Popularity of patterns over d-equivalence classes of words and permutations

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November 13, 2019

Abstract

Two same length words are d-equivalent if they have same descent set and same underlying alphabet. In particular, two same length permutations are d-equivalent if they have same descent set. The popularity of a pattern in a set of words is the overall number of copies of the pattern within the words of the set. We show the far-from-trivial fact that two patterns are d-equivalent if and only if they are equipopular over any d-equivalence class, and this equipopularity does not follow obviously from a trivial equidistribution.

1 Introduction and notation

We consider words over the set of positive integers and permutations are particular words. For $q \geq 1$, $[q]$ denotes the alphabet $\{1, 2, \ldots, q\}$ and the *underlying alphabet* of a word is the set of symbols occurring in the word. For instance, the 4-ary words 4313 and 4212 have different underlying alphabet, namely $\{1, 3, 4\}$ and $\{1, 2, 4\}$. A *descent* in a word $w_1 w_2 \ldots w_n$ is a position i with $w_i > w_{i+1}$ and the *descent set* of w, denoted $\mathrm{Des}(w)$, is the set of all such i; *ascent* and *ascent set*, $\mathrm{Asc}(w)$, are defined similarly. Two same length words are *d-equivalent* if they have the same descent set and the same underlying alphabet. For instance the words 31443 and 21332 have the same descent set but not the same underlying alphabet (thus they are not d-equivalent), whereas 31443 and 41131 are d-equivalent (the common underlying alphabet is $\{1, 3, 4\}$). If v and w are d-equivalent and v is a permutation, it follows that so is w. A *d-equivalence class* is a maximal set of d-equivalent words. For a word $w_1w_2\ldots w_n$ the *reverse* $\mathrm{r}(w)$ of w is the word $w = w_nw_{n-1}\ldots w_1$; and the *complement* $\mathrm{c}(w)$ of w is the word $(q-w_1+1)(q-w_2+1)\ldots(q-w_n+1)$ with q the maximal entry of w.

A *pattern* is a word with the property that if i occurs in it, then so does j, for any j with $1 \leq j \leq i$, and the *reduction* of w, $red(w)$, is the unique pattern order isomorphic with w. The pattern $\pi = \pi_1\pi_2\ldots\pi_k$ occurs in the word $w = w_1w_2\ldots w_n$, $k \leq n$, if w has a subword $w_{i_1}w_{i_2}\ldots w_{i_k}$ order isomorphic with π, see for instance Kitaev's seminal book [6] on this topic. The number of occurrences of the pattern π in the word w is denoted by $(\pi)w$. For a set of words S, $(\pi)w$ becomes an integer valued statistic on S and the overall number of occurrences of π within the words of S is called the *popularity* of π in S; more formally, the popularity of π in S is $\sum_{w\in S}(\pi)w$.

The equidistribution of two patterns implies their equipopularity and recently a growing interest is shown in patterns that have same popularity but not same distribution on particular classes of words or permutations, see for instance [1, 2, 5, 8].

In this paper we show that two patterns are d-equivalent if and only if they have the same popularity on any d-equivalence class. Specializing to permutations, we obtain that two same length permutations have same descent set if and only if they have the same popularity on any descent-set equivalence class of permutations.

2 Preliminary notions and results

2.1 d-equivalence vs. f-equivalence

Here we show that any two d-equivalent patterns can be obtained from each other by a sequence of f-transformations, which are 'small changes' preserving the d-equivalence class.

Lexicographically smallest pattern in a d-equivalence class

For our purposes we need the lexicographically minimal pattern d-equivalent with a given pattern π, which in turn requires two other particular patterns α and ω that we define below.

The *descent word* of a length n word w is the binary word $b = b_1b_2\ldots b_n$ where $b_i = 1$ if and only if i is a descent in w (and so, b_n is a redundant 0). The minimal arity of a pattern having descent word b is one more than the maximal number of consecutive 1s in b, and we denote by $\alpha(b)$ the lexicographically smallest pattern having minimal arity and descent word b. It is easy to see that the pattern $\alpha = \alpha(b)$ is defined as: for each i, $1 \leq i \leq n$,

$$\alpha_i = \min\{j : j \geq i,\ b_j = 0\} - i + 1.$$

Example 1. With $n = 9$ and $\pi = \mathbf{43}2\mathbf{4}112\mathbf{3}1$, we have $b = 110100010$ and $\alpha = \mathbf{32}1\mathbf{2}111\mathbf{2}1$ (the descents are in bold).

The maximal arity of a pattern having descent word $b = b_1 b_2 \dots b_n$ is n, and we denote by $\omega(b)$ the lexicographically smallest n-ary pattern having descent word b, which is necessarily a length n permutation. We divide the descent word b into *runs*: the length maximal factors of the form $11 \cdots 10$ with at least one occurrence of 1 are *descent runs* and, for convenience, we call the remaining length maximal 0s factors (if any) *ascent runs*. We define an order relation on $\{1, 2, \dots, n\}$: for two integers i and j, $1 \leq i, j \leq n$, we say that i *precedes* j, with respect to the binary word b, if

- b_i and b_j are in two distinct runs in b, and $i < j$, or
- b_i and b_j are in the same ascent run in b, and $i < j$, or
- b_i and b_j are in the same descent run in b, and $i > j$.

The desired permutation $\omega = \omega(b)$ is precisely that induced by this order relation:

$$\omega_i = \text{the rank of } i \text{ in } \{1, 2, \dots, n\},\ \text{in the precedence order},$$

and ω is at the same time the lexicographically minimal word of maximal arity (that is n) having descent word b and, as we will see below, defines an order in which we cover the entries of a pattern with descent word b.

