Motzkin subposet and Motzkin geodesics in Tamari lattices

J.-L. Baril and J.-M. Pallo
LE2I, UMR 5158, Université de Bourgogne
B.P. 47870, F21078 DIJON-Cedex FRANCE
e-mail:{barjl,pallo}@u-bourgogne.fr

August 26, 2013

Abstract

The Tamari lattice of order n can be defined by the set $\mathcal{D}_n$ of Dyck words endowed with the partial order relation induced by the well-known rotation transformation. In this paper, we study this rotation on the restricted set of Motzkin words. An upper semimodular join semilattice is obtained and a shortest path metric can be defined. We compute the corresponding distance between two Motzkin words in this structure. This distance can also be interpreted as the length of a geodesic between these Motzkin words in a Tamari lattice. So, a new upper bound is obtained for the classical rotation distance between two Motzkin words in a Tamari lattice. For some specific pairs of Motzkin words, this bound is exactly the value of the rotation distance in a Tamari lattice. Finally, enumerating results are given for join and meet irreducible elements, minimal elements and coverings.

Keywords: Lattices; Dyck words; Motzkin words; Tamari lattice; Metric; Geodesic.

1 Introduction and notations

The set $\mathcal{D}$ of Dyck words over $\{(,)\}$ is the language defined by the grammar $S \rightsquigarrow \lambda|(S)|SS$ where λ is the empty word, *i.e.* the set of well-formed parentheses strings. Let $\mathcal{D}_n$ be the set of Dyck words of length $2n$, *i.e.* with n open and n close parentheses. The cardinality of $\mathcal{D}_n$ is the nth Catalan number $c_n = (2n)!/(n!(n+1)!)$ (see A000108 in [23]). For instance, $\mathcal{D}_3$ consists of the five words $()()()$, $(())()$, $()(())$, $(()())$ and $((()))$. A large number of various classes of combinatorial objects are enumerated by the Catalan sequence. This is the case, among others, for ballot sequences, planar trees, binary rooted trees, nonassociative products, stack sortable permutations, triangulations of polygons, and Dyck paths. See [24] for a compilation of such Catalan sets.

Some of them are endowed with a partial ordering relation [1, 2, 3, 10, 20, 21]. For instance, the coverings of the so-called Tamari lattices [11, 13, 14, 16, 18, 22, 25] can be defined by different elementary transformations depending on the Catalan set considered. The most known is the semi-associative law $x(yz) \longrightarrow (xy)z$ for well-formed parenthesized expressions involving n variables. Also, the Tamari lattice of order n can be defined on the set $\mathcal{T}_n$ of binary rooted trees with $n+1$ leaves. Indeed, from a well-formed parenthesized expression on n variables, we consider the bijection that recursively constructs the binary rooted tree where the left (resp. right) subtree is defined by the left (resp. right) part of the expression. For example, the binary rooted trees associated to the two expressions $x(yz)$ and $(xy)z$ are illustrated in Figure 1. Moreover, the semi-associative law on parenthesized expressions is equivalent to the well-known left-rotation on binary trees showed in Figure 1.

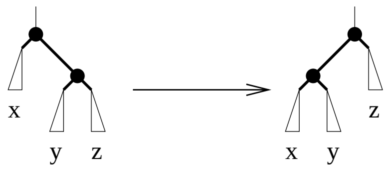

Figure 1: The left-rotation transformation on binary trees.

Now, let $T \in \mathcal{T}_n$ be a binary tree with $n+1$ leaves. Reading T in prefix order and replacing each internal node (resp. each leaf except the last) with an open (resp. a close) parenthesis, we obtain a bijection from $\mathcal{T}_n$ to $\mathcal{D}_n$ that translates the left-rotation transformation into the elementary transformation $(u)(\longrightarrow ((u)$ where u is a Dyck word. This transformation will be also called left-rotation. More precisely, we say that $d' \in \mathcal{D}_n$ is obtained from $d \in \mathcal{D}_n$ by a left-rotation if $d = \alpha(u)(\beta$ and $d' = \alpha((u)\beta$ where u is a Dyck word and α (resp.β) is some prefix (resp. suffix) of some Dyck word. The inverse transformation is called right-rotation. For instance, (())((()(())())()) is obtained from (())(()(())())(()) by a left-rotation.

We define the rotation distance between two Dyck words as the minimum number of left- and right-rotations necessary to transform one word into the other. There remains today an open problem whether the rotation distance can be computed in polynomial time. Previous works on rotation distance have focused on approximation algorithms [4, 7, 18, 19].

In this paper, we study this rotation on the restricted set $\mathcal{M}$ of Motzkin words defined by the grammar $S \rightsquigarrow \lambda|(SS)$. In Section 2, an upper semimodular join semilattice is constructed. In this structure, we compute the length of a shortest path between two Motzkin words. This distance can also be interpreted as the length of a geodesic between these two Motzkin words in a Tamari lattice, *i.e.* the length of a shortest path between them by browsing through Motzkin words only. So, a new upper bound is obtained for the classical rotation distance between two Motzkin words in a Tamari lattice. For some specific pairs of Motzkin words, this bound is exactly the value of the well-known rotation distance in a Tamari lattice. In Section 3, enumerating results are given for join and meet irreducible elements, minimal elements and coverings.

