Enumeration of Łukasiewicz paths modulo some patterns

Jean-Luc Baril, Sergey Kirgizov and Armen Petrossian
LE2I, Université de Bourgogne
B.P. 47 870, 21078 DIJON-Cedex France

e-mail: {barjl, armen.petrossian}@u-bourgogne.fr,
kerzolster@gmail.com

December 5, 2018

Abstract

For any pattern $\alpha$ of length at most two, we enumerate equivalence classes of Łukasiewicz paths of length $n \geq 0$ where two paths are equivalent whenever the occurrence positions of $\alpha$ are identical on these paths. As a byproduct, we give a constructive bijection between Motzkin paths and some equivalence classes of Łukasiewicz paths.

Keywords: Łukasiewicz path, Dyck path, Motzkin path, equivalence relation, patterns.

1 Introduction and notations

In the literature, lattice paths are widely studied. Their enumeration is a very active field in combinatorics, and they have many applications in other research domains as computer science, biology and physics [18, 19]. Dyck and Motzkin paths are the most often considered. This is partly due to the fact that they are respectively counted by the famous Catalan and Motzkin numbers (see A000108 and A001006 in the on-line encyclopedia of integer sequences [28]). Almost always, these paths are enumerated according to several parameters and statistics (see for instance [6, 14, 15, 17, 20, 21, 24, 25, 29] for Dyck paths and [4, 5, 7, 8, 16, 22, 26] for Motzkin paths). Also, many one-to-one correspondences have been found between lattice paths and some combinatorial objects such as Young tableaux, pattern avoiding permutations, bargraphs, RNA shapes and so on [30]. Recently a new approach has been introduced for studying statistics on lattice paths. It consists in determining the cardinality of the quotient set generated by an equivalence relation based on the positions of a given pattern: two paths belong to the same equivalence class whenever the positions of

1
occurrences of a given pattern are identical on these paths. Enumerating results are provided for the quotient sets of Dyck, Motzkin and Ballot paths for patterns of length at most three (see respectively [2], [3] and [13]). The purpose of the present paper is to extend these studies for Lukasiewicz paths that naturally generalizes Dyck and Motzkin paths. As a byproduct, we show how Motzkin paths are in one-to-one correspondence with some equivalence classes of Lukasiewicz paths.

Throughout this paper, a lattice path is defined by a starting point \( P_0 = (0,0) \), an ending point \( P_n = (n,0) \), it consists of steps lying in \( S = \{(1,i), i \in \mathbb{Z}\} \), and it never goes below the \( x \)-axis. The length of a path is the number of its steps. We denote by \( \epsilon \) the empty path, i.e., the path of length zero. Constraining the steps to lie into \( \{(1,1),(1,-1)\} \) (resp. \( \{(1,1),(1,0),(1,-1)\} \)), we retrieve the well known definition of Dyck paths (resp. Motzkin paths). Lukasiewicz paths are obtained when the steps belong to \( \{(1,i) \in S, i \geq -1\} \). We refer to [10, 23, 30, 32, 33] for some combinatorial studies on Lukasiewicz paths. Let \( \mathcal{L}_n \), \( \mathcal{D}_n \), \( \mathcal{M}_n \), \( n \geq 0 \), respectively, be the sets of Lukasiewicz, Dyck and Motzkin paths of length \( n \), and \( \mathcal{L} = \cup_{n \geq 0} \mathcal{L}_n \), \( \mathcal{D} = \cup_{n \geq 0} \mathcal{D}_n \), \( \mathcal{M} = \cup_{n \geq 0} \mathcal{M}_n \). For convenience, we set \( D = (1,-1) \), \( F = U_0 = (1,0) \), \( U = U_1 = (1,1) \) and \( U_i = (1,i) \) for \( i \geq 2 \). See Figure 1 for an illustration of Dyck, Motzkin and Lukasiewicz paths of length 18. Note that Lukasiewicz paths can be interpreted as an algebraic language of words \( w \in \{x_0,x_1,x_2,\ldots\}^* \) such that \( \delta(w) = -1 \) and \( \delta(w') \geq 0 \) for any proper prefix \( w' \) of \( w \) where \( \delta \) is the map from \( \{x_0,x_1,x_2,\ldots\}^* \) to \( \mathbb{Z} \) defined by \( \delta(w_1w_2\ldots w_n) = \sum_{i=1}^n \delta(w_i) \) with \( \delta(x_i) = i-1 \) (see [11, 27]).

![Figure 1](image1.png)

Figure 1: From left to right, we show a Dyck path \( A = UUDDUUDDUDUUDDDDDDUD \), a Motzkin path \( B = UUFDDFUDDUFUDDFFUD \) and a Lukasiewicz path \( C = U_6DDFFDU_2DDDU_2FU_2DDDD \).

Any non-empty Lukasiewicz path \( L \in \mathcal{L} \) can be decomposed (see [9]) into one of the two following forms: (1) \( L = FL' \) with \( L' \in \mathcal{L} \), or (2) \( L = U_kL_1DL_2D\ldots L_kDL' \) with \( k \geq 1 \) and \( L_1,L_2,\ldots,L_k,L' \in \mathcal{L} \) (see Figure 2).

![Figure 2](image2.png)

Figure 2: The two forms of the decomposition of a non-empty Lukasiewicz path.

Due to this decomposition, the generating function \( L(x) \) for the cardinalities of the sets \( \mathcal{L}_n \), \( n \geq 0 \), satisfies the functional equation \( L(x) = 1 + xL(x) + \sum_{k \geq 1} x^{k+1}L(x)^{k+1} \), or equivalently, \( L(x) = \frac{1}{1-xL(x)} \). Then, \( L(x) = \frac{1-\sqrt{1-4x}}{2x} \) and the coefficient of \( x^n \) in the series
expansion of $L(x)$ is given by the $n$-th Catalan number $\frac{1}{n+1} \binom{2n}{n}$ (see sequence A000108 in [28]).

A pattern of length one (resp. two) in a lattice path $L$ consists of one step (resp. two consecutive steps). We will say that an occurrence of a pattern is at position $i \geq 1$ in $L$, whenever the first step of this occurrence appears at the $i$-th step of the path. The height of an occurrence is the minimal ordinate reached by its points. For instance, the path $C = U_5D^2FDU_2DDDUDU_2FU_2DDD$ (see Figure 1) contains one occurrence of the pattern $FD$ at position 5 and of height 2.