Example 2. If b is as in the previous example, then $b = 110.10.00.10$ (runs are separated by dots) and $\omega = \mathbf{32}1\mathbf{5}467\mathbf{9}8$ (the descents are in bold).

Now for an arbitrary arity $q \leq n$ (not necessarily its minimal, or its maximal value n), we construct the lexicographically minimal pattern $\beta = \beta(q, b)$ where each symbol in $[q]$ occurs at least once and i is a descent in β if and only if $b_i = 1$, and in this construction the above defined patterns α and ω are involved. Moreover, if q reaches its minimal value, then $\beta = \alpha$ and if $q = n$, then $\beta = \omega$. The pattern β is obtained by covering its entries in $\beta_{\omega_1}, \beta_{\omega_2}, \dots, \beta_{\omega_n}$ order and the first entries (in this order) are taken from α and the last ones are increasing integers to guarantee that all symbols in $[q]$ occur in β. Formally:

$$\beta_{\omega_i} = \begin{cases} \alpha_{\omega_i} & \text{if} \max\{\alpha_{\omega_1}, \alpha_{\omega_2}, \ldots, \alpha_{\omega_i}\} \geq q - (n-i) \\ q - (n-i) & \text{elsewhere.} \end{cases}$$

With these notations, it follows that if $\beta \neq \alpha$ and $k = \min\{i : \beta_{\omega_i} \neq \alpha_{\omega_i}\}$ then

$$\beta_{\omega_i} = \begin{cases} \alpha_{\omega_i} & \text{if } i < k \\ q - (n-i) & \text{elsewhere,} \end{cases}$$

and the entries $\beta_{\omega_k}, \beta_{\omega_{k+1}}, \ldots, \beta_{\omega_n}$ are consecutive integers in increasing order.

Example 3. Continuing the previous example with $b = 110100010$, if $q = 7$, then the above construction gives $\beta = \mathbf{32121}4576$; and if $q = 8$, then it gives $\beta = \mathbf{321}4\mathbf{1}5687$ (the entries taken from α are in bold).

For a pattern π with descent word b, by a slight abuse of notation we denote by $\alpha(\pi)$ the pattern $\alpha(b)$, by $\omega(\pi)$ the pattern (permutation) $\omega(b)$; and in addition if π has arity q, then we denote by $\beta(\pi)$ the pattern $\beta(q, b)$. Note that

- the pattern $\alpha(\pi)$ is lexicographically minimal in its d-equivalence class and so are $\beta(\pi)$ and $\omega(\pi)$,
- the four patterns π, $\alpha(\pi)$, $\beta(\pi)$ and $\omega(\pi)$ have the same descent set,
- π and $\beta(\pi)$ are d-equivalent, but π, $\alpha(\pi)$ and $\omega(\pi)$ are not necessarily d-equivalent since they can have different underlying alphabet (or equivalently in this case, different arity).

f-equivalent patterns

For later use we need the following rather technical notion: for two d-equivalent patterns π and σ we say that σ is an *f-transformation* of π if σ can be obtained from π by either

- increasing or decreasing by 1 an entry in π, or
- interchanging in π two entries with consecutive values.

Actually, the f-transformation is a symmetric binary relation on a set of d-equivalent patterns and two patterns are said *f-equivalent* if they belong to the same equivalence class with respect to the transitive closure of f-transformation. Below we prove that the notions of d-equivalence and f-equivalence coincide, which is stated in Corollary 1 of Theorem 1.

The order induced by $\omega(\pi)$ is related to the descent word of π, however we have the following.

Proposition 1. *Let π be a pattern. If $\omega = \omega(\pi)$ and i, k are such that $\pi_i > \pi_k$, then $\omega_i < \omega_k$ implies $i < k$.*

Proof. If $\pi_i > \pi_k$ and $\omega_i < \omega_k$, then ω_i and ω_k are not in the same descent run of π, so $i < k$. □

The next proposition says that, under certain conditions, decreasing an entry in the pattern π produces a d-equivalent pattern lexicographically smaller than π.

Proposition 2. *Let π be a pattern. If $\omega = \omega(\pi)$, $\beta = \beta(\pi)$ and*

- *there is an i such that $\pi_{\omega_j} = \beta_{\omega_j}$ for any j, $1 \leq j < i$,*
- *$\pi_{\omega_i} > \beta_{\omega_i}$,*
- *the entry π_{ω_i} occurs at least twice in π,*

then the word σ with $\sigma_{\omega_j} = \pi_{\omega_j}$ for any j except $\sigma_{\omega_i} = \pi_{\omega_i} - 1$ is a pattern d-equivalent with π, lexicographically smaller than π.

Proposition 3. *Let π be a length n pattern. If $\omega = \omega(\pi)$, $\beta = \beta(\pi)$ and*

- *there is an i such that $\pi_{\omega_j} = \beta_{\omega_j}$ for any j, $1 \leq j < i$,*
- *$\pi_{\omega_i} > \beta_{\omega_i}$,*
- *the entry π_{ω_i} occurs once in π,*
- *the entry $\pi_{\omega_i} - 1$ occurs at least once in the set $\{\pi_{\omega_{i+1}}, \pi_{\omega_{i+2}}, \ldots, \pi_{\omega_n}\}$,*

then there is a k with $\omega_k \in \{\omega_{i+1}, \omega_{i+2}, \ldots, \omega_n\}$ and $\pi_{\omega_k} = \pi_{\omega_i} - 1$, and the word σ with $\sigma_{\omega_j} = \pi_{\omega_j}$ for any j, except $\sigma_{\omega_i} = \pi_{\omega_i} - 1$ and $\sigma_{\omega_k} = \pi_{\omega_k} + 1$ ($= \pi_{\omega_i}$) is a pattern d-equivalent with π, lexicographically smaller than π.

