Cyclic and lift closures for $k\ldots21$-avoiding permutations

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Abstract
We prove that the cyclic closure of the permutation class avoiding the pattern $k(k-1)\ldots 21$ is finitely based. The minimal length of a minimal permutation is $2k-1$ and these basis permutations are enumerated by $(2k-1)c_k$ where $c_k$ is the $k$th Catalan number. We also define lift operations and give similar results. Finally, we consider the toric closure of a class and we propose some open problems.

Keywords: Pattern avoiding permutation class; cyclic, lift and toric closures.

1 Introduction and notation
Let $S$ (resp. $S_n$) be the class of permutations (resp. of length $n$). We represent permutations in one-line notation, i.e. if $i_1, i_2, \ldots, i_n$ are $n$ distinct values in $[n] = \{1, 2, \ldots, n\}$, we denote the permutation $\sigma \in S_n$ by the sequence $i_1 i_2 \ldots i_n$ if $\sigma(k) = i_k$ for $1 \leq k \leq n$. For instance, the identity permutation of length $n$, $Id_n$, will be written $1 2 \ldots n$. A segment of a permutation $\sigma \in S_n$ contains the pattern $\pi \in S_k$, $k \geq 2$, if and only if a sequence of indices $1 \leq i_1 < i_2 < \ldots < i_k \leq n$ exists such that $\sigma(i_1)\sigma(i_2)\ldots\sigma(i_k)$ is ordered as $\pi$. We write $\pi \prec \sigma$ to denote that $\pi$ is a pattern of $\sigma$. A permutation $\sigma$ that does not contain $\pi$ as a pattern is said to avoid $\pi$. Let $B$ be a set of permutations. We denote by $Av_n(B)$ (resp. $Av(B)$) the set (resp. the class) of permutations in $S_n$ (resp. $S$) avoiding all patterns $\pi \in B$. For example, $25314 \not\in Av_5(123)$ but $43152 \in Av_5(123)$. See for instance [2, 3]. In the following, a subsequence of a permutation $\sigma$ order-isomorphic to $D_k = k(k-1)\ldots 21$ will be called a $D_k$-pattern of $\sigma$. A class $C$ of permutations is closed (or stable) for the involvement relation $\prec$ if, for any $\sigma \in C$, for
any $\pi \prec \sigma$, then we also have $\pi \in \mathcal{C}$. Such a class can always be defined by a class $\text{Av}(B)$ of pattern-avoiding permutations. In the case where $B$ is minimal (relatively to the relation $\prec$), $B$ is called a basis.

Let $\sigma = \sigma_1\sigma_2\ldots\sigma_n$ be a permutation in $S_n$. We define two bijections $r$ (cyclic rotation) and $\ell$ (lift rotation) on $S_n$ as: $r(\sigma) = \sigma_n\sigma_1\sigma_2\ldots\sigma_{n-1}$ and $\ell(\sigma) = (\sigma_1 \mod (n) + 1)(\sigma_2 \mod (n) + 1)\ldots(\sigma_n \mod (n) + 1)$. For the sake of simplicity, we will say that a permutation $\sigma' \in S_n$ is a cyclic rotation (resp. lift rotation) of a permutation $\sigma \in S_n$ if $\sigma' = r^k(\sigma)$ (resp. $\sigma' = \ell^k(\sigma)$) for some $k \in [1..n]$. For instance, 231 and 312 are both cyclic rotations of 123 and 132 are both lift rotations of 321 in $S_3$. The cyclic closure $\text{cc}(\mathcal{C})$ (resp. lift closure $\text{cl}(\mathcal{C})$) of a class $\mathcal{C}$ is the class of permutations that can be obtained by a cyclic rotation (lift rotation) from a permutation in $\mathcal{C}$. Finally, the toric closure $\text{tr}(\mathcal{C})$ of $\mathcal{C}$ is defined by $\text{cc}(\text{cl}(\mathcal{C}))$ which is also equal to $\text{cl}(\text{cc}(\mathcal{C}))$.

Given a closed class of permutations, two well-studied problems are that of finding its enumeration and its basis. We are interested in these questions for cyclically closed classes, introduced by Albert et al. (See [1]).

Let $\mathcal{C}$ be the class $\text{Av}(B)$ where $B$ is a set of patterns. We know that $\text{cc}(\mathcal{C})$ is a pattern-avoiding class where its basis is the set of permutations that are minimal with respect to not lying in $\text{cc}(\mathcal{C})$. Let $\sigma$ be a permutation in $\mathcal{C}$ and $\theta$ be a pattern of $\sigma$, i.e, a subsequence of $\sigma$ order-isomorphic to a member of $B$. Modulo a cyclic rotation of $\sigma$, we assume that $\sigma = \alpha\beta\sigma'$ where $\alpha$ (resp. $\beta$) is the first (resp. last) value of $\theta$. The witnessed segment $W(\theta)$ of $\theta$ is the segment $\alpha\beta$. Albert et al. give a necessary and sufficient condition for $\sigma$ to be (or not) in the cyclic closure of $\mathcal{C}$.

