Asymptotic bit frequency in Fibonacci words

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Abstract

It is known that binary words containing no k consecutive 1s are enumerated by k-step Fibonacci numbers. In this note we discuss the expected value of a random bit in a random word of length n having this property. This expectation can reveal new properties of some telecommunication protocols or interconnection networks.

For $n \geqslant 0$ and $k \geqslant 2$, we denote by $\mathcal{B}_n(1^k)$ the set of length n binary words avoiding k consecutive 1s. For example, we have

$$\mathcal{B}_4(11) = \{0000, 0001, 0010, 0100, 0101, 1000, 1001, 1010\}, \text{ and}$$
$$\mathcal{B}_4(111) = \{0000, 0001, 0010, 0011, 0100, 0101, 0110, 1000, 1001, 1010, 1011, 1100, 1101\}.$$

It is well known, see Knuth [12, p. 286], that $\mathcal{B}_n(1^k)$ is enumerated by the k-step Fibonacci numbers, precisely $|\mathcal{B}_n(1^k)| = f_{n+k,k}$, where $f_{n,k}$ is defined, following Miles [14] as

$$f_{n,k} = \begin{cases} 0 & \text{if } 0 \leqslant n \leqslant k-2, \\ 1 & \text{if } n = k-1, \\ \sum_{i=1}^{k} f_{n-i,k} & \text{otherwise.} \end{cases}$$

Denote by $v_{n,k}$ the *popularity* of 1s in $\mathcal{B}_n(1^k)$, i.e. the total number of 1s in all words of $\mathcal{B}_n(1^k)$. For instance, $v_{4,2} = 10$ and $v_{4,3} = 22$. The ratio of popularity of 1s to the overall number of bits in words of $\mathcal{B}_n(1^k)$ is

$$\alpha_{n,k} = \frac{v_{n,k}}{n \cdot |\mathcal{B}_n(1^k)|},$$

and it equals the expected value of a random bit in a random word from $\mathcal{B}_n(1^k)$. In [2] the authors left without proof the fact that $\lim_{n\to\infty} \alpha_{n,k}$ converges to a non-zero value as n grows. This note is devoted to clarifying this fact, which apart from its interest *en soi* has practical counterparts. For instance, the expectation mentioned above can give hints on the entropy and efficiency of telecommunication frame synchronization protocols based on words in $\mathcal{B}_n(1^k)$, see for example [1, 3, 5], or graph theoretical properties of Fibonacci-like cubes [8].

Our discussion is based on the bivariate generating function

$$F_k(x, y) = \sum_{n=0}^{\infty} \sum_{m=0}^{k-1} a_{n,m} x^n y^m$$

whose coefficient $a_{n,m}$ equals the number of words from $\mathcal{B}_n(1^k)$ containing exactly m 1s. For $k = 2$ and $k = 3$, Table 1 presents some values of $a_{n,m}$ for small n and m.

$m\backslash n$	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
0	1	1	1	1	1	1	1	1	1
1	1	2	3	4	5	6	7	8	9
2			1	3	6	10	15	21	28
3					1	4	10	20	35
4							1	5	15
5									1

$m\backslash n$	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
0	1	1	1	1	1	1	1	1	1
1	1	2	3	4	5	6	7	8	9
2		1	3	6	10	15	21	28	36
3				2	7	16	30	50	77
4					1	6	19	45	90
5							3	16	51

Table 1: First few values of $a_{n,m}$ for $k = 2$ (left) and $k = 3$.

We recall a result from [2] (Proposition 1 below), and calculate the generating functions for the popularity of 1s and for the overall number of bits in $\mathcal{B}_n(1^k)$ by means of classic generating functions manipulations (Propositions 2 and 3). Then we apply Theorem 4.1 from [16], after ensuring that its conditions are satisfied, and obtain the main result of this note, Theorem 1. The evolution of the random bit expectation for $k = 2$ and $k = 3$ is presented on Figure 1 for small values of n. And numerical estimations for the limit value ($n \to \infty$) of the random bit expectation, for small values of k are given in Table 2.

