Vertex Distinguishing Edge- and Total-Colorings of Cartesian and other Product Graphs

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Abstract

This paper studies edge- and total-colorings of graphs in which (all or only adjacent) vertices are distinguished by their sets of colors. We provide bounds for the minimum number of colors needed for such colorings for the Cartesian product of graphs along with exact results for generalized hypercubes. We also present general bounds for the direct, strong and lexicographic products.

Keywords: graph, edge-coloring, vertex-distinguishing, adjacent vertex-distinguishing, total coloring, total adjacent vertex-distinguishing, graph products.

1 Introduction

All the graphs we deal with are simple, finite and with no component \(K_2\), where \(K_n\) stands for the complete graph of order \(n\). Let \(G = (V, E)\), be a graph with vertex set \(V\) and edge set \(E\). An edge between vertex \(x\) and vertex \(y\) is denoted by \(xy\). Let \(\Delta(G)\) be the maximum degree of the graph.

A proper edge-coloring \(c\) is a mapping from \(E\) to \(\mathbb{N}\) such that edges incident with the same vertex receive distinct values (or colors). For any vertex \(x\) of \(G\), let \(S(x)\) denote the set of the colors of all edges incident to \(x\) (if necessary, we write \(S_c(x)\) to indicate which coloring is used). A proper edge coloring is said to be

- **vertex distinguishing (VD)** if \(S(x) \neq S(y), \forall x, y \in V, x \neq y;\)
- **adjacent vertex distinguishing (AVD)** if \(S(x) \neq S(y), \forall xy \in E.\)

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A total coloring of a graph $G$ is a mapping from $V \cup E$ to $\mathbb{N}$ such that neighboring elements receive distinct colors. For a total coloring, let $S(x)$ be the set of the colors of all edges incident to $x$ plus the color of $x$: $S(x) = \{c(e) \mid e = xy\} \cup \{c(x)\}$. A total adjacent vertex distinguishing (TAVD) coloring is a total coloring satisfying $S(x) \neq S(y), \forall xy \in E$.

The minimum number of colors among all VD-colorings, AVD-colorings and TAVD-colorings respectively of a graph $G$ will be called the VD-chromatic index, AVD-chromatic index and TAVD-chromatic index denoted by $\chi'_s(G)$, $\chi'_a(G)$ and $\chi''_a(G)$ respectively.

The notation $\chi'_s(G)$, $\chi(G)$ and $\chi''(G)$ is used to represent respectively the chromatic index, the chromatic number and the total chromatic number of $G$, as usual. A coloring using the least number of colors with respect to the given constraints will be called a minimal coloring.

The notion of VD-coloring was introduced in [BS97] and independently in [ČHS96] where $\chi'_s(G)$ is called the observability.

A lower bound for the VD-chromatic index is given by $\pi(G) = \min\{k : \binom{k}{d} \geq n_d \text{ for } 1 \leq d \leq \Delta(G)\}$ where $n_d$ is the number of vertices of degree $d$. Moreover, it is conjectured in [BS97] that $\chi'_s(G) \leq \pi(G) + 1$ for any graph $G \neq K_2$. This conjecture has been solved for graphs of maximum degree two [BBS02] and for graphs verifying $\Delta(G) \geq \sqrt{2|V(G)|} + 4$ and $\delta(G) \geq 5$ [BKLS04]. It was proved in [BHBLW99] that $\chi'_a(G) \leq |V(G)| + 1$ and in [BHBLW01] the authors obtained $\chi'_a(G) \leq \Delta(G) + 5$ if $\delta(G) \geq \frac{n}{3}$.

The study of AVD-colorings is more recent. In [BGLS07], the authors proved that $\chi'_a(G) \leq 5$ for graphs of maximum degree 3 and $\chi'_a(G) \leq \Delta(G) + 2$ for bipartite graphs. In [BKT06], the AVD-chromatic index of multidimensional meshes was determined. AVD-colorings are also studied in [EHW06, GR06] under the name of neighbour-distinguishing edge colorings. The bound $\chi'_a(G) \geq \Delta(G)$ is trivial. Moreover if $G$ contains two adjacent vertices of degree $\Delta(G)$ then $\chi'_a(G) \geq \Delta(G) + 1$. The following conjecture was made in [ZLW02]:

**Conjecture 1 ([ZLW02])** Let $G \neq C_5$ be a graph of maximum degree $\Delta$, then

$$\Delta \leq \chi'_a(G) \leq \Delta + 2.$$
paper about the AVD-chromatic index are also new results about the total chromatic number while some other were already known earlier, see [KM03, ZZ04].

Total adjacent vertex distinguishing colorings were considered in [ZCL+05, LWZW06] in which the authors conjecture that $\chi''_a(G) \leq \Delta + 3$.

In this paper we shall consider VD, AVD and TAVD-colorings of products of graphs (see definitions below), trying to derive bounds for $\chi'_s$, $\chi'_a$ and $\chi''_a$ of the product of two graphs in terms of the value of the same parameter on the factors. In Section 2, we present general bounds for the Cartesian product. As an application, we determine in Sections 3 and 4, the AVD and TAVD-chromatic index of the generalized hypercube and present in Section 5 tight lower and upper bounds for its VD-chromatic index. Section 6 provides bounds for VD, AVD and TAVD-chromatic indices of direct, strong and lexicographic products.

We use the following notation from [IK00] for the standard graph products. Let $G \square H$, $G \times H$, $G \boxtimes H$ and $G \circ H$ be the Cartesian, direct (also called Kronecker or categorical), strong and lexicographic product of $G$ and $H$ respectively. The vertex set of any of these products is $V(G) \times V(H)$ and the edge sets are defined below:

$E(G \square H) = \{(a, x)(b, y), ab \in E(G) \text{ and } x = y \text{ or } xy \in E(H)\}$.
$E(G \times H) = \{(a, x)(b, y), ab \in E(G) \text{ and } xy \in E(H)\}$.
$E(G \boxtimes H) = E(G \square H) \cup E(G \times H)$.
$E(G \circ H) = \{(a, x)(b, y), ab \in E(G) \text{ or } a = b \text{ and } xy \in E(H)\}$.

