Cyclic and lift closures for $k\dots21$-avoiding permutations

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Abstract

We prove that the cyclic closure of the permutation class avoiding the pattern $k(k-1)\dots21$ is finitely based. The minimal length of a minimal permutation is $2k-1$ and these basis permutations are enumerated by $(2k-1).c_k$ where c_k is the kth Catalan number. We also define lift operations and give similar results. Finally, we consider the toric closure of a class and we propose some open problems.

Keywords: Pattern avoiding permutation class; cyclic, lift and toric closures.

1 Introduction and notation

Let S (resp. S_n) be the class of permutations (resp. of length n). We represent permutations in one-line notation, *i.e.* if $i_1, i_2, \dots, i_n$ are n distinct values in $[n] = \{1, 2, \dots, n\}$, we denote the permutation $\sigma \in S_n$ by the sequence $i_1\ i_2\ \dots\ i_n$ if $\sigma(k) = i_k$ for $1 \leq k \leq n$. For instance, the identity permutation of length n, Id_n, will be written $1\ 2\ \dots\ n$. A *segment* of a permutation σ is a subsequence of the form $\sigma_i\sigma_{i+1}\dots\sigma_{j-1}\sigma_j$ or $\sigma_j\sigma_{j+1}\dots\sigma_n\sigma_1\dots\sigma_{i-1}\sigma_i$ where $1 \leq i \leq j \leq n$. The *cardinality* (or *length*) of a segment W will be denoted $|W|$. A permutation $\sigma = \sigma_1\sigma_2\dots\sigma_n \in S_n$ contains the *pattern* $\pi \in S_k$, $k \geq 2$, if and only if a sequence of indices $1 \leq i_1 < i_2 < \dots < i_k \leq n$ exists such that $\sigma(i_1)\sigma(i_2)\dots\sigma(i_k)$ is ordered as π. We write $\pi \prec \sigma$ to denote that π is a pattern of σ. A permutation σ that does not contain π as a pattern is said to *avoid* π. Let B be a set of permutations. We denote by $\mathrm{Av}_n(B)$ (resp. $\mathrm{Av}(B)$) the set (resp. the class) of permutations in S_n (resp. S) avoiding all patterns $\pi \in B$. For example, $25314 \notin \mathrm{Av}_5(123)$ but $43152 \in \mathrm{Av}_5(123)$. See for instance [2, 3]. In the following, a subsequence of a permutation σ order-isomorphic to $D_k = k(k-1)\dots21$ will be called a D_k-pattern of σ. A class $\mathcal{C}$ of permutations is *closed* (or *stable*) for the involvement relation $\prec$ if, for any $\sigma \in \mathcal{C}$, for any $\pi \prec \sigma$, then we also have $\pi \in \mathcal{C}$. Such a class can always be defined by a class $\mathrm{Av}(B)$ of pattern-avoiding permutations. In the case where B is minimal (relatively to the relation $\prec$), B is called a *basis*.

Let $\sigma = \sigma_1\sigma_2\ldots\sigma_n$ be a permutation in S_n. We define two bijections r (*cyclic rotation*) and ℓ (*lift rotation*) on S_n as: $r(\sigma) = \sigma_n\sigma_1\sigma_2\ldots\sigma_{n-1}$ and $\ell(\sigma) = (\sigma_1 \mod (n)+1)(\sigma_2 \mod (n)+1)\ldots(\sigma_n \mod (n)+1)$. For the sake of simplicity, we will say that a permutation $\sigma' \in S_n$ is a *cyclic rotation* (resp. *lift rotation*) of a permutation $\sigma \in S_n$ if $\sigma' = r^k(\sigma)$ (resp. $\sigma' = \ell^k(\sigma)$) for some $k \in [1..n]$. For instance, 231 and 312 are both cyclic rotations of $123 \in S_3$; 213 and 132 are both lift rotations of $321 \in S_3$. The *cyclic closure* $\mathrm{cc}(\mathcal{C})$ (resp. *lift closure* $\mathrm{cl}(\mathcal{C})$) of a class $\mathcal{C}$ is the class of permutations that can be obtained by a cyclic rotation (lift rotation) from a permutation in $\mathcal{C}$. Finally, the *toric closure* $\mathrm{tr}(\mathcal{C})$ of $\mathcal{C}$ is defined by $\mathrm{cc}(\mathrm{cl}(\mathcal{C}))$ which is also equal to $\mathrm{cl}(\mathrm{cc}(\mathcal{C}))$.

Given a closed class of permutations, two well-studied problems are that of finding its enumeration and its basis. We are interested in these questions for cyclically closed classes, introduced by Albert et al. (See [1]).

Let $\mathcal{C}$ be the class $\mathrm{Av}(B)$ where B is a set of patterns. We know that $\mathrm{cc}(\mathcal{C})$ is a pattern-avoiding class where its basis is the set of permutations that are minimal with respect to not lying in $\mathrm{cc}(\mathcal{C})$. Let σ be a permutation in $\mathcal{C}$ and θ be a pattern of σ, *i.e*, a subsequence of σ order-isomorphic to a member of B. Modulo a cyclic rotation of σ, we assume that $\sigma = s\alpha t\beta$ where α (resp. β) is the first (resp. last) value of θ. The *witnessed segment* $W(\theta)$ of θ is the segment $s\alpha$. Albert et al. give a necessary and sufficient condition for σ to be (or not) in the cyclic closure of $\mathcal{C}$.

