Equipopularity of descent-equivalent patterns over descent-equivalence classes of words and permutations

Jean-Luc Baril and Vincent Vajnovszki

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Graph Theory
Combinatorics and Applications
Al-Ain, U.A.E., October 2019
words, patterns, patterns, statistics, popularity

previous works

main results statement

proof preliminaries
  \( f \)-equivalence
  bijection \( \psi \)
  pattern trace

sketch of the proofs, examples
Outline

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The **underlying alphabet** of a word is the set of symbols occurring in the word. The **reduction** of a word $w$, denoted $\text{red}(w)$, is the word order isomorphic with $w$ on the smallest arity alphabet. A **descent** in a word $w_1 w_2 \ldots w_n$ is an $i$ with $w_i > w_{i+1}$ and the **descent set** of $w$ is the set of all such $i$.

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<thead>
<tr>
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<th>underlying alphabet</th>
<th>$\text{red}(w)$</th>
<th>Des($w$)</th>
</tr>
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<tbody>
<tr>
<td>41341</td>
<td>${1, 3, 4}$</td>
<td>31231</td>
<td>${1,4}$</td>
</tr>
<tr>
<td>11334</td>
<td>${1, 3, 4}$</td>
<td>11223</td>
<td>$\emptyset$</td>
</tr>
<tr>
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A **pattern** is a word with the property that if $i$ occurs in it, then so does $j$, for any $j$ with $1 \leq j \leq i$.

Equivalently, $\pi$ is a pattern iff $\pi = \text{red}(\pi)$.

The word $w$ **contains** the pattern $\pi$ if $w$ has a (not necessarily contiguous) subword whose terms are order isomorphic to (i.e., have same relative ordering as) $\pi$.

<table>
<thead>
<tr>
<th>pattern</th>
<th>word</th>
<th>$(\pi)w$</th>
</tr>
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<tbody>
<tr>
<td>$\pi$</td>
<td>$w$</td>
<td>$(\pi)w$</td>
</tr>
<tr>
<td>212</td>
<td>43143</td>
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<td>$w$</td>
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<tr>
<th>pattern $\pi$</th>
<th>word $w$</th>
<th>$(\pi)w$</th>
</tr>
</thead>
<tbody>
<tr>
<td>212</td>
<td>43143</td>
<td>3</td>
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<td>123</td>
<td>231434</td>
<td>3</td>
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A combinatorial statistic over the set $S$ is simply an integer valued function on $S$.
The number of occurrences of a pattern becomes a statistic over the set $S$ of words:

$$|\{w : (\pi)w = p\}|.$$  

The popularity of a pattern is the overall number of copies of the pattern within the words of the set

$$\text{the popularity of } \pi \text{ in } S = \sum_{w \in S} (\pi)w,$$
**Example**

$S$ be the set of 5-permutations with descent set $\{1, 4\}$.

<table>
<thead>
<tr>
<th>$w \in S$</th>
<th>$(213)w$</th>
</tr>
</thead>
<tbody>
<tr>
<td>21354</td>
<td>3</td>
</tr>
<tr>
<td>21453</td>
<td>3</td>
</tr>
<tr>
<td>31254</td>
<td>4</td>
</tr>
<tr>
<td>31452</td>
<td>2</td>
</tr>
<tr>
<td>32451</td>
<td>2</td>
</tr>
<tr>
<td>41253</td>
<td>2</td>
</tr>
<tr>
<td>41352</td>
<td>2</td>
</tr>
<tr>
<td>42351</td>
<td>2</td>
</tr>
<tr>
<td>$\ldots$</td>
<td>0</td>
</tr>
<tr>
<td>popularity</td>
<td>20</td>
</tr>
</tbody>
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Main results statement

Theorem

For two patterns $\pi$ and $\sigma$ the following statements are equivalent

- $\pi$ and $\sigma$ are $d$-equivalent
- $\pi$ and $\sigma$ have the same popularity on any class of $d$-equivalent words
### Example

<table>
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<tr>
<th>$w \in S$</th>
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<td>3</td>
<td>0</td>
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<tr>
<td>51243</td>
<td>0</td>
<td>5</td>
</tr>
<tr>
<td>51342</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>52341</td>
<td>0</td>
<td>3</td>
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<td>\ldots</td>
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Let $\pi$ and $\sigma$ be two $d$-equivalent patterns. $\sigma$ is an $f$-transformation of $\pi$ if:

Informally: $\pi$ and $\sigma$ differ little from each other.