**Lemma 1** [1] $\sigma \notin \text{cc}(\mathcal{C})$ if and only if the witnessed segments cover $\sigma$.

Moreover, they obtain a sufficient condition for the class $\text{cc}(\mathcal{C})$ to be finitely based.

**Proposition 1** [1] Let $\mathcal{C} = \text{Av}(B)$, where $B$ is finite and suppose there is a bound $\Delta$ depending on $B$ alone such that, for all $\sigma \notin \text{cc}(\mathcal{C})$, there is a collection of at most $\Delta$ witnessed segments that cover $\sigma$. Then $\text{cc}(\mathcal{C})$ is finitely based.

Albert et al. deduce from Proposition 1 that $\text{cc}(\text{Av}(321))$, $\text{cc}(\text{Av}(231))$ and $\text{cc}(\text{Av}(4321))$ are finitely based. They also show that the cyclic closure of a finitely based class can fail to be finitely based (it is the case for the class $\text{Av}(265143)$). They provide several results about enumeration.

This paper is organized as follows. In Section 2, we prove that the cyclic closure of $\text{Av}(k(k-1)\ldots21)$ has a finite basis $B$. Moreover, we prove that the smallest length of a minimal permutation of $B$ is $2k - 1$, and that these basis permutations are enumerated by $(2k-1)c_k$ where $c_k$ is the $k$th Catalan number. We also characterize these permutations. In Section 3, we investigate the lift closure of closed classes. We show a duality with the cyclic closure which allows to prove that the lift closure of $\text{Av}(k(k-1)\ldots21)$ is finitely based. In Section 4, we study the toric closure. We give a sufficient condition for the class $\text{tr}(\mathcal{C})$ to be finitely based. We have obtained a proof that $\text{tr}(\text{Av}(321))$ is finitely based.
based, but we do not give it. Indeed, the proof requires many cases to check and it does not contain any interesting concept. We finish the paper by presenting several open problems.

2 Cyclic closure

Let \( \sigma = \sigma_1 \ldots \sigma_n \) be a permutation of length \( n \) such that \( \sigma \not\in \mathrm{cc}(\mathrm{Av}(D_k)) \) for \( k \geq 3 \). For \( x \in \{1, \ldots, n\} \), we define \( \mathcal{W}_\sigma(x) \) to be the set of subsequences \( \theta = \theta_1 \theta_2 \ldots \theta_k \) of \( \sigma \) (considered cyclically), order-isomorphic to \( D_k = k(k-1) \ldots 21 \) and such that \( x \in W(\theta) \). Let \( \mathcal{W}_\sigma \) be the union of all \( \mathcal{W}_\sigma(x) \) for \( 1 \leq x \leq n \). In the following, we will omit the subscript \( \sigma \) for \( \mathcal{W}_\sigma(x) \) and \( \mathcal{W}_\sigma \) since it should be clear from context. For \( \theta, \theta' \in \mathcal{W} \), we define an order relation \( \leq \) by: \( \theta' \leq \theta \) if and only if (a) \( |W(\theta')| < |W(\theta)| \); or (b) \( |W(\theta')| = |W(\theta)| \) and the associated word \( \theta'_1 \theta'_2 \ldots \theta'_k \) is smaller than \( \theta_1 \theta_2 \ldots \theta_k \) in lexicographic order. Thus \( \mathcal{W} \) is a non-empty finite totally-ordered set. Let \( \theta \) be its maximal element. Then, \( \sigma \) has a special structure illustrated in Figure 1 and described below.

Indeed, \( \sigma \) can be written (modulo rotation):

\[
\sigma = \theta_1 \alpha_1 \theta_2 \alpha_2 \ldots \theta_{k-1} \alpha_{k-1} \theta_k \alpha_k
\]

where \( W(\theta) = \alpha_k \theta_1 \), \( \theta_i \in \{1, 2, \ldots, n\} \) and \( \alpha_i \) are segments of \( \sigma \) for \( i \in [1, k] \).

By convenience, we set \( \theta_{k+1} = 0 \) and \( \theta_0 = n + 1 \). For \( i \in [1, k-1] \), we define the subsequence \( \alpha_i^- \) (resp. \( \alpha_i^+ \)) of elements in \( \alpha_i \) that are smaller than \( \theta_{i+1} \) (resp. greater than \( \theta_i \)). On the other hand, \( \alpha_{i,j}^-, i \in [1, k-2], j \in [i+2, k+1] \), (resp. \( \alpha_{i,j}^+, i \in [2, k-1], j \in [0, i-2] \)) is the subsequence of \( \alpha_i^- \) (resp. \( \alpha_i^+ \)) constituted by all values greater than \( \theta_j \) (resp. smaller than \( \theta_j \)).