Lemma 1 *[1] $\sigma \notin cc(\mathcal{C})$ if and only if the witnessed segments cover σ.*

Moreover, they obtain a sufficient condition for the class $\mathrm{cc}(\mathcal{C})$ to be finitely based.

Proposition 1 *[1] Let $\mathcal{C} = Av(B)$, where B is finite and suppose there is a bound Δ depending on B alone such that, for all $\sigma \notin cc(\mathcal{C})$, there is a collection of at most Δ witnessed segments that cover σ. Then $cc(\mathcal{C})$ is finitely based.*

Albert et al. deduce from Proposition 1 that $\mathrm{cc}(\mathrm{Av}(321))$, $\mathrm{cc}(\mathrm{Av}(231))$ and $\mathrm{cc}(\mathrm{Av}(4321))$ are finitely based. They also show that the cyclic closure of a finitely based class can fail to be finitely based (it is the case for the class $\mathrm{Av}(265143)$). They provide several results about enumeration.

This paper is organized as follows. In Section 2, we prove that the cyclic closure of $\mathrm{Av}(k(k-1)\ldots 21)$ has a finite basis B. Moreover, we prove that the smallest length of a minimal permutation of B is $2k-1$, and that these basis permutations are enumerated by $(2k-1).c_k$ where c_k is the kth Catalan number. We also characterize these permutations. In Section 3, we investigate the lift closure of closed classes. We show a duality with the cyclic closure which allows to prove that the lift closure of $\mathrm{Av}(k(k-1)\ldots 21)$ is finitely based. In Section 4, we study the toric closure. We give a sufficient condition for the class $\mathrm{tr}(\mathcal{C})$ to be finitely based. We have obtained a proof that $\mathrm{tr}(\mathrm{Av}(321))$ is finitely based, but we do not give it. Indeed, the proof requires many cases to check and it does not contain any interesting concept. We finish the paper by presenting several open problems.

2 Cyclic closure

Let $\sigma = \sigma_1 \dots \sigma_n$ be a permutation of length n such that $\sigma \notin \mathrm{cc}(\mathrm{Av}(D_k))$ for $k \geq 3$. For $x \in \{1, \dots, n\}$, we define $\mathcal{W}_\sigma(x)$ to be the set of subsequences $\theta = \theta_1\theta_2 \dots \theta_{k-1}\theta_k$ of σ (considered cyclically), order-isomorphic to $D_k = k(k-1) \dots 21$ and such that $x \in W(\theta)$. Let $\mathcal{W}_\sigma$ be the union of all $\mathcal{W}_\sigma(x)$ for $1 \leq x \leq n$. In the following, we will omit the subscript σ for $\mathcal{W}_\sigma(x)$ and $\mathcal{W}_\sigma$ since it should be clear from context. For $\theta, \theta' \in \mathcal{W}$, we define an order relation $\trianglelefteq$ by: $\theta' \trianglelefteq \theta$ if and only if (a) $|W(\theta')| < |W(\theta)|$; or (b) $|W(\theta')| = |W(\theta)|$ and the associated word $\theta'_1\theta'_2 \dots \theta'_k$ is smaller than $\theta_1\theta_2 \dots \theta_k$ in lexicographic order. Thus $\mathcal{W}$ is a non-empty finite totally-ordered set. Let θ be its maximal element. Then, σ has a special structure illustrated in Figure 1 and described below.

Indeed, σ can be written (modulo rotation):

$$\sigma = \theta_1\alpha_1\theta_2\alpha_2 \dots \theta_{k-1}\alpha_{k-1}\theta_k\alpha_k$$

where $W(\theta) = \alpha_k\theta_1$, $\theta_i \in \{1, 2, \dots, n\}$ and α_i are segments of σ for $i \in [1, k]$. By convenience, we set $\theta_{k+1} = 0$ and $\theta_0 = n+1$. For $i \in [1, k-1]$, we define the subsequence α_i^- (resp. α_i^+) of elements in α_i that are smaller than θ_{i+1} (resp. greater than θ_{i-1}). On the other hand, $\alpha_{i,j}^-$, $i \in [1, k-2]$, $j \in [i+2, k+1]$, (resp. $\alpha_{i,j}^+$, $i \in [2, k-1]$, $j \in [0, i-2]$,) is the subsequence of α_i^- (resp. α_i^+) constituted by all values greater than θ_j (resp. smaller than θ_j).

With these hypotheses, σ verifies the following structural properties.

Fact 1. For $i \in [1, k-1]$, α_i does not contain any value in the interval $[\theta_{i+1}, \theta_i]$.

If there exists $y \in \alpha_i$ such that $y \in [\theta_{i+1}, \theta_i]$, then $\theta' = \theta_1 \dots \theta_i y \theta_{i+1} \dots \theta_{k-1}$ is a subsequence of σ belonging to $\mathcal{W}$ such that $|W(\theta')| > |W(\theta)|$ which is a contradiction with the maximality of θ.