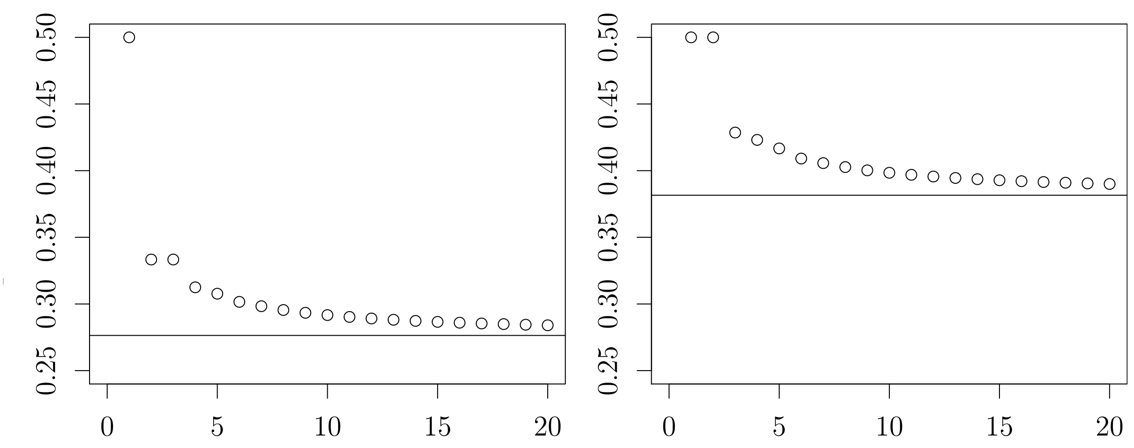
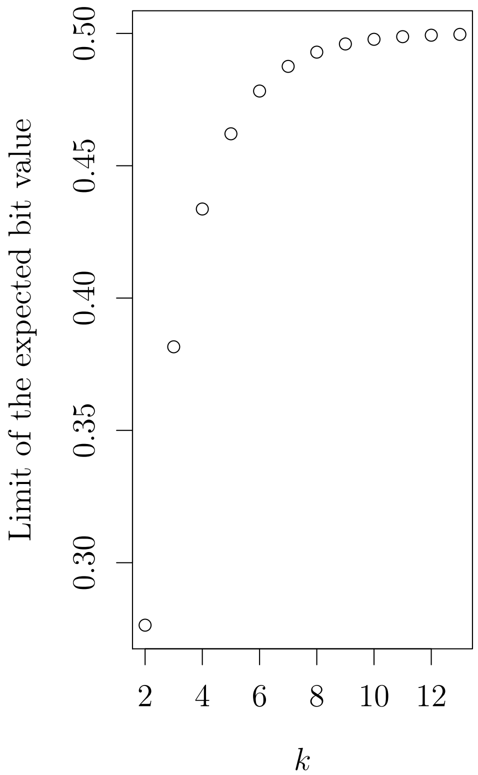


Figure 1: Expected value of a random bit in a random word from $\mathcal{B}_n(1^2)$ (left) and $\mathcal{B}_n(1^3)$ for small values of n.



k	Limit of the expected bit value
2	0.276393202250021
3	0.381580077680607
4	0.433657112297348
5	0.462073883180840
6	0.478227505713290
7	0.487545982771861
8	0.492928265543398
9	0.496019724266083
10	0.497779940783496
11	0.498772398758879
12	0.499326557312936
13	0.499633184444604

Table 2: Numerical estimations for the limit of the expected value of a random bit in a random word from $\mathcal{B}_n(1^k), n \to \infty$.

Proposition 1 (p. 7 in [2])**.**

$$F_k(x,y) = \frac{y\left(1-(xy)^k\right)}{y - xy^2 - xy + (xy)^{k+1}}.$$

Proof. The set $\mathcal{B}(1^k) = \bigcup_{n=0}^{\infty} \mathcal{B}_n(1^k)$ respects the following recursive decomposition

$$\mathcal{B}(1^k) = \mathbb{1}_{k-1} \cup \left(\bigcup_{i=0}^{k-1} \left(1^i 0 \cdot \mathcal{B}(1^k) \right) \right)$$

where $\mathbb{1}_{k-1} = \bigcup_{i=0}^{k-1} \{1^i\}$ is the set of words in $\mathcal{B}(1^k)$ containing no 0s, and $\cdot$ denotes the concatenation. Note that the empty word also lies in $\mathbb{1}_{k-1}$. The claimed generating function is the solution of the following functional equation