Following the recent studies [2, 3, 13], we define an equivalence relation on the set $L$ for a given pattern $\alpha$: two Lukasiewicz paths of the same length are $\alpha$-equivalent whenever the occurrences of the pattern $\alpha$ appear at the same positions in the two paths. For instance, $UFFFDUUDFFFFUDFF$ is $FD$-equivalent to the path $C$ in Figure 1 since the only one occurrence of the pattern $FD$ (in boldface) appear at the same position in the two paths. Note that the height of the occurrences of $\alpha$ is not involved in this definition.

In this paper, for any pattern $\alpha$ of length at most two, we consider the above equivalence relation on the set $L$, and for each of them we provide the cardinality of the quotient set with respect to the length. Three general methods are used:

- $(M_0)$ we prove that any $\alpha$-equivalence class contains at least one Motzkin path. Using $M_n \subset L_n$ for $n \geq 0$, we deduce that the number of $\alpha$-equivalence classes in $M_n$ is equal to that of $L_n$. Since two of the authors have already determined this number for $M_n$ (see [3]), we can conclude,

- $(M_1)$ we define a set $A_n$ of representatives of length $n \geq 0$ and we calculate $a_n = |A_n|$,  
- $(M_2)$ we exhibit a one-to-one correspondence between a subset of Lukasiewicz paths (subset of representatives of the classes) and the set of equivalence classes by using combinatorial reasonings, and then, we evaluate algebraically the generating function for this subset.

The paper is organized as follows. In Section 2, we consider the case of patterns $\alpha$ studied using method $(M_0)$, i.e., $\alpha \in \{U,UU,UD,UF,DU,FU\}$. In Section 3, we focus on these ones that can be dealt using method $(M_1)$, i.e., $\alpha \in \{F,D,FD,DF,DD\}$. In Section 4, we complete our study by the remaining cases which are obtained using method $(M_2)$. We refer to Table 1 for an exhaustive list of our enumerative results.

## 2 Modulo $\alpha \in \{U,UU,UD,UF,DU,FU\}$

In this section, we focus on the patterns that can be dealt using method $(M_0)$.

**Lemma 1** For $n \geq 0$, let $L$ be a Lukasiewicz path in $L_n$ and $\alpha \in \{U,UU,UD,UF,DU,FU\}$. Then, there exists a Motzkin path $M \in M_n$ such that $M$ and $L$ are $\alpha$-equivalent.

**Proof.** Let us assume that $\alpha \in \{U,UU,UD\}$. Any non-empty Lukasiewicz path can be decomposed into one of the two following forms: (i) $L = F'L'$ with $L' \in L$, and (ii) $L = U_kL_1DL_2D\ldots L_kDL'$ with $k \geq 1$ and $L_1, L_2, \ldots, L_k, L' \in L$. For $n \geq 0$, we recursively define a map $\phi$ from $L_n$ to $M_n$ as follows:
<table>
<thead>
<tr>
<th>Pattern $\alpha$</th>
<th>Sequence</th>
<th>Sloane</th>
<th>$a_n, 1 \leq n \leq 10$</th>
<th>Method</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U$</td>
<td>$\left(\begin{array}{c} n \ 2 \end{array}\right)$</td>
<td>A001405</td>
<td>1, 2, 3, 6, 10, 20, 35, 70, 126, 252</td>
<td>$M_0$</td>
</tr>
<tr>
<td>$UU$</td>
<td>$\frac{1-2x+x^2-\sqrt{(x^2+1)(1-3x^2)}}{2x(-1+2x-x^2+x^3)}$</td>
<td>A191385</td>
<td>1, 1, 1, 2, 3, 5, 7, 12, 18, 31</td>
<td>$M_0$</td>
</tr>
<tr>
<td>$UD$</td>
<td>Shift of Fibonacci</td>
<td>A000045</td>
<td>1, 2, 3, 5, 8, 13, 21, 34, 55, 89</td>
<td>$M_0$</td>
</tr>
<tr>
<td>$UF, FU$</td>
<td>$\frac{2}{1-2x+\sqrt{1-4x^2}}$</td>
<td>A165407</td>
<td>1, 1, 2, 3, 4, 7, 11, 16, 27, 43</td>
<td>$M_0$</td>
</tr>
<tr>
<td>$DU$</td>
<td>Shift of Fibonacci</td>
<td>A000045</td>
<td>1, 1, 1, 2, 3, 5, 8, 13, 21, 34</td>
<td>$M_0$</td>
</tr>
<tr>
<td>$F$</td>
<td>$2^n - n$</td>
<td>A000325</td>
<td>1, 2, 5, 12, 27, 58, 121, 248, 503, 1014</td>
<td>$M_1$</td>
</tr>
<tr>
<td>$D$</td>
<td>$2^{n-1}$</td>
<td>A011782</td>
<td>1, 1, 2, 4, 8, 16, 32, 64, 128, 256</td>
<td>$M_1$</td>
</tr>
<tr>
<td>$FD, DF$</td>
<td>Fibonacci</td>
<td>A000045</td>
<td>1, 1, 2, 3, 5, 8, 13, 21, 34, 55</td>
<td>$M_1$</td>
</tr>
<tr>
<td>$DD$</td>
<td>$\frac{1-x}{1-2x+x^2-x^3}$</td>
<td>A005251</td>
<td>1, 1, 2, 4, 7, 12, 21, 37, 65, 114</td>
<td>$M_1$</td>
</tr>
<tr>
<td>$U_k$</td>
<td>Motzkin</td>
<td>A001006</td>
<td>1, 2, 4, 9, 21, 51, 127, 323, 835, 2188</td>
<td>$M_2$</td>
</tr>
<tr>
<td>$FF$</td>
<td>$\frac{1-3x+2x^2-5x^3+7x^4-7x^5+6x^6-3x^7+x^8}{(1-2x+x^2-x^3)(1-x)^2}$</td>
<td>New</td>
<td>1, 2, 5, 9, 17, 32, 59, 107, 192</td>
<td>$M_2$</td>
</tr>
<tr>
<td>$FU_k, U_kF$</td>
<td>$\frac{1-x-x^2-\sqrt{1-2x+x^2-4x^3}}{2x^2}$</td>
<td>A023431</td>
<td>1, 1, 2, 4, 7, 13, 26, 52, 104, 212</td>
<td>$M_2$</td>
</tr>
<tr>
<td>$U_kD$</td>
<td>$\frac{1-x-x^2-\sqrt{1-2x+x^2-2x^3+x^4}}{2x^3}$</td>
<td>A292460</td>
<td>1, 2, 4, 8, 17, 37, 82, 185, 423, 978</td>
<td>$M_2$</td>
</tr>
<tr>
<td>$DU_k$</td>
<td>$\frac{1+x-x^2-\sqrt{1-2x+x^2-2x^3+x^4}}{2x^2}$</td>
<td>A004148</td>
<td>1, 1, 1, 2, 4, 8, 17, 37, 82, 185</td>
<td>$M_2$</td>
</tr>
</tbody>
</table>