Formally: $\sigma$ can be obtained from $\pi$ by either
- increasing or decreasing by 1 an entry in $\pi$, or
- interchanging in $\pi$ two entries with consecutive values.

**Example**
- $\pi = 211$ and $\sigma = 212$ are $f$-equivalent
- $\pi = 1332$ and $\sigma = 2331$ are $f$-equivalent

$f$-transformation is a symmetric binary relation on a set of $d$-equivalent patterns and two patterns are said $f$-equivalent if they belong to the same equivalence class.
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Preliminaries : $f$-equivalence

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*Two patterns are $f$-equivalent if and only if they are $d$-equivalent.*

Example 5124332 and 3115241 are $d$-equivalent

\[
\begin{align*}
5124332 \\
4125332 \\
4125342 \\
3125342 \\
2125342 \\
2115342 \\
2115341 \\
3115241
\end{align*}
\]
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2115341
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We need a bijection $\psi$ from $[q]^n$ into itself satisfying:

(a) $\psi$ preserves the underlying alphabet, and

(b) $\psi$ transforms descent set into ascent set: for any $w \in [q]^n$
   \[ \text{Des } w = \text{Asc } \psi(w). \]

$\psi$ based on the bijection $\phi$ on words defined in [Kitaev, V., 2016],
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(ii) $\text{Des } w = \{n - i : i \in \text{Des } \phi(w)\}$, and

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$$\psi = r \circ \phi$$

Example

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Let $\pi = \pi_1 \pi_2 \ldots \pi_k$ be a pattern and $t = t_1 t_2 \ldots t_k$ be a length $k$ word over $[q] \cup \{\square\}$, $q \geq 1$. We say that $t$ is a trace of $\pi$ if $t_i$ and $t_j$ have the same relative order ($<$, $=\$, or $>$) as $\pi_i$ and $\pi_j$ whenever $t_i, t_j \neq \square$.

**Example**

- $t = \square 44 \square$ is a trace of the pattern $\pi = 1332$

- In $w = 15415432$
  - 1443 is an occurrence of $\pi = 1332$ with trace $t = \square 44 \square$ in
  - at $A = \{3, 6\}$

- In $15415432$
  - 1552 is an occurrence of $\pi = 1332$ with trace $t = \square 55 \square$ at
  - $A = \{2, 5\}$

We denote by $(t, A, \pi)_w$ the number of occurrences of $\pi$ in a word $w$ with trace $t$ at $A$. 
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Equipopularity of descent-equivalent patterns
What happens if we fix the trace $t$ and the position set $A$?
For $\sigma$ is an $f$-transformation of $\pi$, then

$$(t, A, \pi) \text{ and } (t, A, \sigma)$$

have the same distribution on any $d$-equivalence class.
\( A = \{3, 6\}, \quad t = \Box 44 \Box \)

<table>
<thead>
<tr>
<th>( w \in S )</th>
<th>( \pi = 1332 ) ( (t, A, \pi)w )</th>
<th>( \sigma = 2331 ) ( (t, A, \sigma)w )</th>
</tr>
</thead>
<tbody>
<tr>
<td>15415432</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>15425431</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>15425432</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>15435421</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>15435432</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>25415431</td>
<td>1</td>
<td>1</td>
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<tr>
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<td>35435421</td>
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</tr>
<tr>
<td>...</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
Lemma

Let $\pi = \pi_1 \pi_2 \ldots \pi_k$ and $\sigma = \sigma_1 \sigma_2 \ldots \sigma_k$ be two $d$-equivalent patterns with $\pi_\ell = \sigma_\ell$ for any $\ell$, except $\pi_i = \sigma_i + 1$ for some $i$. Let also $t$ be a trace of both $\pi$ and $\sigma$ with one $\square$ symbol and $A$ be a subset of $\{1, 2, \ldots, n\}$ of cardinality $k - 1$. Then on any $d$-equivalent class $(t, A, \pi)$ and $(t, A, \sigma)$ have the same distribution.
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Equipopularity of descent-equivalent patterns
Lemma

Let $\pi = \pi_1 \pi_2 \ldots \pi_k$ and $\sigma = \sigma_1 \sigma_2 \ldots \sigma_k$ be two d-equivalent patterns such that there are $i$ and $j$, $i < j$, with

- $\pi_\ell = \sigma_\ell$ for any $\ell$, except $\pi_i = \sigma_j$ and $\pi_j = \sigma_i$,
- $\pi_i = \pi_j + 1$,
- each of $\pi_i$ and $\pi_j$ occur once in $\pi$ (or, equivalently, $\sigma_i$ and $\sigma_j$ occur once in $\sigma$).