With these hypotheses, \( \sigma \) verifies the following structural properties.

**Fact 1.** For \( i \in [1, k-1] \), \( \alpha_i \) does not contain any value in the interval \( [\theta_{i+1}, \theta_i] \).

If there exists \( y \in \alpha_i \) such that \( y \in [\theta_{i+1}, \theta_i] \), then \( \theta' = \theta_1 \ldots \theta_i y \theta_{i+1} \ldots \theta_{k-1} \) is a subsequence of \( \sigma \) belonging to \( \mathcal{W} \) such that \( |W(\theta')| > |W(\theta)| \) which is a contradiction with the maximality of \( \theta \).

**Fact 2.** For \( i \in [1, k-1] \), \( \alpha_i \) does not contain any value in \( [\theta_i, \theta_{i-1}] \).

If there exists \( y \in \alpha_i \) such that \( y \in [\theta_i, \theta_{i-1}] \), then \( \theta' = \theta_1 \ldots \theta_{i-1} y \theta_{i+1} \ldots \theta_k \) is a subsequence of \( \sigma \) belonging to \( \mathcal{W} \) such that \( \theta \leq \theta' \) which is a contradiction.

**Fact 3.** For \( i \in [1, k-2], j \in [i+2, k] \), \( \alpha_{i,j}^- \) lies in \( \mathrm{Av}(D_{j-i}) \) and \( \alpha_i^- \) lies in \( \mathrm{Av}(D_{k-i}) \).

If \( \alpha_{i,j}^- \) contains a subsequence \( \rho_1 \rho_2 \ldots \rho_{j-i} \) order-isomorphic to \( D_{j-i} \), then \( \theta' = \theta_1 \ldots \theta_i \rho_1 \ldots \rho_{j-i} \theta_{j-i} \ldots \theta_k \) belongs to \( \mathcal{W} \) and verifies \( |W(\theta')| > |W(\theta)| \) which is a contradiction.

**Fact 4.** For \( i \in [2, k-1], j \in [0, i-2] \), \( \alpha_{i,j}^+ \) lies in \( \mathrm{Av}(D_{i-j}) \).
If $\alpha_{i,j}^+$ contains a subsequence $\rho_1, \rho_2, \ldots, \rho_{j-1}$ order-isomorphic to $D_{i-j}$, then $\theta' = \theta_1 \ldots \theta_j \rho_1 \ldots \rho_{j-1} \theta_{i+1} \ldots \theta_{k-1}$ belongs to $W$ and verifies $\theta \leq \theta'$ which is a contradiction.

The following facts are simple generalizations of Facts 3 and 4. The proofs are obtained *mutatis mutandis*.

**Fact 3'.** For $i \in [1, k-2]$, $j \in [i+2, k]$, $\ell \in [i+1, k-1]$, $\alpha_{i,j} \theta_{i+1} \alpha_{i+1,j} \theta_{i+2} \ldots \alpha_{\ell-1,j} \theta_\ell$ lies in $\text{Av}(D_{j-i})$ and $\alpha_{i,j}^- \theta_{i+1} \alpha_{i+1,j}^- \theta_{i+2} \ldots \alpha_{\ell-1,j}^- \theta_\ell$ lies in $\text{Av}(D_{k-i})$.

**Fact 4'.** For $i \in [2, k-1]$, $\ell \in [i+1, k]$, $j \in [0, \ell-3]$, $\alpha_{i,j}^+ \theta_{i+1} \alpha_{i+1,j}^+ \theta_{i+2} \alpha_{i+2,j}^+ \ldots \alpha_{\ell-1,j}^+ \theta_\ell$ lies in $\text{Av}(D_{\ell-j-1})$.

Figure 1: The special structure of a permutation $\sigma \notin \text{Av}(D_k)$. A filled area does not contain any point of the form $(i, \sigma_i)$, $1 \leq i \leq n$. A sequence of $j$ increasing diagonal in an area means that it does not contain a pattern $(j+1)j\ldots 21$.

**Theorem 1** Let $k \geq 3$ be an integer. Then $cc(\text{Av}(D_k))$ is finitely based.
Proof. Let $\sigma = \sigma_1 \ldots \sigma_n$ be a permutation of length $n$ which does not belong to $cc(\Av(D_k))$. The crucial point of the proof is to find a bounded number (independently of $n$) of witnessed segments that cover $\sigma$. Finally, we will conclude through Proposition 1.

Let $\theta = \theta_1 \ldots \theta_k$ be the maximal element of $W$. Modulo cyclic rotation, $\sigma$ can be decomposed as follows:

$$\sigma = \theta_1 \alpha_1 \theta_2 \alpha_2 \ldots \alpha_{k-1} \theta_k \alpha_k.$$

Notice that $\alpha_k$ and $\theta_1 = \sigma_1$ are obviously covered by $W(\theta)$ since $W(\theta) = \alpha_k \theta_1$. Thus, it suffices to prove that $\alpha_i \theta_{i+1}$ for $1 \leq i \leq k-1$ are covered by a bounded number of witnessed segments.