Table 1: Number of $\alpha$-equivalence classes for Lukasiewicz paths. The last three sequences are recorded in OEIS [28] as generalized Catalan sequences.

\[
\begin{align*}
\phi(\epsilon) &= \epsilon, \\
\phi(FL') &= F\phi(L'), \\
\phi(UL'1DL') &= U\phi(L1)L\phi(L'), \\
\phi(U_kL1DL2D...LkDL') &= F\phi(L1)L\phi(L2)L...\phi(Lk)L\phi(L') \\
&\quad \text{for } k \geq 2.
\end{align*}
\]

Clearly, $\phi(L)$ is a Motzkin path in $M_n$, and whenever $\alpha \in \{U, UU, UD\}$ the occurrence positions of $\alpha$ in $L$ and $\phi(L)$ are identical. Then, the equivalence class of $L$ contains a Motzkin path $\phi(L)$.

Let us assume that $\alpha = DU$. Any non-empty Lukasiewicz path $L$ can be written as follows:

\[
L = K_0 \prod_{i=1}^{r} (DU)^{a_i} K_i
\]

with $r \geq 0$, $a_i \geq 1$ for $1 \leq i \leq r$, and where $K_i$, $0 \leq i \leq r$, are some parts that do not contain any pattern $DU$. Note that $K_0$ and $K_r$ necessarily contain at least one step. From $L \in L_n$, we define the Motzkin path

\[
M = UF^{b_0-1} \left( \prod_{i=1}^{r-1} (DU)^{a_i} F^{b_i} \right) (DU)^{a_r} DF^{b_r-1} \in M_n
\]
where \( b_i = |K_i| \) for \( 0 \leq i \leq r \). Since the occurrence positions of \( DU \) in \( L \) and \( M \) are identical, \( M \) is a Motzkin path in the same class as \( L \).

Let us assume that \( \alpha \in \{FU,UF\} \). Any non-empty Lukasiewicz path \( L \) can be written as follows:

\[
L = K_0 \prod_{i=1}^{r} \alpha^{a_i} K_i
\]

with \( r \geq 0 \), \( a_i \geq 1 \) for \( 1 \leq i \leq r \), and where \( K_i \), \( 0 \leq i \leq r \), are some parts that do not contain any pattern \( \alpha \). From \( L \in L_n \), we define the Motzkin path

\[
M = F^{b_0} \prod_{i=1}^{r} \alpha^{a_i} D^{c_i} F^{b_i-c_i} \in M_n
\]

where \( b_0 = |K_0| \), and for \( 1 \leq i \leq r \), \( b_i = |K_i| \) and \( c_i = \min\{b_i, a_i + \sum_{j=1}^{i-1} (a_j - c_j)\} \). Less formally, for \( i \) from 0 to \( r \), \( K_i \) is replaced with \( D^{c_i} F^{b_i-c_i} \) where the value \( c_i \) is the maximal number of down steps \( D \) that can be placed so that \( M \) remains a lattice path. This ensures that \( M \) has the same occurrence positions of \( \alpha \) as \( L \), which means that \( M \) is a Motzkin path in the same class as \( L \).

Using Lemma 1 and the fact that \( M \subset L \), we directly deduce the following theorem.

**Theorem 1** For \( \alpha \in \{U,UU,UD,UF,DU,FU\} \) and \( n \geq 0 \), the number of \( \alpha \)-equivalence classes in \( L_n \) also is that of \( M_n \).

Since two of the authors have already determined the number of \( \alpha \)-equivalence classes in \( M_n \), we refer to their paper [3] for a detailed description of the different proofs, and we report the results in Table 1.

### 3 Modulo \( \alpha \in \{F, D, FD, DF, DD\} \)

In this section, we focus on the patterns that can be dealt with method (\( M_1 \)) which consists in defining a set \( A_n \) of representatives and counting directly \( a_n = |A_n| \).

**Theorem 2** The number of \( F \)-equivalence classes in \( L_n \), \( n \geq 0 \), is given by \( 2^n - n \) (see sequence A000325 in [28]).

**Proof.** We define the set \( A_n \) as the set of Lukasiewicz paths of length \( n \) where the first non-\( F \) step (if any) is of the form \( U_i, i \geq 1 \), and the rest non-\( F \) steps are \( D \) steps. Since \( A_n \) is in one-to-one correspondence with binary words of length \( n \) without a single one, we deduce that \( a_n = 2^n - n \).

**Theorem 3** The number of \( D \)-equivalence classes in \( L_n \), \( n \geq 1 \), is given by \( 2^{n-1} \) (see sequence A011782 in [28]).
Proof. We define the set $\mathcal{A}_n$ of Lukasiewicz paths of length $n$ where the first step is of the form $U_i$, $i \geq 0$, and the rest of non-$D$ steps are $F$’s. Since $\mathcal{A}_n$ is in one-to-one correspondence with binary words of length $n - 1$, we deduce that $a_n = 2^{n-1}$.

**Theorem 4** The number of $FD$-equivalence (resp. $DF$-equivalence) classes in $\mathcal{L}_n$, $n \geq 0$, is given by the Fibonacci number $f_n$ defined by $f_0 = 1$, $f_1 = 1$, $f_2 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$ (see sequence A000045 in [28]).