2 Motzkin geodesics

Let $\mathcal{M}$ be the set of Motzkin words, *i.e* the language over $\{(,)\}$ defined by the grammar $S \rightsquigarrow \lambda|(SS)$. Let $\mathcal{M}_n$ be the set of Motzkin words of length $2n$, *i.e.* with n open and n close parentheses. The cardinality of $\mathcal{M}_n$ is the nth term of the Motzkin sequence A001006 in [23]. For example, $\mathcal{M}_4 = \{(((()))), ((()())), (()(())), ((())())\}$. Obviously we have $\mathcal{M}_n \subseteq \mathcal{D}_n$ for $n \geq 1$. We refer to [8] and [24] for several combinatorial classes enumerated by the Motzkin numbers.

The following lemma shows how the left-rotation between two Motzkin words can be expressed by a natural transformation $\longrightarrow$ in $\mathcal{M}_n$.

Lemma 1 *Let $m, m' \in \mathcal{M}_n$. Then, the word m' is obtained from m by a left-rotation if and only if m' is obtained from m by a left-transformation $u(v) \longrightarrow (uv)$ where u and v are two Motzkin words, i.e. $m = \alpha u(v)\beta$ and $m' = \alpha(uv)\beta$ where $u, v \in \mathcal{M}$ and α (resp. β) is some prefix (resp. suffix) of some Motzkin word.*

Proof. Let m and m' be two Motzkin words such that m' is obtained from m by a left-rotation. Thus, there is a Dyck word u such that $m = \alpha(u)(\beta$ and $m' = \alpha((u)\beta$ where α and β are some prefix and some suffix of m and m'. Since m' is a Motzkin word, it necessarily is of the form $m' = \alpha((u)v)\beta'$ where (u) and v are Motzkin words with $\beta = v)\beta'$. We deduce that $m = \alpha(u)(v)\beta'$. Furthermore, the fact that (v) and (u) are Motzkin words necessarily implies that m is of the form $m = \alpha'((u)(v))\beta''$ and thus, $m' = \alpha'(((u)v))\beta''$ for some α' and β''. Thus the left-rotation between two Motzkin words is equivalent to the transformation $w(v) \longrightarrow (wv)$ where $w = (u) \neq \lambda$ and v are two Motzkin words. Conversely, let us assume that m' is obtained from m by a transformation $u(v) \longrightarrow (uv)$ where u and v are two Motzkin words. Since, a Motzkin word is obtained by the grammar $S \rightsquigarrow \lambda|(SS)$, we necessarily have $m = \alpha(u(v))\beta$ and $m' = \alpha((uv))\beta$ for some prefix and some suffix α and β which directly induces that m' is obtained from m by a left-rotation. $\square$

In the remainder of the paper, given $m, m' \in \mathcal{M}_n$, we write $m \longrightarrow m'$ if m' is obtained from m by the left-rotation defined in the previous lemma. The right-rotation will be the inverse of $\longrightarrow$. Let $\stackrel{*}{\longrightarrow}$ denote the reflexive and transitive closure of the rotation transformation $\longrightarrow$ in $\mathcal{M}_n$.

In order to characterize this left-rotation $\longrightarrow$, we exhibit a bijection between Motzkin words and Motzkin paths. A Motzkin path of length n is a lattice path starting at $(0, 0)$, ending at $(n, 0)$, and never going below the x-axis, consisting of up steps $U = (1, 1)$, horizontal steps $H = (1, 0)$, and down steps $D = (1, -1)$. Let $\mathcal{P}_n$ be the set of Motzkin path of length $n-1$. It is well-known that Motzkin paths are enumerated by the Motzkin numbers (A001006 in [23]).

Let ϕ be the bijection between $\mathcal{M}_n$ and the set $\mathcal{P}_n$ of Motzkin paths of length $n-1$ defined as follows:

- if $m = ()$ then $\phi(m) = \lambda$;
- if $m = (uv)$ where u, v are two non-empty Motzkin words, then $\phi(m) = U\phi(v)D\phi(u)$;
- if $m = (u)$ where u is a non-empty Motzkin word, then $\phi(m) = H\phi(u)$.

For instance, if $m = (()((()())))$ then $\phi(m) = UHUDD$.

Proposition 1 *Let m and m' be two Motzkin words in $\mathcal{M}_n$. Then $m \longrightarrow m'$ if and only if $\phi(m')$ is obtained from $\phi(m)$ by applying one of the two following transformations: $UH \longrightarrow HU$ and $UD \longrightarrow HH$.*

Proof. Let m and m' be two Motzkin words where m' is obtained from m by a left-rotation in the Tamari lattice of order n. By Lemma 1, we deduce that $m = \alpha(u(v))\beta$ and $m' = \alpha((uv))\beta$ where α and β are some prefix and some suffix of m and m'. Therefore $\phi(m')$ is obtained from $\phi(m)$ by replacing the factor $\phi((u(v)))$ with $\phi(((uv)))$. If v is empty, then we have $\phi((u(v))) = UD\phi(u)$, $\phi(((uv))) = HH\phi(u)$ and $\phi(m')$ is obtained from $\phi(m)$ by a transformation $UD \longrightarrow HH$. If v is not empty, then we have $\phi((u(v))) = UH\phi(v)D\phi(u)$, $\phi(((uv))) = HU\phi(v)D\phi(u)$ and $\phi(m')$ is obtained from $\phi(m)$ by a transformation $UH \longrightarrow HU$. This reasoning can also be considered for the converse. $\square$

