. Bacher, C. Banderier, and B. Gittenberger, *Analytic combinatorics of lattice paths with forbidden patterns, the vectorial kernel method, and generating functions for pushdown automata*, Algorithmica 82(3) (2019), 386–428.

[4] A. Asinowski, C. Banderier, and V. Roitner, *Generating functions for lattice paths with several forbidden patterns*, In: Proceedings of Formal Power Series and Algebraic Combibatorics (FPSAC'20), Sém. Lothar. Combin. 84B (2020), Article #95.

[5] A. Asinowski and C. Banderier, *On lattice paths with marked patterns: generating functions and multivariate gaussian distribution*, In: 31st International Conference on Probabilistic, Combinatorial and Asymptotic Methods for the Analysis of Algorithms (AofA 2020), Schloss Dagstuhl-Leibniz-Zentrum für Informatik, (2020) 1:1–1:16.

[6] C. Banderier and P. Flajolet, *Basic analytic combinatorics of directed lattice paths*, Theoret. Comput. Sci. 281 (2002), 37–80.

[7] C. Banderier and M. Wallner, *Lattice paths with catastrophes*, Discrete Math. Theor. Comput. Sci. 19 (1) (2017), Àrticle #23.

[8] E. Barcucci, A. Del Lungo, S. Fezzi, and R. Pinzani, *Nondecreasing Dyck paths and q-Fibonacci numbers*, Discrete Math. 170 (1997), 211–217.

[9] J.-L. Baril and A. Petrossian, *Equivalence classes of Dyck paths modulo some statistics*, Discrete Math. 338 (2015), 655–660.

[10] J.-L. Baril, D. Bevan, and S. Kirgizov, *Bijections between directed animals, multisets and Grand-Dyck paths*, Electron. J. Combin. 27 (2020), #P2.10.

[11] J.-L. Baril and J. L. Ramírez, *Knight's paths towards Catalan numbers*, Discrete Math. 346(6) (2023), 113372.

[12] M. Bóna and A. Knopfmacher, *On the probability that certain compositions have the same number of parts*, Ann. Comb. 14 (2010), 291–306.

[13] H. Cho, J. Huh, H. Nam, and J. Sohn, *Combinatorics on bounded free Motzkin paths and its applications*, Electron. J. Combin. 30(1) (2023), #P1.52.

[14] E. Deutsch, *Dyck path enumeration*, Discrete Math. 204 (1999), 167–202.

[15] P. Flajolet and R. Sedgewick, *Analytic Combinatorics*. Cambridge University Press, 2005.

[16] R. Flórez, J. F. González, M. Matijasevick, C. J. Pardo, J. L. Ramírez, L. M. Simbaqueba, and F. A. Velandia, '*Lattice paths in corridors and cyclic corridors*, Accepted in Contributions to Discrete Mathematics.

[17] R. Flórez and J. L. Ramírez, *Enumerating symmetric and asymmetric peaks in Dyck paths*, Discrete Math. 343 (2020), 112118.

[18] N. Gu and H. Prodinger, *Combinatorics on lattice paths in strips*, European J. Combin. 94 (2021), 103310.

[19] D. E. Knuth, *The Art of Computer Programming*, Vol. III: Sorting and Searching (2nd ed.), Addison-Wesley, p. 48, "Such arrangements were introduced by Alfred Young in 1900," 1973.

[20] J. Labelle and Y.-N. Yeh, *Dyck paths of knight moves*, Discrete Appl. Math. 24 (1989), 213–221.

[21] T. Mansour, *Statistics on Dyck paths*, J. Integer Seq. 9 (2006), Article 06.1.5.

[22] D. Merlini, R. Sprugnoli, and M. C. Verri, *Some statistics on Dyck paths*, J. Statis. Plann. Inference 101 (2002), 211–227.

[23] H. Prodinger, *The kernel method: A collection of examples*, Sém. Lothar. Combin. (2004), Article B50f.

[24] A. Sapounakis, I. Tasoulas, and P. Tsikouras, *Counting strings in Dyck paths*, Discrete Math. 307 (2007), 2909–2924.

[25] N. J. A. Sloane, OEIS Foundation Inc., *The On-line Encyclopedia of Integer Sequences*, available electronically at `http://oeis.org`.

[26] R. P. Stanley, *Enumerative Combinatorics*, Vol. 2, Cambridge University Press, 1999.

[27] Y. Sun, *The statistic "number of udu's" in Dyck paths*, Discrete Math. 287 (2004), 177–186.

[28] Y. Sun, W. Shi, and D. Zhao, *Symmetric and asymmetric peaks or valleys in (partial) Dyck paths*, Enumer. Combin. Appl. 2(3) (2022), Article S2R24.

Discrete Applied Mathematics 364 (2025) 1–15

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

The combinatorics of Motzkin polyominoes

Jean-Luc Baril [a,*], Sergey Kirgizov [a], José L. Ramírez [b], Diego Villamizar [c]

[a] LIB, Université de Bourgogne Franche-Comté, B.P. 47 870, 21078, Dijon Cedex, France
[b] Departamento de Matemáticas, Universidad Nacional de Colombia, Bogotá, Colombia
[c] Department of Mathematics, Xavier University of Louisiana, New Orleans, United States

ARTICLE INFO

Article history:
Received 11 January 2024
Received in revised form 21 June 2024
Accepted 3 December 2024
Available online xxxx

Keywords:
Motzkin word
Generating function
Bijections

ABSTRACT

A word $w = w_1 \cdots w_n$ over the set of positive integers is a Motzkin word whenever $w_1 = 1$, $1 \le w_k \le w_{k-1} + 1$, and $w_{k-1} \ne w_k$ for $k = 2, \ldots, n$. It can be associated to a n-column Motzkin polyomino whose i-th column contains w_i cells, and all columns are bottom-justified. We reveal bijective connections between Motzkin paths, restricted Catalan words, primitive Łukasiewicz paths, and Motzkin polyominoes. Using the aforementioned bijections together with classical one-to-one correspondence with Dyck paths avoiding *UDU*s, we provide generating functions with respect to the length, area, semiperimeter, value of the last symbol, and number of interior points of Motzkin polyominoes. We give asymptotics and closed-form expressions for the total area, total semiperimeter, sum of the last symbol values, and total number of interior points over all Motzkin polyominoes of a given length. We also present and prove an engaging trinomial relation concerning the number of cells lying at different levels and first terms of the expanded $(1 + x + x^2)^n$.

© 2024 The Author(s). Published by Elsevier B.V. This is an open access article under the CC BY license (http://creativecommons.org/licenses/by/4.0/).

1. Introduction

In the literature, Dyck and Motzkin paths play a crucial role in many problems from combinatorial theory (see [19] and the references wherein). Remember that a *Dyck path* of semilength n is a lattice path of $\mathbb{Z} \times \mathbb{Z}$ running from $(0, 0)$ to $(2n, 0)$ that never passes below the x-axis and whose permitted steps are $U = (1, 1)$ and $D = (1, -1)$. A *Motzkin path* of length n is a lattice path in the first quadrant made of n steps U, D, and $F = (1, 0)$, from the origin to the point $(n, 0)$. These paths are enumerated by the well-known Catalan and Motzkin numbers (respectively, A000108 and A001006 in the On-line Encyclopedia of Integer Sequences [29]).

These sequences have a similar behavior since, almost always, they appear together, and a large number of similar types of combinatorial classes are enumerated by these numbers. For Catalan numbers, we refer to the catalog of Catalan combinatorial classes compiled by Stanley [30]. Similarly, there are several combinatorial classes where the Motzkin family arises. For example, the rooted plane trees are a classical Catalan family, but if we add loops, yields a Motzkin family [14]. Additional examples of Motzkin families can be found in [4,8] and the references therein.

In this paper, we focus on the class of Motzkin words introduced by Mansour and Ramírez in [24]. Specifically, a word $w = w_1 w_2 \cdots w_n$ of length n over the set of positive integers is called a *Motzkin word* whenever $w_1 = 1$, $1 \le w_k \le w_{k-1} + 1$, and $w_{k-1} \ne w_k$ for $k = 2, \ldots, n$. The empty word ϵ is the only one word of length 0. For $n \ge 0$, let $\mathcal{M}_n$ denote the set of Motzkin words of length n. For example,

* Corresponding author.
E-mail addresses: barjl@u-bourgogne.fr (J.-L. Baril), sergey.Kirgizov@u-bourgogne.fr (S. Kirgizov), jlramirezr@unal.edu.co (J.L. Ramírez), dvillami@xula.edu (D. Villamizar).

https://doi.org/10.1016/j.dam.2024.12.002
0166-218X/© 2024 The Author(s). Published by Elsevier B.V. This is an open access article under the CC BY license (http://creativecommons.org/licenses/by/4.0/).

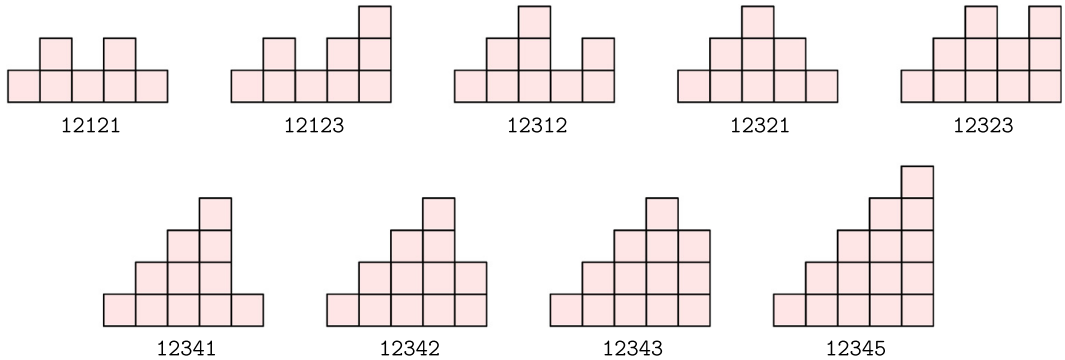


Fig. 1. Motzkin words of length 5 and their associated Motzkin polyominoes.

$$\mathcal{M}_5 = \{12121,\ 12123,\ 12312,\ 12321,\ 12323,\ 12341,\ 12342,\ 12343,\ 12345\}.$$

In [24], the authors prove that the cardinality of the set $\mathcal{M}_n$ is given by the Motzkin number m_{n-1}, where m_n is defined by the combinatorial sum (see [8,14])

$$m_n = \frac{1}{n+1}\sum_{i\geq 0}\binom{n+1}{i}\binom{n+1-i}{i+1}, \quad n \geq 0.$$

The number m_n corresponds to the n-th term of the generating function

$$M(x) := \sum_{n\geq 0} m_n x^n = \frac{1-x-\sqrt{1-2x-3x^2}}{2x^2},$$

and the first few values of the Motzkin numbers are

$$1,\quad 1,\quad 2,\quad 4,\quad 9,\quad 21,\quad 51,\quad 127,\quad 323,\quad 835,\quad 2188,\ldots$$

For all $1 \leq k \leq n$, we denote by $\mathcal{M}_{n,k}$ the set of the Motzkin words of length n whose last symbol is k. Let $m(n,k)$ denote the number of Motzkin words in $\mathcal{M}_{n,k}$. It is clear that $m_n = \sum_{k\geq 1} m(n,k)$. Moreover, in [24] the authors proved that

$$m(n,k) = \frac{k}{n}\sum_{j=k}^{n}(-1)^{n-j}\binom{n}{j}\binom{2j-k-1}{j-1}. \tag{1}$$

A Motzkin word $w = w_1\cdots w_n$ can also be viewed as a polyomino (also called bargraph) whose i-th column contains w_i cells for $1 \leq i \leq n$, and where all columns are bottom-justified (i.e. the bottom-most cells of all of its columns are in the same row). The polyomino associated to a Motzkin word of length n is called a *Motzkin polyomino* of length n. In Fig. 1, we show all Motzkin polyominoes of length 5.

In this work, we focus on the class of Motzkin polyominoes, and we investigate enumerative problems about several statistics on these objects.

Let w be a Motzkin word and $P(w)$ its associated polyomino. We denote by $\mathtt{area}(w)$ the number of cells (or the *area*) of $P(w)$, which also is the sum of all w_i for $1 \leq i \leq n$. The *semiperimeter* of $P(w)$, denoted $\mathtt{sper}(w)$, is half of the *perimeter* of $P(w)$, while the perimeter of $P(w)$ is the number of cell borders that do not touch another cell of $P(w)$. An *interior-vertex* of $P(w)$ is a point that belongs to exactly four cells of $P(w)$. We denote by $\mathtt{inter}(w)$ the number of interior points of $P(w)$. For instance, if $w = 12341$, then $\mathtt{area}(w) = 11$, $\mathtt{sper}(w) = 9$, and $\mathtt{inter}(w) = 3$. We refer to [18] for a historical review on polyominoes, and to [26] for the definitions of many statistical and enumerative methods over polyominoes. Additionally, analogous results are known for bargraphs [9,12], compositions [10], set partitions [20,21], Catalan words [13,23,25], inversion sequences [1], and words [11,22].

Motivation and outline of the paper. The motivation of this work is to provide enumerative results on Motzkin words according to several parameters (area, semiperimeter, and number of interior points) defined on the associated polyominoes. Except for the first section where we exhibit bijections between Motzkin words and other classical combinatorial classes counted by the Motzkin numbers, all other results are obtained algebraically by considering functional equations for multivariate generating functions. In Section 2 we give three bijections between Motzkin words and three known combinatorial classes counted by the Motzkin numbers. In Section 3, we focus on the two statistics of the area and the semiperimeter. We provide the trivariate generating function, where the coefficient of $x^n p^k q^\ell$ is the number of Motzkin polyominoes $P(w)$ of length n, and satisfying $\mathtt{sper}(w) = k$ and $\mathtt{area}(w) = \ell$. From this, we deduce the generating function for the total sum of the last symbol over all Motzkin words of length n, and we give asymptotic approximation for the coefficient of x^n. In Section 4, we focus on the semiperimeter statistic. We give the generating function for nonempty Motzkin polyominoes with respect to the length and the semiperimeter. We exhibit a bijection

$\psi(12123453412) = UDUFUFDFDF$

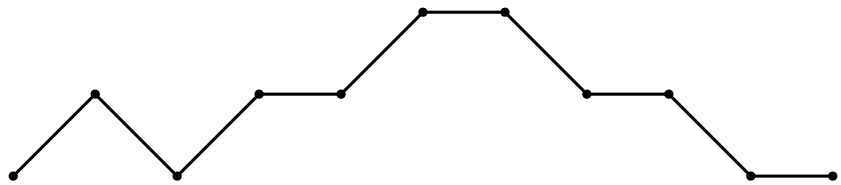


Fig. 2. Bijection between Motzkin paths and Motzkin words.

between Motzkin polyominoes of length n with semiperimeter $2n-k$, and Motzkin paths of length $n-1$ with exactly k up steps. We deduce a closed-form expression for the total semiperimeter over all Motzkin polyominoes of length n. In Section 5, we make a similar study as in Section 4 by considering the area instead of the semiperimeter. Finally, Section 6 is dedicated to the statistic of the number of interior points of the Motzkin polyominoes, and we exhibit a closed-form expression for the total number of interior points over all Motzkin polyominoes of length n.

2. Links between Motzkin words and other Motzkin classes

Below, we present three bijections between Motzkin words (or equivalently Motzkin polyominoes) and three known combinatorial classes counted by the Motzkin numbers.

First, let us recursively define the bijection ψ between $\mathcal{M}_n$ and the set of Motzkin paths of length $n-1$, i.e. lattice paths in the first quadrant, starting at the origin, ending on the x-axis, and made of $(n-1)$ steps $F=(1,0)$, $U=(1,1)$, and $D=(1,-1)$:

$$\psi(w) = \begin{cases} \epsilon & \text{if } w = 1, \\ F\psi(u) & \text{if } w = 1(1+u), \text{ where } u \neq \epsilon, \\ U\psi(u)D\psi(v) & \text{if } w = 1(1+u)v, \text{ where } u, v \neq \epsilon, \end{cases}$$

where u and v are Motzkin words and $(1+u)$ corresponds to the word obtained from u by increasing by one all letters of u. For instance, we have

$$\begin{aligned} \psi(12123453412) &= U\psi(1)D\psi(123453412) = UDU\psi(123423)D\psi(12) \\ &= UDUF\psi(12312)DF\psi(1) = UDUFU\psi(12)D\psi(12)DF = UDUFUFDFDF. \end{aligned}$$

In Fig. 2 we show the corresponding Motzkin path.

On the other hand, in a previous paper [6], the authors study the avoidance of patterns in Catalan words of length n, i.e. words $w = w_1w_2\cdots w_n$ satisfying $w_1 = 0$ and $0 \le w_i \le w_{i-1}+1$. They focus on Catalan words of length n avoiding the pattern $(\geq, \geq)$, that is those that do not contain occurrences $w_i \geq w_{i+1} \geq w_{i+2}$, or equivalently those avoiding the consecutive patterns 100, 000, 110, and 210. They prove algebraically that these words are counted by the Motzkin number m_n. Below, we give a bijection ϕ establishing a link between this class of Catalan words and $\mathcal{M}_n$.

Let w be a Catalan word avoiding $(\geq, \geq)$, then ϕ is recursively defined as follows.

$$\phi(w) = \begin{cases} 1 & \text{if } w = \epsilon, \\ 1(1+\phi(u)) & \text{if } w = 1(1+u), \\ 1(1+\phi(\bar{u}))\phi(v) & \text{if } w = 1(1+u)v, \quad u, v \neq \epsilon, \\ 1(1+\phi(u))1 & \text{if } w = 11(1+u), \\ 1(1+\phi(\bar{u}))\phi(v)1 & \text{if } w = 11(1+u)v, \quad u, v \neq \epsilon, \end{cases}$$

where u and v are some Catalan words, and $\bar{u}$ is obtained from u by deleting the last symbol, and where $(1+u)$ corresponds to the word obtained from u by increasing by one all letters of u. For instance, we have

$$\phi(1123231231) = 1\phi(121)\phi(1231)1 = 12323123121.$$

We leave it as an exercise to check that ϕ is a bijection.

Link with primitive Łukasiewicz paths. A *Łukasiewicz path* of length $n \geq 0$ is a lattice path in the first quadrant of the xy-plane starting at the origin $(0,0)$, ending on the x-axis, and consisting of n steps lying in the set $\{(1,k) : k \geq -1\}$. It is well-known [5,16,30] that these paths are enumerated by the Catalan numbers $c_n = \frac{1}{n+1}\binom{2n}{n}$. A *primitive* Łukasiewicz path is one with exactly one return on the x-axis (necessarily at the end). The empty path is not primitive. Let $\mathcal{L}_n$ be the family of primitive Łukasiewicz paths of length n without horizontal steps.

There is a bijection between the sets $\mathcal{M}_n$ and $\mathcal{L}_{n+1}$. Indeed, for $1 \le i \le n$ the i-th entry of the Motzkin word is exactly the y-coordinate of the final point of the i-th step of the corresponding Łukasiewicz path (see Fig. 3), and the avoidance of horizontal steps becomes from the avoidance of two consecutive identical entries in the Motzkin word.

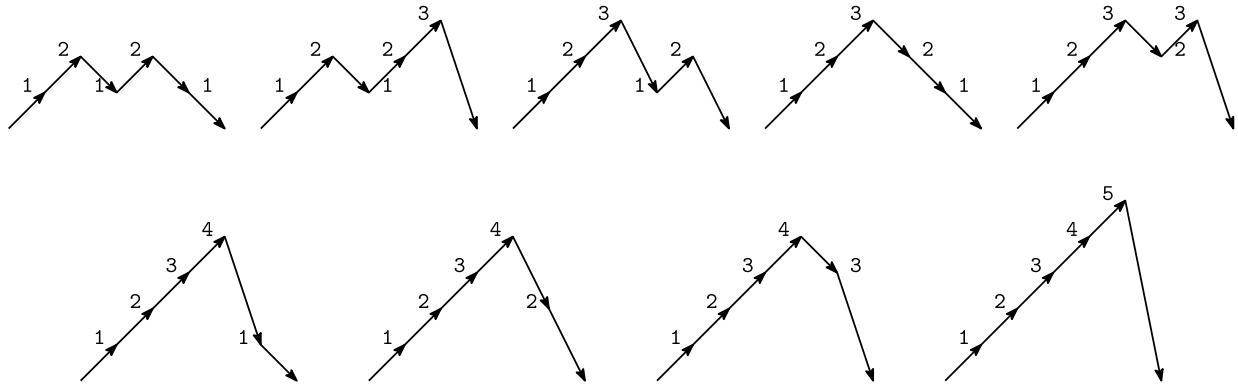


Fig. 3. Bijection between Motzkin words and primitive Łukasiewicz paths.

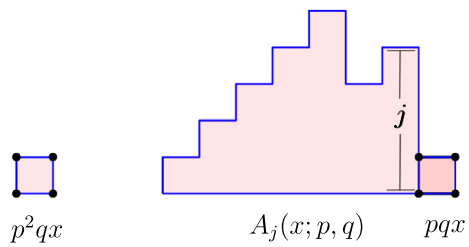


Fig. 4. Motzkin polyominoes whose last column is of height 1.

3. Area and semiperimeter statistics

In this section, we study the area and semiperimeter statistics on Motzkin words (or equivalently Motzkin polyominoes). Recall that, for all $1 \leq i \leq n$, $\mathcal{M}_{n,i}$ is the set of Motzkin words of length n whose last symbol is i. We define the generating functions

$$A_i(x;p,q) := \sum_{n\geq 1} x^n \sum_{w\in\mathcal{M}_{n,i}} p^{\mathtt{sper}(w)} q^{\mathtt{area}(w)}.$$

and

$$A(x;p,q;v) := \sum_{i\geq 1} A_i(x;p,q) v^{i-1}.$$

Theorem 3.1. *The generating function $A(x;p,q;v)$ satisfies the functional equation*

$$A(x;p,q;v) = p^2qx + \frac{pqx}{1-qv} A(x;p,q;1) + \left(p^2q^2xv - \frac{pqx}{1-qv}\right) A(x;p,q;qv). \tag{2}$$

Proof. Let w be a Motzkin word in $\mathcal{M}_n$. We distinguish two cases: (i) $w \in \mathcal{M}_{n,1}$ and (ii) $w \in \mathcal{M}_{n,i}$ for $i \geq 2$.

Case (i). The last symbol of w is 1. Hence, we have either $w = 1$ or $w = w'1$ where $w' \in \mathcal{M}_{n-1,j}$, with $n \geq 2$ and $j \geq 2$. See Fig. 4 for a graphical representation of these two cases. The generating function for this case is given by

$$A_1(x;p,q) = p^2qx + pqx \sum_{j\geq 2} A_j(x;p,q).$$

From this relation it is clear that

$$A_1(x;p,q) = p^2qx + pqx(A(x;p,q;1) - A_1(x;p,q)). \tag{3}$$

Case (ii). The last symbol of w is at least 2. From the definition of Motzkin word we have the decomposition $w = w'i$, with $w' \in \mathcal{M}_{n-1,j}$, and with either $j = i-1 \geq 1$ or $j > i \geq 2$. See Fig. 5 for a graphical representation of these two cases.

In terms of generating functions, we deduce the functional equations:

$$A_i(x;p,q) = p^2q^ixA_{i-1}(x;p,q) + pq^ix \sum_{j\geq i+1} A_j(x;p,q), \quad \text{for} \quad i \geq 2. \tag{4}$$

By multiplying Eq. (4) by v^{i-1} and summing over $i \geq 2$, we obtain

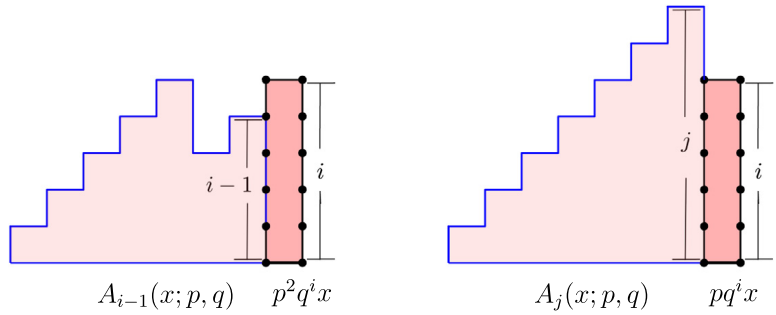


Fig. 5. Motzkin polyominoes whose last column is of height ≥ 2.

$$\begin{aligned}
A(x;p,q;v)-A_1(x;p,q) &= \sum_{i\geq 2} p^2q^ixA_{i-1}(x;p,q)v^{i-1} + \sum_{i\geq 2} pq^ix \sum_{j\geq i+1} A_j(x;p,q)v^{i-1}\\
&= p^2q^2xvA(x;p,q;qv) + px\sum_{j\geq 3} A_j(x;p,q)\sum_{i=2}^{j-1} q^iv^{i-1}\\
&= p^2q^2xvA(x;p,q;qv) + px\sum_{j\geq 3} A_j(x;p,q)\frac{q^2v-q^jv^{j-1}}{1-qv}\\
&= p^2q^2xvA(x;p,q;qv) + \frac{pq^2xv}{1-qv}(A(x;p,q;1)-A_1(x;p,q)-A_2(x;p,q))\\
&\qquad - \frac{pqx}{1-qv}(A(x;p,q;qv)-A_1(x;p,q)-qvA_2(x;p,q))\\
&= p^2q^2xvA(x;p,q;qv) + \frac{pq^2xv}{1-qv}(A(x;p,q;1)-A_1(x;p,q))\\
&\qquad - \frac{pqx}{1-qv}(A(x;p,q;qv)-A_1(x;p,q)).
\end{aligned}$$

Simplifying the last expression using (3), we obtain the desired result. □

Note that when $p=q=1$, we have

$$\left(1-xv+\frac{x}{1-v}\right)A(x;1,1;v) = x+\frac{x}{1-v}A(x;1,1;1).$$

The generating function $A(x;1,1;1)$ corresponds to the generating function of the nonempty Motzkin words, so $A(x;1,1;1)=xM(x)$. Thus, we deduce the following.