Fact 2. For $i \in [1, k-1]$, α_i does not contain any value in $[\theta_i, \theta_{i-1}]$.

If there exists $y \in \alpha_i$ such that $y \in [\theta_i, \theta_{i-1}]$, then $\theta' = \theta_1 \dots \theta_{i-1} y \theta_{i+1} \dots \theta_k$ is a subsequence of σ belonging to $\mathcal{W}$ such that $\theta \trianglelefteq \theta'$ which is a contradiction.

Fact 3. For $i \in [1, k-2]$, $j \in [i+2, k]$, $\alpha_{i,j}^-$ lies in $\mathrm{Av}(D_{j-i})$ and α_i^- lies in $\mathrm{Av}(D_{k-i})$.

If $\alpha_{i,j}^-$ contains a subsequence $\rho_1\rho_2 \dots \rho_{j-i}$ order-isomorphic to D_{j-i}, then $\theta' = \theta_1 \dots \theta_i \rho_1 \dots \rho_{j-i} \theta_j \dots \theta_{k-1}$ belongs to $\mathcal{W}$ and verifies $|W(\theta')| > |W(\theta)|$ which is a contradiction.

Fact 4. For $i \in [2, k-1]$, $j \in [0, i-2]$, $\alpha_{i,j}^+$ lies in $\mathrm{Av}(D_{i-j})$.

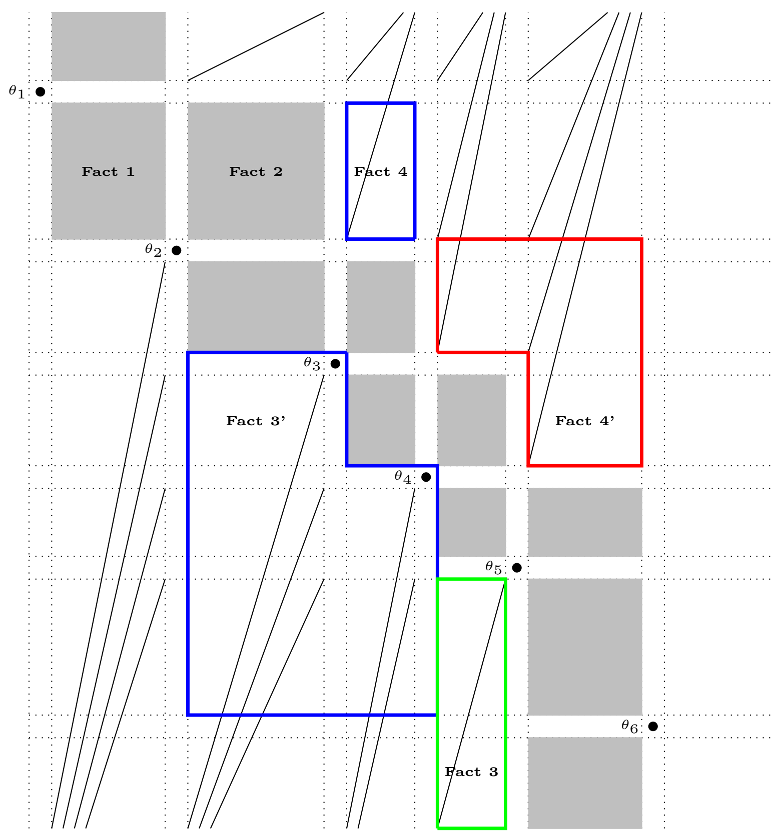
If $\alpha^+_{i,j}$ contains a subsequence $\rho_1\rho_2\dots\rho_{j-i}$ order-isomorphic to D_{i-j}, then $\theta' = \theta_1\dots\theta_j\rho_1\dots\rho_{i-j}\theta_{i+1}\dots\theta_{k-1}$ belongs to $\mathcal{W}$ and verifies $\theta \trianglelefteq \theta'$ which is a contradiction.

The following facts are simple generalizations of Facts 3 and 4. The proofs are obtained *mutatis mutandis*.

Fact 3'. For $i \in [1, k-2]$, $j \in [i+2, k]$, $\ell \in [i+1, k-1]$, $\alpha^-_{i,j}\theta_{i+1}\alpha^-_{i+1,j}\theta_{i+2}\dots\alpha^-_{\ell-1,j}\theta_\ell$ lies in $\mathrm{Av}(D_{j-i})$ and $\alpha^-_i\theta_{i+1}\alpha^-_{i+1}\theta_{i+2}\dots\alpha^-_{\ell-1}\theta_\ell$ lies in $\mathrm{Av}(D_{k-i})$.

Fact 4'. For $i \in [2, k-1]$, $\ell \in [i+1, k]$, $j \in [0, \ell-3]$, $\alpha^+_{i,j}\theta_{i+1}\alpha^+_{i+1,j}\theta_{i+2}\alpha^+_{i+2,j}\dots\alpha^+_{\ell-1,j}\theta_\ell$ lies in $\mathrm{Av}(D_{\ell-j-1})$.