The $d$-dimensional generalized hypercube (also known as Hamming graph) $K^d_n$ is the Cartesian product of the complete graph $K_n$ by itself $d$ times: $K^d_n = K_n \square K_n \square \ldots \square K_n$. The hypercube $Q_d$ is the graph $K^2_d$.

2 General results for Cartesian products

We first present general results for the VD, AVD and TAVD-coloring of the Cartesian product of graphs.

For AVD-colorings of the Cartesian product of a graph and a path or a cycle, the following results were proved in [BKT06]:

**Theorem 1 ([BKT06])** Let $d \geq 2$ be an integer and let $G$ be a graph of maximum degree $\Delta \leq d - 1$. If there exists an AVD-coloring of $G$ with $d$ colors, then

$$\chi'_a(G \square P_2) \leq d + 1,$$

$$\chi'_a(G \square P_k) \leq d + 2,$$ for $k \geq 3$.  

Theorem 2 ([BKT06]) Let \( d \geq 2 \) be an integer and let \( G \) be a graph of maximum degree \( \Delta \leq d - 1 \) and of minimum degree \( \delta \geq 2 \). If there exists an AVD-coloring of \( G \) with \( d \) colors, then

\[
\chi_a'(G \square C_k) \leq d + 2.
\]

The following theorem gives an upper bound on \( \chi_s' \) and \( \chi_a' \) for the Cartesian product of general graphs.

Theorem 3 For any two graphs \( G \) and \( H \) different from \( K_2 \), the following hold

\[
\begin{align*}
\chi_s'(G \square H) &\leq \chi_s'(G) + \chi_s'(H), \\
\chi_a'(G \square H) &\leq \chi_a'(G) + \chi_a'(H).
\end{align*}
\]

Proof : Note that the product \( G' = G \square H \) consists of \( |V(G)| \) copies of \( H \); and there is a perfect matching between any two copies of \( H \) if the corresponding vertices of \( G \) are adjacent. By symmetry, \( G' \) also contains \( |V(H)| \) copies of \( G \). Let \( c_G \) and \( c_H \) be two minimal VD- (AVD-)colorings of \( G \) and \( H \) respectively such that the colors of \( c_G \) are different from those used by \( c_H \). A VD-coloring (AVD-coloring) \( c' \) of \( G' \) is obtained as follows: each copy of \( G \) in \( G' \) is colored by \( c_G \) and each copy of \( H \) in \( G' \) is colored by \( c_H \).

Indeed, let \((a, x)\) be a vertex of \( G' \). We have \( S_{c'}((a, x)) = S_{c_G}(a) \cup S_{c_H}(x) \) and so, \((a, x)\) is distinguished from another (adjacent for AVD-coloring) vertex \((b, y)\) because \( a \) is distinguished from \( b \) in \( G \) or \( x \) is distinguished from \( y \) in \( H \). Notice that, despite the proof of the above theorem is quite simple, it allows to find the exact value of the AVD-chromatic index for graphs \( G \) and \( H \) verifying \( \chi_a'(G) = \Delta(G) \) and \( \chi_a'(H) = \Delta(H) \) (for instance, this is the case for trees with no two adjacent vertices of maximum degree [ZLW02]).

For the TAVD-chromatic index of the Cartesian product of two graphs, we have the following.

Theorem 4 Let \( G \) and \( H \) be two graphs different from \( K_2 \) such that \( \chi(H) \leq \chi_a''(G) \), then

\[
\chi_a''(G \square H) \leq \chi_a''(G) + \chi_a'(H).
\]

Proof : Let \( G' = G \square H \), \( c_G \) be a minimal TAVD-coloring of \( G \) and \( c_H \) be a minimal AVD-coloring of \( H \) such that the colors of \( c_G \) are different from those used by \( c_H \). We also color the vertices of \( H \) with the colors \( 0, 1, \ldots, \chi(H) - 1 \). Let \( \alpha = \chi_a''(G) \) and denote by \( \sigma_i \) the permutation on \( 0, 1, \ldots, \alpha - 1 \) defined by \( \sigma_i(k) = (k + i) \mod \alpha \), for \( 0 \leq k \leq \alpha - 1 \) and
0 \leq i \leq \chi(H) - 1. By extension, the total coloring \( c'_G = \sigma_i(c_G) \) is defined by \( c'_G(x) = \sigma_i(c_G(x)) \) \( \forall x \in V(G) \cup E(G) \).

In order to obtain an AVD-coloring \( c' \) of \( G' \), we first use \( \sigma_i(c_G) \) to color each copy \( G_j \) of \( G \) in \( G' \), where \( i \) is the color (given by the proper vertex coloring of \( H \)) of the vertex corresponding to the copy \( G_j \). Since \( \chi(H) \leq \alpha \), all permutations \( \sigma_i \) are pairwise different.

A TAVD-coloring \( c' \) of \( G' = G \square H \) is obtained as follows: each copy \( G_j \) of \( G \) in \( G' \) is totally colored by \( \sigma_i(c_G) \), where \( i \) is the color (given by the proper vertex coloring of \( H \)) of the vertex of \( H \) corresponding to copy \( G_j \) and each copy of \( H \) in \( G' \) is colored by \( c_H \).

Notice that since \( \chi(H) \leq \alpha \), all permutations \( \sigma_i \) are pairwise different; thus the colors assigned to vertices of \( G' \) form a proper coloring. Moreover, as for the previous theorem, the fact that \( c' \) is a TAVD-coloring is easily shown since the colors on the vertices induce a proper vertex coloring and adjacent vertices are distinguished either by their sets of colors from \( c_G \) or by their sets of colors from \( c_H \).

\[ \square \]

We now propose two better results in more specific cases for the AVD and TAVD-chromatic indices.

**Theorem 5** Let \( G \) be a graph such that the degree of each vertex is relatively prime with \( \chi'_a(G) \), and let \( H \) be a graph verifying \( \chi(H) \leq \chi'_a(G) \) then

\[ \chi'_a(G \square H) \leq \chi'_a(G) + \Delta(H). \]

**Proof**: Let \( G' = G \square H \), \( c_G \) be an AVD-coloring of \( G \) in \( \chi'_a(G) \) colors and \( \gamma_H \) be a proper edge coloring of \( H \) in \( \chi'(H) \) colors distinct from those used by \( c_G \). We also color the vertices of \( H \) with the colors \( 0, 1, \ldots, \chi(H) - 1 \).