Corollary 3.2. *The generating function for the number of nonempty Motzkin words with respect to the length and the value of the last symbol is*

$$A(x;1,1;v)\cdot v = \frac{xv(1-v)+x^2vM(x)}{1-v+x-vx+v^2x}. \tag{5}$$

The first terms of the series expansion are

$$xv+v^2x^2+\left(v^3+v\right)x^3+\left(v^4+2v^2+v\right)x^4+\left(v^5+\mathbf{3v^3}+2v^2+3v\right)x^5+$$
$$\left(v^6+4v^4+3v^3+7v^2+6v\right)x^6+\left(v^7+5v^5+4v^4+12v^3+14v^2+15v\right)x^7+O(x^8).$$

There are three polyominoes of length 5 ending with a column of height three (see Fig. 1). By differentiating $A(x;1,1;v)\cdot v$ at $v=1$, we deduce:

Corollary 3.3. *The generating function for the total sum g_n of the last symbol in all Motzkin words of length n is*

$$\frac{1-x-2x^2-\sqrt{1-2x-3x^2}}{2x^2}.$$

An asymptotic approximation for the coefficient g_n of x^n is given by

$$\frac{3\sqrt{3}\,\left(\frac{1}{n}\right)^{\frac{3}{2}}3^n}{2\sqrt{\pi}}.$$

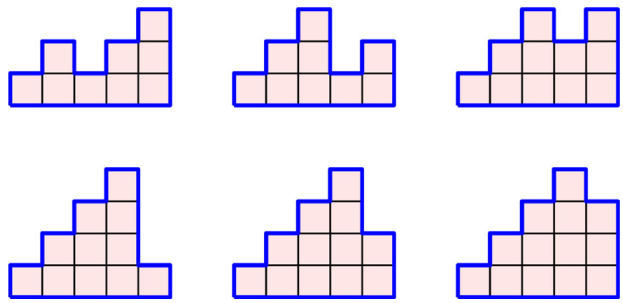


Fig. 6. Motzkin polyominoes of length 5 and semiperimeter 9.

The asymptotic approximation is easily obtained with a singularity analysis (see [15,27]).

The first terms of g_n, $1 \le n \le 10$, are

$$1, \quad 2, \quad 4, \quad 9, \quad 21, \quad 51, \quad 127, \quad 323, \quad 835, \quad 2188, \ldots.$$

This sequence corresponds to the Motzkin numbers A001006 in the OEIS [29]. Finally, the expected value of the last symbol is m_n/m_{n-1}, and an asymptotic is 3.

Notice that we can retrieve Corollary 3.3 as follows. From a Motzkin word $w \in \mathcal{M}_{n-1,k}$, we can construct k Motzkin words $w' = wx \in \mathcal{M}_{n,x}$ by adding a letter $x \in [1, k-1]\backslash\{k\}$. Conversely, any Motzkin word of length n can be uniquely constructed with this process. This means that $\sum_{k=1}^{n-1} km(n-1, k) = m_n$ which gives a simple combinatorial interpretation of Corollary 3.3.

4. The semiperimeter statistic

In this section we study the semiperimeter statistic on Motzkin polyominoes. By (2) with $q = 1$, we obtain

$$\left(1 - p^2xv + \frac{px}{1-v}\right) A(x; p, 1; v) = p^2x + \frac{px}{1-v} A(x; p, 1; 1). \tag{6}$$

In order to compute $S(x, p) := A(x; p, 1; 1)$, we use the kernel method (see [2,28]). The method consists in cancelling the coefficient of $A(x; p, 1; v)$ by taking the small root v_0 (the ones going to 0 for $x \sim 0$) of $1 - p^2xv_0 + \frac{px}{1-v_0}$, namely

$$v_0 = \frac{1 + p^2x - \sqrt{1 - 2p^2x - 4p^3x^2 + p^4x^2}}{2p^2x}.$$

When the factor on the left-hand side equals zero (and thus satisfies the equation), the right-hand side also equals zero and so the solution v_0 can be used on the right-hand side to obtain $A(x; p, 1, 1)$ (see Theorem 4.1).

Theorem 4.1. *The generating function for the number of nonempty Motzkin polyominoes according to the length and the semiperimeter is given by*

$$S(x, p) = \frac{1 - p^2x - \sqrt{1 - 2p^2x - 4p^3x^2 + p^4x^2}}{2px}.$$

The first terms of the series expansion of $S(x, p)$ are

$$p^2x + p^4x^2 + (p^5 + p^6)x^3 + (3p^7 + p^8)x^4 + (2p^8 + \mathbf{6p^9} + p^{10})x^5 + O(x^6).$$

Fig. 6 yields the 6 Motzkin polyominoes of length 5 and semiperimeter 9.

Theorem 4.2. *The map ψ (defined in Section 2) induces a bijection between the Motzkin polyominoes (or equivalently words) in $\mathcal{M}_n$ of semiperimeter $2n - k$ for $k = 1, 2, \ldots, \lfloor \frac{n-1}{2} \rfloor$, and the Motzkin paths of length $n - 1$ with exactly k up steps.*

Proof. Let us prove the statement by induction on n. The case $n = 0$ is trivial since $\psi(1) = \epsilon$. Let us assume the statement is true for $m \le n$, and let us prove it for $n + 1$. Let $w \in \mathcal{M}_{n+1}$ with semiperimeter $2(n + 1) - k$. We denote by $\mathrm{up}(M)$ the number of up-steps of the Motzkin path M.

If $w = 1(1+u)$, $u \ne \epsilon$, then we have $\mathtt{sper}(u) = 2n-k$, $\psi(w) = F\psi(u)$, and $\mathrm{up}(\psi(w)) = \mathrm{up}(\psi(u))$. Using the induction hypothesis on u, we have $\mathrm{up}(\psi(u)) = k$, and then $\mathrm{up}(\psi(w)) = k$ as expected.

If $w = 1(1 + u)v$, $u, v \ne \epsilon$, then we have $\mathtt{sper}(u) = s_1$ and $\mathtt{sper}(v) = s_2$ with $s_1 + s_2 + 1 = 2(n + 1) - k$, $\psi(w) = U\psi(u)D\psi(v)$, and $\mathrm{up}(\psi(w)) = 1 + \mathrm{up}(\psi(u)) + \mathrm{up}(\psi(v))$. Using the induction hypothesis on u and v, we have

$\mathrm{up}(\psi(u)) = 2|u| - s_1$, and $\mathrm{up}(\psi(v)) = 2|v| - s_2$, where $|u|$ is the length of u. Hence, this implies that $\mathrm{up}(\psi(w)) = 1 + 2|u| - s_1 + 2|v| - s_2 = 2 + 2|u| + 2|v| - 2(n+1) + k = k$, as expected.

Considering these two cases, the induction is completed. □

Using relation (6), we deduce the following.

Theorem 4.3. *The generating function for the number of nonempty Motzkin polyominoes according to the length, the semiperimeter, and the value of the last letter is given by*

$$A(x; p, 1; v) = \frac{1 + p^2(x - 2vx) - \sqrt{1 - 2p^2x - 4p^3x^2 + p^4x^2}}{2(1 + px + p^2v^2x - v(1 + p^2x))}.$$

The first terms of the series expansion of $A(x; p, 1; v)$ are

$$p^2x + p^4vx^2 + (p^5 + p^6v^2)x^3 + (p^7 + 2p^7v + p^8v^3)x^4 + (2p^8 + \boldsymbol{p^9} + \boldsymbol{2p^9v} + \boldsymbol{3p^9v^2} + p^{10}v^4)x^5 + O(x^6).$$

From Fig. 6 we can verify that there are 1, 2, and 3 Motzkin polyominoes of length 5 and semiperimeter 9, whose value of the last symbol (height of last column) is 1, 2, and 3, respectively.

Let $s(n, i)$ denote the sum of the semiperimeters of all Motzkin polyominoes of length n whose last column has height i. The first few values are

$$[s(n,i)]_{n,i\geq 1} = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 4 & 0 & 0 & 0 & 0 & \cdots \\ 5 & 0 & 6 & 0 & 0 & 0 & \cdots \\ 7 & 14 & 0 & 8 & 0 & 0 & \cdots \\ 25 & 18 & 27 & 0 & 10 & 0 & \cdots \\ 61 & 72 & 33 & 44 & 0 & 12 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Notice that from the decomposition given in Fig. 5, we have for $n \geq 2$ and $2 \leq i \leq n$,

$$s(n,i) = s(n-1, i-1) + 2m(n-1, i-1) + \sum_{j=i+1}^{n-1} (s(n-1, j) + m(n-1, j)),$$

where $m(n, i)$ is given in (1). If we consider the difference $s(n, i) - s(n, i-1)$, then for $n \geq 2$ and $3 \leq i \leq n$, we obtain the recurrence relation

$$\begin{aligned} s(n,i) &= s(n, i-1) + s(n-1, i-1) - s(n-1, i-2) - s(n-1, i) \\ &\quad + 2(m(n-1, i-1) - m(n-1, i-2)) - m(n-1, i). \end{aligned}$$

Let $s(n)$ be the total semiperimeter over all Motzkin polyominoes of length n. By Theorem 4.3 at $v = 1$, we deduce:

Corollary 4.4. *The generating function of the sequence $s(n)$ is given by*

$$\frac{1 + x^2 - (1+x)\sqrt{1 - 2x - 3x^2}}{2x\sqrt{1 - 2x - 3x^2}}.$$

An asymptotic approximation of $s(n)$ is given by

$$\frac{5\sqrt{3}\sqrt{\frac{1}{n}}\,3^n}{6\sqrt{\pi}}.$$

The first few values for $1 \leq n \leq 10$ are

$$2, \quad 4, \quad 11, \quad 29, \quad 80, \quad 222, \quad 624, \quad 1766, \quad 5030, \quad 14396, \ldots$$

This sequence does not appear in the OEIS. The expected value of the semiperimeter is $5n/3$. In Corollary 4.5 we give an explicit relation for the sequence $s(n)$ using the *central trinomial coefficient* T_n, that is the coefficient of x^n in the expansion $(x^2 + x + 1)^n$. From the multinomial theorem we see that

$$T_n = \sum_{k=0}^{n} \binom{n}{k}\binom{n-k}{k}.$$

Moreover, the generating function of the central trinomial coefficients is given by

$$T(x) := \sum_{n\geq 0} T_n x^n = \frac{1}{\sqrt{1 - 2x - 3x^2}}. \tag{7}$$

Corollary 4.5. *The total semiperimeter over all Motzkin polyominoes of length n is given by*

$$s(n) = T_n + 2T_{n-1} - m_{n-1},$$

where m_n is the n-th Motzkin number.

Proof. By differentiating $S(x, p)$ at $p = 1$, Theorem 4.1 gives

$$\begin{aligned}\frac{\partial}{\partial p}S(x,p)|_{p=1} &= \frac{1+x^2-(1+x)\sqrt{1-2x-3x^2}}{2x\sqrt{1-2x-3x^2}}\\ &= -1+T(x)+2xT(x)-xM(x).\end{aligned}$$

Comparing the n-th coefficient we obtain the desired result. □

We leave as an open question to find a combinatorial interpretation of the equation obtained in Corollary 4.5.

5. The area statistic

The goal of this section is to analyze the area statistic on Motzkin polyominoes. By (2) with $p = 1$ we obtain

$$A(x;1,q;v) = qx + \frac{qx}{1-qv}A(x;1,q;1) - \frac{qx(1-qv+q^2v^2)}{1-qv}A(x;1,q;qv). \tag{8}$$

By iterating this equation an infinite number of times (here we assume $|x| < 1$ or $|q| < 1$), we obtain

$$A(x;1,q;v) = \sum_{j\geq 1}(-1)^{j-1}q^jx^j\left(1+\frac{A(x;1,q;1)}{1-q^jv}\right)\prod_{i=1}^{j-1}\frac{1-q^iv+q^{2i}v^2}{1-q^iv}.$$

By setting $v = 1$, and solving for $A(x; 1, q; 1)$, we can state the following result.

Theorem 5.1. *The generating function $U(x, q) := A(x; 1, q; 1)$ for the number of nonempty Motzkin polyominoes according to the length and the area is given by*

$$U(x,q) = \frac{\sum_{j\geq 1}(-1)^{j-1}q^jx^j\prod_{i=1}^{j-1}\frac{1-q^i+q^{2i}}{1-q^i}}{1-\sum_{j\geq 1}(-1)^{j-1}\frac{q^jx^j}{1-q^j}\prod_{i=1}^{j-1}\frac{1-q^i+q^{2i}}{1-q^i}}.$$

The first terms of the series expansion of $U(x, q)$ are

$$\begin{aligned}&qx+q^3x^2+(q^4+q^6)x^3+(q^6+q^7+q^8+q^{10})x^4+\\ &\qquad+(\boldsymbol{q^7+3q^9+2q^{11}+q^{12}+q^{13}+q^{15}})x^5+O(x^6).\end{aligned}$$

We refer to Fig. 7 for an illustration of the polyominoes of length 5.

Theorem 5.2. *The generating function for the number of Motzkin polyominoes, including the empty word, according to the length and area is given by the infinite continued fraction*

$$1+U(x,q) = \cfrac{1}{1-\cfrac{qx}{1+qx-\cfrac{(1+qx)q^2x}{1+q^2x-\cfrac{(1+q^2x)q^3x}{\ddots}}}}.$$

Proof. First note that the Motzkin words of length n are in bijection with Dyck paths of length $2n$ avoiding the consecutive subword UDU. Indeed, given a Dyck path avoiding UDU, we associate a Motzkin word formed by the y-coordinate of each final point of the up steps. For example, in Fig. 8 we show the Dyck path associated to the Motzkin word $123212343 \in \mathcal{M}_9$. We denote by $\mathcal{D}$ the set of Dyck paths avoiding UDU.

From this bijection, it is clear that the area of a Motzkin polyomino is equivalent to count the area of a Dyck path avoiding the subword UDU (the area of a Dyck path is the area of the region bounded by the path and the x-axis). Given a Dyck path P in $\mathcal{D}$ from the first return decomposition it can be one of the following options: ϵ, UD, $UP'DP''$, where $P', P'' \in \mathcal{D}$ and $P' \neq \epsilon$. Let $J(x, q)$ be the bivariate generating function for the Dyck paths in $\mathcal{D}$ according to the length and the area. From the decomposition we obtain the functional equation

$$J(x,q) = 1 + xq + xq(J(qx,q)-1)J(x,q).$$

Therefore, we deduce

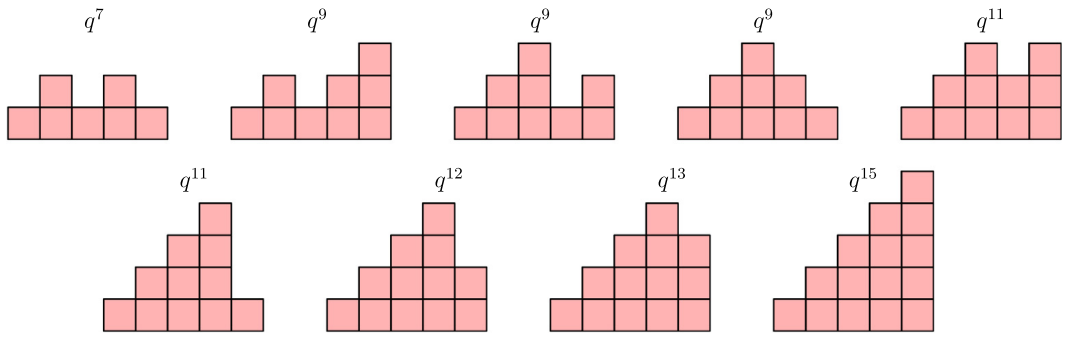


Fig. 7. Motzkin polyominoes of length 5 and their weighted area.

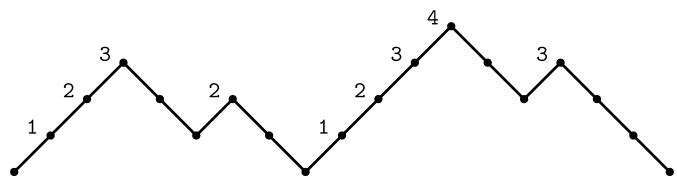


Fig. 8. Dyck path associated to the Motzkin word 123212343.

$$J(x,q) = \frac{1+xq}{1-xq(J(qx,q)-1)}.$$

Iterating this expression we obtain the desired result. □

Let $u(n,i)$ denote the total area of the Motzkin polyominoes of length n that end with a column of height i. The first few values are

$$[u(n,i)]_{n,i\geq 1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 3 & 0 & 0 & 0 & 0 & \cdots \\ 4 & 0 & 6 & 0 & 0 & 0 & \cdots \\ 7 & 14 & 0 & 10 & 0 & 0 & \cdots \\ 27 & 21 & 33 & 0 & 15 & 0 & \cdots \\ 75 & 89 & 45 & 64 & 0 & 21 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

From the decomposition given in Fig. 5, we have for $n \geq 2$ and $2 \leq i \leq n$,

$$u(n,i) = u(n-1,i-1) + i \cdot m(n-1,i-1) + \sum_{j=i+1}^{n-1} \left(u(n-1,j) + i \cdot m(n-1,j)\right),$$

where $m(n,i)$ is given in (1). If we consider the difference $u(n,i) - u(n,i-1)$, then for $n \geq 2$ and $3 \leq i \leq n$, we obtain the recurrence relation

$$\begin{aligned} u(n,i) &= u(n,i-1) + u(n-1,i-1) - u(n-1,i) - u(n-1,i-2) \\ &\quad + i \cdot m(n-1,i-1) - (i-1)m(n-1,i-2) - i \cdot m(n-1,i) + \sum_{j=i}^{n-1} m(n-1,j). \end{aligned}$$

Let $u(n)$ be the total area of over all Motzkin polyominoes of length n. The first few values are

$$1, \quad 3, \quad 10, \quad 31, \quad 96, \quad 294, \quad 897, \quad 2727, \quad 8272, \quad 25048, \ldots$$

This sequence corresponds to the sequence A055217 in the OEIS [29]. Recently, Goy and Shattuck [17] gave a combinatorial interpretation for this sequence by using marked Motzkin paths. Our interpretation is different and probably new. In the following theorem we give an interesting combinatorial formula to calculate the sequence $u(n)$.

Theorem 5.3. *The total area over all Motzkin polyominoes of length n is given by*

$$u(n) = \frac{1}{2}\left(3^n - T_n\right) = \frac{1}{2}\left(3^n - \sum_{k=0}^{n} \binom{n}{k}\binom{n-k}{k}\right),$$

and an asymptotic is $3^n/2$.

Proof. Let $B(x;v)=\frac{\partial}{\partial q}A(x;1,q;v)\,|_{q=1}$. Then by differentiating (8) with respect to q, we obtain

$$B(x;v)=x+\frac{x^2M(x)}{(1-v)^2}+\frac{x}{1-v}B(x;1)$$
$$-\frac{x(1-2v+4v^2-2v^3)}{(1-v)^2}A(x;1,1;v)-\frac{x(1-v+v^2)}{1-v}(B(x;v)+v\frac{\partial}{\partial v}A(x;1,1;v)).$$

By using (5), we obtain

$$\frac{\partial}{\partial v}A(x;1,1;v)=\frac{\partial}{\partial v}\left(\frac{x(1-v)+x^2M(x)}{1-v+x-vx+v^2x}\right) \tag{9}$$
$$=\frac{1-2vx-x^2-2vx^2+2v^2x^2-(1+x-2vx)\sqrt{1-2x-3x^2}}{2(1-v+x-vx+v^2x)^2}. \tag{10}$$

Therefore

$$\frac{(1-v+x-vx+v^2x)^3}{1-v}B(x;v)=x(1-v+x-vx+v^2x)^2+\frac{x^2(1-v+x-vx+v^2x)^2M(x)}{(1-v)^2}$$
$$+\frac{x(1-v+x-vx+v^2x)^2}{1-v}B(x;1)$$
$$-\frac{x(1-2v+4v^2-2v^3)(1-v+x-vx+v^2x)}{(1-v)^2}\left(x(1-v)+x^2M(x)\right)$$
$$-\frac{xv(1-v+v^2)}{2(1-v)}\left(1-2vx-x^2-2vx^2+2v^2x^2-(1+x-2vx)\sqrt{1-2x-3x^2}\right).$$

By twice differentiating this equation with respect to v and taking $v=xM(x)+1$, we obtain that

$$B(x,1)=\frac{1+x-\sqrt{1-2x-3x^2}}{2-4x-6x^2}=\frac{1}{2}\left(\frac{1}{1-3x}-\frac{1}{\sqrt{1-2x-3x^2}}\right).$$

Comparing the n-th coefficient we obtain the desired result. □

5.1. Link with trinomial coefficients

The sequence $u(n)$ (A055217) corresponds to the sum of the first n coefficients of $(1+x+x^2)^n$. Enigmatically, the monomials from the first part of the expansion of $(1+x+x^2)^n$ can be literally written on the cells of all Motzkin polyominoes of size n in a simple one-to-one manner. Term x^k goes onto a cell of height $n-k$, see Fig. 9. This subsection is devoted to explain this fact, which is easy to state, but not easy to prove. Theorem 5.3 follows from the results of this section if we replace x by 1, consider the symmetry of coefficients of $(1+x+x^2)^n$, and subtract the central trinomial coefficient.

Let $T(n,i)$ be the i-th coefficient in the expansion of $(1+x+x^2)^n$. It is not difficult to prove that

$$T(n,i)=T(n-1,i)+T(n-1,i-1)+T(n-1,i-2),\quad 0\le i\le n-2, \tag{11}$$

and $T(n,n-1)=T_{n-1}+T(n-1,n-2)+T(n-1,n-3)$, where T_n is the central trinomial coefficient, that is $T_n=T(n,n)$. We also have $T(n,n-1)=n|\mathcal{M}_n|=nm_{n-1}$, see A005717 in the OEIS for more information. If $i>n-1$ or $i<0$ we define $T(n,i)=0$, moreover $T(1,1)=1$. The first few values of this array are

$$[T(n,i)]_{n\ge1,i\ge0}=\begin{pmatrix}
1&0&0&0&0&0&0&0&\cdots\\
1&2&0&0&0&0&0&0&\cdots\\
1&3&6&0&0&0&0&0&\cdots\\
1&4&10&16&0&0&0&0&\cdots\\
1&5&15&30&45&0&0&0&\cdots\\
1&6&21&50&90&126&0&0&\cdots\\
1&7&28&77&161&266&357&0&\cdots\\
1&8&36&112&266&504&784&1016&\cdots\\
\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\ddots
\end{pmatrix}.$$

Let $h(n,i)$ be the total number of cells of height i in $\mathcal{M}_n$. From Fig. 9, we have $h(5,1)=45$, $h(5,2)=30$, $h(5,3)=15$, $h(5,4)=5$, $h(5,5)=1$. Notice that these numbers are related to the first 5 coefficients of the expansion of

$$(1+x+x^2)^5=\mathbf{1}+\mathbf{5x}+\mathbf{15x^2}+\mathbf{30x^3}+\mathbf{45x^4}+51x^5+45x^6+30x^7+15x^8+5x^9+x^{10}.$$

Let w be a Motzkin word. We denote by $\mathrm{h}_i(w)$ the number of cells of height i in the Motzkin polyomino associated with w. We introduce the following generating functions

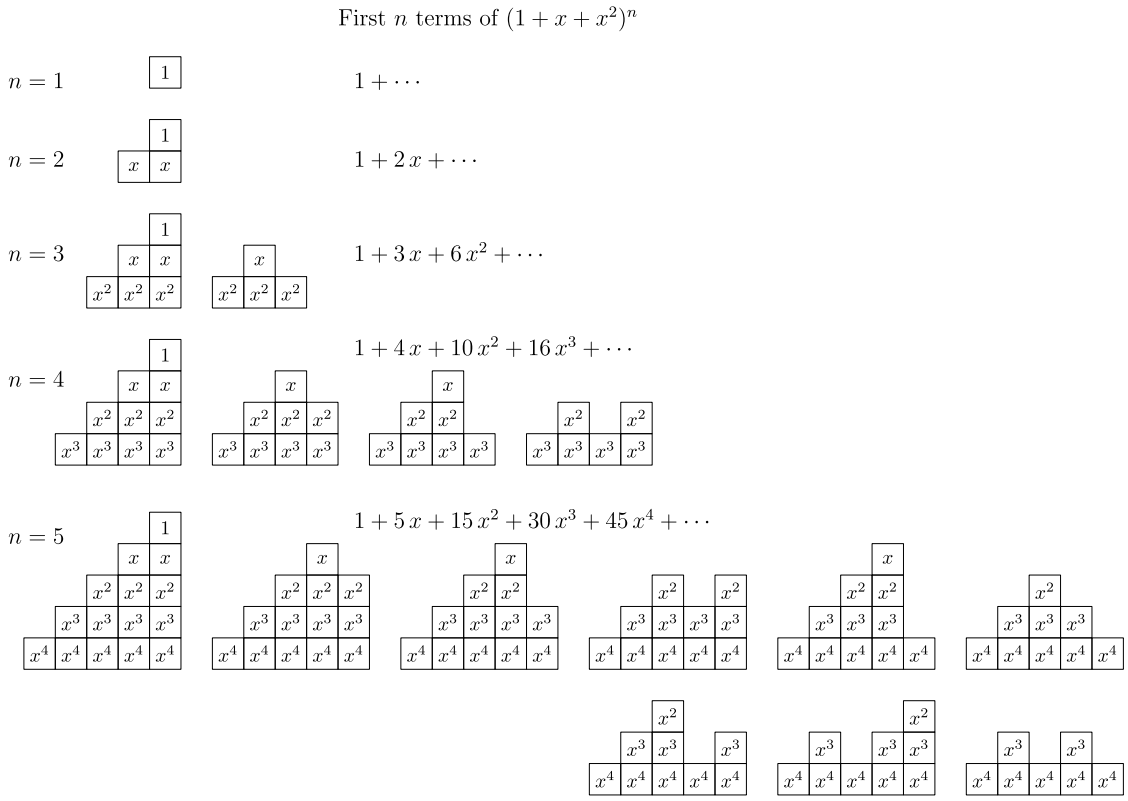


Fig. 9. Correspondence between Motzkin polyominoes of size n and first n terms of the expansion of $(1+x+x^2)^n$.

$$H_i(x,q) := 1 + \sum_{w\in\mathcal{M}} x^{|w|} q^{\mathrm{h}_i(w)}$$

and

$$B_i(x) := \left.\frac{\partial H_i(x,q)}{\partial q}\right|_{q=1}.$$

From the definition it is clear that $[x^n]B_i(x) = h(n,i)$.