Proof. Let ω_a be the largest element of the set $\{\omega_{i+1}, \omega_{i+2}, \ldots, \omega_n\}$ with $\pi_{\omega_a} = \pi_{\omega_i} - 1$. It is enough to choose $k = a$. □

Note that, in the two propositions above σ is obtained by an f-transformation of π.

Proposition 4. *Let π be a q-ary length n pattern. If $\omega = \omega(\pi)$, $\beta = \beta(\pi)$ and i is such that each of $\beta_{\omega_i}, \beta_{\omega_{i+1}}, \dots, \beta_{\omega_n}$ occurs once in β, then*

1. *$\beta_{\omega_i}, \beta_{\omega_{i+1}}, \dots, \beta_{\omega_n}$ is a sequence of consecutive integers ending by q.*

In addition, if $\pi_{\omega_j} = \beta_{\omega_j}$ for any j, $1 \leq j < i$, then

2. *each of $\pi_{\omega_i}, \pi_{\omega_{i+1}}, \dots, \pi_{\omega_n}$ occurs once in π.*

Proof. If condition 1. is violated, then β is not the lexicographically smallest pattern in its d-equivalence class, which is a contradiction. If condition 2. is violated, then π cannot be a q-ary pattern, again a contradiction. □

Proposition 5. *Let π be a length n pattern. If $\omega = \omega(\pi)$, $\beta = \beta(\pi)$ and*

- *there is an i such that $\pi_{\omega_j} = \beta_{\omega_j}$ for any j, $1 \leq j < i$,*
- *$\pi_{\omega_i} > \beta_{\omega_i}$,*
- *the entry π_{ω_i} occurs once in π,*
- *the entry $\pi_{\omega_i} - 1$ does not occur in the set $\{\pi_{\omega_{i+1}}, \pi_{\omega_{i+2}}, \dots, \pi_{\omega_n}\}$,*

then there is a pattern σ which is f-equivalent with π and lexicographically smaller than π.

Proof. First we prove that at least one of the entries $\pi_{\omega_{i+1}}, \pi_{\omega_{i+2}}, \dots, \pi_{\omega_n}$ occurs at least twice in π. Indeed, if these entries occur once in π so are the entries in the set $P = \{\pi_{\omega_i}, \pi_{\omega_{i+1}}, \pi_{\omega_{i+2}}, \dots, \pi_{\omega_n}\}$. But since π and β have the same arity and $\pi_{\omega_j} = \beta_{\omega_j}$ for any $j < i$ it follows that the two sets P and $\{\beta_{\omega_i}, \beta_{\omega_{i+1}}, \dots, \beta_{\omega_n}\}$ are equal, and by the point 1. of Proposition 4, they are formed by consecutive integers. This is a contradiction since π_{ω_i} is not the minimal element of P (otherwise $\pi_{\omega_i} = \beta_{\omega_i}$) and $\pi_{\omega_i} - 1$ does not occur in P.

Now we prove the statement according to the following two (non exclusive) cases: (i) there is an integer larger than π_{ω_i} in the set $\{\pi_{\omega_{i+1}}, \pi_{\omega_{i+2}}, \dots, \pi_{\omega_n}\}$ that occurs at least twice in π, or (ii) there is at least one integer smaller than π_{ω_i} in the set $\{\pi_{\omega_{i+1}}, \pi_{\omega_{i+2}}, \dots, \pi_{\omega_n}\}$.
Case (i). If there is an integer larger than π_{ω_i} in the set $\{\pi_{\omega_{i+1}}, \pi_{\omega_{i+2}}, \dots, \pi_{\omega_n}\}$ that occurs twice in π, let $v \neq \pi_{\omega_i}$ be the smallest of them and let $\{\omega_{k_1}, \omega_{k_2}, \dots\} \subset \{\omega_1, \omega_2, \dots, \omega_n\}$ with $k_1 < k_2 \cdots$ be the set of occurrences of v in π. The entry $v - 1$ occurs once in π and let define a as: if $\pi_{\omega_{k_1}}$ is not in the same descent run as $v - 1$, then $a = \omega_{k_1}$, and $a = \omega_{k_2}$ otherwise. It follows that the

pattern σ with $\sigma_{\omega_j} = \pi_{\omega_j}$ for all j, except $\sigma_{\omega_a} = \pi_{\omega_a} - 1$, is lexicographically smaller than π and is obtained from π by an f-transformation.
Case (ii). If there is an integer smaller than π_{ω_i} in the set $\{\pi_{\omega_{i+1}}, \pi_{\omega_{i+2}}, \ldots, \pi_{\omega_n}\}$, then let $v \neq \pi_{\omega_i}$ be the largest of them and let ω_a be the largest element of $\{\omega_{i+1}, \omega_{i+2}, \ldots, \omega_n\}$ with $\pi_{\omega_a} = v$. Necessarily v occurs at least twice in π; otherwise, since $v+1$ does not occur in $\{\pi_{\omega_{i+1}}, \pi_{\omega_{i+2}}, \ldots, \pi_{\omega_n}\}$, the pattern σ with with $\sigma_{\omega_j} = \pi_{\omega_j}$ for any j, except $\sigma_{\omega_a} = v+1$ and $\sigma_{\omega_k} = v$ for an appropriate $k < i$ with $\pi_{\omega_k} = v + 1$, is d-equivalent with π and lexicographically smaller than π, which is in contradiction with $\pi_{\omega_j} = \beta_{\omega_j}$ for any $j < i$.