Similarly with the proof of the previous theorem, let \( \alpha = \chi'_a(G) \) and denote by \( \sigma_i \) the permutation on \( 0, 1, \ldots, \alpha - 1 \) defined by \( \sigma_i(k) = (k + i) \mod \alpha \), for \( 0 \leq k \leq \alpha - 1 \) and \( 0 \leq i \leq \chi(H) - 1 \). Let also \( c'_G = \sigma_i(c_G) \) be the coloring defined by \( c'_G(e) = \sigma_i(c_G(e)) \) \( \forall e \in E(G) \).

In order to obtain an AVD-coloring \( c' \) of \( G' \), we first use \( \sigma_i(c_G) \) to color each copy \( G_j \) of \( G \) in \( G' \), where \( i \) is the color (given by the proper vertex coloring of \( H \)) of the vertex corresponding to the copy \( G_j \).

Since \( \chi(H) \leq \alpha \), all permutations \( \sigma_i \) are pairwise different; moreover, if \( x \) is a vertex of \( G \), and \( x_i \) (resp. \( x_j \)) its corresponding vertex in \( G_i \) (resp. \( G_j \)), then the color set of \( x_i \) in \( G_i \) is different from the color set of \( x_j \) in \( G_j \). Then, we have two cases to consider:

**Case 1**: \( \chi'(H) = \Delta(H) \)

In this case, we use the proper coloring \( \gamma_H \) of \( H \) to color each copy of \( H \) in \( G' \). Let \( (a, x) \) and \( (b, y) \) be two adjacent vertices of \( G' \). Without
loss of generality, we have $S_c((a, x)) = S_{cG}(a) \cup S_{\gamma_H}(x)$ and $S_c((b, y)) = \sigma_i(S_{cG}(b)) \cup S_{\gamma_H}(y)$, for some $i$.

If $x = y$ (i.e. $(a, x)$ and $(b, y)$ are in the same copy of $G$), then $i = 0$ and $S_c((a, x)) \neq S_c((b, y))$ since $S_{cG}(a) \neq S_{cG}(b)$.

If $x \neq y$, as $(a, x)$ and $(b, y)$ are adjacent then so are $x$ and $y$ in $H$, thus $x$ and $y$ have different colors and so $i \neq 0$. If $S_{\gamma_H}(y) \neq S_{\gamma_H}(x)$, we have immediately $S_c((a, x)) \neq S_c((b, y))$. If $S_{\gamma_H}(y) = S_{\gamma_H}(x)$, we prove that $S_{cG}(a) \neq \sigma_i(S_{cG}(a))$. Indeed, by contradiction, let $d$ be the degree of $a$ in $G$; if $S = S_{cG}(a) = \sigma_i(S_{cG}(a)) = \{s_1, \ldots, s_d\}$, then we have modulo $\alpha = \chi'_a(G)$: $s_j + i = s_k \in S$ for each $j, 1 \leq j \leq d$. If we sum all equalities, we obtain $d \cdot i = 0 \mod \alpha$. So, the hypothesis that $d$ and $\alpha$ are relatively prime gives $\sigma_i = \sigma_0 = Id$ which is a contradiction.

Therefore, when $\chi'(H) = \Delta(H)$ we have $\chi'_a(G \square H) \leq \chi'_a(G) + \Delta(H)$.

**Case 2:** $\chi'(H) = \Delta(H) + 1$

Remark that in this case, for each vertex $x$ of $H$, there exists (at least) one color $j$ such that $j \not\in S_{\gamma_H}(x)$ (the missing color). In order to complete the coloring $\gamma'$ of $G'$, we use the proper edge-coloring $\gamma_H$ to color each copy of $H$. Then, from the above remark, for each copy $G_i$ of $G$ in $G'$, there is a color that is not used by any of the edges incident with any vertex of $G_i$. So we modify the coloring $\sigma_i(cG)$ of each copy $G_i$ of $G'$ by changing the color zero by this missing color. With a similar proof as for the first case, we can show that $\chi'_a(G \square H) \leq \chi'_a(G) + \Delta(H)$. □

**Theorem 6** Let $G$ be a graph such that the degree of each vertex plus one is relatively prime with $\chi''_a(G)$, and let $H$ be a graph verifying $\chi(H) \leq \chi''_a(G)$ then

$$\chi''_a(G \square H) \leq \chi''_a(G) + \Delta(H).$$

**Proof**: We modify the previous proof as follow: each AVD-coloring is replaced by a TAVD-coloring; thus $\chi'_a(G)$ is changed in $\chi''_a(G)$; each set of $d$ colors $\{s_1, s_2, \ldots, s_d\}$ is changed in a set of $d + 1$ colors $\{s_1, s_2, \ldots, s_d, s_{d+1}\}$ and the equality $d \cdot i = 0 \mod \alpha$ becomes $(d + 1) \cdot i = 0 \mod \alpha$. □

Now, if $\alpha$ is taken to be the smallest prime number greater than $\chi'_a(G)$ (respectively $\chi''_a(G)$) then we obtain the two following corollaries.

**Corollary 1** Let $G$ be a graph and let $p$ be the smallest prime number greater than or equal to $\chi'_a(G)$. If $H$ is a graph verifying $\chi(H) \leq p$ then

$$\chi'_a(G \square H) \leq p + \Delta(H).$$

**Corollary 2** Let $G$ be a graph and let $p$ be the smallest prime number greater than or equal to $\chi''_a(G)$. If $H$ is a graph verifying $\chi(H) \leq p$ then

$$\chi''_a(G \square H) \leq p + \Delta(H).$$
3 AVD-coloring of the generalized hypercube

In this section, we determine the AVD-chromatic index of the generalized hypercube $K^d_n$. We first need to compute the AVD-chromatic index of $K_{2p} \sqcap K_2$.

Lemma 1 For $p \geq 2$

$$\chi'_a(K_{2p} \sqcap K_2) = 2p + 1.$$  

Proof: In order to construct the graph $K_{2p} \sqcap K_2$, we consider two copies $K$ and $K'$ of $K_{2p}$. Let $V(K) = \{x_0, x_1, \ldots, x_{2p-1}\}$ and let $V(K') = \{x'_0, x'_1, \ldots, x'_{2p-1}\}$. Let $c$ be the coloring of $K_{2p}$ defined by:

$$c(x_ix_j) = i + j \mod (2p + 1) \text{ with } 0 \leq i < j \leq 2p - 1.$$  

In the following, each integer is considered modulo $(2p + 1)$.