Theorem 5.4. *For $i \geq 2$, we have*

$$H_i(x,q) = \frac{1+x}{1-(H_{i-1}(x,q)-x)},$$

and $H_1(x,q) = 1 + qxM(xq)$, where $M(x)$ is the generating function of the Motzkin numbers.

Proof. Motzkin words of length n are in bijection with Dyck paths of length $2n$ avoiding the subword UDU (the i-th entry of the word w corresponds to the y-coordinate of the endpoint of the i-th up-step of its corresponding Dyck path). From this bijection it is clear that

$$H_1(x,q) = 1 + xq + xq(H_1(x,q)-1)H_1(x,q).$$

Solving this equation, we obtain that $H_1(x,q) = 1 + qxM(xq)$. Analogously, from the decomposition given in the proof of Theorem 5.2 we obtain the functional equation

$$H_i(x,q) = 1 + x + x(H_{i-1}(x,q)-1)H_i(x,q), \quad i \geq 2. \quad \square$$

For example, for $i = 3$ we have the expression

$$\begin{aligned} H_3(x,q) &= \frac{2-x-q(2+x^2)+x\sqrt{(1+qx)(1-3qx)}}{2(1-q-x+q(1-x)x)} \\ &= 1 + x + x^2 + (1+q)x^3 + (1+2q+q^2)x^4 + (\mathbf{1+3q+3q^2+2q^3})x^5 + O(x^6). \end{aligned}$$

Fig. 10 yields the Motzkin polyominoes of length 5 and the cells of height 3.

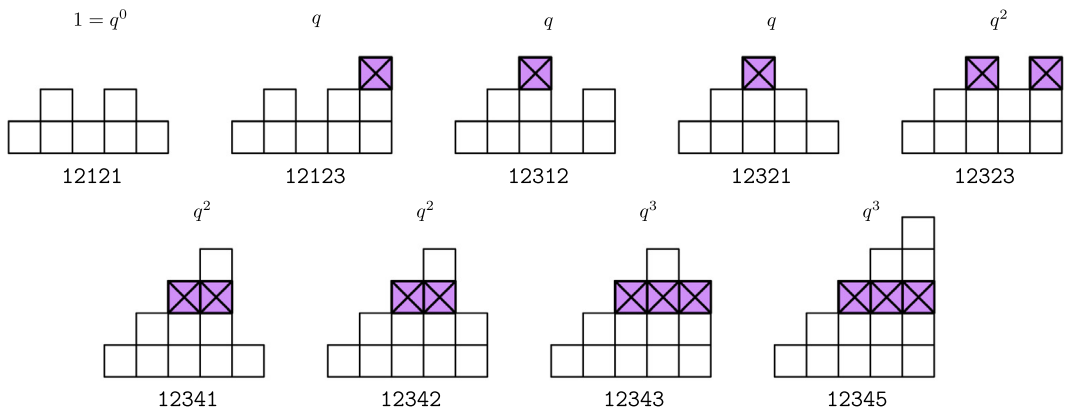


Fig. 10. Motzkin polyominoes of length 5 and the cells of height 3.

Proposition 5.5. *For $i \geq 1$, we have*

$$B_i(x) = \frac{x^i M^i(x)}{1 - x - 2x^2 M(x)},$$

where $M(x)$ is the generating function of the Motzkin numbers.

Proof. We proceed by induction on i. The identity clearly holds for $i = 1$; see Theorem 5.4. Now suppose that the result is true for $i - 1$. We prove it for i:

$$B_i(x) = \left.\frac{\partial H_i(x,q)}{\partial q}\right|_{q=1} = \frac{\partial}{\partial q}\left(\frac{1+x}{1-(H_{i-1}(x,q)-x)}\right)\Bigg|_{q=1} = \frac{\partial}{\partial q}\left(\frac{1}{1-\frac{x}{1-x}H_{i-1}(x,q)}\right)\Bigg|_{q=1}$$

$$= \frac{\frac{x}{1+x}\frac{x^{i-1}M^{i-1}(x)}{1-x-2x^2M(x)}}{\left(1-\frac{x}{1+x}(1+xM(x))\right)^2} = \frac{(1+x)x^iM^{i-1}(x)}{(1-x^2M(x))^2} = \frac{x^iM^i(x)}{1-x-2x^2M(x)}. \quad \square$$

A *Motzkin walk* of length n is a Motzkin path prefix without the condition that never passes below the x-axis (see for instance [3]). A Motzkin walk ending on x-axis is also known as *Grand Motzkin path*. Let $g(n, i)$ be the number of Motzkin walks of length n ending at height i. It is clear that $g(n, i) = g(n-1, i-1) + g(n-1, i) + g(n-1, i+1)$. We define the generating function

$$G_i(x) = \sum_{n \geq 0} g(n, i)x^n.$$

Theorem 5.6. *For all $i \geq 0$, $G_i(x) = B_i(x)$. Moreover, For $n > 0$, $i > 1$, we have*

$$h(n, i) = h(n-1, i-1) + h(n-1, i) + h(n-1, i+1),$$

$h(n, 1) = n \cdot [x^{n-1}]M(x)$.

Proof. Let P be a Motzkin walk ending at height i. It can be decomposed as $GUM_1UM_2U \cdots UM_i$, where G is a Motzkin walk ending at height zero and M_j is a Motzkin path for all $1 \leq j \leq i$. Therefore, we have the generating function $G_i(x) = x^iG_0(x)M^i(x)$, where $M(x)$ is the generating function of the Motzkin numbers. Notice that $G_0(x) = 1 + xG_0(x) + 2x^2M(x)G_0(x)$, because a nonempty Motzkin walk ending at height zero is either FG_0', or $UMDG_0'$, or $D\bar{M}UG_0'$ where G_0' is a Motzkin walk ending at height zero, M is a Motzkin path (ending on the x-axis) and $\bar{M}$ is a Motzkin walk obtained from a Motzkin path (ending on the x-axis) after exchanging U and D. So, we have $G_0(x) = 1/(1 - x - 2x^2M(x))$. Therefore,

$$G_i(x) = \frac{x^iM^i(x)}{1 - x - 2x^2M(x)} = B_i(x).$$

Now, it is clear that the sequences $h(n, i)$ and $g(n, i)$ are the same, therefore $h(n, i)$ satisfies the desired recurrence relation. $\square$

From Theorem 5.6 and (11), it is possible to verify that the sequence $T(n, i)$ and $h(n, n-i)$ satisfy the same recurrence relation with the same initial values, therefore they are the same.

Theorem 5.7. *The number of cells of height $n - i$ ($0 \leq i < n$) in all Motzkin polyominoes of length n is given by the trinomial coefficient $T(n, i)$.*

Now we can give an alternative proof of the combinatorial identity given in Theorem 5.3.

Second proof of Theorem 5.3. Notice that $\sum_{i=0}^{n-1} T(n,i) = u(n)$. But the trinomial coefficients are palindromic, that is $T(n,i) = T(n, 2n-i)$, then

$$\sum_{i=0}^{2n} T(n,i) = 2u(n) - T(n,n) = 2u(n) - T_n.$$

Notice that we have to subtract the central trinomial coefficient. But it is clear that this sum is 3^n, therefore $3^n = 2u(n) - T_n$.

We end this section by asking the following (open) questions: can we obtain an interpretation moving down a level to binomial coefficients? Can we obtain a generalization of this result by using tetranomial coefficients (i.e. coefficients of the polynomials $(1+x+x^2+x^3)^n$), and for which polyomino classes?

6. The interior points statistic

In this section, we study the statistic of the number of interior points on Motzkin polyominoes. As in the previous section, for all $1 \le i \le n$, $\mathcal{M}_{n,i}$ is the set of the Motzkin words of length n whose last symbol is i, and we define the generating functions

$$A_i(x;q) := \sum_{n\ge 1} x^n \sum_{w\in\mathcal{M}_{n,i}} q^{\mathtt{inter}(w)}.$$

and

$$A(x;q;v) := \sum_{i\ge 1} A_i(x;q) v^{i-1}.$$

Theorem 6.1. *The generating function $A(x;q;v)$ is given by*

$$A(x;q;v) = \sum_{j\ge 1} x^j \left(1 + A(x;q;1)\frac{1}{1-q^j v}\right) \prod_{i=1}^{j-1} \left(xq^{i-1}v - \frac{x}{1-q^i v}\right).$$

Proof. According to the decomposition given in Figs. 4 and 5, we obtain the relations

$$A_1(x;q) = x + x\sum_{j\ge 2} A_j(x;q),$$

$$A_i(x;q) = xq^{i-2}A_{i-1}(x;q) + xq^{i-1}\sum_{j\ge i+1} A_j(x;q).$$

By multiplying the last equation by v^{i-1} and summing over $i \ge 2$, we obtain the functional equation

$$\begin{aligned} A(x;q;v) - A_1(x;q) = {} & \frac{xqv}{1-qv}A(x;q;1) + \left(xv - \frac{x}{1-qv}\right)A(x;q;qv) \\ & - \left(\frac{xqv}{1-qv} - \frac{x}{1-qv}\right)A_1(x;q). \end{aligned}$$

Simplifying this expression we obtain the equation

$$A(x;q;v) = x + \frac{x}{1-qv}A(x;q;1) + \left(xv - \frac{x}{1-qv}\right)A(x;q;qv). \tag{12}$$

By iterating the last equation an infinite number of times (here we assume $|x| < 1$ or $|q| < 1$), we obtain the desired result. □

By setting $v = 1$ in Theorem 6.1, and solving for $A(x;q;1)$ we can state the following result.

Corollary 6.2. *The generating function $H(x,q) := A(x;q;1)$ for the number of nonempty Motzkin polyominoes according to the length and the number of interior points is given by*

$$H(x,q) = \frac{\sum_{j\ge 1} x^j \prod_{i=1}^{j-1}\left(q^{i-1} - \frac{1}{1-q^i}\right)}{1 - \sum_{j\ge 1}\frac{x^j}{1-q^j}\prod_{i=1}^{j-1}\left(q^{i-1} - \frac{1}{1-q^i}\right)}.$$

The first terms of the series expansion of $H(x,q)$ are

$$\begin{aligned} H(x,q) = {} & x + x^2 + (1+q)x^3 + (1+q+q^2+q^3)x^4 \\ & + (\mathbf{1} + \mathbf{2q} + \mathbf{q^2} + \mathbf{2q^3} + \mathbf{q^4} + \mathbf{q^5} + \mathbf{q^6})x^5 + O(x^6). \end{aligned}$$

We refer to Fig. 11 for an illustration of the polyominoes of length 5.

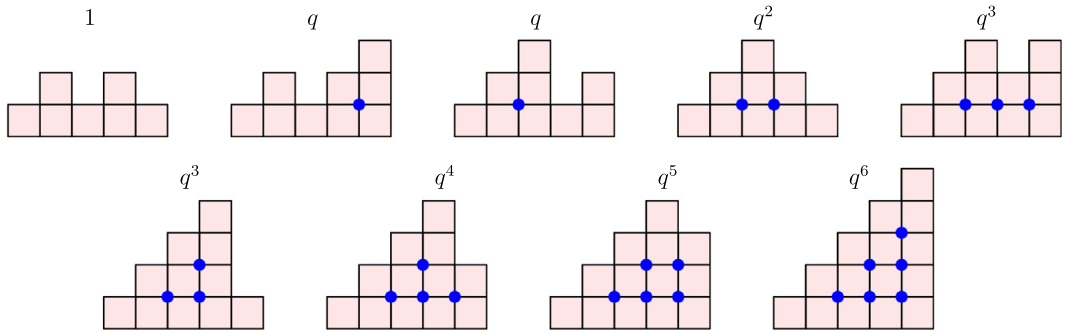


Fig. 11. Motzkin polyominoes of length 5 and their weighted interior points.

Corollary 6.3. *The generating function of the number of interior points over all Motzkin polyominoes of length n is*

$$\frac{2-3x-5x^2-(2-x-2x^2)\sqrt{1-2x-3x^2}}{2x(1+x)(1-3x)},$$

and an asymptotic for the n-th coefficient of the series expansion is $3^n/2$*. The expected value of the number of interior points is* $\sqrt{\frac{\pi}{3}}n^{3/2}$*.*

Proof. Let $B(x;v)=\frac{\partial}{\partial q}A(x;q;v)\mid_{q=1}$. Then by differentiating (12) with respect to q, we obtain

$$B(x;v)=\frac{vx^2M(x)}{(1-v)^2}+\frac{x}{1-v}B(x;1)$$
$$-\frac{xv}{(1-v)^2}A(x;1,v)+\left(xv-\frac{x}{1-v}\right)\left(B(x;v)+v\frac{\partial}{\partial v}A(x;1;v)\right).$$

From (9), we know that

$$\frac{\partial}{\partial v}A(x;1;v)=\frac{1-2vx-x^2-2vx^2+2v^2x^2-(1+x-2vx)\sqrt{1-2x-3x^2}}{2(1-v+x-vx+v^2x)^2}.$$

Therefore

$$\frac{(1-v+x-vx+v^2x)^3}{1-v}B(x;v)=\frac{x(1-v+x-vx+v^2x)^2}{1-v}B(x;1)$$
$$+\frac{v}{2}\left(1-x-vx-3x^2+2vx^2-v^2x^2-x^3+3vx^3-3v^2x^3+2v^3x^3\right.$$
$$\left.-(1-xv-x^2+vx^2-v^2x^2)\sqrt{1-2x-3x^2}\right).$$

By twice differentiating this equation with respect to v and taking $v=xM(x)+1$, we obtain that

$$B(x,1)=\frac{2-3x-5x^2-(2-x-2x^2)\sqrt{1-2x-3x^2}}{2x(1+x)(1-3x)}.$$

Comparing the n-th coefficient we obtain the desired result. □

As an application of Pick's theorem (cf. [7, pp. 40]), one may establish a relation between the area, semiperimeter, and number of interior points of a Motzkin polyomino. Pick's theorem says that the area of a simple polygon with integer vertex coordinates is equal to $I+B/2-1$, where I is the number of interior points and B is the number of points lying on the boundary. Let $\mathtt{int}(n)$ be the sum of the interior points over all Motzkin polyominoes of length n.

Corollary 6.4. *The number of interior points over all Motzkin polyominoes of length n is*

$$\frac{1}{2}\left(3^n-3T_n\right)-2T_{n-1}+2m_{n-1}.$$

Proof. From Pick's theorem we can obtain the relation

$$u(n)=\mathtt{int}(n)+s(n)-m_{n-1}.$$

Now the identity follows from Corollary 4.5 and Theorem 5.3. □

The first few values of the sequence $\mathtt{int}(n)$ for $n\geq 3$ are

$$1,\quad 6,\quad 25,\quad 93,\quad 324,\quad 1088,\quad 3565,\quad 11487,\ldots$$

This sequence does not appear in the OEIS.

Acknowledgments

The authors are grateful to the referees for the detailed comments and corrections that helped us improve the paper. José L. Ramírez was partially supported by Universidad Nacional de Colombia. The third author would like to thank the Laboratoire d'Informatique de Bourgogne for the warm hospitality during his visit at Université de Bourgogne where part of this work was done. This work was supported in part by ANR-22-CE48-0002 funded by l'Agence Nationale de la Recherche and by the project ANER ARTICO funded by Bourgogne-Franche-Comté region (France).

Data availability

No data was used for the research described in the article.

References

[1] M. Archibald, A. Blecher, A. Knopfmacher, Parameters in inversion sequences, Math. Slovaca 73 (3) (2023) 551–564.
[2] C. Banderier, M. Bousquet-Mélou, A. Denise, P. Flajolet, D. Gardy, D. Gouyou-Beauchamps, Generating functions for generating trees, Discrete Math. 246 (2002) 29–55.
[3] C. Banderier, P. Flajolet, Basic analytic combinatorics of directed lattice paths, Theoret. Comput. Sci. 281 (2002) 37–80.
[4] J.-L. Baril, J.-M. Pallo, Motzkin subposets and Motzkin geodesics in Tamari lattices, Inform. Process. Lett. 114 (2014) 31–37.
[5] J.-L. Baril, H. Prodinger, Enumeration of partial Łukasiewicz paths, Enumer. Combin. Appl. 3 (1) (2023) S2R2.
[6] J.-L. Baril, J.L. Ramírez, Descent distribution on Catalan words avoiding ordered pairs of relations, Adv. Appl. Math. 149 (2023) 102551.
[7] M. Beck, S. Robins, Computing the Continuous Discretely: Integer Point Enumeration in Polyhedra, in: Undergraduate Texts in Mathematics, Springer, Berlin, 2007.
[8] F.R. Bernhart, Catalan, Motzkin, and Riordan numbers, Discrete Math. 204 (1999) 73–112.
[9] A. Blecher, C. Brennan, A. Knopfmacher, Combinatorial parameters in bargraphs, Quaest. Math. 39 (2016) 619–635.
[10] A. Blecher, C. Brennan, A. Knopfmacher, The site-perimeter of compositions, Discrete Math. Appl. 32 (2) (2022) 75–89.
[11] A. Blecher, C. Brennan, A. Knopfmacher, T. Mansour, The perimeter of words, Discrete Math. 340 (10) (2017) 2456–2465.
[12] M. Bousquet-Mélou, A. Rechnitzer, The site-perimeter of bargraphs, Adv. in Appl. Math. 31 (2003) 86–112.
[13] D. Callan, T. Mansour, J.L. Ramírez, Statistics on bargraphs of Catalan words, J. Autom. Lang. Comb. 26 (2021) 177–196.
[14] R. Donaghey, L.W. Shapiro, Motzkin numbers, J. Combin. Theory Ser. A 23 (3) (1977) 291–301.
[15] P. Flajolet, R. Sedgewick, Analytic Combinatorics, Cambridge University Press, 2009.
[16] I.M. Gessel, S. Ree, Lattice paths and faber polynomials, in: Advances in Combinatorial Methods and Applications to Probability and Statistics, Birkhauser Verlag, Boston, 1997.
[17] T. Goy, M. Shattuck, Determinant identities for the Catalan, Motzkin and Schröder numbers, Art Disc. Appl. Math. 7 (2024) #P1.09.
[18] A.J. Guttmann (Ed.), Polygons, Polyominoes and Polycubes, in: Lecture Notes in Physics, vol. 775, Springer, Heidelberg, Germany, 2009.
[19] C. Krattenthaler, Lattice path enumeration, in: M. Bóna (Ed.), Handbook of Enumerative Combinatorics, CRC Press, 2015.
[20] T. Mansour, Interior vertices in set partitions, Adv. Appl. Math. 101 (2018) 60–69.
[21] T. Mansour, The perimeter and the site-perimeter of set partitions, Electron. J. Combin. 26 (2) (2019) #P2.30.
[22] T. Mansour, Semi-perimeter and inner site-perimeter of k-ary words and bargraphs, Art Disc. Appl. Math. 4 (2021) P1.06.
[23] T. Mansour, J.L. Ramírez, Enumerations on polyominoes determined by Fuss–Catalan words, Australas. J. Combin. 81 (2021) 447–457.
[24] T. Mansour, J.L. Ramírez, Exterior corners on bargraphs of motzkin words, in: F. Hoffman, S. Holliday, Z. Rosen, F. Shahrokhi, J. Wierman (Eds.), Combinatorics, Graph Theory and Computing 2021, in: Springer Proceedings in Mathematics & Statistics, vol 448, Springer, 2024.
[25] T. Mansour, J.L. Ramírez, D.A. Toquica, Counting lattice points on bargraphs of catalan words, Math. Comput. Sci. 15 (2021) 701–713.
[26] T. Mansour, A. Sh. Shabani, Enumerations on bargraphs, Discrete Math. Lett. 2 (2019) 65–94.
[27] A.G. Orlov, On asymptotic behavior of the Taylor coefficients of algebraic functions, Sib. Math. J. 25 (5) (1994) 1002–1013.
[28] H. Prodinger, The kernel method: a collection of examples, Sém. Lothar. Combin. 50 (2004) Paper B50f.
[29] N.J.A. Sloane, The on-line encyclopedia of integer sequences, http://oeis.org/.
[30] R.P. Stanley, Catalan Numbers, Cambridge University Press, 2015.

The ascent lattice on Dyck paths

Jean-Luc Baril[a] Mireille Bousquet-Mélou[b] Sergey Kirgizov[a]
Mehdi Naima[c]

Submitted: Sep 24, 2024; Accepted: Feb 17, 2025; Published: TBD
© The authors. Released under the CC BY-ND license (International 4.0).

Abstract

In the Stanley lattice defined on Dyck paths of size n, cover relations are obtained by replacing a valley DU by a peak UD. We investigate a greedy version of this lattice, first introduced by Chenevière, where cover relations replace a factor DU^kD by U^kD^2. By relating this poset to another poset recently defined by Nadeau and Tewari, we prove that this still yields a lattice, which we call the *ascent lattice* $\mathbb{D}_n$. We then count intervals in $\mathbb{D}_n$. Their generating function is found to be algebraic of degree 3. The proof is based on a recursive decomposition of intervals involving two *catalytic* parameters. The solution of the corresponding functional equation is inspired by recent work on the enumeration of walks confined to a quadrant. We also consider the order induced in $\mathbb{D}_{mn}$ on *m-Dyck paths*, that is, paths in which all ascent lengths are multiples of m, and on *mirrored m-Dyck paths*, in which all descent lengths are multiples of m. The first poset $\mathbb{D}_{m,n}$ is still a lattice for any m, while the second poset $\mathbb{D}'_{m,n}$ is only a join semilattice when $m > 1$. In both cases, the enumeration of intervals is still described by an equation in two catalytic variables. Interesting connections arise with the *sylvester congruence* of Hivert, Novelli and Thibon, and again with walks confined to a quadrant. We combine the latter connection with probabilistic results to give asymptotic estimates of the number of intervals in both $\mathbb{D}_{m,n}$ and $\mathbb{D}'_{m,n}$. Their form implies that the generating functions of intervals are no longer algebraic, nor even D-finite, when $m > 1$.

Mathematics Subject Classifications: 05A19, 06A07, 06A11, 05A15

Keywords— Exact enumeration — Algebraic series — Posets — Intervals — Lattice walks — Sylvester classes — m-parking functions.

[a]JLB and SK: LIB, Université Bourgogne Europe, B.P. 47 870, 21078 Dijon Cedex France, ({barjl,sergey.kirgizov}@u-bourgogne.fr).
[b]MBM: CNRS, LaBRI, Université de Bordeaux, 351 cours de la Libération, F-33405 Talence Cedex, France (bousquet@labri.fr).
[c]MN: LIP6, Sorbonne Université, 4 place Jussieu, 75005 Paris, France, (mehdi.naima@lip6.fr).

1 Introduction and main results

In recent years, several orders have been studied on the set of Dyck paths of fixed size n, with a special attention to the number of their intervals. The most natural of these posets is the so-called Stanley lattice (or Dyck lattice [30]), where a Dyck path is smaller than another one if it lies (weakly) below it [5]. Its cover relations are obtained by replacing a *valley* by a *peak*; in symbols, $DU \to UD$, where U and D stand for up and down steps, respectively. All other Dyck orders that have been studied are included in this one, in the sense that if a path is less than another one, it lies below it. Let us cite the Kreweras lattice [46, 5], the Tamari lattice [40, 44], its greedy version due to Dermenjian [32], the alt-Tamari lattices of Chenevière [29], or the *pyramid* lattice introduced by three of the authors of the present paper [2]. Figure 1 presents the subposet-inclusion structure of these posets. Interesting connections arise between intervals in these posets and various families of planar maps [5, 21, 18, 26, 37, 35, 36, 38], and, at least conjecturally, with certain quotient rings of polynomials [4, 17, 42].

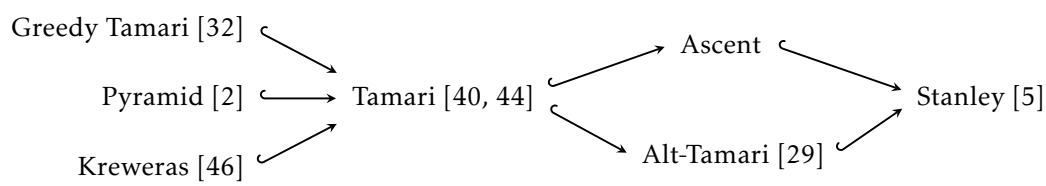


Figure 1: Subposet-inclusion structure of some orders on Dyck paths.

In this paper we consider yet another order — in fact, a lattice — on the set of Dyck paths of size n. This poset was first considered by Chenevière in his thesis [30], following a suggestion of Nadeau. The enumeration of its intervals reveals connections with 2-dimensional walks confined to a cone, and, on a more algebraic side, with classes of the *sylvester congruence* on words [43, 49].

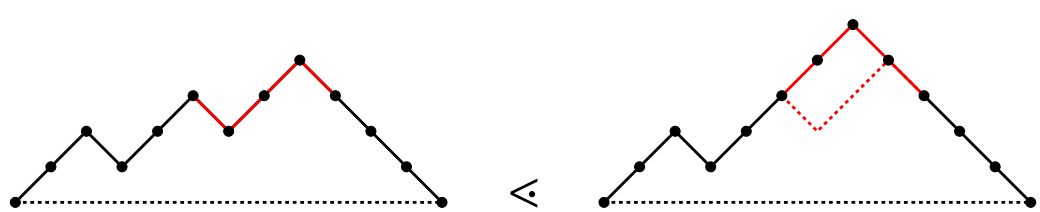


Figure 2: A cover relation between two Dyck paths of size 6.