A sequence of non-negative integers $\chi = \chi(0)\chi(1)\ldots\chi(n-1)$ will be called *height sequence* of length n if $\chi(0) = \chi(n-1) = 0$ and $|\chi(i) - \chi(i-1)| \leq 1$ for $1 \leq i \leq n-1$. Obviously, there is a one-to-one correspondence between Motzkin words in $\mathcal{M}_n$ and height sequences of length n: if $m \in \mathcal{M}_n$, then we associate the height sequence χ_m defined by $\chi_m(0) = 0$ and for $i \geq 1$, $\chi_m(i) - \chi_m(i-1) = 1$ (resp. -1, 0) if the ith step is U (resp. D and H) in the Motzkin path $\phi(m)$. For instance, if $m = (()((()(())))) \in \mathcal{M}_6$, then $\phi(m) = UUHDD$ and $\chi_m = 012210$. Notice that the height sequence of a Motzkin word m corresponds to the ordinates of the different steps in the path $\phi(m)$.

Proposition 2 *Let m and m' be two Motzkin words in $\mathcal{M}_n$. Then $m \longrightarrow m'$ if and only if there exists i, $1 \leq i \leq n-1$, such that $\chi_{m'}(i-1) = \chi_{m'}(i) = \chi_m(i)-1$ and for all $j \neq i$, $\chi_{m'}(j) = \chi_m(j)$.*

Proof. Using Proposition 1, $m \longrightarrow m'$ if and only if $\phi(m')$ is obtained from $\phi(m)$ by one of the two transformations $UH \longrightarrow HU$ and $UD \longrightarrow HH$. Considering the height sequences χ_m and $\chi_{m'}$, this implies the existence of some i such that $\chi_m(j) = \chi_{m'}(j)$ for $j \neq i$, and $\chi_{m'}(i-1) = \chi_{m'}(i) = \chi_m(i) - 1$. For the converse, let us assume that the height sequences χ_m and $\chi_{m'}$ satisfy for some i, $\chi_{m'}(j) = \chi_m(j)$ for $j \neq i$, and $\chi_{m'}(i-1) = \chi_{m'}(i) = \chi_m(i) - 1$. Since χ_m is a height sequence, we have $|\chi_m(i+1) - \chi_m(i)| \leq 1$. This implies that

$|\chi_{m'}(i+1) - \chi_{m'}(i) - 1| \leq 1$ and thus $\chi_{m'}(i+1) - \chi_{m'}(i) \geq 0$. Finally, if $\chi_{m'}(i+1) = \chi_{m'}(i)$ (resp. $\chi_{m'}(i+1) = \chi_{m'}(i)+1$) then $\phi(m')$ is obtained from $\phi(m)$ by the transformation $UD \longrightarrow HH$ (resp. $UH \longrightarrow HU$). □

Proposition 3 *The poset $(\mathcal{M}_n, \stackrel{*}{\longrightarrow})$ is graded by the rank function $r(m) = \rho - \sum_{i=0}^{n-1} \chi_m(i)$ where $\rho = \lfloor \frac{(n-1)^2}{4} \rfloor$.*

Proof. The poset $(\mathcal{M}_n, \stackrel{*}{\longrightarrow})$ is graded by the rank function r whenever $m \stackrel{*}{\longrightarrow} m'$ and $r(m') = r(m) + 1$ if and only if $m \longrightarrow m'$. So, Proposition 2 induces that $(\mathcal{M}_n, \stackrel{*}{\longrightarrow})$ is graded by the rank function $r(m) = \rho - \sum_{i=0}^{n-1} \chi_m(i)$ where the parameter $\rho = \lfloor \frac{(n-1)^2}{4} \rfloor$ is the maximal value of $\sum_{i=0}^{n-1} \chi_m(i)$ among all Motzkin words m. That is, we have $\rho = \sum_{i=0}^{n-1} \chi_m(i)$ with $m = 0123\ldots\frac{n-1}{2}\ldots3210$ if n is odd, and $m = 0123\ldots\frac{n}{2}-1\frac{n}{2}-1\ldots3210$ otherwise. For the two cases, a simple calculation provides $\rho = \lfloor \frac{(n-1)^2}{4} \rfloor$. □

Remark 1 In the previous proposition, we use the definition of a graded poset given by Grätzer (see [12], p. 233). Notice that other authors like Stanley do not use the same definition (see [24], p. 99). Indeed they require that the minimal elements need to have the same rank (see Figure 2).

Proposition 4 *For m, $m' \in \mathcal{M}_n$, we have $m \stackrel{*}{\longrightarrow} m'$ if and only if the sequences χ_m and $\chi_{m'}$ satisfy the two conditions:*

(a) $\chi_{m'}(i) \leq \chi_m(i)$ for all $0 \leq i \leq n-1$, and

(b) there does not exist i, $1 \leq i \leq n-1$, such that $\chi_{m'}(i-1) > \chi_{m'}(i) < \chi_m(i)$.