Proof. For the pattern $FD$ (the pattern $DF$ is similar), we define the set $\mathcal{A}_n$, $n \geq 3$, as the set of Lukasiewicz paths of length $n$ where the first step is of the form $U_i$, $i \geq 0$, and the rest of the steps are either $F$ or $D$ steps provided that every $D$ step is immediately preceded by an $F$ step. Clearly, $\mathcal{A}_n$ is decomposed into two sets of paths starting with $U_iFF$ or with $U_iFD$, which are counted by $a_{n-1}$ and $a_{n-2}$ respectively.

**Theorem 5** The number of $DD$-equivalence classes in $\mathcal{L}_n$, $n \geq 0$, is given by the $n$-th term $g_n$ of the sequence defined by $g_0 = 1$, $g_1 = 1$, $g_2 = 1$, $g_3 = 2$ and $g_n = g_{n-1} + g_{n-2} + g_{n-4}$ for $n \geq 4$ (see sequence A005251 in [28]).

Proof. We define the set $\mathcal{A}_n$, $n \geq 4$, as the set of Lukasiewicz paths of length $n$ where the first step is of the form $U_i$, $i \geq 0$, and the rest of the steps are either $F$ or $D$ steps and there are no isolated $D$ steps. Clearly, $\mathcal{A}_n$ is decomposed into sets of paths starting with $U_iF$, or with $U_iDDDF$ (together with the path $U_3DDD$ for $n = 4$), or with $U_iD^j$, $j \geq 2$, $j \neq 3$ which are counted by $a_{n-1}$, $a_{n-4}$ and $a_{n-2}$ respectively.

**4 Other patterns**

In this section, we extend the definition of $\alpha$-equivalence whenever $\alpha$ is a set $S$ of patterns: two paths $L$ and $L'$ are $S$-equivalent if for any pattern $\alpha \in S$, the occurrence positions of $\alpha$ are the same in $L$ and $L'$. We investigate the cases where $S$ is $\{U_k, k \geq 1\}$, $\{DU_k, k \geq 1\}$, $\{U_kD, k \geq 1\}$, $\{FU_k, k \geq 1\}$, and $\{U_kF, k \geq 1\}$. For short, these $S$-equivalences will be written $U_k$-equivalence (resp. $DU_k$-, $U_kD$-, $FU_k$-, $U_kF$-equivalence). Also, we study the $FF$-equivalence relation in $\mathcal{L}$. For all these cases, we use the method ($M_2$) that consists in exhibiting subsets of representatives of equivalence classes, and determining algebraically their cardinalities.

**4.1 Modulo $S = \{U_k, k \geq 1\}$**

Let $\mathcal{B}$ be the set of Lukasiewicz paths without any flat steps at positive height. For instance, we have $U_3DDDFUD \in \mathcal{B}$ and $U_3FDDDU \notin \mathcal{B}$. Let $\mathcal{B} \subset \mathcal{B}$ be the set of Lukasiewicz paths without any flat steps.

**Lemma 2** There is a bijection between $\mathcal{B}$ and the set of $U_k$-equivalence classes of $\mathcal{L}$.
Proof. Let $L$ be a non-empty Łukasiewicz path in $\mathcal{L}$. Let us prove that there exists a Łukasiewicz path $L' \in \mathcal{B}$ (with the same length as $L$) such that $L$ and $L'$ are equivalent. We write

$$L = K_0 \prod_{i=1}^{r} \alpha_i K_i$$

with $r \geq 0$, where $K_i$ is a part that does not contain any up steps for $0 \leq i \leq r$, and $\alpha_i \in \{U_k, k \geq 1\}$ for $1 \leq i \leq r$. From $L \in \mathcal{L}$, we define the Łukasiewicz path

$$L' = F^{b_0} \prod_{i=1}^{r} \alpha_i D^{c_i} F^{b_i - c_i}$$

with $b_0 = |K_0|$, and for $1 \leq i \leq r$, $b_i = |K_i|$, $c_i = \min\{b_i, a_i + \sum_{j=1}^{i-1} (a_j - c_j)\}$ where $\alpha_i = U_{a_i}$. Less formally, for $i$ from $0$ to $r$, $K_i$ is replaced with $D^{c_i} F^{|K_i| - c_i}$ where $c_i$ is the maximal number of down steps that can be placed so that $L'$ remains a Łukasiewicz path. Clearly, $L'$ belongs to $\mathcal{B}$ (it does not contain any flat at positive height), and for any $k \geq 1$ the occurrence positions of $U_k$ are the same as for $L$, i.e., $L' \in \mathcal{B}$ is in the same class as $L$. For instance, if $L = U_3 D U D F F F U U D D F F D D F F$, then we obtain $L' = U_3 D U D D F F U U D D F F F F F F$ (see Figure 3 for an illustration of this example).

Figure 3: Illustration of the example described in the proof of Lemma 2.

Since the positions of the up steps $U_k$, $k \geq 1$, remain fixed inside a class, and any flat of $L' \in \mathcal{B}$ lies necessarily on the $x$-axis, there are no other paths in $\mathcal{B}$ in the same class as $L$. □

**Theorem 6** The generating function for the set of $U_k$-equivalence classes of $\mathcal{L}$ with respect to the length is given by

$$\frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x^2},$$

which generates the Motzkin numbers ($A001006$ in [28]).