The cover relations in this lattice, called the *ascent lattice* and denoted by $\mathbb{D}_n$, are easy to describe. They consist in swapping a down step with the *ascent* that follows it, where we call *ascent* a (non-empty) maximal sequence of up steps; see Figure 2. In symbols, $DU^kD \longrightarrow U^kDD$ for any $k \geqslant 1$. Roughly speaking, one applies at once as many cover relations of the Stanley lattice as possible, using always the same down step. This explains why this poset is called *greedy Dyck poset* in [30, Def. 7.2.5]. Figure 3 shows the whole lattice for $n = 4$. After establishing some properties of the lattice $\mathbb{D}_n$, we exhibit a recursive construction of its intervals, which can be described by a *generating tree* with two labels. This tree also describes a family of lattice

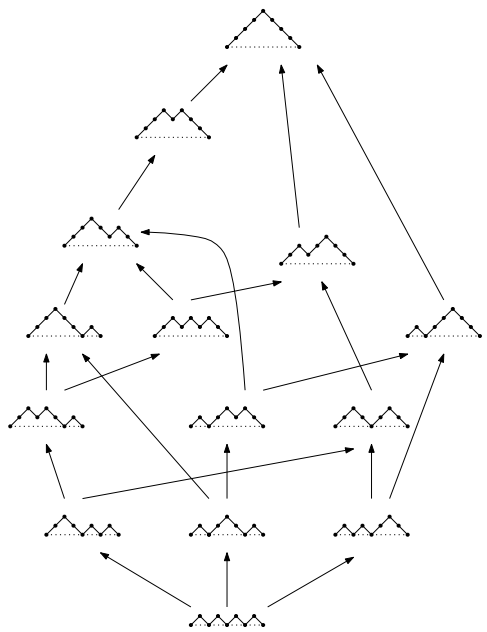


Figure 3: The Hasse diagram of $\mathbb{D}_4 = \mathbb{D}_{1,4} = \mathbb{D}'_{1,4}$.

walks confined to the quadrant, in which an *infinite family of steps* is allowed. This construction translates, for the associated generating function, into a linear equation with two additional (or: *catalytic*) variables. An important literature has been devoted, in the past 20 years, to the solution of similar-looking equations, corresponding to quadrant walks in which only *finitely many* steps are allowed [8, 10, 20, 33, 51]. We solve our equation by adapting the *invariant* approach of [8], and conclude that the generating function of intervals in the ascent lattices $\mathbb{D}_n$ is an algebraic (cubic) series. The asymptotic behaviour of the corresponding numbers, in $\mu^n n^{-7/2}$, is far less common in enumerative combinatorics than the tree behaviour in $\mu^n n^{-3/2}$, or the (rooted) map behaviour in $\mu^n n^{-5/2}$. This puts ascent intervals in the same universality class as unrooted planar maps [59, 53], or discrete versions of the Brownian motion confined to a wedge of angle $2\pi/5$ [31, 8].

Theorem 1. *Let $g(n)$ be the number of intervals in the ascent lattice $\mathbb{D}_n$. The associated generating function $G := \sum_{n\geqslant 1} g(n)t^n$ is*

$$G = Z(1 - 2Z + 2Z^3),$$

where Z is the only formal power series in t satisfying $Z = t(1+Z)(1+2Z)^2$. In particular, the series G is algebraic of degree 3 over $\mathbb{Q}(t)$.

As n tends to infinity, the number of intervals in $\mathbb{D}_n$ is equivalent to

$$\kappa\, \mu^n n^{-7/2},$$

with

$$\mu = \frac{11+5\sqrt{5}}{2}, \qquad \kappa = \frac{3}{8}\sqrt{\frac{275+123\sqrt{5}}{10\pi}}.$$

After completing this paper, we learnt that this result had been conjectured by Nadeau and Tewari.

Figure 4: From left to right, a 2-Dyck path of size 3 (element of $\mathcal{D}_{2,3}$), and a mirrored 2-Dyck path of size 3 (element of $\mathcal{D}'_{2,3}$).

Inspired by earlier papers on other Dyck lattices [21, 18, 4, 17], we also consider the family of *m-Dyck paths* of size n, in which the n up steps now have height m instead of 1 (Figure 4, left). They can also be seen as Dyck paths of size mn in which the length of each ascent is a multiple of m. They form an upper ideal, and an interval, in $\mathbb{D}_{mn}$, and thus a new lattice that we denote by $\mathbb{D}_{m,n}$ (Figure 5, left). Its cover relations are still described by $D\boldsymbol{U}^k D \longrightarrow \boldsymbol{U}^k DD$, where now $\boldsymbol{U}$, in boldface, stands for a *large* up step of height m. We extend to this lattice the construction of intervals found for $m = 1$, and obtain again a bijection with certain quadrant walks with infinitely many allowed steps. However, in contrast with other lattices defined on m-Dyck paths [21, 18], as soon as $m \geqslant 2$ the associated generating function stops being algebraic, or even *D-finite* (i.e., solution of a linear differential equation with polynomial coefficients). We prove this by determining the asymptotic behaviour of the number of intervals, which, due to deep number theoretic results [11], rules out the possibility of D-finiteness.

Proposition 2. *Let us fix $m \geqslant 1$. The number $g_m(n)$ of intervals in the ascent lattice $\mathbb{D}_{m,n}$ satisfies, as n tends to infinity,*

$$g_m(n) \sim \kappa \mu^n n^\alpha,$$

for some positive constant κ, where

$$\mu = \frac{m\sqrt{m^2+4}+m^2+2}{2} \cdot \left(\frac{2+\sqrt{m^2+4}}{m}\right)^m$$

and

$$\alpha = -1 - \pi/\arccos(-c) \qquad \textit{with} \qquad c = -\sqrt{\frac{m^2+2-\sqrt{m^2+4}}{2m^2+6}}.$$

For $m = 1$ we have $\alpha = -7/2$, but for $m > 1$ the exponent α is irrational. This implies that for $m > 1$ the generating function of intervals in $\mathbb{D}_{m,n}$ is not D-finite.

Finally, we also consider the class of *mirrored* m-Dyck paths of size n: now, the up steps have height 1 and the n down steps have height m (Figure 4, right). Alternatively, these paths can be seen as Dyck paths of size mn in which the length of each descent is a multiple of m. For

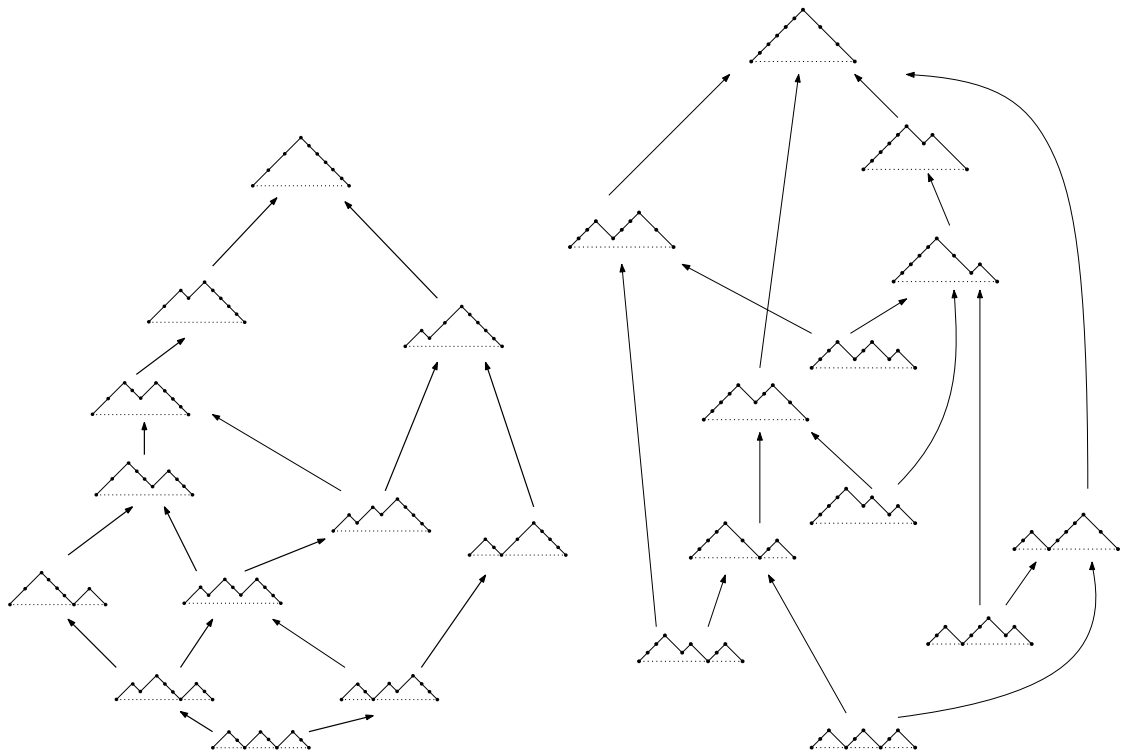


Figure 5: On the left, the Hasse diagram of $\mathbb{D}_{2,3}$, and on the right, the Hasse diagram of $\mathbb{D}'_{2,3}$.

$m > 1$ the order induced by $\mathbb{D}_{mn}$ on these paths is no longer a lattice, and in particular it has several minimal elements. But it is still a join semilattice, which we denote by $\mathbb{D}'_{m,n}$ (Figure 5, right). Its cover relations are still described by $\boldsymbol{D}U^k\boldsymbol{D} \longrightarrow U^k\boldsymbol{D}\boldsymbol{D}$, where $\boldsymbol{D}$, in boldface, stands for a large down step. Working with these mirrored m-Dyck paths is less standard than working with m-Dyck paths, but turns out to be rewarding: when investigating the number of intervals in $\mathbb{D}'_{m,n}$, we found them in the On-line Encyclopedia of Integer Sequences [55], as the number of *sylvester classes of m-parking functions* [43, 49]. We first establish a bijection between these classes and intervals in $\mathbb{D}'_{m,n}$. Then, we describe a recursive construction of intervals of $\mathbb{D}'_{m,n}$, which, when $m = 1$, is *not* the same as the earlier one. It gives again a bijection with quadrant walks with infinitely many allowed steps, and a linear equation in two catalytic variables for the generating function. This time we also have an alternative interpretation in terms of quadrant walks with finitely many (weighted) allowed steps. We determine the asymptotic behaviour of the number of intervals, which again rules out the possibility of D-finiteness for $m \geqslant 2$.

Proposition 3. *Let us fix $m \geqslant 1$. The number $g'_m(n)$ of intervals in the ascent poset $\mathbb{D}'_{m,n}$ satisfies, as n tends to infinity,*

$$g'_m(n) \sim \kappa \mu^n n^\alpha,$$

for some positive constant κ, where

$$\mu = \left(2m + \sqrt{1+4m^2}\right)\left(\frac{1+\sqrt{1+4m^2}}{2m}\right)^{2m}$$

and

$$\alpha = -1 - \pi/\arccos(-c) \qquad \textit{with} \qquad c = -\sqrt{\frac{1+2m^2 - m\sqrt{1+4m^2}}{2(3m^2+1)}}.$$

For $m = 1$ we have $\alpha = -7/2$, but for $m > 1$ the exponent α is irrational. This implies that for $m > 1$ the generating function of ascent intervals in $\mathbb{D}'_{m,n}$ is not D-finite.

Outline of the paper. We begin in Section 2 with various definitions, in particular of the ascent posets $\mathbb{D}_{m,n}$ and $\mathbb{D}'_{m,n}$. We give a characterization of the ascent order that reveals a connection with another poset defined on nonincreasing sequences, recently introduced by Nadeau and Tewari [48]. We rely on their results to conclude that $\mathbb{D}_{m,n}$ is a lattice, and $\mathbb{D}'_{m,n}$ a join semilattice. In Section 3, we show how to transform bijectively intervals of the Nadeau-Tewari poset into words on the alphabet $\mathbb{Z}$ that *avoid* two patterns, and are representatives for classes of the *sylvester congruence* introduced by Hivert *et al.* [43]. In particular, we exhibit sylvester classes that are in bijection with intervals of $\mathbb{D}_{m,n}$ and $\mathbb{D}'_{m,n}$. In Section 4, we describe recursive constructions of these intervals, and convert them into functional equations for the associated generating functions. To write these equations, one needs to record two additional (or: *catalytic*) variables. In Section 5 we solve the two equations obtained for $m = 1$, and establish Theorem 1 and refinements of it. Section 6 is devoted to the asymptotic estimates of Theorems 2 and 3. We conclude with a few remarks.

2 Ascent posets

2.1 Definitions

We begin with precise definitions of the objects and notions discussed more informally in the introduction.

Dyck paths. Let us first recall that a *Dyck path* P is a 2-dimensional path starting at the origin $(0,0)$, consisting of up steps $U = (1,1)$ and down steps $D = (1,-1)$, that ends on the x-axis and never goes strictly below the x-axis. The *size* of P is the number n of up steps. We denote by $\mathcal{D}_n$ the set of such paths. An *ascent* of a path P is a maximal, non-empty sequence of consecutive up steps. A *descent* is defined similarly using down steps. A *factor* of P is a non-empty sequence of consecutive steps. A *peak* is a factor UD, while a *valley* is a factor DU. The *ascent composition* of a Dyck path P is the composition $c(P) = (c_1, \ldots, c_k)$, where the part $c_i > 0$ is the length of the ith ascent of P. Clearly, the c_i's sum to n, so that $c(P)$ is a composition of the integer n if $P \in \mathcal{D}_n$.

For $m \geqslant 1$, we call *m-Dyck path* of size n any path of $\mathcal{D}_{mn}$ in which all ascent lengths are multiples of m, and denote the corresponding set by $\mathcal{D}_{m,n}$. Analogously, we call *mirrored m-Dyck path* of size n any path of $\mathcal{D}_{mn}$ in which all descent lengths are multiples of m, and denote the corresponding set by $\mathcal{D}'_{m,n}$. We sometimes consider m-Dyck paths of size n as sequences of *large* up steps $\boldsymbol{U} = (m,m)$ and (unit) down steps D, and analogously for mirrored m-Dyck paths,

which have large down steps $\boldsymbol{D} = (m, -m)$ and unit up steps. The number of m-Dyck paths of size n is the Fuss-Catalan number (see A355262 in the OEIS [55]):

$$C_m(n) = \frac{1}{mn+1}\binom{(m+1)n}{n}. \tag{1}$$

Orders on Dyck paths. There exists on $\mathcal{D}_n$ a classical order, called the Stanley order (or lattice, in fact), for which P is less than or equal to Q if it lies (weakly) below Q. By this, we mean that for any ℓ, the prefix of P of length ℓ contains at most as many up steps as the prefix of length ℓ of Q. A path Q covers a path P in this order if and only if Q is obtained by replacing a valley DU of P into a peak UD.

In this paper we consider a greedy version of this order, which we call the *ascent order*. It is described by its cover relations: we say that Q covers P (denoted $P \lessdot Q$) if there exists in P a factor DU^kD such that Q is obtained from P by replacing this factor by U^kDD. Observe that in this case P lies below Q. In particular, this relation is irreflexive and anti-symmetric. Its transitive closure is thus an order relation on $\mathcal{D}_n$, denoted $\leqslant$, and one easily checks that the cover relations of this order are indeed those described above. We denote by $\mathbb{D}_n$ the corresponding poset. Since $\mathcal{D}_{m,n}$ and $\mathcal{D}'_{m,n}$ are subsets of $\mathcal{D}_{mn}$, we can consider the orders induced by $\leqslant$ on these subsets. We denote by $\mathbb{D}_{m,n}$ and $\mathbb{D}'_{m,n}$ the corresponding posets.

Formal power series. For a ring R, we denote by $R[t]$ (resp. $R[[t]]$) the ring of polynomials (resp. formal power series) in t with coefficients in R. If R is a field, then $R(t)$ stands for the field of rational functions in t. This notation is generalized to several variables. For instance, in Section 4 we consider the generating function $G_m(t;x,y)$ of intervals in the ascent lattices $\mathbb{D}_{m,n}$, $n \geqslant 1$, where t records the size (the number n of up steps), and x (resp. y) the length of the final descent of the smaller (resp. larger) path; this series belongs to $\mathbb{Q}[x,y][[t]]$. We often omit in our notation the dependency in t of our series, writing for instance $G_m(x,y)$ instead of $G_m(t;x,y)$.

2.2 A characterisation of the ascent order

Let us first recall that a composition c *refines* another composition c' if we can write $c = (c_{1,1}, c_{1,2}, \dots, c_{1,j_1}, \dots, c_{i,1}, c_{i,2}, \dots, c_{i,j_i})$ and $c' = (c'_1, \dots, c'_i)$ where $c'_k = c_{k,1} + c_{k,2} + \cdots + c_{k,j_k}$, for $1 \leqslant k \leqslant i$. For instance, the paths of Figure 2 have ascent compositions $c(P) = (2,2,2)$ and $c(Q) = (2,4)$, and we observe that $c(P)$ refines $c(Q)$. Note that refinement is an order relation on compositions of n, for each integer n. It is isomorphic to the Boolean lattice on $(n-1)$ elements. The cover relations are obtained by merging two consecutive parts.

Proposition 4. *In the ascent poset $\mathbb{D}_n$, we have $P \leqslant Q$ if and only if P lies (weakly) below Q and the ascent composition $c(P)$ refines $c(Q)$.*

Graphically, the second condition means that for every descent of Q, there is a descent of P lying on the same line (of slope -1).

Proof. Let us first prove that if $P \leqslant Q$ in $\mathbb{D}_n$, the other two conditions hold. By transitivity, it suffices to prove them if $P \lessdot Q$. By definition, this means that one obtains Q from P by replacing a factor DU^kD by U^kDD. This shows at once that Q is above P. Moreover, $c(Q)$ is either $c(P)$, or is obtained from $c(P)$ by merging two consecutive parts. It follows that $c(P)$ is a refinement of $c(Q)$.

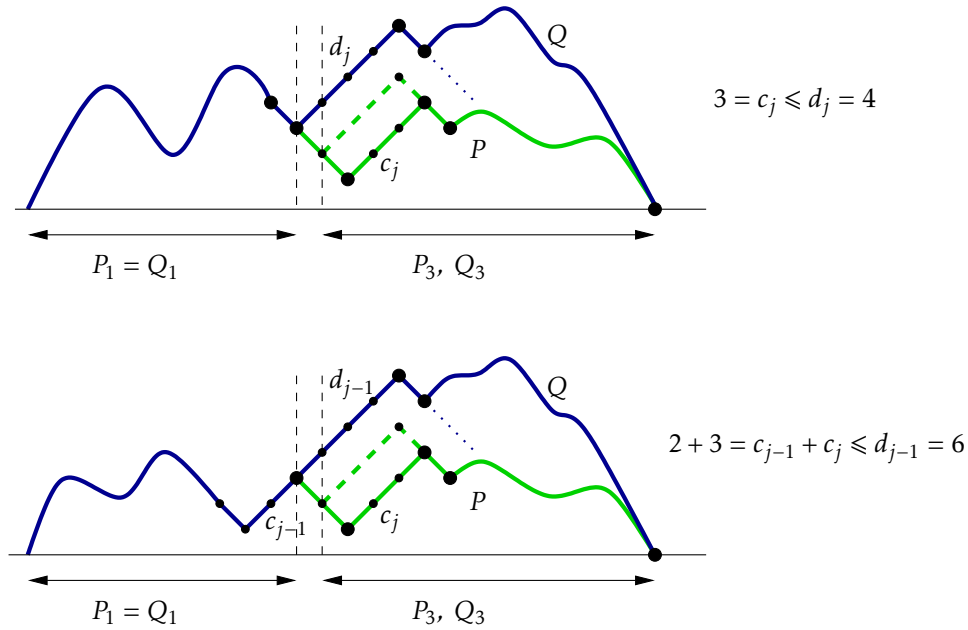


Figure 6: Applying the transformation $DU^kD \to U^kDD$ at the first valley of P that is not a valley of Q. Top: P_1 ends with D. Bottom: P_1 ends with U.

Conversely, let us take a pair (P, Q) in $\mathbb{D}_n$ such that P lies below Q and $c(P)$ refines $c(Q)$. We will argue by induction on the area lying between P and Q. If this area is zero, $P = Q$ and there is nothing to prove. Otherwise, let us write $P = P_1P_2$ and $Q = Q_1Q_2$ where $P_1 = Q_1$ is the longest common prefix of P and Q. Then $P_2 = DP_3$ and $Q_2 = UQ_3$, because P is below Q (Figure 6). Note that P_3 contains at least one U step. Let $c = (c_1, \ldots, c_k)$ be the ascent composition of P, and say that the first ascent of P_3 is the jth ascent of P, of length c_j. If $P_1 = Q_1$ ends with a D step (Figure 6, top), then the first up step of Q_2 is the start of the jth ascent of Q, which has length d_j if $d = (d_1, \ldots, d_\ell)$ is the ascent composition of Q. Since c refines d and the first $(j-1)$ parts of c and d coincide, we have $c_j \leqslant d_j$. If $P_1 = Q_1$ ends with a U step (Figure 6, bottom), then this step belongs to the $(j-1)$th ascents of P and Q, which have lengths c_{j-1} in P and d_{j-1} in Q. Moreover, $d_{j-1} > c_{j-1}$ because this ascent of Q also includes the first step of Q_2 (but definitely not the first step of P_2, which is D). In this case the first $(j-2)$ parts of c and d coincide, and we have $d_{j-1} \geqslant c_{j-1} + c_j$ since c refines d. In both cases, let P' be obtained by applying the cover relation at the first valley of P that follows P_1. The above observations imply that P' lies below Q, and that its composition refines $c(Q)$: in the first case, $c(P') = c(P)$, and in the second case, either $c(P') = c(P)$ or $c(P')$ is obtained by merging the parts c_{j-1} and c_j of $c(P)$. Also, the area between P' and Q is less than the area between P and Q. By the induction hypothesis, we thus have $P' \leqslant Q$. But P' covers P, hence $P \leqslant Q$.

The second statement of the proposition is a simple observation. □

It will be worth keeping in mind the following result, established in the second part of the above proof.

Corollary 5. *Let $P \lneq Q$. Applying the cover relation at the first valley of P that is not a valley of Q gives a path P' that covers P and satisfies $P' \leqslant Q$.*

Remark 6. The characterization of Proposition 4 implies that if Q covers P in the Tamari lattice, then Q is larger than P in the ascent lattice. This inclusion appears in Figure 1.

2.3 The posets $\mathbb{D}_{m,n}$ and $\mathbb{D}'_{m,n}$

We now fix $m \geqslant 1$. We consider the ascent order on $\mathcal{D}_{mn}$, and the induced orders on the set $\mathcal{D}_{m,n}$ of m-Dyck paths of size n, on the first hand, and on the set $\mathcal{D}'_{m,n}$ of mirrored m-Dyck paths of size n, on the other hand. Recall that we sometimes consider these paths as having large up (resp. down) steps, of height m, and that these large steps are denoted by $\boldsymbol{U}$ and $\boldsymbol{D}$, respectively.

Proposition 7. *The poset $\mathbb{D}_{m,n}$ is the interval of $\mathbb{D}_{mn}$ with minimum element $(U^mD^m)^n$ and maximum element $U^{mn}D^{mn}$. Its cover relations are still given by the transformation $DU^\ell D \to U^\ell DD$ (where $\ell = mk$ is necessarily a multiple of m), or equivalently by $D\boldsymbol{U}^kD \to \boldsymbol{U}^kDD$.*

The poset $\mathbb{D}'_{m,n}$ has maximum element $U^{mn}D^{mn}$, but several minimal elements if $m \geqslant 2$. Their number is the Fuss-Catalan number $C_{m-1}(n)$ (see (1)). The cover relations are still given by the transformation $\boldsymbol{D}U^k\boldsymbol{D} \to U^k\boldsymbol{DD}$.

Proof. Recall that the cover relations in $\mathbb{D}_{mn}$ either merge two consecutive parts of the ascent composition, or leave it unchanged. Hence, if $P \leqslant Q$ in $\mathbb{D}_{mn}$ and $P \in \mathcal{D}_{m,n} \subset \mathcal{D}_{mn}$, then $Q \in \mathcal{D}_{m,n}$ as well. Thus $\mathbb{D}_{m,n}$ forms an upper ideal in $\mathbb{D}_{mn}$. Moreover, Proposition 4 implies that the m-Dyck path $(U^mD^m)^n$ is smaller than (or equal to) any other m-Dyck path. This proves the first statement. The second then follows, because Q covers P in $\mathbb{D}_{m,n}$ if and only if Q covers P in $\mathbb{D}_{mn}$.

Let us now consider the subset $\mathcal{D}'_{m,n}$ of $\mathcal{D}_{mn}$. It contains $U^{mn}D^{mn}$, so this path remains the (unique) maximal element in $\mathbb{D}'_{m,n}$. Let us skip for the moment the results on minimal elements, and focus on cover relations. If a path Q is obtained from another path $P \in \mathcal{D}'_{m,n}$ by a transformation $\boldsymbol{D}U^k\boldsymbol{D} \to U^k\boldsymbol{DD}$, then $Q \in \mathcal{D}'_{m,n}$ and $P \leqslant Q$ by Proposition 4. Moreover, any mirrored m-Dyck path that lies above P and below Q is obtained from P by replacing the same factor $\boldsymbol{D}U^k\boldsymbol{D}$ by $U^\ell\boldsymbol{D}U^{k-\ell}\boldsymbol{D}$. By Proposition 4, it is larger than or equal to P in $\mathbb{D}_{mn}$ only if $\ell = 0$ or $\ell = k$. Hence Q covers P. Conversely, if Q covers P in $\mathbb{D}'_{m,n}$, then in particular $P \leqslant Q$ in $\mathbb{D}_{mn}$, and Proposition 4 implies that P lies below Q and $c(P)$ refines $c(Q)$. We then apply m times the construction of Corollary 5: that is, we perform, in the first valley of P that is not a valley of Q, a transformation $\boldsymbol{D}U^k\boldsymbol{D} \to U^k\boldsymbol{DD}$ that gives a path P' lying in $[P, Q]$. Since $P' \neq P$ and P' still belongs to $\mathcal{D}'_{m,n}$, the fact that Q covers P in $\mathbb{D}'_{m,n}$ implies that $P' = Q$. Hence Q is indeed obtained via the claimed cover relation.