Let $i$ such that $1 \leq i \leq k - 1$ and $\sigma^i$ be the rotation of $\sigma$ defined by $\sigma^i = \alpha_i \theta_{i+1} \ldots \theta_k \alpha_k \theta_1 \alpha_i \ldots \alpha_{i-1} \alpha_{i-1} \theta_i$. By hypothesis (Remark 3), there is a subsequence $\rho$ of $\sigma^i$ (considered cyclically) such that $\theta_{i+1} \in W(\rho)$. If $\alpha_i$ is empty, there is nothing to do since $\alpha_i \theta_{i+1}$ is covered by $W(\rho)$. Now, let us suppose that $\alpha_i$ is not empty, and let us consider the first value $x_1$ of $\alpha_i$. Then, there is a subsequence $\rho = \rho_1 \rho_2 \ldots \rho_k$ of $\sigma^i$ order-isomorphic to $D_k$ such that $x_1 \in W(\rho)$. We take for $\rho$ the maximal element of $W(x_1)$.

We discuss on the first value $\rho_1$ of $\rho$: (i) $\rho_1 \notin \alpha_i$; (ii) $\rho_1 \in \alpha_i^-$ and (iii) $\rho_1 \in \alpha_i^+$. The case (i) is straightforward since this means that $\alpha_i \theta_{i+1}$ is included into $W(\rho)$.

Let us examine (ii): $\rho_1 \in \alpha_i^-$. Notice that $\rho_k$ appears on the right of $\theta_k$ (in $\sigma^i$), since otherwise $\theta$ would not be maximal in $W$ (because $\theta_1 \in W(\rho)$ and $|W(\rho)| > |W(\theta)|$). So, let $j$ be the smallest integer, $1 \leq j \leq k - 1$, such that $\rho_j$ belongs to $\alpha_i^- \alpha_{i+1} \ldots \alpha_{k-1}$ but not $\rho_{j+1}$. Then, the special structure (particularly Fact 3') of $\sigma$ necessarily induces that the subsequence $\rho' = \theta_{i+1} \theta_{i+2} \ldots \theta_{i+j} \rho_{j+1} \rho_{j+2} \ldots \rho_k$ is order-isomorphic to $D_k$ and $x \in W(\rho')$. This gives a contradiction with the fact that $\rho$ is the maximal element of $W(x_1)$. Therefore, the case (ii) never occurs.

Let us examine (iii): $\rho_1 \in \alpha_i^+$. If there exists $j$, $1 \leq j \leq k - 1$ such that $\rho_1 \in \alpha_i^+$ and $\rho_{j+1} \in \alpha_i^-$, then we consider $\ell$ being the smallest integer $j + 1 \leq \ell \leq k - 1$, such that $\rho_{\ell}$ belongs to $\alpha_i^- \ldots \alpha_{k-1}$ but not $\rho_{\ell+1}$ ($\ell$ exists since $\rho_k$ is on the right of $\theta_k$). So, with the same argument as for (ii), Fact 3' necessarily induces that $\rho' = \rho_1 \ldots \rho_j \theta_{i+1} \theta_{i+2} \ldots \theta_{i+\ell-j} \rho_{\ell+1} \ldots \rho_k$ is a subsequence in $W(x_1)$ such that $\rho \leq \rho'$. This is a contradiction. Therefore, the subsequence $\rho' = \rho$ verifies the property that any value of $\rho'$ is necessarily in $\alpha_i^+ \theta_{i+1} \alpha_{i+1}^+ \ldots \alpha_{k-1}^+ \theta_k \alpha_k \theta_1 \alpha_1 \theta_2 \alpha_2 \ldots \alpha_{i-1} \theta_i$. Less formally, we say that $\rho_1$ lies over $\theta$ in $\sigma^i$. Since this property is crucial for the following of the proof, we call this property the domination property of $\rho$ over $\theta$.

Now we consider the smallest $j_1$, $1 \leq j_1 \leq k - 1$, such that $\rho_1^{j_1}$ belongs to $\alpha_i$ but not $\rho_1^{j_1+1}$. By considering Fact 4 (or 4') with the domination property of $\rho$ over $\theta$, we deduce $j_1 < i - 1$.