Proof. Using Lemma 2, it suffices to obtain the generating function $B(x)$ for the set $\mathcal{B}$. A non-empty Łukasiewicz path $L \in \mathcal{B}$ can be written either $L = FL'$ where $L' \in \mathcal{B}$, or $L = U_k L_1 D L_2 D \ldots L_k D L'$ for $k \geq 1$ where $L_1, L_2, \ldots, L_k \in \mathcal{B}$ are some Łukasiewicz paths without flats, and $L' \in \mathcal{B}$. So we obtain the functional equation $B(x) = 1 + xB(x) + xB(x) \sum_{k \geq 1} x^k \tilde{B}(x)^k$ where $\tilde{B}(x)$ is the generating function for the set $\mathcal{B}$ of Łukasiewicz paths without flats. Using the classical decomposition of a Łukasiewicz path, $\tilde{B}(x)$ satisfies $\tilde{B}(x) = 1 + \sum_{k \geq 2} x^k \tilde{B}(x)^k$, or equivalently $\tilde{B}(x) = \frac{1}{(1+x)(1-xB(x))}$. A simple calculation provides the result. □
Let us define recursively a map \( \psi \) from \( \mathcal{L} \) to the set of Motzkin paths \( \mathcal{M} \) as follows:

\[
\begin{align*}
\psi(\epsilon) &= \epsilon, \\
\psi(FL) &= F\psi(L), \\
\psi(U_kL_1DL_2D\ldots L_kDL) &= U\psi(L_1)F\psi(L_2)F\ldots\psi(L_k)D\psi(L),
\end{align*}
\]

where \( L, L_1, L_2, \ldots, L_k \) are some Lukasiewicz paths. See Figure 4 for an illustration of the bijection \( \psi \). For instance, the image by \( \psi \) of \( U_4FU_2DFDDDFU_2UDDFDFUFFUFDD \) is \( UFUFFDFFUUUDFDDFDFUFFUFDD \). Obviously, the map \( \psi \) preserves the length of the paths.

![Figure 4: Illustration of the map \( \psi \) from \( \mathcal{L} \) to \( \mathcal{M} \).](image)

We easily deduce the two following facts.

**Fact 1** If \( L, L' \in \mathcal{L} \), then \( LL' \in \mathcal{L} \) and we have \( \psi(LL') = \psi(L)\psi(L') \).

**Fact 2** There is a one-to-one correspondence between:

(a) steps \( \{U_k, k \geq 1\} \) in \( L \) and steps \( U \) in \( \psi(L) \);

(b) \( \{U_kU\ell, k, \ell \geq 1\} \) in \( L \) and \( UU \) in \( \psi(L) \);

(c) steps \( F \) on the \( x \)-axis in \( L \) and steps \( F \) on the \( x \)-axis in \( \psi(L) \);

(d) \( \{U_kD, k \geq 2\} \cup \{U_kF, k \geq 1\} \) in \( L \) and \( UF \) in \( \psi(L) \);

(e) peaks \( UD \) in \( L \) and peaks \( UD \) in \( \psi(L) \).

**Theorem 7** For any \( n \geq 0 \), the map \( \psi \) induces a bijection from \( \mathcal{B}_n \) to \( \mathcal{M}_n \).

**Proof.** We proceed by induction on \( n \). Obviously, for \( n = 0 \) we have \( \psi(\epsilon) = \epsilon \). We assume that \( \psi \) is a bijection from \( \mathcal{B}_k \) to \( \mathcal{M}_k \), \( 0 \leq k \leq n \), and we prove the result for \( n + 1 \). Using the enumerating result of Theorem 6, it suffices to prove that \( \psi \) is surjective. So, let \( M \) be a Motzkin path in \( \mathcal{M}_{n+1} \). We distinguish two cases:

8
(i) $M = FM'$ with $M' \in \mathcal{M}_n$. Using the recurrence hypothesis, there is $L' \in \mathcal{B}_n$ such that $M' = \psi(L')$. So, the Lukasiewicz path $L = FL'$ lies into $\mathcal{B}_{n+1}$ and satisfies $\psi(L) = M$ which proves that $M$ belongs to the image by $\psi$ of $\mathcal{B}_{n+1}$.

(ii) $M = UM'DM''$ where $M'$ and $M''$ are two Motzkin paths in $\mathcal{M}$. We can uniquely write $M' = M_0 \prod_{i=1}^r FM_i$ with $r \geq 0$ and where $M_i$ is a (possibly empty) Motzkin path without flat $F$ on the $x$-axis. Using the recurrence hypothesis, there are $B_0, B_1, \ldots, B_r \in \mathcal{B}$ such that $\psi(B_i) = M_i$, $0 \leq i \leq r$. Also let $B \in \mathcal{B}$ such that $\psi(B) = M''$. Since $B_i$ (resp. $B$) belongs to $\mathcal{B}$, it does not contain any flat at positive height. Since $M_i = \psi(B_i)$, Fact 2(c) implies that $B_i$ does not contain any flat on the $x$-axis. So, $B_i$ does not contain any flat steps. So, let us define

$$L = U_{r+1} \left( \prod_{i=0}^r B_i D \right) B.$$ 

Clearly, $L$ lies in $\mathcal{B}_{n+1}$ and satisfies $\psi(L) = M$; then, $M$ belongs to the image by $\psi$ of $\mathcal{B}_{n+1}$, and the map $\psi$ from $\mathcal{B}_n$ to $\mathcal{M}_n$ is a bijection. \qed

4.2 Modulo $S = \{U_kD, k \geq 1\}$, and $S = \{DU_k, k \geq 1\}$

Let $\mathcal{C} \subset \mathcal{B}$ be the set of Lukasiewicz paths without any flat steps at positive height and such that any up step $U_k$, $k \geq 1$, is immediately followed by a down step $D$. For instance, we have $U_3DDDFFUD \in \mathcal{C}$ and $U_3FDDDUD \notin \mathcal{C}$. We set $\bar{\mathcal{C}} = \mathcal{B} \cap \mathcal{C}$.

Lemma 3 There is a bijection between $\mathcal{C}$ and the set of $U_kD$-equivalence classes of $\mathcal{L}$.

Proof. Let $L$ be a non-empty Lukasiewicz path in $\mathcal{L}$. Let us prove that there exists a Lukasiewicz path $L' \in \mathcal{C}$ (with the same length as $L$) such that $L$ and $L'$ belong to the same class. We write

$$L = K_0 \prod_{i=1}^r (U_{k_i} DK_i),$$

where $k_i \geq 1$ for $1 \leq i \leq r$, and $K_0, K_1, K_2, \ldots, K_r, r \geq 0$, are some parts (possibly empty) without pattern $U_kD$ for any $k \geq 1$.