Now that we have described the cover relations in $\mathbb{D}'_{m,n}$, let us finally return to its minimal elements. We assume that $m \geqslant 2$. If Q covers some path P, then Q contains a factor $\boldsymbol{DD}$. Conversely, if $Q = Q_1U\boldsymbol{DD}Q_2$, then Q covers $P := Q_1\boldsymbol{D}U\boldsymbol{D}Q_2$. Thus the minimal elements of $\mathbb{D}'_{m,n}$ are those that contain no factor $\boldsymbol{DD}$. We claim that these paths are in bijection with mirrored $(m-1)$-Dyck paths of the same size. The bijection simply consists in replacing every factor $U\boldsymbol{D} = UD^m$ by D^{m-1}. This concludes the proof. □

2.4 Lattice properties

The characterization of the ascent order in Proposition 4 reveals a connection with another order, defined on nonincreasing sequences of integers, recently introduced by Nadeau and Tewari [48].

This connection, already observed by Chenevière [30, p. 147], will imply our main structural result.

Definition 8. [48, Def. 5.4] Let $n \geqslant 1$, and $u = (u_1, \dots, u_n)$, $v = (v_1, \dots, v_n)$ be two nonincreasing sequences of integers. Then u is smaller than or equal to v in the Nadeau-Tewari poset $\mathcal{P}_n$, denoted $u \trianglelefteq v$, if u is smaller than or equal to v componentwise, and every descent of v is a descent of u. In symbols:

- $u_i \leqslant v_i$ for all i,
- if $v_i > v_{i+1}$ then $u_i > u_{i+1}$.

Proposition 9. *[48, Prop. 5.5] The poset $\mathcal{P}_n$ is a lattice, called (here) the NT lattice.*

In particular, the join of two sequences u and v is the (componentwise) smallest sequence w whose descent set is contained in the intersection of the descents sets of u and v. For instance, if $u = (4, 4, 2, 2)$ and $v = (4, 4, 3, 1)$ then $w = (4, 4, 3, 3)$.

We can encode an m-Dyck path of $\mathcal{D}_{m,n}$ by a nonincreasing sequence $(u_1, \dots, u_n)$ of integers, where u_i is the number of D steps that occur after the ith large up step $\boldsymbol{U}$. For instance, the encoding of the leftmost path of Figure 4 is $(u_1, u_2, u_3) = (6, 5, 3)$. The encoding sends bijectively $\mathcal{D}_{m,n}$ on the set of nonincreasing sequences u of length n such that for all i,

$$m(n - i + 1) \leqslant u_i \leqslant mn. \tag{2}$$

Analogously, we encode a mirrored m-Dyck path of $\mathcal{D}'_{m,n}$ by the nonincreasing sequence $(u_1, \dots, u_{mn})$ such that u_i is the number of $\boldsymbol{D}$ steps that occur after the ith up step U. For instance, the encoding of the rightmost path of Figure 4 is $(u_1, \dots, u_6) = (3, 3, 3, 2, 1, 1)$. This encoding sends bijectively $\mathcal{D}'_{m,n}$ on the set of nonincreasing sequences u of length mn such that for all i,

$$n - \frac{i-1}{m} \leqslant u_i \leqslant n. \tag{3}$$

Proposition 10. *Let $m, n \geqslant 1$. The poset $\mathbb{D}_{m,n}$ is a lattice, isomorphic to the order induced by the Nadeau-Tewari order on sequences of $\mathcal{P}_n$ satisfying* (2).

The poset $\mathbb{D}'_{m,n}$ is a join-semilattice, isomorphic to the order induced by the Nadeau-Tewari order on sequences of $\mathcal{P}_{mn}$ satisfying (3).

Proof. The descriptions of $\mathbb{D}_{m,n}$ and $\mathbb{D}'_{m,n}$ as induced posets of $\mathcal{P}_n$ and $\mathcal{P}_{mn}$ come directly from the characterization of the ascent order given in Proposition 4, and the definitions of $\mathbb{D}_{m,n}$ and $\mathbb{D}'_{m,n}$ as subposets of $\mathbb{D}_{mn}$. Let us now address the lattice properties.

In $\mathcal{P}_n$, any sequence u satisfying (2) satisfies, for the NT order:

$$u_{\min} := (mn, m(n-1), \dots, m) \trianglelefteq u \trianglelefteq u_{\max} := (mn, \dots, mn).$$

This is because $u_{\min}$ has a descent at each place, while $u_{\max}$ has no descent at all. Conversely, any sequence u such that $u_{\min} \trianglelefteq u \trianglelefteq u_{\max}$ satisfies (2). This means that our encoding sends $\mathbb{D}_{m,n}$ on the interval $[u_{\min}, u_{\max}]$ of $\mathcal{P}_n$, and thus on a lattice.

Analogously, any sequence u of $\mathcal{P}_{mn}$ satisfying (3) satisfies, for the NT order:

$$u \trianglelefteq u_{\max} := (n, \dots, n),$$

because there is no descent in $u_{\max}$. Moreover, any v such that $u \trianglelefteq v \trianglelefteq u_{\max}$ also satisfies (3). Hence our encoding sends $\mathbb{D}'_{m,n}$ on a union of intervals with maximal element $u_{\max}$, and thus on a join semilattice. □

Examples. Let us first take $m = 1$ and $n = 4$, and construct the join in $\mathbb{D}_n$ of the paths $P = UUDDUUDD$ and $Q = UUDUDDUD$, encoded by $u = (4,4,2,2)$ and $v = (4,4,3,1)$. The join of u and v has already be seen to be $w = (4,4,3,3)$, and thus the join of P and Q is $UUDUUDDD$.

Next let us take $m = 2$ and $n = 3$, and construct the join in $\mathbb{D}'_{m,n}$ of $P = UU\boldsymbol{D}UUU\boldsymbol{D}U\boldsymbol{D}$ and $Q = UUUU\boldsymbol{DD}UU\boldsymbol{D}$. The encodings of P and Q are, respectively, $u = (3,3,2,2,2,1)$ and $v = (3,3,3,3,1,1)$, whose join in $\mathcal{P}_n$ is $w = (3,3,3,3,3,3)$. Hence the join of P and Q in $\mathbb{D}'_{m,n}$ is $U^6\boldsymbol{D}^3$. □

3 Intervals in the Nadeau-Tewari poset and classes of the sylvester congruence

In this section, we exhibit a bijection between (some) intervals of the Nadeau-Tewari lattices $\mathcal{P}_n$ and (some) words on the alphabet $\mathbb{N}^* = \{1,2,\ldots\}$ that *avoid the patterns aba* and *acb* (precise definitions will be given below). These words are known to encode classes of the *sylvester congruence* on words, introduced by Hivert et al. [43]. We then specialize our bijection to intervals of $\mathbb{D}_{m,n}$ and $\mathbb{D}'_{m,n}$. In particular, intervals of $\mathbb{D}'_{m,n}$ are sent bijectively to sylvester classes of m-parking functions, considered in [49].

Let us begin with some definitions. Given a word $w = w_1 \cdots w_n$ on the alphabet $\mathbb{N}^*$, we denote its length n by $|w|$. We denote by $\{w\}$ the set of letters occurring in w, and by $\{\{w\}\}$ the multiset of letters of w. We define $\mathtt{NInc}(w)$ (resp. $\mathtt{NDec}(w)$) as the word obtained by reordering the letters of w in nonincreasing (resp. nondecreasing) order. For two words w and w' of the same length, we say that $w \leqslant w'$ if $w_i \leqslant w'_i$ for all i. We define $\mathtt{Low}(w)$ as the only nonincreasing word of length n that has the same left-to-right minima as w, at the same positions. In other words, $\mathtt{Low}(w)$ is the largest nonincreasing word (for the above componentwise order) that is smaller than or equal to w. For instance, if $u = 6\ 8\ 7\ 4\ 5\ 2\ 3\ 1\ 9$ then $\mathtt{Low}(u) = 6\ 6\ 6\ 4\ 4\ 2\ 2\ 1\ 1$.

The *sylvester congruence* on words is generated by the commutation relations

$$ac \cdots b \equiv ca \cdots b, \quad a \leqslant b < c.$$

It is known that the words $w = w_1 \cdots w_n$ *avoiding the patterns aba and acb* form a set of *representatives* of sylvester classes [49, Sec. 2.7]: every class contains such a word, and two distinct words of this type are never congruent. By *pattern avoidance*, we mean that we cannot find $1 \leqslant i < j < k \leqslant n$ such that either $w_i = w_k < w_j$ or $w_i < w_k < w_j$. This congruence arose in connection with binary search trees [43], and the name *sylvester* refers to the forest, *silva* in Latin, rather than to the mathematician James Joseph Sylvester.

Lemma 11. *Let w avoid the patterns aba and acb. Then w can be uniquely reconstructed from the words $w^{(1)} = \mathtt{NInc}(w)$ and $w^{(2)} = \mathtt{Low}(w)$.*

Proof. By construction, the left-to-right minima of w have the same positions and values as those of $w^{(2)}$. Hence, let us write

$$w = w^{(2)}_{i_1} z_1 w^{(2)}_{i_2} z_2 \cdots w^{(2)}_{i_k} z_k = w_{i_1} z_1 w_{i_2} z_2 \cdots w_{i_k} z_k,$$

where $1 = i_1 < i_2 \cdots < i_k$ are the indices of the left-to-right minima of w, and $|z_j| = i_{j+1} - i_j - 1$ for all j. Starting from $w^{(2)}$ and $w^{(1)}$, we thus need to decide how to arrange the letters of

$\{\{w^{(1)}\}\} \setminus \{w^{(2)}_{i_1}, \dots, w^{(2)}_{i_j}\}$ in the words z_j. Obviously, all letters of z_j must be greater than or equal to $w^{(2)}_{i_j}$.

We claim that each z_j is nondecreasing: indeed, any descent of z_j would give rise to a pattern *aba* or *acb*, with first letter $a = w^{(2)}_{i_j}$ at position i_j, and the other two in z_j. So it suffices to determine, inductively on j, which letters of $\{\{w^{(1)}\}\} \setminus \{\{w^{(2)}_{i_1} z_1 \cdots z_{j-1} w^{(2)}_{i_j}\}\}$ go into z_j. We claim that they are the $|z_j|$ smallest, among those that are larger than or equal to $w^{(2)}_{i_j}$. Indeed, imagine that we leave out one of them, say m. Then the final letter M of z_j is larger than m, and m occurs later in w. But then the subword $w^{(2)}_{i_j} M m$ of w has shape *aba* or *acb*. □

Example 12. Take $w = 3\ 2\ 2\ 2\ 2\ 5\ 1\ 1\ 1\ 5$, which avoids *aba* and *acb*. Then $w^{(1)} := \mathtt{NInc}(w) = 5\ 5\ 3\ 2\ 2\ 2\ 2\ 1\ 1\ 1$ and $w^{(2)} := \mathtt{Low}(w) = 3\ 2\ 2\ 2\ 2\ 2\ 1\ 1\ 1\ 1$. To reconstruct w from these two words, we first collect the left-to-right minima of $w^{(2)}$ and leave them in place:

$$w = 3\ 2\ \_\ \_\ \_\ \_ 1 \_\ \_\ \_.$$

The multiset of letters of $w^{(1)}$ that need to fill the gaps is $\{\{5, 5, 2, 2, 2, 1, 1\}\}$. We fill the first gap z_1, in nondecreasing order, with the 4 smallest of these letters that equal at least 2, namely $2, 2, 2$ and 5:

$$w = 3\ 2\ 2\ 2\ 2\ 5\ 1 \_\ \_\ \_.$$

We finally fill the second gap z_2 with the remaining letters, $1, 1$ and 5:

$$w = 3\ 2\ 2\ 2\ 2\ 5\ 1\ 1\ 1\ 5.$$

We have recovered w.

Remark 13. This construction is reminiscent of, but distinct from, the bijection between permutations avoiding the pattern *abc* and those avoiding *acb* found in [54, Prop. 19].

We now consider *positive* nonincreasing sequences $u = (u_1, \dots, u_n)$ of the Nadeau-Tewari poset $\mathcal{P}_n$ (Definition 8), with n fixed. These sequences can be encoded bijectively by nonincreasing words w on the alphabet $\{1, \dots, n\}$ that contain at least one occurrence of the letter 1 (Figure 7): the word $w = W(u)$ has $u_i - u_{i+1}$ occurrences of the letter $n + 1 - i$, for $1 \leqslant i \leqslant n$, where we take $u_{n+1} = 0$. Graphically, if we represent u by a path with East steps at heights $u_1, \dots, u_n, u_{n+1} = 0$, joined by South steps, then w_i is the number of East steps after the ith South step. It can also be seen as the abscissa of the ith South step, if abscissas are numbered from right to left (Figure 7). Note that $W(u)$ has length u_1. We call it the *vertical encoding* of u. Note also that for two sequences u and v in $\mathcal{P}_n$ such that $u_1 = v_1$, we have $u_i \leqslant v_i$ for all i if and only if $W(v) \leqslant W(u)$ componentwise (so the order is reversed). Of course the sequence u and the word w play essentially symmetric roles. We choose to represent u by a sequence and w by a word to keep them distinct. We denote by $\mathcal{W}_n$ the set of words on the alphabet $\{1, \dots, n\}$ that contain at least one occurrence of the letter 1. Nonincreasing words of $\mathcal{W}_n$ are in bijection with positive sequences of $\mathcal{P}_n$.

Proposition 14. *Fix $n \geqslant 1$ and let $w \in \mathcal{W}_n$. Then $\mathtt{NInc}(w)$ and $\mathtt{Low}(w)$ are nonincreasing words of the same length in $\mathcal{W}_n$. Let u (resp. v) be the positive sequence of $\mathcal{P}_n$ such that $W(u) = \mathtt{NInc}(w)$ (resp.*

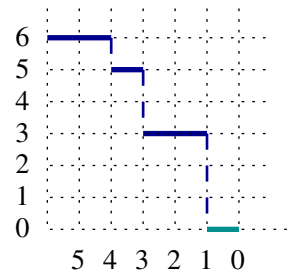


Figure 7: The positive sequence $u = (6,6,5,3,3) \in \mathcal{P}_5$ is encoded by the word $w = 4\ 3\ 3\ 1\ 1\ 1\ \in \mathcal{W}_5$, of length $u_1 = 6$.

$W(v) = \mathtt{Low}(w)$). Then $u_1 = v_1 = |w|$ and $u \trianglelefteq v$ in the Nadeau-Tewari poset $\mathcal{P}_n$. Define $\Phi_n(w)$ to be the interval $[u,v]$.

The restriction of Φ_n to words of $\mathcal{W}_n$ avoiding the patterns aba and acb is a bijection between these words and intervals $[u,v]$ of positive words such that $u_1 = v_1$ in the Nadeau-Tewari poset $\mathcal{P}_n$.

Proof. Let us first observe that, for two positive sequences u and v in $\mathcal{P}_n$ such that $u_1 = v_1$, the condition $u \trianglelefteq v$ translates in terms of the words $w^{(1)} := W(u)$ and $w^{(2)} := W(v)$ as follows: $w^{(2)} \leqslant w^{(1)}$ (componentwise), and the set $\{w^{(2)}\}$ of letters of $w^{(2)}$ is included in $\{w^{(1)}\}$.

Now let $w = w_1 \cdots w_M$ be a word of length M in $\mathcal{W}_n$, and take $w^{(1)} = \mathtt{NInc}(w)$ and $w^{(2)} = \mathtt{Low}(w)$. Write $w^{(k)} = w_1^{(k)} \cdots w_M^{(k)}$ for $k = 1,2$. Given that the letters of $w^{(2)}$ are the values of the left-to-right minima of w, while $w^{(1)}$ is just a reordering of w, we have $\{w^{(2)}\} \subset \{w^{(1)}\}$. Let us now prove that $w^{(2)} \leqslant w^{(1)}$ componentwise (note that for the moment we do not assume that w avoids any pattern). For any letter $a \in \mathbb{N}$, we have, by definition of $w^{(1)} = \mathtt{NInc}(w)$:

$$w_i^{(1)} \geqslant a \Leftrightarrow \sharp\{j : w_j \geqslant a\} \geqslant i.$$

Hence $w_i^{(2)} \leqslant w_i^{(1)}$ if and only if at least i letters of w are larger than or equal to $w_i^{(2)}$. Assume that $m \leqslant i < m'$, where m and m' are the positions of two consecutive left-to-right minima of w. Then $w_i^{(2)} = w_m$, and by definition of $w^{(2)} = \mathtt{Low}(w)$, the letters $w_1, \ldots, w_{m'-1}$ are larger than or equal to w_m. Since there are $m'-1$ of them, and $i \leqslant m'-1$, this proves that $w_i^{(2)} \leqslant w_i^{(1)}$. We conclude that the positive sequences u and v of $\mathcal{P}_n$ given by $W(u) = w^{(1)}$ and $W(v) = w^{(2)}$ form an interval for the NT order. They also satisfy $u_1 = v_1 = M$. These properties hold in particular when w avoids the two forbidden patterns. Moreover, the map Φ_n is injective on those words, by Lemma 11.

Let us now prove surjectivity. Let $u \trianglelefteq v$ in $\mathcal{P}_n$, with u and v positive and $u_1 = v_1 := M$. Let $w^{(1)} = W(u)$ and $w^{(2)} = W(v)$. As already observed, $u \trianglelefteq v$ means that $w^{(2)} \leqslant w^{(1)}$ (componentwise) and $\{w^{(2)}\} \subset \{w^{(1)}\}$. The proof of Lemma 11 tells us how to reconstruct a word w that avoids the two patterns and satisfies $\Phi(w) = (u,v)$ — if such a word exists! So let us try to apply this construction to $w^{(1)}$ and $w^{(2)}$: we keep the left-to-right minima of $w^{(2)}$, denoted $w_{i_j}^{(2)}$, in place, and fill the gaps z_j with the remaining letters of $w^{(1)}$. Let us first explain that the construction succeeds, that is, produces a word w. We will then explain why w does not contain the forbidden patterns. The only thing that could go wrong in the construction of w is that we could run out of letters when filling, with letters of $\{\{w^{(1)}\}\} \setminus \{\{w_{i_1}^{(2)} z_1 \cdots z_{j-1} w_{i_j}^{(2)}\}\}$, the gap following the jth

left-to-right minimum of w. This only happens if the number of letters of $w^{(1)}$ that are larger than or equal to $w^{(2)}_{i_j}$ is less than $i_{j+1}-1$ (with $i_{j+1}=M+1$ if $w^{(2)}_{i_j}$ is the rightmost left-to-right minimum). However, the number of such letters in $w^{(2)}$ is precisely $i_{j+1}-1$, and $w^{(2)} \leqslant w^{(1)}$ componentwise: hence there are at least $i_{j+1}-1$ letters larger than or equal to $w^{(2)}_{i_j}$ in $w^{(1)}$, and one is never stuck in the construction of w. Finally, w avoids *aba* and *acb*: indeed, if there was one of these patterns in w, there would be one where the first a is occurs at a left-to-right minimum, hence $a = w^{(2)}_{i_j}$: but the construction has been designed to guarantee that there is no $w_k > w_\ell \geqslant a$ with $i_j < k < \ell$. Hence the map Φ_n is surjective, and the proposition is proved. □

Example 15. Take $n = 6$, and choose $w = 3\ 2\ 2\ 2\ 2\ 5\ 1\ 1\ 1\ 5$ as in Example 12. Then $w^{(1)} := \mathtt{NInc}(w) = 5\ 5\ 3\ 2\ 2\ 2\ 2\ 1\ 1\ 1$ and $w^{(2)} := \mathtt{Low}(w) = 3\ 2\ 2\ 2\ 2\ 2\ 1\ 1\ 1\ 1$. The positive nonincreasing sequences u and v of $\mathcal{P}_n$ such that $W(u) = w^{(1)}$ and $W(v) = w^{(2)}$ are $u = (10,10,8,8,7,3)$ and $v = (10,10,10,10,9,4)$, and they form an interval in $\mathcal{P}_n$ (Figure 8).

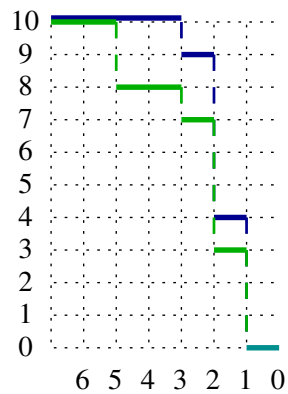


Figure 8: The sequences $u = (10,10,8,8,7,3)$ and $v = (10,10,10,10,9,4)$ form an interval in $\mathcal{P}_6$ such that $u_1 = v_1$. Their vertical encodings are $w^{(1)} = 5\ 5\ 3\ 2\ 2\ 2\ 2\ 1\ 1\ 1$ and $w^{(2)} = 3\ 2\ 2\ 2\ 2\ 2\ 1\ 1\ 1\ 1$, respectively, and the associated pattern avoiding word is $w = 3\ 2\ 2\ 2\ 2\ 5\ 1\ 1\ 1\ 5$ (Example 15).

It follows from the above proposition that, given a positive sequence $u^{(0)}$ of $\mathcal{P}_n$, the map $\Psi_n := \Phi_n^{-1}$ sends bijectively intervals $[u,v]$ of $\mathcal{P}_n$ such that $u_1 = v_1 = u^{(0)}_1$ and $u_i \geqslant u^{(0)}_i$ for all $i \leqslant n$ onto

- words w of $\mathcal{W}_n$ avoiding *aba* and *acb* and satisfying $\mathtt{NInc}(w) \leqslant W(u^{(0)})$, componentwise,
- or equivalently sylvester classes of words w of $\mathcal{W}_n$ such that $\mathtt{NInc}(w) \leqslant W(u^{(0)})$.

Returning to Proposition 10, we can use this to exhibit families of words (or sylvester classes) in bijection with intervals of $\mathbb{D}_{m,n}$ and $\mathbb{D}'_{m,n}$.

Corollary 16. *Upon encoding m-Dyck paths and mirrored m-Dyck paths by nonincreasing sequences (see Section 2.4), the maps Ψ_n and Ψ_{mn} induce respectively bijections between*

- *intervals of* $\mathbb{D}_{m,n}$ *and sylvester classes of words* w *of* $\mathcal{W}_n$ *of length* mn *such that* $\mathtt{NInc}(w) \leqslant n^m(n-1)^m \cdots 1^m$*, where* i^m *represents a sequence of* m *copies of* i*;*
- *intervals of* $\mathbb{D}'_{m,n}$ *and sylvester classes of words* w *of* $\mathcal{W}_{mn}$ *of length* n *such that* $\mathtt{NInc}(w) \leqslant ((n-1)m+1)\cdots(2m+1)(m+1)1$.

Note that the two statements coincide when $m = 1$. The second one explains why we found the sequences $(g'_m(n))_{n>0}$ counting intervals of $\mathbb{D}'_{m,n}$, for $1 \leqslant m \leqslant 5$, in the OEIS: they appear at the very end of [49] in Table 21, as counting sylvester classes of m-parking functions (of size n). These functions are defined in [49, Sec. 6.1] as positive words w of length n such that $\mathtt{NDec}(w) \leqslant 1(m+1)\cdots((n-1)m+1)$, which is equivalent to saying that $\mathtt{NInc}(w) \leqslant ((n-1)m+1)\cdots(m+1)1$ as above. The sylvester classes involved in the first part of the corollary do not seem to have been considered so far.

Example 17. Let us take $m = 1$ and $n = 4$, and the interval $[P, Q]$ given by $P = UDUUDDUD$ and $Q = UUUDDUDD$. We encode P and Q by nonincreasing sequences u and v as described in Section 2.4, with $u = (4,3,3,1)$ and $v = (4,4,4,2)$. Note that $u \trianglelefteq v$ for the NT order, as used in the proof of Proposition 10. The vertical encodings of u and v are $W(u) = 4\ 2\ 2\ 1$ and $W(v) = 2\ 2\ 1\ 1$. The unique word w avoiding *aba* and *acb* such that $\mathtt{NInc}(w) = W(u)$ and $\mathtt{Low}(w) = W(v)$ is $w = 2\ 2\ 1\ 4$ and is the canonical representative of the sylvester class associated with $[P, Q]$.

Remark 18. Words avoiding the patterns *aba* and *acb* have recently been considered for their own sake. See [56, Sec. 4.7], where a formula is given for the number of such words of length n on a k letter alphabet. However, the underlying recursive construction does not seem compatible with the conditions of Corollary 16.

4 Recursive construction of intervals in $\mathbb{D}_{m,n}$ and $\mathbb{D}'_{m,n}$

In this section, we describe recursive constructions of the intervals of the posets $\mathbb{D}_{m,n}$ and $\mathbb{D}'_{m,n}$, and convert them into functional equations for the associated generating functions.

4.1 Intervals in $\mathbb{D}_{m,n}$

Lemma 19. *Let* $m, n \geqslant 1$*, and let* $[P', Q']$ *be an interval of* $\mathbb{D}_{m,n}$*. Delete in* P' *and* Q' *the final (large) peak* $\boldsymbol{U}D^m$*, to obtain two paths* P *and* Q *of* $\mathbb{D}_{m,n-1}$*. Then they form an interval, that is,* $P \leqslant Q$*.*

Conversely, start from an interval $[P, Q]$ *in* $\mathbb{D}_{m,n-1}$*. Let* $a \leqslant b$ *be the lengths of the final descents of* P *and* Q*, respectively. Inserting final large peaks* $\boldsymbol{U}D^m$ *in the final descents of* P *and* Q*, starting at respective heights* $a' \in [\![0, a]\!]$ *and* $b' \in [\![0, b]\!]$*, yields an interval* $[P', Q']$ *if and only if* $a' \leqslant b'$ *and* b' *is maximal as soon as* a' *is maximal. That is,* $a' = a$ *implies* $b' = b$*.*

Proof. The first part directly follows from Proposition 4: deleting the final large peak preserves the relation "lying below", and subtracts m from the final part of the ascent composition (parts of size 0 are then discarded), thus preserving refinement.