Since v occurs at least twice in π it follows that the pattern τ with $\tau_{\omega_j} = \pi_{\omega_j}$ for any j, except $\tau_{\omega_a} = \pi_{\omega_a} + 1$, is d-equivalent with π (and lexicographically larger than π) and the entry τ_{ω_a} occurs at least twice in τ. Now two subcases can occur: $\tau_{\omega_a} = \pi_{\omega_i} - 1$ $(= \pi_{\omega_a} + 1)$ or $\tau_{\omega_a} < \pi_{\omega_i} - 1$.
When $\tau_{\omega_a} = \pi_{\omega_i} - 1$, since $\tau_{\omega_i} = \pi_{\omega_i}$ it follows that τ is a pattern satisfying Proposition 3, and the pattern σ with $\sigma_{\omega_j} = \tau_{\omega_j}$ for any j, except $\sigma_{\omega_i} = \tau_{\omega_i} - 1$ $(= \pi_{\omega_i} - 1)$ and $\sigma_{\omega_a} = \tau_{\omega_a} + 1$ $(= \pi_{\omega_i})$, is d-equivalent with τ (and thus with π) and is lexicographically smaller than π. The patterns π and σ are f-equivalent, and the statement holds.
When $\tau_{\omega_a} < \pi_{\omega_i} - 1$, since the entry τ_{ω_a} occurs twice in τ we can find as previously a pattern $\tau^{(2)}$ d-equivalent with τ (and so with π) where the entry $\tau_{\omega_a} + 1$ occurs twice in $\tau^{(2)}$. Iterating this procedure, we obtain a sequence of d-equivalent patterns $\tau = \tau^{(1)}, \tau^{(2)}, \ldots, \tau^{(k)}$ with $\tau^{(k)}_{\omega_j} = \pi_{\omega_j}$ for all $j \leq i$ and $\tau^{(k)}_{\omega_b} = \pi_{\omega_i} - 1$ for some $b > i$. As above, the pattern σ with $\sigma_{\omega_j} = \tau^{(k)}_{\omega_j}$ for any j, except $\sigma_{\omega_i} = \tau^{(k)}_{\omega_i} - 1$ $(= \pi_{\omega_i} - 1)$ and $\sigma_{\omega_b} = \tau^{(k)}_{\omega_b} + 1$ $(= \pi_{\omega_i})$ is d-equivalent with $\tau^{(k)}$ (and thus with π) and lexicographically smaller than π. Moreover, each $\tau^{(p)}$ is obtained from $\tau^{(p-1)}$ by an f-transformation and the statement holds. □

By Propositions 2, 3 and 5 we have the following theorem.

Theorem 1. *Any pattern π is f-equivalent with $\beta(\pi)$.*

Proof. Let π be a pattern with $\pi \neq \beta(\pi)$. Then π is in one of the cases stated in Propositions 2, 3 or 5, and according to these propositions there exists a pattern f-equivalent (and thus d-equivalent) with π and lexicographically smaller than π, and eventually π is f-equivalent with the lexicographically smallest pattern in its d-equivalence class, that is with $\beta(\pi)$. □

Corollary 1. *Two patterns are d-equivalent if and only if they are f-equivalent.*

Proof. By definition, f-equivalence implies d-equivalence. Conversely, if two patterns π and σ are d-equivalent, then $\beta(\pi) = \beta(\sigma)$, and by the previous theorem π is f-equivalent with $\beta(\pi)$ and σ is f-equivalent with $\beta(\sigma)$. Finally π and σ are f-equivalent. $\square$

2.2 Bijection ψ

In the following we need a bijection on $[q]^n$ onto itself, that we denote by ψ, and satisfying:

(a) ψ preserves the underlying alphabet,

(b) the number of occurrences of the largest entry is the same in w and in $\psi(w)$, and the same holds for the smallest entry in w and in $\psi(w)$,

(c) ψ transforms descent set into ascent set, that is, for any word w $\operatorname{Des} w = \operatorname{Asc} \psi(w)$.

In particular when w is a permutation, the complement transformation c satisfies the three properties above, which in general is not longer true for arbitrary words, and we propose a bijection ψ which satisfies these properties for any words, not necessarily permutations. Its construction is based on the bijection ϕ on words defined in [4], which in turn is built on Foata and Schützenberger [3] bijection j on permutations. The bijection $\phi : [q]^n \to [q]^n$ in [4] satisfies for any word w:

(i) $\phi(w)$ is a rearrangement of the symbols of w,

(ii) $\operatorname{Des} w = \{n - i \,:\, i \in \operatorname{Des} \phi(w)\}$, and

(iii) $\operatorname{Ides} w = \operatorname{Ides} \phi(w)$.

See [7] for the definition of the set valued statistic Ides that we will not use here and for a weaker version of ϕ satisfying only (ii) and (iii) above. Note that from (i) it follows that ϕ preserves the underlying alphabet.

Based on the properties (i) and (ii) of ϕ it is easy to check that $\psi : [q]^n \to [q]^n$ defined as

$$\psi = \mathrm{r} \circ \phi \tag{1}$$

satisfies the above desiderata (a)–(c). Indeed, properties (a) and (b) follow from (i), and property (c) follows from (ii). Property (iii) is a deep and remarkable feature of ϕ (that we will not make use of it) and in some sense our bijection ψ is over endowed. For instance, $\phi(1321) = 3211$, $\phi(1232) = 3122$ and $\phi(3321) = 3213$ (see [4]), and thus $\psi(1321) = 1123$, $\psi(1232) = 2213$ and $\psi(3321) = 3123$.

2.3 Pattern trace and word substitution

For a word $w = w_1w_2\dots w_n$ and a set $S = \{i_1, i_2, \dots, i_p\} \subseteq \{1, 2, \dots, n\}$ we denote by w_S the subword $w_{i_1}w_{i_2}\dots w_{i_p}$ of w.