We define the Lukasiewicz path

$$L' = F^{b_0} \prod_{i=1}^r (U_{k_i} DD^{a_i} F^{b_i - a_i}),$$

with $b_i = |K_i|$, $0 \leq i \leq r$, and for $1 \leq i \leq r$, $a_i = \min\{b_i, k_i - 1 + \sum_{j=1}^{i-1} (k_j - 1 - a_j)\}$. Less formally, for $i$ from 0 to $r$, $K_i$ is replaced with $D^{a_i} F^{b_i - a_i}$, where the value $a_i$ is the maximal number of down steps that can be placed between the two occurrences $U_{k_i}D$ and $U_{k_{i+1}}D$ so that $L'$ remains a lattice path. Clearly, $L' \in \mathcal{C}$ and $L'$ belongs to the same class as $L$.

For instance, from $L = U_3 DUDFFFUUDDFDDFF$, we obtain the path $L' = U_3 DUDDFDDUFDDDF$ (see Figure 5 for an illustration of this example).
Now we will prove that any $U_kD$-equivalence class contains at most one element in $C$. For a contradiction, let $L$ and $L'$ be two different Lukasiewicz paths in $C$ belonging to the same class. We write $L = QR$ and $L' = QS$ where $R$ and $S$ start with two different steps. Since $L$ and $L'$ lie in the same class, the two first steps of $R$ and $S$ cannot be $U_kD$ for $k \geq 1$. Moreover, since $L$ (resp. $L'$) lies into $C$, the two first steps of $R$ (resp. $S$) cannot constitute a pattern $U_kF$ for $k \geq 1$. Then, $R$ and $S$ cannot start with any up step $U_k$, $k \geq 1$.

Without loss of generality, let us assume that the first step of $R$ is a down step $D$ and then, the first step of $S$ is a flat step $F$. This means that the last point of $Q$ has its ordinate equal to zero (otherwise $L'$ could not belong to $C$). As the first step of $R$ is $D$, the height of this step is $-1$ which gives a contradiction and completes the proof. \hfill \Box

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{example.png}
\caption{Illustration of the example described in the proof of Lemma \ref{lem:example}.}
\end{figure}

**Theorem 8** The generating function for the set of $U_kD$-equivalence classes of $\mathcal{L}$ with respect to the length is given by

$$
1 - x - x^2 - \sqrt{1 - 2x - x^2 - 2x^3 + x^4} \\
2x^3,
$$

which generates the sequence \textit{A292460} in \cite{OEIS} that is a shift of the generalized Catalan sequence defined by $g_0 = 1$, and $g_{n+1} = g_n + \sum_{k=1}^{n-1} g_k g_{n-k}$ for $n \geq 0$ (see \textit{A004148} in \cite{OEIS}).

**Proof.** Using Lemma \ref{lem:example}, it suffices to obtain the generating function $C(x)$ for the set $C$. A non-empty Lukasiewicz path $L \in C$ can be written either $L = FL'$ where $L' \in C$, or $L = U_kDL_1DL_2D \ldots L_{k-1}DL'$ for $k \geq 1$ where $L_1, L_2, \ldots, L_{k-1}$ are some Lukasiewicz paths without flats in $\bar{C}$, and $L' \in \bar{C}$. So we obtain the functional equation $C(x) = 1 + xC(x) + x^2C(x) \sum_{k \geq 0} x^k \bar{C}(x)^k$ where $\bar{C}(x)$ is the generating function for the set $\bar{C}$ of Lukasiewicz paths without flats in $C$. On the other hand, every $L \in C$ is decomposed uniquely according to the number $k \geq 0$ of its flat steps as follows: $L = L_0 \prod_{i=1}^{k} (FL_i)$, where $L_i \in \bar{C}$ for $i \geq 0$. Then by adding a peak $U_{k+1}D$ at the beginning of $L$ and changing each $F$ to $D$, we obtain a unique element of $\bar{C} \setminus \{e\}$, viz. $U_{k+1}DL_0 \prod_{i=1}^{k} (DL_i)$. So, we have $\bar{C}(x) - 1 = x^2C(x)$ and a simple calculation provides the result. \hfill \Box

**Theorem 9** For any $n \geq 0$, the map $\psi$ induces a bijection from $C_n$ to the set of Motzkin paths in $\mathcal{M}_n$ that avoid the pattern $UU$.

**Proof.** Theorem 7 ensures that $\psi$ is a bijection from $B_n$ to $\mathcal{M}_n$ for $n \geq 0$. We have $C \subset B$, and the paths in $C$ are those in $B$ that avoid the patterns $U_kU_\ell$, $k, \ell \geq 1$ and $U_kF$, $k \geq 1$. 

10
Using Fact 2(b), the map \( \psi \) transforms occurrences of \( U_kU_\ell, k, \ell \geq 1 \), into occurrences of \( UU \). Since a path in \( C \) avoids any occurrence of \( U_kF \) for \( k \geq 1 \), the image by \( \psi \) of \( C_n \) is the subset of Motzkin paths in \( M_n \) that avoid any occurrence of the pattern \( UU \). \( \square \)

Let \( \mathcal{L}'_n, n \geq 2 \), be the set of Lukasiewicz paths of length \( n \) starting by \( U \) and ending by \( D \). For \( n \geq 0 \), we define the bijection \( \theta \) from \( \mathcal{L}_n \) to \( \mathcal{L}'_{n+2} \) as follows: \( \theta(L) \) is obtained from \( L \) by replacing any occurrence \( U_kD \) by an occurrence \( DU_k \) for \( k \geq 1 \), and by adding a step \( U \) at the beginning and a step \( D \) at the ending. It is straightforward to verify that \( \theta \) induces a bijection \( \bar{\theta} \) between the set of \( U_kD \)-equivalence classes of \( \mathcal{L}_n \) and the set of \( DU_k \)-equivalence classes of \( \mathcal{L}'_{n+2} \). Then, Theorem 10 is deduced from Theorem 8.

**Theorem 10** The generating function for the set of \( DU_k \)-equivalence classes of \( \mathcal{L} \) with respect to the length is given by

\[
1 + x + x^2 - \frac{\sqrt{1 - 2x - x^2 - 2x^3 + x^4}}{2x},
\]

which generates the generalized Catalan sequence defined by \( u_0 = u_1 = 1 \), and for \( n \geq 2 \) \( u_n = g_{n-2} \) where \( g_n \) is defined in Theorem 8. (see A004148 in [28]).