Conversely, inserting new large peaks in the final descents of P and Q, at heights a' and b' respectively, preserves the relation "lying below" if and only if $a' \leqslant b'$. This operation adds to $c(P)$ a new part equal to m if $a' < a$, and adds m to the final part of $c(P)$ if $a' = a$. An analogous statement holds for Q. Since we need $c(P')$ to refine $c(Q')$, these observations imply the final condition of the lemma. □

Let $G_m(t;x,y)$ be the generating function of intervals $[P,Q]$ in the ascent lattices $\mathbb{D}_{m,n}$, counted by the size $n>0$ (variable t), the length of the final descent of P (variable x) and of Q (variable y). Then $G_m(t;x,y)$ are formal power series in t whose coefficients are polynomials in x and y. For $m=1,2$, they start as follows:

$$G_1(t;x,y) = xyt + xy\,(xy+y+1)\,t^2 + xy\Big(x^2y^2+2x\,y^2+2xy+2y^2+3y+3\Big)t^3 + \mathcal{O}(t^4),$$

$$\begin{aligned} G_2(t;x,y) = x^2y^2t + x^2y^2\Big(x^2y^2+xy^2+xy+y^2+y+1\Big)t^2 + x^2y^2\Big(x^4y^4+2x^3y^4+2x^3y^3+3x^2y^4 \\ +4x^2y^3+3xy^4+4x^2y^2+5xy^3+3y^4+6xy^2+5y^3+6xy+6y^2+6y+6\Big)t^3 + \mathcal{O}(t^4). \end{aligned}$$

Clearly, the exponent of y is at least equal to the exponent of x in each monomial of $G_m(t;x,y)$, and in addition all terms in G_m are multiples of tx^my^m. So we define a new series $Q_m(t;x,y)$ by

$$G_m(t;x,y) = tx^my^mQ_m(t;xy,y), \quad \text{or equivalently} \quad Q_m(t;x,y) = G_m(t;x/y,y)/(tx^m). \tag{4}$$

Proposition 20. *For $m \geqslant 1$, the generating function $G_m(t;x,y) \equiv G_m(x,y)$ is characterized by the following functional equation:*

$$G_m(x,y) = tx^my^m + tx^my^mG_m(x,y) + tx^m\,y^{m+1}\,\frac{G_m(x,y)-G_m(1,y)}{(x-1)(y-1)} - tx^my^m\,\frac{G_m(xy,1)-G_m(1,1)}{(y-1)(xy-1)}.$$

Equivalently, $Q_m(t;x,y) \equiv Q_m(x,y)$ is characterized by

$$Q_m(x,y) = 1 + tx^mQ_m(x,y) + ty^2\,\frac{x^mQ_m(x,y)-y^mQ_m(y,y)}{(x-y)(y-1)} - t\,\frac{x^mQ_m(x,1)-Q_m(1,1)}{(x-1)(y-1)}.$$

Proof. We construct intervals of $\mathbb{D}_{m,n}$ recursively on the size n, starting from the only interval of size 1, namely $[\boldsymbol{U}D^m,\boldsymbol{U}D^m]$, and inserting a final peak as in Lemma 19. We use the notation of this lemma. The rule that describes the final descent lengths of paths $P' \leqslant Q'$ obtained from $[P,Q] \in \mathbb{D}_{m,n-1}$ in terms of the final descent lengths a and b of P and Q is

$$(a,b) \to \begin{cases} (m+a',m+b'), & \text{for } \ 0 \leqslant a' < a \ \text{ and } \ a' \leqslant b' \leqslant b, \\ (m+a,m+b). \end{cases} \tag{5}$$

In other words, intervals of $\mathbb{D}_{m,n}$ are in bijection with nodes at height $(n-1)$ in the *generating tree* having root (m,m) and the above rewriting rule (see for instance [61, 13] for the hopefully intuitive notion of generating trees). In terms of generating functions, if we write $G_m(x,y) = \sum_{0\leqslant a\leqslant b} G_{m,a,b}x^ay^b$, so that $G_{m,a,b}$ is the series (in t) counting intervals with final descent lengths a and b, the above construction gives:

$$\begin{aligned} G_m(x,y) &= tx^my^m + t\sum_{0\leqslant a\leqslant b} G_{m,a,b}\left(x^{m+a}y^{m+b} + \sum_{a'=0}^{a-1}\sum_{b'=a'}^{b} x^{m+a'}y^{m+b'}\right) \\ &= tx^my^m + tx^my^mG_m(x,y) + tx^my^m\sum_{0\leqslant a\leqslant b} G_{m,a,b}\sum_{a'=0}^{a-1} x^{a'}\cdot\frac{y^{b+1}-y^{a'}}{y-1} \\ &= tx^my^m + tx^my^mG_m(x,y) + tx^my^m\sum_{0\leqslant a\leqslant b} G_{m,a,b}\,\frac{x^a-1}{x-1}\,\frac{y^{b+1}}{y-1} \end{aligned}$$

$$- tx^m y^m \sum_{0 \leqslant a \leqslant b} G_{m,a,b} \frac{x^a y^a - 1}{(xy-1)(y-1)},$$

which is the announced equation for G_m. It is straightforward to convert it into an equation for Q_m, using (4). □

The rewriting rule (5) also describes the construction of certain lattice walks confined to the first quadrant of the plane, which are thus in bijection with ascent intervals. This will be combined in Section 6 to general probabilistic results on quadrant walks to obtain the asymptotic result of Proposition 2.

Corollary 21. *Let $m, n \geqslant 1$. There is a bijection between intervals in $\mathbb{D}_{m,n}$ and walks in the quarter plane $\mathbb{N}^2$ that start from $(0,0)$ and consist of $n-1$ steps taken from the following subset of $\mathbb{Z}^2$:*

$$\mathcal{S}_m = \{(m,0)\} \cup \Big\{(\delta_x, \delta_y) : \delta_x < m \text{ and } \delta_x + \delta_y \leqslant m\Big\}.$$

More precisely, a walk ending at (i,j) corresponds to an interval $[P,Q]$ where P and Q have final descent lengths $m+i$ and $m+i+j$, respectively.

This also yields a bijection between intervals in $\mathbb{D}_{m,n}$ and quadrant walks of length n starting and ending at $(0,0)$, still taking their steps in $\mathcal{S}_m$. We call such walks excursions.

Proof. There are two (essentially equivalent) ways of proving this result.

The first one starts from the rewriting rule (5) and rewrites the label (a,b) as $(m+i, m+i+j)$, so that the conditions $m \leqslant a \leqslant b$ become $i, j \geqslant 0$. The transformed tree has root $(0,0)$ and rewriting rule

$$(i,j) \to \begin{cases} (k,\ell), & \text{for } 0 \leqslant k < m+i \text{ and } 0 \leqslant \ell \leqslant m+i+j-k, \\ (m+i, j). \end{cases} \tag{6}$$

In other words, from the point (i,j) in the nonnegative quadrant $\mathbb{N}^2$, we can move to another point (k,ℓ) of the quadrant by appending a step $(\delta_x, \delta_y) = (k-i, \ell-j)$ satisfying

$$(\delta_x, \delta_y) = (m,0) \quad \text{or} \quad \Big(\delta_x < m \text{ and } \delta_x + \delta_y \leqslant m\Big), \tag{7}$$

which is indeed the collection of steps $\mathcal{S}_m$. The second statement follows by adding to a walk of length $n-1$ ending at (i,j) the step $(-i,-j)$, which is indeed in $\mathcal{S}_m$.

The second way to prove the corollary is to start from the description of the quadrant walk and to recover the equation on Q_m obtained in Proposition 20. Since handling quadrant walks with infinitely many steps is uncommon in the quadrant walks literature, let us do this. Let us denote by $Q(t;x,y) \equiv Q(x,y) = \sum_{i,j \geqslant 0} Q_{i,j} x^i y^j$ the generating function of quadrant walks with steps in $\mathcal{S}_m$ (starting from $(0,0)$), counted by the length (i.e., the number of steps; variable t), and the coordinates of the endpoint (variables x and y). We construct walks step-by-step, using the description (6) of the endpoint rather than the description (7) of the steps:

$$\begin{aligned} Q(x,y) &= 1 + tx^m Q(x,y) + t \sum_{i,j \geqslant 0} Q_{i,j} \left(\sum_{k=0}^{m+i-1} \sum_{\ell=0}^{m+i+j-k} x^k y^\ell \right) \\ &= 1 + tx^m Q(x,y) + t \sum_{i,j \geqslant 0} Q_{i,j} \left(\sum_{k=0}^{m+i-1} x^k \cdot \frac{y^{m+i+j-k+1} - 1}{y-1} \right) \\ &= 1 + tx^m Q(x,y) + \frac{t}{y-1} \sum_{i,j \geqslant 0} Q_{i,j} \left(y^{m+i+j+1} \frac{(x/y)^{m+i} - 1}{x/y - 1} - \frac{x^{m+i} - 1}{x-1} \right), \end{aligned}$$

which gives the equation obtained in Proposition 20 for the series Q_m. □

4.2 Intervals in $\mathbb{D}'_{m,n}$

Let us now turn our attention to mirrored m-Dyck paths. First, observe that the final descent $\boldsymbol{D}^h$ of such a path is not necessarily preceded by at least m up steps, so there is not always a final (large) peak. It makes sense to consider instead the *first* (large) peak $U^m\boldsymbol{D}$, which always exists. We will use a new parameter on intervals: when $P \leqslant Q$, with associated ascent compositions $c(P) = (c_1, c_2, \ldots)$ and $c(Q) = (d_1, d_2, \ldots)$, we know that $c(P)$ refines $c(Q)$. We define $r(P,Q)$ to be the unique integer r such that $c_1 + \cdots + c_r = d_1$.

Lemma 22. *Let $m, n \geqslant 1$, and let $[P', Q']$ be an interval of $\mathbb{D}'_{m,n}$. Delete in P' and Q' the first (large) peak $U^m\boldsymbol{D}$, to obtain two paths P and Q of $\mathbb{D}'_{m,n-1}$. Then they form an interval, that is, $P \leqslant Q$.*

Conversely, start from an interval $[P,Q]$ in $\mathbb{D}'_{m,n-1}$. Let $c(P) = (c_1, c_2, \ldots)$ and $c(Q) = (d_1, d_2, \ldots)$ be the corresponding ascent compositions. The length of the first ascent in P (resp. Q) is thus c_1 (resp. d_1). Let $r = r(P,Q)$. Inserting a first large peak $U^m\boldsymbol{D}$ in the first ascents of P and Q, starting at heights $a' \in [\![0, c_1]\!]$ and $b' \in [\![0, d_1]\!]$ respectively, yields an interval if and only if

- *either b' is one of $c_1, c_1 + c_2, \ldots, c_1 + \cdots + c_r = d_1$ and a' is any element of $[\![0, c_1]\!]$,*
- *or $0 \leqslant a' = b' < c_1$.*

Proof. The first part follows from Proposition 4: when deleting the first peaks, one just subtracts m from the first part in the ascent compositions of P' and Q'.

Conversely, inserting new large peaks in the first ascents of P and Q at heights a' and b' respectively, preserves the relation "lying below" if and only if $a' \leqslant b'$. We want to choose a' and b' so that the resulting paths P' and Q' are such that $c(P')$ refines $c(Q')$. Recall that this means that every descent of Q' lies on the same line (of slope -1) as a descent of P'. This gives the conditions stated in the lemma; see Figure 9. □

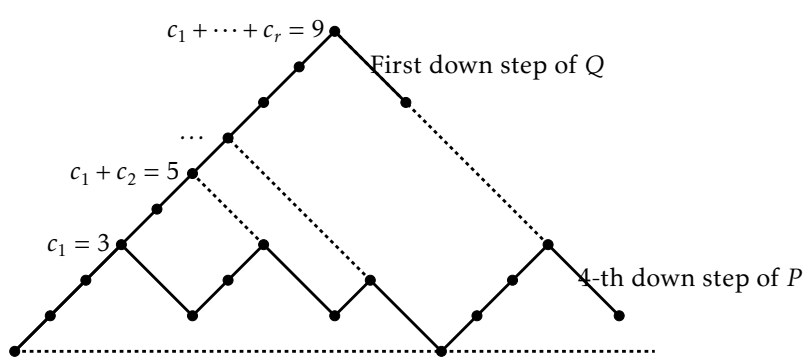


Figure 9: An example where $r(P,Q) = 4$ for an interval $[P,Q]$ in $\mathbb{D}'_{2,n-1}$. A first peak can be inserted in Q at heights $b' = 0, 1, \ldots, c_1, c_1 + c_2, \ldots, c_1 + \cdots + c_r$. If $b' < c_1$, then the peak inserted in P must start at height b' as well.

Let $G'_m(t; x, y)$ be the generating function of intervals $[P,Q]$ in the ascent lattice $\mathbb{D}'_{m,n}$, counted by the size n (variable t), the length of the first ascent of P (variable x) and the statistic $r(P,Q)$

(variable y). Then $G'_m(t;x,y)$ is a formal power series in t whose coefficients are polynomials in x and y. For $m=1,2$, they start as follows:

$$G'_1(t;x,y) = xyt + xy\,(x+y+1)\,t^2 + xy\left(x^2+2xy+y^2+3x+3y+3\right)t^3 \\ + xy\left(x^3+3x^2y+3xy^2+y^3+6x^2+10xy+6y^2+13x+13y+13\right)t^4 + \mathcal{O}\left(t^5\right),$$

$$G'_2(t;x,y) = x^2yt + x^2y\left(x^2+xy+x+y+1\right)t^2 + \\ + x^2y\left(x^4+2x^3y+x^2y^2+3x^3+4x^2y+2xy^2+5x^2+5xy+2y^2+5x+5y+5\right)t^3 + \mathcal{O}(t^4).$$

Note that $G'_1(t;x,y)$ does not coincide with $G_1(t;x,y)$, except at the point $x=y=1$, since these series record different statistics. We will see below that $G'_1(t;x,y)$ is symmetric in x and y, a result that calls for a direct combinatorial explanation.

Since all monomials are multiples of tx^my, we introduce the series $Q'_m(t;x,y)$ defined by

$$G'_m(t;x,y) = tx^m y Q'_m(t;x,y). \tag{8}$$

Proposition 23. *The generating function $G'_m(t;x,y) \equiv G'_m(x,y)$ is characterized by the following functional equation:*

$$G'_m(x,y) = tx^my + tx^my\frac{G'_m(x,y)-G'_m(x,1)}{y-1} + tx^m y^2\,\frac{G'_m(x,y)-G'_m(1,y)}{(x-1)(y-1)} - tx^my\,\frac{G'_m(x,1)-G'_m(1,1)}{(x-1)(y-1)}.$$

Equivalently, $Q'_m(t;x,y) \equiv Q'_m(x,y)$ is characterized by

$$Q'_m(x,y) = 1 + tx^m\frac{yQ'_m(x,y)-Q'_m(x,1)}{y-1} + ty^2\,\frac{x^mQ'_m(x,y)-Q'_m(1,y)}{(x-1)(y-1)} - t\,\frac{x^mQ'_m(x,1)-Q'_m(1,1)}{(x-1)(y-1)}.$$

Remark 24. When $m=1$, the equation for G'_1 can be rewritten as

$$G'_1(x,y) = txy + txy\frac{(x+y-1)G'_1(x,y)-xG'_1(x,1)-yG'_1(1,y)+G(1,1)}{(x-1)(y-1)}.$$

It is symmetric in x and y, and characterizes $G'_1(x,y)$: we conclude that this series is symmetric in x and y. Since the equation reflects a (symmetric) recursive construction of intervals, it can be used to define a recursive involution on ascent intervals that exchanges the length of the first ascent of the bottom path and the parameter $r(\cdot,\cdot)$. See Section 7 for details. Note that another symmetry property was discovered in Tamari intervals [18], and explained in [27, 50].

Proof of Proposition 23. We construct intervals of $\mathbb{D}'_{m,n}$ recursively in the size n, starting from the only interval of size 1, namely $[U^m\boldsymbol{D}, U^m\boldsymbol{D}]$, and inserting peaks as in Lemma 22. We use the notation of this lemma. Let us examine the value of the parameter $r(P',Q')$, for paths $P' \leqslant Q'$ obtained from $[P,Q] \in \mathbb{D}'_{m,n-1}$:

- if $b' = c_1 + \cdots + c_s$, then $r(P',Q') = s$ if $a' = c_1$, and $r(P',Q') = s+1$ otherwise,
- if $b' = a' < c_1$, then $r(P',Q') = 1$.

Hence, if we record for each interval $[P,Q]$ the pair $(c_1(P), r(P,Q))$, we obtain a generating tree with root labelled $(m,1)$ and rewriting rule

$$(a,r) \to \begin{cases} (m+a,s), & \text{for } 1 \leqslant s \leqslant r, \\ (m+a',s+1), & \text{for } 0 \leqslant a' < a \text{ and } 1 \leqslant s \leqslant r, \\ (m+a',1), & \text{for } 0 \leqslant a' < a. \end{cases} \tag{9}$$

Note that the last two lines can be merged into a single one, by allowing s to be 0.

In terms of generating functions, if we write $G'_m(x,y) = \sum_{a,r} G'_{m,a,r} x^a y^r$, the above construction gives:

$$\begin{aligned} G'_m(x,y) &= tx^m y + t \sum_{a,r} G'_{m,a,r} \left(x^{m+a} \sum_{s=1}^{r} y^s + \sum_{a'=0}^{a-1} \sum_{s=0}^{r} x^{m+a'} y^{s+1} \right) \\ &= tx^m y + tx^m \sum_{a,r} G'_{m,a,r} \left(x^a \cdot \frac{y^{r+1}-y}{y-1} + \frac{x^a-1}{x-1} \cdot \frac{y^{r+2}-y}{y-1} \right), \end{aligned}$$

which gives the first equation of Proposition 23. It is straightforward to convert it into an equation for Q'_m, using (8). □

Corollary 25. *Let $m,n \geqslant 1$. There is a bijection between intervals in $\mathbb{D}'_{m,n}$ and walks in the quarter plane $\mathbb{N}^2$ that start from $(0,0)$ and consist of $n-1$ steps taken from*

$$\mathcal{S}'_m = \Big(\{m\} \times (\!|-\infty, 0]\!]\Big) \cup \Big((\!|-\infty, m-1]\!] \times (\!|-\infty, 1]\!]\Big),$$

where $(\!|-\infty, a]\!] := \mathbb{Z} \cap (-\infty, a]$. Moreover, a walk ending at (i,j) corresponds to an interval $[P,Q]$ where P has a first ascent of length $m+i$ and $r(P,Q) = 1+j$.

This also yields a bijection between intervals in $\mathbb{D}'_{m,n}$ and quadrant walks of length n starting and ending at $(0,0)$, still taking their steps in $\mathcal{S}'_m$.

Proof. Let us start from the rewriting rule (9) and rewrite the label (a,r) as $(m+i, 1+j)$, so that the conditions $m \leqslant a, 1 \leqslant r$ become $i,j \geqslant 0$. The transformed tree has root $(0,0)$ and rewriting rule

$$(i,j) \to \begin{cases} (m+i,\ell) & \text{for } \ 0 \leqslant \ell \leqslant j \\ (k,\ell) & \text{for } \ 0 \leqslant k < i+m \ \text{ and } \ 0 \leqslant \ell \leqslant 1+j. \end{cases} \tag{10}$$

In other words, from the point (i,j) in the quadrant, we can move to another point (k,ℓ) of the quadrant by appending a step $(\delta_x, \delta_y) = (k-i, \ell-j)$ satisfying

$$\Big(\delta_x = m \text{ and } \delta_y \leqslant 0\Big) \quad \text{or} \quad \Big(\delta_x < m \text{ and } \delta_y \leqslant 1\Big), \tag{11}$$

which is indeed the collection of steps $\mathcal{S}'_m$. The second statement follows by adding to a walk of length $n-1$ ending at (i,j) the step $(-i,-j)$, which is indeed in $\mathcal{S}'_m$. □

There is an alternative description of the series $G'_m(1,1)$ in terms of quadrant walks involving *finitely many weighted steps*, or equivalently steps from a *finite multiset*.

Corollary 26. *Let $m,n \geqslant 1$. The number of intervals in the poset $\mathbb{D}'_{m,n}$ is the number of quadrant walks of length $n-1$ starting and ending at the origin, for which the multiset $\overline{\mathcal{S}}_m$ of allowed steps has generating polynomial*

$$\overline{S}_m(u,v) := \sum_{(a,b) \in \overline{\mathcal{S}}_m} u^a v^b = \frac{(1+u)^m (1+v)(1+u+v)}{uv}.$$

Proof. Let us introduce a new trivariate series $R_m(t;u,v) \equiv R_m(u,v)$ defined by $R_m(u,v) := Q'_m(1+u,1+v)$. Observe in particular that $R_m(0,0) = Q'_m(1,1) = G'_m(1,1)/t$ is the generating function of intervals in the posets $\mathbb{D}'_{m,n}$, where t records the size of the paths, minus 1. The equation of Proposition 23 gives:

$$R_m(u,v) = 1+t\frac{(1+u)^m(1+v)(1+u+v)}{uv}R_m(u,v)-t\frac{(1+v)^2}{uv}R_m(0,v)-t\frac{(1+u)^{m+1}}{uv}R_m(u,0)+t\frac{R(0,0)}{uv}.$$

This equation precisely describes (see [9]) the generating function of quadrant walks with steps in $\overline{\mathcal{S}}_m$ starting from the origin, where t records the length (number of steps), and (u,v) the final coordinates. □

In particular, for $m = 1$ the multiset of allowed steps has generating polynomial

$$\overline{S}_1(u,v) = 3 + \frac{1}{uv} + \frac{2}{u} + \frac{2}{v} + \frac{u}{v} + \frac{v}{u} + u + v,$$

and, up to the (harmless) constant term 3, coincides with the steps of an already solved quadrant model [8, Sec. A.7]. In the next section, we take advantage of this earlier result to prove Theorem 1. We also explain how the principles of [8] can be used to solve, instead of the equation obtained for $R_1(u,v)$ (or equivalently, for $G'_1(x,y)$), the equation obtained in Proposition 20 for $G_1(x,y)$.

5 Exact enumeration of intervals in $\mathbb{D}_n$

In this section we focus on the case $m = 1$, and drop all indices m in our generating functions, writing for instance $G(x,y)$ instead of $G_1(x,y)$. Our aim is to prove, and actually refine, Theorem 1 that gives the size generating function of intervals in the lattices $\mathbb{D}_n$. We can choose between two starting points to address this question. Either we start from the equation for $Q(x,y) = G(x/y,y)/(tx)$ in Proposition 20:

$$Q(x,y) = 1 + txQ(x,y) + ty^2\,\frac{xQ(x,y)-yQ(y,y)}{(x-y)(y-1)} - t\,\frac{xQ(x,1)-Q(1,1)}{(x-1)(y-1)}, \tag{12}$$

or from the equation for $Q'(x,y) = G'(x,y)/(txy)$ in Proposition 23:

$$Q'(x,y) = 1 + tx\frac{yQ'(x,y)-Q'(x,1)}{y-1} + ty^2\,\frac{xQ'(x,y)-Q'(1,y)}{(x-1)(y-1)} - t\,\frac{xQ'(x,1)-Q'(1,1)}{(x-1)(y-1)}.$$

Recall from Corollary 26 that up to the change of variables $(x,y) \mapsto (1+x,1+y)$, the latter equation is equivalent to

$$R(x,y) = 1 + t\frac{(1+x)(1+y)(1+x+y)}{xy}R(x,y) - t\frac{(1+y)^2}{xy}R(0,y) - t\frac{(1+x)^2}{xy}R(x,0) + t\frac{R(0,0)}{xy}. \tag{13}$$

Again, note that this equation is symmetric in x and y, and hence the same holds for the series $R(x,y)$ and $G'(x,y)$.

In all three cases (and in fact also for $m > 1$) we have a linear equation, relating a main unknown series $F(x,y)$ to some specializations involving at most one of the two *catalytic* variables x

and y: for instance $F(x,1)$, $F(1,1)$, $F(y,y)$ or $F(x,0)$. Historically, the first (non-linear) equations of this type appeared in the seventies in Tutte's work on the enumeration of properly coloured planar maps [58, 57]. He devoted ten years and ten papers to the solution of just one of them (see [60] and references therein). More recently, similar equations appeared in more contexts, like the enumeration of certain classes of permutations [13, 24], or of lattice walks confined to a quadrant [8, 10, 11, 20, 33, 51], or of some classes of maps [1, 6, 7, 23]. The solutions of these equations are not systematically algebraic (nor even D-finite as will be proved here for $m > 1$), but our series Q, Q' and R *will* be proved to be algebraic. Several, often *ad hoc* approaches have been designed to prove algebraicity for such equations [10, 14, 15, 13, 12, 20, 47]. The most systematic one, based on a certain notion of *invariants*, is based on ideas developed by Tutte in his enumeration of coloured maps. In particular, the invariant approach has already solved in [8] an equation that is extremely close to the above equation defining $R(x,y)$, and we will rely on this to solve (13) and determine $G'(x,y)$ (Proposition 27). Next, we will see that an invariant approach can also be applied to determine $Q(x,y)$ (Proposition 28), even if Eq. (12), due to the terms $Q(y,y)$ and the denominator $(x-y)$, looks rather different from (13).

5.1 The first ascent of P, and the statistics $r(P,Q)$

Proposition 27. *The series $G'(t;x,y)$ that counts intervals $[P,Q]$ in the lattices $\mathbb{D}_n$ by the size, the height of the first ascent of P and the statistics $r(P,Q)$ defined above Lemma 22 is symmetric in x and y. It is algebraic of degree 12 over $\mathbb{Q}(t,x,y)$, and can be expressed as follows.*

Let Z be the only series in t with constant term 0 satisfying

$$Z = t(1+Z)(1+2Z)^2.$$

Then the size generating function of intervals in the ascent lattices $\mathbb{D}_n$ is the following cubic series:

$$G'(1,1) = Z(1-2Z+2Z^3). \tag{14}$$

More generally the bivariate series $G'(x,1)$ is

$$G'(x,1) = G'(1,x) = \frac{C_0(x) - C_1(x)\sqrt{\Delta(x)}}{2x^3(1-x)Z^2},$$

where

$$\Delta(x) = (1+Z)^2(1+2Z)^2 - 2Z(Z+1)\left(2Z^2+4Z+1\right)x + Z^2x^2,$$
$$C_1(x) = \left((1+2Z)^2 - 2Z^2x - Zx^2\right)\left((1+Z)(1+2Z) - 2(1+Z)^2x + x^2\right),$$

and

$$C_0(x) = (1+Z)^2(1+2Z)^4 - (1+Z)(1+2Z)^2\left(8Z^3+16Z^2+9Z+2\right)x - Z(1+Z)(1+2Z)^2x^4$$
$$+ Z^2x^5 + (1+Z)\left(18Z^5+46Z^4+48Z^3+25Z^2+7Z+1\right)x^2 - Z\left(2Z^5-6Z^3-7Z^2-3Z-1\right)x^3.$$

An algebraic expression of $G'(x,y)$ in terms of x,y and Z can then be obtained from the functional equation of Proposition 23.