Remark that the equality $2i = i - 1$ has no solution for $i \in [0..2p-1]$, and it is easy to see that the color set $S(x_i)$ of each vertex $x_i$ is exactly $S(x_i) = \{0, 1, \ldots, 2p\} \setminus \{2i, i - 1\}$.

Moreover, for $i \in [0..2p-1]$, $i - 1 \in [0..2p]/\{2p - 1\}$ and $2i \in [0..2p]/\{2p - 1\}$. So, the color $2p - 1$ is the only one which appears in each set $S(x_i)$. Moreover, if we suppose $i \neq j$ then the two equalities $2j = i - 1$ and $2i = j - 1$ induce that $3(i + 1) = 0$ which is impossible when $3$ does not divide $2p + 1$. This proves that $c$ is an AVD-coloring when $3$ does not divide $2p + 1$.

Now we define another coloring $c'$ for the second copy $K'$ of $K_{2p}$ by setting:

$$c'(x'_ix'_j) = \sigma(c(x_ix_j)) \text{ with } 0 \leq i, j \leq 2p - 1,$$

where $x'_i$ is the corresponding vertex of $x_i$ in the second copy $K'$ of $K_{2p}$ and the permutation $\sigma$ is defined by:

$$\sigma(i - 1) = 2i, \text{ with } 1 \leq i \leq 2p.$$  

An illustration of the colorings $c$ and $c'$ is given in Appendix for $p = 3$.

As above, remark that if $3$ does not divide $2p + 1$ and if $i \neq 2p - 1$ then $\sigma^2(i) = i$ has no solution. This proves that $\sigma$ has no cycle of length two in its decomposition into a product of disjoint cycles and $c'$ is also an AVD-coloring if $3$ does not divide $2p + 1$.

In this case ($3$ does not divide $2p + 1$), we give the color $2i$ to each edge $x_ix'_i$ in $K_{2p} \sqcap K_2$. So, in $K_{2p} \sqcap K_2$, the vertex $x_i$ has no incident edge of color $i - 1$, and the vertex $x'_i$ has no incident edge of color $\sigma(2i)$. Since $\sigma$ has no cycle of length $2$ in its decomposition then $\sigma(2i) \neq i - 1$. Consequently, $S(x_i) \neq S(x'_i)$ in $K_{2p} \sqcap K_2$. 

So, we have obtained an AVD-coloring of $K_{2p} \square K_2$ with $2p + 1$ colors when 3 does not divide $2p + 1$ (see Figure 1 for an AVD-coloring of $K_4 \square K_2$).

In the case where 3 divides $2p + 1$, we modify the coloring $c$ into an AVD-coloring $d$ as follows:

Recall that $\sigma$ has a unique cycle of length two in its disjoint cycles decomposition. Let $(a, b)$ be this cycle where $a > b$. We have necessarily $a = 2 \frac{2p+1}{3} - 2$ and $b = 2 \frac{2p+1}{3} - 2$ since $\sigma^2(i) = i$ has only one solution.

Remark also that $a = 2(b + 1) = \sigma(b)$.

We define the coloring $d$ by:

\[
\begin{align*}
  d(x_{a+1+k}x_{2p-1-k}) &= a & \text{with } 0 \leq k \leq 2p - 2 - a, \\
  d(x_{a+1+k}x_{2p-k}) &= a - 1 & \text{with } 1 \leq k \leq 2p - 2 - a,
\end{align*}
\]

and in the other cases

\[d(x, y) = c(x, y).\]

This coloring is also an AVD-coloring for $K_{2p}$ with $2p + 1$ colors. It is easy to see that the color set $S(x_i)$ of each vertex $x_i$ is exactly $S(x_i) = \{0, 1, \ldots, 2p\}\{2i, i - 1\}$ for $i \neq a + 1$, $i \neq a/2 + p$, and $S(x_{a+1}) = \{0, 1, \ldots, 2p\}\{b, a - 1\}$, $S(x_{a/2+p}) = \{0, 1, \ldots, 2p\}\{a, a/2 + p - 1\}$

Recall that we had previously with the coloring $c$

$S(x_{a+1}) = \{0, 1, \ldots, 2p\}\{b, a\}$ and $S(x_{a/2+p}) = \{0, 1, \ldots, 2p\}\{a - 1, a/2 + p - 1\}$

As above, excepted the color $2p - 1$, all colors appear in at least one color set.

Now, we define another AVD-coloring $d'$ for the second copy of $K_{2p}$ by setting:

\[d'(x_i'x_j') = \sigma'(d(x_i, x_j)) \quad \forall \ 0 \leq i, j \leq 2p - 1 \text{ with }\]

\[
\begin{align*}
  \sigma'(i) &= \sigma(i) & \forall i \neq b \text{ and } i \neq a/2 + p - 1 \\
  \sigma'(b) &= a - 1 \\
  \sigma'(a/2 + p - 1) &= a
\end{align*}
\]

An illustration of the colorings $d$ and $d'$ is given in Appendix for $p = 4$.

By construction, $\sigma'$ has no cycle of length 2 in its decomposition into a product of disjoint cycles. Indeed, we have ‘broken’ the cycle $(a, b)$ of length 2 in $\sigma$.

In $K_{2p} \square K_2$, for each $i$, the color sets $S(x_i)$ of $x_i$ and $S(x_i')$ of $x_i'$ are distinct and verify $S(x_i') = \sigma'(S(x_i))$. Moreover, as for the previous case,
for each \( i \) there exists a color which appears neither in \( S(x'_i) \) nor in \( S(x_i) \). We give this color to the edge \( x_i x'_i \) for each vertex \( x_i \) of \( K_{2p} \). Thus, we obtain an AVD-coloring for \( K_{2p} \circ K_2 \) in \( 2p + 1 \) colors when \( 3 \) divides \( 2p + 1 \).