Let also $t = t_1 t_2 \ldots t_k$ be a trace of both $\pi$ and $\sigma$ with two symbols. If $A$ is a subset of $\{1, 2, \ldots, n\}$ of cardinality $k - 2$, then on any d-equivalent class the statistics $(t, A, \pi)$ and $(t, A, \sigma)$ have the same distribution.
Jean-Luc Baril and Vincent Vajnovszki

Equipopularity of descent-equivalent patterns
What happens if we fix only the position set $A$?

For $\sigma$ is an $f$-transformation of $\pi$, then $\pi$ and $\sigma$ have the same popularity on any $d$-equivalence class.
\[ A = \{3, 4\}, \quad t = \boxed{t_1 t_2} \text{ some common trace} \]

<table>
<thead>
<tr>
<th>( w \in S )</th>
<th>( \pi = 2123 )</th>
<th>( \sigma = 3122 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 14245 )</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( 15244 )</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>( 24125 )</td>
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<tr>
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<tr>
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<td>( 25144 )</td>
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<tr>
<td>( 45122 )</td>
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<td>2</td>
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<tr>
<td>( \ldots )</td>
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<td>0</td>
</tr>
<tr>
<td>\text{popularity}</td>
<td>( \sum_w \sum_t (t, A, \pi) w = 4 )</td>
<td>( \sum_w \sum_t (t, A, \sigma) w = 4 )</td>
</tr>
</tbody>
</table>
What happens if we fix only the trace $t$?

For $\sigma$ is an $f$-transformation of $\pi$, then $\pi$ and $\sigma$ have the same popularity on any $d$-equivalence class.
\[ t = 42 \] common trace

<table>
<thead>
<tr>
<th>( w \in S )</th>
<th>( \pi = 211 ) (( \sum_A(t, A, \pi)w ))</th>
<th>( \sigma = 212 ) (( \sum_A(t, A, \sigma)w ))</th>
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</thead>
<tbody>
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<td>2</td>
</tr>
<tr>
<td>...</td>
<td>0</td>
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</tr>
</tbody>
</table>

popularity: \[ \sum_w \sum_A(t, A, \pi)w = 8 \]

[Jean-Luc Baril and Vincent Vajnovszki] Equipopularity of descent-equivalent patterns
What happens if neither the trace \( t \) nor the position are fixed

For \( \sigma \) is an \( f \)-transformation of \( \pi \), then \( \pi \) and \( \sigma \) have the same popularity on any \( d \)-equivalence class.
\[ w \in S \]

\[
\begin{array}{c|c|c}
\pi = 213 & \sigma = 312 \\
\sum_A \sum_t (t, A, \pi) w & \sum_A \sum_t (t, A, \sigma) w \\
\hline
21354 & 3 & 0 \\
21453 & 3 & 0 \\
31254 & 4 & 1 \\
31452 & 2 & 1 \\
32451 & 2 & 0 \\
41253 & 2 & 0 \\
41352 & 2 & 2 \\
42351 & 2 & 1 \\
51243 & 0 & 0 \\
51342 & 0 & 0 \\
52341 & 0 & 0 \\
\ldots & \ldots & \ldots \\
\text{popularity} & \sum_w \sum_A \sum_t (t, A, \pi) w = 20 & \sum_w \sum_A \sum_t (t, A, \sigma) w = 20 \\
\end{array}
\]
If $\pi$ is a $f$-transformation of $\sigma$, then $\pi$ and $\sigma$ have the same popularity on any $d$-equivalence class.

\[\Downarrow\]

If $\pi$ and $\sigma$ are $f$-equivalent, then $\pi$ and $\sigma$ have the same popularity on any $d$-equivalence class.

\[\Downarrow\]

If $\pi$ and $\sigma$ are $d$-equivalent, then $\pi$ and $\sigma$ have the same popularity on any $d$-equivalence class.
Thank you for your attention!