$$F_k(x,y) = \sum_{i=0}^{k-1} x^i y^i + F_k(x,y) \sum_{i=0}^{k-1} x^{i+1} y^i.$$

□

Proposition 2. *Generating function $P_k(x)$ where the coefficient of x^n is the popularity of 1s in $\mathcal{B}_n(1^k)$ is given by*

$$P_k(x) = \left.\frac{\partial F_k(x,y)}{\partial y}\right|_{y=1} = \frac{x(kx^k - kx^{k-1} - x^k + 1)}{(x^{k+1} - 2x + 1)^2},$$

and factorizing and simplifying by $(x-1)^2$, we have

$$P_k(x) = \frac{x \cdot \sum_{i=0}^{k-2}(i+1)x^i}{(x^k + x^{k-1} + \cdots + x^2 + x - 1)^2}.$$

Proposition 3. *Generating function $T_k(x)$ where the coefficient of x^n equals the total number of all bits in $\mathcal{B}_n(1^k)$ is*

$$T_k(x) = x\frac{\partial F_k(x,1)}{\partial x} = \frac{x(kx^k - kx^{k-1} + x^{2k} - 3x^k + 2)}{(x^{k+1} - 2x + 1)^2},$$

and factorizing and simplifying by $(x-1)^2$, we have

$$T_k(x) = \frac{x\left(\sum_{i=0}^{k-2}(2i+2)x^i + \sum_{i=k-1}^{2k-2}(2k-i-1)x^i\right)}{(x^k + x^{k-1} + \cdots + x^2 + x - 1)^2}.$$

We recall two classical propositions consorted by short proofs.

Proposition 4. *The smallest by modulus root of the polynomial*

$$g_k(x) = x^k + x^{k-1} + \cdots + x^2 + x - 1$$

is unique, real, and lies between $1/2$ and $1/\varphi$, where $\varphi = (1+\sqrt{5})/2$ is the golden ratio.

Proof. It is easy to see that $g_k(1/2) = -2^{-k} < 0$, and $g_k(1/\varphi) = 0$ for $k = 2$, and $g_k(1/\varphi) > 0$ for $k > 2$. Also $g_k'(x)$ is positive for $x \geqslant 0$. So, there is only one real root in $(1/2, 1/\varphi]$. The uniqueness of the root inside a disc of radius r, $1/\varphi < r < 1$, directly follows from Rouché's Theorem [18, p. 217] applied to $x^{k+1} - 2x + 1 = (x-1)g_k(x)$, since $2r > 1 + r^{k+1}$ for any $r \in (1/\varphi, 1)$. □

Every root r of a polynomial $h(x)$ of degree n with a non-zero constant term corresponds to the root $1/r$ of its negative reciprocal $-x^n h(1/x)$. The negative reciprocal of $x^k + x^{k-1} + \cdots + x^2 + x - 1$ is $x^k - x^{k-1} - \cdots - x^2 - x - 1$ which is known in the literature as Fibonacci polynomial, see for instance [6, 7, 9, 10, 11, 13, 14, 15, 19] and references therein. In particular, Dubeau proved [7, Theorem 1] that its largest by modulus root is $\varphi_k = \lim_{n\to\infty} f_{n+1,k}/f_{n,k}$, the generalized golden ratio, and φ_k approaches 2 when $k \to \infty$ [7, Theorem 2]. Wolfram [19, Lemma 3.6] showed that any other root r of the Fibonacci polynomial satisfies $3^{-1/k} < |r| < 1$. See Figure 2 for an illustration of this fact.

Proposition 5. *The polynomial $g_k(x) = x^k + x^{k-1} + \cdots + x^2 + x - 1$ is irreducible over $\mathbb{Q}$.*