Finally, as \( K_{2p} \circ K_2 \) is regular of degree \( 2p \), we obtain \( \chi'_a(K_{2p} \circ K_2) \geq 2p + 1 \) and thus \( \chi'_a(K_{2p} \circ K_2) = 2p + 1 \).

\[ \Box \]

**Theorem 7** For any integers \( n \geq 2 \) and \( d \geq 2 \),

\[ \chi'_a(K_{2^n}^d) = d(n - 1) + 1. \]

**Proof:** It is known that \( \chi'_a(K_n) = \chi'_a(K_n) = n + 1 - \epsilon(n) \), where \( \epsilon(n) = 1 \) for odd \( n \) and \( 0 \) for even \( n \).

When \( n \) is odd, \( n = 2p + 1 \) for some \( p \), we proceed by induction on \( d \). The result is true for \( d = 1 \). Assume that \( \chi'_a(K_{2p+1}^{d-1}) = 2p(d - 1) + 1 \), thus \( \chi'_a(K_{2p+1}^{d-1}) \) and \( \Delta(K_{2p+1}^{d-1}) = 2p(d - 1) \) are relatively prime and using Theorem 5 with \( G = K_{2p+1}^{d-1} \) and \( H = K_{2p+1} \), we have that \( \chi'_a(K_{2p+1}^{d}) = 2pd + 1 \).

When \( n \) is even, we obtain \( d(2p-1) + 1 \leq \chi'_a(K_{2p}^d) \leq d(2p-1) + 2 \). We show that \( \chi'_a(K_{2p}^d) = 2p + 1 \) by induction on \( d \).

When \( d = 2 \), by Lemma 1, there exists an AVD-coloring of \( K_{2p} \circ K_2 \) with \( 2p + 1 \) colors, and applying Theorem 5 with \( G = K_{2p} \circ K_2 \) and \( H = K_p \), we conclude that \( \chi'_a((K_{2p} \circ K_2) \circ K_p) = 3p \). Now, we add \( p - 1 \) perfect matchings to \( (K_{2p} \circ K_2) \circ K_p \) in order to obtain \( K_{2p}^2 \), all the edges of each perfect matching being colored with a new color. We therefore obtain \( \chi'_a(K_{2p}^2) = 4p - 1 \).

We suppose by induction that \( \chi'_a(K_{2p}^d) = d(2p-1) + 1 \) for \( d \geq 2 \) and prove that \( \chi'_a(K_{2p}^{d+1}) = (d + 1)(2p-1) + 1 \). As \( K_{2p}^{d+1} = K_{2p} \circ K_{2p} \), \( K_{2p}^{d+1} \) contains \( 2p \) copies of \( K_{2p}^d \). Let \( \{x'_i, 0 \leq i \leq (2p)^d - 1\} \) be the set of vertices of the \( j \)th copy (\( 1 \leq j \leq 2p \)). So, we color each copy with the different AVD-colorings \( c_j, 1 \leq j \leq 2p \) such that: \( c_1 \) is an AVD-coloring of \( K_{2p}^d \) in \( d(2p-1) + 1 \) colors, and if \( j \geq 2 \), we define \( c_j(x'_i x'_j) = c_1(x'_{i+j-1} x'_{k+j-1}) \) where the subscripts are modulo \( (2p)^d \). In order to obtain \( K_{2p}^{d+1} \), we add \( 2p - 1 \) perfect matchings between the \( 2p \) copies of \( K_{2p}^d \), all the edges of each perfect matching being colored with a new color. So, \( \chi'_a(K_{2p}^{d+1}) = d(2p-1) + 1 + (2p-1) = (d + 1)(2p-1) + 1 \).

\[ \Box \]
4 TAVD-coloring of the generalized hypercube

In this section, we determine the TAVD-chromatic index of the generalized hypercube $K_n^d$. In order to do that, we first need to compute the TAVD-chromatic index of $K_{2p+1}^2$.

**Lemma 2** For $p \geq 2$

$$\chi''(K_{2p+1}^2) = 4p + 2.$$  

**Proof**: Let $V(K_{2p+1}^2) = \{x_j^i | 0 \leq i, j \leq 2p \}$ and $E(K_{2p+1}^2) = \{x_j^i x_{j'}^{i'} | 0 \leq i, i', j, j' \leq 2p, i \neq i' \} \cup \{x_j^i x_{j'}^{i'} | 0 \leq i, i', j \leq 2p, i \neq j \}$. Define a total coloring $c$ of $K_{2p+1}^2$ by

$$
\begin{align*}
  c(x_j^i) &= 2i + j \mod (4p + 2), \\
  c(x_j^i x_{j'}^{i'}) &= i + i' + j \mod (4p + 2), \\
  c(x_j^i x_{j'}^{i'}) &= i + j + j' + 2p + 1 \mod (4p + 2).
\end{align*}
$$

We now show that $c$ is a TAVD-coloring. By the above definition, $S(x_j^i) = \{0, \ldots, 4p + 1 \} \setminus \{i + j + 2p + 1 \mod (4p + 2) \}$. Hence, for $i' \neq i$ and $j' \neq j$, we have $S(x_j^i) \neq S(x_{j'}^{i'})$ and $S(x_j^i) \neq S(x_i^{j'})$. \hfill $\square$

**Theorem 8** For any integers $n \geq 2$ and $d \geq 2$,

$$\chi''(K_n^d) = (n - 1)d + 2.$$  

Proof: It is known [ZCL+05] that \( \chi''_a(K_n) = \chi''_s(K_n) = n + 2 - \epsilon(n) \), where \( \epsilon(n) = 0 \) for odd \( n \) and 1 for even \( n \).

We proceed by induction on \( d \), by considering two cases depending on the parity of \( n \).

When \( n \) is even, \( n = 2p \) for some \( p \), the result is true for \( d = 1 \). Assume that \( \chi''_a(K_{2p}^{d-1}) = (2p - 1)(d - 1) + 2 \), thus \( \chi''_a(K_{2p}^{d-1}) \) and \( \Delta(K_{2p}^{d-1}) + 1 = (2p - 1)(d - 1) + 1 \) are relatively prime and using Theorem 6 with \( G = K_{2p}^{d-1} \) and \( H = K_{2p} \), we have that \( \chi''_a(K_{2p}^d) = (2p - 1)d + 2 \).