Proof. Let m and m', $m \neq m'$, such that their corresponding height sequences χ_m and $\chi_{m'}$ satisfy the conditions (a) and (b). Since $m \neq m'$, there exists some i such that $\chi_{m'}(i) < \chi_m(i)$. We choose the rightmost i with this property. We necessarily have $\chi_{m'}(i+1) - \chi_{m'}(i) = \chi_m(i+1) - \chi_{m'}(i) > \chi_m(i+1) - \chi_m(i) \geq -1$ and thus $\chi_{m'}(i+1) \geq \chi_{m'}(i)$. Now, there exists some j, $j \leq i$, such that $\chi_{m'}(j) \leq \chi_{m'}(j-1)$. By contradiction, if there does not exist such a j then $\chi_{m'}(j) > \chi_{m'}(j-1)$ for $j \leq i$. With $\chi_{m'}(0) = 0$, we obtain $\chi_{m'}(i) = i$ and $\chi_m(i) > \chi_{m'}(i) = i$ gives a contradiction. So, we have $\chi_{m'}(k) + 1 = \chi_{m'}(k+1)$ for $j \leq k \leq i-1$, which implies $\chi_{m'}(j) = \chi_{m'}(i) - (i-j)$. On the other hand, we necessarily have $\chi_m(j) \geq \chi_m(i) - (i-j)$. Using $\chi_{m'}(i) < \chi_m(i)$, we deduce that $\chi_{m'}(j) < \chi_m(j)$ with the condition $\chi_{m'}(j) \leq \chi_{m'}(j-1)$. Using Proposition 2, there exists a Motzkin word m_1 satisfying $m_1 \longrightarrow m'$ and such that $\chi_{m_1}(k) = \chi_{m'}(k)$ for $k \neq j$ and $\chi_{m_1}(j) = \chi_{m'}(j) + 1$. By construction, the two sequences χ_{m_1} and $\chi_{m'}$ also satisfy (a) and (b). Repeating iteratively this process for m and m_i, $i \geq 1$, there is a positive integer r so that $m \longrightarrow m_r \longrightarrow \cdots \longrightarrow m_2 \longrightarrow m_1 \longrightarrow m'$.

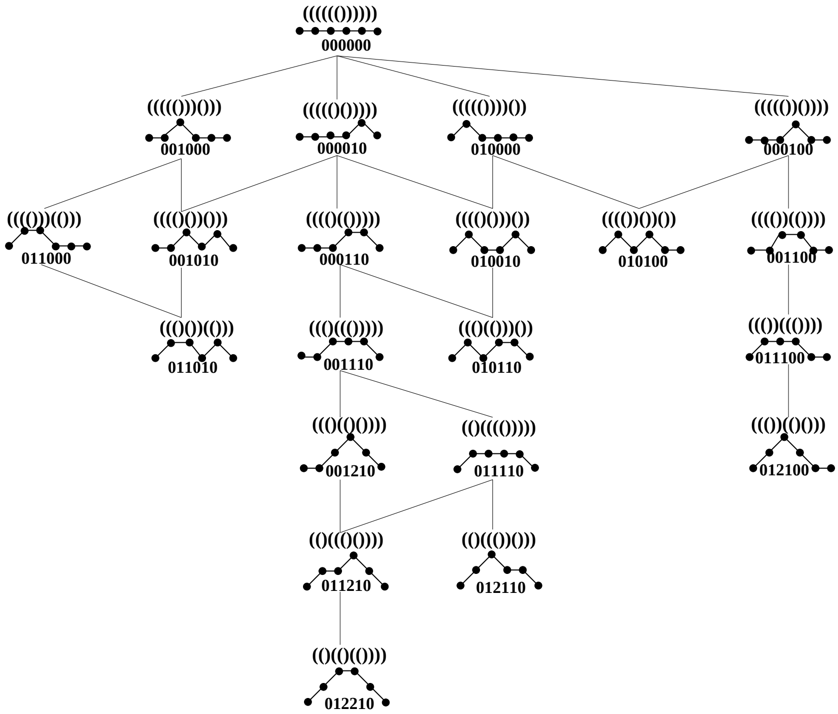

Figure 2: The Motzkin semilattice $\mathcal{M}_6$. Each Motzkin word m is associated with its Motzkin path $\phi(m)$ and its height sequence χ_m.

Conversely, let us assume that $m \xrightarrow{*} m'$. Using Proposition 2, it is straightforward to see that the height sequences χ_m and $\chi_{m'}$ satisfy (a). Moreover, if we have the path $m \longrightarrow m_1 \longrightarrow \cdots \longrightarrow m_r = m'$ for some integer $r \geq 1$, Proposition 2 ensures that the height sequence of m_1 is obtained from χ_m by decreasing by one an entry $\chi_m(i)$ such that $\chi_m(i+1) \leq \chi_m(i) > \chi_m(i-1)$. Therefore, m and m_1 satisfy also (b), and a simple induction proves that m and $m_r = m'$ satisfy (b). □

By construction, the poset $(\mathcal{M}_n, \xrightarrow{*})$ is included in the Tamari lattice of order n (see Figure 3). Moreover the previous result proves that it is contained into the Motzkin lattice defined by Ferrari and Munarini in [9]. More precisely, the elements are the same, the partial order is dual but our poset has less covering relations.

Theorem 1 *The poset* $(\mathcal{M}_n, \xrightarrow{*})$ *is a join semilattice with* $\mathbf{1} = (((\ldots)))$ *as maximum element.*

Proof. Obviously, Proposition 4 induces that $\mathbf{1} = (((\ldots)))$ is the maximum element (its height sequence is $0\ldots0$). In order to prove that the poset is a join semilattice, we show that any two elements of $\mathcal{M}_n$ have a least upper bound.