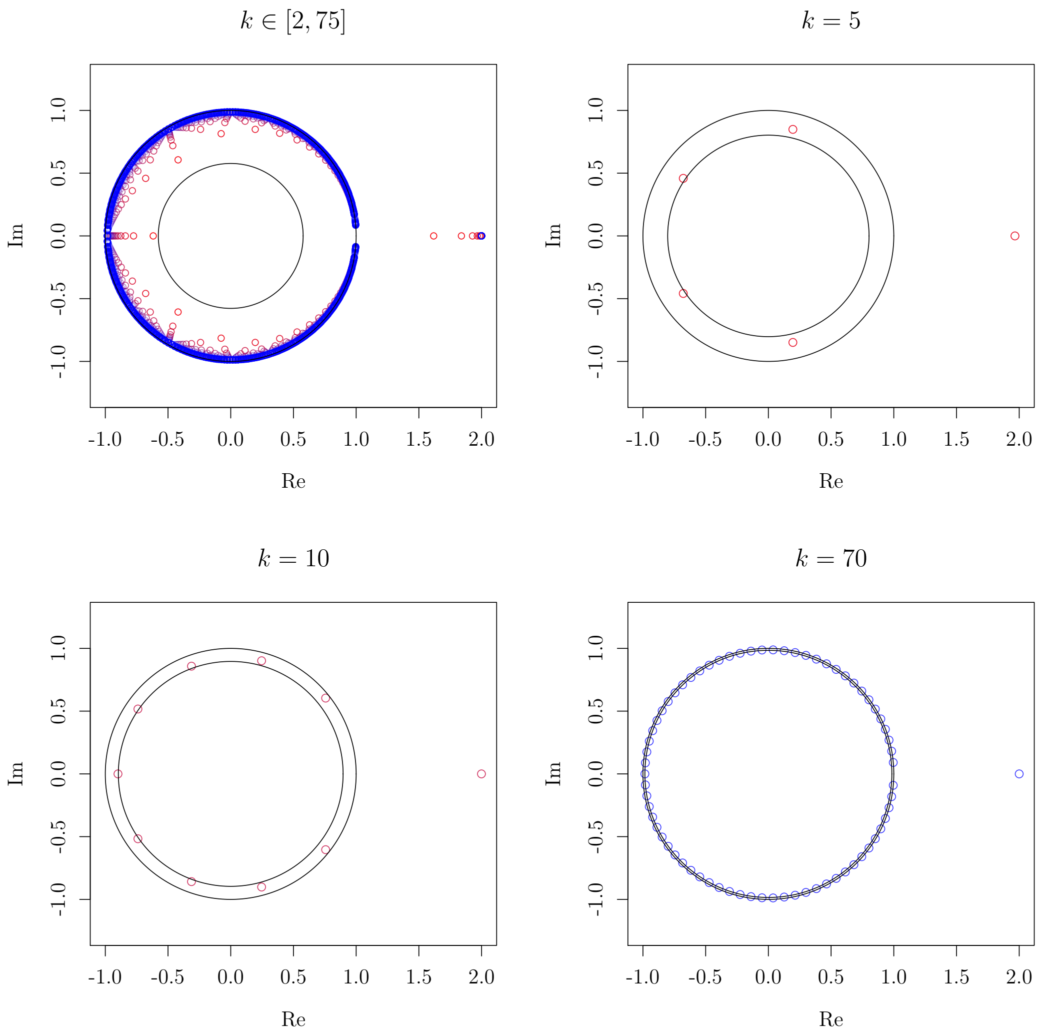


Figure 2: Roots of the polynomial $x^k - x^{k-1} - \cdots - x^2 - x - 1$ (the negative reciprocal of $g_k(x)$) for certain values of k.

Proof. We apply Perron-Selmer result [17, Theorem 2] to $x^{k+1} - 2x + 1$ or Brauer's criterion [4, Theorem 2] to its negative reciprocal $x^k - x^{k-1} - \cdots - x^2 - x - 1$. □

Another proof of Proposition 5 is presented by Wolfram [19, Corollary 3.8].

The next lemma says that after simplifying by $(x-1)^2$ both $P_k(x)$ and $T_k(x)$, the obtained fractions are irreducible.

Lemma 1. *The polynomials* $\sum_{i=0}^{k-2}(i+1)x^i$ *and* $x^k + x^{k-1} + \cdots + x^2 + x - 1$ *are relatively prime; and so are* $\sum_{i=0}^{k-2}(2i+2)x^i + \sum_{i=k-1}^{2k-2}(2k-i-1)x^i$ *and* $x^k + x^{k-1} + \cdots + x^2 + x - 1$.