Let $t = t_1t_2\dots t_k$ be a length k word over $[q] \cup \{\square\}$, $q \geq 1$, and $I(t)$ be the set $\{\ell : 1 \leq \ell \leq k,\ t_\ell \neq \square\}$. We say that t is *a trace* of the pattern $\pi = \pi_1\pi_2\dots\pi_k$ if t_i and t_j have the same relative order ($<$, $=$, or $>$) as π_i and π_j have whenever $i, j \in I(t)$. Equivalently, t is a trace of π if the words $t_{I(t)}$ and $\pi_{I(t)}$ are order-isomorphic. In particular, when t does not contain $\square$, then $red(t) = \pi$; and t formed only by $\square$'s is a trace of any pattern. It can happen that t is a trace of several patterns. For instance, for two same length patterns π and σ, if the trace t of π is such that $\pi_\ell \neq \sigma_\ell$ implies $t_\ell = \square$, then t is a trace of σ as well.

With t a trace of a pattern π and $I(t)$ as above, for a word $w = w_1w_2\dots w_n$ and a set $A \subset \{1, 2, \dots, n\}$ of positions in w we say that t is a *trace of* π *in* w *at* A if $t_{I(t)} = w_A$ (and so, $|I(t)| = |A|$), and a trace t of π in w at A can be seen as a partial occurrence of the pattern π in w with $\square$ playing the role of 'wild' symbol. It can happen that several occurrences of π in a word w have trace t at A, and we denote by $(t, A, \pi)w$ the number of these occurrences, and thus (t, A, π) becomes an integer valued statistic on words.

Example 4. If $\pi = 1332$ and $\sigma = 2331$ are two patterns, then $t = \square 44 \square$ and $t' = \square 55 \square$ are traces of both π and σ. Furthermore, if $w = 154543$, then

- 1443 is an occurrence in w of π with trace $t = \square 44 \square$ at $A = \{3, 5\}$,
- 1554 and 1553 are occurrences in w of π with trace $t' = \square 55 \square$ at $A = \{2, 4\}$, and $(t', A, \pi)w = 2$.

See Table 1 in Appendix for other examples.

For a word $w = w_1w_2\dots w_n$ and two pairs of integer $a < b$ and $c < d$ we denote by $w \,|\, ([a, b], [c, d])$ the length-maximal subword $w_{i_1}w_{i_2}\dots w_{i_k}$ of w with $\{i_1, i_2, \dots, i_k\} \subseteq [a, b]$ and $\{w_{i_1}, w_{i_2}, \dots, w_{i_k}\} \subseteq [c, d]$. Alternatively, $w \,|\, ([a, b], [c, d])$ is the length-maximal subword of $w_{[a,b]}$ with entries in $[c, d]$.

If the word $w_{i_1}w_{i_2}\dots w_{i_k} = w \,|\, ([a, b], [c, d])$ has m different symbols and $u = u_1u_2\dots u_k$ is a word with the underlying alphabet $[m]$, then there is a unique word $v = v_1v_2\dots v_n$ with

- $v_\ell = w_\ell$ for any ℓ with $\ell \notin [a, b]$ or $w_\ell \notin [c, d]$, and
- $red(v \,|\, ([a, b], [c, d]) = u$, and $v \,|\, ([a, b], [c, d])$ and $w \,|\, ([a, b], [c, d])$ have the same underlying alphabet.

Indeed, v is obtained from w by replacing the subword $w_{i_1}w_{i_2}\dots w_{i_k}$ of w by an appropriate word order-isomorphic with u. With these notations we call v the $([a,b],[c,d])$-*substitution by u in w*. In particular, if $u = red(w\,|\,([a,b],[c,d])$, then the $([a,b],[c,d])$-substitution by u in w is w itself, and we have the following easy to understand fact.

Fact 1. If $red(w\,|\,([a,b],[c,d])$ and u are two d-equivalent words, then so are w and the $([a,b],[c,d])$-substitution by u in w.

See Example 5 where the $([3,7],[1,4])$-substitution by 3321 in $w = 21\mathit{143}6\mathit{1}5441$ is $v = 21\mathit{443}6\mathit{1}5441$ (the replaced elements are in italic and represented by $\times$ in the corresponding diagrams).

3 Proof of the main results

The main result of this article is Theorem 4. Prior to its proof, Lemmata 1 and 2 below establish some equidistribution results and the Corollary 2 of Theorems 2 and 3 says that if two patterns are an f-transformation of each other, then the patterns have the same popularity on any d-equivalence class.

Lemma 1. *Let $\pi = \pi_1\pi_2\dots\pi_k$ and $\sigma = \sigma_1\sigma_2\dots\sigma_k$ be two d-equivalent patterns with $\pi_\ell = \sigma_\ell$ for any ℓ, except $\sigma_i = \pi_i + 1$ for some i. Let also t be a trace of both π and σ with one $\square$ symbol and A be a $k-1$ element subset of $\{1,2,\dots,n\}$. Then on any d-equivalence class the statistics (t,A,π) and (t,A,σ) have the same distribution.*

Proof. For any d-equivalence class we give a bijection $w \mapsto v$ with $(t,A,\pi)w = (t,A,\sigma)v$.

Since π and σ differ in position i it follows that $t_i = \square$. In addition, since π and σ are patterns, the entry π_i occurs at least twice in π (otherwise $\sigma = \pi_1\dots(\pi_i+1)\dots\pi_k$ is not longer a pattern) and so does σ_i in σ. Let x be the symbol in t playing the role of π_i and y be that playing the role of σ_i, and we define the interval $[c,d] = [x,y]$ and the interval $[a,b]$ as follows. If $A = \{p_1,p_2,\dots,p_{k-1}\} \subset \{1,2,\dots,n\}$, then

- if $i = 1$, then $[a,b] = [1,p_1-1]$,
- if $i = k$, then $[a,b] = [p_{k-1}+1,n]$,
- elsewhere, $[a,b] = [p_{i-1}+1,p_i-1]$.