Now, we replace $x_1$ by the value $x_2$ of $\alpha_i$ just after $\rho_1^{j_1}$ (if it exists; otherwise, we take $x_2 = \theta_{i+1}$). By hypothesis, $x_2$ is also covered by a witnessed segment $W(\rho^2)$ where $\rho^2$ is another pattern order-isomorphic to $D_k$, and such
Proof. The proof is straightforward for $2 \leq \ell \leq k - 1$, i.e., $\ell \geq k$. In the following of the proof, we take $\ell$ such that $2 \leq \ell \leq k - 1$ and we establish the result by contradiction. So, let us assume that there exists $\sigma$ of length $2k - \ell$ such that $\sigma \notin \text{cc}(\text{Av}_{2k-\ell}(D_k))$ with $2 \leq \ell \leq k - 1$. Modulo cyclic rotation, we set $\sigma_1 = 2k - \ell$. Let $\theta = \theta_1 \ldots \theta_k$ be the leftmost subsequence of $\sigma$ order-isomorphic to $D_k$. We have $\theta_1 = 2k - \ell$ and we discuss on the value of $\theta_k$: (i) $\theta_k = k - \ell + 1$ and (ii) $\theta_k \leq k - \ell$.

Case (i). We deduce $\theta_1 = 2k - \ell, \theta_2 = 2k - \ell - 1, \ldots, \theta_k = k - \ell + 1$. The cyclic rotation $\sigma^1$ of $\sigma$ beginning with $\theta_k$ also contains a subsequence $\theta^1$ order-isomorphic to $D_k$. As $\theta$ is the leftmost pattern in $\sigma$, $\theta^1$ does not begin after $\theta_1$ in $\sigma^1$. This means that $\theta^1_k$ is at most $k - \ell + 1$, and thus $\theta^1_k \leq k - \ell + 1 + k - 1 = 2 - \ell \leq 0$ which induces a contradiction.

Case (ii). We have $\theta_k \leq k - \ell$. The cyclic rotation $\sigma^1$ of $\sigma$ beginning with $\theta_k$ also contains a subsequence $\theta^1$ order-isomorphic to $D_k$. We consider the leftmost subsequence in $\sigma^1$. We necessarily have $\theta^1_k$ on the left of $\theta_1$ and $\theta^1_k$ on the right of $\theta_1$ (in $\sigma^1$). Moreover $\theta^1$ and $\theta$ have a nonempty intersection (since $2k - \ell < 2k$). Let $\sigma^2$ be the cyclic rotation of $\sigma$ beginning with $\theta^1_k$ and $\theta^2$ be the leftmost subsequence in $\sigma^2$ order-isomorphic to $D_k$. We necessarily have: $\theta^2_k$ is on the left of $\theta_k$ and after $\theta^1_k$ (in $\sigma^2$). $\theta^2$ necessarily contains at least one element $x$ of $\theta^1_1 \ldots \theta^1_{k-1}$. We discuss on the position of $x$ relatively to $\theta_1$.

If $x$ is after $\theta^1_k$ in $\sigma^2$, then we also have $\theta^2_k$ after $\theta^1_k$.

If $x$ is before $\theta^1_k$ in $\sigma^2$. As $\theta^2$ also contains an element $y$ of $\theta$ such that $y$ appears after $x$, we deduce that $\theta^2_k$ is on the right of $\theta^1$ in $\sigma^2$. With the same reasoning, we construct an infinite sequence of $D_k$-patterns $\theta^i$ different verifying the property that $\theta_i$ is between $\theta^1_k$ and $\theta^2_k$ which provides a contradiction. \hfill $\square$

Theorem 2 For $2 \leq \ell \leq 2k - 1$, $\text{cc}(\text{Av}_{2k-\ell}(D_k)) = S_{2k-\ell}$.

Proof. It suffices to prove that permutations of length $2k - 1$ such that $\sigma \notin \text{cc}(\text{Av}_{2k-1}(D_k))$ and $\sigma_{i-1} = 2k - 1$ are enumerated by $c_{k-1}$. Let $\sigma = \sigma_1 \sigma_2 \ldots \sigma_{2k-1}$ be a permutation that does not belong to $\text{cc}(\text{Av}_{2k-1}(D_k))$ and such that $\sigma_1 =$
2k − 1. Obviously σ contains the pattern \( D_k \). We consider the leftmost pattern \( \theta_{i_1} \theta_{i_2} \ldots \theta_{i_k} \) in σ (we necessarily have \( \theta_{i_1} = 2k − 1 \)).

Now we discuss on the value of \( \theta_{i_k} \).

Case 1: \( \theta_{i_k} = k \). This means that \( \theta_{i_j} = 2k − j \) for \( 1 \leq j \leq k \), \( \sigma_i \leq k − 1 \) for \( i > i_k \) and σ contains a decreasing subsequence order-isomorphic to \( D_k \). Let \( \sigma^1 \) be the cyclic rotation of \( \sigma \) beginning with \( k \). By hypothesis, \( \sigma^1 \) contains a subsequence order-isomorphic to \( D_k \). The only one possibility is that this subsequence is exactly \( k(k−1)\ldots 21 \).