Proof. The polynomial $x^k + x^{k-1} + \cdots + x^2 + x - 1$ is irreducible due to Proposition 5. It does not divide $\sum_{i=0}^{k-2}(i+1)x^i$ as it has a greater degree. And it also cannot divide $\sum_{i=0}^{k-2}(2i+2)x^i + \sum_{i=k-1}^{2k-2}(2k-i-1)x^i$ as the latter does not have any positive real roots. □

From Propositions 2, 3, 4, Dubeau results [7], and Lemma 1 we have:

Lemma 2. *Both generating functions $P_k(x)$ and of $T_k(x)$ have the same and unique pole of smallest modulus with multiplicity 2. The pole equals $1/\varphi_k$, where φ_k is the generalized golden ratio.*

For our main result of this note we need the next theorem.

Theorem 4.1 ([16]) (Asymptotics for linear recurrences) *Assume that a rational generating function $\frac{f(x)}{g(x)}$, with $f(x)$ and $g(x)$ relatively prime and $g(0) \neq 0$, has a unique pole $1/\beta$ of smallest modulus. Then, if the multiplicity of $1/\beta$ is ν, we have*

$$[x^n]\frac{f(x)}{g(x)} \sim \nu \frac{(-\beta)^\nu f(1/\beta)}{g^{(\nu)}(1/\beta)} \beta^n n^{\nu-1}.$$

Both $P_k(x)$ and $T_k(x)$ are rational generating functions, and by Lemmas 1 and 2 they fulfill the conditions in the above theorem, so

$$[x^n]P_k(x) \sim 2n\varphi_k^{n+2} \cdot \left. \frac{x\left(\sum_{i=0}^{k-2}(i+1)x^i\right)}{\left((x^k + x^{k-1} + \cdots + x^2 + x - 1)^2\right)''} \right|_{x=1/\varphi_k}$$

$$[x^n]T_k(x) \sim 2n\varphi_k^{n+2} \cdot \left. \frac{x\left(\sum_{i=0}^{k-2}(2i+2)x^i + \sum_{i=k-1}^{2k-2}(2k-i-1)x^i\right)}{\left((x^k + x^{k-1} + \cdots + x^2 + x - 1)^2\right)''} \right|_{x=1/\varphi_k}.$$

The expected value of a random bit in a random word from $\mathcal{B}_n(1^k)$ is $\frac{[x^n]P_k(x)}{[x^n]T_k(x)}$. Taking the limit, we obtain:

Theorem 1. *The expected value of a random bit in a random word from $\mathcal{B}_n(1^k)$ tends to*

$$\left. \frac{kx^k - kx^{k-1} - x^k + 1}{kx^k - kx^{k-1} + x^{2k} - 3x^k + 2} \right|_{x=1/\varphi_k} \quad \textit{when } n \to \infty,$$

where $\varphi_k = \lim_{n\to\infty} f_{n+1,k}/f_{n,k}$ is the generalized golden ratio, in particular φ_2 is the golden ratio.

More than 20 years ago it was conjectured by Wolfram [19] that the Galois group of the polynomial $x^k - x^{k-1} - \cdots - x^2 - x - 1$ is the symmetric group S_k, and so there is no algebraic expression for φ_k (the largest by modulus root of this polynomial) when $k \geqslant 5$. In case of even or prime k the conjecture was settled by Martin [13]. Cipu and Luca [6] showed that φ_k cannot be constructed by ruler and compass for $k \geqslant 3$. Nevertheless, good approximations are available, for instance Hare, Prodinger and Shallit [11] expressed φ_k and $1/\varphi_k$ in terms of rapidly converging series.

The generalized golden ratio φ_k tends to 2 as k grows, and we deduce the following.

Corollary 1. *The limit of the expected bit value of binary words avoiding k 1s, whose length tends to infinity, approaches $1/2$ as k grows:*

$$\lim_{k\to\infty} \lim_{n\to\infty} \frac{v_{n,k}}{n \cdot |\mathcal{B}_n(1^k)|} = \frac{1}{2}.$$

Finally, note that other sets of restricted binary words are counted by the generalized Fibonacci numbers, for instance q-decreasing words [2] for $q \geqslant 1$. In this case every length maximal factor of the form $0^a 1^b$ satisfies $a = 0$ or $q \cdot a > b$. Theorem 1 and Corollary 1 apply to this case (with the same limit, see [2, Corollary 5]) by setting $k = q + 1$.

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