When \( n = 2p + 1 \) is odd, the result is true for \( d = 2 \) by Lemma 2. Assume that \( \chi''_a(K_{2p+1}^{d-1}) = 2pd + 2 \). Thus \( \chi''_a(K_{2p+1}^{d-1}) \) and \( \Delta(K_{2p+1}^{d-1}) + 1 = 2pd + 1 \) are relatively prime and using Theorem 6 with \( G = K_{2p+1}^{d-1} \) and \( H = K_{2p+1} \), we have that \( \chi''_a(K_{2p+1}^d) = 2pd + 2 \).

Therefore, we have proved that \( \chi''_a(K_n^d) = (n - 1)d + 2 \).

\[ \square \]

5 VD-coloring of the generalized hypercube

Finding a minimal VD-coloring for the product of a graph by \( K_2 \) seems to be difficult, nevertheless we present two simple upper bounds.

Theorem 9 For any graph \( G \), \( \chi'_s(G \square K_2) \leq 2\chi'_s(G) + 1 \).

Proof: A VD-coloring of \( G \square K_2 \) can be simply obtained by coloring the edges of each of the two copies of \( G \) with two VD-colorings using \( \chi'_s(G) \) colors distinct from each other and by coloring the edges between the two copies by a new color.

Given a graph \( G \) with an edge-coloring \( c \), we say that a color \( j \) touches a vertex \( x \) if \( j \in S_c(x) \).

Theorem 10 If there exists a VD-coloring of \( G \) with \( d \) colors such that one color touches each vertex, then there exists a VD-coloring of \( G \square K_2 \) with \( d + 2 \) colors such that one of them touches each vertex.

Proof: Assume that \( c \) is a VD-coloring of \( G \) with colors \( \{0, 1, \ldots, d - 1\} \) such that the color 0 touches each vertex of \( G \). Color each of the two copies of \( G \) in \( G \square K_2 \) with the coloring \( c \) and replace the color 0 by the color \( d \) in the first copy and by the color \( d + 1 \) in the second copy. Now, give the color 0 to the edges of the perfect matching between the two copies. The coloring obtained is clearly VD since the coloring in each copy is VD and a vertex of a copy is distinguished with a vertex of the other copy since different colors touch them. Moreover, the color 0 touches each vertex of \( G \square K_2 \).

\[ \square \]
This theorem allows to obtain the known fact that \( \chi_s'(Q_n) \leq 2n \) (see [ČHS96]). Notice that the authors of [ČHS96] have given the asymptotic value of \( \chi_s'(Q_n) \) but finding the exact value still remains an open problem.

**Theorem 11** For any integers \( n \geq 3 \) and \( d \),

\[
d(n - 1) \leq \chi_s'(K_n^d) \leq d(2\lfloor \frac{n}{2} \rfloor + 1).
\]

**Proof**: Obviously, \( \chi_s'(K_n^d) \geq \Delta(K_n^d) + 1 = d(n - 1) + 1 \). The relation is true for \( d = 1 \). By recurrence on \( d \) and with Theorem 3, when \( n = 2p + 1 \),

\[
\chi_s'(K_{2p+1}^d) \leq \chi_s'(K_{2p+1}^{d-1}) + \chi_s'(K_{2p+1}) \leq d(2p + 1).
\]

The case \( n = 2p \) is similar.

**Theorem 12** For any integers \( p \geq 1 \) and \( d \geq 1 \),

- if \( d \leq \ln(2p + 1) + 1 \) then \( \chi_s'(K_{2p+1}^d) = d(2p + 1) \),
- if \( d \leq \ln(2p) \) then \( \chi_s'(K_{2p}^d) \geq 2dp - 1 \).

**Proof**:

For the first assertion, in order to see if \( d(2p + 1) \) colors are sufficient to obtain a VD-coloring, we compare \( (2p + 1)^{d-1} \) with \( (2p + 1)^d \).

Let us define:

\[
A = \frac{(2p+1)^{d-1}}{(2p+1)^{d-1}} = \frac{(2p+1)^d - 1}{(2p+1)^d} = \frac{(2p+1)^d}{(2p+1)^d} - \frac{1}{(2p+1)^d}.
\]

Thus \( \ln(A) = \frac{d-1}{\ell=1} \ln(\frac{2p+1+\ell}{2p+1}) = \frac{d-1}{\ell=1} \ln(\frac{2p+1+\ell}{2p+1}) - \frac{d-1}{\ell=1} \ln(\ell) \).

With the well-known Darboux sums inequalities, we obtain

\[
\ln(A) \leq \int_{\ell=1}^{d} \ln(\frac{2p+1+\ell}{2p+1}) d\ell - \int_{\ell=1}^{d-1} \ln(\ell) d\ell \quad \text{and (with Maple),}
\]

\[
\leq (1 + 2dp) \ln(1 + \frac{d}{2p+1}) + (d - 1) \ln(\frac{d}{d-1}) - 1,
\]

\[
< d - 1 \quad \text{(since \( \ln(1 + x) < x \) for \( x \neq 0 \)).}
\]

So, if \( d \leq \ln(2p + 1) + 1 \) then \( \ln(A) < \ln(2p + 1) \) and \( A < 2p + 1 \). Thus when \( d \leq \ln(2p + 1) + 1 \), \( \chi_s'(K_{2p+1}^d) \geq d(2p + 1) \). With the previous theorem, we obtain the result.

The proof of the second point is similar.

\[\square\]

6 Other products

We first present a simple fact about TAVD-colorings that will be useful.
Fact 1 For any graph $G$,

$$
\chi''(G) \leq \chi'(G) + \chi(G).
$$

Actually, coloring separately the edges and the vertices of the graph gives a proper total coloring and the colors assigned to vertices of $G$ distinguish adjacent vertices.

As a consequence, the TAVD-chromatic index of a bipartite graph $B$ with two adjacent vertices of maximum degree is $\Delta(B) + 2$ since $\Delta(B) + 2$ colors are necessary to color and distinguish two adjacent vertices of degree $\Delta(B)$.