Now let u be the word $red(w\,|\,([a,b],[c,d])$ and u' be the word $(\mathrm{c}\circ\psi)(u)$, with ψ defined in relation (1) and c the complement operation. The desired word v is the $([a,b],[c,d])$-substitution by u' in w. Indeed, u and u' have the same descent set and same underlying alphabet, and thus they are d-equivalent. By Fact 1 the transformation $w \mapsto v$ turns the word $w = w_1w_2\dots w_n$ into a d-equivalent word $v = v_1v_2\dots v_n$. In addition, by property (b) of ψ, it follows that the number of the largest entries in u is the same as that of the smallest entries in $u' = (\mathrm{c}\circ\psi)(u)$, and vice versa. Thus, for any j, $p_{i-1} < j < p_i$ (with the convention $p_0 = 0$ and $p_k = n+1$) $w \mapsto v$ transforms any occurrence

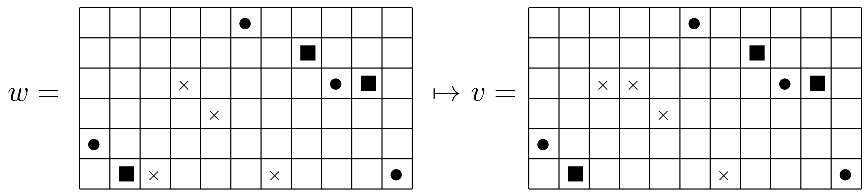
$$w_{p_1}\dots w_{p_{i-1}}w_jw_{p_i}\dots w_{p_{k-1}} = t_1\dots t_{i-1}xt_{i+1}\dots t_k$$

of π in w with trace t at A into an occurrence

$$v_{p_1}\dots v_{p_{i-1}}v_jv_{p_i}\dots v_{p_{k-1}} = t_1\dots t_{i-1}yt_{i+1}\dots t_k,$$

of σ in v with trace t at A. This transformation is reversible, indeed the $([a,b],[c,d])$-substitution by u in v gives the word w, and so it is a bijection. □

Example 5. We represent words as diagrams identifying words $w = w_1w_2\dots w_n$ with the set of points $\{(i,w_i)\,:\,1\le i\le n\}$. Let w be the word 21143615441 (the left-hand side diagram in this representation), $\pi = 1132$ and $\sigma = 1232$ be two patterns as in Lemma 1, $t = 1\square54$ be a common trace of π and σ, and A be the set $\{2,8,10\}$. In the diagram representation of w, the entries 1, 5 and 4 of t occurring in positions belonging to A are represented by ■ symbol. Following the notations in the proof of Lemma 1, the interval $[c,d]$ is $[1,4]$, $[a,b]$ is $[3,7]$, $w\,|\,([a,b],[c,d])$ is the subword 1431 (represented by $\times$ symbols in the left-hand side diagram), $u = red(w\,|\,([a,b],[c,d])$ is the word 1321, $u' = (\mathrm{c}\circ\psi)(u)$ is $\mathrm{c}(1123) = 3321$, see the examples at the end of the Section 2.2. Finally, $v = 21443615441$ in the right-hand side diagram is the image of w through the bijection in the proof of Lemma 1, and we have $(t,A,\pi)w = (t,A,\sigma)v$. Indeed, π occurs twice in w with trace t at A, namely in positions $2,3,8,10$ and in positions $2,7,8,10$; and so does σ in v with trace t at A, namely in positions $2,3,8,10$ and in positions $2,4,8,10$.



The next lemma is the counterpart of Lemma 1 where the patterns differ in two positions, with the additional requirement that the two different entries occur once in each pattern.

Lemma 2. *Let $\pi = \pi_1\pi_2\dots\pi_k$ and $\sigma = \sigma_1\sigma_2\dots\sigma_k$ be two d-equivalent patterns such that there are i and j, $i < j$, with*

- *$\pi_\ell = \sigma_\ell$ for any ℓ, except $\pi_i = \sigma_j$ and $\pi_j = \sigma_i$,*
- *$\pi_j = \pi_i + 1$,*
- *each of π_i and π_j occurs once in π (or, equivalently, σ_i and σ_j occur once in σ).*

Let also $t = t_1t_2\dots t_k$ be a trace of both π and σ with two $\square$ symbols and A be a subset of $\{1, 2, \dots, n\}$ of cardinality $k-2$. Then on any d-equivalence class the statistics (t, A, π) and (t, A, σ) have the same distribution.

Proof. To a certain extent the proof is similar to that of Lemma 1 by giving a bijection $w \mapsto v$ with $(t, A, \pi)w = (t, A, \sigma)v$ on any d-equivalence class.
Since π and σ differ in positions i and j and t contains two $\square$ symbols, it follows that $t_i = t_j = \square$, and since π and σ are d-equivalent i and j are not consecutive positions in t. We define three intervals $[a, b]$, $[a', b']$ and $[c, d]$.
Let $A = \{p_1, p_2, \dots, p_{k-2}\}$ be the $k-2$ element subset.

- If π_i is the smallest entry in π then $c = 1$. Otherwise let π_u be the largest entry in π smaller than π_i, and x be the entry in t playing the role of π_u, and finally $c = x+1$. Similarly, if π_j is the largest entry in π then $d = n$. Otherwise let π_u be the smaller entry in π larger than π_j, and x be the entry in t playing the role of π_u, and finally $d = x-1$.
- If $i = 1$, then $a = 1$, otherwise $a = p_{i-1} + 1$; and $b = p_i - 1$.
- $a' = p_{j-2} + 1$; and if $j = k-2$, then $b' = n$, otherwise $b' = p_{j-1} - 1$.