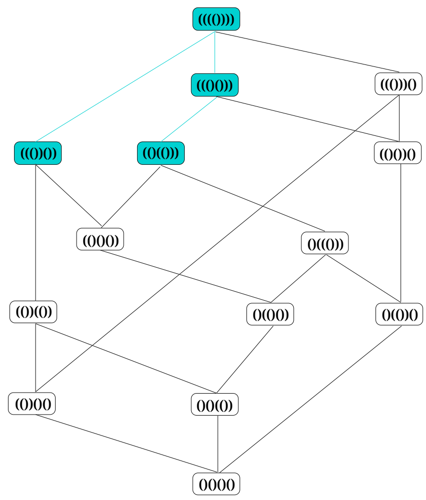

Figure 3: The Motzkin semilattice $\mathcal{M}_4$ included into the Tamari lattice of order 4.

Let m and m' be two different Motzkin words and χ_m and $\chi_{m'}$ their height sequences. Now, we construct algorithmically a Motzkin word $m'' \in \mathcal{M}_n$ (or equivalently its height sequence $\chi_{m''}$) that is a candidate to be the join element of m and m', and we will show that this element really is the join.
Join algorithm The inputs are the height sequences of m and m' and the output is the height sequence of the join $m \vee m'$.

For example, if we perform Algorithm 1 for $\chi_m = 0121112110$ and $\chi_{m'} = 0101111010$, Part I gives $\chi_{m''} = 0101111010$, and Part II modifies $\chi_{m''}$ into $\chi_{m''} = 0000000010$.

Part I of Algorithm 1 computes for all i, $0 \le i \le n-1$, $\chi_{m''}(i) = \min\{\chi_m(i), \chi_{m'}(i)\}$. Since the statements of Part II do not increase any value $\chi_{m''}(i)$, the two following conditions $\chi_{m''}(i) \le \chi_m(i)$ and $\chi_{m''}(i) \le \chi_{m'}(i)$, $0 \le i \le n-1$, remain true throughout Algorithm 1.

Statements of Part II modifies $\chi_{m''}$ so that there does not exist i such that $\chi_{m''}(i-1) > \chi_{m''}(i)$ and $(\chi_{m''}(i) < \chi_m(i)$ or $\chi_{m''}(i) < \chi_{m'}(i))$. For this, we traverse $\chi_{m''}$ from right to left and for each i such that $\chi_{m''}(i-1) > \chi_{m''}(i)$ and $(\chi_{m''}(i) < \chi_m(i)$ or $\chi_{m''}(i) < \chi_{m'}(i))$, we replace $\chi_{m''}(j)$ with $\chi_{m''}(i)$ from $j = i-1$ down to $j_0 + 1$ where j_0 is the rightmost index $j \le i-1$ satisfying $\chi_{m''}(j) = \chi_{m''}(i)$.

At the end of Algorithm 1, Proposition 4 ensures that $\chi_{m''}$ is a height sequence of a Motzkin word m'' so that $m \xrightarrow{*} m''$ and $m' \xrightarrow{*} m''$.

Now, we will prove that m'' really is the least upper bound of m and m'. Let s be a Motzkin word in $\mathcal{M}_n$ such that $m \xrightarrow{*} s$ and $m' \xrightarrow{*} s$ and let us prove that $m'' \xrightarrow{*} s$. Proposition 4 implies that we have (a) $\chi_s(i) \le \chi_m(i)$ and

Algorithm 1 Join algorithm.

```
procedure Join(χ_m, χ_{m'})
  // Part I
  for i ← 0 to n − 1 do
    χ_{m''}(i) ← min{χ_m(i), χ_{m'}(i)}
  end for
  // Part II
  i ← n − 1
  while i > 0 do
    if χ_{m''}(i) < χ_{m''}(i − 1) and (χ_{m''}(i) < χ_m(i) or χ_{m''}(i) < χ_{m'}(i)) then
      x ← χ_{m''}(i)
      i ← i − 1
      while χ_{m''}(i) > x do
        χ_{m''}(i) ← x
        i ← i − 1
      end while
    else
      i ← i − 1
    end if
  end while
  return χ_{m''}
```

$\chi_s(i) \leq \chi_{m'}(i)$ for $i \geq 0$, and (b) there does not exist i such that $\chi_s(i-1) > \chi_s(i)$ and $(\chi_s(i) < \chi_m(i)$ or $\chi_s(i) < \chi_{m'}(i))$.

We distinguish two cases,

- $\chi_s(i) \leq \chi_{m''}(i)$ for all i, $0 \leq i \leq n-1$; then for all j satisfying $\chi_s(j) < \chi_{m''}(j) \leq \min\{\chi_m(j), \chi_{m'}(j)\}$, there is a Motzkin word w ($w = m$ or $w = m'$) such that $\chi_s(j) < \chi_w(j)$. Since we have $w \xrightarrow{*} s$, Proposition 4 implies that $\chi_s(j-1) \leq \chi_s(j) < \chi_{m''}(j)$. Proposition 4 allows to conclude that $m'' \xrightarrow{*} s$.
- There exists i such that $\chi_s(i) > \chi_{m''}(i)$. Let us recall that $\chi_s(i) \leq \min\{\chi_m(i), \chi_{m'}(i)\}$. Then we have $\chi_{m''}(i) < \min\{\chi_m(i), \chi_{m'}(i)\}$ which means that $\chi_{m''}(i)$ was obtained from $\min\{\chi_m(i), \chi_{m'}(i)\}$ using Part II of Algorithm 1. More precisely, there is $i_1 > i$ such that $\chi_{m''}(i_1) = \chi_{m''}(j) < \min\{\chi_m(j), \chi_{m'}(j)\}$ for $i \leq j \leq i_1-1$, and $\chi_{m''}(i_1) = \min\{\chi_m(i_1), \chi_{m'}(i_1)\}$, and $\chi_{m''}(i_1-1) > \chi_{m''}(i_1)$, and $(\chi_{m''}(i_1) < \chi_m(i_1)$ or $\chi_{m''}(i_1) < \chi_{m'}(i_1))$.