6.1 Direct product

Theorem 13 For any two graphs $G$ and $H$ different from $K_2$, the following hold

$$
\chi'_a(G \times H) \leq \chi'_s(G) \chi'_s(H),
$$

$$
\chi'_a(G \times H) \leq \min\{\chi'(G)\chi'_a(H), \chi'_a(G)\chi'(H)\},
$$

$$
\chi''_a(G \times H) \leq \min\{\chi'(G)\chi''_a(H), \chi''_a(G)\chi'(H)\}.
$$

Proof: Let $G' = G \times H$. For the first inequality, given two minimal VD-colorings $c_G$ and $c_H$ of $G$ and $H$ respectively, color each edge $(a, x)(b, y)$ of $G'$ with the color $(c_G(ab), c_H(xy))$. The color set of a vertex $(a, x)$ of $G'$ is then $S((a, x)) = S_{c_G}(a) \times S_{c_H}(x)$. Hence, for any two distinct vertices $(a, x)$ and $(b, y)$ of $G'$, we have $S((a, x)) \neq S((b, y))$ since either $S_{c_G}(a) \neq S_{c_G}(b)$ or $S_{c_H}(x) \neq S_{c_H}(y)$, or both.

For the second inequality, we can suppose without loss of generality that $\min\{\chi'(G)\chi'_a(H), \chi'_a(G)\chi'(H)\} = \chi'_a(G)\chi'(H)$. Let $c_G$ be an AVD-coloring of $G$ with colors $0, 1, \ldots, \alpha - 1$, where $\alpha = \chi'_a(G)$. Let $\gamma_H$ be a proper edge coloring of $H$ in $\chi'(H)$ colors.

An AVD-coloring $c'$ of $G'$ is obtained by setting for each $ab \in E(G)$, $xy \in E(H)$:

$$
c'(ab, xy) = (c_G(ab), \gamma_H(xy)).
$$

The fact that $c'$ is AVD is easily seen: we have $S_{c'}((a, x)) = S_{c_G}(a) \times S_{\gamma_H}(x)$. Hence, for any two adjacent vertices $(a, x)$ and $(b, y)$ of $G'$, we have $S((a, x)) \neq S((b, y))$ since $S_{c_G}(a) \neq S_{c_G}(b)$ ($a$ and $b$ are adjacent vertices of $G'$).

The third inequality can be shown in a same way, replacing $\chi'_a$ by $\chi''_a$, AVD by TAVD and for each vertex $(a, x)$ of $G'$, setting $c'(a, x) = (c_G(a), 0)$.

\[\square\]
Theorem 14 For any graph $G$, the following holds

$$\chi'_a(G \times K_2) \leq \chi'_a(G).$$

Moreover, equality holds if $G$ is bipartite, or if $G$ is regular and $\chi'_a(G) = \Delta(G) + 1$, or if $\chi'_a(G) = \chi'_a(G) = \Delta(G)$.

Proof: An A VD-coloring $c'$ of $G \times K_2$ is obtained by setting for $ab \in E(G)$, $xy \in E(K_2)$:

$$c'((a, x)(b, y)) = c_G(ab).$$

In other words, we give the color of the edge $ab$ to the edge $(a, x)(b, y)$. So, if $S_{c_G}(a) = \{a_1, a_2, \ldots, a_k\}$ then we have also $S_{c'}((a, x)) = \{a_1, a_2, \ldots, a_k\}$ and it is clear that $c'$ is an A VD-coloring of $G \times K_2$. In the case where $G$ is bipartite, then $G \times K_2$ consists in two disconnected copies of $G$ and so $\chi'_a(G \times K_2) = \chi'_a(G)$. In the case where $G$ is regular of degree $\Delta$ and $\chi'_a(G) = \Delta + 1$, then $G \times K_2$ is also regular and we have $\chi'_a(G \times K_2) \geq \Delta + 1$ and $\chi'_a(G \times K_2) \leq \chi'_a(G) = \Delta + 1$. So, $\chi'_a(G \times K_2) = \Delta + 1$. In the case where $\chi'_a(G) = \Delta$, then clearly, $\chi'_a(G \times K_2) = \Delta$.

Theorem 15 For any $m \geq 3$, $n \geq 3$,

$$\chi'_a(P_m \times P_n) = \begin{cases} 4 & \text{if } m = n = 3 \\ 5 & \text{otherwise.} \end{cases}$$

Proof: If $m = n = 3$, then $P_m \times P_n$ is the disjoint union of a 4-cycle and a 4-star. Hence four colors suffice to AVD-color $P_3 \times P_3$.

If $m \geq 3$, $n \geq 3$ and $m + n > 6$ then, in [BKT06], we proved that $\chi'_a(P_m \Box P_n) = 5$ and the coloring used is such that the four edges of each 4-length cycle in $P_m \Box P_n$ have pairwise different colors. Now, it is easily seen that $P_m \times P_n$ consists in two connected isomorphic components with two adjacent vertices of degree 4 (thus $\chi'_a(P_m \times P_n) \geq 5$) that are induced subgraphs of some grid $P_m' \Box P_n'$. Thus, two adjacent vertices of degree 4 in $P_m \times P_n$ have obviously different sets of colors. Moreover, two adjacent vertices of degree 2 have different sets of colors since they belong to a cycle of length 4 in $P_m' \Box P_n'$.

6.2 Strong product

Theorem 16 For any two graphs $G$ and $H$ different from $K_2$, the following hold

$$\chi'_a(G \boxtimes H) \leq \min\{\chi'_a(G \Box H) + \chi'_a(G \times H), \chi'_a(G \Box H) + \chi'_a(G \times H)\},$$
Figure 2: The product $P_b \times P_4$ with a component (in bold) included in $P_b \Box P_b$.

$$
\chi'_a(G \boxtimes H) \leq \chi'_a(G \boxdot H) + \chi'_a(G \times H),
$$

$$
\chi''_a(G \boxtimes H) \leq \chi''_a(G \boxdot H) + \chi_a(G)\chi(H).
$$

Proof: Remember that the edge set of $G \boxtimes H$ is the union of the edge set of $G \boxdot H$ and of $G \times H$. To obtain a VD-coloring of $G \boxtimes H$, VD-color the edges of $G \boxdot H$ in $\chi'_a(G \boxdot H)$ colors and properly color the edges of $G \times H$ in $\chi'(G \times H)$ new colors. Then the coloring is clearly proper and the vertices are distinguished by the VD-coloring of $G \boxdot H$. The same goes when exchanging the roles of $G$ and $H$. We then obtain $\chi'_a(G \boxtimes H) \leq \min\{\chi'_a(G \boxdot H) + \chi'(G \times H), \chi'(G \boxdot H) + \chi'_a(G \times H)\}$.