**4.3 Modulo \( S = \{U_kF, k \geq 1\} \), and \( S = \{FU_k, k \geq 1\} \)**

Let \( \mathcal{E} \subset \mathcal{L} \) be the set of Lukasiewicz paths such that any up step \( U_k, k \geq 1 \), is immediately followed by a flat step \( F \), and any flat step \( F \) of positive height belongs to a pattern \( U_kF, k \geq 1 \). For instance, we have \( U_3FDDFDUD \in \mathcal{E} \) and \( U_3DDDFUFDF \notin \mathcal{E} \). Let \( \bar{\mathcal{E}} \subset \mathcal{E} \) be the set of Lukasiewicz paths in \( \mathcal{E} \) without flat step on the \( x \)-axis.

**Lemma 4** There is a bijection between \( \mathcal{E} \) and the set of \( U_kF \)-equivalence classes of \( \mathcal{L} \).

**Proof.** The proof is obtained *mutatis mutandis* as for Lemma 3 by replacing \( U_kD \) with \( U_kF \). \( \square \)

**Theorem 11** The generating function for the set of \( U_kF \)-equivalence classes of \( \mathcal{L} \) with respect to the length is given by

\[
1 - x - \frac{\sqrt{1 - 2x + x^2 - 4x^3}}{2x^3},
\]

which generates the generalized Catalan sequence defined by \( h_0 = 1 \), and for \( n \geq 0 \), \( h_{n+1} = h_n + \sum_{k=0}^{n-2} h_k h_{n-2-k} \) (see A023431 in [28]).

**Proof.** Using Lemma 4, it suffices to obtain the generating function \( E(x) \) for the set \( \mathcal{E} \). A non-empty Lukasiewicz path \( L \in \mathcal{E} \) can be written either \( L = FL' \) where \( L' \in \mathcal{E} \), or \( L = U_kFL_1DL_2D \ldots L_kDL' \) for \( k \geq 1 \) where \( L_1, L_2, \ldots, L_k \) are some Lukasiewicz paths in \( \bar{\mathcal{E}} \), and \( L' \in \mathcal{E} \). So we obtain the functional equation \( E(x) = 1 + xE(x) + x^3E(x) \sum_{k \geq 0} x^k \bar{E}(x)^{k+1} \) where \( \bar{E}(x) \) is the generating function for the set \( \bar{\mathcal{E}} \). Using the classical decomposition of a
Lukasiewicz path, we have \( E(x) = 1 + \sum_{k \geq 3} x^k E(x)^{k-1} \). A simple calculation provides the result. \( \square \)

Let \( \xi \) be the map from \( \mathcal{L} \) to himself where \( \xi(L) \) is obtained from \( L \) by replacing any occurrence \( U_k F \) by an occurrence \( FU_k \) for \( k \geq 1 \). It is straightforward to verify that \( \xi \) induces a bijection \( \xi \) between the set of \( U_k F \)-equivalence classes and the set of \( FU_k \)-equivalence classes. Then, we have Theorem 12.

**Theorem 12** The generating function for the set of \( FU_k \)-equivalence classes of \( \mathcal{L} \) with respect to the length also is the generating function given in Theorem 11.

**Theorem 13** For \( n \geq 0 \), the map \( \psi \) is a bijection from \( \mathcal{E}_n \) to the subset \( \mathcal{M}_n' \) of Motzkin paths in \( \mathcal{M}_n \) that avoid \( UD \) and \( UU \).

**Proof.** We proceed by induction on \( n \). Obviously, for \( n = 0 \), we have \( \psi(\epsilon) = \epsilon \). For \( 0 \leq k \leq n \), we assume that \( \psi \) is a bijection from \( \mathcal{E}_k \) to the subset \( \mathcal{M}_k' \) and we prove the result for \( n + 1 \). Since the set of length \( n \) Motzkin paths avoiding \( UU \) and \( UD \) is enumerated by the value \( h_n \) defined in Theorem 11 (see A023431 in [28]), it suffices to prove that \( \psi \) is surjective. So, let \( M \) be a Motzkin path in \( \mathcal{M}_{n+1}' \). We distinguish two cases:

(i) \( M = FM' \) with \( M' \in \mathcal{M}_n' \). Using the recurrence hypothesis, there is \( L' \in \mathcal{E}_n \) such that \( M' = \psi(L') \). So, the Lukasiewicz path \( L = FL' \in \mathcal{E}_{n+1} \) satisfies \( \psi(L) = M \) which proves that \( M \) belongs to the image by \( \psi \) of \( \mathcal{E}_{n+1} \).

(ii) \( M = UM'DM'' \) where \( M' \) and \( M'' \) are two Motzkin paths in \( \mathcal{M}' \). Since \( M \in \mathcal{M}_{n+1}' \), we have \( M' \neq \epsilon \) and \( M' \) does not start with \( U \), which implies that \( M' \) starts with \( F \). Using the recurrence hypothesis, there are \( L' \in \mathcal{E} \) and \( L'' \in \mathcal{E} \) such that \( \psi(L') = M' \) and \( \psi(L'') = M'' \). Since \( M' \) starts with a flat step, \( L' \) also starts with a flat step. So, \( L = UL'DL'' \) belongs to \( \mathcal{E}_{n+1} \) and satisfies \( \psi(L) = M \) which proves that \( \psi \) from \( \mathcal{B}_n \) to \( \mathcal{M}_n \) is bijective. \( \square \)

### 4.4 Modulo \( FF \)

Let \( \mathcal{F} \) be the set consisting of the union of \( \{ \epsilon, F \} \) with the set of Lukasiewicz paths containing at most one up step \( U_k \), \( k \geq 1 \) and such that any flat step \( F \) is contained into a pattern \( FF \). For instance, \( FFU_3DFDFDDFDFF \in \mathcal{F} \) and \( FFU_3DFDFDDFDFF \notin \mathcal{F} \).

**Lemma 5** There is a bijection between \( \mathcal{F} \) and the set of \( FF \)-equivalence classes of \( \mathcal{L} \).