  Thus, since $\chi_s(i) > \chi_{m''}(i)$, $\chi_s(j) \leq \min\{\chi_m(j), \chi_{m'}(j)\}$ for all j, $\chi_{m''}(i_1) = \min\{\chi_m(i_1), \chi_{m'}(i_1)\}$ and $\chi_{m''}(i_1) = \chi_{m''}(j)$ for $i \leq j \leq i_1 - 1$, there exists i_2, $i < i_2 \leq i_1$ such that $\chi_s(i_2) = \chi_{m''}(i)$ with $\chi_s(i_2-1) > \chi_s(i_2)$ and $(\chi_s(i_2) < \chi_m(i_2)$ or $\chi_s(i_2) < \chi_{m'}(i_2))$ which is a contradiction with the fact that $m \xrightarrow{*} s$ and $m' \xrightarrow{*} s$. Therefore, this case does not occur.

Finally, we obtain $m'' \xrightarrow{*} s$ and m'' is really the least upper bound of m and

m'. $\square$

Proposition 5 *The semilattice* $(\mathcal{M}_n, \xrightarrow{*})$ *is upper semimodular, i.e. for all* $m_1, m_2, m_3 \in \mathcal{M}_n$ *with* $m_1 \neq m_2$, $m_3 \longrightarrow m_1$ *and* $m_3 \longrightarrow m_2$, *there exists* $m_4 \in \mathcal{M}_n$ *such that* $m_1 \longrightarrow m_4$ *and* $m_2 \longrightarrow m_4$.

Proof. Let χ_{m_i} be the height sequences of m_i, $1 \leq i \leq 4$. With Proposition 2, the sequence χ_{m_1} (resp. χ_{m_2}) is obtained from χ_{m_3} by decreasing by one the value $\chi_{m_3}(j)$ (resp. $\chi_{m_3}(k)$) with $j < k$, $\chi_{m_3}(j) - 1 \geq \chi_{m_3}(j-1)$ and $\chi_{m_3}(k) - 1 \geq \chi_{m_3}(k-1)$. In the case where $k \geq j+2$, the sequence χ_{m_4} obtained from m_1 by decreasing by one $\chi_{m_1}(k)$ is clearly a height sequence of a Motzkin word m_4 satisfying $m_1 \longrightarrow m_4$ and $m_2 \longrightarrow m_4$. The case $k = j+1$ does not occur. Indeed, we necessarily have the two conditions $\chi_{m_3}(j) - 1 \geq \chi_{m_3}(j-1)$ and $\chi_{m_3}(j+1) - 1 \geq \chi_{m_3}(j)$ that imply $\chi_{m_1}(j) = \chi_{m_3}(j) - 1 \leq \chi_{m_3}(j+1) - 2 = \chi_{m_1}(j+1) - 2$ which contradicts the fact that χ_{m_1} is an height sequence of the Motzkin word m_1. $\square$

Let m and m' be two Motzkin words in $\mathcal{M}_n$. A *geodesic* between m and m' in the Tamari lattice of order n is a shortest path between them browsing through only some Motzkin words, *i.e.* lying in $\mathcal{M}_n$ only. Let $d(m, m')$ be the length of a geodesic between m and m'. Equivalently, $d(m, m')$ is the minimum of left- and right-rotations needed to transform m into m' in $(\mathcal{M}_n, \xrightarrow{*})$. Obviously, $d(m, m')$ is an upper bound of the classical rotation distance between m and m' in the Tamari lattice of order n.

Theorem 2 *Let* m *and* m' *be two Motzkin words in* $\mathcal{M}_n$. *Then, we have* $d(m, m') = \sum_{i=0}^{n-1} (\chi_m(i) + \chi_{m'}(i) - 2\chi_{m \vee m'}(i))$.

Proof. $(\mathcal{M}_n, \xrightarrow{*})$ is an upper semimodular join-semilattice, with a maximal element and graded by the rank function $r(m) = \lfloor \frac{(n-1)^2}{4} \rfloor - \sum_{i=0}^{n-1} \chi_m(i)$ (see Proposition 3, Theorem 1 and Proposition 5). Then, from [5, 6, 15] we have: $d(m, m') = 2r(m \vee m') - r(m) - r(m')$ and thus $d(m, m') = \sum_{i=0}^{n-1} (\chi_m(i) + \chi_{m'}(i) - 2\chi_{m \vee m'}(i))$. $\square$

Remark 2 A consequence of Theorem 2 is that the classical rotation distance (in a Tamari lattice) between two Motzkin words m and m' is less than or equal to $d(m, m') = \sum_{i=0}^{n-1} (\chi_m(i) + \chi_{m'}(i) - 2\chi_{m \vee m'}(i))$ (see [4, 7, 17, 19] for bounds of the rotation distance). Moreover, this bound can give the exact value of the classic rotation distance for some particular pairs of Motzkin words. Let us define $m = (()(()(() \ldots (()) \ldots))) = \alpha^n \beta^n$ and $m' = ((()(() \ldots (()()) \ldots))) = (\alpha^{n-1}()\beta^n$ where $\alpha = (()$ and $\beta =)$. A simple calculation proves that $d(m, m') = n-1$ which also is the classic rotation distance between m and m' in a Tamari lattice. For instance, if $n = 3$ then we have $m = (()(()(()))))$, $m' = ((()(()()))))$, $\chi_m = 012210$, $\chi_{m'} = 001210$ and $d(m, m') = 2$; which is exactly the rotation distance in a Tamari lattice between m and m' (see [4]).