For the AVD-coloring of $G \boxtimes H$, AVD-color the edges of $G \boxdot H$ in $\chi'_a(G \boxdot H)$ colors and AVD-color the edges of $G \times H$ in $\chi'_a(G \times H)$ new colors. This coloring is clearly proper and two adjacent vertices are distinguished either by the colors of the edges of $G \boxdot H$ or by the colors of the edges of $G \times H$.

The third inequality is a direct consequence of Claim 1 since $\chi(G \boxtimes H) \leq \chi(G)\chi(H)$ (see [IK00], page 240).

\subsection{Lexicographic product}

Theorem 17 For any two graphs $G$ and $H$ different from $K_2$, the following hold

$$
\chi'_a(G \circ H) \leq \chi'_a(G) + \chi'_a(H) + (|V(H)| - 1)\chi'(G),
$$
χ′_a(G ∘ H) ≤ χ′_a(G) + χ′_a(H) + (|V(H)| − 1)χ′(G),
χ''_a(G ∘ H) ≤ χ′(G ∘ H) + χ(G)χ(H).

**Proof:** Assume G is of order n ≥ 3 and H is of order m ≥ 3. The graph G ∘ H consists in n copies of H, two copies being linked by a complete bipartite graph K_{m,m} if the corresponding vertices of G are adjacent. Thus, the edges between two copies of H can be decomposed into m perfect matchings. To obtain a VD-coloring of G ∘ H:

- VD-color the edges of each copy of H with χ′_a(H) colors,
- for any edge e of G, color one of the m perfect matchings between the two copies of H corresponding to e with the color of e in a VD-coloring of G in χ′_a(G) new colors,
- properly color the edges of the remaining perfect matchings with (m − 1)χ′(G) new colors.

For the AVD-coloring, the proof is the same as above, replacing VD by AVD and χ′ by χ′_a.

For the TAVD-coloring, as for the strong product, the third inequality follows from Claim 1 since χ(G ∘ H) ≤ χ(G)χ(H) ([IK00], page 246).

7 Concluding remarks

We have obtained bounds for the VD, AVD and TAVD-chromatic indices of the Cartesian product of two graphs and we have shown that some of these bounds are optimal since they allow to determine the AVD and TAVD-chromatic indices of generalized hypercubes. General bounds for the direct, strong and lexicographic products have also been determined. Notice that, despite Theorems 3, 13, 16, 17 have quite simple proofs, they give the exact value of the adjacent vertex distinguishing chromatic index if the factors G and H are such that χ′_a(G) = ∆(G) and χ′_a(H) = ∆(H) (for instance, this is the case for trees with no two adjacent vertices of maximum degree [ZLW02]). However, the bounds we obtained for the TAVD-chromatic index of the direct, strong and lexicographic products do not seem to be very tight. It could be interesting to investigate this more in details.

References


Appendix

\[
c(K_6) = \begin{pmatrix}
x_0 & x_1 & x_2 & x_3 & x_4 & x_5 & \text{Missing colors} \\
x_0 & 1 & 2 & 3 & 4 & 5 & 0,6 \\
x_1 & 1 & 3 & 4 & 5 & 6 & 0,2 \\
x_2 & 2 & 3 & 5 & 6 & 0 & 1,4 \\
x_3 & 3 & 4 & 5 & 0 & 1 & 2,6 \\
x_4 & 4 & 5 & 6 & 0 & 2 & 1,3 \\
x_5 & 5 & 6 & 0 & 1 & 2 & 3,4 \\
\end{pmatrix}
\]

\[
c'(K'_6) = \begin{pmatrix}
x_0' & x_1' & x_2' & x_3' & x_4' & x_5' & \text{Missing colors} \\
x_0' & 4 & 6 & 1 & 3 & 5 & 0,2 \\
x_1' & 4 & 1 & 3 & 5 & 0 & 2,6 \\
x_2' & 6 & 1 & 5 & 0 & 2 & 3,4 \\
x_3' & 1 & 3 & 5 & 2 & 4 & 0,6 \\
x_4' & 3 & 5 & 0 & 2 & 6 & 1,4 \\
x_5' & 5 & 0 & 2 & 4 & 6 & 1,3 \\
\end{pmatrix}
\]

\[
d(K_8) = \begin{pmatrix}
x_0 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & \text{Missing colors} \\
x_0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 0,8 \\
x_1 & 1 & 3 & 4 & 5 & 6 & 7 & 8 & 0,2 \\
x_2 & 2 & 3 & 5 & 6 & 7 & 8 & 0 & 1,4 \\
x_3 & 3 & 4 & 5 & 7 & 8 & 0 & 1 & 2,6 \\
x_4 & 4 & 5 & 6 & 7 & 0 & 1 & 2 & 3,8 \\
x_5 & 5 & 6 & 7 & 8 & 0 & 2 & 4 & 1,3 \\
x_6 & 6 & 7 & 8 & 0 & 1 & 2 & 3 & 4,5 \\
x_7 & 7 & 8 & 0 & 1 & 2 & 4 & 3 & 5,6 \\
\end{pmatrix}
\]
\[ d'(K'_8) = \begin{pmatrix}
  x'_0 & x'_1 & x'_2 & x'_3 & x'_4 & x'_5 & x'_6 & x'_7 & \text{Missing colors} \\
  x'_0 & 3 & 6 & 8 & 1 & 4 & 5 & 7 & 0,2 \\
  x'_1 & 3 & 8 & 1 & 4 & 5 & 7 & 0 & 2,6 \\
  x'_2 & 6 & 8 & 4 & 5 & 7 & 0 & 2 & 1,3 \\
  x'_3 & 8 & 1 & 4 & 7 & 0 & 2 & 3 & 5,6 \\
  x'_4 & 1 & 4 & 5 & 7 & 2 & 3 & 6 & 0,8 \\
  x'_5 & 4 & 5 & 7 & 0 & 2 & 6 & 1 & 3,8 \\
  x'_6 & 5 & 7 & 0 & 2 & 3 & 6 & 8 & 1,4 \\
  x'_7 & 7 & 0 & 2 & 3 & 6 & 1 & 8 & 4,5 \\
\end{pmatrix} \]