Figure 1: The special structure of a permutation $\sigma \notin \mathrm{Av}(D_k)$. A filled area does not contain any point of the form (i, σ_i), $1 \le i \le n$. A sequence of j increasing diagonal in an area means that it does not contain a pattern $(j+1)j\dots21$.



Theorem 1 *Let $k \ge 3$ be an integer. Then $cc(Av(D_k))$ is finitely based.*

Proof. Let $\sigma = \sigma_1 \ldots \sigma_n$ be a permutation of length n which does not belong to $\text{cc}(\text{Av}(D_k))$. The crucial point of the proof is to find a bounded number (independently of n) of witnessed segments that cover σ. Finally, we will conclude through Proposition 1.

Let $\theta = \theta_1 \ldots \theta_k$ be the maximal element of $\mathcal{W}$. Modulo cyclic rotation, σ can be decomposed as follows:

$$\sigma = \theta_1\alpha_1\theta_2\alpha_2 \ldots \alpha_{k-1}\theta_k\alpha_k.$$

Notice that α_k and $\theta_1 = \sigma_1$ are obviously covered by $W(\theta)$ since $W(\theta) = \alpha_k\theta_1$. Thus, it suffices to prove that $\alpha_i\theta_{i+1}$ for $1 \le i \le k-1$ are covered by a bounded number of witnessed segments.

Let i such that $1 \le i \le k-1$ and σ^i be the rotation of σ defined by $\sigma^i = \alpha_i\theta_{i+1} \ldots \theta_k\alpha_k\theta_1\alpha_1 \ldots \theta_{i-1}\alpha_{i-1}\theta_i$. By hypothesis (Remark 3), there is a subsequence ρ of σ^i (considered cyclically) such that $\theta_{i+1} \in W(\rho)$. If α_i is empty, there is nothing to do since $\alpha_i\theta_{i+1}$ is covered by $W(\rho)$. Now, let us suppose that α_i is not empty, and let us consider the first value x_1 of α_i. Then, there is a subsequence $\rho = \rho_1\rho_2 \ldots \rho_k$ of σ^i order-isomorphic to D_k such that $x_1 \in W(\rho)$. We take for ρ the maximal element of $\mathcal{W}(x_1)$.

We discuss on the first value ρ_1 of ρ: (i) $\rho_1 \notin \alpha_i$; (ii) $\rho_1 \in \alpha_i^-$ and (iii) $\rho_1 \in \alpha_i^+$. The case (i) is straightforward since this means that $\alpha_i\theta_{i+1}$ is included into $W(\rho)$.

Let us examine (ii): $\rho_1 \in \alpha_i^-$. Notice that ρ_k appears on the right of θ_k (in σ^i), since otherwise θ would not be maximal in $\mathcal{W}$ (because $\theta_1 \in W(\rho)$ and $|W(\rho)| > |W(\theta)|$). So, let j be the smallest integer, $1 \le j \le k-1$, such that ρ_j belongs to $\alpha_i^-\alpha_{i+1}^- \ldots \alpha_{k-1}^-$ but not ρ_{j+1}. Then, the special structure (particularly Fact 3') of σ necessarily induces that the subsequence $\rho' = \theta_{i+1}\theta_{i+2} \ldots \theta_{i+j}\rho_{j+1}\rho_{j+2} \ldots \rho_k$ is order-isomorphic to D_k and $x \in W(\rho')$. This gives a contradiction with the fact that ρ is the maximal element of $\mathcal{W}(x_1)$. Therefore, the case (ii) never occurs.

Let us examine (iii): $\rho_1 \in \alpha_i^+$. If there exists j, $1 \le j \le k-1$ such that $\rho_j \in \alpha_i^+$ and $\rho_{j+1} \in \alpha_i^-$, then we consider ℓ being the smallest integer $j+1 \le \ell \le k-1$, such that ρ_ℓ belongs to $\alpha_i^- \ldots \alpha_{k-1}^-$ but not $\rho_{\ell+1}$ (ℓ exists since ρ_k is on the right of θ_k). So, with the same argument as for (ii), Fact 3' necessarily induces that $\rho' = \rho_1 \ldots \rho_j\theta_{i+1}\theta_{i+2} \ldots \theta_{i+\ell-j}\rho_{\ell+1} \ldots \rho_k$ is a subsequence in $\mathcal{W}(x_1)$ such that $\rho \trianglelefteq \rho'$. This is a contradiction. Therefore, the subsequence $\rho^1 = \rho$ verifies the property that any value of ρ^1 is necessarily in $\alpha_i^+\theta_{i+1}\alpha_{i+1}^+ \ldots \alpha_{k-1}^+\theta_k\alpha_k\theta_1\alpha_1^-\theta_2\alpha_2^- \ldots \alpha_{i-1}^-\theta_i$. Less formally, we say that ρ_1 lies over θ in σ^i. Since this property is crucial for the following of the proof, we call this property the *dominance property* of ρ^1 over θ.

Now we consider the smallest j_1, $1 \le j_1 \le k-1$, such that $\rho^1_{j_1}$ belongs to α_i but not $\rho^1_{j_1+1}$. By considering Fact 4 (or 4') with the dominance property of ρ^1 over θ, we deduce $j_1 < i-1$.