3 Some properties of $(\mathcal{M}_n, \xrightarrow{*})$

In this part, we present several enumerating results for some characteristic elements of the semilattice $(\mathcal{M}_n, \xrightarrow{*})$.

Proposition 6 *The generating function for the number of minimal elements in $(\mathcal{M}_n, \xrightarrow{*})$ is given by $\frac{1-\sqrt{1-4x^2-4x^3}}{2x^2}$ (see A007477, [23]).*

Proof. A minimal element in $(\mathcal{M}_n, \xrightarrow{*})$ is a Motzkin word m such that its associated path $\phi(m)$ satisfies the property that each horizontal step H is either followed by a down step D or ends the path. We distinguish three cases: (i) $\phi(m)$ is empty; (ii) $\phi(m) = H$; and (iii) $\phi(m) = U\phi(m_1)D\phi(m_2)$ where m_1 and m_2 are two minimal Motzkin words. Thus, the generating function for the number of minimal Motzkin paths is $A(x) = 1 + x + x^2A(x)^2$ which gives $A(x) = \frac{1-\sqrt{1-4x^2-4x^3}}{2x^2}$. □

Recall that $m \in \mathcal{M}_n$ is a join (resp. meet) irreducible element if $m = a \vee b$ (resp. $m = a \wedge b$) implies $m = a$ or $m = b$. Since the set $\mathcal{M}_n$ is finite, join (resp. meet) irreducible elements are elements that have a unique lower (resp. upper) cover.

Proposition 7 *The generating function for the number of meet-irreducible elements in $(\mathcal{M}_n, \xrightarrow{*})$ is given by $\frac{x}{(1-x-x^2)(1-x)^2}$ (see A001924, [23]).*

Proof. A meet-irreducible element in $(\mathcal{M}_n, \xrightarrow{*})$ is a Motzkin word $m \in \mathcal{M}_n$ such that its associated Motzkin path $\phi(m)$ contains only one occurrence of UH or UD. Let a_n (resp. b_n) be the cardinality of the set of meet-irreducible elements (resp. starting with U) in $\mathcal{M}_n$. Moreover, if $\phi(m)$ starts with U then it can be written $\phi(m) = U^kM$ where $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$ and M is a word of length $n-k$ consisting of k down steps and $n-2k$ horizontal steps. Thus, we obtain $b_n = \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k}$ that is the sequence $A000071$ in [23]. Notice that this sequence are Fibonacci numbers minus 1. Finally, since $a_n = a_{n-1} + b_n$ we have a recurrence which gives precisely sequence A001924 in [23]. □

Proposition 8 *The generating function for the number of join-irreducible elements in $(\mathcal{M}_n, \xrightarrow{*})$ is given by $\frac{1-2x^2-\sqrt{1-4x^2-4x^3}}{2x\sqrt{1-4x^2-4x^3}}$.*

Proof. A join-irreducible element in $(\mathcal{M}_n, \xrightarrow{*})$ is a Motzkin word $m \in \mathcal{M}_n$ such that its associated Motzkin path $\phi(m)$ contains only one occurrence of HU or HH. Let $A(x)$ (resp. $B(x)$) be the generating function for the join-irreducible elements in $\mathcal{M}_n$ (resp. for the minimal elements in $\mathcal{M}_n$). An element $m \in \mathcal{M}_n$ is join-irreducible if and only if $\phi(m)$ can be written in one of the three following forms: (i) $U\phi(m')D\phi(m'')$ where m' is join-irreducible and m'' is a minimal element; (ii) $U\phi(m')D\phi(m'')$ where m' is a minimal element and m'' is join-irreducible; and (iii) $H\phi(m')$ where m' is a non-empty minimal element. We deduce $A(x) = 2x^2A(x)B(x) + x(B(x) - 1)$ and since $B(x) = \frac{1-\sqrt{1-4x^2-4x^3}}{2x^2}$, we obtain $A(x) = \frac{1-2x^2-\sqrt{1-4x^2-4x^3}}{2x\sqrt{1-4x^2-4x^3}}$. The first values of this sequence are $0, 0, 1, 1, 4, 7, 18, 39, 90, 206, 470, 1085, 2492, 5762$ for $1 \leq n \leq 14$. □

Proposition 9 *The generating function for the number of coverings in* $(\mathcal{M}_n, \xrightarrow{*})$ *is given by* $\frac{(1-x)(1-x-2x^2-\sqrt{1-2x-3x^2})}{2x(1-2x-x^2)}$.

Proof. Let $C(x) = \sum_{n\geq 0} c_n x^n$ be the generating function for the number of coverings of $(\mathcal{M}_n, \xrightarrow{*})$. In order to enumerate the coverings $m \longrightarrow m'$ in $(\mathcal{M}_n, \xrightarrow{*})$, we count the possible elementary transformations $UH \longrightarrow HU$ and $UD \longrightarrow HH$ for Motzkin paths $M = \phi(m)$. So, we distinguish two cases: (i) $M = HM_1$ where M_1 is a Motzkin path; and (ii) $M = UM_1DM_2$ where M_1 and M_2 are two Motzkin paths.