Moreover, remark that the value just after \( k+1 \) is necessarily at least \( k \), i.e., \( k + 1 \) and \( k \) are consecutive in \( \sigma \). This is due to the hypothesis that \( \theta \) is the leftmost pattern in \( \sigma \). Moreover, for the same reason, two successive decreasing values can not appear between \( k + 2 \) and \( k + 1 \). More generally, there does not exist \( i \) decreasing values after \( k + i \) for \( 0 \leq i \leq k − 1 \) which characterizes the Catalan numbers. Conversely, such a permutation does not belong to \( cc(Av_{2k−1}(D_k)) \). See Figure 2 for \( k = 4 \).

Case 2: \( \theta_{i_k} \leq k − 1 \). We repeat \textit{mutatis mutandis} the reasoning of the case (ii) in the proof of Theorem 2.

Figure 2: The five permutations \( \sigma \) beginning with 7 such that \( \sigma \notin cc(Av_{7}(D_4)) \):
\textbf{7654321, 7216543, 7165432, 7615432, 7261543}. The leftmost pattern \( \theta \) is illustrated in boldface for the one-line notation of \( \sigma \) and with empty points in the representations below. The corresponding well-formed parentheses are respectively \(((())), (())(), (()()), ()(())) and ()()().

Notice that a \((2k − 1)\)-length permutation \( \sigma \notin cc(Av(D_k)) \) beginning with \( 2k − 1 \) does not contain the pattern 123. Moreover, the previous proof induces a constructive bijection between the set of well-formed parentheses of size \( 2k − 2 \) and the set of permutations of length \( 2k − 1 \), beginning with \( 2k − 1 \), and containing a pattern \( D_k \) in each of their cyclic rotations. Indeed, we consider the binary representation \( b = b_1 \ldots b_{2k−2} \) of a well-formed parentheses, i.e., \( b_i = 0 \) if the \( i \)th parenthesis is a closed parenthesis, \( b_i = 1 \) otherwise). For
convenience we add \( b_{2k-1} = 0 \) on the right of \( b \). Now, let \( j, 1 \leq j \leq 2k - 1 \), be the rank of the rightmost zero on the left of the rightmost one in \( b \) (if \( j \) does not exist we set \( j = 2k - 1 \)). Then, we traverse \( b \) from right to left (from indices \( j \) to 1), and we label each zero in increasing order from 1. Finally, we continue to label (in increasing order) the unlabeled elements of \( b \) from right to left. For instance, the parenthesis \(((())(((())(())())())())\) has a binary representation 1011101001110000 where \( j = 9 \) (in boldface). Its corresponding permutation 17 4 16 15 14 3 13 21 12 11 10 9 8 7 6 5 contains the pattern \( D_9 \) in each cyclic rotation. See Figure 2 when \( k = 4 \).

Theorem 1 shows that the basis \( B \) of \( \text{cc}(\text{Av}(D_k)) \) is finite. Theorems 2 and 3 imply that the smallest length of a minimal permutation is \( 2k - 1 \), and that these permutations are enumerated by \( (2k - 1) \cdot c_{k-1} \). It remains to characterize the other elements of \( B \). Notice that Albert et al. have experimentally obtained the basis elements of \( \text{cc}(\text{Av}(321)) \) which are 15432, 14235, 164253, 163254 and 1472536. We have experimentally checked that the basis of \( \text{cc}(\text{Av}(4321)) \) contains 5 minimal permutations of length 7, 32 of length 8, 54 of length 9, and 136 of length 10.

**Theorem 4** The inversions of length \( 2k - 1 \) which do not lie in \( \text{cc}(\text{Av}(D_k)) \) are enumerated by \( \frac{k(k+1)}{2} \).

*Proof.* Let \( \sigma \) be an involution of length \( 2k - 1 \) that does not belong to \( \text{cc}(\text{Av}(D_k)) \). We distinguish two cases: (i) there exists \( i, k \leq i \leq 2k - 2 \), such that \( \sigma_{2k-i} = 2k-1 \); and (ii) there exists \( i, 1 \leq i \leq k - 1 \), such that \( \sigma_{2k-1} = 2k-1 \).

The first case (i) implies that there is , \( k \leq i \leq 2k - 2 \), such that \( \sigma_{2k-i} = 2k-1 \) and \( \sigma_{2k-1} = 2k-i \). Moreover, if we take \( j, 1 \leq j < 2k - i \), such that \( \sigma_{2k-j} = 1 \) then this induces \( \sigma_1 = 2k-j \). This means that the cyclic rotation of \( \sigma \) beginning with \( 2k - 1 \), contains from right to left the subsequence 1, \( 2k-i \), \( 2k-j \) which contradicts the remark just after Theorem 3. Thus, there exists \( j, 1 \leq j < 2k-i \), such that \( \sigma_j = 1 \) and \( \sigma_1 = j \). A similar argument as above allows us to conclude that \( \sigma \) is necessarily \( 2k - i - 1, \ldots, 1, 2k - 1, \ldots, 2k - i \).