Now, we replace $x_1 = x$ by the value x_2 of α_i just after ρ^1_1 (if it exists; otherwise, we take $x_2 = \theta_{i+1}$). By hypothesis, x_2 is also covered by a witnessed segment $W(\rho^2)$ where ρ^2 is another pattern order-isomorphic to D_k, and such that ρ_k^2 belongs in σ^i on the right of ρ_k^1 or on the left of x_2. With the same argument as above, ρ^2 has the dominance property over ρ^1, *i.e*, less formally, the sequence ρ^2 lies over ρ^1. Therefore, ρ^2 necessarily contains at most $j_2 \leq j_1 - 1$ values in α_i^+.

By iterating the reasoning for the value just after the first value of ρ^2, we construct a subsequence ρ^3 such that the number j_3 of its values in α_i^+ verifies $j_3 \leq j_2 - 1$.

The process finishes after at most $i-1$ steps. This means that α_i can be covered by the witnessed segments $W(\rho^\ell)$ for some ℓ, such that $1 \leq \ell \leq i-2 \leq k-2$, *i.e.* with at most $i-2$ witnessed segments, which is a bound depending on k alone. Therefore, $\alpha_i\theta_{i+1}$ can be covered by at most $i-1$ witnessed segments. Thus, σ is covered by at most $1 + \sum_{\ell=1}^{k-1} \ell = 1 + \frac{k(k-1)}{2}$ witnessed segments. □

Theorem 2 *For* $2 \leq \ell \leq 2k-1$*,* $cc(Av_{2k-\ell}(D_k)) = S_{2k-\ell}$*.*

Proof. The proof is straightforward for $2k - \ell \leq k$, *i.e.*, $\ell \geq k$. In the following of the proof, we take ℓ such that $2 \leq \ell \leq k-1$ and we etablish the result by contradiction. So, let us assume that there exists σ of length $2k-\ell$ such that $\sigma \notin \mathrm{cc}(\mathrm{Av}_{2k-\ell}(D_k))$ with $2 \leq \ell \leq k-1$. Modulo cyclic rotation, we set $\sigma_1 = 2k-\ell$. Let $\theta = \theta_1 \dots \theta_k$ be the leftmost subsequence of σ order-isomorphic to D_k. We have $\theta_1 = 2k-\ell$ and we discuss on the value of θ_k: (i) $\theta_k = k-\ell+1$ and (ii) $\theta_k \leq k-\ell$.

Case (i). We deduce $\theta_1 = 2k-\ell, \theta_2 = 2k-\ell-1, \dots,$ and $\theta_k = k-\ell+1$. The cyclic rotation σ^1 of σ beginning with θ_k also contains a subsequence θ^1 order-isomorphic to D_k. As θ is the leftmost pattern in σ, θ^1 does not begin after θ_1 in σ^1. This means that θ_1^1 is at most $k-\ell+1$, and thus $\theta_k^1 \leq k-\ell+1-k+1 = 2-\ell \leq 0$ which induces a contradiction.

Case (ii). We have $\theta_k \leq k-\ell$. The cyclic rotation σ^1 of σ beginning with θ_k also contains a subsequence θ^1 order-isomorphic to D_k. We consider the leftmost subsequence in σ^1. We necessarily have θ_1^1 on the left of θ_1 and θ_k^1 on the right of θ_1 (in σ^1). Moreover θ^1 and θ have a nonempty intersection (since $2k-\ell < 2k$). Let σ^2 be the cyclic rotation of σ beginning with θ_k^1 and θ^2 be the leftmost subsequence in σ^2 order-isomorphic to D_k. We necessarily have: θ_1^2 is on the left of θ_k and after θ_k^1 (in σ^2). θ^2 necessarily contains at least one element x of $\theta_1^1 \dots \theta_{k-1}^1$. We discuss on the position of x relatively to θ_1.

If x is after θ_1 in σ^2, then we also have θ_k^2 after θ_1.

If x is before θ_1 in σ^2. As θ^2 also contains an element y of θ such that y appears after x, we deduce that θ_k^2 is on the right of θ_1 in σ^2. With the same reasoning, we construct an infinite sequence of D_k-patterns θ^i different verifying the property that θ_1 is between θ_1^i and θ_k^i which provides a contradiction. □

Theorem 3 *The set* $cc(Av_{2k-1}(D_k))$ *is enumerated by* $(2k-1)! - (2k-1)c_{k-1}$ *where* $(c_k)_{k\geq 1}$ *is the well-known Catalan sequence.*

Proof. It suffices to prove that permutations of length $2k-1$ such that $\sigma \notin \mathrm{cc}(\mathrm{Av}_{2k-1}(D_k))$ and $\sigma_1 = 2k-1$ are enumerated by c_{k-1}. Let $\sigma = \sigma_1\sigma_2 \dots \sigma_{2k-1}$ be a permutation that does not belong to $\mathrm{cc}(\mathrm{Av}_{2k-1}(D_k))$ and such that $\sigma_1 =$

$2k-1$. Obviously σ contains the pattern D_k. We consider the leftmost pattern $\theta_{i_1}\theta_{i_2}\dots\theta_{i_k}$ in σ (we necessarily have $\theta_{i_1} = 2k-1$).