For the case (i), the generating function for coverings whose lower path M has the form $M = HM_1$ is clearly $xC(x)$.

Now let us consider the case (ii) $M = UM_1DM_2$ where M_1 and M_2 are two Motzkin paths. A covering derived from M can be of three different forms:

- it can be derived from a covering of M_1 or M_2; thus there are $\sum_{i=0}^{n-2}(c_i + c_{n-2-i})$ possible coverings of this form, and the corresponding generating function is $2\frac{x^2C(x)}{1-x}$;

- it can be of the form $M = UHM_1'DM_2 \longrightarrow M' = HUM_1'DM_2$ where M_1' and M_2 are two Motzkin paths; thus the generating function is $x^3M(x)^2$ where $M(x) = \frac{1-x-\sqrt{1-2x-3x^2}}{2x^2}$ is the generating function for Motzkin numbers;

- it can be of the form $M = UDM_2 \longrightarrow M' = HHM_2$ where M_2 is a Motzkin path. Thus the generating function is $x^2M(x)$.

Putting this together, the generating function $C(x)$ satisfies the equation:

$$C(x) = xC(x) + x^3M(x)^2 + 2\frac{x^2C(x)}{1-x} + x^2M(x)$$

and we obtain $C(x) = \frac{(1-x)(1-x-2x^2-\sqrt{1-2x-3x^2})}{2x(1-2x-x^2)}$.

The first values of this sequence are $0, 0, 1, 3, 9, 26, 73, 202, 553, 1504, 4073, 11003, 29689, 80094$ for $1 \leq n \leq 14$. □

Acknowledgements

We would like to thank the anonymous referees for their constructive remarks which have greatly improved this paper.

References

[1] E. Barcucci, A. Bernini, L. Ferrari, M. Poneti: A distributive lattice structure connecting Dyck paths, noncrossing partitions and 312-avoiding permutations, Order, 22(4)(2005), 311-328.

[2] J.L. Baril, J.M. Pallo: The phagocyte lattice of Dyck words, Order, 23(2-3)(2006), 97-107.

[3] J.L. Baril, J.M. Pallo: The pruning-grafting lattice of binary trees, Theoretical Computer Science, 409(3)(2008), 382-293.

[4] J.L. Baril, J.M. Pallo: Efficient lower and upper bounds of the diagonal-flip distance between triangulations, Inform. Process. Lett., 100(2006), 131-136.

[5] J.P. Barthélemy, B. Leclerc, B. Monjardet, On the use of ordered sets on problems of comparison and consensus of classifications, Journal of Classification, 3(1986), 187-224.

[6] S.A. Boorman, D.C. Olivier: Metrics on spaces of finite trees, Journal of Mathematical Psychology, 10(1973), 26-59.

[7] A. Bonnin, J.-M. Pallo: A shortest path metric on unlabeled binary trees, Pattern Recognition Letters, 13(1992), 411-415.

[8] R. Donaghey and L.W. Shapiro: Motzkin numbers. Journal of Combinatorial Theory, Series A, 23(1977), 291301.

[9] L. Ferrari, E. Munarini: Lattices of paths: representation theory and valuations. J. Comb. 2(2)(2011), 265291.

[10] L. Ferrari, R. Pinzani: Lattices of lattice paths, J. Stat. Plan. Inference, 135(2005), 77-92.

[11] H. Friedman, D. Tamari: Problèmes d'associativité: une structure de treillis fini induite par une loi de demi-associative, J. Comb. Theory, 2(1967), 215-242.

[12] G. Grätzer: General Lattice Theory, Second edition, Birkhäuser, 1998.

[13] S. Hanke, T. Ottmann, S.J. Schuierer: The edge-flipping distance of triangulations, J. Univers. Comp. Sci., 2(1996), 570-579.

[14] S. Huang and D. Tamari: Problems of associativity: A simple proof for the lattice property of systems ordered by a semi-associative law, J. Combinat. Th. Series A, 13 (1972), 7-13.

[15] B. Monjardet: Metrics on partially ordered sets - A survey, Discrete Math., 35(1981), 173-184.

[16] F. Müller-Hoissen, J.M. Pallo, J. Stasheff (eds): Associahedra, Tamari Lattices and Related Structures, Progress in Mathematics vol. 299, Birkhäuser, 2012.

[17] J.M. Pallo: On the rotation distance in the lattice of binary trees, Inform. Process. Lett., 25(1987), 369-373.

[18] J.M. Pallo: An efficient upper bound of the rotation distance of binary trees, Inform. Process. Lett., 73(2000), 87-92.

[19] J.M. Pallo: Right-arm rotation distance between binary trees, Inform. Process. Lett., 87(2003), 173-177.

[20] R. Simion, D. Ullman: On the structure of lattice of noncrossing partitions, Discrete Math., 98(1991), 193-206.

[21] R. Simion: Noncrossing partitions, Discrete Math., 217(2000), 367-409.

[22] D.D. Sleator, R.E. Tarjan, W.P. Thurston: Rotation distance, triangulations and hyperbolic geometry, J. Am. Math. Soc., 1(1988), 647-681.

[23] N.J.A. Sloane: The On-line Encyclopedia of Integer Sequences, available electronically at http://oeis.org.

[24] R.P. Stanley: Enumerative Combinatorics, vol. 2. Cambridge University Press, 1999.

[25] D. Tamari: The algebra of bracketings and their enumeration, Nieuw Archief voor Wiskunde, 10 (1962), 131-146.