Now we define the announced bijection $w \mapsto v$, where v is obtained by constructing the words w', w'' and $w''' = v$ by applying the following steps.

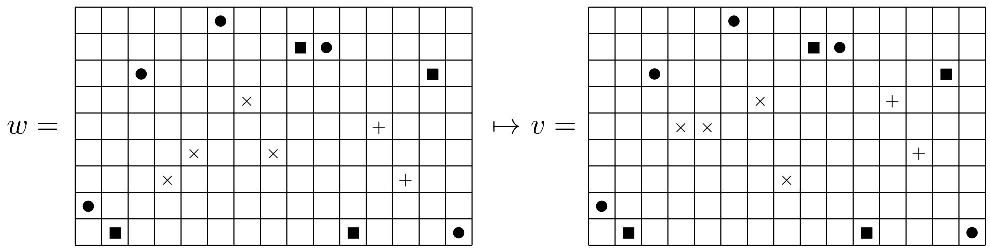
1. Let u be the word $red(w \,|\, ([a,b],[c,d])$ and $u' = \psi(u)$, with ψ defined in relation (1), and w' be the $([a,b],[c,d])$-substitution by u' in w;
2. let u be the word $red(w' \,|\, ([a',b'],[c,d])$ and $u' = \psi(u)$, and w'' be the $([a',b'],[c,d])$-substitution by u' in w';
3. let u be the word $red(w'' \,|\, ([a,b'],[c,d])$ and $u' = \mathrm{c}(u)$, with c the complement operation, and w''' be the $([a,b'],[c,d])$-substitution by u' in w'';

and finally $v = w'''$. Note that the first two steps can be performed in arbitrary order since the substitution operations act on different entries of w (on the disjoint intervals $[a,b]$ and $[a',b']$). As in the proof of Lemma 1, taking in consideration the properties of ψ, $w \mapsto v$ transforms any occurrence of π in w with trace t at A into an occurrence of σ in v with trace t at A, and $w \mapsto v$ is reversible and so it is a bijection. □

Note that in the previous proof, unlike in that of Lemma 2, the property (b) of the bijection ψ is not used.

See Table 1 in Appendix for an example of the equidistribution stated in Lemma 2.

Example 6. Let w be the word 217349648815371 (the left-hand side diagram below), $\pi = 125134$ and $\sigma = 135124$ be two patterns as in Lemma 2, $t = 1\square81\square7$ be a common trace of π and σ, and A be the set $\{2, 9, 11, 14\}$. In the diagram representation of w, the entries $1, 8, 1, 7$ of t occurring in positions belonging to A are represented by ■ symbols. Following the notations in the proof of Lemma 2, the interval $[a,b]$ is $[4,8]$, $[a',b']$ is $[12,13]$, $[c,d]$ is $[3,6]$, $w \,|\, ([a,b],[c,d])$ is the subword 3464 (represented by $\times$ symbols in the left-hand side diagram), $w \,|\, ([a',b'],[c,d])$ is the subword 53 (represented by $+$ symbols), $red(w \,|\, ([a,b],[c,d]) = red(3464)$ is 1232 and $\psi(1232) = 2213$, $red(w \,|\, ([a',b'],[c,d]) = red(53)$ is 21 and $\psi(21) = 12$. Finally, $v = 217559638816471$ in the left-hand side diagram is the image of w through the bijection in the proof of Lemma 2, and we have $(t, A, \pi)w = (t, A, \sigma)v$. Indeed, π occurs three times in w with trace t at A, namely in positions $2, 4, 9, 11, 12, 14$, in positions $2, 5, 9, 11, 12, 14$, and in positions $2, 8, 9, 11, 12, 14$; and so does σ in v with trace t at A, namely in positions $2, 4, 9, 11, 13, 14$, in positions $2, 5, 9, 11, 13, 14$, and in positions $2, 7, 9, 11, 13, 14$.



Theorem 2. *Let $\pi = \pi_1\pi_2\ldots\pi_k$ and $\sigma = \sigma_1\sigma_2\ldots\sigma_k$ be two d-equivalent patterns with $\pi_\ell = \sigma_\ell$ for any ℓ, except $\sigma_i = \pi_i + 1$ for some i. Then π and σ have the same popularity on any d-equivalence class.*

Proof. By Lemma 1, for any

- integer p,
- trace t with one $\square$ symbol in position i of both π and σ, and
- cardinality $k-1$ subset A of $\{1, 2, \ldots, n\}$,

on any d-equivalence class we have

$$|\{w : (t, A, \pi)w = p\}| = |\{w : (t, A, \sigma)w = p\}|.$$

For t and A fixed, summing over all w in a d-equivalence class we have

$$\sum_w (t, A, \pi)w = \sum_w (t, A, \sigma)w.$$

Further, for a fixed A, summing over all possible traces t at A of both π and σ we have

$$\sum_t \sum_w (t, A, \pi)w = \sum_t \sum_w (t, A, \sigma)w.$$

Note that in this equality there are no 'double counting' since different traces result in occurrences of π and σ with different values for the entries. Finally summing over all cardinality $k-1$ set A we have

$$\sum_A \sum_t \sum_w (t, A, \pi)w = \sum_A \sum_t \sum_w (t, A, \sigma)w.$$

Again, there are no 'double counting' since different sets A result in occurrences of π and σ in different positions. The two sides of the last equality give precisely the popularity of π and σ, respectively, on a d-equivalence class, and the statement follows. □