Remarks

1. The above expression of $G'(1,1)$ coincides with the value of $\sum_n g(n)t^n$ given in Theorem 1. Deriving from this the asymptotic behaviour of the numbers $g(n)$ is a routine task, following the principles of *singularity analysis* [39, Sec. VII.7]. The series Z is found to have a unique dominant singularity, of the square root type, located at $t_c := 1/\mu$ where μ is given in Theorem 1. One needs to expand Z around t_c up to the order of $(1-\mu t)^{5/2}$ to obtain from (14) the singular behaviour of $G'(1,1)$, in the form

$$G'(1,1) = c_0 + c_1(1-\mu t) + c_2(1-\mu t)^2 + c_{5/2}(1-\mu t)^{5/2} + \mathcal{O}\Big((1-\mu t)^3\Big).$$

The constant κ of Theorem 1 is then $c_{5/2}/\Gamma(-5/2)$.

2. Since any algebraic series is also D-finite, the series $G'(1,1) = \sum_n g(n)t^n$ also satisfies a linear differential equation. The corresponding linear recurrence relation reads

$$(n+4)(2n+7)g(n+2) = 2\Big(11n^2+44n+42\Big)g(n+1) + n(2n+1)g(n).$$

One may wonder whether this can be explained combinatorially directly on intervals.

3. In the same way we have parametrized rationally t and $G'(1,1)$ by Z, we can write x and $G'(x,1)$ as rational functions in Z and the unique series $U(x)$ with constant term 0 satisfying

$$U(x) = tx(1+U(x))\Big(1+3Z+Z^2+Z(1+Z)U(x)\Big).$$

Note that $U(1) = Z/(1+Z)$. One can then write x rationally in terms of U and Z. In particular, the discriminant $\Delta(x)$ becomes a square, and finally

$$G'(x,1) = G'(1,x) = \frac{tx(1+U)}{(1+2Z)^2(1+3Z+Z^2-Z^2U)}\,P(Z,U)$$

with

$$\begin{aligned} P(Z,U) = {} & -Z^7(1+Z)^3U^4 - Z^5(1+Z)^2\Big(2Z^3+3Z^2-4Z-2\Big)U^3 \\ & + Z^3(1+Z)\Big(13Z^5+49Z^4+62Z^3+24Z^2-1\Big)U^2 \\ & + Z^2\Big(2Z^8+35Z^7+140Z^6+221Z^5+120Z^4-48Z^3-80Z^2-31Z-4\Big)U \\ & + \Big(Z^2+3Z+1\Big)\Big(Z^8+15Z^7+31Z^6+10Z^5-19Z^4-7Z^3+9Z^2+6Z+1\Big). \end{aligned}$$

Proof of Proposition 27. As discussed at the end of Section 4, the series $R(t;x,y) = G'(t;1+x,1+y)/(t(1+x)(1+y))$ counts, by length and final coordinates, quadrant walks with step polynomial

$$\overline{S}(x,y) = 3 + \frac{1}{xy} + \frac{2}{x} + \frac{2}{y} + \frac{x}{y} + \frac{y}{x} + x + y.$$

These walks have already been counted in [8], but without the empty step, which has multiplicity 3 here. This means that the series R is related to the series denoted Q in [8] and Q^{old} here, to avoid confusion, by

$$R(t;x,y) = \frac{1}{1-3t}Q^{\text{old}}\Big(\frac{t}{1-3t};x,y\Big).$$

We now dig in the details of the solution presented in [8, App. A.7]. The series denoted $Z(t)$ in Eq. (A.8) of [8], and $Z^{\text{old}}(t)$ in the present paper, is related to the series $Z(t)$ of the proposition by

$$Z(t) = 2 \times Z^{\text{old}}\left(\frac{t}{1-3t}\right).$$

Using Eq. (A.9) of [8], which gives the expression of $Q^{\text{old}}(t;0,0)$, we then obtain

$$G'(t;1,1) = tR(t;0,0) = \frac{1}{1-3t} Q^{\text{old}}\left(\frac{t}{1-3t};0,0\right) = Z(1-2Z+2Z^3).$$

Then Eq. (A.10) in [8] gives a quadratic equation for $Q^{\text{old}}(t;0,y)$, or equivalently $Q^{\text{old}}(t;y,0)$, with coefficients that are rational expressions in t, y and two series A_1 and A_2, having themselves rational expressions in Z^{old}. From this we derive a quadratic equation for $G'(t;x,1)$ in terms of x and Z (details are given in our Maple session). Solving this equation gives the announced rational expression of $G'(t;x,1)$ in terms of x, Z and $\sqrt{\Delta(x)}$. □

5.2 The finals descents of P and Q

We will now solve the equation (12) defining $Q(x,y)$, and thus count intervals $[P,Q]$ by the heights of the final descents of P and Q. Our solution is inspired from the *invariant approach* used in [8] to solve, among other equations, the equation (13) defining $R(x,y)$ (or more precisely, its variant with no empty step). The proof is detailed in Appendix A. The specialists of equations in two catalytic variables may be interested in the fact that it uses both an *additive* and a *multiplicative decoupling*.

Proposition 28. *The series $G(t;x,y)$ that counts intervals $[P,Q]$ in the lattices $\mathbb{D}_n$ by the size, the height of the last descent of P and the height of the last descent of Q, is algebraic of degree 12 over $\mathbb{Q}(t,x,y)$, and can be expressed as follows.*

Let Z be the only series in t with constant term 0 satisfying

$$Z = t(1+Z)(1+2Z)^2.$$

Then

$$G(x,1) = \frac{C_0(x) - C_1(x)\sqrt{\Delta_1(x)}}{2x^2Z^2},$$

with

$$\Delta_1(x) = (1+Z-xZ)\Big((1+2Z)^2(1+Z) - xZ\Big),$$
$$C_1(x) = (x-1)\Big((1+2Z)^2 - 2xZ\Big)(2(1+Z)(1+2Z) - x)$$

and

$$C_0(x) = -2(1+Z)^2(1+2Z)^4 + 3(1+Z)(1+2Z)^2\Big(4Z^3+8Z^2+6Z+1\Big)x -$$
$$\Big(12Z^6+64Z^5+132Z^4+134Z^3+70Z^2+16Z+1\Big)x^2 + Z\Big(12Z^3+24Z^2+16Z+3\Big)x^3 - 2Z^2x^4.$$

Analogously,

$$G(1,y) = \frac{D_0(y) - D_1(y)\sqrt{\Delta_2(y)}}{2yZ(1-y)^2},$$

with

$$\Delta_2(y) = (1+2Z)^2 - 4yZ(1+Z),$$
$$D_1(y) = (1+2Z-y)(2+2Z-y(1+2Z)),$$

and

$$D_0(y) = -2Z\,y^3 + \Big(4Z^3+8Z^2+10Z+1\Big)y^2 - \Big(12Z^3+24Z^2+16Z+3\Big)y + 2\,(1+2Z)^2\,(1+Z).$$

An algebraic expression of $G(x,y)$ in terms of x, y and Z can then be obtained from the functional equation of Proposition 20.

Remarks
1. Note that the second factor in $\Delta_1(x)$ is simply $Z(1-tx)/t$.
2. Again we have rational parametrisations for the series $G(x,1)$ and $G(1,y)$, by series $U \equiv U(x)$ and $V \equiv V(y)$ (with no constant term in t) satisfying

$$U = \frac{xZ(1+U)}{(1+Z)(1+2Z-Z^2U)}, \quad \text{and} \quad V = \frac{yZ(1+Z)(1+V)^2}{(1+2Z)^2},$$

respectively. Indeed,

$$G(x,1) = U\frac{(1+Z)\,P(Z,U)}{(1+2Z-Z^2U)^2},$$

with

$$\begin{aligned} P(Z,U) &= 2Z^6\,(1+Z)\,U^3 + Z^4\Big(8Z^3+4Z^2-8Z-5\Big)U^2 \\ &\quad + Z^2\Big(6Z^5-6Z^4-20Z^3-5Z^2+9Z+4\Big)U - (1+2Z)\Big(4Z^5+4Z^4-2Z^3-3Z^2+Z+1\Big), \end{aligned}$$

and

$$G(1,y) = V\,\frac{2Z^2\,(1+Z)^2\,V^2 + Z\Big(4Z^3+4Z^2-4Z-3\Big)V + (1+Z)\Big(2Z^3-2Z^2+1\Big)}{(1+Z-ZV)^2}.$$

6 Asymptotic enumeration of intervals

Our aim in this section is to prove Propositions 2 and 3, which give asymptotic estimates for the interval numbers in $\mathbb{D}_{m,n}$ and $\mathbb{D}'_{m,n}$ and establish non-D-finiteness results. The key ingredient is the existence of bijections between intervals and quadrant walks, described in Section 4, combined with general asymptotic results on such walks by Denisov and Wachtel [31] and their application to enumeration by Bostan et al. [11]. So far these results do not seem to have been applied to walks with infinitely many allowed steps, but, as will shall see by following the arguments of [11], this does not raise difficulties as long as the step generating function converges in a sufficiently large domain.

6.1 Intervals of $\mathbb{D}_{m,n}$

Proof of Proposition 2. According to the second part of Corollary 21, the number $g_m(n)$ of intervals in the ascent lattice $\mathbb{D}_{m,n}$ is also the number of quadrant excursions of length n taking their steps in the set $\mathcal{S}_m$. The generating function of this set is

$$S_m(x,y) = x^m + \sum_{i=-\infty}^{m-1} x^i \sum_{j=-\infty}^{m-i} y^j = x^m + \sum_{i=-\infty}^{m-1} x^i \frac{y^{m-i}}{1-1/y} = \frac{x^m(xy-x+y)}{(x-y)(y-1)}.$$

It converges absolutely for $1 < |y| < |x|$. We now follow the probabilistic arguments of [11, Sec. 2.3] (see also [31, Sec. 1.5]). We consider a random walk $(Y_1(n), Y_2(n))_{n\geqslant 0}$ in $\mathbb{Z}^2$, starting at $(0,0)$ and taking its steps in $\mathcal{S}_m$, where each step (i,j) occurs with a probability ${x_0}^i {y_0}^j / S_m(x_0,y_0)$, for some (x_0,y_0) chosen such that $1 < y_0 < x_0$. We moreover require that

$$\frac{\partial S_m}{\partial x}(x_0,y_0) = \frac{\partial S_m}{\partial y}(x_0,y_0) = 0,$$

as this choice guarantees that the walk (Y_1,Y_2) has no drift, that is, the average displacement is zero. This gives

$$x_0 = \frac{2+\sqrt{m^2+4}}{m}, \qquad y_0 = \frac{\sqrt{m^2+4}-m+2}{2} = \frac{m}{2}(x_0-1),$$

and one can check that indeed, $1 < y_0 < x_0$. The next step is to apply to the walk $Y = (Y_1,Y_2)$ a linear transformation so that the covariance matrix of the resulting walk, denoted $Z = (Z_1,Z_2)$, is the identity. As in [11], the image by this transformation of the first quadrant ends up being a wedge W_c of opening $\arccos(-c)$, where

$$c = \frac{\frac{\partial^2 S_m}{\partial x \partial y}}{\sqrt{\frac{\partial^2 S_m}{\partial x^2}\frac{\partial^2 S_m}{\partial y^2}}}(x_0,y_0).$$

This coincides with the value of c given in Proposition 2. Now the probability that the walk (Z_1,Z_2) visits $(0,0)$ at time n without exiting the wedge W_c before is

$$p_m(n) := \frac{g_m(n)}{S_m(x_0,y_0)^n},$$

as each of the corresponding trajectories has probability $1/S_m(x_0,y_0)^n$. Now the random walk Z satisfies the conditions of [31]: its steps are not contained in any (linear) half-plane, it is aperiodic (since the step $(0,0)$ is allowed), has no drift, its covariance matrix is the identity, and there are finite moments of any order. Moreover, the point $(0,0)$ can be reached from infinity (an assumption that seems to be missing in [31], see [9, Sec. 3.3]). By [31, Thm. 6], there exists a positive constant κ such that

$$p_m(n) \sim \kappa n^{-1-\pi/\arccos(-c)}.$$

Combining this with the previous identity, and using

$$\mu := S_m(x_0,y_0) = \frac{m\sqrt{m^2+4}+m^2+2}{2} \cdot \left(\frac{2+\sqrt{m^2+4}}{m}\right)^m,$$

yields the announced asymptotic estimate of $g_m(n)$.

Let us now discuss the implications of this result on the nature of the generating function of these numbers.

When $m = 1$, we obtain $c = (1 - \sqrt{5})/4$, so that the uncorrelated random walk Z lives in a cone of opening $\arccos(-c) = 2\pi/5$, and the exponent α is $-7/2$, as already established in the previous section.

Let us now prove that $m = 1$ is the only integer value of m for which $\pi/\arccos(-c)$ is rational. By [11, Thm. 3], this implies that the series $\sum_n g_m(n)t^n = G_m(1,1)$ is not D-finite when $m > 1$. If $\arccos(-c)$ was a rational multiple of π, say of the form $\pi - \theta$, then we would have $c = \cos\theta = (z + 1/z)/2$ for $z = e^{i\theta}$ a root of unity. Since c is a root of the polynomial

$$P_m(u) := 4\left(m^2+3\right)u^4 - 4\left(m^2+2\right)u^2 + m^2,$$

the two solutions z and $1/z$ of $c = (z + 1/z)/2$ are roots of

$$\overline{P}_m(v) := \left(m^2+3\right)v^8 + 4v^6 + 2\left(m^2+1\right)v^4 + 4v^2 + m^2 + 3.$$

So we only have to prove that this polynomial admits no root of unity — that is, no cyclotomic factor — for $m > 1$. There are exactly 18 cyclotomic polynomials of degree at most 8. The first one is of course $\phi_1(v) = v - 1$, and the last one is $\phi_{30}(v) = v^8 + v^7 - v^5 - v^4 - v^3 + v + 1$. We then take each of these polynomials $\phi(v)$ one by one, and reduce $\overline{P}_m(v)$ modulo $\phi(v)$ to detect if it could be a multiple of $\phi(v)$ for some values of m. But the leading coefficient of the remainder (which is a polynomial in m) never vanishes when m is an integer larger than 1. For instance, for $\phi = \phi_{30}$, we find

$$\overline{P}_m(v) \mod \phi_{30} = -\left(m^2+3\right)z^7 + 4z^6 + \left(m^2+3\right)z^5 + \left(3m^2+5\right)z^4 + \left(m^2+3\right)z^3 + 4z^2 - \left(m^2+3\right)z,$$

and the leading coefficient $-(m^2+3)$ has no integer root. We conclude that no root of $\overline{P}_m(v)$ is a root of unity, so that the exponent α is irrational, and the generating function of the numbers $g_m(n)$ is not D-finite. □

6.2 Intervals of $\mathbb{D}'_{m,n}$

Proof of Proposition 3. We now argue in a similar fashion for mirrored m-Dyck paths. According to the second part of Corollary 25, the number $g'_m(n)$ of intervals in the ascent poset $\mathbb{D}'_{m,n}$ is also the number of quadrant excursions of length n taking their steps in the set $\mathcal{S}'_m$. The generating function of this set is

$$S'_m(x,y) = x^m \sum_{j=-\infty}^{0} y^j + \sum_{i=-\infty}^{m-1} x^i \sum_{j=-\infty}^{1} y^j = \frac{x^m}{1-1/y} + \frac{x^{m-1}}{1-1/x}\cdot\frac{y}{1-1/y} = \frac{x^m y\,(x+y-1)}{(x-1)(y-1)}.$$

It converges absolutely for $1 < |x|$ and $1 < |y|$. The critical point (x_0, y_0) is now

$$x_0 = \frac{2m^2+1+\sqrt{4m^2+1}}{2m^2}, \qquad y_0 = \frac{2m+1+\sqrt{4m^2+1}}{2m} = 1 + m(x_0 - 1),$$

and one can check that indeed, $1 < x_0$ and $1 < x_0$. The opening angle is found to be $\arccos(-c)$, where c takes the value of Proposition 3. One obtains the asymptotic estimate of $g'_m(n)$ as before,

using

$$\mu := S'_m(x_0, y_0) = \left(2m + \sqrt{1+4m^2}\right)\left(\frac{1+\sqrt{1+4m^2}}{2m}\right)^{2m}.$$

It remains to prove that the exponent α is irrational for $m > 1$. The argument is the same as in the previous subsection, with now

$$\overline{P}_m(v) = \left(3m^2+1\right)v^8 + 4m^2v^6 + 2\left(m^2+1\right)v^4 + 4m^2v^2 + 3m^2 + 1.$$

□

Remark 29. One could alternatively establish the asymptotic estimate of $g'_m(n)$ via the bijection with weighted walks mentioned in Corollary 26. Recall that these walks use finitely many steps only.

7 Final comments

Several questions that have been investigated on other Dyck posets may be asked for the ascent poset. For instance, what is the height of a random vertex of P or Q in an interval $[P, Q]$ (see [28])? Can one write a q-analogue of our functional equations recording the size of the longest chain from P to Q (see [18, 21])?

Conversely, it seems to be the first time that an order induced on *mirrored* m-Dyck paths is examined. For the ascent orders as for the other Dyck orders (e.g., Tamari) this seems less natural than studying m-Dyck paths because mirrored paths do not form an upper ideal (while m-Dyck paths do). However, this is how we discovered here the connection with sylvester classes of m-parking functions, and were able to count these, at least asymptotically. Would interesting results arise from other orders?

The most immediate question raised by this paper is probably to explain bijectively, and non-recursively, the symmetry in x and y of the series $G'_1(x, y)$ that counts ascent intervals $[P, Q]$ in $\mathbb{D}_1$ by the height $a(P)$ of the first ascent of P (variable x) and the statistics $r(P, Q)$ defined at the beginning of Section 4.2 (variable y). Recall that this is directly related to the fact that the quadrant walks that encode the recursive construction of these intervals (Corollary 25) have a step set $\mathcal{S}'_1$ that is x/y-symmetric. On these walks, the involution is obvious and consists in a reflection in the first diagonal. In recursive terms, if w' is a quadrant walk and w is the walk obtained by deleting its final step (δ_x, δ_y), then the image of w' by the involution is obtained by appending the step (δ_y, δ_x) at the end of the image of w. Recall the quadrant walk associated with an interval $[P, Q]$ such $a(P) = a$ and $r(P, Q) = r$ ends at $(a-1, r-1)$. Hence we can define an involution f on intervals $[P', Q']$, recursively on their size n as follows:

- $f([UD, UD]) = [UD, UD]$
- for $n > 1$, let $b = a(P')$ and $s = r(P', Q')$, and let $[P, Q]$ be the interval of size $n-1$ obtained by deleting the initial peaks of P' and Q'. Then $f([P', Q'])$ is the only interval $[\bar{P}', \bar{Q}']$ obtained by insertion of peaks in $f([P, Q])$ such that $a(\bar{P}') = s$ and $r(\bar{P}', \bar{Q}') = b$. An example is given is Figure 10.

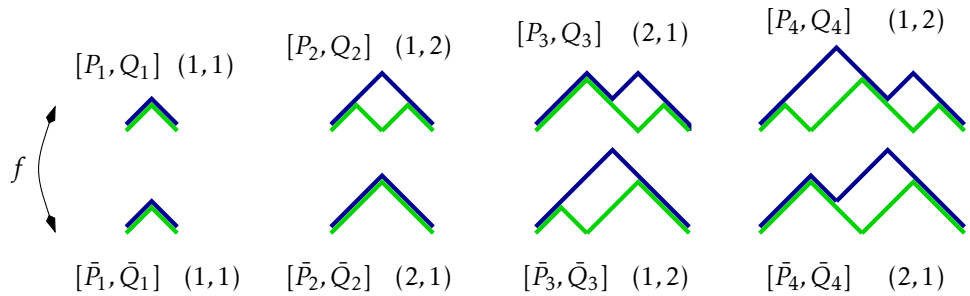


Figure 10: Recursive construction of an interval $[P_4, Q_4]$ in $\mathbb{D}_4$, and of its image $[\bar{P}_4, \bar{Q}_4]$ by the involution f. For each interval $[P, Q]$ we give the statistics $(a(P), r(P, Q))$.

Acknowledgements

We thank Clément Chenevière, Philippe Nadeau, and Benjamin Testart for interesting comments.
MBM was partially supported by the ANR projects DeRerumNatura (ANR-19-CE40-0018), Combiné (ANR-19-CE48-0011), and CartesEtPlus (ANR-23-CE48-0018). JLB and SK were supported in part by ANR PiCs (ANR-22-CE48-0002) and ANER ARTICO funded by Bourgogne-Franche-Comté region.

A Proof of Proposition 28

Here we solve the equation (12) defining $Q(x, y)$, by adapting the *invariant approach* of [8]. All details of the calculations can be followed on an accompanying Maple session, available on the second author's webpage.

In Eq. (12), let us group the terms involving $Q(x, y)$, and multiply out by $(y-1)$. We thus obtain

$$K(x,y)(y-1)Q(x,y) = y - 1 - t\,\frac{xQ(x,1) - Q(1,1)}{x-1} - \frac{ty^3}{x-y}Q(y,y), \tag{15}$$

where

$$K(x,y) = 1 - tx - \frac{txy^2}{(x-y)(y-1)} \tag{16}$$

is the *kernel* of the equation.

A.1 Invariants

The notion of invariants is related to the expansion (in t) of $1/K(x, y)$. Namely,

$$\frac{1}{K(x,y)} = \sum_{n \geqslant 0} t^n \left(x + \frac{xy^2}{(x-y)(y-1)} \right)^n.$$

This is a series in t, with coefficients in $\mathbb{Q}(x, y)$. The denominators of these coefficients are powers of $(x-y)(y-1)$. Seen as fractions in y, they have a pole at x and a pole at 1, and the orders of these two poles increase with the exponent of t. Let us now introduce a notion of series with *poles of bounded order*.

Definition 30. Let $F(t;x,y) = \sum_n t^n f_n(x,y)$ be a Laurent series in t with coefficients in $\mathbb{Q}(x,y)$. We say that F has *poles of bounded order at $y=1$ and $y=x$* if there exists an integer m such that the coefficients of $(x-y)^m(y-1)^m F(t;x,y)$, seen as rational functions in y, have no pole at $y=1$ nor $y=x$. We will often say, for short, that F has *poles of bounded order*.

Clearly, the above series $1/K(x,y)$ does *not* have poles of bounded order.

Definition 31. A pair $(I(x),J(y))$ of Laurent series in t with coefficients in $\mathbb{Q}(x)$ and $\mathbb{Q}(y)$, respectively, is *a pair of invariants* (for the kernel $K(x,y)$) if the ratio

$$\frac{I(x)-J(y)}{K(x,y)},$$

expanded as a series in t with coefficients in $\mathbb{Q}(x,y)$, has poles of bounded order.

Note that in this case, $J(y)$ itself has no pole at $y=x$ (because it does not depend on x), but may have a pole (of bounded order) at $y=1$. The following lemma allows us to build new pairs of invariants from old. Its proof mimics the proof of Lemma 2.8 in [16].

Lemma 32. *The componentwise sum of two pairs of invariants $(I_1(x),J_1(y))$ and $(I_2(x),J_2(y))$ is still a pair of invariants. The same holds for their componentwise product.*

The following lemma will be key to construct equations (for $Q(x,1)$, or $Q(y,y)$) in a *single* catalytic variable — hence with an algebraic solution, by [19].

Lemma 33. *Let $(I(x),J(y))$ be a pair of invariants such that the ratio*

$$\frac{I(x)-J(y)}{(x-y)(y-1)K(x,y)},$$

expanded in powers of t, has coefficients with no pole at $y=1$ nor $y=x$ (when seen as rational series in y). Then $I(x)$ and $J(y)$ are equal, and in particular, they depend on t only.