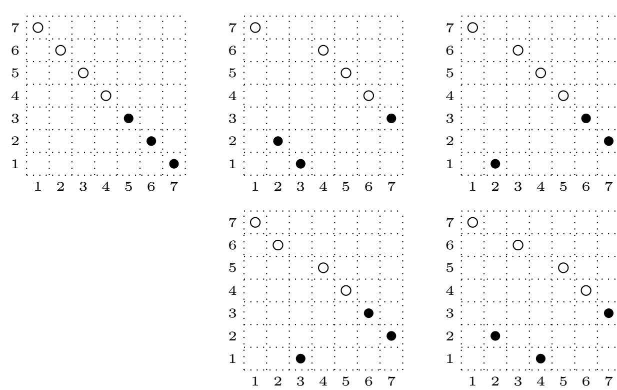
Now we discuss on the value of θ_{i_k}.

Case 1: $\theta_{i_k} = k$. This means that $\theta_{i_j} = 2k-j$ for $1 \leq j \leq k$, $\sigma_i \leq k-1$ for $i > i_k$ and σ contains a decreasing subsequence order-isomorphic to D_k. Let σ^1 be the cyclic rotation of σ beginning with k. By hypothesis, σ^1 contains a subsequence order-isomorphic to D_k. The only one possibility is that this subsequence is exactly $k(k-1)\dots 21$.

Moreover, remark that the value just after $k+1$ is necessarily at least k, *i.e*, $k+1$ and k are consecutive in σ. This is due to the hypothesis that θ is the leftmost pattern in σ. Moreover, for the same reason, two successive decreasing values can not appear between $k+2$ and $k+1$. More generally, there does not exist i decreasing values after $k+i$ for $0 \leq i \leq k-1$ which characterizes the Catalan numbers. Conversely, such a permutation does not belong to $\text{cc}(\text{Av}_{2k-1}(D_k))$. See Figure 2 for $k=4$.

Case 2: $\theta_{i_k} \leq k-1$. We repeat *mutatis mutandis* the reasoning of the case (ii) in the proof of Theorem 2. □

Figure 2: The five permutations σ beginning with 7 such that $\sigma \notin \text{cc}(\text{Av}_7(D_4))$: **7654**321, **7**21**654**3, **7**1**654**32, **76**1**54**32, **7**2**6**1**54**3. The leftmost pattern θ is illustrated in boldface for the one-line notation of σ and with empty points in the representations below. The corresponding well-formed parentheses are respectively ((())), (())(), (()()), ()(()) and ()()().



Notice that a $(2k-1)$-length permutation $\sigma \notin \text{cc}(\text{Av}(D_k))$ beginning with $2k-1$ does not contain the pattern 123. Moreover, the previous proof induces a constructive bijection between the set of well-formed parentheses of size $2k-2$ and the set of permutations of length $2k-1$, beginning with $2k-1$, and containing a pattern D_k in each of their cyclic rotations. Indeed, we consider the binary representation $b = b_1\dots b_{2k-2}$ of a well-formed parentheses, *i.e.* $b_i = 0$ if the ith parenthesis is a closed parenthesis, $b_i = 1$ otherwise). For convenience we add $b_{2k-1} = 0$ on the right of b. Now, let j, $1 \leq j \leq 2k-1$, be the rank of the rightmost zero on the left of the rightmost one in b (if j does not exist we set $j = 2k-1$). Then, we traverse b from right to left (from indices j to 1), and we label each zero in increasing order from 1. Finally, we continue to label (in increasing order) the unlabeled elements of b from right to left. For instance, the parenthesis ()((()())((()))) has a binary representation 10111010**0**11100000 where $j = 9$ (in boldface). Its corresponding permutation 17 4 16 15 14 3 13 21 12 11 10 9 8 7 6 5 contains the pattern D_9 in each cyclic rotation. See Figure 2 when $k = 4$.

Theorem 1 shows that the basis B of $\mathrm{cc}(\mathrm{Av}(D_k))$ is finite. Theorems 2 and 3 imply that the smallest length of a minimal permutation is $2k-1$, and that these permutations are enumerated by $(2k-1) \cdot c_{k-1}$. It remains to characterize the other elements of B. Notice that Albert et al. have experimentally obtained the basis elements of cc(Av(321)) which are 15432, 14325, 164253, 163254 and 1472536. We have experimentaly checked that the basis of cc(Av(4321)) contains 5 minimal permutations of length 7, 32 of length 8, 54 of length 9, and 136 of length 10.

Theorem 4 *The inversions of length $2k-1$ which do not lie in $cc(Av(D_k))$ are enumerated by $\frac{k(k+1)}{2}$.*

Proof. Let σ be an involution of length $2k-1$ that does not belong to $\mathrm{cc}(\mathrm{Av}(D_k))$. We distinguish two cases: (i) there exists i, $k \leq i \leq 2k-2$, such that $\sigma_{2k-i} = 2k-1$; and (ii) there exists i, $1 \leq i \leq k-1$, such that $\sigma_{2k-i} = 2k-1$.