Theorem 3. *Let $\pi = \pi_1\pi_2\ldots\pi_k$ and $\sigma = \sigma_1\sigma_2\ldots\sigma_k$ be two d-equivalent patterns with $\pi_\ell = \sigma_\ell$ for any ℓ, except $\pi_i = \sigma_j$ and $\pi_j = \sigma_i$ for some i and j, and $\pi_j = \pi_i + 1$. Then π and σ have the same popularity on any d-equivalence class.*

Proof. We distinguish two cases: at least one of the symbols π_i and π_j occurs twice in π, or each of these symbols occurs exactly once in π.
In the first case, suppose that π_i occurs twice in π and let $\tau = \tau_1\tau_2\ldots\tau_k$ be the pattern with $\tau_\ell = \tau_\ell$ for any ℓ, except $\tau_i = \pi_i + 1$ ($= \pi_j = \sigma_i$). Since π and σ are d-equivalent so are π and τ (and thus τ and σ). By Theorem 2 it follows that τ has the same popularity as π. But $\tau_\ell = \sigma_\ell$ for any ℓ, except $\tau_j = \sigma_j + 1$ ($= \pi_j$) and again by Theorem 2 it follows that τ has the same popularity as σ, and the statement follows.
In the second case (π_i and π_j occur once in π), applying Lemma 2 and reasoning as in the proof of Theorem 2, we have the desired equipopularity. □

Recall that an f-transformation turns a pattern π into another d-equivalent one σ by making 'small changes' as in Theorems 2 and 3, and we have the next consequence of these theorems.

Corollary 2. *If the pattern σ is an f-transformation of the pattern π, then π and σ have the same popularity on any d-equivalence class.*

See Table 2 in Appendix for an example of the equipopularity stated in Corollary 2. Combining Corollaries 1 and 2 we obtain the next theorem.

Theorem 4. *Two patterns are d-equivalent if and only if they have the same popularity on any d-equivalence class.*

Proof. '⇒' If the patterns π and σ are d-equivalent, then they are f-equivalent, and thus there is a sequence of patterns $\pi = \tau^{(1)}, \tau^{(2)}, \ldots, \tau^{(k)} = \sigma$ such that $\tau^{(p+1)}$ is an f-transformation of $\tau^{(p)}$, $1 \le p \le k-1$. Thus $\tau^{(p+1)}$ and $\tau^{(p)}$ are equipopular on any d-equivalence class and so are π and σ.
'⇐' By contraposition: if the patterns π and σ are not d-equivalent, then within the words of the d-equivalence class containing (once) π the pattern σ does not occur, or vice versa. Indeed π and σ differ by their length and/or

their arity, and/or their descent set; and so there is a d-equivalence class on which π and σ are not equipopular. $\square$

Two same length words are *descent-equivalent* if they have same descent set (and not necessarily same underlying alphabet), and so d-equivalence implies descent-equivalence. A q-ary descent-equivalence class is a maximal set of same length descent-equivalent q-ary words, for instance $\{121, 131, 132, 231, 232\}$ is a 3-ary descent-equivalence class. And we have the next easy to see corollary.

Corollary 3. *Two patterns are d-equivalent if and only if they have the same popularity on any q-ary descent-equivalence class.*

Finally, permutations are particular words (and particular patterns) for which the notions of d-equivalence and descent-equivalence coincide. Specializing the previous results to permutations we have the following straightforward result.

Corollary 4. *Two permutations are descent-equivalent if and only if they have the same popularity on any descent-equivalence class of permutations.*

See Table 2 in Appendix for an example of equipopularity of two descent-equivalent permutations.

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Appendix

Here we give two examples of equidistribution and equipopularity considered through this article.

w	$(t, A, \pi)w$	$(t, A, \sigma)w$
15**4**15**4**32	2	0
15**4**25**4**31	1	0
15**4**25**4**32	2	0
15**4**35**4**21	1	0
15**4**35**4**32	2	0
25**4**15**4**31	1	1
25**4**15**4**32	1	0
25**4**25**4**31	1	1
25**4**35**4**21	0	1
25**4**35**4**31	1	1
35**4**15**4**21	0	2
35**4**15**4**32	0	1
35**4**25**4**21	0	2
35**4**25**4**31	0	1
35**4**35**4**21	0	2
...	0	0

Table 1: The equidistribution of the statistics (t, A, π) and (t, A, σ) over the set of length eight words with underlying alphabet $\{1, 2, \ldots, 5\}$ and descent set $\{2, 3, 5, 6, 7\}$, for: $\pi = 1332$, $\sigma = 2331$, $t = \square\mathbf{44}\square$ and $A = \{3, 6\}$. Only words w with $(t, A, \pi)w \neq 0$ or $(t, A, \sigma)w \neq 0$ are shown. The occurrences of the symbols **4** in positions belonging to A (and playing the role of 3 in the occurrences of π and of σ) are in bold. There are six words w for which $(t, A, \pi)w = 1$, as many as for which $(t, A, \sigma)w = 1$; and there are three words w for which $(t, A, \pi)w = 2$, as many as for which $(t, A, \sigma)w = 2$.

w	$(213)w$	$(312)w$
21354	3	0
21453	3	0
31254	4	1
31452	2	1
32451	2	0
41253	2	3
41352	2	2
42351	2	1
51243	0	5
51342	0	4
52341	0	3
...	0	0
popularity	20	20

Table 2: The equipopularity of the patterns 213 and 312 over the set of length five words with underlying alphabet $\{1, 2, \ldots, 5\}$ (that is, length five permutations) and descent set $\{1, 4\}$. Only words w with $(213)w \neq 0$ or $(312)w \neq 0$ are shown. The two patterns are not equidistributed over this set.