Proof. Assume on the contrary that $I(x) \neq J(y)$, and write

$$I(x)-J(y) = (x-y)(y-1)K(x,y)H(x,y),$$

where $H(x,y)$ is a non-zero series in t, with coefficients in $\mathbb{Q}(x,y)$ and no pole at $y=1$ nor $y=x$. Let m be the valuation of $H(x,y)$ in t, and denote by $h_m(x,y) \neq 0$ the coefficient of t^m. Then, given that $K(x,y) = 1+\mathcal{O}(t)$,

$$I(x)-J(y) = (x-y)(y-1)h_m(x,y)t^m + \mathcal{O}(t^{m+1}).$$

Let us write $I(x) = \sum_n i_n(x)t^n$ and $J(y) = \sum_n j_n(y)t^n$. The above identity gives

$$i_m(x) - j_m(y) = (x-y)(y-1)h_m(x,y).$$

By assumption, $h_m(x,y)$, seen as a fraction in y, has no pole at $y=1$ nor at $y=x$. Hence the same holds for $j_m(y)$. Evaluating the above identity at $y=1$ shows that $i_m(x) = j_m(1)$, so that $i_m(x)$ does not depend on x. We now have

$$j_m(1) - j_m(y) = (x-y)(y-1)h_m(x,y).$$

Evaluating this at $y=x$ gives $j_m(x) = j_m(1) = i_m(x)$, but this forces $h_m(x,y)=0$, a contradiction. □

A.2 A finite group, and rational invariants

To the kernel K, given by (16), one can associate as in [20] a group of birational transformations of pairs (u,v) that leave the value $K(u,v)$ unchanged. Solving $K(u,v)=K(u',v)$ for u' gives $u'=u$ or $u'=v(uv-u+v)/(u-v)/(v-1)$. Solving $K(u,v)=K(u,v')$ for v' gives $v'=v$ or $v'=uv/(uv-u+v)$. We introduce accordingly two transformations Φ and Ψ defined by

$$\Phi(u,v)=\left(\frac{v(uv-u+v)}{(u-v)(v-1)},v\right),\qquad \Psi(u,v)=\left(u,\frac{uv}{uv-u+v}\right).$$

One can check that they are involutions, and generate a group $\mathcal{G}$ of order 10. The orbit of (x,y) under the action of $\mathcal{G}$ is

$$\begin{aligned}(x,y)\overset{\Phi}{\longleftrightarrow}&\left(\frac{y(xy-x+y)}{(x-y)(y-1)},y\right)\overset{\Psi}{\longleftrightarrow}\left(\frac{y(xy-x+y)}{(x-y)(y-1)},\frac{xy-x+y}{x(y-1)}\right)\overset{\Phi}{\longleftrightarrow}\left(\frac{xy-x+y}{y(y-1)},\frac{xy-x+y}{x(y-1)}\right)\overset{\Psi}{\longleftrightarrow}\\ &\left(\frac{xy-x+y}{y(y-1)},\frac{xy-x+y}{y^2}\right)\overset{\Phi}{\longleftrightarrow}\left(\frac{x(xy-x+y)}{y(x-y)},\frac{xy-x+y}{y^2}\right)\overset{\Psi}{\longleftrightarrow}\left(\frac{x(xy-x+y)}{y(x-y)},\frac{x}{x-y}\right)\overset{\Phi}{\longleftrightarrow}\\ &\left(\frac{xy}{(y-1)(x-y)},\frac{x}{x-y}\right)\overset{\Psi}{\longleftrightarrow}\left(\frac{xy}{(y-1)(x-y)},\frac{xy}{xy-x+y}\right)\overset{\Phi}{\longleftrightarrow}\left(x,\frac{xy}{xy-x+y}\right)\overset{\Psi}{\longleftrightarrow}(x,y).\end{aligned}$$

Groups of order 10 also appear in the enumeration of several families of walks in the quadrant, including of course those used in the previous subsection [8, 45]. The group $\mathcal{G}$ can be used to construct rational invariants. Guided by Theorem 4.6 in [8], we take any rational function $H(u,v)$, and compute the sum of its values over all pairs (u,v) of the above orbit. Denoting this sum by $H_\sigma(x,y)$, we now define $I_0(x)=H_\sigma(x,Y)$ and $J_0(y)=H_\sigma(X,y)$, where X (resp. Y) is a root of $K(\cdot,y)$ (resp. $K(x,\cdot)$). We could adapt the proof of Theorem 4.6 in [8] to prove that $(I_0(x),J_0(y))$ then forms a pair of rational invariants. But this is not really needed, as we can simply apply the above recipe with some rational function $H(u,v)$, and check that the resulting pair is indeed a pair of invariants.

So let us start for instance with $H(u,v)=u$. The above recipe gives $I_0(x)=J_0(y)=-2+4/t$, which is a trivial pair of invariants. Starting instead from $H(u,v)=v$, we obtain a non-trivial pair, which, after dividing by 2 and subtracting 2, reads

$$I_0(x)=\frac{1}{1-tx}-\frac{1}{tx^2}+\frac{1+t}{tx}+x(1-t)-tx^2,\qquad J_0(y)=-\frac{t}{(y-1)^2}+\frac{1-t}{y-1}-\frac{1}{ty^2}+\frac{1+t}{yt}+y.$$

Indeed, one can check that

$$\frac{I_0(x)-J_0(y)}{K(x,y)}=\frac{(x-y)(1-y+txy)(x+y-xy-xyt(1+x-xy))}{x^2y^2t(xt-1)(y-1)},\tag{17}$$

which has poles of bounded order (at $y=1$ and $y=x$). Note that the pair (I_0,J_0) does not satisfy the conditions of Lemma 33.

A.3 Decouplings, and a new pair of invariants

Let us now return to the functional equation (15) that defines $Q(x,y)$. We would like to derive from it a new pair of invariants, that is, to transform it into an identity of the form

$$K(x,y)H(x,y)=I(x)-J(y),\tag{18}$$

where $H(x,y)$ has poles of bounded order. But there is an obstacle on our way, since the term $ty^3/(x-y)Q(y,y)$ does not depend on y only, but also on x. A similar difficulty also arises, for instance, when counting quadrant walks weighted by interactions with the coordinates axes [3], or walks avoiding a quadrant [16, 34, 52], or in some continuous probabilistic models [22].

We are going to remedy this difficulty using a *multiplicative decoupling* (and later, an *additive decoupling*). First, we observe that by definition (16) of the kernel,

$$\frac{ty^3}{x-y} = \frac{1-tx}{x} \cdot y(y-1) - \frac{y(y-1)}{x} K(x,y).$$

This is what we call a *multiplicative decoupling*: we have written the problematic term $ty^3/(x-y)$ as the *product* of a series in x and a series in y, modulo the kernel K. This would not be possible for any term of course. This allows us to rewrite (15) as follows:

$$\frac{K(x,y)(y-1)}{1-tx}\Big(xQ(x,y) - yQ(y,y)\Big) = \frac{x(y-1)}{1-tx} - \frac{tx}{1-tx}\frac{xQ(x,1)-Q(1,1)}{x-1} - y(y-1)Q(y,y), \quad (19)$$

and our problem is solved, since the term in $Q(y,y)$ no longer involves x.

However, we have created a second difficulty, and our equation still does not look like (18): the constant term, which was formerly $(y-1)$ in (15), now mixes x and y. This new problem will be solved as well if we can find rational functions $A(x)$, $B(y)$ and $H(x,y)$ such that

$$\frac{x(y-1)}{1-tx} = A(x) + B(y) + K(x,y)H(x,y), \quad (20)$$

where $H(x,y)$ does not contain a factor $K(x,y)$ in its denominator. This is what we call an *additive decoupling*. Again, an arbitrary rational function does not have, in general, an additive decoupling.

There are two ways to look for a solution (A,B) of (20) (see [8, Sec. 4.2] and our Maple session for details). The first one is by guessing (say, the fraction $B(y)$) and checking. Let us explain how this works. Denoting by Y_0 and Y_1 the two roots of $K(x,\cdot)$, we observe that (20) implies that

$$\frac{x}{1-tx} = \frac{B(Y_0)-B(Y_1)}{Y_0-Y_1}.$$

We can then start from an Ansatz on the form of $B(y)$ (fixing the number of poles and their orders, but not their values), form the divided difference $(B(Y_0)-B(Y_1))/(Y_0-Y_1)$, write it as a fraction in x and t (since it is a symmetric function of the two roots Y_0 and Y_1), and solve the resulting identity for the coefficients occurring in the Ansatz. This approach readily gives a solution,

$$B(y) = -\frac{y}{t} + \frac{1}{t(y-1)} - \frac{1}{t^2 y},$$

from which we derive

$$A(x) = \frac{2+x}{t} + \frac{1}{t^2 x} + \frac{1}{t(tx-1)}.$$

We can now check that

$$\frac{x(y-1)}{1-tx} = A(x) + B(y) + K(x,y)\frac{(x-y)(1-txy)}{xyt^2(1-tx)}. \quad (21)$$

The second approach is constructive, and consists in applying Theorem 4.11 in [8] (even though our kernel is *not* of the same form as the kernels studied in [8]). This gives an alternative solution, differing from the above solution (A,B) by a pair of invariants. More precisely, this new solution reads:

$$A(x)+\frac{9}{5t}I_0(x)-\frac{7}{5t^2}(1+3t), \qquad B(y)-\frac{9}{5t}J_0(y)+\frac{7}{5t^2}(1+3t).$$

Let us now combine the functional equation (19) with the decoupling relation (21): we obtain

$$\frac{(x-y)(y-1)K(x,y)}{1-tx}\left(\frac{xQ(x,y)-yQ(y,y)}{x-y}-\frac{1-txy}{t^2xy(y-1)}\right)=I_1(x)-J_1(y), \tag{22}$$

with

$$I_1(x)=\frac{2+x}{t}+\frac{1}{t^2x}+\frac{1}{t(tx-1)}-\frac{tx}{1-tx}\frac{xQ(x,1)-Q(1,1)}{x-1}, \quad J_1(y)=\frac{y}{t}-\frac{1}{t(y-1)}+\frac{1}{t^2y}+y(y-1)Q(y,y).$$

That is, we have found a second pair (I_1,J_1) of invariants, this time in terms of the unknown series Q. It does not satisfy the conditions of Lemma 33.

A.4 Equations in one catalytic variable

Our aim is now to combine polynomially our two pairs of invariants (I_0,J_0) and (I_1,J_1), thanks to Lemma 32, to form a new pair $(I(x),J(y))$ satisfying the conditions of Lemma 33. This will imply that $I(x)$ and $J(y)$ are in fact both equal to a series C depending on t only. We will thus obtain an equation in only one catalytic variable for each of the two series $Q(x,1)=G(x,1)/(tx)$ and $Q(y,y)=G(1,y)/t$, just by writing $I(x)=C=J(y)$.

Let us examine the ratios

$$\frac{I_0(x)-J_0(y)}{(x-y)(y-1)K(x,y)} \qquad \text{and} \qquad \frac{I_1(x)-J_1(y)}{(x-y)(y-1)K(x,y)},$$

derived from (17) and (22), respectively. The coefficient of t^n in each of these two series has no pole at $y=x$. This is obvious for the first ratio, and for the second, this relies on the fact that the divided difference

$$\frac{xQ(x,y)-yQ(y,y)}{x-y}$$

has polynomial coefficients in x and y. However, the coefficient of t^n in the first (resp. second) ratio has a double (resp. simple) pole at $y=1$. Accordingly, we observe a double (resp. simple) pole at $y=1$ in $J_0(y)$ (resp. $J_1(y)$). More precisely, the singular expansions at $y=1$ of these two series are respectively

$$J_0(y)=-\frac{t}{(y-1)^2}+\frac{1-t}{y-1}+\mathcal{O}(1) \qquad \text{and} \qquad J_1(y)=-\frac{1}{t(y-1)}+\frac{1+t}{t^2}+\mathcal{O}(y-1).$$

It is then natural to introduce the series

$$J(y)=J_0(y)+t^3J_1(y)^2-t(1+3t)J_1(y),$$

which has no pole at $y = 1$. By Lemma 32, the pair $(I(x), J(y))$ forms a pair of invariants, if we define $I(x)$ by

$$I(x) = I_0(x) + t^3 I_1(x)^2 - t(1+3t)I_1(x).$$

Using (17) and (22), we can then check that this new pair of invariants satisfies the condition of Lemma 33. This implies that $I(x) = J(y) = C$ for some series C. The latter series is easily identified by expanding the expression of $J(y)$ at $y = 1$. This gives

$$I(x) = 2 - 4t - 2t^2 Q(1,1) = J(y).$$

We can now replace $I(x)$ and $J(y)$ by their expressions in terms of $(xQ(x,1) - Q(1,1))/(x-1)$ and $Q(y,y)$, respectively. This gives two polynomial equations,

$$\mathrm{Pol}_1(Q(x,1), Q(1,1), t, x) = 0, \qquad \text{and} \qquad \mathrm{Pol}_2(Q(y,y), Q(1,1), t, y) = 0.$$

Each of them is a polynomial equation in one catalytic variable only, quadratic in the main series (that is, $Q(x,1)$ or $Q(y,y)$).

A.5 Algebraicity

It remains to solve these two equations in one catalytic variable. We will work, say, with the equation for $Q(y,y)$, which is a bit lighter than the other. It reads:

$$\mathrm{Pol}_2(Q(y,y), Q(1,1), t, y) = 0, \tag{23}$$

where

$$\mathrm{Pol}_2(q, q_1, y, t) = y^2 t^2 (y-1)^2 q^2 + \Big(y\Big(2y^2 - 5y + 1\Big)t - (y-1)(y-2)\Big)q + 2tq_1 + (y-1)(y-2) = 0.$$

This is a very simple instance of a polynomial equation in one catalytic variable [19]: it is (only) quadratic in $Q(y,y)$, and involves a single additional unknown series depending on t only, namely $Q(1,1)$. In this case, the machinery of [19] reduces to Brown's quadratic method [25, 41]. We first examine whether there exists formal power series $Y \equiv Y(t)$ such that the first derivative of the above polynomial Pol_2, evaluated at $(Q(Y,Y), Q(1,1), t, Y)$, vanishes. That is,

$$(Y-1)(Y-2) = 2t^2 Y^2 (Y-1)^2 Q(Y,Y) + tY(2Y^2 - 5Y + 1).$$

The form of this equation shows that two such series exist, with constant terms 1 and 2, respectively. Then Brown's result tells us that each of these two series is a double root of the discriminant of Pol_2 with respect to its first variable, evaluated at $(Q(1,1), t, y)$. That is, a double root of

$$\begin{aligned} -4ty^5 + \Big(-8Q_{11}t^3 - 7t^2 + 22t + 1\Big)y^4 + \Big(16Q_{11}t^3 + 18t^2 - 40t - 6\Big)y^3 \\ + \Big(-8Q_{11}t^3 - 7t^2 + 26t + 13\Big)y^2 - 4(t+3)y + 4, \end{aligned}$$

where we have written $Q(1,1) = Q_{11}$. Since this polynomial in y has multiple roots, its discriminant vanishes. This gives for Q_{11} the following cubic equation:

$$64t^6 Q_{11}^3 + 16t^3\Big(11t^2 - 18t - 1\Big)Q_{11}^2 + \Big(161t^4 - 452t^3 + 238t^2 - 28t + 1\Big)Q_{11} + 49t^3 - 167t^2 + 25t = 1.$$

One can now check that if we introduce the series Z of Theorem 1, the above equation factors and yields $Q_{1,1} = (1+Z)(1+2Z)^2(1-2Z+2Z^3)$, so that $G(1,1) = tQ(1,1) = Z(1-2Z+2Z^3)$.

Let us now return to (23). We can express t and Q_{11} as fractions in Z, and we now have a quadratic equation for $Q(y,y)$ with coefficients in $\mathbb{Q}(Z,y)$. Solving it then gives the announced expression of $G(1,y) = tyQ(y,y)$ in Proposition 28.

We now proceed similarly with the other equation in one catalytic variable, $\mathrm{Pol}_1(Q(x,1), Q(1,1), t, x) = 0$, which is quadratic in $Q(x,1)$. We replace t and Q_{11} by their rational expressions in Z, solve the resulting equation, and obtain the announced expression of $G(x,1) = txQ(x,1)$. $\square$

References

[1] M. Albenque, L. Ménard, and G. Schaeffer. Local convergence of large random triangulations coupled with an Ising model. *Trans. Amer. Math. Soc.*, 374(1):175–217, 2021. arXiv:1812.03140.

[2] J.-L. Baril, S. Kirgizov, and M. Naima. A lattice on Dyck paths close to the Tamari lattice. arXiv:2309.00426v1, 2023.

[3] N. R. Beaton, A. L. Owczarek, and A. Rechnitzer. Exact solution of some quarter plane walks with interacting boundaries. *Electron. J. Combin.*, 26(3):Paper No. 3.53, 38, 2019.

[4] F. Bergeron and L.-F. Préville-Ratelle. Higher trivariate diagonal harmonics via generalized Tamari posets. *J. Comb.*, 3(3):317–341, 2012. arXiv:1105.3738 [doi].

[5] O. Bernardi and N. Bonichon. Intervals in Catalan lattices and realizers of triangulations. *J. Combin. Theory Ser. A*, 116(1):55–75, 2009. arXiv:0704.3731 [doi].

[6] O. Bernardi and M. Bousquet-Mélou. Counting colored planar maps: algebraicity results. *J. Combin. Theory Ser. B*, 101(5):315–377, 2011. arXiv:0909.1695.

[7] O. Bernardi and M. Bousquet-Mélou. Counting coloured planar maps: differential equations. *Comm. Math. Phys.*, 354(1):31–84, 2017. arXiv:1507.02391.

[8] O. Bernardi, M. Bousquet-Mélou, and K. Raschel. Counting quadrant walks via Tutte's invariant method. *Comb. Theory*, 1:Paper No. 3, 77 p., 2021. arXiv:1708.08215.

[9] A. Bostan, M. Bousquet-Mélou, and S. Melczer. Counting walks with large steps in an orthant. *J. Eur. Math. Soc. (JEMS)*, 23(7):2221–2297, 2021. arXiv:1806.00968.

[10] A. Bostan and M. Kauers. The complete generating function for Gessel walks is algebraic. *Proc. Amer. Math. Soc.*, 138(9):3063–3078, 2010. With an appendix by M. van Hoeij. arXiv:0909.1965.

[11] A. Bostan, K. Raschel, and B. Salvy. Non-D-finite excursions in the quarter plane. *J. Combin. Theory Ser. A*, 121:45–63, 2014. arXiv:1205.3300.

[12] M. Bousquet-Mélou. Counting walks in the quarter plane. In *Mathematics and computer science 2, (Versailles, 2002)*, Trends Math., pages 49–67. Birkhäuser, Basel, 2002. arXiv:1708.06192.

[13] M. Bousquet-Mélou. Four classes of pattern-avoiding permutations under one roof: generating trees with two labels. *Electron. J. Combin.*, 9(2):Research paper 19, 31 pp. (electronic), 2002/03.

[14] M. Bousquet-Mélou. Walks in the quarter plane: Kreweras' algebraic model. *Ann. Appl. Probab.*, 15(2):1451–1491, 2005. arXiv:math/0401067.

[15] M. Bousquet-Mélou. An elementary solution of Gessel's walks in the quadrant. *Adv. Math.*, 303:1171–1189, 2016. arXiv:1503.08573.

[16] M. Bousquet-Mélou. Enumeration of three-quadrant walks via invariants: some diagonally symmetric models. *Canad. J. Math.*, 75(5):1566–1632, 2023. arXiv:2112.05776.

[17] M. Bousquet-Mélou, G. Chapuy, and L.-F. Préville-Ratelle. The representation of the symmetric group on m-Tamari intervals. *Adv. Math.*, 247:309–342, 2013. arXiv:1202.5925 [doi].

[18] M. Bousquet-Mélou, E. Fusy, and L.-F. Préville-Ratelle. The number of intervals in the m-Tamari lattices. *Electron. J. Combin.*, 18(2):Paper 31, 26, 2011. arXiv:1106.1498 [doi].

[19] M. Bousquet-Mélou and A. Jehanne. Polynomial equations with one catalytic variable, algebraic series and map enumeration. *J. Combin. Theory Ser. B*, 96:623–672, 2006. arXiv:math/0504018.

[20] M. Bousquet-Mélou and M. Mishna. Walks with small steps in the quarter plane. In *Algorithmic probability and combinatorics*, volume 520 of *Contemp. Math.*, pages 1–39. Amer. Math. Soc., Providence, RI, 2010. arXiv:0810.4387.

[21] M. Bousquet-Mélou and F. Chapoton. Intervals in the greedy Tamari posets. *Combin. Theory*, 4(1):paper no. 5, 2024. arxiv:2303.18077.

[22] M. Bousquet-Mélou, A. Elvey Price, S. Franceschi, C. Hardouin, and K. Raschel. The stationary distribution of reflected brownian motion in a wedge: differential properties. arXiv:2101.01562, 2021-2022.

[23] M. Bousquet-Mélou, E. Fusy, and K. Raschel. Plane bipolar orientations and quadrant walks. *Sém. Lothar. Combin.*, 81:Paper B81l, 2021. arXiv:1905.04256.

[24] M. Bouvel, V. Guerrini, A. Rechnitzer, and S. Rinaldi. Semi-Baxter and strong-Baxter: two relatives of the Baxter sequence. *SIAM J. Discrete Math.*, 32(4):2795–2819, 2018. arXiv:1702.04529.

[25] W. G. Brown. On the existence of square roots in certain rings of power series. *Math. Ann.*, 158:82–89, 1965.

[26] F. Chapoton. Sur le nombre d'intervalles dans les treillis de Tamari. *Sém. Lothar. Combin.*, 55:Art. B55f, 18 pp. (electronic), 2006.

[27] F. Chapoton, G. Châtel, and V. Pons. Two bijections on Tamari intervals. In *26th International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2014)*, Discrete Math. Theor. Comput. Sci. Proc., AT, pages 241–252. Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2014. [doi].

[28] G. Chapuy. On the scaling of random Tamari intervals and Schnyder woods of random triangulations (with an asymptotic D-finite trick). arXiv:2403.18719.

[29] C. Chenevière. Linear intervals in the Tamari and the Dyck lattices and in the alt-Tamari posets. arXiv:2209.00418.

[30] C. Chenevière. *Enumerative study of intervals in lattices of Tamari type*. PhD thesis, IRMA, Université de Strasbourg, 2023. On-line version.

[31] D. Denisov and V. Wachtel. Random walks in cones. *Ann. Probab.*, 43(3):992–1044, 2015. Arxiv:1110.1254.

[32] A. Dermenjian. Maximal degree subposets of ν-Tamari lattices. *Electron. J. Combin.*, 30(2):Paper No. 2.43, 40, 2023. arXiv:2208.11417 [doi].

[33] T. Dreyfus, C. Hardouin, J. Roques, and M. F. Singer. On the nature of the generating series of walks in the quarter plane. *Invent. Math.*, 213(1):139–203, 2018. arXiv:1702.04696.

[34] T. Dreyfus and A. Trotignon. On the nature of four models of symmetric Walks avoiding a quadrant. *Ann. Comb.*, 25(3):617–644, 2021. arXiv:2006.02299.

[35] W. Fang. Planar triangulations, bridgeless planar maps and Tamari intervals. *European J. Combin.*, 70:75–91, 2018. arXiv:1611.07922 [doi].

[36] W. Fang. A trinity of duality: non-separable planar maps, $\beta(1,0)$-trees and synchronized intervals. *Adv. in Appl. Math.*, 95:1–30, 2018. arXiv:1703.02774 [doi].

[37] W. Fang. Bijective link between Chapoton's new intervals and bipartite planar maps. *European J. Combin.*, 97:Paper No. 103382, 15, 2021. arXiv:2001.04723 [doi].

[38] W. Fang, E. Fusy, and P. Nadeau. Tamari intervals and blossoming trees. *Comb. Theory*, 5(1):Paper No. 4, 41, 2025. arXiv:2312.13159.

[39] P. Flajolet and R. Sedgewick. *Analytic combinatorics*. Cambridge University Press, Cambridge, 2009. Available online: http://algo.inria.fr/flajolet/Publications/book.pdf.

[40] H. Friedman and D. Tamari. Problèmes d'associativité: Une structure de treillis finis induite par une loi demi-associative. *J. Combinatorial Theory*, 2:215–242, 1967.

[41] I. P. Goulden and D. M. Jackson. *Combinatorial enumeration*. John Wiley & Sons Inc., New York, 1983. Wiley-Interscience Series in Discrete Mathematics.

[42] M. D. Haiman. Conjectures on the quotient ring by diagonal invariants. *J. Algebraic Combin.*, 3(1):17–76, 1994. [doi].

[43] F. Hivert, J.-C. Novelli, and J.-Y. Thibon. The algebra of binary search trees. *Theoretical Computer Science*, 339:129–165, 2005. arXiv:math/0401089.

[44] S. Huang and D. Tamari. Problems of associativity: A simple proof for the lattice property of systems ordered by a semi-associative law. *J. Combin. Theory Ser. A*, 13(1):7–13, 1972. [doi].

[45] M. Kauers and R. Yatchak. Walks in the quarter plane with multiple steps. In *Proceedings of FPSAC 2015*, Discrete Math. Theor. Comput. Sci. Proc., pages 25–36. Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2015. arXiv:1411.3537.

[46] G. Kreweras. Sur les partitions non croisées d'un cycle. *Discrete Math.*, 1(4):333–350, 1972.

[47] M. Mishna. Classifying lattice walks restricted to the quarter plane. *J. Combin. Theory Ser. A*, 116(2):460–477, 2009. arXiv:math/0611651.

[48] P. Nadeau and V. Tewari. P-partitions with flags and back stable quasisymmetric functions. *Comb. Theory*, 4(2):Paper No. 4, 22, 2024. arXiv:2303.09019.

[49] J.-C. Novelli and J.-Y. Thibon. Hopf algebras of m-permutations, $(m+1)$-ary trees, and m-parking functions. *Advances in Applied Mathematics*, 117, 2020. arXiv:math/1403.5962.

[50] V. Pons. The rise-contact involution on Tamari intervals. *Electron. J. Combin.*, 26(2):Paper No. 2.32, 53, 2019. [doi].

[51] K. Raschel. Counting walks in a quadrant: a unified approach via boundary value problems. *J. Eur. Math. Soc. (JEMS)*, 14(3):749–777, 2012. arXiv:1003.1362.

[52] K. Raschel and A. Trotignon. On walks avoiding a quadrant. *Electron. J. Combin.*, 26(3):Paper 3.31, 34 pp., 2019. arXiv:1807.08610.

[53] L. B. Richmond and N. C. Wormald. Almost all maps are asymmetric. *J. Combin. Theory Ser. B*, 63(1):1–7, 1995. [doi].

[54] R. Simion and F. W. Schmidt. Restricted permutations. *Europ. J. Combin.*, 6:383–406, 1985. [doi].

[55] N. J. A. Sloane and T. O. F. Inc. The on-line encyclopedia of integer sequences, 2020.

[56] B. Testart. Completing the enumeration of inversion sequences avoiding one or two patterns of length 3. arXiv:2407.07701, 2024.

[57] W. T. Tutte. Dichromatic sums for rooted planar maps. In *Combinatorics (Proc. Sympos. Pure Math., Vol. XIX, Univ. California, Los Angeles, Calif., 1968)*, pages 235–245. Amer. Math. Soc., Providence, R.I., 1971.

[58] W. T. Tutte. Chromatic sums for rooted planar triangulations: the cases $\lambda = 1$ and $\lambda = 2$. *Canad. J. Math.*, 25:426–447, 1973. [doi].

[59] W. T. Tutte. On the enumeration of convex polyhedra. *J. Combin. Theory Ser. B*, 28(2):105–126, 1980. [doi].

[60] W. T. Tutte. Chromatic sums revisited. *Aequationes Math.*, 50(1-2):95–134, 1995. [on-line].

[61] J. West. Generating trees and forbidden subsequences. *Discrete Math.*, 157(1-3):363–374, 1996. [doi].