The first case (i) implies that there is , $k \leq i \leq 2k-2$, such that $\sigma_{2k-i} = 2k-1$ and $\sigma_{2k-1} = 2k-i$. Moreover, if we take j, $1 \leq j < 2k-i$, such that $\sigma_{2k-j} = 1$ then this induces $\sigma_1 = 2k-j$. This means that the cyclic rotation of σ beginning with $2k-1$, contains from right to left the subsequence 1, $2k-i$, $2k-j$ which contradicts the remark just after Theorem 3. Thus, there exists j, $1 \leq j < 2k-i$, such that $\sigma_j = 1$ and $\sigma_1 = j$. A similar argument as above allows us to conclude that σ is necessarily $2k-i-1, \ldots, 1, 2k-1, \ldots, 2k-i$.

For the second case, there exists i, $1 \leq i \leq k-1$, such that $\sigma_{2k-i} = 2k-1$ and $\sigma_{2k-1} = 2k-i$. In the same way as above, we deduce that there does not exist j, $j > 2k-i$, such that $\sigma_j = 1$. Thus, let us consider j, $j < 2k-i$, such that $\sigma_j = 1$ and $\sigma_1 = j$. If $j = 2k-i-1$ then it is straightforward to see that we necessarily have $\sigma = (2k-2)\ldots 1(2k-1)\ldots(2k-i)$. If $j \in [k-i, 2k-i-2]$ then we easily see that there does not exist some involutions verifying this hypothesis. If $j \in [1..k-i-1]$ then the only one involution is $\sigma = j \ldots 1(2k-i-1)\ldots(j+1)(2k-1)\ldots(2k-i)$. To summarize, there are $\sum_{i\in[1..k-1]}(k-i-1) + k = \frac{k(k+1)}{2}$ such involutions. □

For $k = 3$ there are 6 involutions of length 5 that do not lie in $\mathrm{cc}(\mathrm{Av}(D_3))$: 54321, 15432, 21543, 32154, 43215 and 14325.

3 Lift closure

In this section, we study the lift closure $\mathrm{cl}(\mathcal{C})$ of a closed class $\mathcal{C}$, *i.e.*, the class of permutations that can be obtained by a lift of a permutation in $\mathcal{C}$. We provide several general results which make links between cyclic and lift closures.

Lemma 2 *If X is a set of permutations then $\pi \in cc(X) \Longleftrightarrow \pi^{-1} \in cl(X^{-1})$. Consequently, if $X = Av(B)$ then $cc(Av(B)) = cl(Av(B^{-1}))^{-1}$. Moreover, if the basis B is stable by inversion, i.e. $B = B^{-1}$, then $cc(Av(B)) = cl(Av(B))^{-1}$.*

Proof. Indeed, $\pi \in \mathrm{cc}(X)$ means there exists $\sigma = \sigma_1 \dots \sigma_n \in X$ such that $\pi = r^k(\sigma) = \sigma_k \dots \sigma_n \sigma_1 \dots \sigma_{k-1}$. With $r(\sigma)^{-1} = \ell(\sigma^{-1})$ we deduce $\pi^{-1} = \ell^k(\sigma^{-1})$. Thus $\pi^{-1} \in \mathrm{cl}(\mathrm{Av}(X^{-1}))$. By symmetry, we conclude $\pi^{-1} \in \mathrm{cl}(\mathrm{Av}(X^{-1}))$ implies $\pi \in \mathrm{cc}(X)$. The straightforward relation $\mathrm{Av}(B^{-1}) = \mathrm{Av}(B)^{-1}$ induces the consequences. □

Now we set $\mathcal{C} = \mathrm{Av}(B)$ where B is a basis. As for the case of the cyclic rotation, $\mathrm{cl}(\mathcal{C})$ also is a class of avoiding permutations. Let σ be a permutation in $\mathcal{C}$ and θ be a pattern (in B) of σ. We define by *witnessed interval* of σ the cyclic interval $W'(\theta) = [n] \backslash] \min(\theta) .. \max(\theta)]$. Notice that $W'(\theta)$ is an interval in $[n]$ considered cyclically, but it is not necessarily a segment in σ. We can easily remark that $W'(\theta)$ is also the witnessed segments of σ^{-1} (considered cyclically) relatively to the pattern θ^{-1}. Two similar results of Lemma 1 and Proposition 1 can be deduced below.

Lemma 3 *$\sigma \notin cl(\mathcal{C})$ if and only if the witnessed intervals $W'(\theta)$ cover $[n]$ (or equivalently σ).*

Proof. By applying Lemmas 1 and 2, we obtain: $\sigma \notin \mathrm{cl}(\mathcal{C})$ if and only if $\sigma^{-1} \notin \mathrm{cc}(\mathcal{C}^{-\infty})$, *i.e.* iff the witnessed segments (relatively to θ^{-1} and σ^{-1}) cover σ^{-1} (or equivalently $[n]$). With the remark above, this is equivalent to the covering of $[n]$ by the witnessed intervals $W'(\theta)$ of σ. □

Proposition 2 *Let $\mathcal{C} = Av(B)$, where B is finite and suppose there is a bound Δ depending on B alone such that, for all $\sigma \notin cl(X)$, there is a collection of at most Δ witnessed intervals that cover σ (or equivalently $[n]$). Then $cl(X)$ is finitely based.*