For the second case, there exists \( i, 1 \leq i \leq k - 1 \), such that \( \sigma_{2k-i} = 2k-1 \) and \( \sigma_{2k-1} = 2k-i \). In the same way as above, we deduce that there does not exist \( j, j > 2k-i \), such that \( \sigma_j = 1 \). Thus, let us consider \( j, j < 2k-i \), such that \( \sigma_j = 1 \) and \( \sigma_1 = j \). If \( j = 2k-i-1 \) then it is straightforward to see that we necessarily have \( \sigma = (2k - 2) \ldots (2k - 1) \ldots (2k - i) \). If \( j \in [k - i, 2k - i - 2] \) then we easily see that there does not exist some involutions verifying this hypothesis. If \( j \in [1..k-i-1] \) then the only one involution is \( \sigma = j \ldots 1(2k-i-1) \ldots (j+1)(2k-i-1) \ldots (2k-i) \). To summarize, there are \( \sum_{i \in [1..k-1]} (k-i-1) + k = \frac{k(k+1)}{2} \) such involutions.

For \( k = 3 \) there are 6 involutions of length 5 that do not lie in \( \text{cc}(\text{Av}(D_3)) \): 54321, 15432, 21543, 32154, 43215 and 14325.
3 Lift closure

In this section, we study the lift closure \( \text{cl}(C) \) of a closed class \( C \), i.e., the class of permutations that can be obtained by a lift of a permutation in \( C \). We provide several general results which make links between cyclic and lift closures.

**Lemma 2** If \( X \) is a set of permutations then \( \pi \in \text{cc}(X) \iff \pi^{-1} \in \text{cl}(X^{-1}) \).

Consequently, if \( X = \text{Av}(B) \) then \( \text{cc}(\text{Av}(B)) = \text{cl}(\text{Av}(B^{-1}))^{-1} \). Moreover, if the basis \( B \) is stable by inversion, i.e. \( B = B^{-1} \), then \( \text{cc}(\text{Av}(B)) = \text{cl}(\text{Av}(B))^{-1} \).

**Proof.** Indeed, \( \pi \in \text{cc}(X) \) means there exists \( \sigma = \sigma_1 \ldots \sigma_n \in X \) such that \( \pi = r^k(\sigma) = \sigma_k \ldots \sigma_1 \sigma_n \sigma_1 \ldots \sigma_{k-1} \). With \( r(\sigma)^{-1} = \ell(\sigma^{-1}) \) we deduce \( \pi^{-1} = \ell^k(\sigma^{-1}) \).

Thus \( \pi^{-1} \in \text{cl}(\text{Av}(X^{-1})) \). By symmetry, we conclude \( \pi^{-1} \in \text{cl}(\text{Av}(X^{-1})) \) implies \( \pi \in \text{cc}(X) \). The straightforward relation \( \text{Av}(B^{-1}) = \text{Av}(B)^{-1} \) induces the consequences.

Now we set \( C = \text{Av}(B) \) where \( B \) is a basis. As for the case of the cyclic rotation, \( \text{cl}(C) \) also is a class of avoiding permutations. Let \( \sigma \) be a permutation in \( C \) and \( \theta \) be a pattern (in \( B \)) of \( \sigma \). We define by witnessed interval of \( \sigma \) the cyclic interval \( W'(\theta) = [n] \setminus \text{min}(\theta) \ldots \text{max}(\theta) \). Notice that \( W'(\theta) \) is an interval in \([n]\) considered cyclically, but it is not necessarily a segment in \( \sigma \). We can easily remark that \( W'(\theta) \) is also the witnessed segments of \( \sigma^{-1} \) (considered cyclically) relatively to the pattern \( \theta^{-1} \). Two similar results of Lemma 1 and Proposition 1 can be deduced below.

**Lemma 3** \( \sigma \notin \text{cl}(C) \) if and only if the witnessed intervals \( W'(\theta) \) cover \([n]\) (or equivalently \( \sigma \)).

**Proof.** By applying Lemmas 1 and 2, we obtain: \( \sigma \notin \text{cl}(C) \) if and only if \( \sigma^{-1} \notin \text{cc}(C^{-\infty}) \), i.e. iff the witnessed segments (relatively to \( \theta^{-1} \) and \( \sigma^{-1} \)) cover \( \sigma^{-1} \) (or equivalently \([n]\)). With the remark above, this is equivalent to the covering of \([n]\) by the witnessed intervals \( W'(\theta) \) of \( \sigma \).

**Proposition 2** Let \( C = \text{Av}(B) \), where \( B \) is finite and suppose there is a bound \( \Delta \) depending on \( B \) alone such that, for all \( \sigma \notin \text{cl}(X) \), there is a collection of at most \( \Delta \) witnessed intervals that cover \( \sigma \) (or equivalently \([n]\)). Then \( \text{cl}(X) \) is finitely based.