**Proof.** Let \( L \) be a non-empty Lukasiewicz path in \( \mathcal{L} \). Let us prove that there exists a Lukasiewicz path \( L' \in \mathcal{F} \) (with the same length as \( L \)) such that \( L \) and \( L' \) belong to the same class. We write

\[
L = K_1F^{a_1}K_2F^{a_2}K_3 \ldots K_rF^{a_r}K_{r+1},
\]

with \( r \geq 0 \) and \( a_i \geq 2 \) for \( 1 \leq i \leq r \), such that \( K_1, K_2, \ldots, K_r, K_{r+1} \) are some parts without pattern \( FF \), \( k \geq 1 \), and \( K_2, \ldots, K_r \) are not empty and do not have any \( F \) in first and last position, and \( K_1 \) has no flat in last position, and \( K_{r+1} \) has no flat in first position.

12
If $L = F^n$ with $n \geq 0$, then its equivalence class is reduced to a singleton. Now let us assume that $L \neq F^n$. We distinguish two cases:

(1) $K_1$ is not empty. We define the Łukasiewicz path
\[
L' = U_b D^{b_1} F^{a_1} D^{b_2} F^{a_2} D^{b_3} \ldots D^{b_r} F^{a_r} D^{b_{r+1}}
\]
with $b_i = |K_i|$, $1 \leq i \leq r + 1$, and $b = \sum_{i=1}^{r+1} b_i - 1$.

(2) $K_1$ is empty which means that $L = F^{a_1} K_2 F^{a_2} K_3 \ldots K_r F^{a_r} K_{r+1}$. We define the Łukasiewicz path
\[
L' = F^{a_1} U_b D^{b_2} F^{a_2} D^{b_3} \ldots D^{b_r} F^{a_r} D^{b_{r+1}}
\]
with $b_i = |K_i|$, $2 \leq i \leq r + 1$, and $b = \sum_{i=2}^{r+1} b_i - 1$.

Less formally, we obtain $L'$ from $L$ by replacing any $K_i$ (excepted the first) with a run of down steps $D^{(K_i)}$, and by replacing the first $K_i$ ($K_1$ or $K_2$ according to the case (1) or (2)) with $U_b D^{(K_i)}$ (or $U_b D^{(K_i-1)}$) where the up step $U_b$ balances all down steps in $L'$, i.e., $b$ is the number of down steps in $L'$. Clearly, $L' \in \mathcal{F}$ and $L'$ belongs to the same class as $L$. For instance, if $L = U_2 D F F U_2 D D F F U_3 D D F F D D F F$, then $L' = U_b D F F D D F F D D F F D D F F$ (see Figure 6).

The definition of $\mathcal{L}$ implies that there is only one path of $\mathcal{F}$ in the same class as $L$, which completes the proof.

![Figure 6: Illustration of the example described in the proof of Lemma 5.](image)

**Theorem 14** The generating function for the set of $FF$-equivalence classes of $\mathcal{L}$ with respect to the length is given by
\[
\frac{1 - 3x + 4x^2 - 5x^3 + 7x^4 - 7x^5 + 6x^6 - 3x^7 + x^8}{(1 - 2x + x^2 - x^3)(1 - x)^2}.
\]

(Note that the associated sequence does not yet appear in [28]).

**Proof.** Using Lemma 5, it suffices to obtain the generating function $F(x)$ for the set $\mathcal{F}$. A non-empty Łukasiewicz path $L \in \mathcal{F}$ can be written either (i) $L = F^k$ for $k \geq 0$, or (ii) $L = F^{j_0} U_k F^{j_1} D^{i_1} F^{j_2} D^{i_2} \ldots F^{j_{\ell+1}} D^{i_{\ell+1}}$ with $\ell \geq 1$, $i_0 = 0$ or $i_0 \geq 2$, $i_1 = 0$ or $i_1 \geq 2$, $i_{\ell+1} = 0$ or $i_{\ell+1} \geq 2$, $i_m \geq 2$ for $2 \leq m \leq \ell$, and $j_m \geq 1$ for $1 \leq m \leq \ell$.

The generating function for the Łukasiewicz paths satisfying (i) is given by $\frac{1}{1-x}$.

For Łukasiewicz paths satisfying (ii), we give the generating function for each part of $L$, and we multiply them:
- For $F_{i_0}$, with $i_0 = 0$ or $i_0 \geq 2$, the generating function is $1 + \frac{x^2}{1-x}$;
- For $F_{i_{\ell+1}}$, with $i_{\ell+1} = 0$ or $i_{\ell+1} \geq 2$, the generating function is $1 + \frac{x^2}{1-x}$;
- For $U_k F_{i_1} D_{j_1}$, with $i_1 = 0$ or $i_1 \geq 2$ and $j_1 \geq 1$, the generating function is $x(1 + \frac{x^2}{1-x})\frac{x}{1-x}$;
- For $F_{i_2} D_{j_2} \ldots F_{i_m} D_{j_m}$, with $i_m \geq 2$, and $j_m \geq 1$, the generating function is $\frac{1}{1-(1-x)^2}$.

Considering all these cases, we deduce:

$$F(x) = \left(1 + \frac{x^2}{1-x}\right)^3 x^2 (1-x)^{-1} \left(1 - \frac{x^3}{(1-x)^2}\right)^{-1} + (1-x)^{-1}$$

which completes the proof. ☐

5 Concluding remarks

Extending recent works on Dyck and Motzkin paths [2, 3], the goal of this paper is to calculate the number of Lukasiewicz paths modulo the positions of a given pattern, i.e. the number of possible sets $I = \{i_1, i_2, \ldots, i_k\}$ where $i_1, i_2, \ldots, i_k$ are the occurrence positions of the pattern in Lukasiewicz paths. Can one do the same study for other lattice paths such as meanders, bridges and excursions, or Schroeder and Riordan paths?

From our study, we can deduce a lower bound for the maximal cardinality of a class by calculating the average of cardinalities of the classes, i.e., the total number of Lukasiewicz paths divided by the number of classes. Is it possible to calculate the exact value of the maximal cardinality for a class, and for which set $I$ it is reached? Also, it would be interesting to study some properties of the number of Lukasiewicz paths (of a given length) having $I$ as set of positions of the pattern. One can think this number is a polynomial with respect to the length $n$. If this is true, then we could give properties of these coefficients and roots, which would be a counterpart for lattice paths of the study of descent polynomial on the symmetric group $S_n$ (see MacMahon [12]).

6 Acknowledgements

We would like to thank the anonymous referees for their very careful reading of this paper and their helpful comments and suggestions.

References