Proof. The proof is a direct consequence of Proposition 1 and Lemma 3. □

Theorem 5 *Let $k \geq 2$ be an integer. Then $cl(Av(D_k))$ is finitely based.*

Proof. This theorem is a consequence of Theorem 1 combined with Lemma 2. Indeed, we have $\mathrm{cl}(\mathrm{Av}(D_k)) = \mathrm{cc}(\mathrm{Av}(D_k))^{-1} = \mathrm{Av}(B)^{-1} = \mathrm{Av}(B^{-1})$ where B is a finite basis. □

All enumeration results of [1] for the cyclic closure are also valid for the lift closure.

Theorem 6 *$cl(Av_{2k-1}(D_k))$ is enumerated by $(2k-1)! - (2k-1)c_{k-1}$ where $(c_k)_{k \geq 1}$ is the well-known Catalan sequence.*

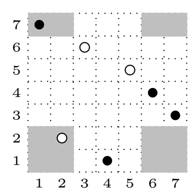
Proof. This corollary is deduced from Theorem 3. □

4 Going further and conclusion

In this section, we give some general results about the toric closure $\mathrm{tr}(\mathcal{C})$ for a class $\mathcal{C}$ of permutations, *i.e.* $\mathrm{tr}(\mathcal{C}) = \mathrm{cl}(\mathrm{cc}(\mathcal{C})) = \mathrm{cc}(\mathrm{cl}(\mathcal{C}))$. Let $f_{u,v}$, $1 \leq u, v \leq n$, be the function defined on S_n by $f_{u,v}(\sigma) = \sigma'$ where $\sigma'_j = (\sigma_{(u+j-2) \mod (n)+1} - v) \mod (n) + 1$, for $j \in [1..n]$. The set $\{f_{u,v}(\sigma),\ 1 \leq u, v \leq n\}$ contains exactly all toric rotations of σ since the permutation $f_{u,v}(\sigma)$ equals the permutation $r^{n-u+1}(\ell^{n-v+1}(\sigma))$.

Now assume that $\mathcal{C} = \mathrm{Av}(B)$ where B is a finite set of permutations. Let θ be a pattern (relatively to B) in $\sigma \in \mathcal{C}$. The *witnessed area* $W''(\theta)$ of θ in σ is the direct product of $\sigma^{-1}(W(\theta))$ of θ with the witnessed interval $W'(\theta)$: $W''(\theta) = \sigma^{-1}(W(\theta)) \times W'(\theta)$. For instance, the permutation **7**2**6**1**5**43 contains the pattern 132 ($\theta = 265$), $W(\theta)$ is the segment 4372, thus $\sigma^{-1}(W(\theta)) = \{6, 7, 1, 2\}$, $W'(\theta)$ is the interval 712, and $W''(\theta)$ is the area $\{6, 7, 1, 2\} \times \{7, 1, 2\}$. See Figure 3 for an illustration.

Figure 3: The witnessed area $W''(265)$ for the permutation 7261543.



We also obtain similar general results as for the cyclic (or lift) closure.

Lemma 4 *$\sigma \notin tr(\mathcal{C})$ if and only if the witnessed areas cover the area $[n] \times [n]$ where n is the length of σ.*

Proof. If the witnessed areas of σ cover $[n] \times [n]$, then no toric rotation of σ lies in $\mathcal{C}$. Indeed, the toric rotations σ' of σ defined by $\sigma'_j = (\sigma_{(u+j-2) \mod (n)+1} - v) \mod (n) + 1$, for $j \in [1..n]$, where $(u, v) \in W''(\theta)$, all contain θ as a pattern without wrap-around and none of these permutations lie in $\mathcal{C}$. Conversely, if the witnessed areas do not cover a point (u, v) then the toric rotation σ' of σ defined as above, contains no pattern (without wrap-around) and thus lies in $\mathrm{Av}(\mathcal{C})$. Consequently, σ lies in $\mathrm{tr}(\mathrm{Av}(\mathcal{C}))$. □

Proposition 3 *Let $\mathcal{C} = Av(B)$, where B is finite and suppose there is a bound Δ depending on B alone such that, for all $\sigma \notin tr(\mathcal{C})$, there is a collection of at most Δ witnessed areas that cover σ. Then $tr(\mathcal{C})$ is finitely based.*

Proof. The proof are obtained mutatis mutandis as for Proposition 1. □

We have obtained a proof that $\mathrm{tr}(\mathrm{Av}(321))$ is finitely based, but we do not present it because it is very technical and requires a lot of cases. Experimental calculations suggest that this basis contains one element of length 5, 2 of length

7, 2 of length 8, 39 of length 9 and 2 of length 10. Thus the problem of finding a nice proof that tr(Av(321)) is finitely based remains open. We also have the following open questions:
Problem 1: Is tr(Av(231)) finitely based?
Problem 2: Is tr(Av($k \dots 21$)) finitely based for $k \geq 4$?
Problem 3: Given a finite basis B, is it decidable if cc(Av(B)) is finitely based?

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