**Proof.** The proof is a direct consequence of Proposition 1 and Lemma 3.

**Theorem 5** Let \( k \geq 2 \) be an integer. Then \( \text{cl}(\text{Av}(D_k)) \) is finitely based.

**Proof.** This theorem is a consequence of Theorem 1 combined with Lemma 2. Indeed, we have \( \text{cl}(\text{Av}(D_k)) = \text{cc}(\text{Av}(D_k))^{-1} = \text{Av}(B)^{-1} = \text{Av}(B^{-1}) \) where \( B \) is a finite basis.

All enumeration results of [1] for the cyclic closure are also valid for the lift closure.

**Theorem 6** \( \text{cl}(\text{Av}_{2k-1}(D_k)) \) is enumerated by \((2k-1)! - (2k-1)c_{k-1}\) where \((c_k)_{k \geq 1}\) is the well-known Catalan sequence.

**Proof.** This corollary is deduced from Theorem 3.
4 Going further and conclusion

In this section, we give some general results about the toric closure $\text{tr}(C)$ for a class $C$ of permutations, i.e. $\text{tr}(C) = \text{cl}(\text{cc}(C)) = \text{cc}(\text{cl}(C))$. Let $f_{u,v}, 1 \leq u, v \leq n$, be the function defined on $S_n$ by $f_{u,v}(\sigma) = \sigma'$ where $\sigma'_j = (\sigma_{(u+j-2) \mod (n)+1} - v) \mod (n) + 1$, for $j \in [1..n]$. The set $\{f_{u,v}(\sigma), 1 \leq u, v \leq n\}$ contains exactly all toric rotations of $\sigma$ since the permutation $f_{u,v}(\sigma)$ equals the permutation $r_{n-u+1}(\ell_{n-v+1}(\sigma))$. Now assume that $C = \text{Av}(B)$ where $B$ is finite set of permutations. Let $\theta$ be a pattern (relatively to $B$) in $\sigma \in C$. The witnessed area $W''(\theta)$ of $\theta$ in $\sigma$ is the direct product of $\sigma - 1(W(\theta))$ of $\theta$ with the witnessed interval $W'(\theta)$: $W''(\theta) = \sigma^{-1}(W(\theta)) \times W'(\theta)$. For instance, the permutation 7261543 contains the pattern 132 ($\theta = 265$), $W'(\theta)$ is the segment 4372, thus $\sigma^{-1}(W(\theta)) = \{6,7,1,2\}$, $W'(\theta)$ is the interval 712, and $W''(\theta)$ is the area $\{6,7,1,2\} \times \{7,1,2\}$. See Figure 3 for an illustration.

Figure 3: The witnessed area $W''(265)$ for the permutation 7261543.

We also obtain similar general results as for the cyclic (or lift) closure.

Lemma 4 $\sigma \not\in \text{tr}(C)$ if and only if the witnessed areas cover the area $[n] \times [n]$ where $n$ is the length of $\sigma$.

Proof. If the witnessed areas of $\sigma$ cover $[n] \times [n]$, then no toric rotation of $\sigma$ lies in $C$. Indeed, the toric rotations $\sigma'$ of $\sigma$ defined by $\sigma'_j = (\sigma_{(u+j-2) \mod (n)+1} - v) \mod (n) + 1$, for $j \in [1..n]$, where $(u, v) \in W''(\theta)$, all contain $\theta$ as a pattern without wrap-around and none of these permutations lie in $C$. Conversely, if the witnessed areas do not cover a point $(u, v)$ then the toric rotation $\sigma'$ of $\sigma$ defined as above, contains no pattern (without wrap-around) and thus lies in $\text{Av}(\ell_{n-v+1}(\sigma))$. Consequently, $\sigma$ lies in $\text{tr}(\text{Av}(\ell_{n-v+1}(\sigma)))$.

Proposition 3 Let $C = \text{Av}(B)$, where $B$ is finite and suppose there is a bound $\Delta$ depending on $B$ alone such that, for all $\sigma \not\in \text{tr}(C)$, there is a collection of at most $\Delta$ witnessed areas that cover $\sigma$. Then $\text{tr}(C)$ is finitely based.

Proof. The proof are obtained mutatis mutandis as for Proposition 1.
7, 2 of length 8, 39 of length 9 and 2 of length 10. Thus the problem of finding a nice proof that $\text{tr(Av(321))}$ is finitely based remains open. We also have the following open questions:

**Problem 1:** Is $\text{tr(Av(231))}$ finitely based?

**Problem 2:** Is $\text{tr(Av(k ... 21))}$ finitely based for $k \geq 4$?

**Problem 3:** Given a finite basis $B$, is it decidable if $\text{cc(Av(B))}$ is finitely based?

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## References

